Khovanov homology and the unknotting number

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Abstract

KHOVANOV HOMOLOGY AND THE UNKNOTTING NUMBER

LAURA MARINO

The aim of this thesis is to describe a new lower bound λ for the unknotting number.
The unknotting number u is a classical knot invariant, defined as the minimum number of crossing changes that are needed in order to turn a knot into the Unknot (where a knot is the image of a smooth embedding $\mathbb{S}^1 \to \mathbb{R}^3$, and the Unknot is the "unknotted" knot). We have that u is hard to compute, thus one of the goals of knot theory is to find lower bounds for it.
Among the tools that have recently been used to describe lower bounds for u there is Khovanov homology. It is a link invariant, constructed from algebraic structures called Frobenius systems in the following way: given a link diagram D, we associate a cube of (1+1)-cobordisms to it. Then every Frobenius system $\mathcal{F}$ of rank two generates a functor, called TQFT, that associates a chain complex $C_{\mathcal{F}}(D)$ to this cube. Khovanov homology is the homology of $C_{\mathcal{F}}(D)$.
Thus different Frobenius systems $\mathcal{F}$ generate different homology theories $H_{\mathcal{F}}$. Among Frobenius systems, $\mathcal{F}_{Univ}$ is particularly interesting because $H_{\mathcal{F}_{Univ}}$ determines every other Khovanov homology $H_{\mathcal{F}}$.
Alishahi and Dowlin (2017) defined two lower bounds λ_{BN} and λ_{Lee} for the unknotting number using the Khovanov homology theories relative to Frobenius systems $\mathcal{F}_{BN}$ and $\mathcal{F}_{Lee}$. Other than giving a bound for u these bounds have several interesting applications related to the convergence of some spectral sequences and to the Knight Move Conjecture. Using the structures and tools introduced by Alishahi and Dowlin, in this thesis we find a new bound that subsumes λ_{BN} and λ_{Lee}, using the Khovanov homology theory relative to $\mathcal{F}_{Univ}$.

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Introduction

The aim of this thesis is to describe a new lower bound for the unknotting number, obtained using the tools of Khovanov homology. This bound is a generalization of the bounds found by Alishahi and Dowlin in [Al] and [AD]. All the necessary theory will be developed in order to achieve this goal.
One of the areas of interest in knot theory revolves around link invariants, and in particular invariants that detect the Unknot. Khovanov homology is a link invariant with this property. It will be shown in this thesis, following [BN1], [Kh1] and [Kh2], that every graded Frobenius system $\mathcal{F}$ of rank two generates a homology theory for links, called "Khovanov homology relative to $\mathcal{F}$".
The unknotting number u is another classical knot invariant that detects the Unknot, but it is particularly hard to compute. It is thus useful to find lower bounds for u. Among those we mention Rasmussen's invariant, described in [Ra], and the slice genus. In 2017 Alishahi and Dowlin, in [Al] and [AD], developed two lower bounds coming from the Khovanov homology theory relative to Frobenius systems $\mathcal{F}_{BN}$ and $\mathcal{F}_{Lee}$ respectively. These lower bounds have several other interesting applications: they determine the page at which the Lee and the Bar-Natan spectral sequences collapse and give a proof of the Knight Move Conjecture when $u \leq 2$.
Following [Kh1] we will show that all Khovanov chain complexes relative to a rank two Frobenius system can be obtained from the Khovanov chain complex relative to the Frobenius system $\mathcal{F}_{Univ}$ by tensoring with a ring. This makes $\mathcal{F}_{Univ}$ particularly interesting. This thesis generalizes Alishahi and Dowlin's construction: using the Khovanov homology theory relative to $\mathcal{F}_{univ}$ we find a new knot invariant λ. The following theorem is the core of this thesis:

Theorem 3.12. *Let K be a knot. Then $\lambda(K)$ is a lower bound for the unknotting number of K:*

$$\lambda(K) \leq u(K).$$

We have that λ subsumes Alishahi and Dowlin's bounds.

The thesis is structured in the following way. In the first chapter the necessary theory will be provided for the construction of Khovanov homology and the bound λ. We first give an overview of knot theory containing all the basic definitions used throughout the thesis. Some

other, less generic, definitions and statements will be provided in successive chapters, when needed. We then introduce the tools necessary to the construction of Khovanov homology: we define the categories Cob_1 of (1+1)-cobordisms and *R-mod* of graded R-modules (with R a commutative ring), and describe Frobenius systems in detail. Finally, for each graded Frobenius system we define a monoidal functor called TQFT from Cob_1 to *R-mod*.
The goal of the second chapter is to construct a bigraded homology theory for each graded Frobenius system of rank two and show that it is a link invariant, called Khovanov homology. Given a link diagram D we first construct a cube of resolutions in Cob_1, next we associate to it an algebraic structure via the TQFT functor, in order to get an algebraic cube of spaces and maps, and finally we see how to obtain a chain complex $C(D)$ from it. We then prove that this chain complex is a link invariant up to chain homotopy equivalence, and thus its homology is a link invariant. We call it Khovanov homology.
The third chapter describes a new knot invariant λ, defined as the maximum $(2X-h)$-torsion order of an element in H^*, where H^* is the Khovanov homology relative to the universal Frobenius system $\mathcal{F}_{Univ}$. We also prove that λ is a lower bound for the unknotting number. Finally, we define Alishahi and Dowlin's bounds and show their relation to the Bar-Natan and Lee spectral sequences and to the Knight Move Conjecture.
The Appendix provides an overview of spectral sequences: these objects appear often in the context of Khovanov homology and are closely related to it. Moreover, we expect possible developments of this thesis to link the invariant λ to the convergence of some spectral sequence, as for Alishahi and Dowlin's bounds.

Notations

n_+, n_-	number of positive and negative crossings in a link diagram D
D_0, D_1	diagrams obtained from a diagram D by 0- and 1-resolving a crossing c respectively
D_v	complete resolution of a diagram D corresponding to the vertex v
$\|v\|$	sum of the entries of the vertex vector v
k_v	number of connected components of D_v
δ	edge map of the cube of resolutions
$\boldsymbol{\delta}$	bundle of edge maps going from $C(D_0)$ to $C(D_1)$
d	differential of the chain complex $C(D)$
$\{\cdot\}$	quantum shift
$[\cdot]$	homological shift
$\widetilde{C}$	chain complex before the global shifts
$D, \overline{D}$	diagrams related by a crossing change
$\mathcal{F} = (R, A, \Delta, \varepsilon)$	Frobenius system with ground ring R, R-module A, multiplication m, comultiplication Δ, unit ι, counit ε

1 Basic concepts

Throughout this work we will use in a fundamental way the notions of cobordism and Frobenius system and we will go back and forth from one to the other using Topological Quantum Field Theories. We thus provide an overview of these concepts.
We first give a quick review of knot theory, which is the basis of this thesis.

1.1 An overview of knot theory

This section is inspired by lecture notes taken in the Knot Theory course taught by Dr. Lukas Lewark, during the Winter semester 2019/2020, at the University of Regensburg.

Definition 1.1. A *knot* K is the image of a smooth embedding $\varphi\colon \mathbb{S}^1 \to \mathbb{R}^3$ together with an orientation. A *link* L is a union of finitely many disjoint knots, called the *components* of L.

We consider links up to equivalence:

Definition 1.2. Two links L, L' are *equivalent* if there is an orientation preserving diffeomorphism $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi(L) = L'$ and ϕ preserves the orientations of the links.

It is often convenient to work with link diagrams, i.e. images of projections of a link on the plane, rather than with links themselves. Diagrams allow us to treat knot theory using a combinatorial approach, rather than a topological one. In order to do this we want the projections to preserve as much information on (the equivalence class of) the link as possible. This is achieved by perturbing the link in general position before taking the projection.

Definition 1.3. Consider the projection $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $p(x, y, z) = (x, y)$. Let L be a link with components $K_i = \varphi_i(\mathbb{S}^1) \subseteq \mathbb{R}^3$. We say that L is in *general position* if

1. $p \circ \varphi_i$ has nowhere vanishing differential for all i,
2. $p \circ \varphi_i$ and $p \circ \varphi_j$ only intersect transversely for all i, j,
3. $\#(p^{-1}(x) \cap L) \leq 2$, for all $x \in \mathbb{R}^2$.

Remark 1.4. Every link is equivalent to a link in general position.

Definition 1.5. Given a link L in general position, we call $D = p(L) \subseteq \mathbb{R}^2$, together with over-under information at crossings and the orientation induced by L, a *link diagram*.
A *crossing* of D is a point $x \in \mathbb{R}^2$ where $\#(p^{-1}(x) \cap L) = 2$.

Remark 1.6. Every link diagram D has only finitely many crossings. This follows from the fact that by transversality the crossings are isolated, and $\varphi_i(\mathbb{S}^1)$ is compact.

In Figure 1.1 we give a few examples of links (or rather, of link diagrams). The orientations are omitted in this figure.
Note that, in general, identical links with different orientations are non-equivalent, but for example the Unknot and the Trefoils are *invertible*, i.e. they are equivalent to their reverse (the *reverse* of a link L is the link obtained by reversing the orientation on all components of L). The Hopf link is non-invertible.
In addition, we have that the Right-handed Trefoil is not equivalent to the Left-handed Trefoil, but they are the *mirror image* of each-other (i.e. obtained from each-other by reflecting along a plane in $\mathbb{R}^3$). For more details on these concepts see [Ad].

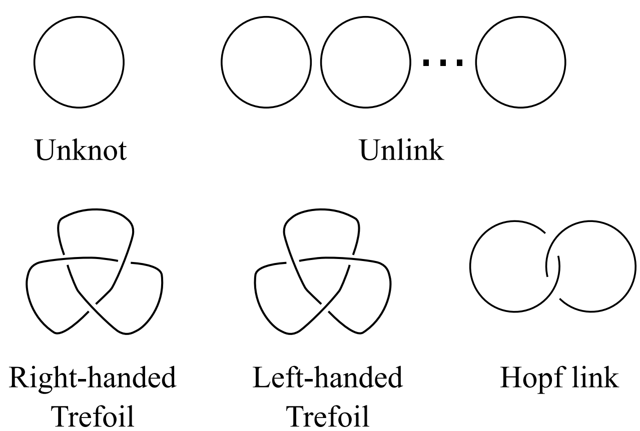


Figure 1.1

Definition 1.7. Two diagrams D, D' are *equivalent* if there is an orientation preserving diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(D) = D'$, and such that ϕ preserves the over-under information and the orientations of the diagrams.

A diagram D of a link L determines L up to (link) equivalence, and equivalent diagrams represent equivalent links. However, every (equivalence class of) link has infinitely many non equivalent diagrams.
The following theorem allows us to overcome this problem:

Theorem 1.8. *Two diagrams D, D' represent equivalent links if and only if they are related by a finite sequence of diagram equivalences and the moves $R1, R2, R3$ shown in Figure 1.2, called Reidemeister moves.*

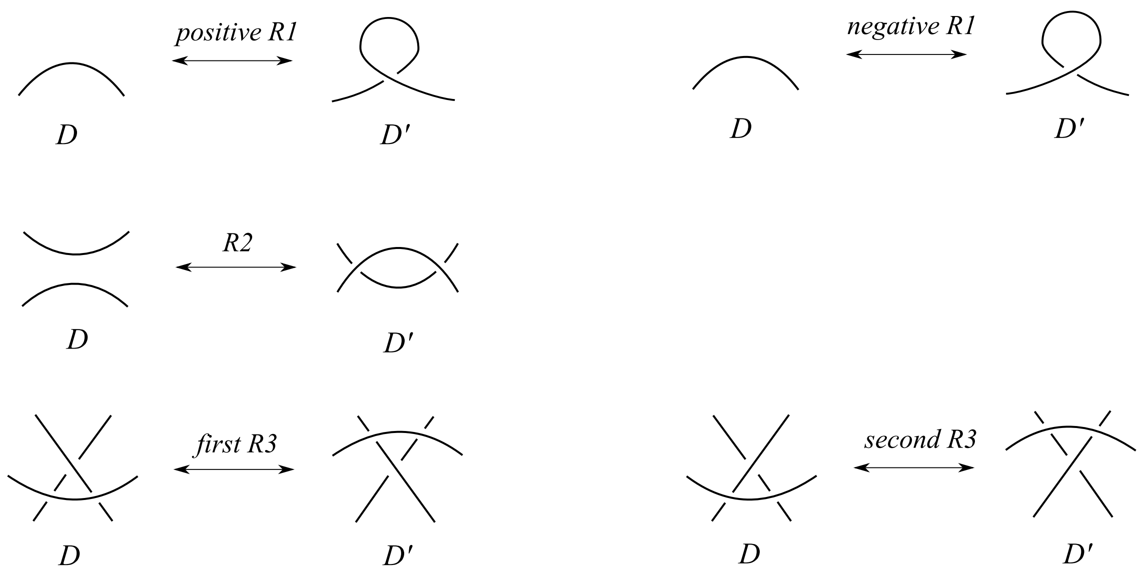


Figure 1.2

Remark 1.9. The moves *negative R1* and *second R3* are redundant, i.e. they follow from the other Reidemeister moves.

Definition 1.10. A *link invariant* is a function $\alpha\colon \{\text{links}\} \to H$ (where $H \in Set$) that is invariant under link equivalence.

One of the goals of knot theory is to find "good" link invariants, that is functions α that distinguish links well and can be computed easily from diagrams. In particular, one of the goals is to find link invariants that detect the Unknot, i.e. functions α such that $\alpha(\text{Unknot}) \neq \alpha(L)$ for all links L non equivalent to the Unknot.
In this thesis we will describe several link and knot invariants, such as the unknotting number, Khovanov homology in Chapter 2 and λ in Chapter 3.

Let us first describe the unknotting number.

Definition 1.11. Two diagrams $D, \overline{D}$ are *related by a crossing change* if they are identical except at one crossing, where they differ by the over-under information, as in Figure 1.3.

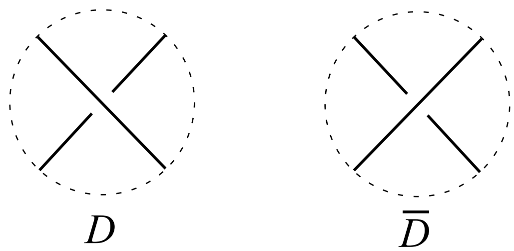


Figure 1.3

Two links $L, \overline{L}$ are *related by a crossing change* if there are diagrams $D, \overline{D}$ representing links equivalent to $L, \overline{L}$ respectively such that D and $\overline{D}$ are related by a crossing change.

For example, the Trefoil knot and the Unknot are related by a crossing change, as shown in Figure 1.4.

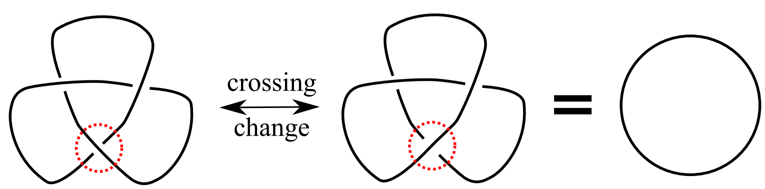


Figure 1.4

Definition 1.12. The *unknotting number* $u(L)$ of a link L is the minimum $n \in \mathbb{N}$ such that there is a sequence $L_0, \dots, L_n$ of links (called *unknotting sequence*) with $L_0 = L, L_n =$ Unlink and L_i, L_{i+1} are related by a crossing change for $0 \le i \le n - 1$.

Remark 1.13. Every link admits a (finite) unknotting sequence, so $u(L)$ is a well-defined link invariant.

Figure 1.4 shows an unknotting sequence of length 1 for the Trefoil. Since the Trefoil is not equivalent to the Unknot (or to any Unlink), its unknotting number must be at least 1. Thus $u(\text{Trefoil}) = 1$.

We observe that Unlinks are the only links that have unknotting number 0, but, as a link invariant, u doesn't detect the Unknot, because it cannot distinguish Unlinks with a different number of components. However, if we only consider u as a knot invariant (i.e. as a function $\{knots\} \to \mathbb{N}$) then it detects the Unknot (the Unknot is the only knot with a crossingless diagram).
The unknotting number is a classical link invariant, and very elementary in its definition. However, it is hard to compute: one can easily find an unknotting sequence for a link, but it is very difficult to determine whether an unknotting sequence is the most efficient (i.e. shortest) one. For this reason one of the goals of knot theory is to find lower bounds for the unknotting number, i.e. link invariants $\beta\colon \{links\} \to \mathbb{N}$ such that $\beta(L) \leq u(L)$ for all links L.
There are several link invariants that provide lower bounds for the unknotting number, such as the splitting number, half of the linking number, the Rasmussen invariant and the slice genus.
In this thesis we use the tools of Khovanov homology to describe several other lower bounds for the unknotting number (as a knot invariant) and show that for some knots these bounds are sharper than the Rasmussen invariant.
Moreover, Khovanov homology itself is a good link invariant: it is proved in [KM] that it detects the Unknot.

1.2 The category of (1+1)-cobordisms

In this section we use definitions and results from [Kh2].

Definition 1.14. A *(n+1)-cobordism* between two closed n-dimensional manifolds M, N is a compact, orientable, differentiable $(n+1)$-dimensional manifold, whose boundary is $M \sqcup N$.

We will only be interested in (1+1)-cobordisms, i.e. compact, orientable, differentiable surfaces whose boundaries are disjoint unions of circles (the closed 1-dimensional manifolds).

Let's define the category Cob_1 of (1+1)-cobordisms.
The objects of this category are closed 1-dimensional manifolds, i.e. disjoint unions of circles. More precisely $Ob(Cob_1) = \{\overline{n} : n \in \mathbb{N}\}$, where $\overline{n} = \{c_{n,1}, \dots, c_{n,n}\}$ represents the disjoint union of n circles in the plane.
A morphism from $\overline{n}$ to $\overline{m}$ is a (1+1)-cobordism S between $\overline{n}$ and $\overline{m}$, embedded in $\mathbb{R}^2 \times [0, 1]$ so that $S \cap \mathbb{R}^2 \times \{0\} = \overline{n}$ and $S \cap \mathbb{R}^2 \times \{1\} = \overline{m}$ (so the domain in on the bottom and the target on the top of the surface S).

Two morphisms $S, T\colon \overline{n} \to \overline{m}$ are equal if the surfaces S and T are diffeomorphic and the diffeomorphism φ fixes every circle of the boundary: $\varphi(c_{n,i}) = c_{n,i}$ and $\varphi(c_{m,i}) = c_{m,i}$ $\forall i$.
The composition of two morphisms is their concatenation, realised by gluing boundary circles.

Cob_1 is a monoidal category: the tensor product of two morphisms S, T is defined as the disjoint union of the surfaces S and T.

Using Morse Theory one can show that every morphism S of Cob_1 is the composition of tensor products of the elementary morphisms shown in Figure 1.5 (i.e. every (1+1)-cobordism is the concatenation of disjoint unions of elementary (1+1)-cobordisms).

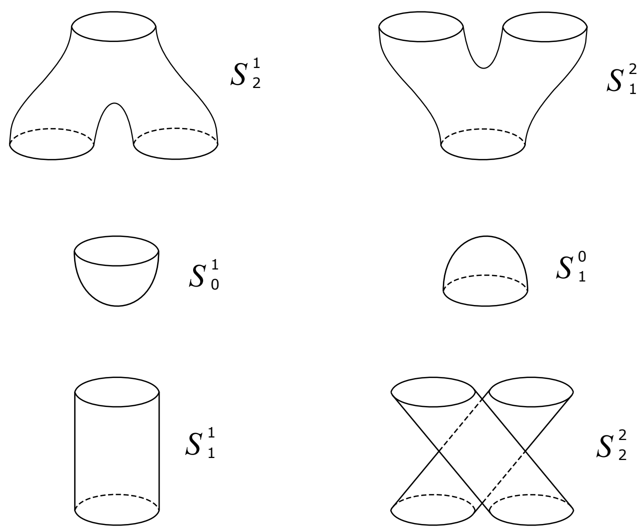


Figure 1.5

Observe that S^1_2 and S^2_1 correspond to attaching a 1-handle to two circles and to one circle respectively, S^0_1 corresponds to attaching a 2-handle to a circle and S^1_0 is a 0-handle.
We call S^1_2 a *merge cobordism* (it merges two circles in one) and S^2_1 a *split cobordism*, since it generates two circles from one. The cobordism S^1_1 is the identity morphism of Cob_1.

1.3 Frobenius systems and the category of R-modules

This section and the following one are mainly based on [Kh1] and [Kh2]. The last part, about the S, T and $4Tu$ relations, is inspired by [BN2].

Before talking about Frobenius systems we recall the definition of cocommutative coalgebra over a ring R. This notion is dual to that of a commutative algebra.
Let us first define a map that permutes two tensor copies of an R-module A:

$$Perm\colon A \otimes_R A \to A \otimes_R A$$

$$a \otimes b \mapsto b \otimes a$$

Definition 1.15. A *cocommutative coalgebra* over a ring R is an R-module A together with R-module maps $\Delta\colon A \to A \otimes_R A$ and $\varepsilon\colon A \to R$ that satisfy the following identities:

- coassociativity:
$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \tag{1.1}$$
- cocommutativity:
$$Perm \circ \Delta = \Delta \tag{1.2}$$
- counit identity:
$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} \tag{1.3}$$

The maps Δ and ε are, respectively, the *comultiplication* and the *counit* of A.

Definition 1.16. A *Frobenius system* is a 4-tuple $\mathcal{F} = (R, A, \Delta, \varepsilon)$ consisting of a commutative unitary ring R and an R-module A that we endow with a commutative algebra and a cocommutative coalgebra structures that are related by the following identity:

$$\Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \tag{1.4}$$

where m is the multiplication of the algebra structure of A, Δ is the comultiplication of the coalgebra structure, and ε is the counit.
We call equation (1.4) the *Frobenius identity.*

In this thesis we will always assume that the algebra A is unitary and that, as an R-module, it is free and of finite rank.
We will often denote the multiplication of two elements $x, y \in A$ by $x \cdot y$ instead of $m(x \otimes y)$. This notation is simpler, and it will come in handy later on (in Chapters 2 and 3), when we introduce the edge maps of the cube of resolutions: then, to avoid confusions, the multiplication map of the algebra A will be denoted by "$\cdot$", while m will be used to indicate the edge map.

Let $\{X_1, \ldots, X_m\}$ be a basis of A as an R-module. The tensor product $A \otimes_R A$ is again a free R-module of finite rank. Let $\{Y_1, \ldots, Y_n\}$ be a basis, with $Y_i = Y_i^1 \otimes Y_i^2$. Then the comultiplication $\Delta\colon A \to A \otimes_R A$ decomposes as $\Delta = \Delta_1 + \ldots + \Delta_n$: if $\Delta(X_j) = a_{j1}Y_1 + \ldots + a_{jn}Y_n$, for $1 \leq j \leq m$ and $a_{ji} \in R$, then we define Δ_i for $1 \leq i \leq n$ as the R-module map

$$\Delta_i(X_j) = a_{ji}Y_i.$$

The map Δ_i, in turn, decomposes as $\Delta_{1i} \otimes \Delta_{2i}$ as follows: $\Delta_i(X_j) = a_{ji}Y_i = a_{ji}Y_i^1 \otimes Y_i^2$, so we define the R-module maps Δ_{1i}, Δ_{2i} as

$$\Delta_{1i}(X_j) = a_{ji}Y_i^1 \qquad \text{and} \qquad \Delta_{2i}(X_j) = Y_i^2.$$

Thus Δ decomposes as $\Delta = \Delta_{11} \otimes \Delta_{21} + \ldots + \Delta_{1n} \otimes \Delta_{2n}$.

We make the following observations:

Remark 1.17. By the cocommutativity property we have that Δ is symmetric in the following sense: there is a permutation σ of $\{1, \ldots, n\}$ such that $\Delta_{1i} \otimes \Delta_{2i} = \Delta_{2\sigma(i)} \otimes \Delta_{1\sigma(i)}$. Thus

$$\Delta = \sum_{i=1}^{n} \Delta_{1i} \otimes \Delta_{2i} = \sum_{i=1}^{n} \Delta_{2i} \otimes \Delta_{1i}.$$

Beside the Frobenius identity (1.4) for Frobenius systems we have:

Lemma 1.18. *The identity* $\Delta \circ m = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id})$ *also holds.*

Proof. Let $\Delta = \sum_{i=1}^{n} \Delta_{1i} \otimes \Delta_{2i} = \sum_{i=1}^{n} \Delta_{2i} \otimes \Delta_{1i}$ (by Remark 1.17). Using the commutativity and cocommutativity properties of m and Δ respectively we have, given $x \otimes y \in A \otimes A$:

$$\begin{aligned}
&\Delta \circ m(x \otimes y) = \Delta \circ m(y \otimes x) = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)(y \otimes x) = \\
&= (m \otimes \mathrm{id}) \circ \left(y \otimes \sum_{i=1}^{n} \Delta_{1i}(x) \otimes \Delta_{2i}(x) \right) = \\
&= (m \otimes \mathrm{id}) \circ \left(y \otimes \sum_{i=1}^{n} \Delta_{2i}(x) \otimes \Delta_{1i}(x) \right) = \\
&= \sum_{i=1}^{n} m(y \otimes \Delta_{2i}(x)) \otimes \Delta_{1i}(x) = \sum_{i=1}^{n} \Delta_{1i}(x) \otimes m(\Delta_{2i}(x) \otimes y) = \\
&= (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id})(x \otimes y).
\end{aligned}$$

□

Lemma 1.19. *Multiplying the left factors or the right factors of $\Delta(x)$ by an element $y \in A$ is equivalent, i.e.*

$$\sum_{i=1}^{n} (y \cdot \Delta_{1i}) \otimes \Delta_{2i} = \sum_{i=1}^{n} \Delta_{1i} \otimes (y \cdot \Delta_{2i}) \, .$$

Proof. Let $\Delta_i(x) = a_i \otimes b_i$, for $1 \leq i \leq n$, then, using the Frobenius identity (1.4) and the commutativity of m we get:

$$\begin{aligned} (y \cdot a_i) \otimes b_i &= (m \otimes \mathrm{id})(\mathrm{id}(y) \otimes \Delta_i(x)) = (m \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_i)(y \otimes x) = \\ &= \Delta_i \otimes m(y \otimes x) = \Delta_i \otimes m(x \otimes y) = \\ &= (\mathrm{id} \otimes m)(\Delta_i \otimes \mathrm{id})(x \otimes y) = (\mathrm{id} \otimes m)(\Delta_i(x) \otimes \mathrm{id}(y)) = a_i \otimes (b_i \cdot y). \end{aligned}$$

□

We can then write $y \cdot \Delta(x)$ to mean equivalently multiplication by y on the right factors of Δ or on the left ones.

By the previous lemma we have:

Remark 1.20. $\Delta(x \cdot y) = x \cdot \Delta(y) = y \cdot \Delta(x)$ for any $x, y \in A$.

We now describe an equivalent way of defining Frobenius systems:
Let $\iota\colon R \to A$ be an inclusion of commutative unitary rings, such that $\iota(1) = 1$. The map ι is a *Frobenius extension* if there exists an A-bimodule map $\Delta\colon A \to A \otimes_R A$ and an R-module map $\varepsilon\colon A \to R$ such that Δ is coassociative and cocommutative, and $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$.
Then a Frobenius system is a Frobenius extension together with a choice of Δ and ε.

The two definitions are equivalent:

- The inclusion ι corresponds to the unit of the algebra structure of A;
- The condition that Δ is coassociative and cocommutative corresponds to the coassociativity and cocommutativity of the coalgebra structure of A;
- The condition that Δ is an A-bimodule map translates to the Frobenius identity and lemmas and remarks from 1.17 to 1.20;
- The formula $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ expresses the fact that ε is the counit of the coalgebra structure of A.

We now describe two operations on Frobenius systems: the base change and the twist.

Base change

Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a Frobenius system. Let R' be a commutative unitary ring and $\varphi\colon R \to R'$ a ring homomorphism such that $\varphi(1) = 1$.
We define

$A' = A \otimes_R R'$,
$\Delta'\colon A' \to A' \otimes A'$ the map obtained from Δ by tensoring with the identity on R',
$\varepsilon'(a \otimes r') = \varphi(\varepsilon(a))r'$, for $a \in A, r' \in R'$.

Then $\mathcal{F}' = (R', A', \Delta', \varepsilon')$ is a Frobenius system, obtained by *base change* from $\mathcal{F}$.

Twisting

Let $\mathcal{F}$ be a Frobenius system and $y \in A$ an invertible element. We can modify the comultiplication and counit of $\mathcal{F}$ as follows:

$$\Delta'(a) = \Delta(y^{-1}a) = y^{-1}\Delta(a), \qquad \varepsilon'(a) = \varepsilon(ya).$$

We get a new Frobenius system $\mathcal{F}' = (R, A, \Delta', \varepsilon')$, obtained by *twisting* from $\mathcal{F}$.

Remark 1.21. Twisting is the only way we can modify the comultiplication and counit of a Frobenius system, that is, given Frobenius systems $\mathcal{F} = (R, A, \Delta, \varepsilon), \mathcal{F}' = (R, A, \Delta', \varepsilon')$ with the same ground ring R and algebra A, we have that $\mathcal{F}'$ is obtained by twisting from $\mathcal{F}$.
Thus, up to twisting, a Frobenius system is completely determined by its algebra structure.

Remark 1.22. The twist is an invertible operation: we always twist by an invertible element, thus if $\mathcal{F}'$ is obtained from $\mathcal{F}$ by twisting by an element y, then if we twist $\mathcal{F}'$ by y^{-1} we obtain again $\mathcal{F}$.
On the other hand the base change operation is not invertible: assume for example that $\mathcal{F}'$ is a base change of $\mathcal{F}$, and that $R = \mathbb{Z}$, $R' = \mathbb{Q}$. There are no unital ring homomorphisms $R' \to R$, thus $\mathcal{F}$ can't be obtained by base change from $\mathcal{F}'$.

Remark 1.23. Let β be a base change and τ a twist. Then $\beta \circ \tau = \tau \circ \beta$.
Moreover, given base changes $\beta_1, \ldots, \beta_n$ and twists $\tau_1, \ldots, \tau_n$ we have that $\beta_1 \circ \ldots \circ \beta_n$ is again a base change and $\tau_1 \circ \ldots \circ \tau_n$ is a twist.

In this work we will mostly deal with graded Frobenius systems. A Frobenius system is graded if A is graded as an R-module and if the structure maps respect this grading.

Definition 1.24.
- A ring R is *graded* if it decomposes as a direct sum $R = \bigoplus_{i=0}^{\infty} R_i$ of additive groups such that $R_i R_j \subseteq R_{i+j}$ for all i, j.
- An R-module A is *graded* if R is a graded ring and A decomposes as $A = \bigoplus_{j=0}^{\infty} A_j$, where A_j are abelian groups, so that $R_i A_j \subseteq A_{i+j}$ for all i, j.
  We define the *quantum degree*[1] of a homogeneous element x in a graded R-module as:
  $$\mathrm{qdeg}(x) = i \quad \text{if} \quad x \in A_i$$
  If A is a graded R-module, the grading on A induces a grading on tensor powers of A: $\mathrm{qdeg}(a_1 \otimes \ldots \otimes a_n) = \mathrm{qdeg}(a_1) + \ldots + \mathrm{qdeg}(a_n)$ (so $A^{\otimes n}$ is again a graded R-module).
- A map $\varphi\colon A \to B$ of graded R-modules is *graded of quantum degree* n if $\varphi(A_i) \subseteq B_{i+n}$ for all i.
- Finally, a Frobenius system $\mathcal{F} = (R, A, \Delta, \varepsilon)$ is *graded* if A is graded as an R-module and if the structure maps ι, m, ε and Δ are graded.

Let R be a graded ring. We denote the category whose objects are graded R-modules and whose morphisms are graded R-module maps by *R-mod*.
This category is not abelian. In fact it isn't even additive: let $\alpha, \beta\colon A \to B$ be graded R-module maps of degrees n, m respectively. If $n \neq m$ then $\alpha + \beta$ is not a graded map.

Let *R-mod*$_0$ be the category whose objects are graded R-modules and whose morphisms are grading-preserving, i.e. graded R-module maps of degree 0. This category is now abelian.

We will later use graded Frobenius systems $\mathcal{F} = (R, A, \Delta, \varepsilon)$ to construct homology theories; we will then need m and Δ to be graded maps of quantum degree 0 (i.e. morphisms of *R-mod*$_0$).
If a map $\varphi\colon A \to B$ of graded R-modules is graded of quantum degree k we can make it grading-preserving (i.e. of degree 0) by shifting the gradings of A or B as follows:

[1] The name "quantum" comes from the fact that, in the context of Khovanov homology, the quantum degree is related to the representation theory of the *quantum sl_2 algebra.*

Definition 1.25. The *quantum degree shift* $\{k\}$ is an automorphism of *R-mod*: if $A = \bigoplus_{j=0}^{\infty} A_j$ is a graded R-module, then the graded R-module $A\{k\}$ is given by

$$A\{k\}_j = A_{j-k}.$$

We also give the following definition:

Definition 1.26. A Frobenius system $\mathcal{F}$ is of *rank two* if there exists $X \in A$ such that $A \cong R1 \oplus RX$.

In this thesis we will encounter several graded Frobenius systems of rank two. We define them here.
Since ε, Δ are R-module maps, it is enough to define them on the basis $1, X$. Note that Δ is also an A-module map, so it would be enough to define it on 1; for clarity we also include its action on X.

Let F be a field. We define $\mathcal{F}_{BN'}$ by:

$$\begin{aligned} R_{BN'} &= F \\ A_{BN'} &= \frac{F[X]}{(X^2)} \\ \Delta_{BN'}\colon \quad 1 &\mapsto 1 \otimes X + X \otimes 1 \\ X &\mapsto X \otimes X \\ \varepsilon_{BN'}\colon \quad 1 &\mapsto 0 \\ X &\mapsto 1 \end{aligned} \tag{1.5}$$

We introduce a grading on the algebra $A_{BN'}$ by setting $\text{qdeg}(X) = 2$ and $\text{qdeg}(1) = 0$.
Then we see that the map $\Delta_{BN'}$ is graded of degree 2.
We can also define another grading on this Frobenius system by setting $\text{qdeg}(X) = -1$ and $\text{qdeg}(1) = 1$. In this case $\Delta_{BN'}$ and $m_{BN'}$ have degree -1.

The second system we introduce is $\mathcal{F}_{BN}$. Let $\mathbb{F}_2$ denote $\mathbb{Z}/2\mathbb{Z}$. Then:

$$\begin{aligned} R_{BN} &= \mathbb{F}_2[h] \\ A_{BN} &= \frac{\mathbb{F}_2[h,X]}{(X^2 - hX)} \\ \Delta_{BN}\colon \quad 1 &\mapsto 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ X &\mapsto X \otimes X \\ \varepsilon_{BN}\colon \quad 1 &\mapsto 0 \\ X &\mapsto 1 \end{aligned} \tag{1.6}$$

If we set $\mathrm{qdeg}(h) = \mathrm{qdeg}(X) = -2$ and $\mathrm{qdeg}(1) = 0$ this becomes a graded Frobenius system.

Next, we define $\mathcal{F}_{Lee}$:

$$\begin{aligned} R_{Lee} &= \mathbb{Q}[t] \\ A_{Lee} &= \frac{\mathbb{Q}[t,X]}{(X^2 - t)} \\ \Delta_{Lee}\colon \quad 1 &\mapsto 1 \otimes X + X \otimes 1 \\ X &\mapsto X \otimes X + t1 \otimes 1 \\ \varepsilon_{Lee}\colon \quad 1 &\mapsto 0 \\ X &\mapsto 1 \end{aligned} \tag{1.7}$$

This system is graded, with $\mathrm{qdeg}(X) = -2$, $\mathrm{qdeg}(1) = 0$ and $\mathrm{qdeg}(t) = -4$.

Finally, let $\mathcal{F}_{Univ}$ be given by:

$$
\begin{aligned}
R_{Univ} &= \mathbb{Z}[h,t] \\
A_{Univ} &= \frac{\mathbb{Z}[h,t,X]}{(X^2 - hX - t)} \\
\Delta_{Univ}\colon \quad 1 &\mapsto 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\
X &\mapsto X \otimes X + t1 \otimes 1 \\
\varepsilon_{Univ}\colon \quad 1 &\mapsto 0 \\
X &\mapsto 1
\end{aligned}
\tag{1.8}
$$

This system is graded, with $\mathrm{qdeg}(X) = \mathrm{qdeg}(h) = -2$, $\mathrm{qdeg}(1) = 0$ and $\mathrm{qdeg}(t) = -4$.

In this work we will mainly use Frobenius system $\mathcal{F}_{Univ}$. This system is particularly interesting because it is "universal" in the following sense:

Proposition 1.27. *Every rank two Frobenius system is obtained from $\mathcal{F}_{Univ}$ by a composition of a base change and a twist.*

Proof. Consider a rank two Frobenius system $\mathcal{F}' = (R', A', \Delta', \varepsilon')$.
The system $\mathcal{F}'$ has rank two, so 1 and X are a basis for A' as an R'-module (for some $X \in A'$).
Thus X^2 is a linear combination of 1 and X:

$$X^2 = h'X + t'1 \tag{1.9}$$

for some $h', t' \in R'$. Moreover, there are $c', a' \in R'$ such that $\varepsilon'(1) = -c'$ and $\varepsilon'(X) = a'$.
A basis for $A' \otimes A'$ as an R'-module is given by $1 \otimes 1, 1 \otimes X, X \otimes 1, X \otimes X$, thus there are $d', e', f' \in R'$ such that

$$\Delta'(1) = d'1 \otimes 1 + e'X \otimes X + f'(1 \otimes X + X \otimes 1)$$

(Δ' is cocommutative, so $1 \otimes X$ and $X \otimes 1$ have the same coefficient, see Remark 1.17).
Note that a', c', e', f', h', t' are uniquely determined.
Now, since Δ' is an A'-bimodule map, and using Remark 1.17, we have

$$(X \otimes 1)\Delta'(1) = (1 \otimes X)\Delta'(1)$$

that is, using equation (1.9):

$$d'X \otimes 1 + e'(h'X \otimes X + t'1 \otimes X) + f'(X \otimes X + h'X \otimes 1 + t'1 \otimes 1) = \\ = d'1 \otimes X + e'(h'X \otimes X + t'X \otimes 1) + f'(h'1 \otimes X + t'1 \otimes 1 + X \otimes X).$$

Looking at the coefficients for $1 \otimes X$ (and $X \otimes 1$) we get that

$$d' = e't' - h'f'. \tag{1.10}$$

Further, ε' and Δ' satisfy the counit identity, so $(\mathrm{id} \otimes \varepsilon') \circ \Delta'(1) = 1$. We obtain (using equation 1.10):

$$-c'e't' + c'h'f' + a'e'X' + a'f' - c'f'X' = 1$$

thus

$$a'e' - c'f' = 0, \\ a'f' + c'h'f' - c'e't' - 1 = 0.$$

Now consider the Frobenius system $\widetilde{\mathcal{F}}$ with:

$$\begin{aligned} \widetilde{R} &= \frac{\mathbb{Z}[a,c,e,f,h,t]}{(ae - cf, af + chf - cet - 1)}, \\ \widetilde{A} &= \frac{\widetilde{R}[X]}{(X^2 - hX - t)}, \\ \widetilde{\Delta}(1) &= (et - hf)1 \otimes 1 + eX \otimes X + f(1 \otimes X + X \otimes 1), \\ \widetilde{\Delta}(X) &= ft1 \otimes 1 + et(1 \otimes X + X \otimes 1) + (f + eh)X \otimes X, \\ \widetilde{\varepsilon}(1) &= -c, \\ \widetilde{\varepsilon}(X) &= a. \end{aligned} \tag{1.11}$$

for some variables a, c, e, f, h, t.
We give this system a grading by setting the degrees of a, c, e, f, h, t, X to be $0, -2, -2, 0, 2, 4, 2$ respectively.

We see that every Frobenius system $\mathcal{F}'$ is obtained from $\widetilde{\mathcal{F}}$ by a base change sending a, c, e, f, h, t to a', c', e', f', h', t' respectively.

If we show that $\mathcal{F}_{Univ}$ can be obtained from $\widetilde{\mathcal{F}}$ by twisting we are done, using Remark 1.22.

Consider the element $f + eX \in \widetilde{A}$. It is invertible with inverse $a + ch - cX$, so we can twist $\widetilde{\mathcal{F}}$ by it. Moreover $\operatorname{qdeg}(f + eX) = 0$, so after twisting the comultiplication and counit maps will have the same grading as in $\widetilde{\mathcal{F}}$. We get:

$$\begin{array}{ccc} \widetilde{\Delta} & \xrightarrow{\text{twist}} & \Delta_{Univ} \\ \widetilde{\varepsilon} & \xrightarrow{\text{twist}} & \varepsilon_{Univ}. \end{array}$$

So after the twist the structure maps only depend on the variables h, t. Thus we can replace $\widetilde{R}, \widetilde{A}$ by R_{Univ}, A_{Univ} respectively.
Now, since the twisting operation is invertible (Remark 1.22) and every Frobenius system is a base change of $\widetilde{\mathcal{F}}$, we proved that every Frobenius system $\mathcal{F}'$ of rank two is obtained from $\mathcal{F}_{Univ}$ by the composition of a base change and a twist.

□

1.4 Topological Quantum Field Theories

We described the categories Cob_1 of (1+1)-cobordisms and *R-mod*, with R a unitary ring. We now want to introduce monoidal functors $F\colon Cob_1 \to R\text{-}mod$ that relate these two categories. Such a functor is called a *Topological Quantum Field Theory* (TQFT). We will see that each Frobenius system of rank two determines a different TQFT.

Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a Frobenius system of rank two.
We define a TQFT $F\colon Cob_1 \to R\text{-}mod$ relative to $\mathcal{F}$. We send the object $\bar{n}$ to the R-module $A^{\otimes n}$, i.e. we assign to each circle of $\bar{n}$ a copy of A.
The empty object of Cob_1, which is the unit of the disjoint union operation, is sent to R, the unit of $\otimes$.
We then describe the action of F on the elementary cobordisms of Figure 1.5:

$$\begin{array}{ll} F(S_2^1) = m, & F(S_1^2) = \Delta, \\ F(S_0^1) = \iota, & F(S_1^0) = \varepsilon, \\ F(S_2^2) = Perm, & F(S_1^1) = \mathrm{id}\,. \end{array} \tag{1.12}$$

Given an arbitrary morphism S of Cob_1, i.e. a (1+1)-cobordism, we divide it in elementary surfaces and apply F to each of them. Thus F is well-defined if and only if for any decomposition of S in elementary surfaces we obtain equivalent maps of R-modules after applying F. This follows from the commutative algebra and cocommutative coalgebra structures on A, and the Frobenius identity, as stated in [Kh2].

We can translate the properties of Frobenius systems into equivalences of cobordisms as in Figure 1.6:

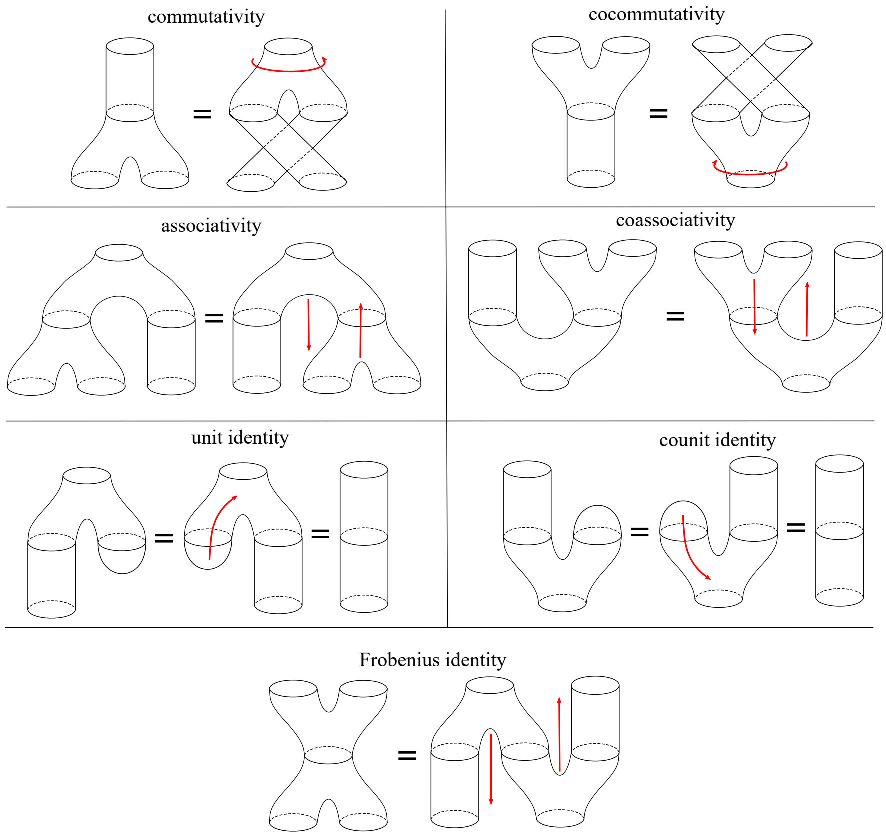


Figure 1.6

Throughout this thesis we will very often go back and forth between cobordisms of Cob_1 and morphisms of *R-mod* using TQFTs.

We end this chapter with three relations between cobordisms in Cob_1 and morphisms in *R-mod*. The first is true for any Frobenius system of rank two, the others hold for $\mathcal{F}_{Univ}$. These relations will be useful in Chapter 2, when proving the invariance of Khovanov homology under the Reidemeister moves.

Lemma 1.28. *(The T relation) If $\mathcal{F}$ is a rank two Frobenius system, the torus cobordism in Cob_1 corresponds to multiplication by 2 in R-mod.*

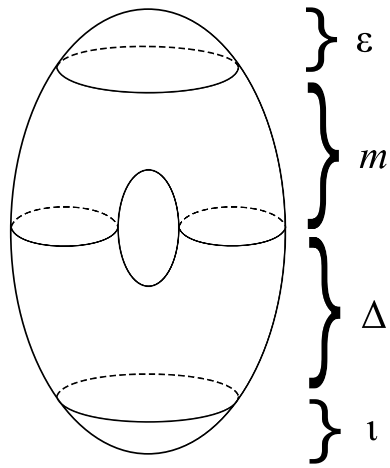


Figure 1.7

Proof. We observe that a torus is the composition of the four elementary cobordisms in Figure 1.7. After applying the TQFT relative to $\mathcal{F}$ we obtain $\varepsilon \circ m \circ \Delta \circ \iota$. Now consider the Frobenius system $\widetilde{\mathcal{F}}$, given by equations (1.11). Since every other rank two Frobenius system is obtained from $\widetilde{\mathcal{F}}$ by twisting, it is enough to compute $\varepsilon \circ m \circ \Delta \circ \iota$ for this system. We have:

$$\Delta(1) = (et - hf)1 \otimes 1 + eX \otimes X + f(1 \otimes X + X \otimes 1)$$
$$m(\Delta(1)) = 2et + 2fX + ehX - hf$$
$$\varepsilon(m(\Delta(1))) = -2cet + 2af + hea + hfc = 2(-cet + af + hfc) = 2 \cdot 1 = 2$$

(remember that in $\widetilde{R}$ we have $ea = fc$ and $-cet + af + hfc = 1$).

□

Lemma 1.29. *(The S relation) Consider the Frobenius system $\mathcal{F}_{Univ}$. The sphere cobordism in Cob_1 corresponds to multiplication by 0 in R-mod.*

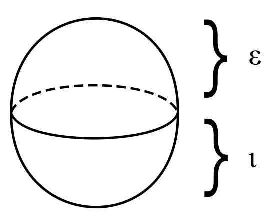


Figure 1.8

Proof. A sphere is the composition of the elementary cobordisms S_0^1 and S_1^0, as in Figure 1.8. After applying the TQFT we obtain $\varepsilon \circ \iota$. Now for $\mathcal{F}_{Univ}$ we have that $\varepsilon(\iota(1)) = \varepsilon(1) = 0$. □

Lemma 1.30. *(The 4Tu relation) Consider a surface S such that the intersection of S with a ball in $\mathbb{R}^3$ looks like the four cylinders of Figure 1.9. Call C_{ij} the cobordism obtained from S by connecting the circles labeled i and j with a tube and "closing" the remaining circles with S_1^0 or S_0^1. After applying the TQFT relative to $\mathcal{F}_{Univ}$ we have that $C_{12}+C_{34} = C_{13}+C_{24}$ (see Figure 1.10).*

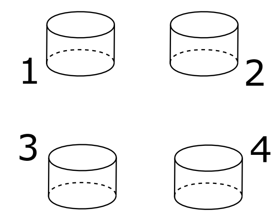


Figure 1.9

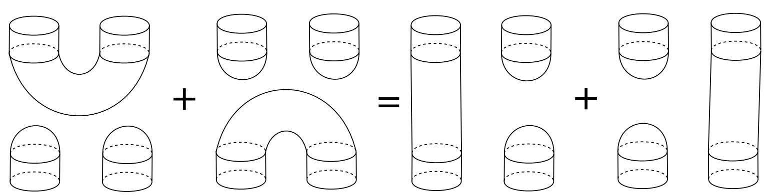


Figure 1.10

Proof. We have:

$$C_{12} \overset{\text{TQFT}}{\longmapsto} \Delta \circ \iota \circ (\varepsilon \otimes \varepsilon)$$
$$C_{34} \overset{\text{TQFT}}{\longmapsto} (\iota \otimes \iota) \circ \varepsilon \circ m$$
$$C_{13} \overset{\text{TQFT}}{\longmapsto} (\mathrm{id} \otimes \iota) \circ (\mathrm{id} \otimes \varepsilon)$$
$$C_{24} \overset{\text{TQFT}}{\longmapsto} (\iota \otimes \mathrm{id}) \circ (\varepsilon \otimes \mathrm{id})$$

Now it is enough to verify that (after applying the TQFT) $C_{12} + C_{34} = C_{13} + C_{24}$ on $1 \otimes 1, 1 \otimes X, X \otimes 1, X \otimes X$. This is an easy exercise. □

2 Khovanov homology

In this chapter we introduce several link invariants (one for each graded Frobenius system of rank two). Let $\mathcal{F}$ be a graded Frobenius system of rank two. Given a link L we can associate to it a "cube of resolutions" whose elements are in the category Cob_1, then, applying the TQFT associated to $\mathcal{F}$ and shifting the degrees appropriately, we obtain a chain complex. The homology of this complex is a link invariant called Khovanov homology.
Khovanov homology has several interesting applications, and in particular it can be used to find a lower bound for the unknotting number, as we will see in Chapter 3.
We will mostly follow [BN1], [BN2], [BN3] and [Tu1]. For the last part, about the relationship between chain complexes relative to different Frobenius systems, we will use [Kh1].

Throughout this chapter, to help understand and visualize the construction of Khovanov homology we will use figures featuring the example of the Trefoil knot. This knot has the advantage of only having 3 crossings, and is thus easy to visualize.

2.1 The cube of resolutions

Let L be a link and let us fix a diagram D for L. We denote the n crossings of D by $c_1, \ldots, c_n$. For every i we can replace a small tangle (i.e. the intersection of D with a small disk) containing c_i with a 0- or 1-resolution of c_i:

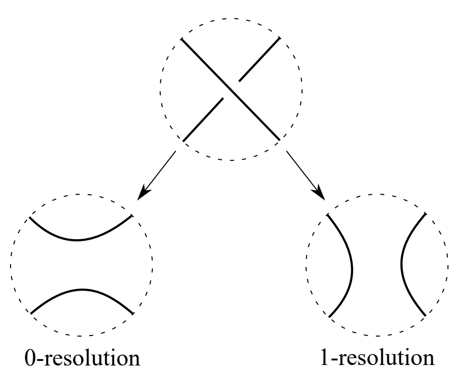


Figure 2.1

Definition 2.1. The *0-resolution* and *1-resolution* of a crossing c_i of D are the tangles shown in Figure 2.1.

Definition 2.2. A *complete resolution* of a diagram D is a diagram where every crossing is replaced by a 0- or 1-resolution.

Let $v = (v_1, \ldots, v_n) \in \{0,1\}^n$. We denote by D_v the complete resolution of D obtained by replacing each crossing c_i with its v_i-resolution (see Figure 2.2).

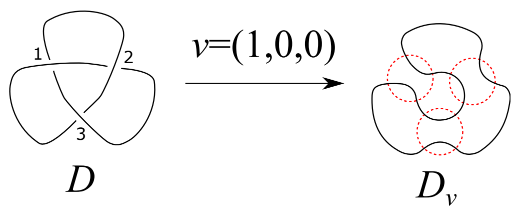


Figure 2.2

Remark 2.3. For all $v \in \{0,1\}^n$ we have that D_v is a disjoint union of circles.
This follows immediately from the fact that D_v is a diagram with zero crossings.
We can then see D_v as an object of the category Cob_1.

For each v we denote the number of connected components of D_v (i.e. the number of circles) by k_v.
Given $v = (v_1, \ldots, v_n) \in \{0,1\}^n$ we further write $|v| = v_1 + \ldots + v_n$, i.e. $|v|$ is the number of ones appearing in v.
If $|u| = |v| + 1$ and D_u only differs from D_v at one crossing (i.e. there is a j such that $u_i = v_i$ for all $i \neq j$ and $u_j = 1, v_j = 0$) we write $v < u$.

Remark 2.4. If $v < u$ then $k_u = k_v \pm 1$: suppose we ($v_i = u_i$)-resolve the crossings of D for all $i \neq j$. Then c_j will connect two arcs as in Figure 2.3 a) or b). The 0-resolution at c_j can then either generate one circle (in a)), or two circles (in b)). If the 0-resolution gives two circles, then 1-resolving c_j will give two circles, and vice-versa.

We now construct an n-dimensional cube having at each vertex $v \in \{0,1\}^n$ the complete resolution D_v. Since all diagrams D_v are disjoint unions of circles, i.e. objects of the category Cob_1, we can consider (1+1)-cobordisms between them. If $v < u$ and they differ at the entry j we connect D_v and D_u by a saddle cobordism S going from the 0- to the 1-resolution of c_j, as in Figure 2.4.

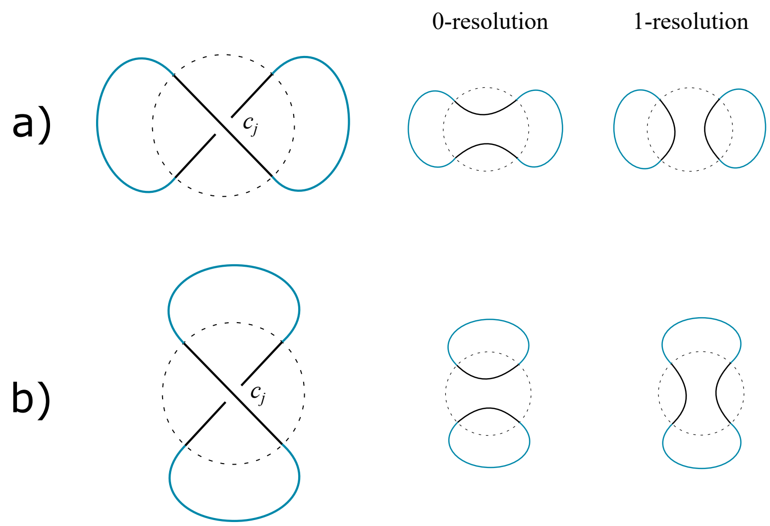


Figure 2.3

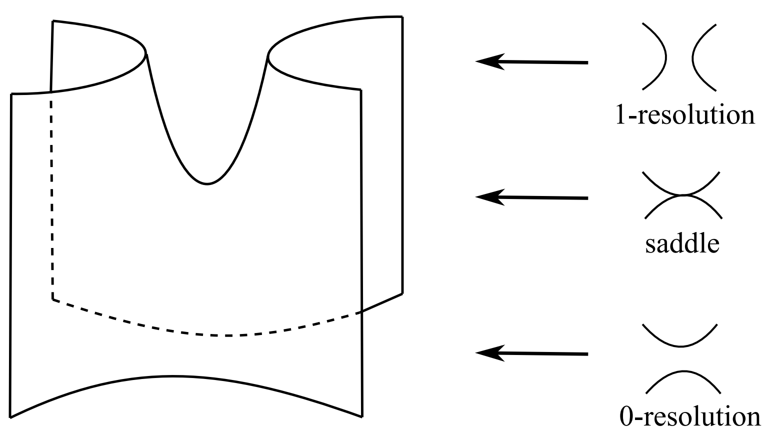


Figure 2.4

By Remark 2.4, the cobordism S between D_v and D_u will either split a circle in two or merge two circles, and we set it to be the identity on all other circles oof D_v. In the first case S will be a split cobordism, in the second case a merge cobordism. The edges of the cube will be then given by merge and split (1+1)-cobordisms.
We call such a cube a *cube of resolutions* of the diagram D. In Figure 2.5 we see the cube of resolution of the trefoil diagram: the leftmost edges are merge cobordisms, all the others are split cobordisms.

Observe that if we reorder the crossings of D we will get a different cube of resolutions.

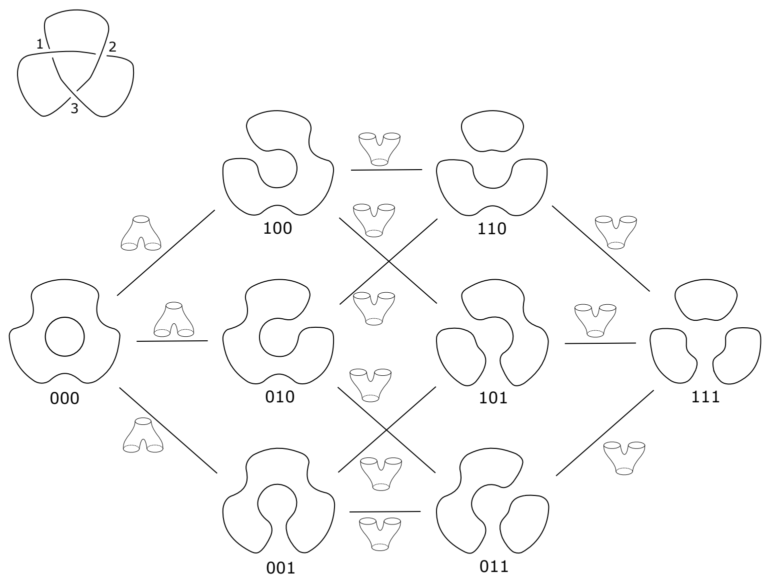


Figure 2.5

Given a cube of resolutions of D our goal is to associate to it an algebraic structure, in order to obtain a chain complex whose homology is invariant under all Reidemeister moves. This will be done by applying a TQFT to the cube of resolutions, and we will see that not all TQFTs (i.e. not all Frobenius systems) generate such a chain complex.

Let us fix a graded rank two Frobenius system $\mathcal{F} = (R, A, \Delta, \varepsilon)$. The R-module A decomposes as $A = R1 \oplus RX$ for some $X \in A$. Up to a global shift and rescaling we can assume $\text{qdeg}(1) = 0$ and $\text{qdeg}(X) = -2$.
Using the TQFT relative to $\mathcal{F}$ we associate to each D_v the R-module $A^{\otimes k_v}$, so we associate a copy of A to each circle of D_v (we see D_v as an object in Cob_1, so the circles are ordered). Let $v < u$: the TQFT sends the cobordism S between vertices v and u to a map $\widetilde{\delta}\colon A^{\otimes k_v} \to A^{\otimes k_u}$. First assume that S is a merge cobordism. Then $\widetilde{\delta}$ acts as the identity on all copies of A except the two corresponding to the circles of D_v that are merged by S. On these two copies $\widetilde{\delta}$ will be given by the multiplication map of $\mathcal{F}$: $m\colon A \otimes A \to A$. Similarly, if S is a split cobordism, $\widetilde{\delta}$ will be the identity on all copies of A except that corresponding to the circle of D_v that is split by S. On that copy $\widetilde{\delta}$ acts as the comultiplication $\Delta\colon A \to A \otimes A$ of $\mathcal{F}$ (see Figure 2.6).

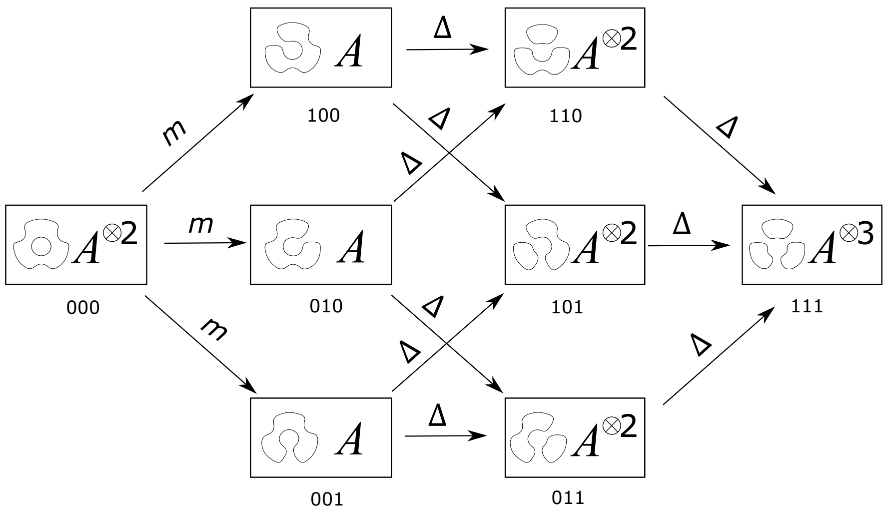


Figure 2.6

2.2 The Chain complex

We now have an algebraic structure on the cube of resolutions. We would like to obtain from it a chain complex: its groups will be obtained by taking vertical direct sums of the vertices and its differentials will be vertical sums of the edges. We will get a chain complex if the differentials d fulfil the two following conditions:

1. d is a graded map of quantum degree 0,
2. $d \circ d = 0$.

In order for these conditions to be satisfied we have to slightly modify the algebraic structure that we gave on the cube of resolutions.
For the first condition to hold we have to make sure that all the edge maps have quantum degree 0, i.e. that $m, \Delta \in R\text{-}mod_0$. This is achieved by shifting, at each vertex v, the quantum grading on $A^{\otimes k_v}$ by $|v| + k_v$, i.e. we replace $A^{\otimes k_v}$ by $A^{\otimes k_v}\{|v| + k_v\}$ (see Definition 1.25 for the definition of quantum shift). We call these shifts *local shifts*.
For the second condition it suffices to make sure that all squares in the cube of resolutions anticommute. We sprinkle signs on the edge maps in the following way: let $v < u$ and $\widetilde{\delta}$ be an edge map between these two vertices. Assume the entry at which $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ differ is the j-th one, i.e. $v_j = 0, u_j = 1$ and $v_i = u_i$ for $i \neq j$. Then we define

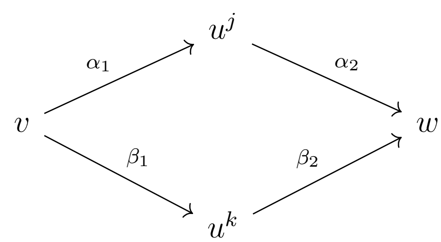
$$\delta = (-1)^{\xi_{vu}}\widetilde{\delta}, \qquad \text{where} \qquad \xi_{vu} = \sum_{i=1}^{j-1} v_i. \tag{2.1}$$

See Figure 2.9 for a representation of the local shifts and the edge maps δ. The maps with a negative sign (i.e. the maps where ξ_{vu} is odd) are marked with a small white square.

We want to prove the following lemma:

Lemma 2.5. *With the new edge maps δ all squares of the cube of resolutions anticommute.*

Proof. Let us first show that before sprinkling the signs all squares commute.
Consider a square of the cube of resolutions



where v, u^j, u^k, w are vertices and α_i, β_i edge maps between them. The vertices v and w differ at two crossings c_j, c_k: they are both 0-resolved in v and both 1-resolved in w. Following the path $\alpha_2 \circ \alpha_1$ we first 1-resolve c_j and then c_k, while following $\beta_2 \circ \beta_1$ we first 1-resolve c_k. Thus the vertices are such that $v < u^j$ and they differ at the j-th entry, $v < u^k$ and they differ at the k-th entry, $u^j, u^k < w$ and they differ from w at entries k and j respectively. Summing up:

$$
\begin{aligned}
&v_j = v_k = u^k_j = u^j_k = 0,\\
&u^j_j = u^k_k = w_j = w_k = 1,\\
&v_i = u^j_i = u^k_i = w_i \text{ for } i \neq j, k.
\end{aligned}
$$

Depending on the position of the crossings c_j, c_k we have to consider eight cases, shown in Figure 2.7. In this figure the black circles represent D_v (where both crossings are 0-resolved) and the red lines show the position of crossings c_j, c_k, or, more precisely, the 1-handles that we need to attach to go from the 0- to the 1-resolution of crossings c_j, c_k.

In cases 1), 2), 5), 6), 8) we clearly have that $\alpha_2 \circ \alpha_1 = \beta_2 \circ \beta_1$.
In case 3) the fact that the square commutes follows from the associativity of m, since $\alpha_2 \circ \alpha_1 = m \circ (m \otimes \mathrm{id})$ while $\beta_2 \circ \beta_1 = m \circ (id \otimes m)$.
In case 4) it follows from the Frobenius identity, as shown in Figure 2.8 , and finally in case 7) it follows from the coassociativity of Δ.

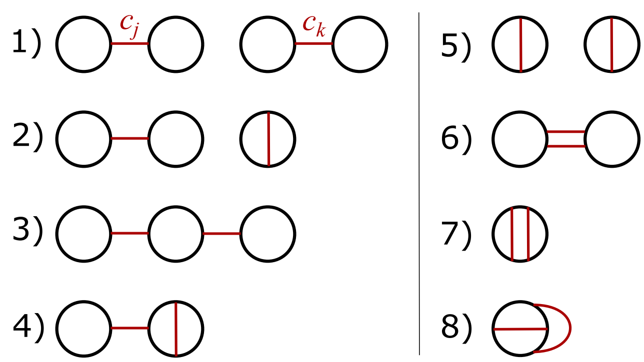


Figure 2.7

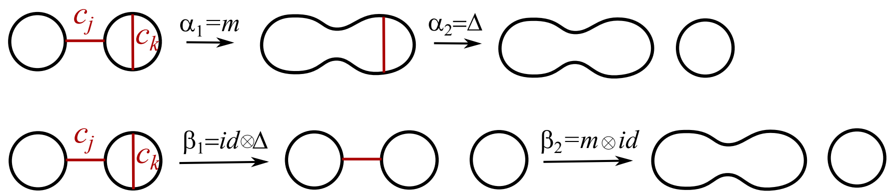


Figure 2.8

Now we show that the squares anticommute after sprinkling the signs.
The square anticommutes if $\alpha_1\alpha_2 = -\beta_1\beta_2$. We alreay showed that before sprinkling the signs the square commutes, so it remains to check that $(-1)^{\xi_{vu^j}+\xi_{u^jw}} = -(-1)^{\xi_{vu^k}+\xi_{u^kw}}$.
Without loss of generality we can assume that $j < k$. Then:

$$\xi_{vu^j} = v_1 + \ldots + v_{j-1}, \qquad \xi_{u^jw} = v_1 + \ldots + v_{j-1} + u^j_j + v_{j+1} + \ldots + v_{k-1},$$
$$\xi_{u^kw} = v_1 + \ldots + v_{j-1}, \qquad \xi_{vu^k} = v_1 + \ldots + v_{j-1} + v_j + v_{j+1} + \ldots + v_{k-1}.$$

Since $u^j_j = 1$ and $v_j = 0$, we have that $\xi_{u^jw} = \xi_{vu^k} + 1$ and $\xi_{vu^j} = \xi_{u^kw}$. This completes the proof.

□

We now construct a chain complex $C^{*,*}$ from the cube of resolutions.
Let first
$$\widetilde{C}^{i,*} = \bigoplus_{|v|=i} A^{\otimes k_v}\{i + k_v\}.$$

(see Figure 2.9)

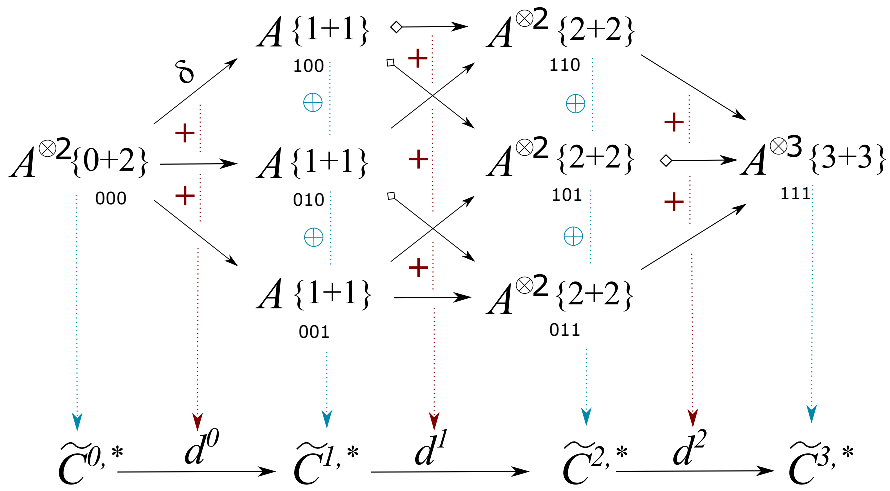


Figure 2.9

Now let us call n_+, n_- respectively the number of positive and negative crossings of D. To obtain $C^{*,*}$ we want to make a homological shift of n_- and a quantum shift of $n_+ - 2n_-$ on $\widetilde{C}^{*,*}$. We will refer to these as *global shifts*, because they act on the entire complex, whereas the local shifts act on the vertices of the cube of resolutions.
We already saw the definition of quantum shift in Chapter 1, so let us define the homological shift $[r]$:

Definition 2.6. Let $\mathcal{C}^*$ be a chain complex. Then a *quantum shift* of $\mathcal{C}^*$ by $[r]$ gives a chain complex $\mathcal{C}^*[r]$ such that

$$\mathcal{C}^n[r] = \mathcal{C}^{n-r}.$$

Then the chain groups of $C^{*,*}$ are given by

$$C^{i,*} = \widetilde{C}^{i,*}[-n_-]\{n_+ - 2n_-\}.$$

We will later see that these shifts ensure that $C^{*,*}$ is invariant (up to chain homotopy equivalence) under the Reidemeister moves.
The chain groups $C^{i,j}$ are bigraded: i represents the homological grading, j the quantum grading.
The differentials $d\colon C^{i,*} \to C^{i+1,*}$ are given by

$$d = \sum_{|v|=i} \delta_{vu}$$

where $\delta_{vu}\colon A^{\otimes k_v} \to A^{\otimes k_u}$ (for some $v < u$) is an edge map. The differentials are bigraded of bidegree (1,0).
Throughout this work we will always write d to denote differentials of $C^{*,*}$ and δ to indicate edge maps of the cube of resolutions.

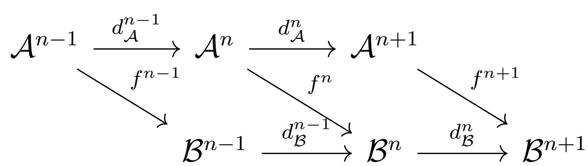
Remark 2.7. Reordering the crossings of D generates a chain complex isomorphic to C. This follows from the fact that if we reorder the crossings we will obtain a cube of resolutions that differs only on the vertical order of the spaces and on the signs of the edge maps: each column $|v| = i$ will contain the same spaces, but in a different order, and the same happens for the maps. Moreover, each sprinkling of the signs that makes all squares of the cube of resolutions anticommute gives chain homotopy equivalent chain complexes.

Then the chain complex associated to a link diagram D is well defined up to chain homotopy equivalence. We denote it by $C(D)$, and we call its homology $H(D)$. Thus each graded Frobenius system $\mathcal{F}$ of rank two defines a bigraded homology theory, called *Khovanov homology relative to* $\mathcal{F}$. We will show that, for every Frobenius system of rank two, $H(D)$ is a link invariant.

Mapping Cone

Let us fix a crossing c of D, and let us reorder the crossings so that c is the last one (for simplicity). Denote by D_0 and D_1 the diagrams obtained from D by 0- and 1-resolving c respectively. Then $\widetilde{C}(D)$ is the mapping cone of a chain map $\boldsymbol{\delta}\colon \widetilde{C}(D_0) \to \widetilde{C}(D_1)\{1\}$.

Definition 2.8. Let $(\mathcal{A}, d_\mathcal{A}), (\mathcal{B}, d_\mathcal{B})$ be chain complexes and $f\colon \mathcal{A} \to \mathcal{B}$ a chain map between them.



The *mapping cone* of f, denoted by $\text{Cone}(f)$, is the chain complex $\mathcal{C}$ whose groups are

$\mathcal{C}^n = \mathcal{A}^n \oplus \mathcal{B}^{n-1}$ (i.e. $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}[1]$) and whose differentials are given by

$$d^n = \begin{pmatrix} d^n_{\mathcal{A}} & 0 \\ (-1)^n f^n & d^{n-1}_{\mathcal{B}} \end{pmatrix}.$$

Let $v = (v_1, \dots, v_{n-1})$ be a vertex of the cube of resolutions of D_0. Observe that v corresponds to the vertex $v^0 = (v_1, \dots, v_{n-1}, 0)$ of the cube of D (i.e. $(D_0)_v = D_{v^0}$). We denote by $\boldsymbol{\delta}_v$ the map corresponding to the saddle cobordism going from the 0- to the 1-resolution of c (as in Figure 2.4), i.e. $\boldsymbol{\delta}_v : (D_0)_v \to (D_1)_v$. We note that $(D_1)_v = D_{v^1}$, where $v^1 = (v_1, \dots, \dots, v_{n-1}, 1)$. So at every vertex v we have either $\boldsymbol{\delta} = m$ or $\boldsymbol{\delta} = \Delta$. Let now $\boldsymbol{\delta} : \widetilde{C}(D_0) \to \widetilde{C}(D_1)\{1\}$ be the chain map given by $\boldsymbol{\delta}^i = \sum_{|v|=i} \boldsymbol{\delta}_v$. Then $\widetilde{C}(D)$ is the mapping cone of $\boldsymbol{\delta}$ (see Figure 2.10).
We can write

$$\widetilde{C}(D) = \left(\widetilde{C}(D_0) \xrightarrow{\boldsymbol{\delta}} \widetilde{C}(D_1)\{1\} \right). \tag{2.2}$$

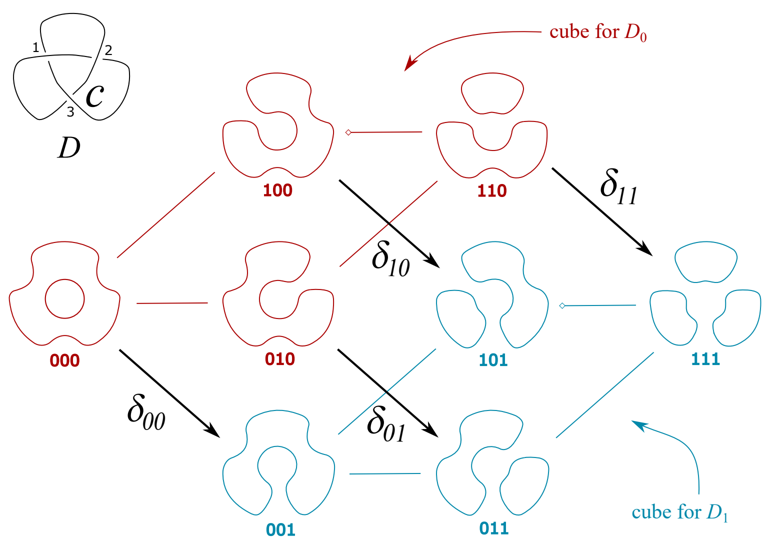


Figure 2.10

We would like to have a similar result for the complex $C(D)$, that is, after the global shifts $[-n_-]\{n_+ - 2n_-\}$. In this case however the situation is slightly more complicated.

Assume c is a positive crossing. Then D_0 inherits an orientation from D (see Figure 2.11) and $n^0_- = n_-, n^0_+ = n_+ - 1$ (where n^0_-, n^0_+ are, respectively, the number of negative and positive crossings of D_0). The diagram D_1 on the other hand doesn't inherit an orientation, so we can't relate n_-^1, n_+^1 to n_-, n_+. We thus assign to D_1 an arbitrary orientation and define $\gamma = n_-^1 - n_-$. Then

$$C(D) = C(D_0)\{1\} \oplus C(D_1)[\gamma + 1]\{3\gamma + 2\}.$$

Let now c be negative. Then D_1 inherits an orientation from D (see Figure 2.11) and $n_-^1 = n_- - 1, n_+^1 = n_+$. D_0 does not inherit an orientation, so we orient it arbitrarily and define $\gamma = n_-^0 - n_-$. Then

$$C(D) = C(D) = C(D_0)[\gamma]\{3\gamma + 1\} \oplus C(D_1)\{-1\}$$

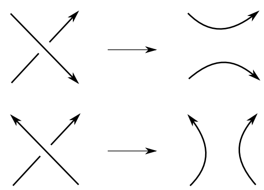


Figure 2.11

Throughout this thesis, in particular in Chapter 3, it will often be convenient to work with $\widetilde{C}(D)$ rather than $C(D)$, to avoid dealing with the shifts mentioned above.

2.3 Invariance under the Reidemeister moves

Our goal is to prove that the homology of the chain complex C is a link invariant. This will follow from the fact that if D and D' are link diagrams related by a Reidemeister move (see Figure 1.2), then $C(D)$ and $C(D')$ are chain homotopy equivalent.
We give the proof of this for the universal Frobenius system $\mathcal{F}_{Univ}$ defined in (1.8). We will see later that from this follows that all Khovanov homology theories coming from rank two Frobenius systems are link invariants. For this section we will use Bar-Natan's article [BN2]. For more details about chain complexes for tangles look at this source.

Invariance under $R1$

In this proof we will often implicitly use the TQFT to go back and forth from spaces and maps in the algebraic cube of resolutions to circles D_v and cobordisms. We make the category Cob_1 pre-additive by allowing formal sums of cobordisms (details can be found in [BN2]).

For simplicity we won't write homological and quantum shifts in the chain complexes, as they only come into play at the end of the proof.

Let D, D' be link diagrams related by a positive $R1$ move as in Figure 1.2.
Let n_+, n_- and n'_+, n'_- be the number of positive and negative crossings of D and D' respectively. Then $n'_+ = n_+ + 1$ and $n'_- = n_-$.

Let now c be the crossing in D' corresponding to the $R1$ move and let D'_0, D'_1 be the diagrams obtained from D' by 0- and 1-resolving c respectively. Thus D'_0 and D'_1 are identical except for a tangle, where $D'_0 =$ and $D'_1 =$ Then the complex $C(D')$ is the mapping cone of the chain map $\boldsymbol{\delta}\colon C(D'_0) \to C(D'_1)$, where $\boldsymbol{\delta}$ is the saddle-move, or rather the bundle of saddle-moves, from the 0- to the 1-resolution at c (so $\boldsymbol{\delta} = m$ at every vertex of the cube of resolutions of D'_0).

Our aim is to find a chain homotopy equivalence $F\colon C(D) \to C(D')$ that has bidegree $(0,0)$. At each vertex v of the cube of resolutions of D the map F will go from a complete resolution D_v of D to a complete resolution D'_u of D'_0. The resolutions D_v and D'_u are identical everywhere except at a tangle (where the $R1$ move happens), where $D_v =$ and $D'_u =$. So we define F as the identity everywhere on D_v except at the tangle , where F acts as the map F^0 defined in Figure 2.12.
We also define a map $G\colon C(D') \to C(D)$ as follows: G is the zero map on the cube of resolutions of D'_1. At each vertex of the cube of D'_0 we define G as the identity everywhere except at the tangle where G acts as the map G^0 of Figure 2.12.

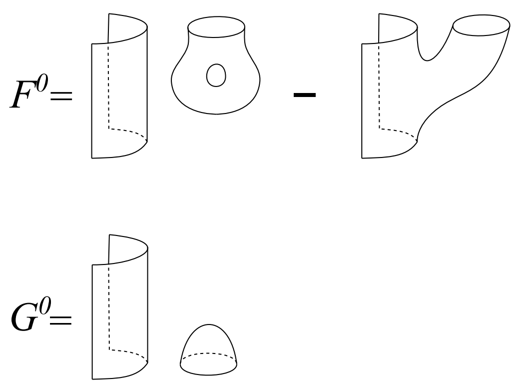


Figure 2.12

We can thus interpret Figure 2.13 as a diagram where each tangle represents a complex: the tangle in the top left corner stands for the complex $C(D)$, the tangle stands for $C(D'_0)$ and the tangle in the bottom right corner stands for $C(D'_1)$. Each map of the diagram then represents a bundle of maps between the vertices of the cubes of resolutions of the complexes $C(D), C(D'_0), C(D'_1)$.

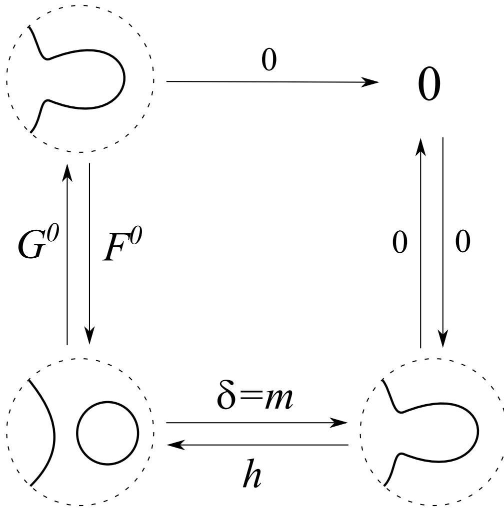
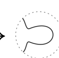


Figure 2.13

We have to prove that F is a chain homotopy equivalence with inverse G. For the proof we will only consider the local tangles where the Reidemeister moves happen (see [BN2] for details). Instead of $C(D)$ and $C(D')$, let us then consider the complexes of tangles

$C\Big($  $\Big)$ and

$C\Big($  $\Big) = \Big(0 \longrightarrow$  $\xrightarrow{m}$  $\longrightarrow 0\Big)$

Then F consists of two maps: $F^0\colon C^0\big($  $\big) \to C^0\big($  $\big)$ and $F^1\colon 0 = C^1\big($  $\big) \to C^1\big($  $\big)$ (that is defined as 0). Similarly, G consists of the maps $G^0\colon C^0\big($  $\big) \to C^0\big($  $\big)$ and $G^1\colon C^1\big($  $\big) \to C^1\big($  $\big) = 0$ defined as 0 (see Figure 2.13).

We have:

- F is a chain map: we only need to check that $\delta F = 0$, but this follows from an isotopy.
- G is a chain map: this is clear.
- $GF = \mathrm{id}$: This follows from the T relation (see Lemma 1.28) and the counit identity (see Figure 1.6), as shown in Figure 2.14.

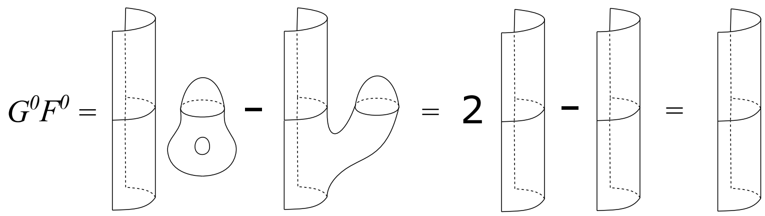


Figure 2.14

- $FG \sim \mathrm{id}$. We define the map $h\colon C^1$ () $\to C^0$ () as in Figure 2.15. We want to show that $FG - \mathrm{id} + h\delta + \delta h = 0$. By the unit identity (see Figure 1.6) we see that $\delta h = \mathrm{id}$, thus $F^1G^1 - \mathrm{id} + \delta h = 0 - \mathrm{id} + \delta h = 0$. It remains to show that $F^0G^0 - \mathrm{id} + h\delta = 0$, and this follows from the $4Tu$ relation (see Lemma 1.30), as shown in Figure 2.16: we have that $F^0G^0 = C_{12} - C_{13}$ and $-\mathrm{id} + h\delta = -C_{24} + C_{34}$.

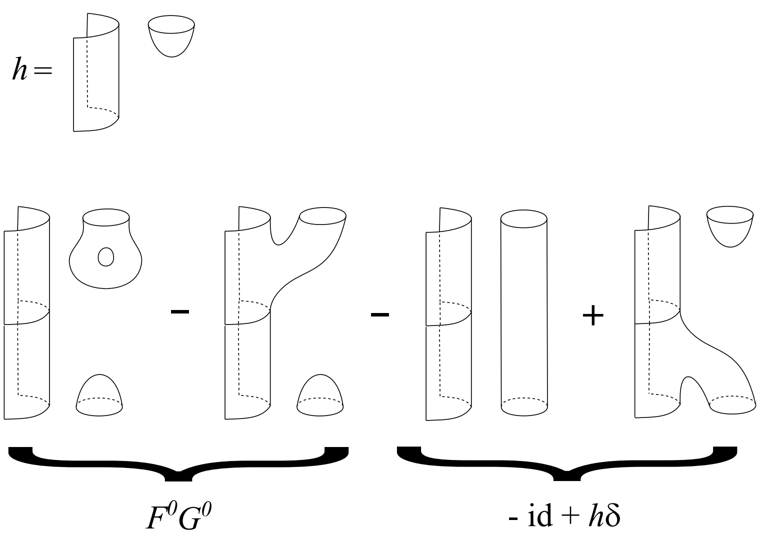


Figure 2.15

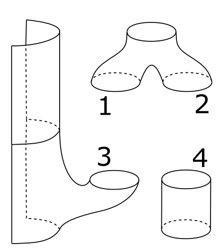


Figure 2.16

Thus, ignoring quantum shifts, the complexes C () and C (), and thus also $C(D)$ and $C(D')$, are chain homotopy equivalent.

The only thing it remains to check to prove that $C(D)$ and $C(D')$ are chain homotopy equivalent is that the maps F, G have quantum degree 0. We observe the following:

Remark 2.9. Let us consider the cubes of resolutions for D and D' before the global shifts (but with the local shifts at each vertex), and let $f\colon D_v \to D'_u$, for some vertices u, v, be a morphism. We have:

(a) If f is the cap morphism (i.e. the counit ε), it sends $X \mapsto 1$, and decreases the number of circles by 1, so it has $\text{qdeg} = 2 - 1 + |u| - |v| = 1 + |u| - |v|$.

(b) If f is the cup (i.e. the unit ι), it increases the number of circles by 1, and sends $1 \mapsto 1$, thus it has $\text{qdeg} = 1 + |u| - |v|$.

(c) If f is the identity morphism, it has $\text{qdeg} = |u| - |v|$.

(d) If f is the merge (i.e. m), it has $\text{qdeg} = -1 + |u| - |v|$ because it decreases the number of circles by 1.

(e) If f is the split (i.e. Δ), it sends $1 \mapsto X \otimes 1 + 1 \otimes X - h1 \otimes 1$ and increases the number of circles by 1, thus it has $\text{qdeg} = -2 + 1 + |u| - |v| = -1 + |u| - |v|$.

By this remark we have that, before global shifts, $\text{qdeg}(F^0) = -1$ and $\text{qdeg}(G^0) = 1$ (since $|u| = |v|$).
Let us now see what happens after the global shifts. Let α be a homogeneous element of D'_v for some v and let $q + n'_+ - 2n'_- = \text{qdeg}(\alpha)$. Then after the global shifts

$$\text{qdeg}(G^0(\alpha)) = q + 1 + n_+ - 2n_- = q + 1 + n'_+ - 1 - 2n'_- = \text{qdeg}(\alpha).$$

Let now β be a homogeneous element of D_v and let $q + n_+ - 2n_- = \text{qdeg}(\beta)$. Then, after the shifts,

$$\text{qdeg}(F^0(\beta)) = q - 1 + n'_+ - 2n'_- = q - 1 + n_+ + 1 - 2n_- = \text{qdeg}(\beta).$$

Thus F, G have quantum degree 0. They also have homological degreee 0 because $n_- = n'_-$.

Remark 2.10. Invariance under a negative $R1$ move follows from the positive $R1$ and from $R2$ (see Remark 1.9).

We note that the fact that $\mathcal{F}_{Univ}$ has rank two is crucial. We have the following:

Remark 2.11. If $\mathcal{F}$ is not of rank two we don't have invariance under $R1$.

We give another proof of R1, this time for a generic Frobenius system $\mathcal{F}$ of rank two, to show more explicitly where the rank of $\mathcal{F}$ comes into play. Again, we ignore homological and quantum shifts.

Let us assume that D, D' are link diagrams related by a positive $R1$ move. It is enough to consider the case where D is the Unknot and D' is a one-crossing diagram. The chain complex $C(D')$ is:

$$0 \longrightarrow A \otimes A \xrightarrow{m} A \longrightarrow 0,$$

while the chain complex $C(D)$ is:

$$0 \longrightarrow A \longrightarrow 0.$$

The complex $C(D')$ splits through the map $\iota \otimes \mathrm{id}$ as

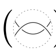
$$0 \longrightarrow A \longrightarrow A \longrightarrow 0$$

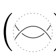
and

$$0 \longrightarrow \ker(m) \longrightarrow 0 \longrightarrow 0.$$

The first complex is contractible, so we only need to show that there is an A-module isomorphism between $\ker(m)$ and A. We have $\ker(m) \cong (A/R) \otimes A$. Since $\mathcal{F}$ is a rank two Frobenius system, i.e. $A \cong R1 \otimes RX$ for some $X \in A$, we have that $A/R \cong R$, i.e. $\ker(m) \cong R \otimes A \cong A$.

Invariance under R2

Assume now that D, D' are related by a $R2$ move, as shown in Figure 1.2. We proceed as in the first proof of $R1$ invariance and work with tangles rather than full link diagrams.
The complex $C(D)$ is replaced by C (), while $C(D')$ is replaced by the complex C () shown on the bottom of Figure 2.17

This time we have $n'_+ = n_+ + 1$ and $n'_- = n_- + 1$. So we have to take into account the fact that the global shift of $[-n_-]$ to obtain $C(D)$ and the global shift of $[-n'_-]$ to obtain $C(D')$ differ by 1. This means that we have to define the map F from $\widetilde{C}(D)$ to $\widetilde{C}(D')[1]$ (or rather, from $\widetilde{C}$ () to $\widetilde{C}$ () $[1]$), so that after the global shift it will have homological degree 0.

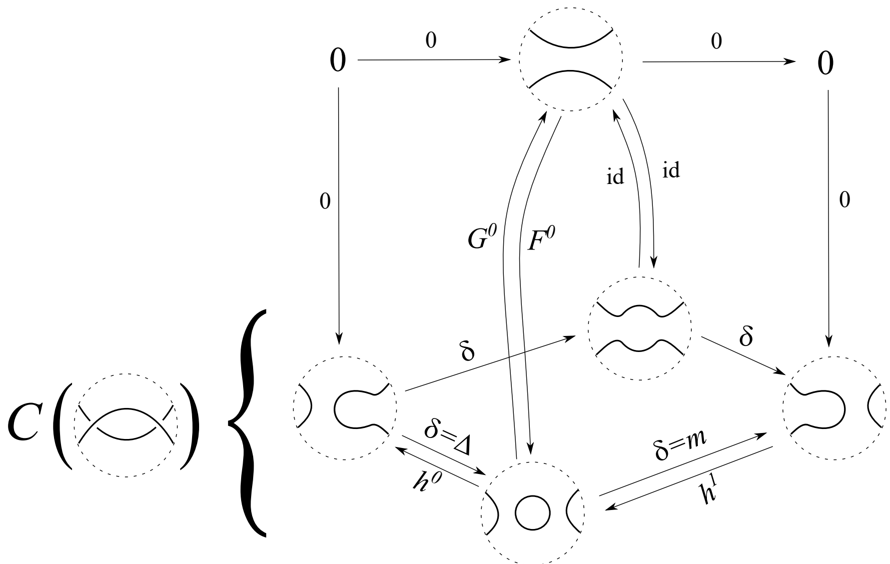


Figure 2.17

We define F and G to be 0 everywhere except for $F^0\colon \widetilde{C}^0\left(\right) \to \widetilde{C}^1\left(\right)$ and $G^0\colon \widetilde{C}^1\left(\right) \to \widetilde{C}^0\left(\right)$, that are defined in Figure 2.18.

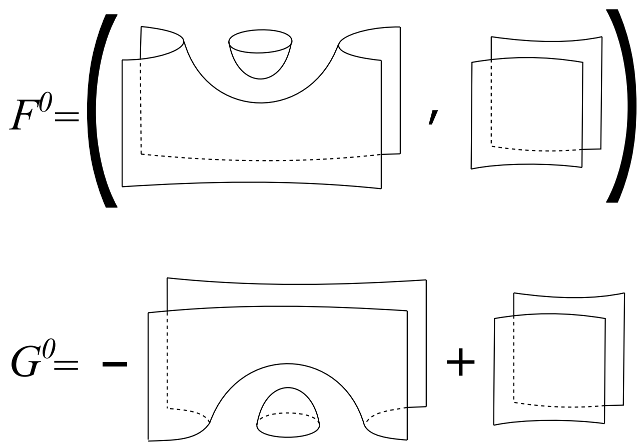


Figure 2.18

We now show that F is a chain homotopy equivalence with inverse G. We have:

- F is a chain map: we only need to check that $\delta F^0 = 0$, and this follows from the unit identity (see Figure 1.6).

- G is a chain map: we only need to check that $G^0\delta = 0$, and this follows from the counit identity (see Figure 1.6).
- $GF = \mathrm{id}$. This follows from the S relation (see Lemma 1.29), as shown in Figure 2.19

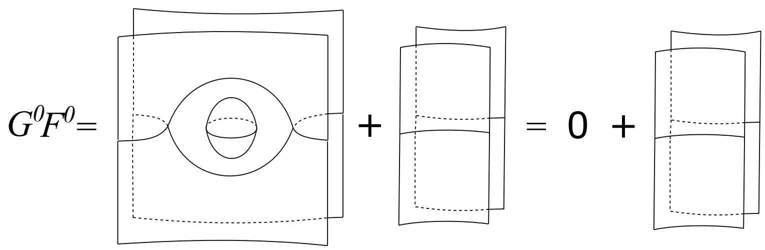
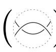
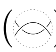


Figure 2.19

- $FG \sim \mathrm{id}$. We define a map $h\colon C$ () $\to C$ () $[1]$ as in Figure 2.20. We want to prove that $FG - \mathrm{id} = h\delta + \delta h$. This follows from the $4Tu$ relation (see Lemma 1.30), as shown in Figure 2.21: $F^0G^0 = -C_{12}, \mathrm{id} = C_{34}, h\delta = -C_{13}, \delta h = -C_{24}$ (up to orientation-preserving diffeomorphisms).

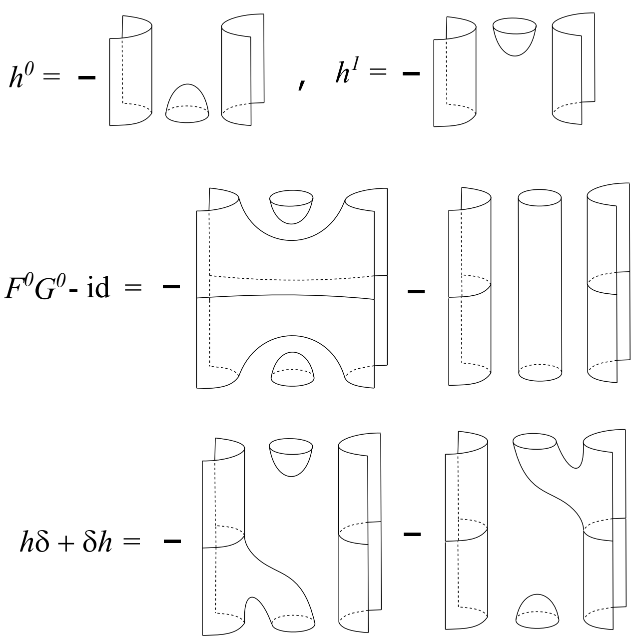


Figure 2.20

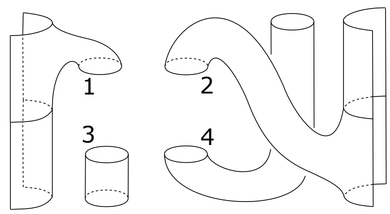


Figure 2.21

This proves that, ignoring the shifts, $C()$ and $C()$ are chain homotopy equivalent, and thus, by an argument similar to the one for $R1$, that $C(D)$ and $C(D')$ are chain homotopy equivalent as well. See Figure 2.17 for an overview.

Let us now check that F and G have quantum degree 0. Before the global shifts F^0 goes from D_v to D'_u for some vertices v of the cube of resolutions of D and u of the cube of D', with $|u| = |v| + 1$. Depending on how D_v looks around the tangle , F^0 is either a split morphism union a cup or a merge morphism union a cup, thus by Remark 2.9 we have $\text{qdeg}(F^0) = -1+1+1 = 1$ (since $|u|-|v| = 1$). But $n'_+ - 2n'_- = n_+ + 1 - 2n_- - 2 = n_+ - 2n_- - 1$ so after the global shifts on $\widetilde{C}(D)$ and $\widetilde{C}(D')$ by $\{n_+ - 2n_-\}$ and $\{n'_+ - 2n'_-\}$ respectively, we have that $\text{qdeg}(F^0) = 1 - 1 = 0$.

On the other hand, before global shifts, G^0 goes from spaces D'_v to spaces D_u with $|u| = |v| - 1$. Depending on the vertex, the map G^0 is either a split union a cap or a merge union a cap. Thus by Remark 2.9 we have $\text{qdeg}(G^0) = -1 + 1 - 1 = -1$ (since $|u| - |v| = -1$). We have that $n_+ - 2n_- = n'_+ - 2n'_- + 1$, thus after the global shifts $\text{qdeg}(G^0) = -1 + 1 = 0$.

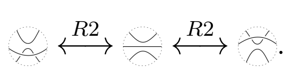
This completes the proof of invariance under $R2$.

Invariance under $R3$

Let D, D' be diagrams related by the first $R3$ move, as in Figure 1.2.

We replace $C(D)$ and $C(D')$ with the complexes $C()$ and $C()$ respectively. These complexes are shown in Figure 2.22: we have that $C()$ is the mapping cone of the map $\boldsymbol{\delta}\colon C() \to C()$, i.e. the map going from the 0- to the 1-resolution of the crossing c. Similarly, $C()$ is the mapping cone of the map $\boldsymbol{\delta'}\colon C() \to C()$.

We observe that the bottom layers of these complexes, i.e. $C()$ and $C()$, are equal. The top layers are also isomorphic (i.e. chain homotopy equivalent): this follows from the fact that and are related by two $R2$ moves:



However, this is not enough to prove that the whole complexes $C()$ and $C()$ are chain homotopy equivalent.

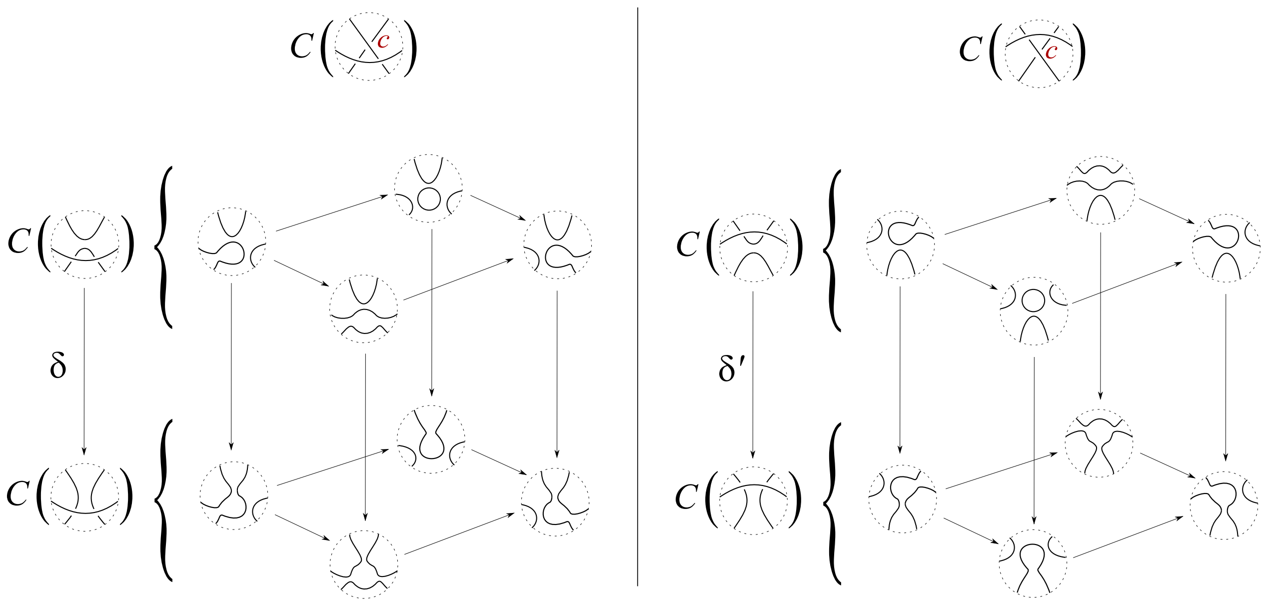


Figure 2.22

We then state the following lemma, which is proved in [BN2]:

Lemma 2.12. *Let $\mathcal{A}', \mathcal{A}, \mathcal{B}$ be complexes and assume that $\mathcal{A}$ is a strong deformation retract of $\mathcal{A}'$, with inclusion map $F\colon \mathcal{A} \to \mathcal{A}'$. Then given a chain map $\varphi\colon \mathcal{A}' \to \mathcal{B}$ we have that Cone(φ) and Cone$(\varphi \circ F)$ are chain homotopy equivalent.*

We recall the definition of strong deformation retract of a complex:

Definition 2.13. Let $\mathcal{A}', \mathcal{A}$ be complexes. We say that $\mathcal{A}$ is a *strong deformation retract* of $\mathcal{A}'$ if there exist chain maps $F\colon \mathcal{A} \to \mathcal{A}'$ and $G\colon \mathcal{A}' \to \mathcal{A}$ and a homotopy $h\colon \mathcal{A}' \to \mathcal{A}'[1]$ such that

- $GF = \mathrm{id}$;
- $hF = 0$;
- $\mathrm{id} - FG = hd + dh$.

The map F is called the *inclusion* and G is the *retract*.

Remark 2.14. If $\mathcal{A}$ is a strong deformation retract of A' then the two complexes are also chain homotopy equivalent.

We observe that the maps F and G defined in Figure 2.18, when proving invariance under $R2$, are respectively an inclusion and a retract, thus the complex $C( )$ is a strong deformation retract of $C( )$ (so they are not only chain homotopy equivalent).
Going back to our complexes $C( )$ and $C( )$, we have that $C( )$ and $C( )$ are strong deformation retracts of $C( )$, thus Lemma 2.12 implies that $C( )$ and $C( )$ are chain homotopy equivalent to the complexes in Figure 2.23 obtained by replacing the top layers with . It is easy to see that these two complexes are chain homotopy equivalent (they are, in fact, equal).

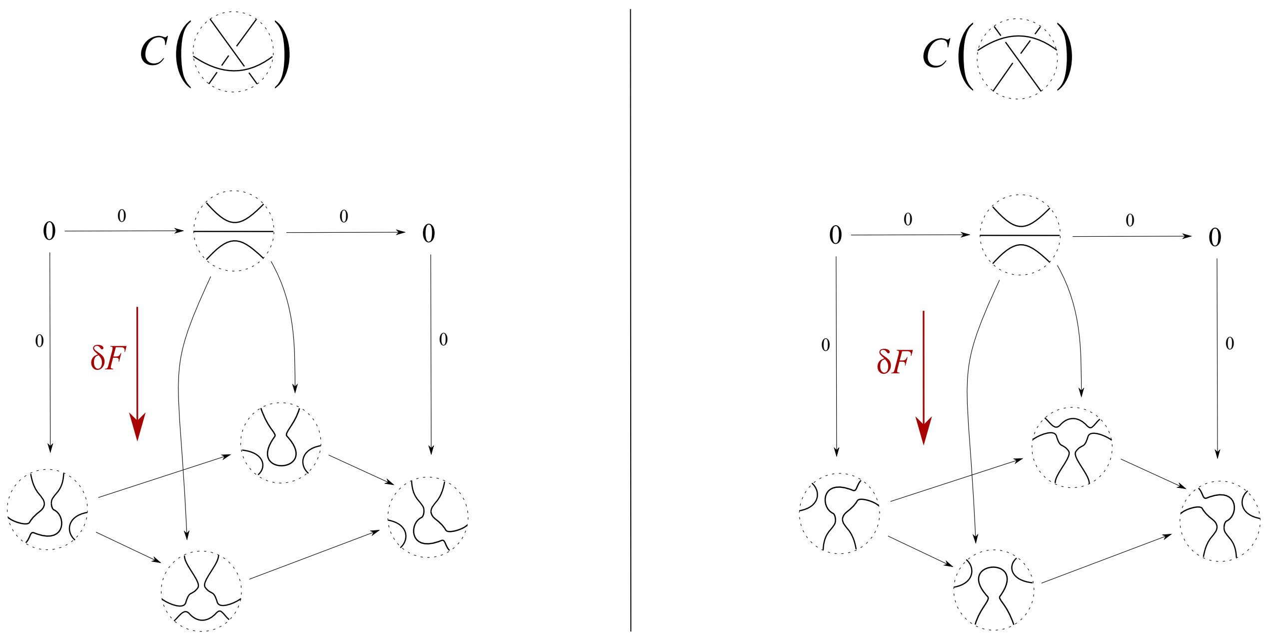


Figure 2.23

We conclude that $C(D)$ and $C(D')$ are chain homotopy equivalent complexes.

Remark 2.15. Invariance under the second $R3$ move follows from invariance under the $R2$ and first $R3$ moves, as stated in Remark 1.9.

We then proved that, given two diagrams D, D' representing equivalent links L, L', the chain complexes $C(D)$ and $C(D')$ relative to $\mathcal{F}_{Univ}$ are chain homotopy equivalent. Thus their homologies $H(D)$ and $H(D')$ are equal. Thus Khovanov homology $H(L)$ is a link invariant.

Homology theories coming from a generic Frobenius system

We saw that the Khovanov homology relative to $\mathcal{F}_{Univ}$ is a link invariant. We would like to say the same about Khovanov homology theories coming from every graded Frobenius system of rank two. We first state some results about chain complexes associated to different Frobenius systems.

Lemma 2.16. *Let $\mathcal{F}, \mathcal{F}'$ be Frobenius systems such that $\mathcal{F}'$ is obtained by base change from $\mathcal{F}$. Let D be a link diagram and denote by $C(D)$ and $C'(D)$ the chain complexes associated to $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then $C'(D) \cong C(D) \otimes_R R'$.*

Proof. This simply follows from the definition of base change, since $A' = A \otimes_R R'$ and each space of $C'(D)$ is just a direct sum of tensor powers of A'.

□

Lemma 2.17. *Let $\mathcal{F}, \mathcal{F}'$ be Frobenius systems such that $\mathcal{F}'$ is obtained by twisting from $\mathcal{F}$. Then for every link diagram D we have that $C(D) \cong C'(D)$.*

For a proof of this lemma see [Kh1].

Let now $\mathcal{F}'$ be a Frobenius system of rank two. We recall that $\mathcal{F}_{Univ}$ is universal, which means that $\mathcal{F}'$ is obtained from $\mathcal{F}_{Univ}$ by a composition of a base change and a twist. We proved that the chain complex associated to $\mathcal{F}_{Univ}$ is invariant under the Reidemeister moves (up to homotopy equivalence), thus by the two lemmas above we have that the chain complex associated to $\mathcal{F}'$ is invariant too.

Summing up, we found that every graded Frobenius system of rank two generates a homology theory for links, called *Khovanov homology*, and each theory is a link invariant.

The following holds:

Theorem 2.18. *Khovanov homology detects the Unknot, i.e. given a link L we have:*

$$H(L) = H(\mathit{Unknot}) \qquad \Longleftrightarrow \qquad L = \mathit{Unknot}.$$

A proof of this result can be found in [KM].

3 Lower bounds for the unknotting number

3.1 A lower bound coming from $\mathcal{F}_{Univ}$

In 2017 [Al] and [AD] found two lower bounds $\lambda_{BN}, \lambda_{Lee}$ for the unknotting number using the Khovanov homology theories relative to Frobenius systems $\mathcal{F}_{BN}$ and $\mathcal{F}_{Lee}$ respectively. In this chapter we describe a new lower bound λ coming from Frobenius system $\mathcal{F}_{Univ}$. The construction of the bound λ, and the proof that it is a lower bound for the unknotting number, follow very closely the constructions and proofs given in [Al] and [AD] for λ_{BN} and λ_{Lee}. We will see that λ is a generalization of λ_{BN} and λ_{Lee} and subsumes them.
For simplicity we will work with the complex $\widetilde{C}$ rather than with C (we will see that the invariant λ does not depend on the grading of the complex).

Let us recall the definition of $\mathcal{F}_{Univ}$.

$$R_{Univ} = \mathbb{Z}[h,t] \qquad\qquad A_{Univ} = \frac{\mathbb{Z}[h,t,X]}{(X^2 - hX - t)}$$

The multiplication and comultiplication are given by:

$$\begin{aligned} m\colon\ & A \otimes A \to A & \qquad \Delta\colon\ & A \to A \otimes A \\ & 1 \otimes 1 \mapsto 1 & & 1 \mapsto 1 \otimes X + X \otimes 1 - h1 \otimes 1 \\ & 1 \otimes X \mapsto X & & X \mapsto X \otimes X + t1 \otimes 1 \\ & X \otimes 1 \mapsto X & & \\ & X \otimes X \mapsto hX + t & & \end{aligned}$$

and the unit and counit are respectively:

$$\iota\colon R \to A \qquad\qquad \varepsilon\colon\ A \to R$$

$$1 \mapsto 1 \qquad\qquad 1 \mapsto 0$$

$$X \mapsto 1$$

To make the computations of this chapter easier we introduce another notation for the algebraic structure given by $\mathcal{F}_{Univ}$ to the cube of resolutions of a diagram D.
Let K be a knot and let us fix a knot diagram D with crossings $c_1, \ldots, c_n$ (where $n = n_+ + n_-$ is the sum of the positive crossings and the negative ones), and edges $e_1, \ldots, e_m$. Define

$$\mathfrak{R} = \frac{\mathbb{Z}[X_1, \ldots, X_m, h, t]}{\{X_i^2 - hX_i - t = 0 \text{ for } 1 \leq i \leq m\}}$$

where, for all i, the variable X_i corresponds to the edge e_i.
Each $v \in \{0,1\}^n$ defines an equivalence relation on the set of edges of D: we say that $e_p \sim_v e_q$ if e_p and e_q lie on the same connected component of D_v.
We then associate to every vertex v the quotient

$$\mathfrak{R}_v = \mathfrak{R}/\{X_p = X_q \text{ if } e_p \sim_v e_q\}.$$

We have that $\mathfrak{R}$ and $\mathfrak{R}_v$, for all v, are commutative unitary rings, with internal multiplication "$\cdot$" given by

$$\mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$$

$$1 \times 1 \mapsto 1$$

$$X_i \times 1 \mapsto X_i$$

$$X_i \times X_j \mapsto X_i X_j$$

They are also R_{Univ}-modules.
We have that $\mathfrak{R}_v$ is naturally isomorphic to $A_{Univ}^{\otimes k_v}$. The element $X_i \in \mathfrak{R}_v$ corresponds to multiplying the i-th factor of $1 \otimes \cdots \otimes 1$ by X in $A_{Univ}^{\otimes k_v}$.

For each crossing c of D consider the four edges e_i, e_j, e_l, e_k around it, as in Figure 3.1.

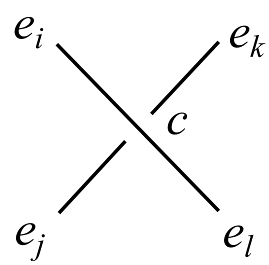


Figure 3.1

We want to express the edge maps in terms of X_i, X_j, X_l, X_k.
Each edge map $\delta\colon \mathfrak{R}_v \to \mathfrak{R}_u$, with $u < v$, goes from the 0- to the 1-resolution of a crossing c. If $\delta = m$ it merges two circles (see Figure 3.2), and the equivalence relations on the edges give:
on the vertex v: $X_i = X_k$, $X_j = X_l$;
on the vertex u: $X_i = X_k = X_j = X_l$.

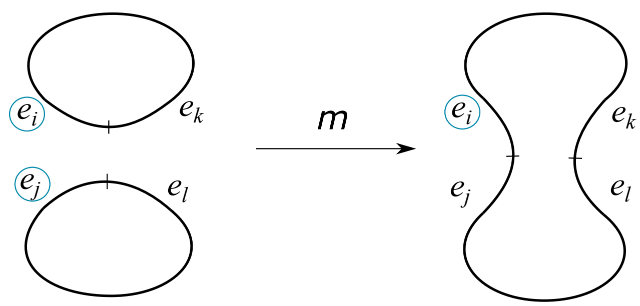


Figure 3.2

If $\delta = \Delta$ it splits a circle in two (see Figure 3.3), so:
on v: $X_i = X_k = X_j = X_l$;
on u: $X_j = X_i$, $X_k = X_l$.

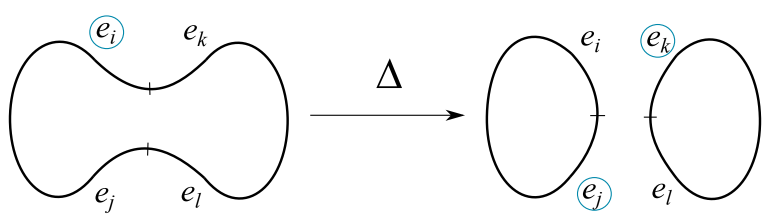


Figure 3.3

Remark 3.1. At every vertex we have $X_j + X_k = X_i + X_l$.

Given vertices $v < u$ we then have:

$$
\begin{aligned}
m\colon \quad \mathfrak{R}_v &\to \mathfrak{R}_u & \qquad \Delta\colon \mathfrak{R}_v &\to \mathfrak{R}_u \\
1 &\mapsto 1 & 1 &\mapsto X_j + X_k - h \\
X_i &\mapsto X_i & X_i &\mapsto X_j X_k + t \\
X_j &\mapsto X_i & & \\
X_i X_j &\mapsto h X_i + t & &
\end{aligned}
$$

We observe that the maps m and Δ (and thus all edge maps of the cube of resolutions) are $\mathfrak{R}$-module maps.

The advantage of this notation is that we don't need to keep track of all the tensor factors of an element, or specify which circles are involved in a multiplication or comultiplication.

Remark 3.2. Δ is multiplication by $X_j + X_k - h$.
Moreover, m is just the quotient map of the relation $\sim_u$.

We recall that the system $\mathcal{F}_{Univ}$, and the chain complex relative to it, are graded. Thus for every v the ring $\mathfrak{R}_v$ is graded: before the shifts made to obtain the chain complex, the variables h, t and X_i (for all $1 \leq i \leq m$) have quantum degrees -2, -4 and -2 respectively, and 1 has quantum degree 0. We then make the shift $\{n_+ - 2n_- + |v| + k_v\}$. So given $r = X_1^{d_1} \cdots X_m^{d_m} h^{d_{m+1}} t^{d_{m+2}} \in \mathfrak{R}_v$ we have:

$$\operatorname{qdeg}(r) = n_+ - 2n_- + |v| + k_v - 2(d_1 + \ldots + d_m) - 2d_{m+1} - 4d_{m+2}.$$

The homological degree of $\mathfrak{R}_v$ after the shifts is $|v| - n_-$.

We now want to define a knot invariant λ coming from the Khovanov homology H relative to $\mathcal{F}_{Univ}$. We first have to state a few results and definitions.

We start by giving an A_{Univ}-module structure on the Khovanov chain complex C. Let D be a knot diagram with edges $e_1, \ldots, e_m$. Let us fix an edge e_p and consider a small Unknot U near this edge. Consider the saddle cobordism S obtained by attaching a 1-handle to connect U and e_p, as in Figure 3.4: on every complete resolution D_v of D the cobordism S will merge U and the circle corresponding to e_p. This induces a map

$$m_p \colon C(D \sqcup U) = C(D) \otimes A_{Univ} \to C(D)$$

defined as follows: let v be a vertex of the cube of resolutions of D and $a = a_1 \otimes \ldots \otimes a_{k_v} \in A_{Univ}^{k_v}$, and let $b \in A_{Univ}$. Then $m_p(a \otimes b) = a_1 \otimes \ldots \otimes a_{p-1} \otimes (a_p \cdot b) \otimes a_{p+1} \otimes \ldots \otimes a_{k_v}$, i.e. it is given by multiplying the p-th factor of a by b. The map m_p gives an A_{Univ}-module structure on $C(D)$. This structure depends on the chosen edge e_p.

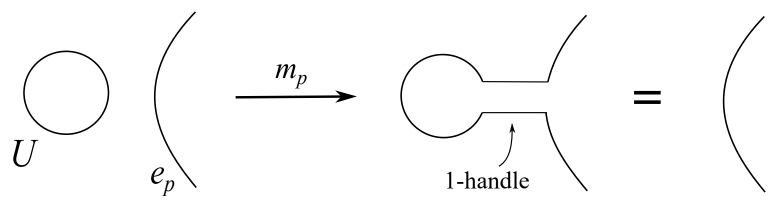


Figure 3.4

We have:

Lemma 3.3. *Let D, D' be knot diagrams related by a Reidemeister move. Then $C(D)$ and $C(D')$ are chain homotopy equivalent as complexes of A_{Univ}-modules (not only as complexes of R_{Univ}-modules). Thus Khovanov homology is invariant also as an A_{Univ}-module.*

For a proof of this statement see [Kh3], Section 3.

Lemma 3.4. *Multiplication by $X_p + X_q$ and multiplication by h are chain homotopic for all p, q such that e_p and e_q are diagonal to one another at a crossing c.*

We will give a proof of this later on.
The following corollary states that in $H(K)$, up to sign, multiplication by $2X_i - h$ is the same for all i.

Corollary 3.5. *In $H(K)$ we have*

$$2X_p - h = \pm(2X_q - h) \quad \textit{for any} \quad 1 \leq p, q \leq m.$$

Proof. Let us first consider p, q such that e_p, e_q are diagonal to one another. Then by Lemma 3.4 we have that $X_p = -X_q + h$ in $H(K)$, and thus $2X_p - h = -2X_q + 2h - h = -(2X_q - h)$. Now consider any $1 \leq p, q \leq m$. Since K is a knot there is a sequence of edges $e_{i_0}, \ldots, e_{i_k}$ with $e_{i_0} = e_p, e_{i_k} = e_q$, such that $e_{i_s}, e_{i_{s+1}}$ are diagonal to one another for all $0 \leq s \leq k-1$. Then

$$2X_p - h = -(2X_{i_1} - h) = 2X_{i_2} - h = \ldots = \pm(2X_q - h).$$

□

More precisely, if there are $2k$ crossings between e_p and e_q, for $k \geq 0$, then $2X_p - h = 2X_q - h$; if there are $2k+1$ crossings then $2X_p - h = -(2X_q - h)$.
(Notice that there are two possible paths of edges in D that we can follow to connect e_p and e_q, but the number of crossings encountered in these paths will have the same parity. This is because, if we fix a point Q on D, the path along D starting and ending at Q will meet each crossing of D twice, thus it meets an even number of crossings.)

Let us write X for any of the X_i.

Definition 3.6. An element $a \in H(K)$ is $(2X - h)$-*torsion* if $(2X - h)^n a = 0$ for some $n \in \mathbb{N}$.

By Corollary 3.5 this definition doesn't depend on the X_i that we choose for X, since $(2X_i - h)^n a = 0 \iff (-(2X_i - h))^n a = 0$.

We call $T(H(K))$ the $(2X - h)$-torsion classes in $H(K)$, i.e.

$$T(H(K)) = \{a \in H(K) \, : \, (2X - h)^n a = 0 \text{ for some } n \geq 0\}.$$

The *order* of a $(2X - h)$-torsion element $a \in T(H(K))$, denoted by $\mathrm{ord}(a)$, is the smallest $n \geq 0$ such that $(2X - h)^n a = 0$.

We can now finally define λ:

Definition 3.7. We define $\lambda(K)$ to be the maximum order of a torsion element in $T(H(K))$:

$$\lambda(K) = \max_{a \in T(H(K))} \mathrm{ord}(a).$$

We have that λ is a well-defined knot invariant, since the Khovanov homology $H(K)$ is, and by Lemma 3.3.

Our next goal is to show that λ is a lower bound for the unknotting number. The core of the proof consists in showing that if two knots $K, \overline{K}$ are related by a crossing change, then $|\lambda(K) - \lambda(\overline{K})| \leq 1$.

Let us then consider two knot diagrams $D, \overline{D}$ related by a crossing change at a crossing c. We will denote by e_i, e_j, e_l, e_k the four edges around c in D (as in Figure 3.1).
We order the crossings of D and $\overline{D}$ so that c is the last crossing.
Let D_0 and D_1 be the diagrams obtained from D by 0- and 1-resolving c respectively. The chain complex $\widetilde{C}(D)$ can be written as (see (2.2)):

$$\widetilde{C}(D) = \left(\widetilde{C}(D_0) \xrightarrow{\boldsymbol{\delta}} \widetilde{C}(D_1)\{1\}\right)$$

i.e. $\widetilde{C}(D) = \mathrm{Cone}(\boldsymbol{\delta})$, where $\boldsymbol{\delta}$ is the bundle of edge maps $\boldsymbol{\delta}_v$ in $\widetilde{C}(D)$ going from the 0- to the 1-resolution of c: $\boldsymbol{\delta}_v$ goes from a vertex $v = (v_1, v_2, \ldots, v_{n-1}, 0)$ to $(v_1, v_2, \ldots, v_{n-1}, 1)$.

We define $\overline{D_0}$, $\overline{D_1}$ and $\overline{\boldsymbol{\delta}}$ analogously, so:

$$\widetilde{C}(\overline{D}) = \left(\widetilde{C}(\overline{D_0}) \xrightarrow{\overline{\boldsymbol{\delta}}} \widetilde{C}(\overline{D_1})\{1\}\right).$$

Lemma 3.8. *We have that $D_1 = \overline{D_0}$ and $D_0 = \overline{D_1}$.*
Moreover, if we let $\widetilde{C}(D)_i$ be the subcomplexes of $\widetilde{C}(D)$ coming from the i-resolution of c (i.e. where c is always i-resolved), we have that $\widetilde{C}(D_0) = \widetilde{C}(D)_0$ and $\widetilde{C}(D_1)[1]\{1\} = \widetilde{C}(D)_1$. Thus $\widetilde{C}(D)_0[1]\{1\} = \widetilde{C}(\overline{D})_1$ and $\widetilde{C}(D)_1 = \widetilde{C}(\overline{D})_0[1]\{1\}$.

Proof. The first statement is clear. For the second we observe that, ignoring the grading, the subcube $\mathfrak{C}(D)_i$ of resolutions (relative to D) where c is always i-resolved is identical to the cube $\mathfrak{C}(D_i)$ of D_i except possibly for the signs of the edge maps. But since c is the last crossing, we have that to a vertex $(v_1, v_2, \ldots, v_{n-1})$ of $\mathfrak{C}(D_i)$ corresponds the vertex $(v_1, v_2, \ldots, v_{n-1}, i)$ of $\mathfrak{C}(D)_i$, thus the signs of the edge maps are sprinkled in the same way in both cubes. The degree shifts in $\widetilde{C}(D_1)[1]\{1\} = \widetilde{C}(D)_1$ come from the fact that in D_1 the crossing c is replaced by its 1-resolution, so it doesn't influence the shifts. See Figure 3.5 to visualize this result.

□

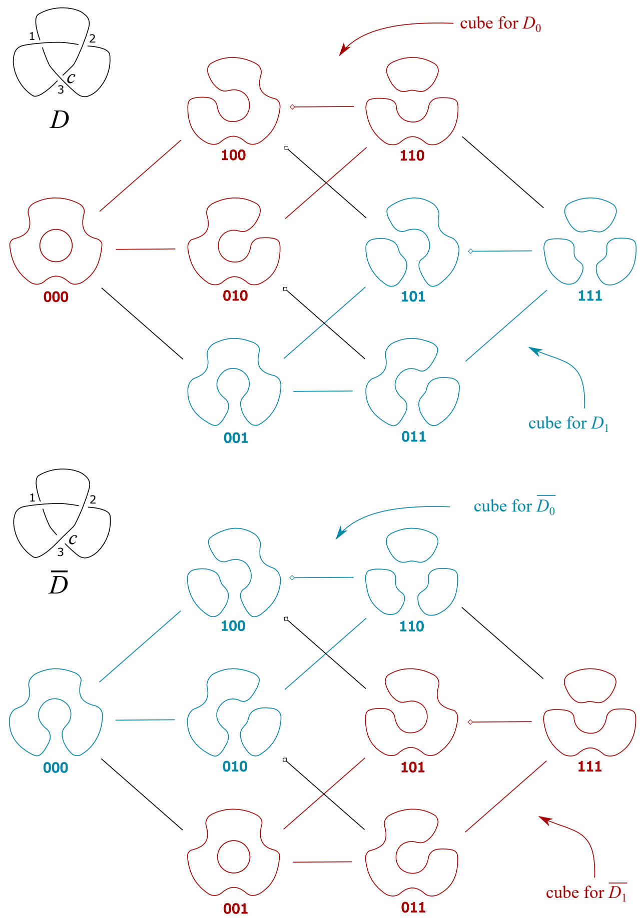


Figure 3.5

We now define a map $\Phi\colon \widetilde{C}(D) \to \widetilde{C}(D)$ as $\Phi(a) = (X_j - X_k)a$, i.e. as multiplication by $X_j - X_k$. Remember that X_j, X_k are the variables associated to the edges e_j, e_k diagonal with respect to the crossing c, as in Figure 3.1. Since the edge maps of the cube of resolutions for D are $\mathfrak{R}$-module maps we have that Φ is a chain map.

We observe that Φ is the composition of two maps $f\colon \widetilde{C}(D) \longrightarrow \widetilde{C}(\overline{D})$ and $g\colon \widetilde{C}(\overline{D}) \longrightarrow \widetilde{C}(D)$ defined as follows:
Given $a \in \widetilde{C}(D) = \widetilde{C}(D_0) \oplus \left(\widetilde{C}(D_1)[1]\{1\}\right)$, we can write $a = (a_0, a_1)$. Then

$$f(a_0, a_1) = ((X_j - X_k)a_1, a_0)$$

Similarly, for $b = (b_0, b_1) \in \widetilde{C}(\overline{D})$ we define

$$g(b_0, b_1) = ((X_j - X_k)b_1, b_0)$$

The maps f and g are $\mathfrak{R}$-module maps, and:

Lemma 3.9. *For any $a \in \widetilde{C}(D)$ we have $(g \circ f)(a) = (X_j - X_k)a = \Phi(a)$.*

Proof. Let $a = (a_0, a_1) \in \widetilde{C}(D)$. Then

$$\begin{aligned}(g \circ f)(a_0, a_1) &= g((X_j - X_k)a_1, a_0) = ((X_j - X_k)a_1, (X_j - X_k)a_0) = \\ &= (X_j - X_k)(a_0, a_1).\end{aligned}$$

□

We observe that $f = \alpha + \beta$ where, given $a = (a_0, a_1) \in \widetilde{C}^n(D)$, we have

$$\begin{aligned}&\alpha\colon \widetilde{C}(D) \to \widetilde{C}(\overline{D})[1] \quad \text{is the map} \quad \alpha(a_0, a_1) = ((X_j - X_k)a_1, 0) \in \widetilde{C}^{n-1}(\overline{D}) \\ &\beta\colon \widetilde{C}(D) \to \widetilde{C}(\overline{D})[-1] \quad \text{is the map} \quad \beta(a_0, a_1) = (0, a_0) \in \widetilde{C}^{n+1}(\overline{D}).\end{aligned}$$

Similarly, given $b = (b_0, b_1) \in \widetilde{C}^n(\overline{D})$ we have

$$g(b_0, b_1) = \overline{\alpha}(b_0, b_1) + \overline{\beta}(b_0, b_1) = ((X_j - X_k)b_1, 0) + (0, b_0) \in \widetilde{C}^{n-1}(D) \oplus \widetilde{C}^{n+1}(D).$$

We have that $\alpha, \beta, \overline{\alpha}$ and $\overline{\beta}$ are chain maps, since the differentials of all chain complexes involved are $\mathfrak{R}$-module maps.

We would like to show that Φ is chain homotopic to $\pm(2X - h)$, where X represents any of $X_1, \ldots, X_m$. For that we will need Lemma 3.4. We can now give a proof of this lemma.

Proof. (of Lemma 3.4). We give the proof for the edges e_j, e_k of D, diagonal with respect to the crossing c. The proof for any other pair of diagonal edges with respect to a crossing ξ is identical, it suffices to replace D_0, D_1 with the 0- and 1-resolutions of ξ.
We define a chain homotopy $\mathcal{H}\colon \widetilde{C}(D) \to \widetilde{C}(D)$. Let $a = (a_0, a_1) \in \widetilde{C}^n(D)$, and remember that $a_1 \in \widetilde{C}(D_1)[1]\{1\} = \widetilde{C}(\overline{D_0})[1]\{1\}$. Then let

$$\mathcal{H}(a_0, a_1) = (\overline{\boldsymbol{\delta}}(a_1), 0) \in \widetilde{C}^{n-1}(D).$$

Figure 3.6 shows the map $\mathcal{H}$ in red, the map $\boldsymbol{\delta}$ in blue and the edge maps within $\widetilde{C}(D_0)$ and $\widetilde{C}(D_1)$ in black. The degree shifts are omitted.

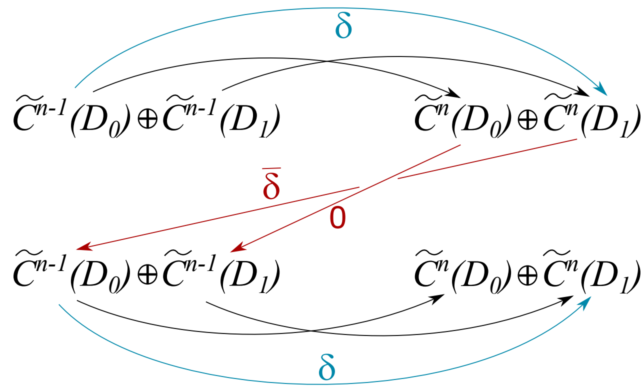


Figure 3.6

Since $\widetilde{C}(D_0) = \widetilde{C}(\overline{D_1})$ and $\widetilde{C}(D_1) = \widetilde{C}(\overline{D_0})$ and all squares anticommute in $\widetilde{C}(\overline{D})$, we see that $\overline{\boldsymbol{\delta}}$ anticommutes with all edge maps δ within $\widetilde{C}(D_0)$ and $\widetilde{C}(D_1)$, i.e. $\overline{\boldsymbol{\delta}}\delta + \delta\overline{\boldsymbol{\delta}} = 0$.
So if d is a differential of $\widetilde{C}(D)$ we have that $(d\mathcal{H} + \mathcal{H}d)(a_0, a_1) = (\overline{\boldsymbol{\delta}}\boldsymbol{\delta}(a_0), \boldsymbol{\delta}\overline{\boldsymbol{\delta}}(a_1)) = \boldsymbol{\delta}\overline{\boldsymbol{\delta}}(a_1) + \overline{\boldsymbol{\delta}}\boldsymbol{\delta}(a_0)$ (see Figure 3.6). We want to show that $\boldsymbol{\delta}\overline{\boldsymbol{\delta}} + \overline{\boldsymbol{\delta}}\boldsymbol{\delta} = X_j + X_k - h$.
Consider $\boldsymbol{\delta}\overline{\boldsymbol{\delta}}$: $\boldsymbol{\delta}$ and $\overline{\boldsymbol{\delta}}$ are (bundles of) edge maps between the same vertices, but going in opposite directions. So on the vertices where $\boldsymbol{\delta}$ is a merge, we have that $\overline{\boldsymbol{\delta}}$ is a split, and when $\boldsymbol{\delta}$ is a split $\overline{\boldsymbol{\delta}}$ is a merge. Assume first that $\boldsymbol{\delta}$ is a merge and $\overline{\boldsymbol{\delta}}$ a split. Then

$$1 \xrightarrow{\overline{\boldsymbol{\delta}}} X_j + X_k - h \xrightarrow{\boldsymbol{\delta}} X_i + X_i - h = X_j + X_k - h$$

because $X_i = X_j = X_k$ after applying $\boldsymbol{\delta}$, and

$$X_i \xrightarrow{\boldsymbol{\delta}\overline{\boldsymbol{\delta}}} (X_j + X_k - h)X_i$$

because $\boldsymbol{\delta}$ and $\overline{\boldsymbol{\delta}}$ are $\mathfrak{R}$-module maps and thus commute with multiplication by X_i.
If $\boldsymbol{\delta}$ is a split, then $\overline{\boldsymbol{\delta}}$ is a merge. Again, since $\boldsymbol{\delta}\overline{\boldsymbol{\delta}}$ is an $\mathfrak{R}$-module map, it is enough to compute its action on 1. We have:

$$1 \xrightarrow{\overline{\boldsymbol{\delta}}} 1 \xrightarrow{\boldsymbol{\delta}} X_j + X_k - h$$

Similar calculations apply to $\overline{\boldsymbol{\delta}}\boldsymbol{\delta}$.
In the end we obtain $(d\mathcal{H} + \mathcal{H}d)(a_0, a_1) = (X_j + X_k - h)a_1 + (X_j + X_k - h)a_0 = (X_j + X_k - h)(a_0, a_1)$.

□

From Lemma 3.4 we then get the following

Corollary 3.10. *For any $1 \leq i \leq m$, the map Φ is chain homotopic to multiplication by $\pm(2X_i - h)$. Thus*

$$\Phi_* = \pm(2X_i - h).$$

Proof. By Lemma 3.4 we have that $X_k \sim -X_j + h$, thus $\Phi = X_j - X_k \sim 2X_j - h$.
By Corollary 3.5 we get $2X_p - h \sim \pm(2X_q - h)$ for all $1 \leq p, q \leq m$, so $\Phi \sim 2X_j - h \sim \pm(2X_i - h)$ for all i.

□

We then have:

Lemma 3.11. *Let $K, \overline{K}$ be knots related by a crossing change. Then*

$$|\lambda(K) - \lambda(\overline{K})| \leq 1.$$

Proof. In what follows X will denote any of the X_i. Let $D, \overline{D}$ be diagrams of $K, \overline{K}$ respectively, related by a crossing change at a crossing c. Let $a \in T(H(K))$. We observe that, if $\mathrm{ord}(a) = n$ and φ is an $\mathfrak{R}$-module map, then $(2X - h)^n \varphi_*(a) = \varphi_*((2X - h)^n a) = \varphi_*(0) = 0$, so $\mathrm{ord}(a) \geq \mathrm{ord}(\varphi_*(a))$. Then

$$\mathrm{ord}(a) \geq \mathrm{ord}(\Phi_*(a)).$$

By Corollary 3.10 we have that $\Phi_*(a) = \pm(2X - h)a$, so $(2X - h)^i \Phi_*(a) = \pm(2X - h)^{i+1} a$ for any i. So

$$\begin{aligned} \mathrm{ord}(\Phi_*(a)) &= \mathrm{ord}(a) - 1 && \text{if } \mathrm{ord}(a) > 0 \\ &= 0 && \text{if } \mathrm{ord}(a) = 0. \end{aligned}$$

Now we remember that Φ decomposes as $g \circ f$. Let $f_* = \alpha_* + \beta_*$ and $g_* = \overline{\alpha}_* + \overline{\beta}_*$.
We then have that

$$\mathrm{ord}(a) \geq \mathrm{ord}(f_*(a)) \geq \mathrm{ord}(g_*(f_*(a))) = \mathrm{ord}(\Phi_*(a)) \geq \mathrm{ord}(a) - 1.$$

Observe that if $a \in T(H(K))$, then $f_*(a) \in T(H(\overline{K}))$, so

$$\max_{b \in T(H(\overline{K}))} \mathrm{ord}(b) \geq \max_{a \in T(H(K))} \mathrm{ord}(f_*(a)).$$

Thus

$$\lambda(\overline{K}) \geq \max_{a \in T(H(K))} \mathrm{ord}(f_*(a)) \geq \max_{a \in T(H(K))} \mathrm{ord}(a) - 1 = \lambda(K) - 1.$$

So $\lambda(K) - \lambda(\overline{K}) \leq 1$.
The inequality $\lambda(\overline{K}) - \lambda(K) \leq 1$ is obtained in the same way, switching K with $\overline{K}$.

□

Theorem 3.12. *Let K be a knot. Then $\lambda(K)$ is a lower bound for the unknotting number of K:*

$$\lambda(K) \leq u(K).$$

Proof. First let's compute $\lambda(\text{Unknot})$. The Unknot has zero crossings and one connected component, so its cochain complex is given by $0 \to A_{Univ} \to 0$. Thus $H(\text{Unknot}) = A_{Univ}$. We notice that A_{Univ} is an integral domain, so $H(\text{Unknot})$ is torsion-free: $\lambda(\text{Unknot}) = 0$.
Let's now call $N =$ unknotting number of K, and consider an unknotting sequence

$$K = K_0, K_1, \ldots, K_N = \text{Unknot}.$$

For all $0 \leq i \leq N-1$ we have that K_i and K_{i+1} are related by a crossing change, so, by Lemma 3.11, $|\lambda(K_i) - \lambda(K_{i+1})| \leq 1$. Thus

$$|\lambda(K_0) - \lambda(K_N)| \leq |\lambda(K_0) - \lambda(K_1)| + |\lambda(K_1) - \lambda(K_2)| + \ldots + |\lambda(K_{N-1}) - \lambda(K_N)| \leq N.$$

Observing that

$$|\lambda(K_0) - \lambda(K_N)| = |\lambda(K) - \lambda(\text{Unknot})| = \lambda(K) - 0 = \lambda(K)$$

we conclude that $\lambda(K) \leq N$.

□

3.2 Other bounds and applications

We now define the lower bounds λ_{BN} and λ_{Lee} found by [Al] and [AD] and provide some applications and relations with other lower bounds for $u(K)$.

Let H_{BN}, H_{Lee} be the Khovanov homology theories coming from Frobenius systems $\mathcal{F}_{BN}$ and $\mathcal{F}_{Lee}$ respectively (see (1.6) and (1.7)).
We observe that $\mathcal{F}_{BN}$ and $\mathcal{F}_{Lee}$ are obtained from $\mathcal{F}_{Univ}$ by base change using the following

ring homomorphisms:

$$\varphi_{BN}\colon R_{Univ} = \mathbb{Z}[h,t] \to \mathbb{F}_2[h] = R_{BN} \qquad \varphi_{Lee}\colon R_{Univ} = \mathbb{Z}[h,t] \to \mathbb{Q}[t] = R_{BN}$$

$$1 \mapsto 1 \qquad 1 \mapsto 1$$

$$h \mapsto h \qquad h \mapsto 0$$

$$t \mapsto 0 \qquad t \mapsto t$$

Then we define

$$\lambda_{BN} = \max_{a \in T(H_{BN}(K))} \mathrm{ord}_{BN}(a), \qquad \lambda_{Lee} = \max_{a \in T(H_{Lee}(K))} \mathrm{ord}_{Lee}(a)$$

where $\mathrm{ord}_{BN}(a)$, is the smallest $n \geq 0$ such that $h^n a = 0$ and $\mathrm{ord}_{Lee}(a)$, is the smallest $n \geq 0$ such that $X^n a = 0$.

It is shown in [Al] and [AD] that these are also lower bounds for the unknotting number. The proof is very similar to the one given in the previous section for λ: for λ_{BN} we replace t with 0 and $\mathbb{Z}$ with $\mathbb{F}_2$, for λ_{Lee} we replace h with 0, and note that the X-torsion and the $2X$-torsion are equivalent, since the ground ring is $R_{Lee} = \mathbb{Q}[t]$.

These lower bounds have interesting applications related to the convergence of some spectral sequences and to the Knight Move conjecture. We briefly describe these applications here. For an overview of spectral sequences see the Appendix.
Consider the Frobenius system $\mathcal{F}_{BN}$. It can be proved that for every knot K the Khovanov homology $H_{BN}(K)$, as an R_{BN}-module (and ignoring the gradings), decomposes as $\mathbb{F}_2[h] \oplus \mathbb{F}_2[h] \oplus T(H_{BN}(K))$, where $T(H_{BN}(K))$ are the torsion elements of $H_{BN}(K)$ (see [Tu2]).
Let us now set $h = 1$. This collapses the grading of $\mathcal{F}_{BN}$, and generates, for every knot diagram, a chain complex $C_{BN}/(h = 1)$ that is not graded anymore, but filtered. Thus we obtain a spectral sequence, called *Bar-Natan spectral sequence*.

Remark 3.13. The first page of the Bar-Natan spectral sequence is the Khovanov homology theory $H_{BN'}$ relative to $\mathcal{F}_{BN'}$, with $F = \mathbb{F}_2$ (see (1.5)). This follows from the fact that the differential d of the chain complex $C_{BN}/(h = 1)$ is now given by the sum of a grading-preserving map and of a map of degree 2:

$$\Delta_{BN}(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1 \quad \xrightarrow{h=1} \quad 1 \otimes X + X \otimes 1 - 1 \otimes 1 \tag{3.1}$$

$$m_{BN}(X \otimes X) = X^2 - hX \quad \xrightarrow{h=1} \quad X^2 - X \tag{3.2}$$

(since $\mathrm{qdeg}(h) = -2$ the red terms above have now quantum degree 2, while the blue terms still have degree 0).
Now the differential d_n of the n-th page of the Bar-Natan spectral sequence has bidegree $(1, 2n)$, thus d_0, i.e. the differential of the associated graded complex for $C_{BN}/(h = 1)$, is only given by the grading-preserving part of the differential d, i.e. the blue terms in (3.1) and (3.2). This is exactly the differential of the chain complex $C_{BN'}$ relative to $\mathcal{F}_{BN'}$ (see (1.5)). It is easy to see that the chain groups of the associated graded complex for $C_{BN}/(h = 1)$ and the groups of $C_{BN'}$ also coincide. Thus the first page of the Bar-Natan spectral sequence is the homology of $C_{BN'}$, i.e. $H_{BN'}$.

The Bar-Natan spectral sequence converges to $H_{BN}(K)/(h = 1) = \mathbb{F}_2 \oplus \mathbb{F}_2$.
We have that:

Lemma 3.14. *If the Bar-Natan spectral sequence collapses at page n then $\lambda_{BN}(K) = n - 1$.*

Similarly, $H_{Lee}(K)$ decomposes as $\mathbb{Q}[t] \oplus \mathbb{Q}[t] \oplus T(H_{Lee}(K))$. Setting $t = 1$ in $\mathcal{F}_{Lee}$ gives rise to the *Lee spectral sequence*. The differentials d_n of the n-th page of this spectral sequence have bidegree $(1, 4n)$ (remember that $\mathrm{qdeg}(t) = -4$), thus the first page is again $H_{BN'}$ (as for the Bar-Natan spectral sequence), this time with $F = \mathbb{Q}$.
The Lee spectral sequence converges to $H_{Lee}(K)/(t = 1) = \mathbb{Q} \oplus \mathbb{Q}$. More precisely, it converges to

$$\mathbb{Q}[0]\{s(K) - 1\} \oplus \mathbb{Q}[0]\{s(K) + 1\},$$

where $s(K)$ is Rasmussen's invariant (for a proof see [Ra]).
Let us now define λ'_{Lee} as the maximum order of t-torsion in H_{Lee}, i.e.

$$\lambda'_{Lee} = \lceil \lambda_{Lee}/2 \rceil$$

(since $X^2 = t$). Then:

Lemma 3.15. *We have that $\lambda'_{Lee}(K) = n - 1$ if and only if the Lee spectral sequence collapses at page n.*

It follows that the pages at which the Bar-Natan and Lee spectral sequences collapse also give a lower bound for the unknotting number.

Using Lemma 3.15 one can prove the Knight Move Conjecture for knots K with unknotting number at most 2. Let us define this conjecture.

Conjecture 3.16. *(Knight Move Conjecture). Let $H_{BN'}$ be the Khovanov homology relative to Frobenius system $\mathcal{F}_{BN'}$, with $F = \mathbb{Q}$. Then, as an $R_{BN'}$-module, $H_{BN'}$ decomposes as a* *pair*

$$\mathbb{Q}[0]\{s-1\} \oplus \mathbb{Q}[0]\{s+1\}$$

called pawn move (for some $s \in \mathbb{N}$), together with a set of pairs of the form

$$\mathbb{Q}[i]\{j\} \oplus \mathbb{Q}[i+1]\{j+4\}$$

called knight moves (for various $i, j \in \mathbb{N}$).

This conjecture has been proved to not hold in general: a counterexample can be found in [MM], for the knot shown in Figure 3.7. However, since at the n-th page of the Lee spectral sequence the differentials d_n have bidegree $(1, 4n)$, a similar result holds, where "longer" knight moves are allowed (see [MM]):

Lemma 3.17. *Khovanov homology $H_{BN'}$ decomposes as a pawn move pair*

$$\mathbb{Q}[0]\{s-1\} \oplus \mathbb{Q}[0]\{s+1\}$$

together with a set of long knight move pairs of the form

$$\mathbb{Q}[i]\{j\} \oplus \mathbb{Q}[i+1]\{j+4n\}.$$

Using the bound λ_{Lee} we obtain the following:

Lemma 3.18. *The Knight Move Conjecture holds for knots K such that $u(K) \leq 2$.*

Proof. If $K =$ Unknot then $H_{BN'}(K) = \mathbb{Q}[0]\{-1\} \oplus \mathbb{Q}[0]\{1\}$, i.e. it consists of a pawn move. Let now K be a knot different from the Unknot, with $u(K) \leq 2$.
Since λ_{Lee} is a lower bound for the unknotting number we also have that $\lambda_{Lee}(K) \leq 2$, and thus $\lambda'_{BN}(K) \leq 1$.
We have that $H_{Lee}(K) = \mathbb{Q}[t] \oplus \mathbb{Q}[t] \oplus T(H_{Lee}(K))$ and $H_{Lee}(\text{Unknot}) = \mathbb{Q}[t] \oplus \mathbb{Q}[t]$, so $H_{Lee}(\text{Unknot})$ has no X-torsion elements other than 0, and $H_{Lee}(K) = H_{Lee}(\text{Unknot}) \oplus T(H_{Lee}(K))$. By Theorem 2.18 Khovanov homology detects the Unknot, thus $T(H_{Lee}(K)) \neq \{0\}$. So $\lambda'_{Lee} > 0$.
It follows that $\lambda'_{Lee} = 1$, so, by Lemma 3.15, the Lee spectral sequence collapses at page 2.
Now we recall that the Lee spectral sequence converges to $\mathbb{Q}[0]\{s(K)-1\} \oplus \mathbb{Q}[0]\{s(K)+1\}$.
The Knight Move Conjecture then follows from the fact that the first page of the Lee spectral sequence is $H_{BN'}$, with differential of bidegree $(1, 4)$.

□

Examples and further developments

We now give a few examples where λ is a sharp lower bound for the unknotting number.

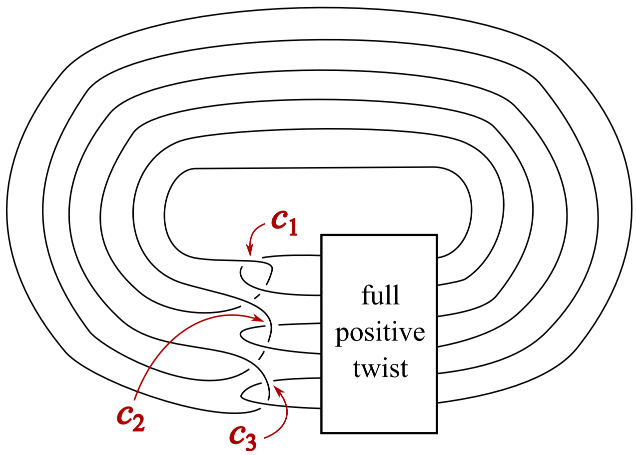


Figure 3.7

Consider the 38-crossing knot K shown in Figure 3.7. It is easy to find an unknotting sequence for K of length 3, by making crossing changes at the crossings labeled c_1, c_2, c_3 in Figure 3.7. So $u(K) \leq 3$.

We now try to find a lower bound for $u(K)$. It is proved in [MM] that K does not satisfy the Knight Move Conjecture 3.16 and that the Lee spectral sequence does not collapse at page 2 (the differential d_2 is non-vanishing). Thus by Lemma 3.15 we have that $\lambda'_{Lee}(K) > 2-1 = 1$, so

$$\lambda(K) \geq \lambda_{Lee}(K) \geq 2\lambda'_{Lee}(K) - 1 \geq 3.$$

Thus $u(K) \geq 3$.

We conclude that $u(K) = \lambda(K) = 3$.

Rasmussen's invariant s, defined in [Ra], gives another lower bound for the unknotting number: $\frac{|s(K)|}{2} \leq u(K)$. In [Al] Alishahi found examples of knots where λ_{BN} (and thus λ) is a sharper bound than Rasmussen's invariant:

For $K = 13n689, 13n1166, 13n2504, 13n2807$ we have:

$$\frac{|s(K)|}{2} = 1 < 2 = \lambda_{BN}(K) \leq \lambda(K).$$

Further developments of this work could aim at finding knots where the invariant λ is strictly

bigger than λ_{BN} and λ_{Lee}. This is not an easy task, since Khovanov homology H_{Univ} relative to Frobenius system $\mathcal{F}_{Univ}$ is hard to compute.
Another interesting development would be to find a spectral sequence E similar to Bar-Natan's and Lee's sequences, this time coming from the chain complex C_{Univ} relative to $\mathcal{F}_{Univ}$: the goal would be to obtain a result similar to Lemmas 3.14 and 3.15, using the invariant λ to determine the convergence of the spectral sequence E.

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Appendix

In Section 3.2 we mention the connection between the lower bounds $\lambda_{BN}, \lambda_{Lee}$ and some spectral sequences. We introduce this concept here briefly.
We follow [Ch] and [MC].

Let us consider a chain complex $(C^*(\_; R), d)$ where R is a commutative ring, and the differential d is such that $d^n : C^n \to C^{n+1}$.

We assume that C^* has a decreasing filtration

$$\{0\} \subseteq \ldots \subseteq C^{n,p} \subseteq C^{n,p-1} \subseteq \ldots \subseteq C^{n,0} \subseteq C^{n,-1} = C^n$$

(the first index represents the homological degree, the second is the filtration degree). The differential respects the filtration, i.e. $dC^{n,p} \subseteq C^{n+1,p}$.
We call $d^{n,p} : C^{n,p} \to C^{n+1,p}$, i.e. $d^{n,p}(C^*) = d(C^{n,p})$.

This filtration induces a filtration on homology:

$$H^{n,p}(C^*, d) = H^n(\operatorname{im}(C^{*,p} \to C^*), d).$$

We observe that $\frac{\ker d^{n,p}}{\operatorname{im} d^{n-1} \cap C^{n,p}} = H^n(\operatorname{im}(C^{*,p} \to C^*), d) \neq H^n(C^{*,p}, d) = \frac{\ker d^{n,p}}{\operatorname{im} d^{n-1,p}}$, because in general $\operatorname{im} d^{n-1,p} \subsetneq \operatorname{im} d^{n-1} \cap C^{n,p}$.

We further assume that the filtration is bounded, that is, for every n there is $s = s(n)$ such that

$$\{0\} \subseteq C^{n,s} \subseteq \ldots \subseteq C^{n,0} \subseteq C^{n,-1} = C^n.$$

For $\infty \geq u \geq s(n)$ we let $C^{n,u} = \{0\}$, and for $-\infty \leq v \leq -1$: $C^{n,v} = C^n$.

Let's give the definition of spectral sequence:

Definition. A *spectral sequence* is a collection of differential bigraded modules $\{E_r^{*,*}, d_r\}$, $r \geq 0$, such that all differentials have bidegree $(1, r)$, i.e. $d_r : E_r^{n,p} \to E_r^{n+1,p+r}$, and $E_{r+1}^{n,p} \cong H^{n,p}(E_r^{*,*}, d_r)$.

Appendix

We now state a theorem on the existence of a spectral sequence that converges to $H(C^*, d)$.

Theorem. *The chain complex* (C^*, d) *determines a spectral sequence* $\{E_r^{*,*}, d_r\}$, $r \geq 0$, *where* $d_r : E_r^{n,p} \to E_r^{n+1,p+r}$, *such that*

$$E_0^{n,p} \cong \frac{C^{n,p}}{C^{n,p+1}} \quad \text{and} \quad E_1^{n,p} \cong H^n\left(\frac{C^{*,p}}{C^{*,p+1}}\right).$$

Moreover this spectral sequence converges to $H(C^*, d)$, *i.e.* $E_\infty^{n,p} \cong \frac{H^{n,p}(C^*,d)}{H^{n,p+1}(C^*,d)}$.

Proof. We define, for $-1 \leq r \leq \infty$,

$$Z_r^{n,p} := (d^{n,p})^{-1}(C^{n+1,p+r}) = d^{-1}(C^{n+1,p+r}) \cap C^{n,p}$$

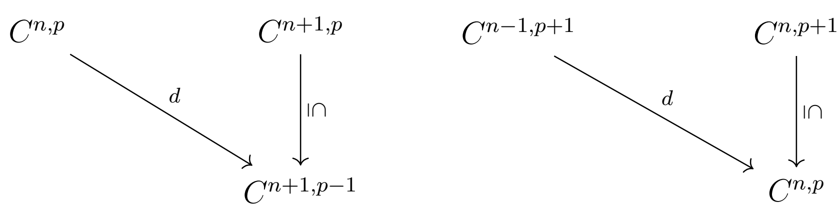
and

$$B_r^{n,p} := d(C^{n-1,p-r}) \cap C^{n,p}.$$

So $Z_r^{n,p}$ is the submodule of $C^{n,p}$ that is mapped to $C^{n+1,p+r}$ and $B_r^{n,p}$ is the submodule of $C^{n,p}$ that is in the image of $C^{n-1,p-r}$.

Let's make some remarks about these two definitions:

1. Since d respects the filtration we have:
   $Z_{-1}^{n,p} = (d^{n,p})^{-1}(C^{n+1,p-1}) = (d^{n,p})^{-1}(C^{n+1,p}) = Z_0^{n,p} = C^{n,p}$
   and $B_{-1}^{n,p} = d(C^{n-1,p+1}) \cap C^{n,p} = d(C^{n-1,p+1})$.

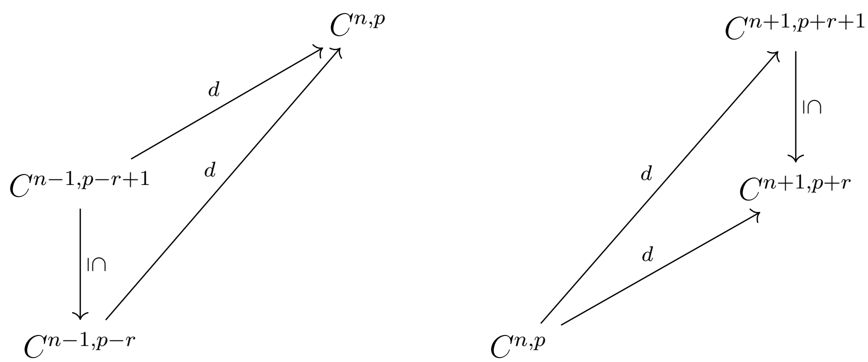


2. The filtration is bounded so there is a t such that $Z_u^{n,p} = Z_t^{n,p}$ and $B_u^{n,p} = B_t^{n,p}$ for all $t \leq u \leq \infty$.

   We have that, for $u \gg 0$, $Z_\infty^{n,p} = Z_u^{n,p} = (d^{n,p})^{-1}(0) = \ker d^{n,p}$ and $B_\infty^{n,p} = B_u^{n,p} = d(C^{n-1,p-u}) \cap C^{n,p} = d(C^{n-1}) \cap C^{n,p} = \operatorname{im} d \cap C^{n,p}$.

3. $B_{-1}^{n,p} \subseteq B_0^{n,p} \subseteq \cdots \subseteq B_\infty^{n,p} \subseteq Z_\infty^{n,p} \subseteq \cdots \subseteq Z_0^{n,p} \subseteq Z_{-1}^{n,p}$.

   The fact that $B_{r-1}^{n,p} \subseteq B_r^{n,p}$ and $Z_{r+1}^{n,p} \subseteq Z_r^{n,p}$ is shown by the following diagrams:



   The fact that $B_\infty^{n,p} \subseteq Z_\infty^{n,p}$ follows from $\operatorname{im} d^{n-1} \subseteq \ker d^n$ and from the definition of $B_\infty^{n,p}$ and $Z_\infty^{n,p}$.

4. $d(Z_r^{n-1,p-r}) = B_r^{n,p}$

   This is because $d(Z_r^{n-1,p-r}) = d(d^{-1}(C^{n,p}) \cap C^{n-1,p-r}) = C^{n,p} \cap d(C^{n-1,p-r}) = B_r^{n,p}$.

We define, for $0 \leq r \leq \infty$,

$$E_r^{n,p} := \frac{Z_r^{n,p}}{Z_{r-1}^{n,p+1} + B_{r-1}^{n,p}}$$

Now we want to define the differential $d_r : E_r^{n,p} \to E_r^{n+1,p+r}$.
Let $\eta_r^{n,p} : Z_r^{n,p} \to E_r^{n,p}$ be the projection. We observe that $\ker \eta_r^{n,p} = Z_{r-1}^{n,p+1} + B_{r-1}^{n,p}$.
Since $d(Z_r^{n,p}) = B_r^{n+1,p+r} \subseteq Z_r^{n+1,p+r}$ (by remarks 3 and 4) and $d(Z_{r-1}^{n,p+1} + B_{r-1}^{n,p}) = d(Z_{r-1}^{n,p+1}) + 0 \subseteq B_{r-1}^{n+1,p+r} \subseteq Z_{r-1}^{n+1,p+r+1} + B_{r-1}^{n+1,p+r}$, we have that d induces a map d_r:

$$\begin{array}{ccc} Z_r^{n,p} & \xrightarrow{\quad d \quad} & Z_r^{n+1,p+r} \\ \downarrow{\scriptstyle \eta_r^{n,p}} & & \downarrow{\scriptstyle \eta_r^{n+1,p+r}} \\ \frac{Z_r^{n,p}}{Z_{r-1}^{n,p+1}+B_{r-1}^{n,p}} = E_r^{n,p} & \xrightarrow{d_r} & E_r^{n+1,p+r} = \frac{Z_r^{n+1,p+r}}{Z_{r-1}^{n+1,p+r+1}+B_{r-1}^{n+1,p+r}} \end{array} \tag{3.3}$$

and the diagram commutes.

We now need to check that $(E_r^{*,*}, d_r)$ is a spectral sequence.

Since $d \circ d = 0$, we have $d_r \circ d_r = 0$.

Let's show that $H^{*,*}(E_r^{*,*}, d_r) \cong E_{r+1}^{*,*}$.
To do that we will find a surjective map $\gamma : Z_{r+1}^{n,p} \to H^{n,p}(E_r^{*,*}, d_r)$ such that $\ker \gamma = Z_r^{n,p+1} + B_r^{n,p}$.

We first show that $\eta_r^{n,p}(Z_{r+1}^{n,p}) = \ker d_r$.
Let's see what $(\eta_r^{n,p})^{-1}(\ker d_r)$ is: consider the commutative diagram (3.3), and remember that $Z_{r+1}^{n,p} \subseteq Z_r^{n,p}$. Given $z \in Z_r^{n,p}$ we have:
$d_r(\eta_r^{n,p} z) = 0 \iff \eta_r^{n+1,p+r}(dz) = 0 \iff dz \in Z_{r-1}^{n+1,p+r+1} + B_{r-1}^{n+1,p+r} \iff z \in Z_r^{n,p} \cap (d^{-1}(Z_{r-1}^{n+1,p+r+1}) + d^{-1}(B_{r-1}^{n+1,p+r}))$.
But $Z_r^{n,p} \cap (d^{-1}(Z_{r-1}^{n+1,p+r+1}) + d^{-1}(B_{r-1}^{n+1,p+r})) \overset{(a)}{=} Z_r^{n,p} \cap (C^n \cap d^{-1}(C^{n+1,p+r+1}) + C^{n,p+1} \cap d^{-1}(C^{n+1,p+r})) \overset{(b)}{=} (Z_r^{n,p} \cap d^{-1}(C^{n+1,p+r+1})) + (Z_r^{n,p} \cap C^{n,p+1}) = Z_{r+1}^{n,p} + Z_{r-1}^{n,p+1}$.
Step (a) follows from the definition of $Z_{r-1}^{n+1,p+r+1}$ and $B_{r-1}^{n+1,p+r}$, and by the fact that $d^{-1}d^{-1}(C^{n+2,q}) = C^n$ for all q.
Step (b) follows from the fact that $Z_r^{n,p} \cap (C^n \cap d^{-1}(C^{n+1,p+r+1}) + C^{n,p+1} \cap d^{-1}(C^{n+1,p+r})) = Z_r^{n,p} \cap (d^{-1}(C^{n+1,p+r+1}) + C^{n,p+1}) = d^{-1}(C^{n+1,p+r}) \cap C^{n,p} \cap (d^{-1}(C^{n+1,p+r+1}) + C^{n,p+1}) = d^{-1}(C^{n+1,p+r}) \cap (C^{n,p} \cap d^{-1}(C^{n+1,p+r+1}) + C^{n,p+1})$ because $C^{n,p+1} \subseteq C^{n,p}$. Then notice that $C^{n,p} \cap d^{-1}(C^{n+1,p+r+1}) \subseteq d^{-1}(C^{n+1,p+r})$.
So $(\eta_r^{n,p})^{-1}(\ker d_r) = Z_{r+1}^{n,p} + Z_{r-1}^{n,p+1}$.
Since $Z_{r-1}^{n,p+1} \subseteq \ker \eta_r^{n,p}$ we have that $\ker d_r = \eta_r^{n,p}(Z_{r+1}^{n,p})$.

We now show that $Z_r^{n,p+1} + B_r^{n,p} = Z_{r+1}^{n,p} \cap ((\eta_r^{n,p})^{-1}(\operatorname{im} d_r))$.
Consider the commutative diagram:

$$\begin{array}{ccc} Z_r^{n-1,p-r} & \xrightarrow{\quad d \quad} & Z_r^{n,p} \\ \Big\downarrow \eta_r^{n-1,p-r} & & \Big\downarrow \eta_r^{n,p} \\ \frac{Z_r^{n-1,p-r}}{Z_{r-1}^{n-1,p-r+1} + B_{r-1}^{n-1,p-r}} = E_r^{n-1,p-r} & \xrightarrow{\quad d_r \quad} & E_r^{n,p} = \frac{Z_r^{n,p}}{Z_{r-1}^{n,p+1} + B_{r-1}^{n,p}} \end{array}$$

So $\operatorname{im} d_r = \eta_r^{n,p}(d(Z_r^{n-1,p-r})) = \eta_r^{n,p}(B_r^{n,p})$.
It follows that $(\eta_r^{n,p})^{-1}(\operatorname{im} d_r) = B_r^{n,p} + \ker \eta_r^{n,p} = B_r^{n,p} + Z_{r-1}^{n,p+1} + B_{r-1}^{n,p} = B_r^{n,p} + Z_{r-1}^{n,p+1}$ (using

remark 3).
We observe that $Z_{r-1}^{n,p+1} \cap Z_{r+1}^{n,p} = (C^{n,p+1} \cap d^{-1}(C^{n+1,p+r})) \cap (C^{n,p} \cap d^{-1}(C^{n+1,p+r+1})) = C^{n,p+1} \cap d^{-1}(C^{n+1,p+r+1}) = Z_r^{n,p+1}$ (because $C^{n,p+1} \subseteq C^{n,p}$ and $C^{n+1,p+r+1} \subseteq C^{n+1,p+r}$).
Moreover $B_r^{n,p} \subseteq Z_{r+1}^{n,p}$ by remark 3.
So $Z_{r+1}^{n,p} \cap ((\eta_r^{n,p})^{-1}(\operatorname{im} d_r)) = Z_r^{n,p+1} + B_r^{n,p}$.

Now we can define $\gamma : Z_{r+1}^{n,p} \xrightarrow{\eta_r^{n,p}} \ker d_r \xrightarrow{\text{projection}} H^{n,p}(E_r^{*,*}, d_r)$.
The map γ is surjective. Let's find $\ker\gamma$.
Let $z \in Z_{r+1}^{n,p}$. $\gamma(z) = 0 \iff \eta_r^{n,p}(z) \in \operatorname{im} d_r$. So $\ker\gamma = Z_{r+1}^{n,p} \cap (\eta_r^{n,p})^{-1}(\operatorname{im} d_r) = Z_r^{n,p+1} + B_r^{n,p}$.
So $E_{r+1}^{n,p} = \frac{Z_{r+1}^{n,p}}{\ker\gamma} \cong H^{n,p}(E_r^{*,*}, d_r)$, that is, $\{E_r^{*,*}, d_r\}$ is a spectral sequence.

The next thing to do is find out what $E_0^{n,p}$ and $E_1^{n,p}$ are.

$E_0^{n,p} = \frac{Z_0^{n,p}}{Z_{-1}^{n,p+1} + B_{-1}^{n,p}} = \frac{C^{n,p}}{C^{n,p+1} + d(C^{n-1,p+1})} = \frac{C^{n,p}}{C^{n,p+1}}$ since d respects the filtration, and using remark 1.

Since d_0 is induced by d we then have $E_1^{n,p} \cong H^{n,p}(E_0^{*,*}, d_0) = H^n\left(\frac{C^{*,p}}{C^{*,p+1}}\right)$.

Finally we show that the spectral sequence converges to $H(C^*, d)$.

Let's construct an isomorphism $d_\infty : E_\infty^{n,p} \xrightarrow{\cong} \frac{H^{n,p}(C^*,d)}{H^{n,p+1}(C^*,d)}$.

First consider the projections $\eta_\infty^{n,p} : Z_\infty^{n,p} \to E_\infty^{n,p} = \frac{Z_\infty^{n,p}}{Z_\infty^{n,p} + B_\infty^{n,p}}$ and $\pi : \ker d \to H(C^*, d)$ (notice that $Z_\infty^{n,p} \subseteq \ker d$).
We observe that the filtration on $H(C^*, d)$ is: $H^{n,p}(C^*, d) = H^n(\operatorname{im}(C^{*,p} \to C^*), d) = \pi(\ker d^{n,p}) = \pi(Z_\infty^{n,p})$.

We have $\pi(\ker \eta_\infty^{n,p}) = \pi(Z_\infty^{n,p+1} + B_\infty^{n,p}) = \pi(Z_\infty^{n,p+1}) + 0 = H^{n,p+1}(C^*, d)$, so π induces a map $d_\infty : E_\infty^{n,p} \to \frac{H^{n,p}(C^*,d)}{H^{n,p+1}(C^*,d)}$.

Now we show that $\ker d_\infty = \{0\}$:
$\ker d_\infty = \eta_\infty^{n,p}(\pi^{-1}(H^{n,p+1}(C^*, d)) \cap Z_\infty^{n,p}) = \eta_\infty^{n,p}((Z_\infty^{n,p+1} + d(C^*)) \cap Z_\infty^{n,p}) \subseteq \eta_\infty^{n,p}(Z_\infty^{n,p+1} + B_\infty^{n,p}) = \{0\}$.

So d_∞ is an isomorphism.

This concludes the proof of the theorem.

Note the filtration on (C^*, d), and thus on $H(C^*, d)$, is finite, so there is a finite sequence of extension problems going from $E_\infty^{*,*}$ to $H(C^*, d)$.