A Function Approximator Approach to Nonlinear Systems Analysis

Oreoluwa Albert Ajayi

A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements

for the Degree of

Master of Applied Science (Electrical and Computer Engineering) at

Concordia University

Montréal, Québec, Canada

August 2020

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: **Oreoluwa Albert Ajayi**

Entitled: **A Function Approximator Approach to Nonlinear Systems Analysis**

and submitted in partial fulfillment of the requirements for the degree of

Master of Applied Science (Electrical and Computer Engineering)

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the Final Examining Committee:

______________________ Chair
Dr. Hassan Rivaz

______________________ External Examiner
Dr. Wenfang Xie

______________________ Examiner
Dr. Hassan Rivaz

______________________ Supervisor
Dr. Amir Aghdam

Approved by ______________________
Yousef R. Shayan, Chair
Department of Electrical and Computer Engineering

__________ 2020

____
Mourad Debbabi, Dean
Faculty of Engineering and Computer Science

Abstract

A Function Approximator Approach to Nonlinear Systems Analysis

Oreoluwa Albert Ajayi

A novel fuzzy inference system is introduced with desirable approximation properties for highly nonlinear systems that can be expressed in linear parameter varying form. This fuzzy inference system uses a hashing function to eliminate unnecessary computations and is compared with existing fuzzy inference systems. Furthermore, an application to nonlinear systems state estimation is provided, and the result is compared with that of the Extended Kalman filter. The main benefit of this novel fuzzy inference system is its suitability to resource-constrained embedded control and estimation applications. Furthermore, multidimensional sampling is applied to the state-space variables and it is shown that (de)fuzzification in control systems and (de)modulation in communication systems are analogous. Finally, the values of fuzzy submodels as quantum mechanical objects are explored, for stability analysis and feedback controller synthesis for a class of nonlinear systems, using artificial intelligence approaches. Simulations confirm the effectiveness of the proposed approach.

Acknowledgments

My deep gratitude goes first to Dr. Amir Aghdam for his expert guidance in navigating complex topics in control theory throughout my graduate studies. His unwavering support and mentoring were vital to the completion of this thesis. I thank the Natural Sciences and Engineering research Council of Canada for providing the funding needed to complete this research.

My appreciation also extends to my family and friends for their unwavering support and suggestions for the last two years. This work would not have been possible without my late Father nurturing my curiosity and inspiring me to follow in his footsteps. Finally, this thesis is dedicated to every researcher dedicated to the pursuit of knowledge above all else, despite seemingly insurmountable hurdles.

Contribution of Authors

Oreoluwa Albert Ajayi conceived of the presented ideas and developed the theory as well as the analytical calculations. Mina Babahaji performed the numerical simulations. Dr. Amir Aghdam supervised the project and verified the analytical methods and numerical simulations.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

Industries that integrate safety-critical applications are typically slower than others in adopting novel technology, due to the rigour required for demonstrating compliance to safety objectives. An example is the aerospace industry, where original equipment manufacturers (OEMs) prefer to use components which have service history and field reliability data available, as this makes quantitative failure analysis feasible. As a result, OEMs must weigh the cost of demonstrating that their products are adequately safe against expected financial returns, when performing design trade studies that incorporate emerging technology.

Prior to the COVID-19 pandemic, many OEMs concluded trade studies and commenced prototype development in urban air mobility and unmanned drone delivery, in order to increase their market capitalization. Aviation experts noted that the profitability of such ventures may be hampered by the ongoing pilot shortage in the aviation industry, due to the high initial and recurrent costs of training commercial pilots on new aircraft types. The aftershocks of the COVID-19 pandemic has reduced air transport demand globally, leaving fewer aircraft for pilots to fly.

In order to maintain adequate safety of flight (SOF) as airline operations recover, flight crew will have to undergo costly recency training. Advancements in artificial intelligence (AI), machine learning (ML) algorithms and computer hardware over the last two decades have made it possible to automate more actions that would normally be performed by a flight crew in an aircraft. Promising results and lessons have been learnt from demonstrator projects such as NASA's learn-to-fly project [1] and the intelligent flight control system flight research project. In this regard, the European Union Aviation Safety Agency (EASA) in references [2] and [3] has highlighted several AI/ML trustworthiness issues such as:

- Traditional development assurance frameworks are not adapted to machine learning.
- Difficulties in keeping a comprehensive description of the intended function.
- Lack of predictability and explainability of the ML application behaviour.
- Lack of guarantee of robustness and of no "unintended function".
- Lack of standardised methods for evaluation of the operational performance.
- Complexity of architectures and algorithms.

Future regulatory frameworks for AI applications in aviation will be structured in a manner that guarantees safety, security and public interest are not compromised. The onus is on OEMs to guarantee the deterministic and explainable behaviour of their systems. Lessons learnt from past aircraft certification programs has shown that explainability tends to reduce with increased complexity of architectures and algorithms. As a result, the tendency to treat complex systems as black boxes also increase, making unintended functionality more likely, even with the implementation of rigorous hardware/software requirement and change management processes.

1.2 Objectives

The objectives of this thesis are as follows:

- Define a strategy for intelligent control of multi-input multi-output (MIMO) nonlinear systems.
- Bridge the gap between research advancements and industry practice.
- Eliminate blackbox impediments to AI verification and validation, by creating a rule-based explainability platform.

1.3 Literature Review

A literature survey provides clues to the following questions:

- How can AI be applied to embedded control applications? Specifically, how can hardware complexity be overcome?

- What design and testing strategy is better for modelling and control? Black box (i.e. model-free) or white box (i.e. model-based)?

In reference [4], the authors highlight that machine learning systems are particularly prone to incurring system-level technical debt, which is difficult to detect. This is consistent with the findings of the authors in reference [3]. To overcome computation complexity in hardware-constrained applications, the authors of reference [5] indicate the need for a better theoretical understanding of hashing neural networks.

In the foreward of reference [6], Lotfi Zadeh states that *"Generally, fuzzy systems work well when we can use experience or introspection to articulate the fuzzy if-then rules. When we cannot do this, we may need neural-network techniques to generate the rules"*. The use of introspection or experience refers to contexts where a mathematical model of physical system behaviour can be developed; indicating that fuzzy methods are better suited to model-based problems, whereas neural networks are suited to model-free problems. Further insight on model-based fuzzy methods is provided in reference [7], where the authors introduce a function approximator known as the Takagi-Sugeno fuzzy inference system (TS-FIS), which approximates a nonlinear system using a weighted sum of linear state-space submodels, provided the local/global sector nonlinearity condition is satisfied over the state domain of interest. The weights of the TS-FIS linear submodels are determined by fuzzy membership functions that are defined for each premise variable.

The author in reference [8] suggests that fuzzy membership functions are quantum values, which indicates that fuzzy systems can be represented as quantum mechanics objects. In reference [9], classical mechanics is defined by the author as a crisp limit of fuzzy quantum mechanics. The quantum representation and implementation of linear time invariant systems in the presence of noise is discussed in reference [10]. Note that approximating a nonlinear system using the TS-FIS will result in approximation error, which is regarded as information loss, and is associated with increased entropy [11]. The author in reference [12] alludes to the wavenumber-limited nature of all physical systems by deriving the Bekenstein bound; which is a finite bound on the entropy-to-energy ratio of physical systems. The Bekenstein bound and the second law of thermodynamics are used in reference [13] to derive the theory of general relativity (an extension of special relativity); which states that space and time are a single entity called space-time, whose curvature is affected by the presence of energy or matter. Reference [13] elaborately defines the Einstein relation as an equation of state. Applications of the theory of general relativity to nonlinear systems control, data mining, and navigational systems are given in references [14], [15] and [16].

In references [17], [18] and [19], the application of multidimensional sampling and reconstruction to wavenumber-limited multivariable functions is demonstrated. The definition of stability for nonlinear state-space models with equilibirum points and limit cycles is provided in reference [20], as well as Lyapunov methods for demonstrating stability. Using Lyapunov stability theory, the authors in reference [21] provide sufficient conditions for the stability of a nonlinear system approximated using the TS-FIS. The authors in reference [22] highlight a relationship between the Lyapunov equation for linear systems and the generalized Einstein relation.

1.4 Contribution

The contributions of this thesis are as follows:

- A computationally efficient fuzzy inference system, that relaxes the sector-nonlinearity constraint of the TS-FIS.
- Use of multidimensional sampling as a systematic (rather than intuitive) method of fuzzy modelling.
- Explainability of fuzzy AI/ML systems from a theoretical physics standpoint, with applications to nonlinear observer-based controller synthesis.
- Reduced algorithm complexity for the mathematical analysis of highly nonlinear systems.

1.5 Thesis Structure

Chapter II introduces a novel model-based fuzzy inference system, which is referred to as the Ore fuzzy inference system (Ore-FIS). The computational superiority of the Ore-FIS over the TS-FIS due to the H(X) algorithm is explained, and proof of the function approximation properties of the Ore-FIS is provided. A nonlinear observer based on the Ore-FIS is derived and extended to discrete-time, by modifying the Kalman filter algorithm, based on similar work in [23]. This extension is called a hashing based neurofuzzy network Kalman filter (HBNN-KF) and is compared with the extended Kalman filter, via simulation, for the state-estimation of a unicycle mobile robot. The EKF is chosen for comparison rather than the unscented Kalman filter (UKF) because:

- Typically, the UKF processing time is significantly larger than that of the EKF, although the UKF estimates better than the EKF using noisy measurements.
- It is more common to use the EKF for nonlinear state estimation in aerospace applications.

In Chapter III, the wavenumber-limited nature of physical nonlinear systems is used to construct a fuzzy space and subsequently determine an appropriate number of fuzzy sets per premise variable. This is done by extending sampling in time to sampling in space. It is shown that fuzzification and defuzzification is the state-space equivalent of modulation and demodulation, and a metric to help determine whether fuzzification or linearization is appropriate for a control problem is derived. Chapter III concludes by providing sufficient stability conditions for linear parameter varying (LPV) nonlinear systems approximated using the Ore-FIS, and discusses considerations for stability analysis of such systems in the presence of time-varying uncertainty. Chapter IV concludes the work presented in the thesis.

Chapter 2

Nonlinear Systems State-Estimation

Overview

A novel fuzzy inference system is introduced with desirable approximation properties for highly nonlinear systems that can be expressed in linear parameter varying form. This fuzzy inference system uses a hashing function to eliminate unnecessary computations and is compared with existing fuzzy inference systems. Furthermore, an application to nonlinear systems state estimation is provided, and the result is compared with that of the extended Kalman filter. The main benefit of this novel fuzzy inference system is its suitability to resource-constrained embedded control and estimation applications.

2.1 Introduction

A critical step in the design of a controller for a system is the development of a suitable model for the system, if it is not available *a priori*. A model can be defined as a mathematical construct intended to quantify the nature of observations of physical phenomena. It serves as a bridge between theoretical abstractions and pragmatic considerations, and is typically constructed based on measurements obtained from the phenomena of interest, in addition to the practitioner's expert knowledge, in order to predict outputs of said phenomena as accurately as desired. Due to the existence of measurement error in the process, any model can be expected to predict phenomena with some uncertainty; hence the common trope that a controller is only as good as the constructed plant model used in its synthesis.

Note that every system identification method relies on measurements [24], [25], which are crisp and inherently incapable of providing complete information about the signal(s). It is therefore important to also model parametric uncertainty using a probabilistic approach [26], [27]. A drawback of this approach, however, is that it is only suited to dealing with uncertainties of a random nature, not systemic-induced uncertainties. In practice, mitigating all systemic uncertainty contributions is as challenging as identifying all possible systemic contributions [28], [29].

Zadeh states in [6] that ”generally, fuzzy systems work well when we can use experience or introspection to articulate the fuzzy if-then rules. When we cannot do this, we may need neural network techniques to generate the rules”. This is exemplified by the authors of [23] in what is commonly referred to as the Takagi-Sugeno Fuzzy inference system (TS-FIS). The TS-FIS can approximate a given nonlinear model to any desired degree of accuracy, provided the sector nonlinearity condition is satisfied over the domain of interest.

In this work, a novel Kalman filter is introduced based on a fuzzy inference system (FIS) that takes the states of a modelled nonlinear system as the premise variables (unlike the TS-FIS which takes functions of the system states as premise variables). This new FIS, which we refer to as the Ore-FIS, is proven to approximate a given nonlinear model to any desired degree of accuracy, provided the Lipschitz condition is satisfied over the domain of interest. Conditions under which the states of a nonlinear system approximated by the Ore-FIS can be accurately estimated from the system's output measurements are derived. [30] analytically compares fuzzy and crisp measurements and shows that random-fuzzy variables contain more information and mitigate uncertainty.

The remainder of this chapter is organized as follows. In Section 2.2, the function approximation problem is formulated. In Section 2.3, the nonlinear observer problem in the context of the Ore-FIS is presented. In Section 2.4, a Kalman filter extension is made and subsequently demonstrated in Section 2.5.

2.2 Problem Formulation

2.2.1 Ore Fuzzy Inference System

Consider the Nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x) \end{aligned} \tag{1}$$

Assume this system can be rewritten as

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u \\ y &= C(x)x \end{aligned} \tag{2}$$

where $A(x) \in R^{n\times n}$, $B(x) \in R^{n\times m}$, $C(x) \in R^{r\times n}$, $x \in R^n$, $u \in R^m$ and $y \in R^r$. Assume also $x_{i,min} \leq x_i \leq x_{i,max}$ for $i \in N_n$. Thus, $A(x)$, $B(x)$ and $C(x)$ have n^2, nm and rn elements, respectively. Therefore, there is a total number of not more than $n^2 + nm + rn$ TS-FIS premise functions for the TS-FIS as it takes the premise variables from the pool of candidate premise functions.

Now, let a subset of the state variables be considered as premise variables. Then, the function approximation properties may hold provided the Lipschitz condition is satisfied on a compact set defined on the state variables of the physical system to be approximated. In this proposed FIS, the number of premise variables is reduced to n in comparison with the TS-FIS.

By reducing the premise variables, the total number of rules/submodels and consequently computation time can be decreased. This proposed FIS will be referred to as the Ore-FIS. It is assumed that there are s fuzzy sets per premise variable in the Ore-FIS, and that $x_1, x_2, \ldots, x_n$ form the pool of candidate premise variables, p of which are selected. The premise variables can be selected such that they are the independent state variables appearing in the TS-FIS pool of candidate premise functions.

Definition 1: F_{ij} is the i-th fuzzy set of x_j and $M_{F_{ij}}$ is the value of membership function of the i-th fuzzy set of premise variable x_j given a crisp value of x_j. In this case, $i \in N_s$, $j \in N_p$, and $M_{F_{ij}}$ has the following properties:

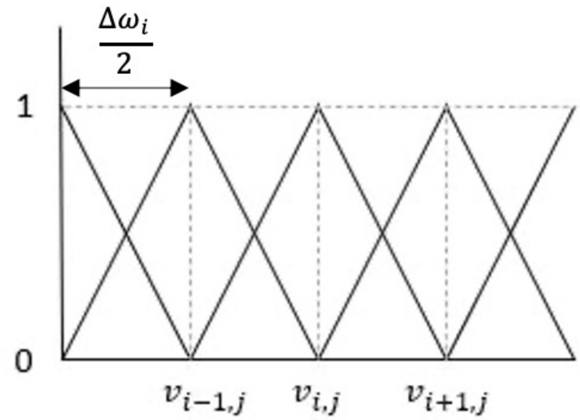

Figure 2.1: The triangular membership functions

(1) $0 \leq M_{F_{ij}} \leq 1, i \neq j$;

(2) the membership functions are triangular, and

(3) the premise variables are a subset of the independent state variables.

Fig. 2.1 illustrates a triangular membership function. As it can be observed, $v_{i,j}$ is the vertex of the i-th fuzzy set of premise variable x_j. Moreover, it can be observed that $M_{F_{i,j}} = 1$ if and only if $x_j = v_{i,j}$.

The width of a fuzzy set of premise variable x_j is shown by $\triangle w_j$. It is straightforward to show that $\triangle w_j \propto \frac{1}{s}$. It is important to note that the inequality $x_{j,min} \leq x_j \leq x_{j,max}$ implies that all values of x_j must lie within at least one fuzzy set. Note also that if there are s fuzzy sets per premise variables, and there are p premise variables, then from basic combinatorics, there will be s^p rules.

In the TS-FIS, all possible fuzzy rules/submodels are computed even though some rules have no weight. If a rule has at least one membership function with a value of zero, the relative weight of that rule will be zero.

Definition 2: An activated membership function is one that has a non-zero value for a given crisp premise variable.

In the Ore-FIS, it is observed that a crisp value of $x_{j,min} \leq x_j \leq x_{j,max}$ cannot belong to more than 2 fuzzy sets (this can be verified by inspecting Fig. 2.1). Hence, each premise variable must activate at least one membership function and a maximum of two membership functions. Therefore, at any point in time, a nonlinear system is represented by at least one submodel (or rule) and at most, 2^p submodels (or rules).

Each fuzzy rule can be described by a 5-tuple $\Psi_i = (h_i(x),\ \overrightarrow{v_i},\ A(\overrightarrow{v_i}),\ B(\overrightarrow{v_i}),\ C(\overrightarrow{v_i}))$, where $\overrightarrow{v_i} \in R^p$. In the above expression, Ψ_i refers to the i-th rule, $h_i(x)$ is a scalar representing the relative weight of rule i, $\overrightarrow{v_i}$ is a column vector constructed such that each element is the vertex of the fuzzy sets of the i-th rule. $A(\overrightarrow{v_i})$ is the matrix obtained by replacing the premise variable in $A(x)$ with the vertices of the i-th rule. Likewise, $B(\overrightarrow{v_i})$ and $C(\overrightarrow{v_i}))$ are obtained by replacing the premise variables in $B(x)$ and $C(x)$ respectively with the vertices of the i-th rule.

Definition 3: Fuzzification in the Ore-FIS is a transformation from the real-space to the fuzzy space using a hashing function $H(x)$. Therefore, models approximated by the Ore-FIS are hashing-based neurofuzzy networks (HBNNs).

Definition 4: The dimension of the fuzzy space is $p + 1$, where the points are $(\overrightarrow{v_i}, h_i(x))$ for $i = 1, \ldots, 2^p$.

Remark 1: The fuzzy rules exist in the fuzzy space and the submodels representing the nonlinear function can be constructed with the information in the fuzzy 5-tuple Ψ_i.

Remark 2: $H(x)$ maps crisp values in the real space to fuzzy submodels associated with $(\overrightarrow{v_i}, h_i(x))$.

Remark 3: $H(x)$ is such that only the activated fuzzy sets are computed during fuzzification.

The benefit of using $H(x)$ is that all computations associated with rules with zero relative weight are avoided. Since there are at least $s^p - 2^p$ submodels, the computational savings associated with $H(x)$ are significant. Given a crisp value of premise variable x_j, it is observed from Fig. 2.1 that the vertices of the fuzzy sets of x_j are ordered in the form of an arithmetic series written below

$$v_{i,j} = v_{1,j} + (i-1)\frac{\Delta w}{2} \tag{3}$$

2.3 Main Results

2.3.1 The $H(x)$ Algorithm

Consider p premise variables and s fuzzy sets per premise variable. Let $C_i(t)$ denote the crisp value of the i-th premise variable at time t. The upper and lower vertices saddling $C_i(t)$ are given by the following formulas

$$
\begin{aligned}
v_{lower} &= v_{1,i} + floor(\tfrac{c_i(t)-v_{1,i}}{\Delta w_i} \times 2) \times \tfrac{\Delta w_i}{2} \\
v_{upper} &= v_{1,i} + ceil(\tfrac{c_i(t)-v_{1,i}}{\Delta w_i} \times 2) \times \tfrac{\Delta w_i}{2}
\end{aligned} \tag{4}
$$

Remark 4: If $floor(\frac{c_i(t)-v_{1,i}}{\Delta w_i} \times 2) = ceil(\frac{c_i(t)-v_{1,i}}{\Delta w_i} \times 2)$ then the crisp value is a member of exactly one fuzzy set.

Across all premise variables, there are a maximum of $2p$ and a minimum of p fuzzy sets to be formed. Consequently, a maximum of 2^p rules and a minimum of one rule are to be formed by combining the activated fuzzy sets of all the premise variables.

If a fuzzy rule is described by $\Psi_i = (h_i(x),\ \overrightarrow{v_i},\ A(\overrightarrow{v_i}),\ B(\overrightarrow{v_i}),\ C(\overrightarrow{v_i}))$, its associated submodel is given by

$$
\begin{aligned}
\dot{x} &= A(v_i)x + B(v_i)u \\
y &= C(v_i)x
\end{aligned} \tag{5}
$$

Recall that v_i is derived from either v_{lower} or v_{upper} calculated for each premise variable.

2.3.2 A Separation Principle for Nonlinear Systems

Consider system (28). This system after fuzzification has the following format with μ submodels having non-zero relative weight

$$
\dot{x} = \sum_{i=1}^{\mu}[A(v_i)x \ + \ B(v_i)u]h(\Psi_i) + e_1(x) \tag{6a}
$$

$$
\begin{aligned}
y &= \textstyle\sum_{i=1}^{\mu} C(v_i)xh(\Psi_i) + e_2(x) \\
y &= \textstyle\sum_{i=1}^{\mu} \widehat{y}_i h(\Psi_i) + e_2(x)
\end{aligned} \tag{6b}
$$

where $1 \leq \mu \leq 2^p$ and $\widehat{y}_i = C(v_i)x$. Note that any arbitrary error $e(x) = \sum_{i=1}^{\mu} e(x)h(\Psi_i)$

since $\sum_{i=1}^{\mu} h(\Psi_i) = 1$. The fuzzified nonlinear system can now be rewritten as

$$\begin{aligned}
\dot{x} &= \textstyle\sum_{i=1}^{\mu}[A(v_i)x \ + \ B(v_i)u + e_1(x)]h(\Psi_i) \\
y &= \textstyle\sum_{i=1}^{\mu}[C(v_i)x + e_2(x)]h(\Psi_i) \\
y &= \textstyle\sum_{i=1}^{\mu}[\widehat{y}_i + e_2(x)]h(\Psi_i)
\end{aligned} \tag{7}$$

The structure of the nonlinear fuzzy observer is assumed to be as follows

$$\dot{\widehat{x}} = \sum_{i=1}^{\mu}[A_c(v_i)\widehat{x} + L(v_i)\widehat{y}_i + z(v_i)]h(\Psi_i) \tag{8}$$

Let $e_0 = x - \widehat{x}$. This implies $\dot{e}_0 = \dot{x} - \dot{\widehat{x}}$.

Theorem 1: If the pair $(C(v_i), A(v_i))$ satisfy the detectability test for all μ submodels, then $\exists L(v_i)\ for\ i = 1, \ldots, \mu$ such that $\dot{e}_0 - e_1(x) \to 0$ as $t \to \infty$.

Proof: Substituting $\dot{x}$ and $\dot{\widehat{x}}$ from (29a) and (8), respectively, the following equation is obtained

$$\begin{aligned}
\dot{e}_0 &= \sum_{i=1}^{\mu}[A(v_i)x + B(v_i)u + e_1(x) \\
&-A_c(v_i)\widehat{x} - L(v_i)\widehat{y} - z(v_i)]h(\Psi_i) \\
\dot{e}_0 &= \sum_{i=1}^{\mu}[(A(v_i) - L(v_i)C(v_i))x - A_c(v_i)\widehat{x} \\
&+B(v_i)u - z(v_i) + e_1(x)]h(\Psi_i)
\end{aligned} \tag{9}$$

Assume $A_c(v_i) = A(v_i) - L(v_i)C(v_i)$ and also $z(v_i) = B(v_i)u$; (9) is then rewritten as

$$\dot{e}_0 = \sum_{i=1}^{\mu}[A_c(v_i)(x - \widehat{x}) + e_1(x)]h(\Psi_i) \tag{10}$$

Remark 5: Theorem 1 implies the HBNN observer output will always contain a bias term $e_1(x)$. By proving the function approximation properties of the Ore-FIS, it follows that the term $e_1(x)$ can be made arbitrarily small.

Theorem 2: Assume that the region of operation defined by the state variables of a physical system are within a compact hyper-rectangle, and that the nonlinear state-space representation satisfies the Lipschitz condition within such a polytope. Then, said physical system can be approximated to any degree of accuracy by the Ore-FIS.

Proof: Consider the nonlinear function $F(x) = A(x)x$, where $x \in R^n$ and $A(x) \in R^{b\times n}$. Let $a_1(x), \ldots, a_b(x)$ be the rows of matrix $A(x)$. Then, $f_i(x) = a_i(x)x \quad, \forall i \in N_b$. If the Ore-FIS is used to approximate $F(x)$, then

$$f_i(x) = \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)x + e_i(x) \tag{11}$$

Equation (11) can be rewritten as

$$f_i(x) - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)x = e_i(x) \tag{12}$$

Taking the norm 2 of both sides of the above equation yields $\left\| f_i(x) - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)x \right\| = \|e_i(x)\|$. The term $\sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)v_j$ is added and subtracted from the left-hand side of (12) to obtain

$$\begin{aligned} \left\| f_i(x) - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)x - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)(v_j - v_j) \right\| \\ = \|e_i(x)\| \end{aligned} \tag{13}$$

The equation above can be rewritten as

$$\begin{aligned} \left\| f_i(x) - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)v_j - \sum_{j=1}^{\mu} h(\psi_j)a_i(v_j)(x - v_j) \right\| \\ = \|e_i(x)\| \end{aligned} \tag{14}$$

Note that $f_i(v_j) = a_i(v_j)v_j$ and also $f_i(x) = \sum_{j=1}^{\mu} h(\psi_j)f_i(x)$ because $\sum_{j=1}^{\mu} h(\psi_j) = 1$. Thus, equation (14) can be rewritten as

$$
\begin{aligned}
&\|e_i(x)\| = \| \textstyle\sum_{j=1}^{\mu} h(\psi_j) f_i(x) \\
&- \textstyle\sum_{j=1}^{\mu} h(\psi_j) f_i(v_j) - \sum_{j=1}^{\mu} h(\psi_j) a_i(v_j)(x - v_j) \;\|
\end{aligned}
\tag{15}
$$

from which, it is straightforward to derive the following inequality

$$
\begin{aligned}
&\|e_i(x)\| \leq \left\| \textstyle\sum_{j=1}^{\mu} h(\psi_j)(f_i(x) - f_i(v_j) \right\| \\
&+ \left\| \textstyle\sum_{j=1}^{\mu} h(\psi_j) a_i(v_j)(x - v_j) \right\|
\end{aligned}
\tag{16}
$$

Furthermore, assume that $f_i(x) : D \to R$, and that it is Lipschitz on D. So $\|f_i(x) - f_i(v_j)\| \leq L\,\|x - v_j\|$, where L is a positive constant. It is concluded that

$$
\begin{aligned}
\|e_i(x)\| &\leq \left\| \textstyle\sum_{j=1}^{\mu} h(\psi_j) L(x - v_j) \right\| + \left\| \textstyle\sum_{j=1}^{\mu} h(\psi_j) a_i(v_j)(x - v_j) \right\| \\
&\leq \textstyle\sum_{j=1}^{\mu} h(\psi_j) L\,\|x - v_j\| + \sum_{j=1}^{\mu} h(\psi_j)\,\|a_i(v_j)\|\,\|x - v_j\|
\end{aligned}
$$

As a result

$$
\|e_i(x)\| \leq \sum_{j=1}^{\mu} h(\psi_j)(L + \|a_i(v_j)\|)(\|x - v_j\|) \tag{17}
$$

Denote by a_{im} the smallest number greater than $\|a_i(v_j)\|$ for all values of v_j, and note that a_{im} is finite. Note also that if a crisp value is a member of a fuzzy set, then the absolute value of the difference between the said crisp value and the vertex of the associated membership function is less than or equal to $\frac{\triangle w}{2}$, where $\triangle w$ is the width of the fuzzy set of the premise variable associated with the crisp value. Since $\triangle w \to 0$ as $s \to \infty$, $\|e_i(x)\| \to 0$ as well. This proves function approximation property of the Ore-FIS.

2.4 HBNN Applications

The modified Kalman filter based on the TS-FIS is presented in [23]. The main difference between the Ore-FIS and TS-FIS is the choice of premise variables. Therefore, a similar algorithm is developed for the Ore-FIS. Consider the system (2), and approximate it by the Ore-FIS as

$$\begin{aligned} \dot{x} &= \textstyle\sum_{i=1}^{\mu} h(\Psi_i)A(v_i)x + \sum_{i=1}^{\mu} h(\Psi_i)B(v_i)u \\ y &= \textstyle\sum_{i=1}^{\mu} h(\Psi_i)C(v_i)x \end{aligned} \tag{18}$$

In discrete time, assume the system is represented by

$$\begin{aligned} x(k+1) &= \textstyle\sum_{i=1}^{\mu} h(\Psi_i)\widehat{A}(v_i)x + \sum_{i=1}^{\mu} h(\Psi_i)\widehat{B}(v_i)u \\ y(k) &= \textstyle\sum_{i=1}^{\mu} h(\Psi_i)\widehat{C}(v_i)x \end{aligned} \tag{19}$$

Define $\bar{A}_k := \sum_{i=1}^{\mu} h(\Psi_i)\widehat{A}(v_i)$, $\bar{B}_k := \sum_{i=1}^{\mu} h(\Psi_i)\widehat{B}(v_i)$ and $\bar{C}_k := \sum_{i=1}^{\mu} h(\Psi_i)\widehat{C}(v_i)$. We can then update the Kalman filter by the following steps:

(1) Calculate priory state estimate for the next iteration using
$\hat{x}^- = \bar{A}_{k-1}\hat{x}_{k-1} + \bar{B}_{k-1}u_{k-1}$

(2) Calculate priory error covariance matrix using
$P_k^- = \bar{A}_{k-1}P_{k-1}\bar{A}_{k-1}^T + Q_{k-1}$

(3) Calculate Kalman gain using
$K_k = P_k^- \bar{C}_k^T \left(\bar{C}_k P_k^- \bar{C}_k^T + R_k\right)^T$

(4) Calculate updated state using
$\hat{x}_k = \hat{x}_k^- + K_k \left(y_k - \bar{C}_k \hat{x}_k^-\right)$

(5) Calculate updated error covariance matrix using
$P_k = \left(I - k_k \bar{C}_k\right) P_k^-$

Remark 6: e_1 is the error due to state function approximation, e_2 is the error due to approximation of the output function, e_0 is the error due to observation and e_i is the error from approximating the i-th row of $F(x) = A(x)x$. Q_t is the covariance of the noise process matrix and R_t is the covariance of the measurement noise, both of which are considered Gaussian in the Kalman filter.

Lets consider an application of the HBNN-KF to the attitude determination of a rigid body (e.g. a small satellite) in earth's orbit.

$$
\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} k_1\omega_2\omega_3 + c_1u_1 \\ k_2\omega_1\omega_3 + c_2u_2 \\ k_1\omega_2\omega_3 + c_3u_3 \end{bmatrix} \tag{20}
$$

It is possible to write this equation in multiple LPV forms such as

$$
\begin{aligned}
\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = &\begin{bmatrix} 0 & 0 & k_1\omega_3 \\ k_2\omega_3 & 0 & 0 \\ 0 & k_3\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \\
&+ \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
\end{aligned} \tag{21}
$$

The applications of small satellite for deterministic space-to-earth and earth-to-space ethernet communications in low earth orbit (LEO) has generated a lot of research interest. LEO satellites cover less of the earths surface when compared with geostationary satellites. Additionally, the low deployment altitude (600Km to 1,500Km) means the time required to complete a full orbit is much shorter. This means that higher accuracy is required for antenna orientation.

Phased-array antennas can be steered electrically (beam-forming) or mechanically using the attitude determination and control system (ADCS). If a suitable model of sensor output as a function of the system states can be constructed using MEMS mathematical modelling [31], then HBNNs may be suitable for observer design and controller synthesis. This can be an alternate adaptive and robust solution for an ADCS.

The feedback gain may be obtained online if a sufficiently fast solution of the Ricatti equation is obtainable for each activated fuzzy submodel [32]. The feedback gain may also be obtained if a linear matrix inequality (LMI) problem[33] is solvable online using interior point methods. This will require hardware in the loop (HIL) simulations with a real time operating system (RTOS). For the purpose, it will be necessary to write interior-point LMI algorithms in C/C++, as this will allow the use of RTOS on miniature satellites for deterministic ethernet, with the OSI physical layer operating at radio frequencies greater than 6.0GHz [34].

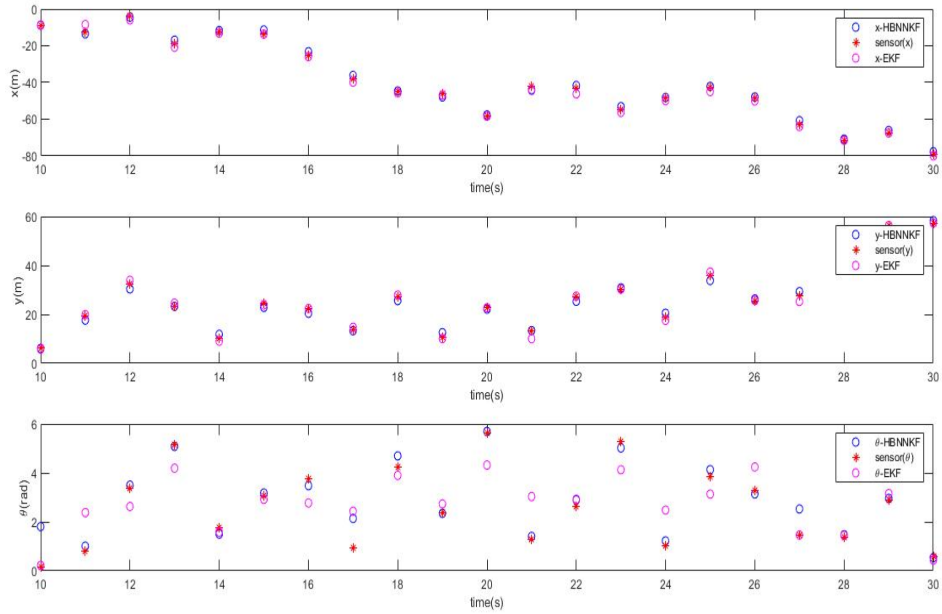

Figure 2.2: The states estimated by the HBNN-KF, EKF and sensor output

2.5 Simulation Results for Mobile Robot Tracking

In this section, a mobile robot system will be tracked using the HBNN-KF and the results are compared with the EKF. Consider the following model of a unicycle mobile robot in [35].

$$\begin{bmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{\theta}_t \end{bmatrix} = \begin{bmatrix} r\frac{v_R+v_L}{2}\cos\theta_t \\ r\frac{v_R+v_L}{2}\sin\theta_t \\ r\frac{v_R-v_L}{b} \end{bmatrix} \tag{22}$$

where v_L and v_R are inputs representing the linear velocity of the left and right wheels, b is the distance between two wheels, and r is the radius of the wheels. Furthermore, x, y and θ are the states of the robot, specifying its location. The discrete time approximation with constant sampling time T is

$$\begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + (T)r\frac{v_R+v_L}{2}\cos\theta_{k-1} \\ y_{k-1} + (T)r\frac{v_R+v_L}{2}\sin\theta_{k-1} \\ \theta_{k-1} + (T)r\frac{v_R-v_L}{b} \end{bmatrix} \tag{23}$$

Assume r and b are equal to $0.5m$ and $1.0m$, respectively. The system (23) can be rewritten the following form

$$x_k = Ax_{k-1} + B(x_{k-1})u \tag{24}$$

where $A = I_{3\times 3}$ and $B(x)$ is equal to

$$B(x) = \begin{bmatrix} \frac{1}{4}T\cos\theta_{k-1} & \frac{1}{4}T\cos\theta_{k-1} \\ \frac{1}{4}T\sin\theta_{k-1} & \frac{1}{4}T\sin\theta_{k-1} \\ 0.5T & -0.5T \end{bmatrix} \tag{25}$$

(note that $u = \begin{bmatrix} v_R \\ v_L \end{bmatrix}$). The location of the robot is considered to be the output, hence $C = I_{3\times 3}$. Since $A(x)$ and $C(x)$ are linear, the premise functions only contain the nonlinear elements of $B(x)$. The vector of premise function is given by

$$F(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} f_1(\theta) \\ f_2(\theta) \end{pmatrix} \tag{26}$$

The premise functions depends on only one variable (i.e. $p = 1$); thus the only premise variable is θ. The simulations are run by MATLAB/SIMULINK on a personal computer with an 4-core i5-5200U processor at 2.2 GHz, and 8GB DDR3-SODIMM. Fig. 3.2 shows the state estimation results using the HBNN-KF and EKF. Table 1 presents the mean square error (MSE) of the x, y and θ for different values of R using the EKF, while Table 2 gives the results for the HBNN-KF (R_0 is assumed [0.2, 0, 0; 0, 0.2, 0; 0, 0, 1]). The lower MSE of the HBNN-KF implies that it is a more accurate state estimator.

Table 2.1: Comparing the MSE of EKF at different sensor noise values

MSE(EKF)	R=0.5I	R=R_0	R=I	R=2R_0	R=5I
x	10.2345	8.5804	13.9509	22.2357	56.5626
y	5.4315	14.2908	15.2189	40.7898	42.3878
θ	1.5102	3.6157	3.0686	4.1798	6.9931

Table 2.2: Comparing the MSE of the HBNN-KF at different sensor noise values

MSE(HBNN -KF)	R=0.5I	R=R_0	R=I	R=2R_0	R=5I
x	2.1317	0.7696	4.1761	1.7628	12.7028
y	2.4964	1.1262	4.4924	2.0267	8.8892
θ	0.6960	1.1852	1.4588	1.2000	3.5040

On the other hand, it is observed that the computation time for the EKF is 0.412s and for HBNN-KF is 0.665s in this application. However, an objective comparison cannot be made based only on the computing time. This is due to the fact that the HBNN-KF code implementation is suboptimal because of rapid prototyping and also because the computation time is application-dependent. An objective comparison of computing time for both the EKF and HBNN-KF must be performed on a real time system with hardware-in-the-loop to give realistic results.

2.6 Conclusion

The function approximation and state estimation properties of the hashing based neurofuzzy network Kalman filter (HBNN-KF) have been demonstrated theoretically and by simulation. One obvious benefit of the HBNN-KF is that stochastic/non-deterministic behavior is consistent with fuzzy if-then rules. The other benefit of this method is the ability to increase information gain by arbitrarily reducing approximation error. It is evident from the simulation case that the HBNN-KF is more robust and has a lower covariance than the extended Kalman filter (EKF). Future work will implement code optimizations for the HBNN-KF and make comparisons with the EKF or UKF for satellite attitude determination applications, with detailed attitude sensor models included in the simulations.

Chapter 3

Nonlinear Systems Stability and Control

Overview

A novel fuzzy inference system is introduced, that has desirable approximation properties for highly nonlinear systems that can be expressed in linear parameter varying form. This novel fuzzy inference system uses a hashing function to eliminate unnecessary computation. Furthermore, multidimensional sampling is applied to the state-space variables and it is shown that (de)fuzzification in control systems and (de)modulation in communication systems are analogous. Finally, the values of fuzzy submodels as quantum mechanical objects are explored, for stability analysis and feedback controller synthesis for a class of nonlinear systems, using artificial intelligence approaches. Simulations confirm the effectiveness of the proposed approach.

3.1 Introduction

The state-space representation is a physical abstraction that can be used to mathematically represent multi-variable systems. Such a model is often nonlinear, but is often linearized around an equilibrium point which may be stable or unstable. Although typically the stability of nonlinear systems is centered about linearized equilibrium points, the presence of phenomena such as non-vanishing perturbations renders stability analysis of this nature inapplicable. To address this hurdle, one can use the Lyapunov stability criterion and directly apply it to the nonlinear system [21] [20]

[22].

An alternative to modelling nonlinear systems around equilibrium points is the use of function approximators (e.g., neural networks and neurofuzzy networks). In practice, traditional artificial intelligence (AI) and machine learning (ML) approaches such as neural networks are prone to incurring technical debt, which is very difficult to detect, as it exists at the system level rather than the code level [4]. This is consistent with the findings in [3], where black-box design methodologies are highlighted as an impediment to AI trustworthiness (which includes verification, validation and explainability). In the foreword of [6], Lotfi Zadeh states that *"Generally, fuzzy systems work well when we can use experience or introspection to articulate the fuzzy if-then rules. When we cannot do this, we may need neural-network techniques to generate the rules"*. The use of introspection or experience refers to contexts where a mathematical model of physical system behaviour can be developed by an expert; indicating that fuzzy methods are better suited to model-based (or white-box) problems, whereas neural networks are suited to model-free (or black-box) problems. Further insight on model-based fuzzy methods is provided in [7], where the authors introduce a function approximator known as the Takagi-Sugeno fuzzy inference system (TS-FIS), which approximates a nonlinear system using a weighted sum of linear state-space submodels, provided the local/global sector nonlinearity condition is satisfied over the state domain of interest.

In [36], a fuzzy inference system known as the Ore fuzzy inference system (Ore-FIS) is introduced and is demonstrated to be computationally superior to the TS-FIS. This new fuzzy inference system is proven to approximate a given nonlinear model to any desired degree of accuracy, provided the Lipschitz condition is satisfied over the domain of interest. The Ore-FIS is particularly useful in embedded applications, where accurate results in real-time are desired, despite measurement uncertainty and hardware constraints imposed by mission requirements.

The rest of this work is structured as follows. In Section 3.2, we formulate the problem of nonlinear systems modelling using fuzzy quantum mechanics and apply the Petersen-Middleton theorem to fuzzification and defuzzification in the Ore-FIS. In Section 3.3, we use fuzzy quantum mechanics as a theoretical framework for the Ore-FIS and mathematically derive stability conditions. In Section 3.4, some practical insights are given, as well as a summary of computational limitations that will be addressed in future work by hardware-in-the-loop simulation. Section 3.5

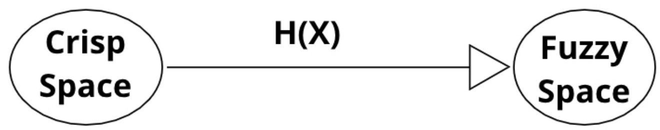

Figure 3.1: Crisp space to fuzzy space transformation

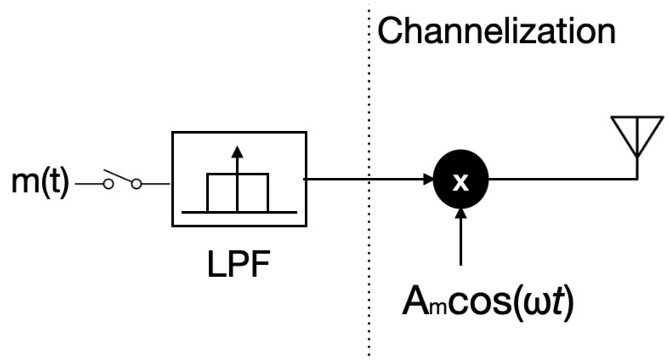

Figure 3.2: Simplified digital modulation

provides simulation results for a translational oscillator with rotational actuator (TORA) approximated using the Ore-FIS, which is stabilized using parallel distributed compensation, exploiting the $H(x)$ algorithm described in [36]. Finally, some concluding remarks are given in Section 3.6.

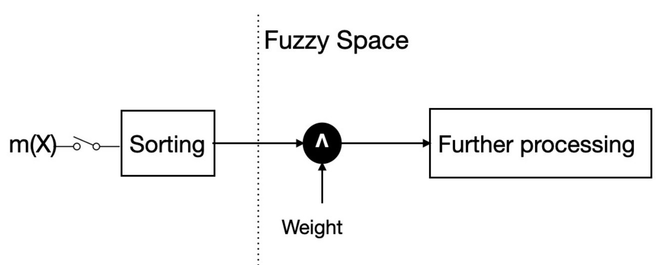

Figure 3.3: Fuzzification of a continuous multivariable function

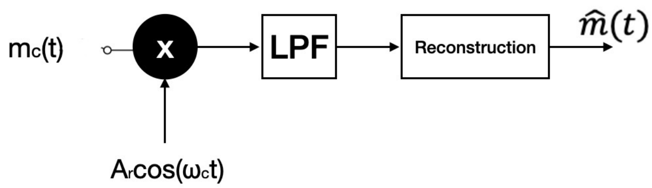

Figure 3.4: Simplified synchronous digital demodulation

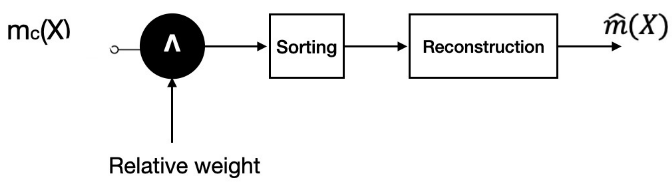

Figure 3.5: Defuzzification of a sampled and processed multivariable function

3.2 Problem Formulation

3.2.1 A Fuzzy Quantum Representation of Nonlinear Systems

Consider the Nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x) \end{aligned} \tag{27}$$

Assume this system can be rewritten as

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u \\ y &= C(x)x \end{aligned} \tag{28}$$

where $A(x) \in R^{n\times n}$, $B(x) \in R^{n\times m}$, $C(x) \in R^{r\times n}$, $x \in R^n$, $u \in R^m$ and $y \in R^r$. Assume also $x_{i,min} \leq x_i \leq x_{i,max}$ for $i \in N_n$.

In [9], the author defines classical mechanics as a crisp limit of fuzzy quantum mechanics. Also, the author in [8] remarks that fuzzy systems can be represented as quantum mechanics objects, using a quantum superposition of membership functions. In between fuzzification and defuzzification, the system in (28) can be written as follows with μ submodels having non-zero relative weight by using the Ore-FIS

$$\dot{x} = \sum_{i=1}^{\mu}[A(v_i)x \ + \ B(v_i)u]h(\Psi_i) + e_1(x) \tag{29a}$$

$$\begin{aligned} y &= \textstyle\sum_{i=1}^{\mu} C(v_i)xh(\Psi_i) + e_2(x) \\ y &= \textstyle\sum_{i=1}^{\mu} \widehat{y}_i h(\Psi_i) + e_2(x) \end{aligned} \tag{29b}$$

where $1 \leq \mu \leq 2^p$ and $\widehat{y}_i = C(v_i)x$.

Definition 1: Each fuzzy rule can be described by a 5-tuple $\Psi_i = (h_i(x),\ \overrightarrow{v_i},\ A(\overrightarrow{v_i}),\ B(\overrightarrow{v_i}),\ C(\overrightarrow{v_i})),$ where $\overrightarrow{v_i} \in R^p$. In the above expression, Ψ_i refers to the i-th rule, $h_i(x)$ is a scalar representing the relative weight of rule i, $\overrightarrow{v_i}$ is a column vector constructed such that each element is the vertex of the fuzzy sets of the i-th rule. $A(\overrightarrow{v_i})$ is the matrix obtained by replacing the premise variable in $A(x)$ with the vertices of the i-th rule. Likewise, $B(\overrightarrow{v_i})$ and $C(\overrightarrow{v_i}))$ are obtained by replacing the premise variables in $B(x)$ and $C(x)$ respectively with the vertices of the i-th rule.

Definition 2: The variables $e_1(x)$ and $e_2(x)$ represent the approximation errors, which can be made arbitrarily small by increasing the number of fuzzy sets per premise variable.

The author of [10] highlights that any arbitrary linear time invariant system can be implemented as a quantum system, provided additional quantum noises are permitted.

Remark 1: Note that any arbitrary error $e(x) = \sum_{i=1}^{\mu} e(x)h(\Psi_i)$ since $\sum_{i=1}^{\mu} h(\Psi_i) = 1$. The fuzzified nonlinear system can now be rewritten as

$$\begin{aligned} \dot{x} &= \textstyle\sum_{i=1}^{\mu}[A(v_i)x \ + \ B(v_i)u + e_1(x)]h(\Psi_i) \\ y &= \textstyle\sum_{i=1}^{\mu}[C(v_i)x + e_2(x)]h(\Psi_i) \\ y &= \textstyle\sum_{i=1}^{\mu} [\widehat{y}_i + e_2(x)]h(\Psi_i) \end{aligned} \tag{30}$$

Remark 2: Approximating a nonlinear system using the Ore-FIS will result in approximation error, which is regarded as information loss and is associated with an increase in entropy [11].

Consider that fuzzy submodels are quantum mechanical objects and that each submodel has a relative weight associated with it.

Definition 3: A wavefunction is a weighted sum or superposition of eigenfunctions.

Definition 4: A fuzzy submodel is an eigenfunction.

Definition 5: The Ore-FIS representation of a LPV nonlinear system is a wavefunction.

The problem posed in this subsection is that of finding a compact set of submodels to represent a nonlinear system at a particular instant in time. The $H(x)$ algorithm in [36] dictates that at any point in time, there exists a finite number of fuzzy submodels with non-zero relative weight. Therefore, the dimension of the inference problem collapses the n-dimensional state-space model to a p-dimensional fuzzy topological space, since p is the number of premise variables and $p \leq n$.

3.2.2 Wavenumber-limited Physical Systems

It is well known that the stability of a system represented by equation (28) can be analyzed using Lyapunov energy functions. A definitiion of observability for the class of nonlinear systems represented by equation (30) is given in [36]. Although a similar notion of nonlinear controllability is desirable, a proper definition requires additional derivation.

As per the second law of thermodynamics, a consequence of the reversibility of a physical process is that the total entropy of an isolated system does not decrease. If controllability of a nonlinear isolated thermodynamic system is assumed to be dependent on the degree of reversibility, a question arises as to the nature of the relationship between the energy and entropy.

By studying black-hole entropy, the author in [12] determines that it would be possible to violate the second law of thermodynamics if the entropy to energy ratio of an isolated system is infinite.

Definition 6: The Bekenstein Bound [37] is expressed as

$$\frac{S}{E} \leq \frac{2\pi R}{\hbar c} \tag{31}$$

where S is entropy, E is energy, R is the effective radius of the system, c is the speed of light and $\hbar$ is Planck's constant.

Remark 3: Einstein's field equations can be mathematically derived [13] by assuming that the law of thermodynamics and the Bekenstein bound are true.

Remark 4: The Bekenstein bound is evidence that any physical process can be represented with a finite amount of information.

Definition 7: A wavenumber limited function $f(x) : R^n \to R$ is the multidimensional equivalent of a scalar band-limited function of time.

We describe physical systems that are subject to the Bekenstein as being wavenumber-limited. Therefore, depending on the uncertainty budget of a design, an appropriate amount of fuzzy sets per premise variable must be found. In Section 3.3, the choice of number of fuzzy sets is shown to be a systematic one, in contrast with a previous methods that depend on an expert's intuition.

3.2.3 Multispectral Analysis of Physical Spaces

Although Lie groups [38] are commonly used in physics and mathematics to study differential equations and fuzzy sets [39], these require a high degree of mathematical understanding. Spectral analysis of multivariable nonlinear equations is a more intuitive approach for a practitioner with a systems background [40], [41].

Definition 8: The points $(\overrightarrow{v_i}, h_i(x))$ for $1 \leq i \leq 2^p$, as defined in [36], constitute the fuzzy space and are mapped from the crisp space by $H(x)$ as shown in Fig. 3.1.

An analysis of fuzzy spaces begins with the observation that fuzzy rules resemble hyper-rectangles in a fuzzy space constructed using the Ore-FIS. If p premise variables are used, then there's a maximum of 2^p rules with non-zero relative weights, since there is at most two active fuzzy sets per premise variable. These 2^p hyper-rectangles are not only neighbours but actually overlap in the fuzzy space.

Remark 5: Consider the system in equation (28). When such a system is fuzzified using the Ore-FIS as shown in (30), the sample points $v_i, \quad \forall i \in N_\mu, \quad where \quad 1 \leq \mu \leq 2^p$, and the weighted versions of $A(v_i)$, $B(v_i)$, and $C(v_i)$ are used to construct the fuzzy submodels in the fuzzy space

Definition 9: The division of a premise variable into s fuzzy sets is the multi-dimensional equivalent of sampling (i.e., multidimensional sampling).

A comparison can be made between transmitting a digital signal and fuzzifying a continuous multivariable system as shown in Figs. 3.2 and 3.3. It is shown that sampling a continuous multivariable signal is necessary for fuzzification. Furthermore, consider defuzzification as a method of multidimensional reconstruction of wavenumber-limited physical systems; it follows from the function approximation principle of the Ore-FIS that each premise variable (which is also a system state) will have several quantized states with varying degrees of membership (i.e., fuzzy sets). Consequently, the process of inferring which submodels are applicable (i.e., the $H(x)$ algorithm) is simply a choice of quantized premise variables with non-zero weight. A comparison between receiving and reconstructing a digital signal and defuzzification is illustrated by Figs. 3.4 and 3.5.

Definition 10: Further processing may refer to any computational actions by a deterministic Turing machine that are related to the $H(x)$ algorithm, observation with the HBNN-KF [36], or state-feedback control signal synthesis.

The concept of multidimensional periodicity in relation to the multivariable Fourier transform is presented in [17]. It is shown below that multispectral analysis can be used to construct the fuzzy space representation of a class of nonlinear system respecting the Bekenstein Bound.To this end, consider a function $a(x) : R^n \to R$ defined $\forall x \in D \subseteq R^n$. The Fourier transform of $a(x)$ as defined in [17] is the following transformation

$$\begin{aligned} a(x) &\xrightarrow{f} A(w) \\ a(x) &\xleftarrow{f^{-1}} A(w) \end{aligned} \tag{32}$$

Definition 11: The points $(w, A(w))$ will be referred to as wavepoints in the spectral domain (or simply wavepoints). These wavepoints lie within a finite spectral support if $A(w)$ has finite signal energy. If $A(w)$ has finite energy but is not a finite signal, it can be truncated and an adequate spectral support can be found.

Remark 6: A sampled multivariable signal with finite energy is periodic in the spectral domain [17]. Let N_n be the set of integers from 1 to n. Assume that the region of operation of a nonlinear system is given by

$$x_{i_{min}} \leq x_i \leq x_{i_{max}}, \quad \forall i \in N_n$$

which is essentially a hyper-rectangle constraint on the state space. The convex nature of this hyper-rectangle implies that it is also a polytope. In Section 3.3, we will use the Nyquist density (in relation to reconstruction by defuzzification)to derive an upper bound on the number of fuzzy sets per premise variable.

3.2.4 Uncertainty in Fuzzy Systems

It is known that observations and control policies can be improved by using more information with less uncertainty. Robustness of any arbitrary system can be loosely defined as a measure of said systems performance in the presence of uncertainty. The lack of robustness guarantees in traditional machine learning frameworks may result in unintended functionality [3], which is a concern in safety-critical applications. This is compounded by a lack of explainability of machine learning behaviour in neural networks.

The limitations of modelling physical systems with crisp variables in the presence of uncertainty was first highlighted by Lotfi Zadeh [42]. Note that probability and possibility theory are two mathematical tools for characterizing measurement uncertainty [30]. A drawback of the probabilistic approach to characterizing uncertainty in a system model is that it is based on boolean operations hence rendering it incapable of representing ignorance[30]. The authors in [30] illustrate with several thought experiments that the inability of probability theory to represent ignorance is because probability theory assumes the true values are known prior to quantifying uncertainties. A question arises as to how likely it is that the results of an experiment match presumptions, if the results are not known prior to the experiment? The authors of [30] state that this is very unlikely, and infer that models that address the practitioners' ignorance are better than those that do not, since the former is richer in information.

In his 1921 German address to the Prussian academy of sciences [43] titled "Geometry and Experience", Albert Einstein questioned how mathematics, a product of human thought (based on experience), can claim to be absolute and indisputable when human experience is not? His answer was that *"as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."* This can be interpreted to mean that if experience is not absolute, then ignorance is a persistent factor in reality. Therefore, it can be argued that any system designed with no consideration for ambiguity or uncertainty is likely to fail in real world applications.

For a nonlinear and imprecise model of a physical system, maintaining stability at linearized equilibrium points is a challenging robust control problem. This difficulty stems primarily from two assumptions made about a physical model; that the model can be adequately represented by a linearized model, and that crisp variables can represent the model. Making these assumptions about a physical system will result in information loss. It is to be noted that similar to reconstruction, defuzzification leads to information loss. In Section 3.3, stability conditions are derived using Lyapunov's stability theory, by treating entropy-induced uncertainties are perturbations of weighted fuzzy submodels.

3.3 Main Results

3.3.1 Multidimensional Fourier Transforms and Fuzzy Spaces

The Nyquist density specifies the minimum rate required for the unaliased reconstruction of a sampled signal's spectrum [17]. If fuzzification is considered as multidimensional sampling, then the sampling density must be greater than or equal to the Nyquist density, in order to reconstruct (via defuzzification) a sampled signal. If such a Nyquist density exists, then the signal must have finite energy. For instance, if $a(x)$ is continuous and belongs to the set of finite energy multivariable signals, the energy of $A(w)$ can be approximately calculated by wavepoints $w \in R^n$ in a spectral domain polytope.

Lemma III.1: Consider a class of nonlinear systems described by

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u \\ y &= C(x)x \end{aligned} \tag{33}$$

Define the non-empty set $M = M_A \cup M_B \cup M_c$, where M_A, M_B, and M_c contain the non-constant elements of $A(x)$, $B(x)$, and $C(x)$, respectively. Let $F(M)$ denote the set of Fourier transforms of each element of M truncated by an appropriate spectral support, with $m_i \in M$, $i \in N_{|M|}$. Let also $n_q(m_i) \geq 0$ be the Nyquist density of m_i, and $p \leq n$ be the number of premise variables. Then the number of fuzzy rules satisfies the inequality below

$$\zeta \leq \sup_{i,j}(n_q(m_i)\times \parallel P_j \parallel)^p \tag{34}$$

Proof: The maximum number of fuzzy sets in a premise variable is the Nyquist density multiplied by the length of the premise variable. If the constraint of an equal number of fuzzy sets per premise variable is imposed, then a sufficient number of fuzzy sets is given by

$$s = \sup_{i,j}(n_q(m_i)\times \parallel P_j \parallel) \tag{35}$$

If a smaller amount of fuzzy sets is used, then the control system must be designed to be robust by accounting for the additional uncertainty. Using combinatronics methods, an upper bound on the number of fuzzy rules can be obtained as

$$\zeta \leq s^p = \sup_{i,j}(n_q(m_i)\times \parallel P_j \parallel)^p \tag{36}$$

3.3.2 Potential Functions

The authors in [22] indicate that a Lyapunov function is a potential function and note that the Lyapunov equation is a specific case of the generalized Einstein relation [44] for linear systems.

Definition 12: The method of defuzzification used in the Ore-FIS is commonly referred to as centre-of-gravity (COG) defuzzification.

Although the stability of individual fuzzy submodels is not sufficient for the stability of the non-linear system, the approximation error in the Ore-FIS can be treated as a perturbation of the stable submodels.In [21], under the assumption of zero approximation error, a sufficient condition for the quadratic stability of a Takagi-Sugeno fuzzy system is given. A similar condition will be derived using Lyapunov stability theory for the Ore-FIS.

Definition 13: By definition and without loss of generality, an equilibrium point $x(0) \in R^n$ is stable if there exists positive scalar values ϵ and $\delta(\epsilon)$ such that the existence of $\parallel x(0) \parallel \leq \delta(\epsilon)$ implies that $\parallel x(t) \parallel \leq \epsilon$ [20].

Theorem III.1: If the approximation error is sufficiently small and can be modelled as a perturbation of the fuzzy submodels, then a sufficient condition for stability of the nonlinear system in a given region is the stability of each weighted fuzzy submodel in that region..

Proof: Consider the LPV nonlinear system defined $\forall x \in D \subseteq R^n$

$$\dot{x} = A(x)x = \sum_{i=1}^{\mu} h(\Psi_i)A(v_i)x + e(x) \tag{37}$$

where v_i is the vertex point of rule Ψ_i, and $A(v_i)$ is Hurwitz $\forall i \in N_\mu$. Let $L(x)$ be a Lyapunov energy function such that:

(1) $L(x) = x^T Px$.

(2) $\lambda_{min}(P) \parallel x \parallel \leq L(x) \leq \lambda_{max}(P) \parallel x \parallel$.

(3) $\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} \sum_{i=1}^{\mu} h(\Psi_i)A(v_i)x \leq -\alpha \parallel x \parallel^2$

(4) $\parallel \frac{\partial L}{\partial x} \parallel \leq 2\lambda_{max}(P) \parallel x \parallel$

Note that

$$\dot{L}(x) = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x}(\sum_{i=1}^{\mu} h(\Psi_i)A(v_i)x + e(x)) \tag{38}$$

then assuming $e(x) = 0$ and since $\frac{\partial L}{\partial t} = 0$

$$\dot{L}(x) = \frac{\partial L}{\partial x}(\sum_{i=1}^{\mu} h(\Psi_i)A(v_i)x) = \sum_{i=1}^{\mu} h(\Psi_i)x^T(P + P^T)A(v_i)x \tag{39}$$

which can then be written in the form

$$\dot{L}(x) = \sum_{i=1}^{\mu} -h(\Psi_i)x^T Q(v_i)x \tag{40}$$

where $-Q(v_i) = 2PA(v_i)$. If P exists such that $Q(v_i)$ is positive definite $\forall i \in N_\mu$, then the nonlinear system is stable.

When the approximation error is non-zero, the quadratic stability conditions above must be revisited. Equation (37) can be rewritten as

$$\dot{x} = A(x)x = \sum_{i=1}^{\mu} h(\Psi_i)(A(v_i)x + e(x)) \tag{41}$$

since $e(x) = \sum_{i=1}^{\mu} h(\Psi_i)e(x)$. In equation (41), it is obvious that the approximation error is a perturbation of the linear Hurwitz submodels. If the approximation error can be modelled as a vanishing perturbation, then we can study the stability of the nonlinear system along several equilibrium points that are defined by the weighted submodels. Specifically, $\forall x \in D$, if $\| e(x) \| \leq \gamma \| x \|$ and $0 \leq \gamma$, then equation (38) can be written as

$$\dot{L}(x) = \sum_{i=1}^{\mu} h(\Psi_i)(-x^T Q(v_i)x + \frac{\partial L}{\partial x} e(x)) \tag{42}$$

and then expressed as

$$\dot{L}(x) = \sum_{i=1}^{\mu} h(\Psi_i)(-x^T Q(v_i)x) + \frac{\partial L}{\partial x} e(x)) \tag{43}$$

Let $\alpha = \sum_{i=1}^{\mu} h(\Psi_i)\lambda_{min}(Q(v_i))$. Then an upper bound on $\dot{L}(x)$ is given by

$$\dot{L}(x) \leq -\alpha \| x \|^2 + \| \frac{\partial L}{\partial x} \| \| e(x) \| \tag{44}$$

Note that

$$-\alpha \| x \|^2 + \| \frac{\partial L}{\partial x} \| \| e(x) \| \leq -\alpha \| x \|^2 + 2\lambda_{max}(P)\gamma \| x \|^2 \tag{45}$$

If $\gamma < \frac{\alpha}{2\lambda_{max}(P)}$, then $\dot{L}(x)$ is negative definite.

On the other hand, modelling the approximation error as a nonvanishing perturbation is a more generic case, since the state space solution may not necessarily approach the equilibrium point (as is the case with limit cycles). By extending Lemma 9.2 in [20], we have that $e(x)$ must satisfy the following bound $\forall x \in D = \{x \in R^n | \| x \| < r\}$.

$$\| e(x) \| < \frac{\alpha}{2\lambda_{max}(P)} \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}} \theta r \tag{46}$$

where $0 \leq \theta < 1$ and r is the effective radius of the system.

Remark 7: It is shown in [36] that the approximation error from the Ore-FIS can be made arbitrarily small.

Remark 8: Note that μ satisfies the bound in equation (36).

Definition 14: Any fuzzy inference system that uses the H(X) algorithm as described in [36] is called a hashing-based neurofuzzy network (HBNN).

3.4 Applicability to Nonlinear Control

The analysis of dynamic nonlinear systems will be incomplete, without considering digital sampling and real-time scheduling effects. In observer-based feedback controller design for applications requiring real-time performance, all of the following factors must be considered:

(1) Sufficiently small approximation errors are acceptable for state-space-time sampling (i.e., fuzzification, and digital sampling) and state space estimation.

(2) The stabilizability and detectability conditions are satisfied for all activated submodels throughout the intended operational envelope of the system under control.

(3) The synthesized controller is robust to variations in estimation/measurements subject to uncertainty of a given magnitude.

(4) A controller can be found for each activated fuzzy submodel, within a fixed time window on a deterministic Turing machine.

(5) The reference state-space trajectory of the system is such that all activated submodels are stabilizable and detectable.

Note that hardware-in-the-loop simulations can provide more insights than theoretical analysis; especially, when uncertainty is being considered. These factors will be the subjects of future work.

3.5 Simulation Results

Example 1. The translational oscillator with a rotational actuator (TORA) is fully described in [21]. Let x_1 and x_2 be the translational position and velocity of the cart, and x_3 and x_4 be the

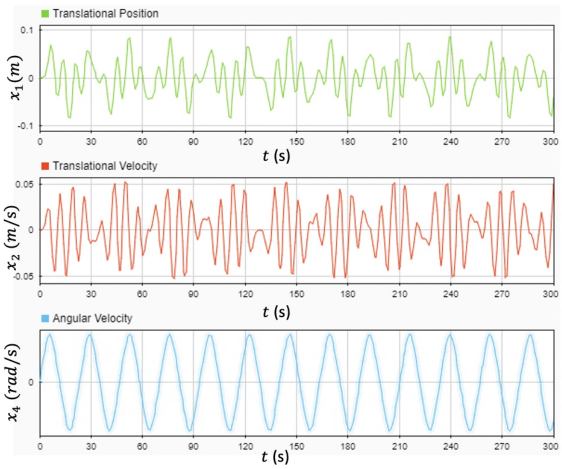

Figure 3.6: TORA dynamics with $q = 0.1$ and $R = 1000$

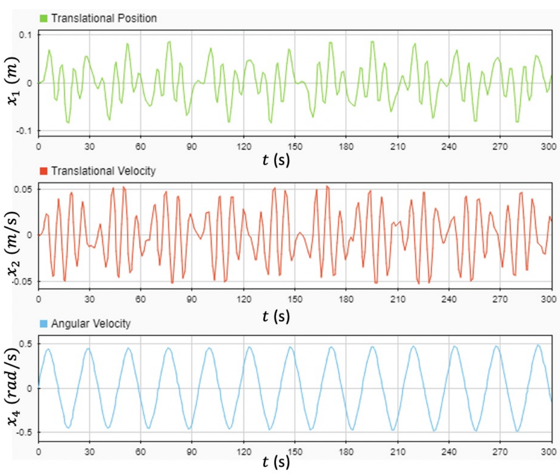

Figure 3.7: TORA dynamics with $q = 1$ and $R = 100$

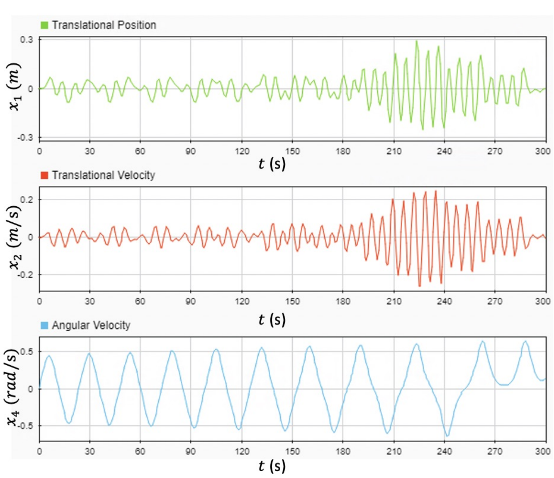

Figure 3.8: TORA dynamics with $q = 10$ and $R = 50$

rotational position and velocity of the mass m respectively. The system dynamics is expressed by the following nonlinear state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{W} & 0 & 0 & \frac{\epsilon \sin(x_3) x_4}{W} \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon \cos(x_3)}{W} & 0 & 0 & \frac{-\epsilon^2 \cos(x_3) \sin(x_3) x_4}{W} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\epsilon \cos(x_3)}{W} \\ 0 \\ \frac{1}{W} \end{bmatrix} \begin{bmatrix} T \end{bmatrix} \quad (47)$$

where T is input torque, $W = 1 - \epsilon^2 \cos^2(x_3)$, and $\epsilon = 0.1$.

The control objective is to stabilize x_1 at the origin using a hashing based neurofuzzy network (HBNN). Based on the Ore-FIS, there are two premise variables (x_3 and x_4), resulting in $2^2 = 4$ parallel distributed compensators (PDC). For simplicity, the feedback gains and the relative weights are made constant based on the initial conditions. This is acceptable if the control objective is to maintain the initial conditions (i.e., the regulator problem).

The feedback gains obtained by solving the LQR problem with a scalar parameter R and the following matrix

$$Q = \begin{bmatrix} q & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix} \quad (48)$$

where q is a scalar. On the other hand, the energy input to the system is given by

$$E = \int_0^{t_f} P\,dt = \int_0^{t_f} \omega T\,dt = \int_0^{t_f} x_4 T\,dt \quad (49)$$

The simulation confirms the main results in Theorem 1; that the nonlinear system is stabilized by considering only the weighted submodels determined by the $H(x)$ algorithm, provided the approximation error is sufficiently small. Based on the results in Figs. 3.6, 3.7 and 3.8, as R increases, the magnitude of translational displacement from the equilibrium point decreases. The translational amplitude and energy consumption for different combinations of R and q illustrated by Figs. 3.6, 3.7 and 3.8 are summarized in the table below.

Table 1. Translational amplitude and energy consumption for different parameter values

q	R	$\lvert x_1 \rvert$ (m)	Energy consumed (J)
0.1	1000	0.08	0.0011
1	100	0.08	0.0257
10	50	0.3	0.2801

3.6 Conclusion

By considering LPV nonlinear systems as a crisp limit of fuzzy quantum mechanics, and treating approximation errors as quantum noises, stability conditions for LPV nonlinear systems are derived. Additionally, concepts in digital telecommunication theory such as (de)modulation are shown to be applicable in the multi-dimensional sense to state-space variables via (de)fuzzification, which means the choice of fuzzy sets and rules can be decided systematically, and not just based on an expert's intuition. We note that the quantum mechanical objects presented (i.e., fuzzy submodels) have a physical realization and, as such, must satisfy the Bekenstein bound. This work is preliminary and leverages the synergy in mathematics, theoretical physics, and control systems engineering, and constitutes a foundation for future work that address the nonlinear control problem in the context of model predictive control, using fuzzy function approximators.

Chapter 4

Conclusions and Future Research Directions

This thesis has introduced novel concepts with regards to intelligent approximation, estimation and control of LPV nonlinear systems. The function approximation property of the Ore-FIS has been mathematically proven to hold, provided the Lipschitz condition is satisfied over the state domain of interest. A discrete time estimator called the HBNN-KF has been derived as a result of the Ore-FIS, and is shown to be more robust than the EKF in a specific application, with comparable processing times.

Furthermore, when defuzzification is considered as a method of multidimensional reconstruction of wavenumber-limited physical systems; it follows from the function approximation principle of the Ore-FIS that each premise variable (which is also a system state) will have several quantized states with varying degrees of membership (i.e. fuzzy sets). Consequently, the process of inferring which submodels are applicable (i.e. the $H(x)$ algorithm) is simply a choice of quantized premise variables with non-zero weight.

It is noted that the dimension of the inference problem collapses the n-dimensional state-space model to a p-dimensional fuzzy topological space, since p is the number of premise variables and $p \leq n$. The 2^p limit on the number of submodels is exploited to prove the quadratic stability of a nonlinear system with fuzzy submodels defined by the Ore-FIS. A deliberate effort is made in this thesis to provide physical meaning to the operations of fuzzy observers and controllers and not just a mathematical interpretation. A practitioner with a background in linear control theory will face much less difficulty in analyzing nonlinear systems using the concepts defined in this thesis. The following applications will be explored and compared with existing literature in future work:

(1) Stability analysis and robust-adaptive controller synthesis for systems with uncertainty.

(2) Online model parameter estimation and controller synthesis in model predictive control applications.

Bibliography

[1] E. Heim., E. Viken., J. Brando., and M. Croom. *NASA's Learn-to-Fly Project Overview*. NASA Langley Research Center. 2018.

[2] EASA. *Artificial Intelligence Roadmap: A Human-centric Approach to AI in Aviation*. Version 1.0. 2020.

[3] EASA AI Task Force and Daedalean AG. *Concepts of Design Assurance for Neural Networks*. Version 1.0. 2020.

[4] D. Sculley, G. Holt, D. Golovin, E. Davydov, T. Phillips, D. Ebner, V. Chaudhary, M. Young, J. Crespo, and D. Dennison. *Hidden Technical Debt in Machine Learning Systems*. NIPS. 2015.

[5] Y. Lin, Z. Song, and L. Yang. *Towards a Theoretical Understanding of Hashing-based Neural Nets*. AISTATS. 2019.

[6] B. Kosko. *Neural Networks and Fuzzy Systems*. Prentice-Hall, 1992.

[7] T. Takagi and M. Sugeno. *Fuzzy Identification of Systems and its Applications to Modeling and Control*. IEEE Transactions on Systems, Man, and Cybernetics, Vol. 15, pp. 116–132. 1985.

[8] D. Chernyshev. *Fuzzy Quantum Control*. International Multi-conference on Industrial Engineering and Modern Technologies. 2018.

[9] A. Granik. *Fuzziness in quantum mechanics*. Proceedings Volume 10277, Adaptive Computing: Mathematics, Electronics, and Optics: A Critical Review. 1994.

[10] S. Vuglar and I. Petersen. *Quantum Noises, Physical Realizability and Coherent Quantum Feedback Control*. International Multi-conference on Industrial Engineering and Modern Technologies. 2018.

[11] S. Russell and P. Norvig. *Artificial Intelligence: A Modern Approach*. Pearson, 2018.

[12] J. Bekenstein. *How Does the Entropy/Information Bound Work?* Foundations of Physics, Vol. 35, Iss. 11, pp. 1805-1823. 2005.

[13] T. Jacobson. *Thermodynamics of Spacetime: The Einstein Equation of state*. Physics Review Letters, Vol. 75, pp. 1260-1263. 1995.

[14] A. Kumar, G. Srungavarapu, H. Beiranvand, and E. Rokrok. *A Novel Approach for Automatic Generation Control of Multi Area Power Systems with Nonlinearity using General Relativity Search Algorithm*. IEEE Annual India Conference. 2016.

[15] S. Dutta, D. Das, and T. Chakraborty. *Modelling Engagement Dynamics of Online Discussions using Relativistic Gravitational Theory*. IEEE International Conference on Data Mining. 2019.

[16] U. Kostic, M. Horvat, and A. Gomboc. *An Autonomous Reference Frame for Relativistic Positioning System*. IEEE Metrology for Aerospace. 2015.

[17] R. Marks II. *Introduction to Shannon Sampling and Interpolation Theory*. Springer-Verlag, 1991.

[18] C. Li, G. Liu, Z. Hao, S. Zu, F. Mi, and X. Chen. *Multidimensional Seismic Data Reconstruction using Frequency-Domain Adaptive Prediction-error Filter*. IEEE Transactions on Geoscience and Remote Sensing, Vol. 56, No. 4. 2018.

[19] S. Wang, V. Patel, and A. Petropulu. *Multidimensional Sparse Fourier Transform Based on the Fourier Projection-slice Theorem*. IEEE Transactions on Signal Processing, Vol. 67, No. 1. 2019.

[20] H. Khalil. *Nonlinear Systems*. Prentice Hall, 1996.

[21] K. Tanaka and H. Wang. *Fuzzy Control Systems Design and Analysis*. John Wiley & Sons, 2001.

[22] Y. Ruoshi, M. Yian, and A. Ping. *Potential Function in Dynamical Systems and the Relation with Lyapunov Function*. Proceedings of the 30th Chinese Control Conference. 2011.

[23] L. Paramo-Carranza, J. Meda-Campana, J. Rubio, R. Tapia-Herrera, A. Curtidor-Lopez, A. Grande-Meza, and I. Cazares-Ramirez. *Discrete-time Kalman Filter for Takagi-Sugeno fuzzy models*. Evolving Systems 8, pp. 211–219. 2017.

[24] E. Morelli. *Transfer Function Identification using Orthogonal Fourier Transform Modeling Functions*. AIAA Atmospheric Flight Mechanics (AFM) Conference. 2013.

[25] E. Morelli. *Real-time Parameter Estimation in the Frequency Domain*. Journal of Guidance, Control and Dynamics. 2000.

[26] E. Morelli and V. Klein. *Determining the Accuracy of Maximum Likelihood Parameter Estimates with Colored Residuals*. NASA Technical Reports Server, 1994.

[27] A. Barron. *Predicted Squared Error: A Criterion for Automatic Model Selection*. Self-Organizing Methods in Modeling. 1984.

[28] E. Morelli. *Multiple Input Design for Real-Time Parameter Estimation in the Frequency Domain*. IFAC Proceedings vol. 36. 2003.

[29] K. Kanth, D. Agrawal, and A. Singh. *Dimensionality Reduction for Similarity Searching in Dynamic Databases*. Proceedings of the 1998 ACM SIGMOD International Conference on Management of Data, pp. 166-176. 1998.

[30] S. Salicone. *Measurement Uncertainty: An approach via the Mathematical Theory of Evidence*. Springer, 2007.

[31] M. Bao. *Analysis and Design Principles of MEMS Devices*. Elsevier, 2005.

[32] K. Tanaka, T. Taniguchi, and H. Wang. *Robust and Optimal Fuzzy Control: A Linear Matrix Inequality Approach*. Beijing IFAC World Congress, pp. 213-218. 1999.

[33] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM. 1994.

[34] *Resolution 155*. World Radiocommunications Conference. Geneva, 2015.

[35] E. Ivanjko, T. Petrinic, and I. Petrovic. *Modelling of Mobile Robot Dynamics*. 7th Eurosim Congress on Modelling and Simulation, Vol. 2. 2010.

[36] O. Ajayi, M. Babahaji, and A. Aghdam. *A neurofuzzy Function Approximator Approach to Nonlinear Systems State-estimation*. IEEE International Conference on Wireless for Space and Extreme Environments. 2020.

[37] H. Casini. *Relative Entropy and the Bekenstein Bound*. Classical and Quantum Gravity, Vol. 25, Iss. 20. 2008.

[38] C. Kim and D. Lee. *Fuzzy Lie Ideals and Fuzzy Lie Subalgebras*. Fuzzy Sets and Systems, Vol. 94, Iss. 1, pp. 101-107. 1995.

[39] T. Yalvaç. *Fuzzy Sets and Functions on Fuzzy Spaces*. Journal of Mathematical Analysis and Applications, Vol. 126 Iss. 2, pp. 409-423. 1987.

[40] A. Oppenheim and A. Willsky. *Signals and Systems*. Prentice-Hall, 1997.

[41] A. Oppenheim and R. Schafer. *Discrete-time Signal Processing*. Prentice-Hall, 2009.

[42] L.A. Zadeh. *Fuzzy sets as a basis for a theory of possibility*. Fuzzy Sets and Systems 1, 3–28. 1978.

[43] A. Einstein. *Geometrie und Erfahrung*. Erweiterte Fassung des Festvortrages Gehalten an der Preussischen Akademie der Wissenschaften zu Berlin. 1921.

[44] A. Einstein. *On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat*. Annalen der Physik 17, pp. 549-560., 1905.