Application of Condition-based Maintenance in Control of a Supply Chain Network under Stochastic Disruption

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Abstract

Application of Condition-based Maintenance in Control of a Supply Chain Network under Stochastic Disruption

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The thesis develops novel and proactive optimal control policies for a partially observable facility, which is subject to stochastic disruptions. Unlike traditional Supply Chain Networks (SCN), where established facilities are considered to be continuously available, a more practical scenario is developed. More specifically, in the proposed frameworks, the aforementioned assumption is relaxed such that the facilities are subject to stochastic disruptions potentially leading to costly failures. In such practical scenarios, it is critical and of paramount importance for the established facilities to operate with the highest achievable reliability in the presence of disruptions and degradation. In this regard, this thesis provides a conceptual framework to obtain an optimal control policy for an already established facility subject to stochastic disruptions/degradation such that the disruptions have a direct effect on the connection links within the SCN. The level of degradation of a facility is modeled as a N state continuous time hidden-Markov process with $N-1$ operational and unobservable states together with one observable failure state. The facility is monitored periodically to observe the level of degradation. If the degradation level exceeds a critical state, a preventive action, namely partial fortification, will be performed. On the other hand, when the degradation level exceeds the failure state, a corrective action, namely full fortification, will be performed which brings the facility to the healthy state. The model is extended to the scenario where an integrated model of Statistical Process Control (SPC) and maintenance planning is considered and the optimal control limit policy is achieved based on a novel Bayesian control chart. The control problems under consideration are formulated in a Partially Observable Markov Decision Process (POMDP) framework to find the optimal preventive level in order to minimize the long-run expected average cost. A comprehensive sensitivity analysis is performed to evaluate the performance of proposed models.

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Chapter 1

Thesis Introduction and Overview

1.1 Motivation

In today's competitive market, it is essential that supply chain network (SCN) operates with full potential and highest achievable reliability in order to keep up with competitors and strongly stay in the business. In reality, facilities in a SCN and the links connecting them are subject to stochastic disruption over time due to several factors including but not limited to poor weather condition, degradation, natural or man-made disasters, or a combination of any other factors. Disruption can affect a SCN in any layer and lead to delays in the process flow of the products or service. Facility disruption affects all the players in the SCN, including the suppliers, manufacturers, distributors, and customers which eventually result in potential major economic losses and customer's dissatisfaction. Therefore, there is a huge need to design a sustainable SCN which is capable to cope with the stochastic and unexpected disruption in an efficient manner. In this regard, supply chain managers are now trying to develop a trade-off between supply chain disruptions and a sustainable system in order to improve the performance of a SCN. As a potential solution, resilience is a new approach to design a sustainable SCN. One promising approach to make a SCN resilience, is to define a proper control and maintenance policy which is the focus and motivation for this thesis.

1.2 Thesis Statement

While most researchers focus their attention on facility location problems assuming no disruption or defining the stochastic disruption on facility itself, we have considered the effect of disruption on distribution links. Accordingly, we develop a novel control policy to minimize the effect of disruption to avoid total failure of a SCN. When a facility experiences a disruption, if no precautionary plans have been set, corrective actions must take place as a mandatory action to ensure operations will resume in the future. However corrective measures are costly. That is why preventive measures are necessary for every facility to prevent any failure and reduce the affect of disruption in a SCN. Preventive actions are defined as actions to bring a system back to a healthy condition. Operating in an unhealthy state results in high operating costs and an increased risk of random failure.

The likelihood of random failures increases with the facility's age and degradation. To assess health condition of a facility, the systems needs to be monitored frequently. This type of maintenance is referred to as Condition-based Maintenance (CBM). The CBM is a maintenance policy that suggests maintenance actions based on the condition monitoring (CM) data with the goal to reduce the system downtime, maintenance costs, and improve the availability and reliability of the system. In many practical applications of CBM, the operational states of the system can not be observed directly. The system is partially observable through specific point of times. In CBM context, the following factors play significant role in total cost of system, failure detection rate and failure detection time:

- Sampling (monitoring or inspection) interval: which affects the cost of monitoring as well as the probability of failure.
- Control limit: setting low control limit leads to performing maintenance actions much more in advance which results in high false alarm rate. On the other hand, setting high control limit leads to missing the opportunity of preventing the system to fail (low detection rate). Both contribute to the high cost of system.

In particular, since the facility/system deterioration increases over time, monitoring the system more frequently while the system is in healthy state leads to increase the total cost.

On the other hand, if we monitor the system less frequently, it leads to an increase in probability of failure. Therefore, it is important to find the optimal monitoring/inspection interval. Similarly, when the control limit is chosen large, the failure probability increases and when the control limit is chosen small, unnecessary preventive maintenance (PM) actions are performed which lead to high costs.

In this regard, this thesis tries to emerge new dimension in SCN literature by developing new control policies to make a trade-off among sampling/inspection intervals, control limit and total cost of the system.

1.3 Objectives and Contributions

The main objective of this thesis is to develop conceptual frameworks to design a reliable and resilient SCN subject to stochastic disruption/degradation in order to minimize the long-run expected average cost per unit time. The thesis is categorized into the following two main parts:

- Develop an optimal control policy for one facility in a SCN.
- Develop an optimal control policy for a SCN consists of multiple facilities by incorporating the appealing concept of integrated model of SPC and CBM.

In summary, the following contributions have been made in this thesis:

(1) We have developed an optimal control policy for one established facility in a SCN. It is assumed that disruption has direct effect on links connected to facility. Facility degradation is model based on 5-state continuous-time hidden-Markov process. We further assume that the deterioration level of facility is hidden and can be known only by inspections performed at discrete equidistant time epochs. The facility will fail if the deterioration level reaches the last state. We show that there is a critical state denoted by N^* such that upon arrival of facility to this state, partially fortification must be performed which result in minimum long-run expected average cost per unit

time. A mathematical model is derived to find the preventive fortification level and the problem is formulated and solved in the semi-Markov decision process (SMDP).

(2) For the proposed model, a closed-form expression is derived for reliability function of a facility.

(3) Since the process parameters such as transition rates and cost can not be estimated accurately, it can lead to in a sub-optimal result. Therefore, we perform misspecification analysis of parameters to see the effects of parameter estimation changes on the long-run expected average cost. The results confirm the robustness of our proposed model.

(4) We have extended the previous development to the scenario when multiple facilities within a SCN are affected by disruption. Each facility is prone to degradation and upon failures of most facilities the network will fail which result in huge cost. In order to model the problem at hand, we proposes an integrated model CBM and SPC for a network subject to stochastic degradation and random failures. In particular, a novel attribute-type Bayesian control chart is designed for monitoring the whole network. The degradation process is modeled based on 3-state continuous-time hidden-Markov process with two operational (healthy and warning) and a failure state. The network is monitored at equidistant sampling epoch and posterior probability is updated based on Bayes' rule. Then, the posterior probabilities are plotted on the Bayesian control chart. If the posterior probability exceeds the control limit, the monitoring process should be stopped and the full inspection should be performed followed possibly by preventive maintenance (PM) action. The Bayesian control chart for proposed model is attribute-type chart such that monitors the posterior probability that the network has shifted to the warning state given the history of the process, when the fraction defective is the quality characteristic of interest. The objective is to find an optimal control limit for PM such that the long-run expected average cost is minimized. In order to find the optimal control limit and monitoring interval, the problem is formulated in SMDP framework and policy-iteration algorithm is applied as a solution

methodology.

(5) The closed-form expression for most significant life characteristics such as reliability and mean remaining useful life (MRL) are also derived for proposed model as functions of posterior probability.

(6) A comprehensive sensitivity analysis including design of experiment (DOE) is performed to see the effects of input parameters on total cost.

1.4 Outline

The rest of the thesis is organized as follows: The following chapter consist of a detailed literature review on the topic of this thesis. Chapter 3 considers a single facility in a SCN and investigates the application of CBM on SCN by defining the problem in SMDP framework. In Chapter 4, the development of integrated model of CBM and SPC is presented for a SCN where multiple facilities are subject to degradation. Finally, the conclusion and future works are presented in Chapter 5.

Chapter 2

Background & Literature Review

2.1 Literature Review

Supply chain network (SCN) is defined as a set of processes, from the receiving of raw materials to eventual consumption of the finished products, linking across supplier-user industries and manufacturing ([Afify et al., 2019]). Therefore, as shown in Figure 2.1, the SCN can be considered as a complex system, which consists of different layers, including suppliers, manufacturing plants, distribution centers, and customers. In an SCN, there are usually several supplier options for purchasing raw materials, various production and manufacturing sectors for producing/assembling the semi-finished and/or final products, and different distribution centers to transport final products to the customers and retails.

Supply chain design is an essential step in strategic planning, which includes different aspects of the SCN, such as inventory management, facility location, assignments, distribution, and production control. A comprehensive review of the strategic planning and design of supply chains is performed by [Lambiase et al., 2013]. Facility location is considered as one of the main steps in supply chain design, such that a growing body of literature is devoted to address facility location problems. In particular, facility location problems deal with the determination of the optimal place to locate the facility to minimize the transportation costs as the facilities and customers are typically distributed in large geographical areas. Facility location decisions are critical elements in strategic planning for a wide range

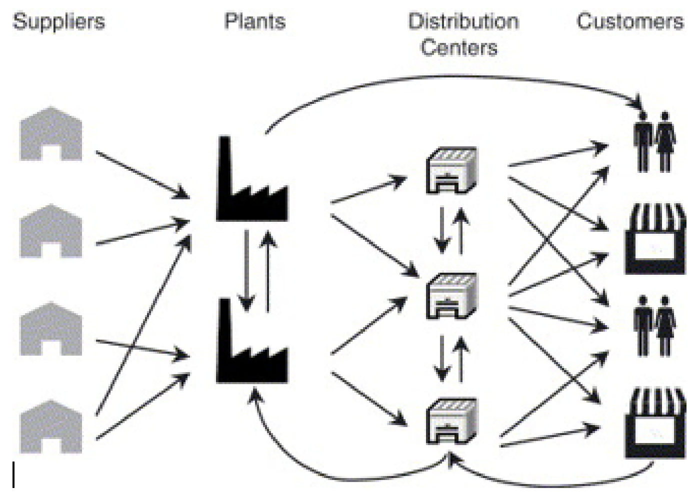

Figure 2.1: Supply Chain Network ([Melo et al, 2005])

of private and public sectors. Most researches on the development of solution approaches for addressing facility location problems in the SCN design assume that the facilities are operating perfectly with their full capacity. In the real world, however, facilities are subject to random failures due to various factors and disruptions, including but not limited to poor inventory management, technical issues, work-related accidents, power outage, road blockage, and natural disasters ([Afify et al., 2019]). Disruptions have a significant effect on the performance of an SCN which could lead to a variety of severe consequences such as higher operational and transportation costs, lost sales, delay in the delivery and dissatisfaction of the customers which are forced to either travel longer distances to obtain service from another facility or give up service which incurs a huge penalty.

According to [Kleindorfer and Saad, 2005], in general, the SCNs are exposed to the following two main types of disruptions/risks:

(i) Risk associated with problems related to the supply and demand ([Chauhan et al., 2007, Stevenson and Spring, 2007, Acar et al., 2010]). This category deals with demand fluctuations, lead-time, resource availability, and inventory that are not constant and changing dynamically, posing significant analytical challenges in both planning and execution control stages ([Ivanov and Sokolov, 2012]).

(ii) Risk associated with disruptions to normal activities. This category, on the other hand, deals with external disruptions such as road blockage and natural disasters that have significant negative effects on the reliability of an SCN.

Item (ii) above is the target area of this thesis. Disruptions are considered as random events that can happen at any level of an SCN, which result in complete or partial failure of an SCN for a certain amount of time, leading to potential major economic losses.

As an example, a study by [Singhal and Hendricks, 2002] shows that at the time disruption is triggered, the average shareholders' return immediately drops at 7.5%. Four months after a disruption, the total loss grows to an average of 18.5%. Recently, as a real case study of disruptions on SCN, [Haraguchi and Lall, 2015] investigated the impact of natural hazards and disaster on the performance of SCNs. They examined Thailand's 2011 flood as a notable example of the impact of floods both on industries and the economy as a whole. The extensive floods affected the primary manufacturing and industrial sectors, such as automotive and electronics, with a destructive impact on the whole economy. Disruptions in any level of SCN do not only increase the total cost of providing the demand but also lead to an increase in the overall customer loss and the cost associated with it. The latter is much harder to quantify but has a much greater effect on total expected cost in case of business interruption. Similarly, [Ivey et al., 2011] estimated the repair cost and downtime of losing a port due to the earthquake. Ports play a critical role in transportation in SCN, which are subject to seismic hazards. The damages and downtime of these structures that result from natural disruption have a direct impact on the port system's ship handling operations and economy. The results of their analysis can be used by decision-makers to make better decisions in advance within the design, reconstruction, and operational stages of the ports.

As a motivated example for this research, we can also refer to power grid networks shown in Figure 2.2 that provide the power to the residential loads through some connection links from a generation plant. They are subject to random failures which are inflicting enormous socioeconomic costs. The failure of power grids is usually referred to as cascading failure which can happen on initial node (generator) or link (line). Node or link failures lead to a large connected component of failed nodes or links affecting the whole network globally which is caused a massive power blackout. When the failure happens in one facility, it forces a redistribution of flows. When a link (line) goes out of service, it can reduce the network's overall capacity which begets power overloads on the remaining links as the power flows are redistributed. A cascade failure generally terminates in a major blackout, with large areas of a network unable to supply demand. An illustration of a general process of a cascade failure in a power grid network is shown in Figure 2.3.

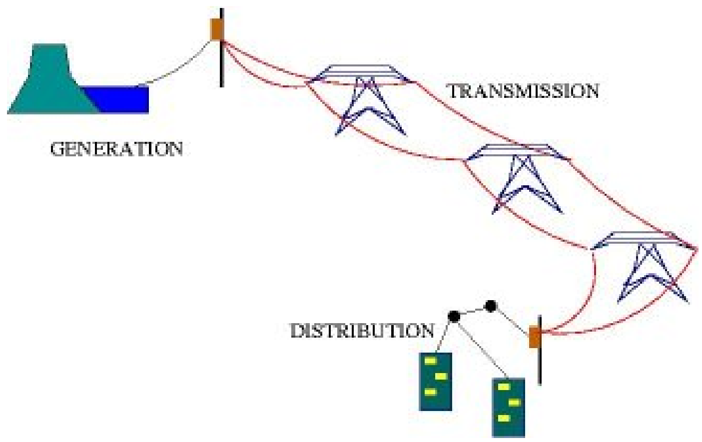

Figure 2.2: Power Grid Networks

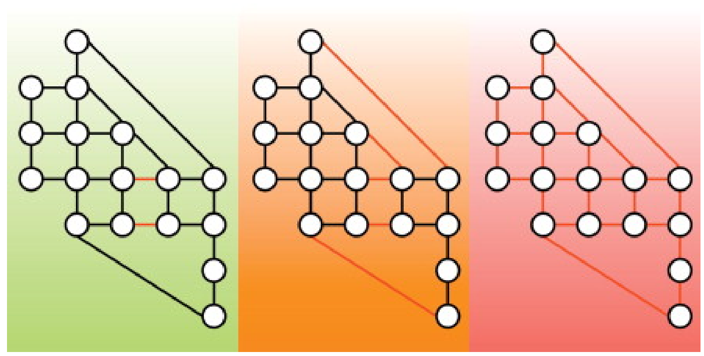

Figure 2.3: An illustration of a cascade failure in a power grid network

The left network in Figure 2.3 represents a healthy network such that almost all the facilities and links are operating in healthy condition. During the time, due to the occurrence of stochastic disruptions and degradation, facilities are loosing their links and gradually the total failure of a network is expected with huge economic loss. In order to mitigate such costly failures, it is critical and of paramount importance that SCNs operate at their full potential with minimum disruptions and failures. In order to mitigate the risks in SCNs, organizations have to invest tremendous effort in learning how to anticipate, absorb, and overcome disruptions ([Pickett, 2006]). Consequently, recent SCN designs with disruptions have attracted a great deal of attention among researchers. Different mathematical models, such as stochastic and probabilistic models, are developed in this context. As an example, [Bozorgi-Amiri and Khorsi, 2016] developed a multi-objective dynamic stochastic programming model for a humanitarian relief logistics problem, which aims to minimize the maximum amount of shortages among the affected areas in all periods, the total travel time, and the sum of pre- and post-disaster costs. Similarly, [Portillo and Carlos, 2008] proposed a robust optimization approach in global SCN design, considered disruption risk as important evaluation criteria, and applied a Lagrangian-based heuristic method as a solution methodology. Along the same path, [Peng et al., 2011] applied p-robustness technique for risk reduction in order to design a reliable logistics network such that facilities are subject to disruptions. [Afify et al., 2019] developed an evolutionary learning technique to solve reliable incapacitated facility location problem and reliable p-median problem. The problem was a case study considering assigning each customer to two facilities, one primary and one back up, with the probabilities of disruption. [Contreras et al., 2012] developed two mixed-integer programming formulations, which were the expansion of typical P-median problem. The model was presented as a combined facility location/network design problem (FLNDP), and it simultaneously considers the location of facilities and its underlying network. [Shishebori and Jabalameli, 2013] considered a reliable facility location-network design problem (RFLNDP) with two kinds of facilities, reliable and unreliable, each with a different cost in which the cheaper one is subjected to failure. Network design and reliability of facilities simultaneously have been considered to develop the mixed-integer nonlinear programming.

Table 2.1 summarizes the literature on SCN design under disruptions. As mentioned earlier and Table 1 shows, disruption can happen at any level of SCN, such as the facility itself, distribution center, or supply and demand. Furthermore, most of the researches assume that the travel times in the SCN network are predetermined and fixed, and disruption is only defined on facilities and suppliers. The problems are formulated based on different mathematical models in which different solution methodologies are applied. More references in this line of research can be found in [Afify et al., 2019], [Contreras et al., 2012], [Esmaeilikia et al., 2016] [Hatefi and Jolai, 2014], [Shishebori and Jabalameli, 2013].

Research Papers	Disruption type	Problem type	Modeling framework
[Drezner, 1987]	Facility	PMP	MILP
[Snyder and Daskin, 2005]	Facility	PMP-RUFL	MILP
[Snyder et al., 2006]	Facility	Facility location-network design	Stochastic MLIP
[Cui et al., 2010]	Facility	RUFL	MLIP and CA
[Li and Ouyang, 2010]	Facility	RUFL	CA
[Aryanezhad et al., 2010]	Distribution center	Integrated inventory-location	NLIP
[Lim et al., 2010]	Facility	Facility location-network design	MILP
[Qi et al., 2010]	Supplier	Integrated inventory-location	NLIP
[Azad and Davoudpour, 2012]	Facility	Distribution Network Design	NLIP
[Shen et al., 2011]	Facility	Facility location	Stochastic MLIP
[Peng et al., 2011]	Facility	Logistic network design	Robust MILP
[Jabbarzadeh et al., 2012]	Distribution center	Supply chain network	NLIP
[Bozorgi-Amiri and Khorsi, 2016]	Distribution center	Supply chain network design	Bi-objective robust optimization
[Ruiz-Torres and Mahmoodi, 2007]	Supplier	Supplier decision-making	unequal failure probability
[Meena et al., 2011]	Supplier	Facility location-network design	Proposed algorythm
[Berger and Zeng, 2006]	Supplier	Supplier decision-making	Decision-tree approach
[Masih-Tehrani et al., 2011]	Facility	Supply chain design	Multivariate dependence theory
[Tomlin and Wang, 2005]	Supplier and demand	Supply chain design	Mix-flexibility & dual-sourcing approach
[Xu and Nozick, 2009]	Supplier	Supplier decision-making	Two-stage stochastic program
[Snyder and Daskin, 2006]	Facility	Stochastic p-robust location problems	Bi-objective robustness measure
[Berman et al., 2011]	Facility	optimal search paths	NP-complete
[Berman et al., 2009]	Facility	RFL	Facility centralization and co-location
[Aboolian et al., 2012]	Facility	RFL	RUFLP

Table 2.1: The relevant reliable models.

2.1.1 Reliable and Resilient SCN Design

In order to improve the performance of an SCN, which is subject to random disruption, a great deal of attention has been devoted to developing a flexible and resilient SCN. Strategic robustness and flexibility through redesigning or re-configuring an existing network can be expensive. Nowadays, the enormous scale and impact of natural disasters and hazards in the SCN is increasing dramatically, which has attracted a great deal of attention to design an SCN in a more resilient way. Resilience of the supply chain is defined as its capability to sustain or recover its performance and functionality following a significant disruption ([Ivanov and Ivanov, 2019], [Gunasekaran et al., 2015], [Tukamuhabwa et al., 2015], [Hosseini et al., 2019], [Dolgui et al., 2018]). Therefore, it is vital, of great practical importance and theoretical significance to develop timely and effective control and recovery policies in SCNs. Recent natural disasters and man-made catastrophes highlighted the high vulnerability of modern SCNs, their disruption risk exposure and the importance of timely and effective recovery policy deployments ([Hosseini et al., 2019], [Dolgui et al., 2019], [Cavalcante et al., 2019], [Macdonald et al., 2018], [Heet al., 2019], [Tang et al., 2016], [Simchi-Levi et al., 2015]).

The SCN resiliency has been widely studied in the literature in different contexts, such as discrete optimization, simulation, and control research ([Ivanov and Ivanov, 2019]). The definition of resilience implies achieving the following three key objectives:

(i) Readiness (being prepared or available for service);

(ii) Response, i.e., reaction to a specific stimulus, and;

(iii) Recovery, which is defined as returning to “normal” stable or steady-state conditions ([Spiegler et al., 2016]).

The latter implies the implementation of proper control and action policies, which is the focus of this thesis. The most recent contribution in this line of research is given by [Ivanov and Ivanov, 2019], where a resilience control model is developed for simultaneous structural and operational control of supply chain dynamics. The proposed model takes into account both structural recovery control in the SCN within a disruption dynamic as a whole, and the corresponding functional recovery control at individual firms in the SCN. The proposed model can improve the firms' operations in terms of demand allocation and re-allocation in case of disruption and change in SCN as well as dynamic analysis of SCN disruption and recovery. [Ivanov and Sokolov, 2012] developed an integrated framework with interconnected SC structures and the consideration of structural dynamics. The goal is to represent an SCN as a multi-structural dynamic system and to develop planning and control models based on the SDC approach. The structure dynamics control is a dynamic interpretation of the supply chain (re)synthesis process and aims at both advancing the supply chain (re)planning domain and enlarging the scope of the supply chain analysis domain. [Ivanov et al., 2016] represent a robust schedule coordination approach in a hybrid discrete/continuous flow shop supply chain with job shop processes at each supplier stage is studied with dynamic optimal control models. They introduced a robust analysis of schedule coordination in the presence of disruptions in capacities and supply and integrate the schedule coordination issues into robustness analysis, exemplify the developed approach for the case of two-stage supply chain coordination, and derived managerial insights for both considered scheduling problem and application of dynamic control methods to supply chain schedule coordination in general. [Sarker and Diponegoro, 2009] prescribe optimal policies for a multi-stage production and procurement for all shipments scheduled over the planning horizon. Numerical examples are also provided to illustrate the system. It addresses an optimal policy for production and procurement in a supply chain system with multiple non-competing suppliers, a manufacturer, and multiple non-identical buyers (optimal production plans and shipment schedules are considered in an SCN with multiple suppliers, one manufacturer and multiple buyers subject to known demands of buyers).

2.1.2 Condition-based Maintenance

By extending recent developments in making SCNs more reliable and resilient, we propose a novel and proactive control policy based on preventive maintenance actions for SCNs subject to stochastic disruptions. More precisely, we concentrate the attention on obtaining an optimal control policy for an already established facility with its distribution network as a system under stochastic disruption and degradation. A widely used control technique in systems under disruption is to implement proper preventive maintenance (PM) actions. Reasonable PM strategies can enhance the reliability of facilities, decrease operation costs, increase availability, and reduce downtime. Among different PM schemes, condition-based maintenance (CBM) is considered as a state-of-the-art maintenance methodology that defines the optimum time for performing PM actions by incorporating condition monitoring (CM) data. In the CBM strategy, the observations (CM data) are collected, and based on the information obtained from CM data, the actual condition of a system/facility can be determined. Finally, a proper maintenance action will be performed. The CBM outperforms the traditional PM strategies in terms of cost reduction by eliminating unnecessary maintenance actions. Therefore, nowadays, CBM is used widely in various industries to reduce risk, minimize maintenance costs, and improve system availability. As an example, [Jeong and Oh, 2003] presents a discrete event simulation framework for the systematic investigation of the effect of a CBM policy on the performance of an aerospace maintenance supply chain (AMSC). Although a growing body of literature is dedicated to implementing CBM in the production lines, which can be considered as the production layer of SCN, few papers implement it in other layers of the SCN. [Jeong and Oh, 2003] shows that in order to maximize the benefits from CBM for the enterprise, it is important to focus on the aftermarket supply chain which results in lower costs and turn around times, and higher asset availability, spare part availability, and fill rates. Some other interesting applications of CBM in SCN can be found in [Reimann et al., 2009], [Prajapati et al., 2012], [Mourtzis and Vlachou, 2018], [Gulledge et al., 2010]. Regardless of application of CBM in different domains, the CBM contains three main steps: (i) Data collection, (ii) Data processing; and, (iii) Decision support system. There are different approaches to model and implement CBM policy including but not limited to statistical models, artificial intelligence (AI), kalman filter, hidden Markov model (HMM), and integrated models [Jardine et al., 2006]. The last two models are the main focus of this thesis which will be discussed in details in the next subsections.

2.1.3 Hidden Markov Models

HMM is an appropriate and widely used model for analyzing event and condition monitoring data together. As an example, [Wang et al., 2019] proposed an integrated algorithm using HMM and improved Gated Recurrent Unit (GRU) network to evaluate the performance degradation of bearing and its health indicator. The results of this experiment showed that the proposed integrated algorithm can successfully evaluate the health condition of the slewing bearing. Similarly, [Kamlu and Laxmi, 2019] proposed a method known as vehicle maintenance scheduling (VMS) to identify the details regarding the type of maintenance required for vehicles within a transport system and the time, in weeks, required for a maintenance plan. The proposed HMM is used for acquiring the probability values of constraints as inputs of VMS. The purpose of this research is to develop an effective maintenance strategy to maintain high quality, safety, and security for a healthy transportation system. The results of the proposed maintenance strategy showed an effective outcome in the area of maintenance scheduling for the development of a healthy transportation system. Another example in this line of research is given by [Zhang and Djurdjanović, 2019] such that the authors proposed a novel degradation method assuming the system is under imperfect maintenance. The purpose of this research is to study the uncertainty in maintenance effectiveness. The proposed novel process monitoring method provides condition indicators every time a new observation is retained from the system under monitoring. A large-scale semiconductor data set was then used to prove the significant improvement in the log-likelihood observed in the HMM using imperfect maintenance. Furthermore, the proposed monitoring method is shown to be capable of significantly reducing the false alarm ratios, compared to the conventional multivariate signature-based methods. Recently, [Salari and Makis, 2017] developed a new optimal opportunistic and preventive maintenance policy for a two-unit model based on HMM. A mathematical model is derived to find the preventive and opportunistic replacement levels for two units such that the long-run expected average cost is minimized. The authors showed the great performance of the model in comparison with other maintenance policies. The application of HMM in SCN has attracted attention among researchers in different aspects. As an example [Arifoğlu and Özekici , 2011] introduce a HMM to identify optimal ordering policies for inventory issues. They use two single-item inventory models with periodic-review and assume supply and demand are random in an environment that is partially observed with imperfect information. Results showed that the inventory models made better decisions with imperfect information as precision in observations increased, and that the model stored more inventory less frequently due to an increase in the uncertainty of demand and decrease in the dependability of the supplier. [Liu et al., 2018] utilized the Markov chain to identify the cause of a system malfunction as well as a hidden Markov model to provide managers with improved failure prevention. The purpose of this research was to promote the reduction of system downtime as a significant factor in the reduction of costs. The results showed 86% accuracy in helping managers improve production efficiency. [Zhao et al., 2017] presented a resilience analysis framework consisting of non-homogeneous Hidden Markov Models for measuring resilience in network infrastructure systems. They also proposed a dispatch strategy that was optimized to maximize system resilience and secure system functionality in the long-run. A case study demonstrated that the proposed framework was effective to assess and measure system resilience. [Dhulipala et al., 2020] proposed a series of semi-Markov processes to identify and model the inter-event dependencies in infrastructure recovery when systems are subjected to multiple hazard events. They developed two processes within the inter-event dependency model, considering the worst and aggregated impacts of two successive hazards. Results displayed that the consideration of inter-event dependencies led to lesser-predicted resilience. [Qiu and Chen, 2018] utilized a hidden Markov model to analyze passenger behavior when purchasing an airline ticket and identify new dynamic pricing strategies for airlines. The research showed that passengers' choices when purchasing a ticket were influenced by an invisible logical chain powered by several key elements.

2.1.4 Integrated Models

Recently, there is a surge of interest in implementing a CBM policy by developing integrated models of maintenance planning (MP) and statistical process control (SPC). By considered the similarity between on-line quality control and CM for maintenance purposes, the researchers are proposing optimal and efficient control policies. For example, [Liu et al., 2017] proposed an integrated model of CBM and SPC for a system subject to deterioration to find out the the maximum number of times that PM action must be performed before system's failure. In particular, an $\bar{X}$ control chart is used to monitor the system performance. An integrated model of maintenance management and quality control policy is presented by [Wan et al., 2018] such that the authors try to find out the relationship between two concepts. They applied an economical design model which minimizes the general cost of implementing the model, along with numerical experiments to study the optimal policy. Based on this research, the results display that the integrated model presents 2.64% economic benefit. Another research is performed by [Wu and Makis, 2008] such that an economic-statistical design of a X^2 chart is designed for a machine deteriorating process that is structured by a three-state continuous time Markov chain (healthy, warning and failure state). The purpose of this research is to identify the optimal control chart parameters that minimize the average maintenance cost using renewal theory. In addition, they utilized a constraint that assures the event of the true alarm signal before system failure. Similarly, [Da Silva et al., 2018] suggests a model that utilizes modified CUSUM and exponentially weighted moving average (EWMA) control charts to monitor online system equipment and prevent quality decline. When using the model, rules and limits would be defined for online vibration monitoring, leading to a reduction in false alarms and early intervention planning. One of the major issues with traditional control charts is that they are not capable to make full use of the previous information. So, Bayesian control chart is introduced in quality control. Generally speaking, Bayesian control chart is considered as an adaptive control chart such that the control scheme relies on continuously updating the information about the system state using Bayes' theorem. The origins of the Bayesian process chats can be traced back to the early theoretical paper of [Girshick and Rubin, 1952]. The Bayesian control chart is proved to be a useful SPC tool to determine the optimal control policy ([Taylor, 1965] and [Taylor, 1967]). The papers developed by [Calabrese, 1995], [Tagaras and Nikolaidis, 2002], [Vaughan, 1993], [Tagaras, 1996], [Makis, 2009], [Nenes and Tagaras, 2007], and [Makis, 2008] are the contributions dealing directly with different aspects of the Bayesian control chart design.

The application of Bayesian control chart is not limited to quality control context. It has been widely applied into CBM area to determine optimal maintenance policy which shows the superior performance of Bayesian control chart for maintenance decision making. In particular, in Bayesian setting and for CBM applications, after each sampling/inspection, the posterior probability of the system being in unhealthy or warning state is calculated which is proved to be a sufficient statistics for decision making. Then, the posterior probability is plotted on the control chart. If the posterior probability exceeds a certain limit which is called control limit, the system should be stopped for inspection which is followed possibly by PM action. The control limit plays a significant role and it should be determined precisely to have an optimal or near-optimal control policy. As an example, we can refer to [Naderkhani and Makis, 2017] which an optimal control policy for a partially observable deteriorating system is derived to minimize the long-run expected average cost based on newly developed Bayesian control chart. Similarly, [Lin et al., 2019] applied a cost-optimal Bayesian control chart for a multivariate observation process of early fault detection of gearboxes. The model presented is designed as three states continuous-time hidden Markov process with two unobservable states and one observable failure state. The control limit is found and the remaining useful life is calculated based on the observed posterior probability when multiple sensors are used for data collection. The parameters are estimated by using the expectation maximization algorithm, and validation is proposed by using real gearbox vibration data from multiple sensors. Recently, [Li et al., 2019] applied the same procedure for early fault detection of gear shaft and improved the model by considering Erlang distribution for modeling the sojourn time in the hidden states, which makes the model closer to reality rather than considering Exponential distribution. Later on, [Li et al., 2018] extended previous research by considering multiple independent failure modes instead of single failure for the gear shafts. An optimal Bayesian control scheme is applied while considering both degradation failure and catastrophic failure to determine the optimal time to perform PM action. The information is obtained from condition monitoring data and the age of the system, and the objective is to maximize the long-run expected average system availability per unit time. The results are compared with previously HMM and age-based approaches to prove the success of the Bayesian approach.

[Duan et al., 2019] further extended this field of research by applying a cost-optimal multivariate Bayesian control technique for early fault detection of a computer numerically controlled (CNC) equipment. The proposed approach is modeled based on vector auto-regressive (VAR) degradation and hidden three states semi Markov decision process (SMDP). The optimal control policy is obtained by analyzing real multivariate CM data such that expectation-maximization (EM) algorithm is applied to estimate the model parameters, and mean residual life is also calculated at the end of each decision period. The efficiency of the proposed model is shown by comparing the results with a multivariate Bayesian control chart based on a hidden Markov model. [Shi et al., 2020] considered a multi-component system which needs to meet a predefined reliability requirement with the objective of minimizing the total maintenance cost over a finite planning horizon. Bayesian updating procedure is conducted to define the deterioration level of the system. Once the reliability does not meet the expectations, a dynamic-priority-based heuristic algorithm is used to decide which component needs to go under PM. One of the recent research in the Bayesian field is performed by [Duan et al., 2020], which considered a two-level Bayesian control chart for an equipment subject to both catastrophic failures and dependent degradation. Marshall-Olkin bivariate exponential distribution is used to model the dependency. The SMDP framework is applied to model the optimization problem, and mean remaining useful life (MRL) is derived by using the Bayesian approach.

The main objective of this thesis is to develop optimal/near-optimal control policies for a SCN subject to stochastic degradation/disruption. This thesis is a step-forward contribution in applying CBM on an already established facility in a SCN network, where the failure can happen both in the facility and the links in the distribution network. The problem is first modeled as a N state continuous-time hidden Markov process with $N-1$ non-observable operational states and one observable failure state for a single facility. The condition of a facility is monitored periodically at equal and constant predetermined intervals. The model is then extended to a complete network where there are more than one operating facilities. To overcome the challenges of the network, Bayesian control charts are applied to monitor the status of the network. The problem is formulated in the semi-Markov decision process (SMDP) framework with 3 states. The objective for both problems is to obtain the optimal/near-optimal control in order to minimize the long-run expected average cost per unit time in infinite time horizon.

Chapter 3

Optimal Contrl of a Single Facility in a SCN

In this chapter, we concentrate the attention on the transportation or distribution network within a SCN, which is usually visualized as a graph with a set of N nodes and L transportation/communication links. The set of nodes represent customers with known demands required to be met by several facilities that are already established in a pre-defined subset of the customers' locations. Each link connecting any pair of nodes in such a graph represents the transportation path with known distance and its associated cost. The objective is to find the optimal control policy of the facility in order to serve the demands such that the expected total cost is minimized.

The organization of the chapter is as follows: Section 3.1 provides problem description. Section 3.2 formulates the problem in SMDP framework.Also, the derivation of the expected costs, transition probabilities and the expected sojourn times are given in this section. In Section 3.3, we provide numerical examples to illustrate the whole procedure, and to show the effectiveness of the newly developed model. Finally, Section 3.4 concludes the chapter.

3.1 Problem Description

In a real-world scenario, the transportation networks are subject to stochastic disruption, including but not limited to severe weather conditions, accidents, and special events, which potentially lead to total failure such that facilities can not satisfy the customer's demands. Therefore, we need to incorporate disruptions accurately into the system model. In this regard, we assume that the disruptions can happen at any time in the network with a direct effect on links between two nodes, which is a reasonable and realistic assumption. In particular, such disruptions can change the travel times on the links of the network due to the occurrence of probabilistic events or can destroy the link between two nodes. For illustration purposes, we start with a simple network (graph), as shown in Figure 3.1, which consists of 5 nodes such that a facility is already established in Node 5 denoted by $N5$. If there is no disruption in the network, the travel times between the two nodes are considered as minimum. However, if there is some disruption in the network, these travel times are accordingly increased. As an illustration, we show these changes with "red" colored values in the links between two nodes in Figure 3.1.

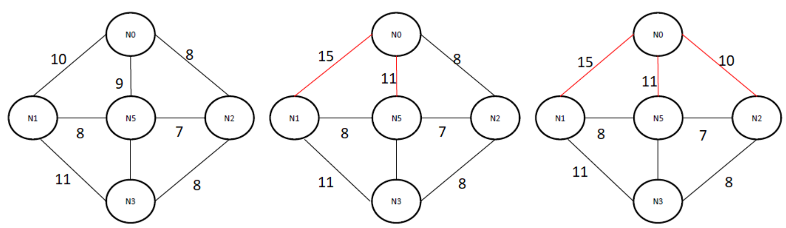

Figure 3.1: Network configurations with the effect of disruptions

Consider the facility in supply chain as a system that is observed frequently at equidistant inspection epoch, and action is made based on the observation. The cost of each action is known and is a consequence of the action taken. This dynamic system is Semi-Markov Decision Process (SMDP) when the following assumptions of Markovian property are satisfied:

If action a is chosen at the decision epoch when the system is in the state i, the expected time and expected cost until the next decision epoch depends only on the present state i and the action a ([Tijms, 2003]). We further assume that the established facility is incapacitated and can satisfy all the demands of the network. Therefore, in this study, we consider that the facility can gradually deteriorate over time, and degradation of the facility can be only observed at equidistant inspection epochs denoted by $\delta, 1\delta, 2\delta, \ldots, n\delta$. The degradation process of the facility can be modeled as a continuous-time homogeneous hidden-Markov chain $\{X_t : t \geq 0\}$ with state-space $S = \{\ 0, 1, 2, ..., N\}$ such that state 0 represents unobservable healthy state, states $\{\ 1, 2, ..., N-1\}$ represent unobservable partially disrupted states, and finally state N represents an absorbing failure (fully disrupted) state. We assume that the facility starts in the healthy state at the time 0, i.e., $X_0 = 0$. The instantaneous transition rates for the degradation process are given by:

$$\begin{aligned} \lambda_{ij} &= \lim_{h \to o} \frac{P(X_h = j \mid X_0 = i)}{h} < +\infty, \qquad i \neq j \\ \lambda_{ii} &= -\sum_{i \neq j} \lambda_{ij}. \end{aligned} \tag{1}$$

We further assume that the state-space of a facility is $S = \{0, 1, 2, 3, 4\}$, such that state 0 represent the healthy state, state 4 is an absorbing failure state and the intermediate states, i.e., $\{1, 2, 3\}$ represent the increasing degree of deterioration of a facility. Therefore, the transition rate matrix can be written as follows:

$$\Lambda = \begin{bmatrix} -(\lambda_{01} + \lambda_{02} + \lambda_{03} + \lambda_{04}) & \lambda_{01} & \lambda_{02} & \lambda_{03} & \lambda_{04} \\ 0 & -(\lambda_{12} + \lambda_{13} + \lambda_{14}) & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & 0 & -(\lambda_{23} + \lambda_{24}) & \lambda_{23} & \lambda_{24} \\ 0 & 0 & 0 & -\lambda_{34} & \lambda_{34} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where λ_{ij} are the instantaneous state transition rates of the Markov process. We assume that the degradation process is non-decreasing with probability 1, i.e., $\lambda_{ij} = 0$ for all $j < i$ and the failure state (state N) is absorbing. The facility can make transitions among its states such that the transition probability matrix, $P = [P_{ij}(t)]$ is obtained by explicitly solving the Kolmogorov-backward differential equations which results in the following transition matrix ([Tijms, 2003]):

$$P_{ij} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} \\ 0 & P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned}
P_{00} &= e^{-v_0 t}, \\
P_{01} &= \frac{v_0}{v_1 - v_0}(e^{-v_0 t} - e^{-v_1 t}), \\
P_{02} &= \frac{v_0 v_1}{(v_0 - v_2)(v_0 - v_1)} e^{-v_0 t} + \frac{v_0 v_1}{(v_2 - v_1)(v_0 - v_1)} e^{-v_1 t} - \frac{v_0 v_1}{(v_2 - v_1)(v_0 - v_2)} e^{-v_2 t}, \\
P_{03} &= \frac{v_0 v_1 v_2}{v_3 - v_2}\Big[\frac{v_2 - v_3}{(v_0 - v_1)(v_0 - v_2)(v_0 - v_3)} e^{-v_0 t} - \frac{v_2 - v_3}{(v_0 - v_1)(v_1 - v_2)(v_1 - v_3)} e^{-v_1 t} \\
&+ \frac{1}{(v_1 - v_2)(v_0 - v_2)} e^{-v_2 t} - \frac{1}{(v_1 - v_3)(v_0 - v_3)} e^{-v_3 t}\Big], \\
P_{04} &= \Big(1 + \frac{v_0^3 - v_0^2 v_1 - v_0^2 v_2 - v_3^2 v_2 + v_0 v_1 v_2 + v_0 v_1 v_3 + v_0 v_2 v_3}{(v_0 - v_1)(v_1 - v_2)(v_1 - v_3)}\Big) \times e^{-v_0 t} \\
&+ \frac{v_0 v_2 v_3}{(v_1 - v_0)(v_1 - v_2)(v_1 - v_3)} \times e^{-v_1 t} - \frac{v_0 v_1 v_3}{(v_0 - v_2)(v_1 - v_2)(v_2 - v_3)} \times e^{-v_2 t} \\
&+ \frac{v_0 v_1 v_2}{(v_0 - v_3)(v_2 - v_3)(v_1 - v_3)} \times e^{-v_3 t} + 1, \\
P_{11} &= e^{-v_1 t}, \\
P_{12} &= \frac{v_1}{v_2 - v_1}(e^{-v_1 t} - e^{-v_2 t}), \\
P_{13} &= \frac{v_1 v_2}{(v_1 - v_3)(v_1 - v_2)} e^{-v_1 t} + \frac{v_1 v_2}{(v_1 - v_2)(v_3 - v_2)} e^{-v_2 t} - \frac{v_1 v_2}{(v_3 - v_2)(v_1 - v_3)} e^{-v_3 t}, \\
P_{14} &= \frac{-v_2 v_3}{(v_1 - v_2)(v_1 - v_3)} e^{-v_1 t} + \frac{v_1 v_3}{(v_1 - v_2)(v_2 - v_3)} e^{-v_2 t} - \frac{v_1 v_2}{(v_2 - v_3)(v_1 - v_3)} e^{-v_3 t} + 1, \\
P_{22} &= e^{-v_2 t}, \\
P_{23} &= \frac{v_2}{v_3 - v_2}(e^{-v_2 t} - e^{-v_3 t}), \\
P_{24} &= 1 + \frac{v_3}{v_2 - v_3} e^{-v_2 t} - \frac{v_2}{v_2 - v_3} e^{-v_3 t}, \\
P_{33} &= e^{-v_3 t}, \\
P_{34} &= 1 - e^{-v_3 t}.
\end{aligned}$$

Please refer to Appendix 1 for detailed derivations of transition probabilities.

Let $\xi = \inf\{t \in R^+ : X_t = N\}$ be the observable failure time of a facility. At each inspection epoch the following three scenarios can happen:

- Upon facility inspection, the facility is found to be in the healthy state. Therefore, no action is required, and the facility continues its operation without any intervention.
- If the deterioration level of the facility reaches or exceeds the preventive fortification level denoted by N^*, partial fortification action is performed to bring back the facility to the healthy condition.
- If the deterioration level of the facility reaches or exceeds the failure state N, the facility is fully fortified such that the facility is renewed.

Figure 3.3 represents the state-space of the proposed model along with the actions.

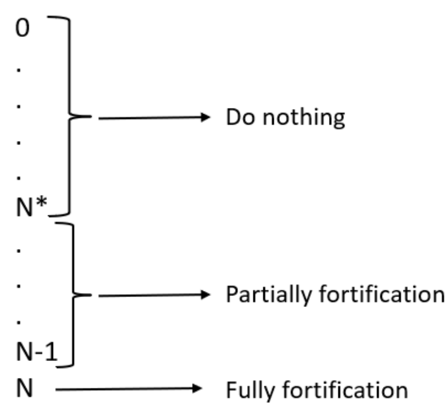

Figure 3.2: Illustration of state-space model

We will show that there exists N^* such that the long-run expected average cost of the transportation network is minimized. In this regard, the following cost components are considered in this research:

- C_I: Inspection cost incurred at each inspection time.
- C_P: Cost of partially fortification of a facility, which takes T_P time units.

- C_F: Cost of fully fortification of a completely disrupted facility, which takes T_F time units.

- C_T^1: Transportation cost when the facility operates in the healthy state.

- C_T^2: Transportation cost when the facility operates in degradation states.

- C_L: Cost rate of lost every time we stop the facility.

It is assumed that $C_F > C_P$, which is a realistic assumption. Note that, in situations where the cost of performing the partial fortification action is greater than the cost of failure, the optimal action is always performing corrective action only upon failure. In order to find the optimal value of N^* to minimize the long-run expected average cost per unit time, the problem is formulated in the SMDP framework, which is discussed in details in the following section.

3.2 Computational Algorithm in SMDP

In this section, we develop a computational algorithm in the SMDP framework. We start monitoring the facility at time epochs δ. At each inspection epoch, the facility can be in state $X_{n\delta} \in \{0, 1, 2, \ldots, N\}$. The SMDP is determined by the following quantities ([Tijms, 2003]):

(1) $P_{r,k}(\eta)$: Probability that the facility will be in state $k \in \zeta$ at the next decision epoch given the current state is $r \in \zeta$.

(2) $\tau_r(\eta)$: The expected sojourn time until the next decision epoch given the current state is $r \in \zeta$.

(3) $C_r(\eta)$: The expected cost incurred until the next decision epoch given the current state is $r \in \zeta$.

Using quantities defined above, for a fixed set of inspection interval, i.e., δ, optimal preventive fortification level N^*, minimizing the long-run expected average cost $g(\eta)$ can be obtained by solving the following system of linear equations:

$$\begin{aligned} u_r &= C_r(\eta) - g(\eta)\tau_r(\eta) + \sum_{k\in\zeta} P_{r,k}(\eta)u_k, \quad \text{for} \quad r \in \zeta \\ u_0 &= 0. \end{aligned} \tag{3}$$

Next, computation of the transition probabilities will be discussed.

3.2.1 Computation of the Transition Probabilities

The SMDP transition probabilities for the states defined above are calculated as follows:

- Assume that the facility's state is $X_{n\delta} = x$ at time $n\delta$ where $X_{n\delta} < N^*$ and in the next inspection epoch, the facility degradation doses not exceed either the preventive fortification level, i.e., $X_{(n+1)\delta} < N^*$ and failure level, then the transition probability is given by:

$$\begin{aligned} P_{x,x'} &= P\Big(X_{(n+1)\delta} = x', \xi > (n+1)\delta \mid X_{n\delta} = x, \xi > n\delta\Big) \\ &= P\Big(X_{(n+1)\delta} = x' \mid \xi > (n+1)\delta, X_{n\delta} = x, \xi > n\delta\Big) \\ &\times\ P\Big(\xi > (n+1)\delta \mid X_{n\delta} = x, \xi > n\delta\Big) \\ &= P_{x,x'}(\delta) \times R(\delta \mid x). \end{aligned} \tag{4}$$

- If the level of degradation exceeds N^* and the facility does not fail, the facility makes transition to any states between N^* and N. In this scenario, the facility enters the partially fortification state denoted by PF such that partially fortification action will be performed. Therefore, the transition probability to the PF state is given by:

$$P_{x,PF} = \sum_{x'=N^*}^{N-1} P\Big(X_{(n+1)\delta} = x' \mid X_{n\delta} = x, \xi > (n+1)\delta\Big) \times R(\delta \mid x). \tag{5}$$

- When the facility is in the PF state, then partially fortification action is performed, which brings the facility back to the healthy state and the transition probability from state PF to state 0 is given by:

$$P_{PF,0} = 1. \tag{6}$$

- Finally, when the failure happens, the mandatory corrective action, i.e., full fortification is performed, which brings the facility to the healthy state 0. Therefore, the remaining transition probabilities are as follows:

$$\begin{aligned} P_{x,F} &= 1 - R(\delta \mid x), \\ P_{F,0} &= 1. \end{aligned} \tag{7}$$

Note that the conditional reliability defined in Eqs. 27-30 can be calculated as follows:

$$R(\delta \mid X_{n\delta} = i) = 1 - P_{i,N}(\delta). \tag{8}$$

This completes the calculations of the transition probabilities. Next, the expected average cost will be calculated.

3.2.2 Computation of the Expected Average Cost

In this sub-section, the expected cost incurred in each state will be developed based on the following cases:

- The expected cost incurred until the next inspection time for state x, where $x < N^*$ when there is no preventive or full fortification action at current inspection time, and the facility will not fail in the next inspection time is given by:

$$\begin{aligned} C_x = E(Cost \mid x) &= E(Cost \mid X_{n\delta} = x, \xi > n\delta) \times P(\xi > n\delta \mid x) \\ &= \Big[C_I + E(\text{Transportation Cost})\Big] \times R(\delta \mid x), \end{aligned} \tag{9}$$

where the expected transportation cost can be calculated as follows:

$$E(\text{Transportation Cost}) = C_T^1 \times \sum_{x'=0}^{N^*} P_{xx'} + C_T^2 \times \sum_{x'=N*}^{N^*-1} P_{xx'}. \tag{10}$$

- The expected cost incurred until the next inspection time for state x, such that $N^* \leq x < N$, when PF action is performed is given by:

$$C_{PF} = E(Cost \mid PF) = C_I + C_P + C_L.T_P. \tag{11}$$

- The expected cost incurred until the next inspection time for failure state is given by:

$$C_F = E(Cost \mid F) = C_F + C_L.T_F. \tag{12}$$

3.2.3 Computation of the Expected Sojourn Time

- The expected sojourn time in state x, such that $x < N^*$ is as follows:

$$\tau_x = \int_0^{\delta} R(t \mid x)dt. \tag{13}$$

- The expected sojourn time for the states above N^* is given by:

$$\tau_x = T_{PF}. \tag{14}$$

Finally, the expected sojourn time for the failure state N when the full fortification action is performed is as follows:

$$\tau_r = T_F. \tag{15}$$

3.2.4 Computation of the Reliability Function

In this sub-section, we derive an explicit formula for the reliability function.

Lemma 3.2.0.1. For any $t \in R+$, the reliability function is given by:

$$R(t) = \sum_{i=0}^{N-1} P_{0,i}^{(n)}(\delta)\Big(1 - P_{i,N}(t)\Big). \tag{16}$$

Proof. By considering the fact that the facility starts at the healthy state at time 0, i.e., $P(X_0 = 0) = 1$ and by conditioning, the reliability function can be obtained as follows:

$$\begin{aligned}
R(t) &= P(\xi > n\delta + t \mid \xi > n\delta, X_{n\delta} = x) \times P(X_{nd} = x) \\
&= P(X_{n\delta+t} \neq N \mid \xi > n\delta, X_{n\delta} = x) \times P(X_{nd} = x) \\
&= P(X_{n\delta+t} = N-1 \mid \xi > n\delta, X_{n\delta} = 0) \times P(X_{n\delta} = 0) \\
&+ P(X_{n\delta+t} = N-2 \mid \xi > n\delta, X_{n\delta} = 0) \times P(X_{n\delta} = 0) + ... \\
&+ P(X_{n\delta+t} = 0 \mid \xi > n\delta, X_{n\delta} = 0) \times P(X_{n\delta} = 0) \\
&+ P(X_{n\delta+t} = N-1 \mid \xi > n\delta, X_{n\delta} = 1) \times P(X_{n\delta} = 1) \\
&+ P(X_{n\delta+t} = N-2 \mid \xi > n\delta, X_{n\delta} = 1) \times P(X_{n\delta} = 1) + ... \\
&+ P(X_{n\delta+t} = 0 \mid \xi > n\delta, X_{n\delta} = 1) \times P(X_{n\delta} = 1) \\
&+ ... + P(X_{n\delta+t} = 0 \mid \xi > n\delta, X_{n\delta} = N-1) \times P(X_{n\delta} = N-1) \\
&= P(X_{n\delta} = 0) \times P_{0,N-1}(t) + P(X_{n\delta} = 0) \times P_{0,N-2}(t) + ... + P(X_{n\delta} = 0) \times P_{0,0}(t) \\
&+ P(X_{n\delta} = 1) \times P_{1,N-1}(t) + P(X_{n\delta} = 1) \times P_{1,N-2}(t) + ... + P(X_{n\delta} = 1) \times P_{1,0}(t) + ... \\
&= \underbrace{P(X_{n\delta} = 0)}_{\text{Term I}} \times \underbrace{\Big[P_{0,N-1}(t) + P_{0,N-2}(t) + P_{0,N-3}(t) + ... + P_{0,0}(t)\Big]}_{\text{Term II}} \\
&+ \underbrace{P(X_{n\delta} = 1)}_{\text{Term III}} \times \underbrace{\Big[P_{1,N-1}(t) + P_{1,N-2}(t) + P_{1,N-3}(t) + ... + P_{1,0}(t)\Big]}_{\text{Term IV}} \\
&+ ... \\
&+ \underbrace{P(X_{n\delta} = N-1)}_{\text{Term N}} \times \underbrace{\Big[P_{N-1,N-1}(t) + P_{N-1,N-2}(t) + P_{N-1,N-3}(t) + ... + P_{N-1,0}(t)\Big]}_{\text{Term N+1}}.
\end{aligned} \tag{17}$$

By further conditioning, Term I can be calculated as follows:

$$\text{Term I} = P(X_{n\delta} = 0 \mid X_0 = 0) \times \underbrace{P(X_0 = 0)}_{=1} = P_{0,0}^{(n)}. \tag{18}$$

The remaining terms, i.e., Term III till Term N, are calculated similarly, which is omitted here to save on space. By knowing the fact that summation of the probabilities in each row of the transition probability matrix is equal to one, Term II can be simplified as follows:

$$\text{Term II} = 1 - P_{0,N}. \tag{19}$$

A similar calculation can be performed for the remaining terms, i.e., Term IV till Term N+1. Therefore, Eq. (17) can be summarized as follows:

$$R(t) = \sum_{i=0}^{N-1} P_{0,i}^{(n)} \Big(1 - P_{i,N}(t)\Big). \tag{20}$$

which completes the proof. □

3.3 Numerical Example

In this section, we illustrate the proposed computational procedure with a numerical example. For the scope of this chapter, the network is reduced to 5 nodes in a complete graph network with known distances. A complete graph is a graph that all the nodes in the network are connected to each other, so we do not need to pass another node to reach the destination city. The facility is located at node five which is serving other nodes/customers at a known cost. The disruptions will result in an increment in travel time between the facility and other nodes, which implies higher transportation costs or totally losing the link between the facility and other nodes.

The problem of finding optimal control policy is formulated as a continuous-time homogeneous hidden-Markov chain represented by $\{X_t : t \geq 0\}$. In this framework, a facility's condition is in one of the finite numbers of states $S = \{\ 0, 1, 2, 3, 4\}$. As mentioned before, the degradation can observe in links, so state 0 represent a healthy facility with no disruption, states 1,2,3 represent a facility with some disruptions; and finally state 4 is the failure state such that all the link connected to the facility are fully disrupted. Since the network is a complete graph as long as a single link is connected to the facility, the facility still has access to other nodes through that single link. The moment that all the links are disrupted or lost, the total failure will occur.

Only failure is observable and absorbing which means after entering this state, the facility cannot exit the state without proper action. The facility will be inspected at equidistant time epoch. In this work, we assume the fixed inspection time as $\delta = 10$. Proper action will be taken based on the current state of the facility. The cost and time associated with these actions are presented along with other inputs parameters in Table 3.1. Note that for simplicity sake, we assume that transportation costs are fixed and will not change over time. However, in order to show the effect of disruption on the network, we consider two different transportation costs denoted by C_T^1 and C_T^2, such that the C_T^1 represent the transportation cost when the facility operates in the healthy state and C_T^2 represent the transportation cost when the facility operates in the warning states.

Furthermore, the instantaneous transition rates matrix for the degradation process is assumed as follow:

$$q_{ij} = \begin{bmatrix} -0.9 & 0.9 & 0 & 0 & 0 \\ 0 & -4.76 & 4.76 & 0 & 0 \\ 0 & 0 & -4.35 & 4.35 & 0 \\ 0 & 0 & 0 & -1.24 & 1.24 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^{-2}.$$

Using Kolmogoroff-backward equations, the transition probability matrix is driven as follows:

$$P_{ij} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} \\ 0 & P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned}
P_{00} &= e^{-0.9\times10^{-2}t}, \\
P_{01} &= 0.23e^{-0.9\times10^{-2}t} - 0.23e^{-4.76\times10^{-2}t}, \\
P_{02} &= 0.32e^{-0.9\times10^{-2}t} + 2.7e^{-4.76\times10^{-2}t} - 3.03e^{-4.35\times10^{-2}t}, \\
P_{03} &= 4.12e^{-0.9\times10^{-2}t} - 3.34e^{-4.76\times10^{-2}t} + 4.24e^{-4.35\times10^{-2}t} - 5e^{-1.24\times10^{-2}t}, \\
P_{04} &= -5.67e^{-0.9\times10^{-2}t} + 0.87e^{-4.76\times10^{-2}t} - 1.21e^{-4.35\times10^{-2}t} + 5e^{-1.24\times10^{-2}t} + 1, \\
P_{11} &= e^{-4.76t}, \\
P_{12} &= -11.61e^{-4.76\times10^{-2}t} + 11.61e^{-4.35\times10^{-2}t}, \\
P_{13} &= 14.35e^{-4.76\times10^{-2}t} - 16.24e^{-4.35\times10^{-2}t} + 1.89e^{-1.24\times10^{-2}t}, \\
P_{14} &= -3.74e^{-4.76\times10^{-2}t} + 4.63e^{-4.35\times10^{-2}t} - 1.89e^{-1.24\times10^{-2}t} + 1, \\
P_{22} &= e^{-4.35\times10^{-2}t}, \\
P_{23} &= -1.4e^{-4.35\times10^{-2}t} + 1.4e^{-1.24\times10^{-2}t}, \\
P_{24} &= 0.4e^{-4.35\times10^{-2}t} - 1.4e^{-1.24\times10^{-2}t} + 1, \\
P_{33} &= e^{-1.24\times10^{-2}t}, \\
P_{34} &= 1 - e^{-1.24\times10^{-2}t}.
\end{aligned}$$

By solving iteratively the system of linear equations, we compute the optimal preventive fortification level N^*, minimizing the long-run expected average cost per unit time. The

Table 3.1: The input parameters.

Input Parameters	Values	Input Parameters	Values
δ	10	C_T^2	100
C_I	50	C_L	30
C_P	150	T_P	5
C_F	1000	T_F	10
C_T^1	70	$t = n\delta$	$10n$

results are presented as follows:

$$N^* = 3; \quad g - value = 14.0519. \tag{21}$$

The optimal policy is to preventively fortify the facility when its degradation level reaches or exceeds $N^* = 3$ which gives the minimum cost of $g(N^*) = 14.0519$ per unit time. We also plot the reliability function over time, i.e., $R(t)$, as shown in Figure 3.3. As the degradation of a facility is increasing over time, the reliability is decreasing.

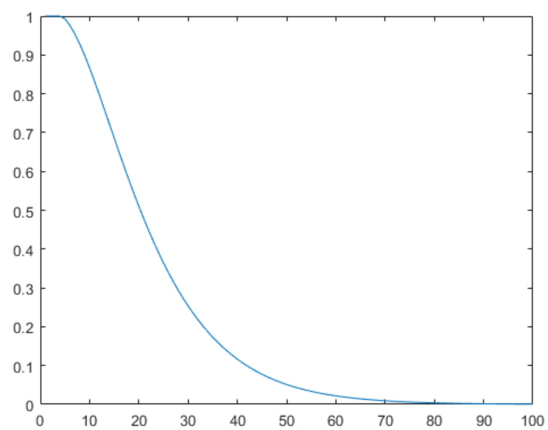

Figure 3.3: The reliability function of the proposed model

3.3.1 Misspecification Analysis

For this subsection, we examine the effect of misspecification of input parameters on the long-run expected average cost. Process and cost parameters are difficult to estimate accurately, therefore it can lead to sub-optimal results. In this regard, we examine the effect of misspecfication of input parameters on the cost to see if the proposed model is robust to changes in input parameters. For this purpose, for each input parameters, we set two hypothetical values that represent scenarios of falling behind estimations, known as overestimation, and scenarios of exceeding estimations, known as underestimation. Over estimations have their actual values increased by 20% while underestimations have their actual values decreased by 20%.

Table 3.2 represents the results obtained from the sensitivity analysis. The costs are displayed according to each parameter with 20% estimation error. From the results, it can be concluded that the difference of the overestimation and underestimation values with the optimum value obtained previously is low, meaning the proposed model is robust in case that the input parameters are estimated inaccurately.

Parameters	Estimation error	cost penalty
C_T^1	+	0.09
	-	0.09
C_T^2	+	0.01
	-	0.01
C_I	+	0.07
	-	0.07
C_P	+	0.01
	-	0.01
C_L	+	0.01
	-	0.01
C_F	+	0.01
	-	0.01
T_P	+	0.01
	-	0.01
T_F	+	0.00
	-	0.00
δ	+	0.14
	-	0.20

Table 3.2: Effect of misspecification on the long-run expected average cost with 20% error

We have extended the analysis to the scenario when 35% error is considered rather than 20%. The results are shown in Table 3.3. As it can be seen form the results, the effect of misspecification is almost the same for all process parameters except for value of δ. Therefore, the value of δ has the most significant effect on the long-run expected average cost. This makes sense as the change does not only influence the monitoring cost, but also influences the detection of failure, which could lead to significant outcomes and effects.

Parameters	Estimation error	cost penalty
C_T^1	+	0.16
	-	0.16
C_T^2	+	0.01
	-	0.01
C_I	+	0.13
	-	0.13
C_P	+	0.02
	-	0.02
C_L	+	0.02
	-	0.02
C_F	+	0.01
	-	0.01
T_P	+	0.01
	-	0.01
T_F	+	0.00
	-	0.00
δ	+	0.21
	-	0.44

Table 3.3: Effect of misspecification on total long-run expected average cost 35% difference

3.4 Conclusion

In this chapter, we have developed the optimal control policy for a facility subject to stochastic degradation due to the occurrence of random disruption with the objective of minimizing the long-run expected average cost per unit time. The state-space of the degradation process is modeled based on continuous-time hidden Markov chain with finite number of states. The facility is inspected at the equidistant time and the level of degradation of the facility is observed. If the level of degradation exceeds level N^*, the partially fortification action is initiated. On the other hand, if the facility found to be in the observable failure state, mandatory fully fortification action will be performed to rebuild the facility. The proposed model is formulated in the SMDP framework and coded in MATLAB in order to obtain the optimal control policy which minimizes the total cost.

Chapter 4

Optimal Control of Multiple Facilitites in a SCN

In the previous chapter, an optimal control policy for a single facility with its distribution network was presented. In this chapter, we extend the previous developments in the maintenance and control framework to a SCN such that more that one facility is under disruption. Each facility is prone to degradation and disruption. More precisely, this chapter proposes an economical design of a Bayesian control chart for monitoring the proportion of failed facilities in a SCN. The control chart parameters are defined based on the posterior probability of the system being in a warning state. The objective is to find an optimal control policy that minimizes the total expected average cost per unit time in the long-run. Similar to the previous chapter, the model is formulated in the SMDP framework, and the control chart parameters are obtained by minimizing the objective.

The organization of the chapter is as follows: Section 4.1 provides a brief review of the Bayesian control chart development. Section 4.2 provides a detailed description of the proposed Bayesian control chart. In section 4.3, we formulate the Bayesian control chart in the SMDP framework. Section 4.4 presents numerical examples to prove the effectiveness of the proposed model. Finally, Section 4.5 concludes the chapter.

4.1 Review of Bayesian Control Chart

In this section, we briefly review the multivariate Bayesian control chart that was originally designed and developed by [Calabrese, 1995]. Since after, the Bayesian control chart has been used widely in lots of practical applications domains which have been completely discussed in Chapter 2. In general, the Bayesian control chart tracks output process by plotting the posterior probability that the process is out of control. After each sample, the posterior probability is updated using the Bayes' Theorem. It has been shown that the posterior probability is a sufficient statistic to be used for decision-making in SPC and CBM domains.

In particular and in SPC domain, the process is defined to be either in in-control state or out-of-control state. The occurrence of an assignable cause can switch the process from an in-control state (state 0) to an out-of-control state (state 1). Based on the Bayesian control chart, it is assumed that the time difference between the occurrence of two consequent assignable causes is exponentially distributed with the mean $1/\lambda$. In attribute-type Bayesian control chart, the fraction defective is a quality characteristic of interest. The fraction defective is described as a function of the state of the process itself. When the process is in state 0, i.e., in-control state, the fraction defective is set to p_0. When the process shifts to state 1, i.e., out-of-control state, the fraction defective is set to p_1 where $p_1 > p_0$. When the process is in in-control state and out-of-control state, the number of defectives in a sample is considered to follow Binomial distribution with mean np_0 and np_1, respectively. Then the posterior probability that the process is out-of-control at time t given the sample observations, i.e., number of defectives denoted by D up to time t is given as follows:

$$\Pi(t) = P(X_t = 1 \mid S_t), \tag{22}$$

where

$$S_t = \Big(\Pi(0), a_0, D_1, a_1, ..., D_{t-1}, a_{t-1}, D_t\Big), \tag{23}$$

such a_t represents an action taken at time t.

The objective is to find optimal control policy that minimizes the total cost. Based on the methodology that presented by [Calabrese, 1995], under standard assumptions, it can be concluded that a control limit policy is optimal. Although the author only considered the application of Bayesian control charts for SPC problems considering two states, we have extended the attribute-type Bayesian control chart in CBM domain for monitoring a SCN consists of multiple facilities such that each facility is subject to degradation. This completes our review on attribute-type Bayesian control chart and in the next section, a detailed description of proposed model will be discussed.

4.2 Model Description

In this section, the extended model of attribute-type Bayesian control chart is presented. As an extension of the previous chapter, a SCN with N facilities is considered, where each facility is prone to degradation and failure. The state of the whole network is defined based on the performance of the facilities within the network. While failure of one facility in a big SCN can be negligible as other facilities can be used as backups, multiple facility failures reduce the efficiency of the whole network which lead to higher service cost, dissatisfaction of customers and unmet demands. Fig 4.1 illustrates the explained degradation in a SCN.

In order to model the problem at hand, we assume that the network starts in a healthy state such that all the facilities are operating in healthy condition and all the links are in as-good-as-new condition. During the time, due to the occurrence of stochastic disruptions and degradation, facilities lose their links, and gradually the total failure of a facility will occur. A failure of a facility can have a direct effect on the failure of the remaining facilities, i.e., cascading failure can happen in the network. In particular, a cascading failure is a process in a system that consists of interconnected parts in which the failure of one or few parts can trigger the failure of other parts.

The network's degradation is modeled by a continuous-time homogeneous hidden-Markov chain $\{X_t : t \geq 0\}$ with state-space $\Omega = \{\ 0, 1, 2\}$ which represents healthy, warning and

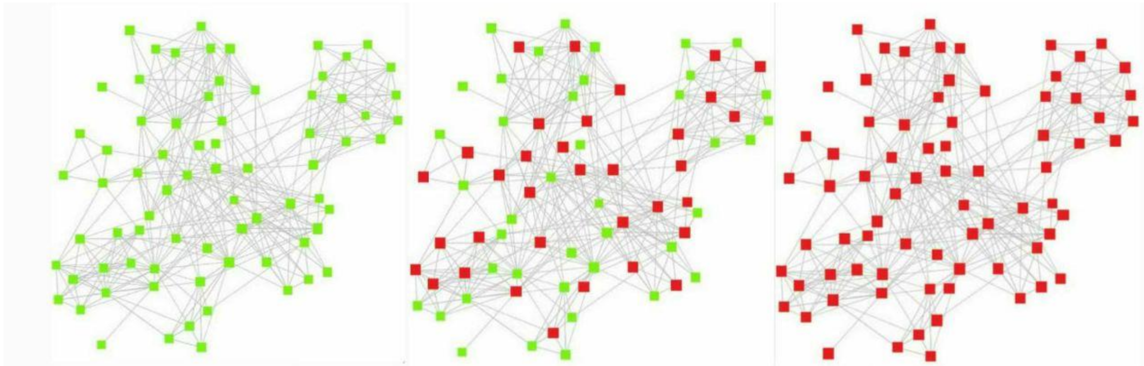

Figure 4.1: An illustration of three state multi facility network

failure states, respectively. It is worth mentioning that describing the system by only two operational states has been proved to be practical and sufficient for most applications ([Tian et al., 2017]). In particular, when the network is in state 0, it is assumed that all of the facilities or most of them are operation in healthy states. The state 1 represents the scenario where certain number of facilities are not working properly, however, the whole network is still working and operational. Finally, the last state represents the scenario such that most of the facilities are failed which result in failure of whole network. The first two states are unsolvable and can be known after the inspection, while the failure state is observable and absorbing.

The instantaneous transition rates for the above-mentioned Markov process of the network are given by:

$$\begin{aligned} \lambda_{ij} &= \lim_{h \to o} \frac{P(X_h = j \mid X_0 = i)}{h} < +\infty, \qquad i \neq j \\ \lambda_{ii} &= -\sum_{i \neq j} \lambda_{ij}. \end{aligned} \tag{24}$$

where λ_{ij} are the instantaneous state transition rates of the Markov process $(i, j \in \{0, 1, 2\})$. We assume that the degradation process is non-decreasing with probability 1, i.e., $\lambda_{ij} = 0$ for all $j < i$ and the failure state is absorbing. Also, it is assumed at time 0, i.e., $X_0 = 0$; the system starts from a healthy state. The system makes transitions among its states, and its transition probability matrix is calculated by explicitly solving the Kolmogorov-backward differential equations ([Tijms, 2003]).

$$P = \begin{bmatrix} e^{-v_0 t} & \frac{\lambda_{01}(e^{-v_1 t} - e^{-v_0 t})}{v_0 - v_1} & 1 - e^{-v_0 t} - \frac{\lambda_{01}(e^{-v_1 t} - e^{-v_0 t})}{v_0 - v_1} \\ 0 & e^{-v_1 t} & 1 - e^{-v_1 t} \\ 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where $v_0 = \lambda_{01} + \lambda_{02}$ and $v_1 = \lambda_{12}$.

At each monitoring epoch, the number of failed facilities in the network denoted by d will be observed. When the network is in the healthy state, the proportion of failed nodes (facilities) is denoted by p_0, and when the network is in the warning state, the proportion of failed nodes is represented by p_1 such that $p_0 < p_1$.

The posterior probability which denotes the probability that the network is in the warning state at the time $t = n\delta$ given the observations up to time t as follows:

$$\Pi_{n\delta} = P\big(X_{n\delta} = 1 \mid \xi > n\delta, D_1, D_2, ..., D_{(n-1)\delta}, D_{n\delta}\big), \tag{26}$$

where D is number of defective (failed) facilities observed

Let $\xi = \inf\{t \in R^+ : X_t = 2\}$ be the observable failure time of the network. Monitoring the system happens at the equidistant time denoted by $n\delta$. At each monitoring epoch, the following three scenarios can happen:

- If the posterior probability is less than the control limit denoted by M, no action is required, and the network continues to operate without any intervention. Next, monitoring will happen in δ time unit.
- If the posterior probability exceeds the control threshold M, the network will be stopped for full inspection followed possibly by preventive maintenance.
- Upon observable network failure, corrective action is performed so that the network is renewed.

It has been shown by [Calabrese, 1995] that the posterior probability that the system is in the warning state is a sufficient statistic for decision making at time t. The posterior probability defined in Eq. (26) can be updated as follows:

$$\begin{aligned}
\Pi_{n\delta} &= P(X_{n\delta} = 1 \mid \xi > n\delta, D_1, D_2, ..., D_{n\delta}) \\
&= \frac{f(D_{n\delta} \mid X_{n\delta} = 1) \times P(X_{n\delta} = 1)}{f(D_{n\delta} \mid X_{n\delta} = 1) \times P(X_{n\delta} = 1) + f(D_{n\delta} \mid X_{n\delta} = 0) \times P(X_{n\delta} = 0)}
\end{aligned} \tag{27}$$

Given that the state is equal to i, where $i = \{1, 2\}$, the conditional density of the observation is calculated based on the Binomial distribution as follows:

$$f(D_{n\delta} \mid i) = \binom{N}{D} P_i^D (1 - P_i)^{N-D}, \tag{28}$$

where D is number of defective facilities in a SCN and N is total number of facilities. The remaining terms in Eq. 27 are calculated as follows:

$$\begin{aligned}
P(X_{n\delta} = 1) &= P(X_{n\delta} = 1 \mid X_{(n-1)\delta} = 0) \times P(X_{(n-1)\delta} = 0) \\
&+ P(X_{n\delta} = 1 \mid X_{(n-1)\delta} = 1) \times P(X_{(n-1)\delta} = 1) \\
&= P_{01}(\delta) \times (1 - \Pi_{(n-1)\delta}) + P_{11}(\delta) \times \Pi_{(n-1)\delta} \\
&= P_{01}(\delta) \times (1 - \pi) + P_{11}(\delta) \times \pi
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
P(X_{n\delta} = 0) &= P(X_{n\delta} = 0 \mid X_{(n-1)\delta} = 0) \times P(X_{(n-1)\delta} = 0) \\
&= P_{00}(\delta) \times (1 - \pi).
\end{aligned} \tag{30}$$

Note that to simplify the notations, we use $\Pi_{(n-1)\delta} = \pi$. Therefore, Eq. 27 can be written as follows:

$$\Pi_{n\delta} = \frac{f(D_{n\delta} \mid 1)\big[P_{01}(\delta) \times (1 - \pi) + P_{11}(\delta) \times \pi\big]}{f(D_{n\delta} \mid 1)\big[P_{01}(\delta) \times (1 - \pi) + P_{11}(\delta) \times \pi\big] + f(D_{n\delta} \mid 0)\big[P_{00}(\delta) \times (1 - \pi)\big]} \tag{31}$$

It is assumed that the SCN starts in healthy state. The SCN is monitored periodically at

δ points time unit, and the posterior probability that the SCN is in the warning state is calculated based on Eq. 31. If the posterior probability exceeds the control threshold, the monitoring is stopped, and a full inspection is triggered. After this inspection, if the system is in the warning state, it is called the true alarm, and the preventive maintenance action (partial fortification) is initiated to bring back the SCN to the healthy state. Otherwise, the false alarm occurs, and the SCN continues functioning without any action. Finally, if the SCN fails, corrective maintenance (full fortification) will be perform which brings the it back to the healthy state. The objective is to find the optimal control chart parameters to minimize the long-run expected average cost. We consider the following cost structure:

- C_M is the fixed monitoring cost.
- C_I is the cost associated with the full inspection, which takes T_I time units.
- C_F is the cost of the corrective action, which takes T_F time units.
- C_{PM} is the cost of the preventive action, which takes T_{PM} time units.
- C_{T_0} is transportation cost per unit time when the SCN is in the healthy state.
- C_{T_1} is the transportation cost per unit time when the SCN is in the warning state.

Due to some number of facilities failure in the warning state, the transportation cost is higher than the healthy state where the customers are assigned to the closest facility ($C_{T_0} < C_{T_1}$). For the development of the computational algorithm in the SMDP framework that is presented in Section 4.3, the calculation of the conditional reliability function is required. which is described in next sub-section.

4.2.1 Calculation of Reliability and Mean Remaining Useful Life

We derive an explicit formula for the reliability function and mean remaining useful life (MRL), which are given by the following lemmas.

Lemma 4.2.0.1. For any $t \in R+$, the reliability function is given by:

$$R(t \mid \Pi_{n\delta}) = (1 - P_{02}(t))(1 - \Pi_{n\delta}) + (1 - P_{12}(t)) \times \Pi_{n\delta}. \tag{32}$$

Proof. By considering the fact that the system starts at the healthy state at time 0, i.e., $P(X_0 = 0) = 1$, and by conditioning, the reliability function can be obtained as follows:

$$
\begin{aligned}
R(t \mid \Pi_{n\delta}) &= P(\xi > n\delta + t \mid \xi > n\delta, D_1, D_2, ..., D_{n\delta}, \Pi_{n\delta}) \\
&= P(X_{n\delta+t} \neq 2 \mid \xi > n\delta, D_1, D_2, ..., D_{n\delta}, \Pi_{n\delta}) \\
&= P(X_{n\delta+t} \neq 2 \mid X_{n\delta} = 0, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 0 \mid D, \Pi_{n\delta}) \\
&+ P(X_{n\delta+t} \neq 2 \mid X_{n\delta} = 1, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 1 \mid D, \Pi_{n\delta}) \\
&= P(X_{n\delta+t} = 0 \mid X_{n\delta} = 0, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 0 \mid D, \Pi_{n\delta}) \\
&+ P(X_{n\delta+t} = 1 \mid X_{n\delta} = 0, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 0 \mid D, \Pi_{n\delta}) \\
&+ P(X_{n\delta+t} = 0 \mid X_{n\delta} = 1, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 1 \mid D, \Pi_{n\delta}) \\
&+ P(X_{n\delta+t} = 1 \mid X_{n\delta} = 1, \xi > n\delta, D, \Pi_{n\delta}) \times P(X_{n\delta} = 1 \mid D, \Pi_{n\delta}) \\
&= P(X_{n\delta} = 0 \mid D, \Pi_{n\delta}) \times \Big[P_{00}(t) + P_{01}(t)\Big] \\
&+ P(X_{n\delta} = 1 \mid D, \Pi_{n\delta}) \times \Big[P_{10}(t) + P_{11}(t)\Big] \\
&= (1 - P_{02}(t))(1 - \Pi_{n\delta}) + (1 - P_{12}(t)) \times \Pi_{n\delta} \qquad (33)
\end{aligned}
$$

□

which completes the proof.

We also derive the explicit formula for the MRL function in terms of the posterior probability statistic. In particular, for any $t \in R+$, MRL is given by the following lemmas.

Lemma 4.2.0.2. For any $t \in R+$, the mean residual life is given by:

$$MRL_{n\delta} = \frac{\Pi_{n\delta}(\lambda_{02} - \lambda_{12}) + \lambda_{01} + \lambda_{12}}{\lambda_{12}(\lambda_{01} + \lambda_{02})} \qquad (34)$$

Proof.

$$\begin{aligned} MRL_{n\delta} &= E\{\xi - n\delta \mid \xi > n\delta, D_1, D_2, ..., D_{n\delta}, \Pi_{n\delta}\} \\ &= \int_0^\infty R(t \mid \Pi_{n\delta})dt \\ &= \int_0^\infty \Big((1 - P_{02}(t))(1 - \Pi_{n\delta}) + (1 - P_{12}(t)) \times \Pi_{n\delta}\Big)dt \\ &= \int_0^\infty \Big[(1 - \Pi_{n\delta}) \times \Big[e^{-v_0 t} + \frac{\lambda_{01}}{v_0 - v_1}\big(e^{-v_1 t} - e^{-v_0 t}\big)\Big] \\ &+ \Pi_{n\delta} e^{-v_1 t}\Big]dt \\ &= \frac{\Pi_{n\delta}(\lambda_{02} - \lambda_{12}) + \lambda_{01} + \lambda_{12}}{\lambda_{12}(\lambda_{01} + \lambda_{02})} \end{aligned} \tag{35}$$

□

In the next section, we formulate the proposed attribute control chart problem in the SMDP framework.

4.3 Computational Algorithm in the SMDP Framework

In this section, we develop an efficient computational algorithm for the proposed model in the SMDP framework. The aim is to find the optimal values of the control chart parameters that minimize the long-run expected average cost per unit time. For this purpose, the posterior probability interval $[0, 1]$ is partitioned into $K \in \mathcal{N}$ disjoint sub-intervals $I_k = [l_k, u_k)$, where $l_k = \frac{k-1}{K}$ and $u_k = \frac{k}{K}$. If the posterior probability surpasses the control limit, the SMDP enters the inspection state denoted by l. In the event of true alarm, SMDP enters the state PM, where preventive maintenance action must be performed to bring the system back to the healthy state, and if it is a false alarm, the system continues without any further action. Finally, if the system enters the failure state F, mandatory corrective maintenance is performed. The coded state space is defined as $\zeta = \{0, 1, ..., K\}$, where 0 represents the new or renewed system, state 1 represents the mid-point of the first interval, and state K represents the mid-point of the last interval, respectively. With the definition of the space state, the SMDP is determined by the following quantities ([Tijms, 2003]):

(1) $P_{r,k}(\eta)$: Probability that the facility will be in state $k \in \zeta$ at the next decision epoch given the current state is $r \in \zeta$.

(2) $\tau_r(\eta)$: The expected sojourn time until the next decision epoch given the current state is $r \in \zeta$.

(3) $C_r(\eta)$: The expected cost incurred until the next decision epoch given the current state is $r \in \zeta$.

Using quantities defined above, for a fixed maintenance limit M, the long-run expected average cost $g(\eta)$ is obtained by solving the set of linear equations defined as follows:

$$
\begin{aligned}
u_r &= C_r(\eta) - g(\eta)\tau_r(\eta) + \sum_{k\in\zeta} P_{r,k}(\eta)u_k, \quad \text{for} \quad r \in \zeta \\
u_0 &= 0.
\end{aligned}
\tag{36}
$$

Next, computation of the transition probabilities will be considered.

4.3.1 Transition Probability

The SMDP transition probabilities for the states defined above are calculated as follows:

- Assume that the posterior probability's state at time $n\delta$ is i, where $i < M$, and in the next sampling epoch the system deterioration does not exceed the maintenance level, i.e., $(n+1)\delta < \xi$ and $k < M$. Then, the transition probability for any i and k when $i, k < M$ is given by:

$$
\begin{aligned}
P_{i,k} &= P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K}, \xi > n\delta \mid \xi > (n-1)\delta, i\Big) \\
&= P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, i\Big) \times P\Big(\xi > n\delta \mid \xi > (n-1)\delta, i\Big) \\
&= P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, i\Big) \times R(\delta \mid i)
\end{aligned}
\tag{37}
$$

The first term on RHS of Eq. (37) is given as follows:

$$\begin{aligned}
P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, i\Big) &= \underbrace{P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, X_{n\delta}=0, i\Big)}_{\text{Term I}} \\
&\times \underbrace{P(X_{n\delta}=0 \mid i)}_{\text{Term II}} \\
&+ \underbrace{P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, X_{n\delta}=1, i\Big)}_{\text{Term III}} \\
&\times \underbrace{P(X_{n\delta}=1 \mid i)}_{\text{Term IV}} \qquad (38)
\end{aligned}$$

Where

$$\begin{aligned}
&\text{Term I} = P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, X_{n\delta}=0, i\Big) \\
&= P\Big(\frac{k-1}{K} \leq \frac{\binom{N}{D} P_1^D (1-P_1)^{N-D}\big[P_{01}(\delta)\times(1-\pi) + P_{11}(\delta)\times\pi\big]}{\binom{N}{D} P_1^D (1-P_1)^{N-D}\big[P_{01}(\delta)\times(1-\pi) + P_{11}(\delta)\times\pi\big] + \binom{N}{D} P_0^D (1-P_0)^{N-D}\big[P_{00}(\delta)\times(1-\pi)\big]} \\
&\qquad < \frac{k}{K} \mid i, X_{n\delta}=0\Big) \\
&= P\Big(\frac{k-1}{K} \leq \frac{P_{01}(\delta)\times(1-\pi) + P_{11}(\delta)\times\pi}{P_{01}(\delta)\times(1-\pi) + P_{11}(\delta)\times\pi + \big(\frac{p_0}{p_1}\big)^D \big(\frac{1-p_0}{1-p_1}\big)^{N-D}\big[P_{00}(\delta)\times(1-\pi)\big]} < \frac{k}{K} \mid i, X_{n\delta}=0\Big)
\end{aligned} \qquad (39)$$

To simplify notation, let $\beta = P_{01}(\delta)\times(1-\pi) + P_{11}(\delta)\times\pi$. Then, the term in Eq. (39) can be written as:

$$\text{Term I} = P\left[\frac{k-1}{K} < \frac{\beta}{\beta + \left[\frac{p_0(1-p_1)}{P_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \left[P_{00}(\delta)\times(1-\pi)\right]} \leq \frac{k}{K} | i, X_{n\delta}=0\right]$$

$$= P\left[\frac{K}{k-1} > \frac{\beta + \left[\frac{p_0(1-p_1)}{P_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \left[P_{00}(\delta)\times(1-\pi)\right]}{\beta} \geq \frac{K}{k} \ | i, X_{n\delta}=0\right]$$

$$= P\left[(\frac{K}{k-1})(\beta) > \beta + \left[\frac{p_0(1-p_1)}{P_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \left[P_{00}(\delta)\times(1-\pi)\right] \geq (\frac{K}{k})(\beta) \ | i, X_{n\delta}=0\right]$$

$$= P\left[(\frac{K}{k-1})(\beta) - \beta > \left[\frac{p_0(1-p_1)}{P_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \left[P_{00}(\delta)\times(1-\pi)\right] \geq (\frac{K}{k})(\beta) - \beta \ | i, X_{n\delta}=0\right]$$

$$= P\left[(\beta)(\frac{K}{k-1} - 1) > \left[\frac{p_0(1-p_1)}{P_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \left[P_{00}(\delta)\times(1-\pi)\right] \geq (\beta)(\frac{K}{k} - 1) \ | i, X_{n\delta}=0\right]$$

$$= P\left[(\frac{\beta}{P_{00}(\delta)\times(1-\pi)})(\frac{K-(k-1)}{k-1}) > \left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right]^D (\frac{1-p_0}{1-p_1})^N \geq (\frac{\beta}{P_{00}(\delta)\times(1-\pi)})(\frac{K-k}{k}) \ | i, X_{n\delta}=0\right]$$

$$= P\left[\frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-(k-1)}{k-1}\right)}{(\frac{1-p_0}{1-p_1})^N} > \underbrace{\left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right]^D}_{<1} \geq \frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-k}{k}\right)}{(\frac{1-p_0}{1-p_1})^N} \ | i, X_{n\delta}=0.\right]$$

We further simplify Term I by taking the logarithm of different terms within the probability notation as follows:

$$\text{Term I} = P\left[\frac{log\left(\frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-(k-1)}{k-1}\right)}{(\frac{1-p_0}{1-p_1})^N}\right)}{log\left(\frac{p_0(1-p_1)}{p_1(1-p_0)}\right)} < D \leq \frac{log\left(\frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-k}{k}\right)}{(\frac{1-p_0}{1-p_1})^N}\right)}{log\left(\frac{p_0(1-p_1)}{p_1(1-p_0)}\right)} \ | \ i, X_{m_1h_1}=0\right] \quad (40)$$

So, the first term of transition probability in Eq. (39) can be written as:

$$\text{Term I} = P\left(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \ | \ \xi > n\delta, X_{n\delta}=0, i\right)$$
$$= P\left[\underbrace{\frac{log\left(\frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-(k-1)}{k-1}\right)}{(\frac{1-p_0}{1-p_1})^N}\right)}{log\left(\frac{p_0(1-p_1)}{p_1(1-p_0)}\right)}}_{\varphi_1} < D \leq \underbrace{\frac{log\left(\frac{\left(\frac{\beta}{P_{00}(\delta)\times(1-\pi)}\right)\left(\frac{K-k}{k}\right)}{(\frac{1-p_0}{1-p_1})^N}\right)}{log\left(\frac{p_0(1-p_1)}{p_1(1-p_0)}\right)}}_{\varphi_2} \ | \ i, X_{m_1h_1}=0\right] \quad (41)$$

So:

$$\begin{aligned}
\text{Term I} &= P\Big(\frac{k-1}{K} \leq \Pi_{n\delta} < \frac{k}{K} \mid \xi > n\delta, X_{n\delta} = 0, i\Big) \\
&= P\Big(\varphi_1 \leq D < \varphi_2 \mid \xi > n\delta, X_{n\delta} = 0, i\Big) \\
&= P(D \leq \varphi_2 \mid \Pi = i, X_{n\delta} = 0) - P(D \leq \varphi_1 \mid \Pi = i, X_{n\delta} = 0) \\
&= \sum_{j=0}^{\varphi_2} \binom{N}{j} p_0^j (1-p_0)^{N-j} - \sum_{j=0}^{\varphi_1} \binom{N}{j} p_0^j (1-p_0)^{N-j}.
\end{aligned}$$

The same calculation is done for Term III. The second term is calculated as follows:

$$\text{Term II} = \frac{P_{00}(\delta) \times (1-\pi)}{P_{00}(\delta) \times (1-\pi) + P_{01}(\delta) \times (1-\pi) + P_{11}(\delta) \times \pi}. \tag{42}$$

Term IV is similarly calculated as follows:

$$\text{Term IV} = \frac{P_{01}(\delta) \times (1-\pi) + P_{11}(\delta) \times \pi}{P_{00}(\delta) \times (1-\pi) + P_{01}(\delta) \times (1-\pi) + P_{11}(\delta) \times \pi} \tag{43}$$

Remaining transition probabilities are calculated similarly which are presented as follows:

- When the posterior probability exceeds the maintenance limit M, the SCN enters state l such that full inspection is performed, and the transition probability is given by:

$$P_{i,l} = P\Big(\frac{l-1}{K} < \pi_{n\delta} \leq \frac{l}{K} \mid i\Big) \times R(\delta \mid i) \tag{44}$$

- When SCN make transition to state l and full inspection happens, two scenarios can occur:
  - Inspection indicates false alarm, i.e., SCN found to be in healthy state. In this case, the corresponding transition probability is given by:

$$P_{l,0} = 1 - \frac{l-0.5}{K} \tag{45}$$

- Inspection indicates true alarm, i.e., SCN found to be in warning state. In this case, the transit to PM state happens, which follows by a preventive maintenance action. The corresponding transition probability is given by:

$$P_{l,PM} = \frac{l-0.5}{K} \tag{46}$$

- When the failure occurs, then mandatory corrective maintenance is performed which brings the SCN to the healthy state with the following transition probability:

$$P_{F,0} = 1. \tag{47}$$

4.3.2 Computing the Expected Costs

In this sub-section, the expected cost incurred until the next inspection time will be developed base on each case:

- The expected cost incurred until the next monitoring time for stat i, where $i < M$, and the system will not fail in the next monitoring time is given by:

$$C_i = E(Cost \mid i) = [C_M + E(\text{Transportation Cost})] \times R(\delta \mid i), \tag{48}$$

where the expected transportation cost can be calculated as follows:

$$E(\text{Transportation Cost}) = \int_0^{\delta} C_{T_0} P(X_{n\delta} = 0 \mid i) + \int_0^{\delta} C_{T_1} P(X_{n\delta} = 1 \mid i). \tag{49}$$

The above probabilities are calculated similarly to Eq. (42) and Eq. (43).

- The expected cost incurred until the next monitoring epoch for state l, where $M < l$, and the full system inspection is performed, is given by:

$$C_l = E(Cost \mid l) = C_I. \tag{50}$$

- The expected cost incurred until the next inspection time for state PM, when PM action is performed is given by:

$$C_P = E(Cost \mid PM) = C_{PM}. \tag{51}$$

- The expected cost incurred until the next inspection time for failure state F is calculated by:

$$C_F = E(Cost \mid F) = C_F. \tag{52}$$

4.3.3 Computing the Sojourn Times

In this sub-section, the expected sojourn time of the proposed model is formulated based on the following scenarios:

- The expected sojourn time before the next monitoring time in state i where $i < M$, and the system will not fail in the next inspection time is as follows:

$$\begin{aligned} \tau_i &= \int_0^{\delta} R(t \mid \Pi_{n\delta})dt \\ &= \int_0^{\delta} (1-\Pi_{n\delta})(1-P_{02}(t)) + \Pi_{n\delta}(1-P_{12}(t)) \\ &= (1-\Pi_{n\delta}) \times \Big[\frac{1-e^{-v_0\delta}}{v_0} + \frac{v_0}{v_0-v_1}\big(\frac{v_0(1-e^{-v_1\delta}) - v_1(1-e^{-v_0\delta})}{v_0v_1}\Big] \\ &+ \Pi_{n\delta}\frac{1-e^{-v_1\delta}}{v_1}. \end{aligned} \tag{53}$$

- The expected sojourn time incurred until the next monitoring epoch for state l, where $M < l$, and the full system inspection is performed, is given by:

$$\tau_l = T_I. \tag{54}$$

- The mean sojourn time when the PM action is performed, is given by:

$$\tau_P \quad = \quad T_{PM}. \tag{55}$$

- Finally, the expected sojourn time for the failure state F when the corrective action is performed is as follows:

$$\tau_F \quad = \quad T_F. \tag{56}$$

This completes the mathematical formulation of the proposed model. Next, a comprehensive numerical analysis is provided to show the performance of proposed model.

4.4 Numerical Example

In this section, we illustrate the proposed computational procedure with a numerical example. First, the performance of proposed model is illustrated in Section 4.4.1 followed by comprehensive design of experiment in Section 4.4.2. In addition, sensitivity analysis on most significant factors are performed in Section 4.4.3. In order to see the effectiveness of the proposed model, a comparison is made with one of the traditional control policy which is represented in Section 4.4.4.

4.4.1 Performance of Proposed Model

A network with 15 facilities is considered, where all the facilities are in healthy condition providing service to the customers. We assume that the degradation of a SCN follows a continuous-time homogeneous hidden-Markov chain $\{X_t : t \geq 0\}$ with state space $\Omega = \{\ 0, 1, 2\}$. States 0 and 1 correspond to the healthy and warning operational states, respectively. State 2 represents a failure state. It is assumed that the operational states are unobservable and the failure state is observable and absorbing. The transition rates between these states are given by$\lambda_{01} = 0.089$, $\lambda_{02} = 0.001$, and $\lambda_{12} = 0.376$. Furthermore,

Input parameter	Value
C_M	5
C_I	50
C_{PM}	100
C_F	10000
C_{T_0}	50
C_{T_1}	100
T_{PM}	10
T_F	50
T_I	5
p_0	0.05
p_1	0.2

Table 4.1: The input parameters.

the instantaneous transition rates matrix for the degradation process is assumed as follow:

$$\Lambda = \begin{bmatrix} -0.09 & 0.089 & 0.001 \\ 0 & -0.376 & 0.376 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

Using Kolmogoroff-backward equations, the transition probability matrix can be obtained accordingly. The SCN is monitored periodically and the number of failed facilities is observed, then the posterior probability is updated at each time epoch. The posterior probability is divided to K intervals and the state space of the posterior probability is formulated as a continuous-time homogeneous hidden-Markov chain. In this framework, the posterior probability can be in one of the finite numbers of states $\zeta = \{\ 0, .., K\}$, where state 0 represents a healthy SCN with no disruption, and state K is the mid point of the last interval. In other words, the posterior probability interval is divided into K sub-intervals and mid-point of each interval is considered as a coded value of the posterior probability. The objective is to define the time between each monitoring epoch (δ), and the maintenance limit (M) which minimize the long-run expected average cost per unit time denoted by $g - value$. Table 4.1 presents all the input parameters of the model. The model is programmed in MATLAB software. It is worth mentioning that we have found in practice that when $K \geq 40$, the value of the cost converges and leads to a high degree of precision, so K does not need to be chosen very large. This makes the computational algorithm in SMDP framework extensively fast, which is an appealing feature for practical applications. Figure 4.2 represents the relation between discritization level and total cost for fixed values of δ and M.

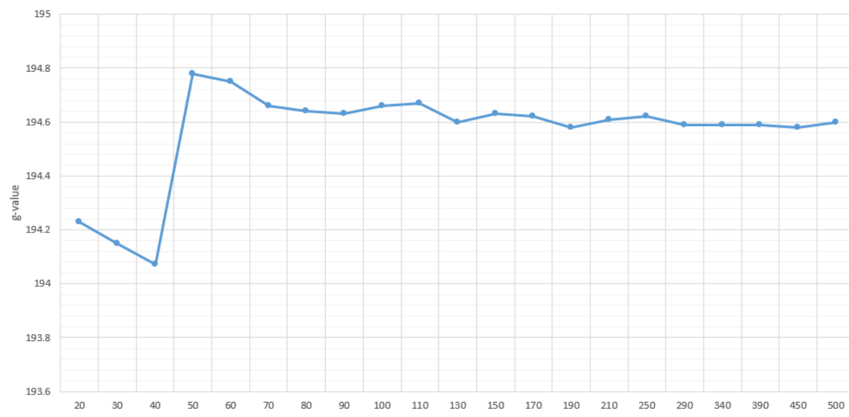

Figure 4.2: The relation between cost and discretization level

For a fixed value of K, the final results are obtained based on comparing over 3780 different potential scenarios. By solving iteratively the system of linear equations, the optimal values of monitoring interval, and the optimal maintenance limit along with minimum long-run expected average cost are obtained. The results are presented in Table 4.2.

Table 4.2: Results for the optimal Bayesian control chart.

Optimal control limit (M)	Optimal sampling interval (δ)	Average cost
0.1	1	98.85

As the results show, the long-run expected average cost for the proposed attribute-type Bayesian control chart is equal to 98.85 per unit time.

4.4.2 Design of Experiment

In this study, a comprehensive design of experiment (DOE) is carried out to study the effect of input parameters on the proposed control model based on attribute-type Bayesian control chart. The response variable is the long-run expected average cost per unit time. The purpose of this sensitivity analysis is to identify and understand which parameters, along with their interactions, have the most and significant effect on the average cost.

In order to do this design experiment, Minitab statistical software 19 is used to investigate the effect of each factor on the total cost, two level is defined for each parameters as a high level (+) and low (-). The value associate with these levels for each parameter is represented in Table 4.3. The second row of the table represent the coded value that represent each factor in the software and in the explanation of this section each parameter is showed by its coded value.

Design Factors	C_F	C_{PM}	C_I	C_M	C_{T_0}	C_{T_1}	T_F	T_{PM}	T_I	p_0	p_1
Coded value	A	B	C	D	E	F	G	H	J	K	L
Low (-)	10000	100	50	5	50	100	50	10	5	0.05	0.2
High (+)	12000	120	70	7	70	120	70	12	7	0.07	0.4

Table 4.3: Factors and levels used in DOE.

By considering all the possibilities for the parameters and considering the fact that each has 2 levels, all the possible scenarios would be equal to $2^{11} = 2048$. Because of the many possibilities and the fact that the long-run expected average cost is deterministic, we only considered a single replication fractional factorial design with resolution of four that gives us a total of 64 runs. To determine which factors and their interactions have the most significant effect, we consider the input parameters for each combination. Using these parameters and MATLAB software, we calculate and get the response variable of interest (the long-run expected average cost per unit time referred as g-value) in the SMDP framework . The results is shown in Table 4.4. Each row corresponds to one of the 64 runs. The results of the NOVA test are also presented in Table 4.3. It can be seen that factors and their interactions both have positive as well as negative impacts on the total cost. The factors with p-value less than the significance level at 0.05 are considered to be significant factors.

In addition to ANOVA test, some important charts and plots are constructed which are described in details as follow:

(1) Normal Plot: A normal plot of effects is also constructed as shown in Figure 4.4. The normal probability plot is a useful approach that determines the magnitude of effects and their significance. According to the plot, the significant effects are represented

Order	A	B	C	D	E	F	G	H	J	K	L	g-value	Order	A	B	C	D	E	F	G	H	J	K	L	g-value
1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1	112.9609	33	-1	-1	-1	-1	-1	1	-1	1	1	-1	-1	82.4053
2	1	-1	-1	-1	-1	-1	-1	-1	1	1	-1	98.8047	34	1	-1	-1	-1	-1	1	-1	-1	-1	-1	1	130.3557
3	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1	104.2696	35	-1	1	-1	-1	-1	1	-1	-1	-1	1	-1	144.0127
4	1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	108.821	36	1	1	-1	-1	-1	1	-1	1	1	1	1	121.2103
5	-1	-1	1	-1	-1	-1	1	-1	-1	1	1	102.653	37	-1	-1	1	-1	-1	1	1	-1	1	-1	-1	77.2897
6	1	-1	1	-1	-1	-1	1	1	1	1	-1	124.2383	38	1	-1	1	-1	-1	1	1	1	-1	-1	1	106.4397
7	-1	1	1	-1	-1	-1	1	1	1	-1	1	86.6587	39	-1	1	1	-1	-1	1	1	1	-1	1	-1	112.8796
8	1	1	1	-1	-1	-1	1	-1	-1	-1	-1	100.7451	40	1	1	1	-1	-1	1	1	-1	1	1	1	110.8265
9	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	86.7494	41	-1	-1	-1	1	-1	1	1	-1	1	1	1	96.0388
10	1	-1	-1	1	-1	-1	1	1	1	-1	1	98.6666	42	1	-1	-1	1	-1	1	1	1	-1	1	-1	132.3738
11	-1	1	-1	1	-1	-1	1	1	1	1	-1	105.3171	43	-1	1	-1	1	-1	1	1	1	-1	-1	1	93.3739
12	1	1	-1	1	-1	-1	1	-1	-1	1	1	118.5771	44	1	1	-1	1	-1	1	1	-1	1	-1	-1	88.3579
13	-1	-1	1	1	-1	-1	-1	1	-1	-1	-1	96.4036	45	-1	-1	1	1	-1	1	-1	1	1	1	1	106.3827
14	1	-1	1	1	-1	-1	-1	-1	1	-1	1	120.5832	46	1	-1	1	1	-1	1	-1	-1	-1	1	-1	170.7834
15	-1	1	1	1	-1	-1	-1	-1	1	1	-1	133.583	47	-1	1	1	1	-1	1	-1	-1	-1	-1	1	116.1481
16	1	1	1	1	-1	-1	-1	1	-1	1	1	132.1584	48	1	1	1	1	-1	1	-1	1	1	-1	-1	96.9217
17	-1	-1	-1	-1	1	-1	1	1	-1	-1	-1	88.3374	49	-1	-1	-1	-1	1	1	1	1	1	1	1	97.2277
18	1	-1	-1	-1	1	-1	1	-1	1	-1	1	110.0571	50	1	-1	-1	-1	1	1	1	-1	-1	1	-1	138.0635
19	-1	1	-1	-1	1	-1	1	-1	1	1	-1	110.1112	51	-1	1	-1	-1	1	1	1	-1	-1	-1	1	105.395
20	1	1	-1	-1	1	-1	1	1	-1	1	1	118.6555	52	1	1	-1	-1	1	1	1	1	1	-1	-1	89.7092
21	-1	-1	1	-1	1	-1	-1	-1	-1	-1	-1	107.1403	53	-1	-1	1	-1	1	1	-1	-1	1	1	1	120.1413
22	1	-1	1	-1	1	-1	-1	1	1	-1	1	120.3923	54	1	-1	1	-1	1	1	-1	1	-1	1	-1	170.0364
23	-1	1	1	-1	1	-1	-1	1	1	1	-1	133.7675	55	-1	1	1	-1	1	1	-1	1	-1	-1	1	116.2171
24	1	1	1	-1	1	-1	-1	-1	-1	1	1	148.7519	56	1	1	1	-1	1	1	-1	-1	1	-1	-1	106.2627
25	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	130.0651	57	-1	-1	-1	1	1	1	-1	-1	1	-1	-1	92.3242
26	1	-1	-1	1	1	-1	-1	1	1	1	-1	156.2376	58	1	-1	-1	1	1	1	-1	1	-1	-1	1	131.1816
27	-1	1	-1	1	1	-1	-1	1	1	-1	1	106.5227	59	-1	1	-1	1	1	1	-1	1	-1	1	-1	144.7561
28	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	105.9135	60	1	1	-1	1	1	1	-1	-1	1	1	1	137.5078
29	-1	-1	1	1	1	-1	1	1	-1	1	1	104.9945	61	-1	-1	1	1	1	1	1	1	1	-1	-1	80.051
30	1	-1	1	1	1	-1	1	-1	1	1	-1	130.017	62	1	-1	1	1	1	1	1	-1	-1	-1	1	120.7091
31	-1	1	1	1	1	-1	1	-1	1	-1	1	98.8047	63	-1	1	1	1	1	1	1	-1	-1	1	-1	118.8242
32	1	1	1	1	1	-1	1	1	-1	-1	-1	102.9111	64	1	1	1	1	1	1	1	1	1	1	1	112.6176

Table 4.4: The results of DOE runs

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		113.605	0.928	122.45	0.000	
A	13.971	6.986	0.928	7.53	0.000	1.00
B	-0.297	-0.149	0.928	-0.16	0.877	1.00
C	3.187	1.593	0.928	1.72	0.124	1.00
D	1.907	0.953	0.928	1.03	0.334	1.00
E	7.397	3.699	0.928	3.99	0.004	1.00
F	1.967	0.984	0.928	1.06	0.320	1.00
G	-16.730	-8.365	0.928	-9.02	0.000	1.00
H	-2.782	-1.391	0.928	-1.50	0.172	1.00
J	-11.378	-5.689	0.928	-6.13	0.000	1.00
K	22.452	11.226	0.928	12.10	0.000	1.00
L	0.075	0.037	0.928	0.04	0.969	1.00
A*B	-15.890	-7.945	0.928	-8.56	0.000	1.00
A*C	2.432	1.216	0.928	1.31	0.226	1.00
A*D	1.352	0.676	0.928	0.73	0.487	1.00
A*E	1.300	0.650	0.928	0.70	0.503	1.00
A*F	2.272	1.136	0.928	1.22	0.256	1.00
A*G	0.920	0.460	0.928	0.50	0.633	1.00
A*H	1.923	0.962	0.928	1.04	0.330	1.00
A*J	-2.002	-1.001	0.928	-1.08	0.312	1.00
A*K	1.475	0.738	0.928	0.80	0.450	1.00
A*L	1.081	0.541	0.928	0.58	0.576	1.00
B*C	-1.589	-0.794	0.928	-0.86	0.417	1.00
B*D	-2.282	-1.141	0.928	-1.23	0.254	1.00
B*E	-2.218	-1.109	0.928	-1.20	0.266	1.00
B*G	-0.961	-0.481	0.928	-0.52	0.618	1.00
B*H	-1.317	-0.659	0.928	-0.71	0.498	1.00
C*D	-2.066	-1.033	0.928	-1.11	0.298	1.00
C*E	-1.338	-0.669	0.928	-0.72	0.491	1.00
C*F	-2.047	-1.023	0.928	-1.10	0.302	1.00
C*G	-2.334	-1.167	0.928	-1.26	0.244	1.00
C*H	-2.230	-1.115	0.928	-1.20	0.264	1.00
C*J	0.799	0.399	0.928	0.43	0.678	1.00
C*K	1.234	0.617	0.928	0.67	0.525	1.00
C*L	-2.411	-1.205	0.928	-1.30	0.230	1.00
D*F	-1.289	-0.645	0.928	-0.69	0.507	1.00
D*H	-1.225	-0.612	0.928	-0.66	0.528	1.00
D*J	2.253	1.127	0.928	1.21	0.259	1.00
D*K	2.211	1.106	0.928	1.19	0.267	1.00
D*L	-1.149	-0.575	0.928	-0.62	0.553	1.00
E*F	-1.446	-0.723	0.928	-0.78	0.458	1.00
E*H	2.378	1.189	0.928	1.28	0.236	1.00
E*J	1.990	0.995	0.928	1.07	0.315	1.00
E*K	1.914	0.957	0.928	1.03	0.333	1.00
E*L	0.224	0.112	0.928	0.12	0.907	1.00
F*G	-2.425	-1.212	0.928	-1.31	0.228	1.00
F*H	-2.171	-1.086	0.928	-1.17	0.276	1.00
F*K	2.582	1.291	0.928	1.39	0.202	1.00
F*L	-1.530	-0.765	0.928	-0.82	0.434	1.00
G*H	-0.890	-0.445	0.928	-0.48	0.644	1.00
G*J	2.897	1.448	0.928	1.56	0.157	1.00
G*K	-3.754	-1.877	0.928	-2.02	0.078	1.00
G*L	-0.342	-0.171	0.928	-0.18	0.858	1.00
H*J	1.742	0.871	0.928	0.94	0.375	1.00
H*K	1.348	0.674	0.928	0.73	0.488	1.00
H*L	-3.794	-1.897	0.928	-2.04	0.075	1.00

Figure 4.3: The results of ANOVA test

by red squares, and blue circles represent the effects that are not significant. The significant effects are plotted further from the noise line indicated by the straight red line, while the effects that are not significant and can be neglected are plotted on the red line. The cut-off point of p-value = 0.05 is used to determine which factors are significant. The significant factors derived from the chart are the main factors of A, E, G, J, K, and AB interaction.

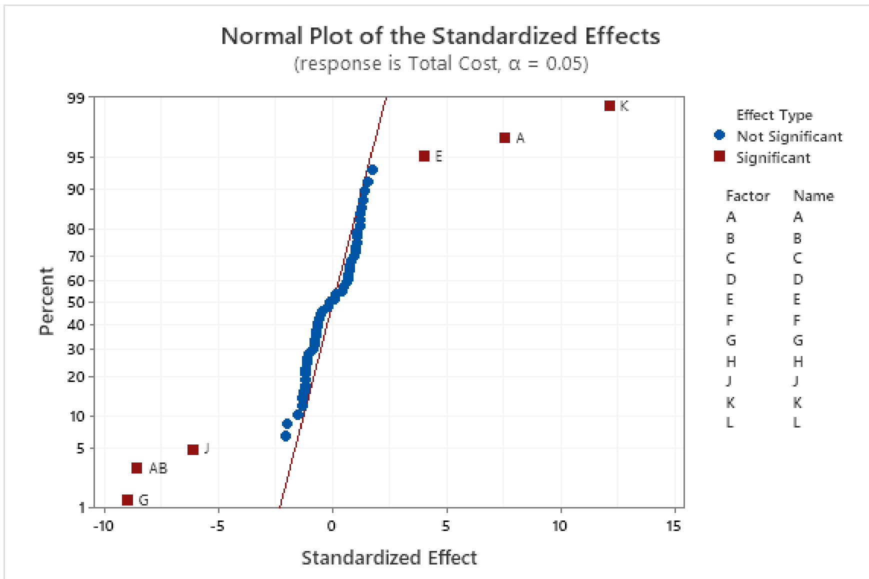

Figure 4.4: Normal probability plot of effects

(2) Pareto Chart: A Pareto chart is a type of bar chart that arranges the absolute values and magnitudes of the standardized effects in descending order, starting from the most significant effect. For this analysis, the Pareto chart is represented in Figure 4.5. The reference standardized effects of 2.31 are represented by a red dashed line. If a factor surpasses the reference standardized effects of 2.31, then it is considered as a significant factor which will affect the results. If the factor does not pass 2.31, then it is considered insignificant at the 0.05 α level with the current model specifications. As an example, main effect K has the most effect on the cost following by the factor G.

(3) Plots of the Main Effects and Interaction Effects The main effects plot is displayed

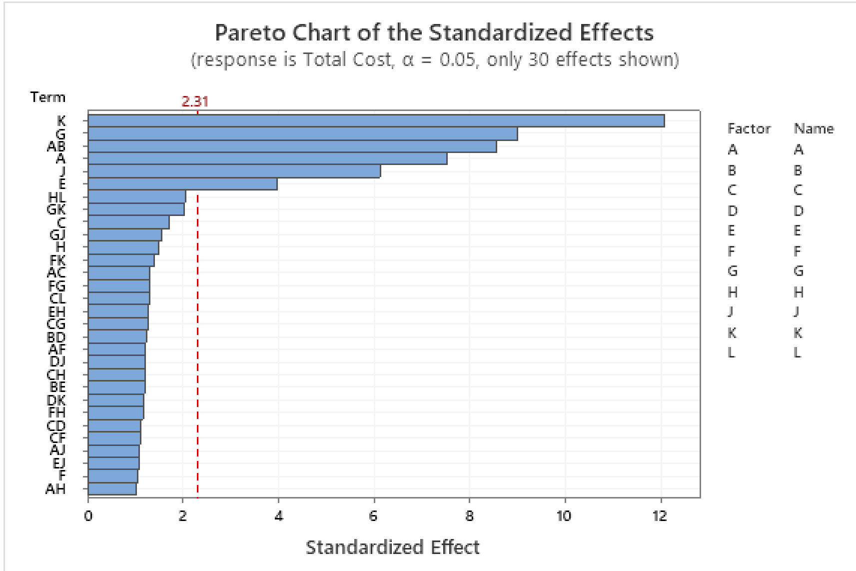

Figure 4.5: Pareto chart

in Figure 4.6. From this plot, it can be observed that factor K $i.e., p_0$ is the most significant factor and has the biggest impact on the total cost, as it is the factor with the highest difference in cost at high (0.05) and low levels (0.07). This means when the value is increased, it will have a direct effect on the cost, causing it to increase, even with the slightest change of value. Factor G $i.e., T_F$ is the second most significant factor, where unlike factor K, the cost decreases when there is an increase in the value, an opposite effect to factor K. The average cost (g-value) is 113.6046 which is shown by the dash line, with a minimum 77.2897 and maximum of 170.7834. This plot also shows that factor L $i.e., p_1$ is the least significant factor, since the cost will almost remain the same regardless of any changes in the value.

(4) Interaction Plots: While the main effect plot shows the relation between the cost and main effects, interaction plot is used to observe the relation between the cost and two factors. An interaction plot is used to represent the relationship between two variables, where one variable is categorized as one factor with a continuous response dependent on the value of the second variable. An interaction plot shows the levels of one variable, plotted on the X axis, with separate lines for the means of each level of the second variable, which is a dependent variable plotted on the Y axis. The

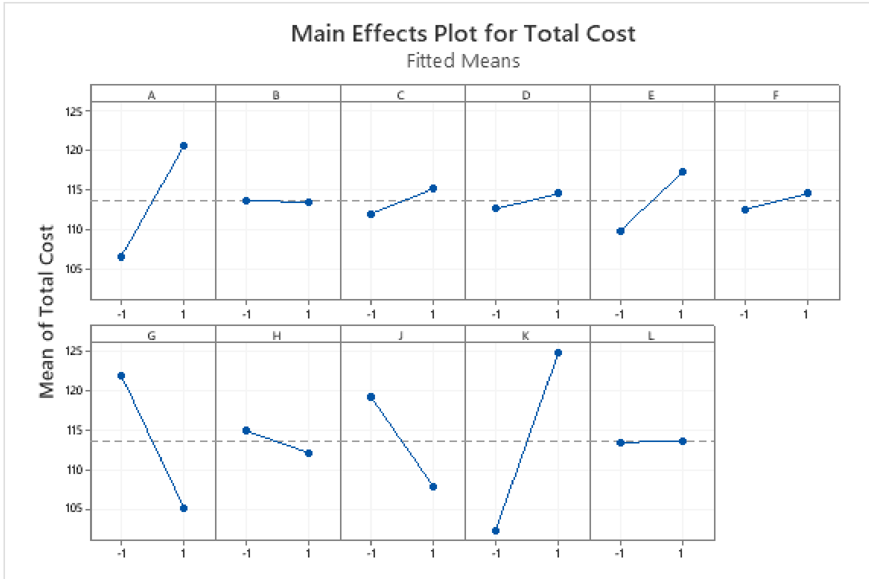

Figure 4.6: Main effects plot

interaction plot is presented in Figure 4.7. Parallel interaction lines, such as factor $E * L$, show that there is no interaction as they follow similar and parallel linear directions. When interaction lines are not similar and have different slopes, like factor $A * B$, the interaction is stronger and significant.

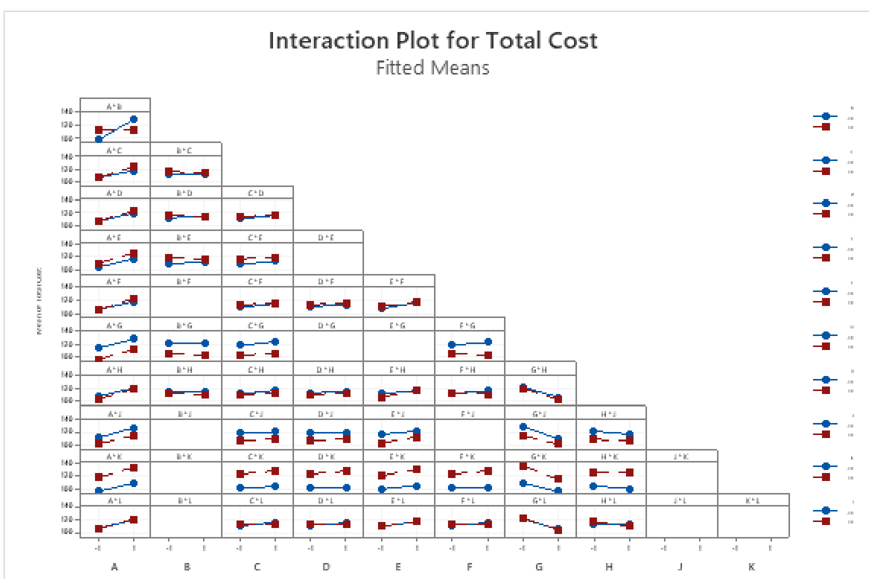

Figure 4.7: Interaction plot

(5) Normal probability Plot of Residuals: Normal probability plot is constructed for residuals as shown in Figure 4.8. In particular, the residuals are equal to the differences between the value of the response variable and the fitted value. It can be seen that these points on the plot that are closely distributed to a straight line. This shows an adequate model as the error terms are normally distributed because the plot is approximately linear.

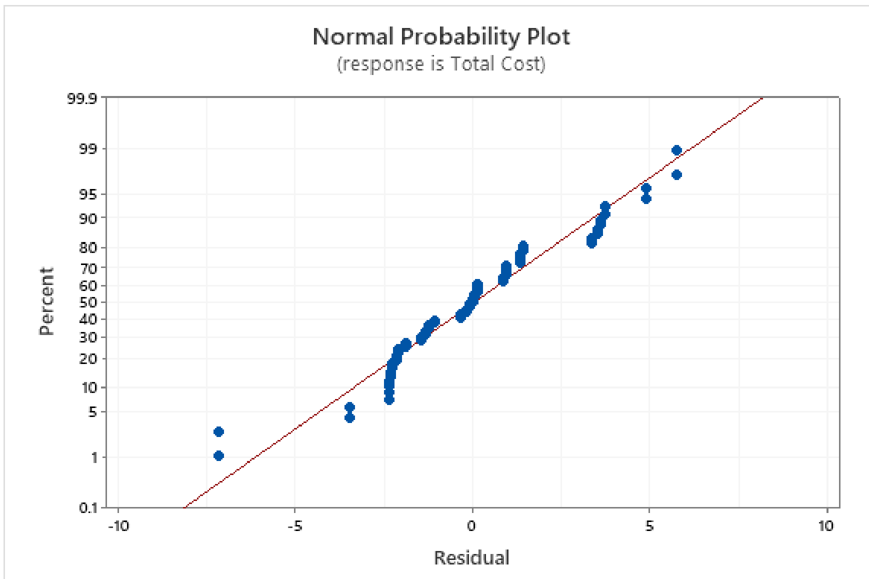

Figure 4.8: Normal probability plot for residuals

Also, based on the ANOVA analysis and by using the regression model, we compute the residuals from the experiment. The regression equation is as follows:

$$\begin{aligned}
\hat{Y} &= 113.605 + 6.986A - 0.149B + 1.593C + 0.953D + 3.699E + 0.984F \\
&- 8.365G - 1.391H - 5.689J + 11.226K + 0.037L - 7.945AB + 1.216AC \\
&+ 0.676AD + 0.650AE + 1.136AF + 0.460AG + 0.962AH - 1.001AJ \\
&+ 0.738AK + 0.541AL - 0.794BC - 1.141BD - 1.109BE - 0.481BG \\
&- 0.659BH - 1.033CD - 0.669CE - 1.023CF - 1.167CG - 1.115CH \\
&+ 0.399CJ + 0.617CK - 1.205CL - 0.645DF - 0.612DH + 1.127DJ \\
&+ 1.106DK - 0.575DL - 0.723EF + 1.189EH + 0.995EJ + 0.957EK \\
&+ 0.112EL - 1.212FG - 1.086FH + 1.291FK - 0.765FL - 0.445GH \\
&+ 1.448GJ - 1.877GK - 0.171GL + 0.871HJ + 0.674HK - 1.897HL \qquad (58)
\end{aligned}$$

As it mentioned before, factor K has the most significant affect on the cost, thus it has the greatest coefficient in regression equation. While factor L has the least effect and its coefficient is minimum between all the main factors. As an another observation, we can refer to the positive coefficient of factor K and negative coefficient of factor G which proved that they have direct and opposite effect on the cost, respectively.

This completes DOE analysis. The result of DOE shows that failure cost has the significant effect on the long-run expected average cost. Therefore, in the next section, we further extend our analysis on the effect of failure cost on the total cost as well as control chart parameters.

4.4.3 Sensitivity Analysis on Failure Cost and Sampling Interval

In this sub-section, we examine the effect of changing the value of failure cost on total cost and chart parameters. In order to perform the analysis, the value of failure cost increases from $C_f = 1000$ to $C_f = 10000$. The results of sensitivity analysis is shown in Table 4.5. The first column of table shows the scenario when the value of δ is fixed as 1 and the failure cost is fixed as 1000. It is observed that the minimum total cost is obtained when the control limit is in the highest level, i.e., $M = 0.8$. On the other hand, if the cost of failure is increased to 10000, the minimum cost is obtained when the control limit is in the lowest level, i.e., $M = 0.1$. This behavior is expected from Bayesian control chart. The rational behind this behavior is that when the failure cost is chosen large, the system avoids to go to failure state due to the huge cost, as such the chart is designed in conservative manner by setting the control limit in lower level. On the other hand, when the failure cost is chosen small, the model allows the system to continue operation, as the difference between the failure cost and preventive maintenance cost is much less. The model automatically postpone preventative maintenance since catastrophic failure is unlikely and the cost of failure is affordable.

In the second scenario which is shown in the second column of Table 4.5, the value of δ is changed to 7. When the C_f is fixed and the time interval between monitoring epoch is chosen big, it results in a higher total cost. We expect this behavior from the proposed model. It is showing that monitoring interval plays an important and critical role in total cost. When the SCN is monitored less frequently, i.e., $\delta = 7$, the probability of failure occurrence between two sampling becomes higher.

Variables	$\delta = 1$			$\delta = 7$		
	$M = 0.1$	$M = 0.4$	$M = 0.8$	$M = 0.1$	$M = 0.4$	$M = 0.8$
$C_F = 1000$	32.53	29.10	27.69	37.27	36.87	36.57
$C_F = 10000$	98.85	169.50	187.50	194.07	202.35	207.64

Table 4.5: Comparison between different scenarios

If we further continue to monitor the SCN more frequently such as $\delta = 0.5$, this leads to higher cost. The results of this experiment is presented in Table 4.6. In this case, the cost of failure and control limit are set to 10000 and 0.1, receptively. The reason for this behavior is that the monitoring is costly itself. Therefore, when the sampling is costly, it is important to determine jointly the optimal times when the CM samples should be collected as well as utilization of the information for maintenance decision-making which is the main objective of this thesis. This completes the sensitivity analysis.

δ	g-value	δ	g-value
0.25	181.32	2.75	175.07
0.5	167.56	3	177.63
0.75	166.87	3.25	180.21
1	161.54	3.5	168.46
1.25	164.71	3.75	171.53
1.5	167.43	4	174.32
1.75	170.26	4.25	176.62
2	173.22	4.5	179.06
2.25	175.85	4.75	181.10
2.5	178.53	5	183.08

Table 4.6: The relation between total cost and monitoring interval.

4.4.4 Comparison to Replace only on Failure (R-O-O-F) Policy

In this sub-section the proposed model is compared to the policy when corrective action is done upon failure reffed to as R-O-O-F policy such that no CM data is collected. In this traditional policy, the SCN does not receive any preventive maintenance and is renewed whenever failure happens. The average expected cost in this method is calculated as follow:

$$C_{avg} = \frac{C_F}{MTTF}, \tag{59}$$

where MTTF referred to as mean time to failure. The value of the MTTF is equivalent to value of MRL at time 0. In other words, MTTF= MRL_0. It is assumed that at the beginning the system is in the healthy state. The posterior probability that the system is in warning state at the time $t = 0$ is equal to 0. By this fact and using Eq. 35, the corresponding MRL in the healthy state and at time 0 is equal to $MRL_0 = 13.74$. By substituting C_F and the value of MTTF in Eq. 59, the total cost for R-O-O-F policy is obtained as 727.80. By comparing this value to the cost of proposed Bayesian model obtained in Section 4.4.1, i.e., 98.85, a huge difference can be observed which validates the superior performance our proposed model.

4.5 Conclusion

In this chapter, a novel application of attribute-type Bayesian control chart in CBM application is presented for control policy for a SCN under disruption and random failure. The network has a deterioration process structured as a 3-state continuous-time hidden-Markov process. The states that are not observable are 0 and 1, and stand for healthy system condition and warning conditions. The failure state 2 is assumed to be observable. When the SCN fails, corrective maintenance is executed to bring it back to a healthy state. The system is subject to CM at discrete time epochs. Suppose that the posterior probability that the SCN is in state 1 exceeds a maintenance limit at a monitoring epoch. In that case, the full system inspection is performed. The optimal control problem has been formulated and solved in the SMDP framework. The suggested optimal maintenance and monitoring policy have been assessed, applying sensitivity analysis on sampling interval and failure cost. We performed a DOE to understand the factors that have significant effect on the cost. The information gathered from a design experiment is crucial to manage and understand process inputs and optimize output. For the purpose of improving future research into this, we propose using a hidden semi-Markov model and dual monitoring interval where a longer sampling interval would be required when the system starts and is in a healthy state. Observations are taken more often if, at a monitoring epoch, the posterior probability will exceed a warning limit.

Chapter 5

Summary and Future Research Directions

The chapter concludes the thesis with a list of important contributions made in the dissertation and some proposed directions for future work.

5.1 Summary of Thesis Contributions

In this thesis, we look into the application of maintenance control problems in a SCN in order to design a reliable and resilience network. The objective of proposed models are to obtain the optimal control policies for a SCN subject to disruption and degradation such that the long-run expected average cost per unit time is minimized. In particular, two control polices are developed in this thesis. In the first control policy, we concentrate the attention on one single facility in a SCN. The control policy is formulated based on HMM considering N states such that $N-1$ states are considered unobservable operational states and one observable failure state. The proposed model is formulated in SMDP framework and policy-iteration algorithm is applied to find the optimal maintenance/control level such that the total cost is minimized.

In addition, the previous analysis is extended to a SCN such that multiple facilities are under disruption. Similarly, we model the multiple facility network as a Markov chain with two operational states, namely, healthy and warning states and one unobservable failure state. By incorporating the appealing concept of integrated model of SPC and CBM, a novel attribute-type Bayesian control chart is designed to monitor the multiple facility network's condition. The posterior probability that the network is in warning state is monitored and updated at each monitoring interval. The posterior probability is divided into 40 sub-intervals representing a SMDP with 40 states. The optimum maintenance limit is obtained for both models using MATLAB programming. A comprehensive sensitivity analysis is provided to evaluate the performance of proposed models.

5.2 Future Research

Below, potential future research directions are discussed to further improve the proposed contributions made throughout this thesis:

- We previously proposed maintenance policy for facilities in SCN for both single and multiple facilities. For future research, this study can be further extend by applying a dual sampling interval. In this case, the monitoring of the SCN starts with a longer sampling interval. If at a sampling epoch, the control statistic reaches a warning limit, observations are taken more frequently and the sampling interval changes to a shorter one. When the statistic control surpasses the control limit, preventive maintenance will be applied. This can be applied to both proposed models in Chapters 3 and 4.

- Additionally, the presence of imperfect preventive maintenance can be considered as future works. In reality, sometimes performing maintenance can not be done perfectly which means that after performing maintenance, the system can not be considered as "as-good-as-new" system. For example, the system undergoing repairs will likely deteriorate faster and have a shortened life span. In industry practices, there is a maximum number of maintenance actions that can be taken in order to bring the system back to the full function. After this maximum number, the system will die.

- Another interesting extension of the current work is to consider a back-up facility for each end node (customer). In this scenario, each customer is assigned to two facilities, and in case of failure of both facilities, the cost of losing customer will happen. This further complicates the problem due to the consideration of the assignment in the network.

- Another fruitful direction for this research is to consider modeling each facility in a SCN based on 3-state Markov process, with two operational states (healthy and warning) and one failure state while the whole system is also formulated in three state SMDP. At each monitoring epoch, the number of failed facilities and number of partially operational facilities are observed. Considering the available budget, not only is the action based on what type of maintenance action needs to be done (Do nothing, PM, and CM), but also the number of facilities that will receive these actions needs to be considered.

Appendix A

Derivation of the Probability Transition Matrix in Chapter 3 using Laplace Transform

The Kolmogorov backward differential equations is given by:

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \quad (60)$$

Please not that $v_i = \sum_{j=0}^{4} q_{ij}$, which in our problem results in:

$$v_0 = q_{01}$$

$$v_1 = q_{12}$$

$$v_2 = q_{23}$$

$$v_4 = q_{34}.$$

A.0.1 Calculation of $P_{00}(t)$

$$P_{00}(t) = q_{01}P_{10}(t) + q_{02}P_{20}(t) + q_{03}P_{30}(t) + q_{04}P_{40}(t) + v_0P_{00}(t). \tag{61}$$

Since it is assumed that the degradation process is non-decreasing with probability 1, for all $j < i$, $q_{ij} = 0$ and $P_{ij} = 0$.

$$P'_{00}(t) = \cancelto{0}{q_{01}P_{10}(t)} + \cancelto{0}{q_{02}P_{20}(t)} + \cancelto{0}{q_{03}P_{30}(t)} + \cancelto{0}{q_{04}P_{40}(t)} - v_0P_{00}(t). \tag{62}$$

$$P'_{00}(t) = -v_0P_{00}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{00}(s) - 1 = -v_0\tilde{P}_{00}(s)$$

$$\tilde{P}_{00}(s) = \frac{1}{s+v_0} \xrightarrow{\mathscr{L}^{-1}} P_{00}(t) = e^{-v_0t} \tag{63}$$

Similarly, by substituting 0 for the terms mentioned, we will have the following set of equations:

A.0.2 Calculation of $P_{11}(t)$

$$P'_{11}(t) = -v_1P_{11}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{11}(s) - 1 = -v_1\tilde{P}_{11}(s)$$

$$\tilde{P}_{11}(s) = \frac{1}{s+v_1} \xrightarrow{\mathscr{L}^{-1}} P_{11}(t) = e^{-v_1t} \tag{64}$$

A.0.3 Calculation of $P_{01}(t)$

$$P_{01}^{'}(t) \quad = \quad q_{01}P_{11}(t) - v_0P_{01}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{01}(s) = v_0\tilde{P}_{11}(s) - v_0\tilde{P}_{01}(s)$$

$$\tilde{P}_{01}(s) \quad = \quad \frac{v_0}{(s+v_0)(s+v_1)} = \frac{a}{(s+v_0)} + \frac{b}{(s+v_1)} = \frac{s(a+b) + av_1 + bv_0}{(s+v_0)(s+v_1)}$$

$$\begin{cases} a + b = 0 \\ \\ av_1 + bv_0 = v_0 \end{cases} \longrightarrow a = \tfrac{v_0}{v_1 - v_0}, b = \tfrac{-v_0}{v_1 - v_0}$$

$$\tilde{P}_{01}(s) \quad = \quad \frac{\frac{v_0}{v_1 - v_0}}{(s+v_0)} - \frac{\frac{v_0}{v_1 - v_0}}{(s+v_1)} \xrightarrow{\mathscr{L}^{-1}} P_{01}(t) = \frac{v_0}{v_1 - v_0}(e^{-v_0t} - e^{-v_1t}) \tag{65}$$

A.0.4 Calculation of $P_{22}(t)$

$$P_{22}^{'}(t) \quad = \quad -v_2P_{22}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{22}(s) - 1 = -v_2\tilde{P}_{22}(s)$$

$$\tilde{P}_{22}(s) \quad = \quad \frac{1}{s+v_2} \xrightarrow{\mathscr{L}^{-1}} P_{22}(t) = e^{-v_2t} \tag{66}$$

A.0.5 Calculation of $P_{12}(t)$

$$P_{12}^{'}(t) \quad = \quad q_{12}P_{22}(t) - v_1P_{12}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{12}(s) = v_1\tilde{P}_{22}(s) - v_1\tilde{P}_{12}(s) \xrightarrow{\tilde{P}_{22}(s) = \frac{1}{s+v_2}}$$

$$\tilde{P}_{12}(s) \quad = \quad \frac{v_1}{(s+v_1)(s+v_2)} = \frac{a}{(s+v_1)} + \frac{b}{(s+v_2)} = \frac{s(a+b)+av_2+bv_1}{(s+v_1)(s+v_2)}$$

$$\begin{cases} a+b=0 \\ \\ av_2+bv_1=v_1 \end{cases} \longrightarrow a = \frac{v_1}{v_2-v_1}, b = \frac{-v_1}{v_2-v_1}$$

$$\tilde{P}_{12}(s) \quad = \quad \frac{\frac{v_1}{v_2-v_1}}{(s+v_1)} - \frac{\frac{v_1}{v_2-v_1}}{(s+v_2)} \xrightarrow{\mathscr{L}^{-1}} P_{12}(t) = \frac{v_1}{v_2-v_1}(e^{-v_1 t} - e^{-v_2 t}) \tag{67}$$

A.0.6 Calculation of $P_{02}(t)$

$$P_{02}^{'}(t) \quad = \quad q_{01}P_{12}(t) - v_0 P_{02}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{02}(s) = v_0\tilde{P}_{12}(s) - v_0\tilde{P}_{02}(s) \xrightarrow{\tilde{P}_{12}(s)=\frac{\frac{v_1}{v_2-v_1}}{(s+v_1)}-\frac{\frac{v_1}{v_2-v_1}}{(s+v_2)}}$$

$$\tilde{P}_{02}(s) \quad = \quad \frac{v_0 v_1}{v_2-v_1}(\underbrace{\frac{1}{(s+v_0)(s+v_1)}}_{\text{Term I}} - \underbrace{\frac{1}{(s+v_0)(s+v_2)}}_{\text{Term II}})$$

$$\text{Term I} \quad = \quad \frac{a}{(s+v_1)} + \frac{b}{(s+v_0)} = \frac{s(a+b)+av_0+bv_1}{(s+v_0)(s+v_1)} \longrightarrow \begin{cases} a = \frac{1}{v_0-v_1} \\ b = \frac{-1}{v_0-v_1} \end{cases}$$

$$\text{Term II} \quad = \quad \frac{a^{'}}{(s+v_2)} + \frac{b^{'}}{(s+v_0)} = \frac{s(a^{'}+b^{'})+a^{'}v_0+b^{'}v_2}{(s+v_0)(s+v_2)} \longrightarrow \begin{cases} a^{'} = \frac{1}{v_0-v_2} \\ b^{'} = \frac{-1}{v_0-v_2} \end{cases}$$

$$\tilde{P}_{02}(s) \quad = \quad \frac{v_0 v_1}{v_2-v_1}\left(\frac{\frac{1}{v_0-v_1}}{s+v_1} - \frac{\frac{1}{v_0-v_1}}{s+v_0} + \frac{\frac{1}{v_0-v_2}}{s+v_0} - \frac{\frac{1}{v_0-v_2}}{s+v_2}\right) \xrightarrow{\mathscr{L}^{-1}}$$

$$P_{02}(t) = \frac{v_0 v_1}{v_2 - v_1}\Big[\frac{1}{v_0 - v_1}\big(e^{-v_1 t} - e^{-v_0 t}\big) + \frac{1}{v_0 - v_2}\big(e^{-v_0 t} - e^{-v_2 t}\big)\Big]$$

$$\begin{aligned} P_{02}(t) &= \frac{v_0 v_1}{(v_0 - v_2)(v_0 - v_1)} e^{-v_0 t} + \frac{v_0 v_1}{(v_2 - v_1)(v_0 - v_1)} e^{-v_1 t} \\ &- \frac{v_0 v_1}{(v_2 - v_1)(v_0 - v_2)} e^{-v_2 t} \end{aligned} \tag{68}$$

A.0.7 Calculation of $P_{33}(t)$

$$P'_{33}(t) = -v_3 P_{33}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{33}(s) - 1 = -v_3 \tilde{P}_{33}(s)$$

$$\tilde{P}_{33}(s) = \frac{1}{s + v_3} \xrightarrow{\mathscr{L}^{-1}} P_{33}(t) = e^{-v_3 t} \tag{69}$$

A.0.8 Calculation of $P_{23}(t)$

$$P'_{23}(t) = q_{23} P_{33}(t) - v_2 P_{23}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{23}(s) = v_2 \tilde{P}_{33}(s) - v_2 \tilde{P}_{23}(s) \xrightarrow{\tilde{P}_{33}(s) = \frac{1}{S + v_3}}$$

$$\tilde{P}_{23}(s) = \frac{v_2}{(s + v_2)(s + v_3)} = \frac{a}{(s + v_2)} + \frac{b}{(s + v_3)} = \frac{s(a + b) + av_3 + bv_2}{(s + v_2)(s + v_3)}$$

$$\begin{cases} a + b = 0 \\ \\ av_3 - bv_2 = v_2 \end{cases} \longrightarrow a = \frac{v_2}{v_3 - v_2}, b = \frac{-v_2}{v_3 - v_2}$$

$$\tilde{P}_{23}(s) = \frac{\frac{v_2}{v_3 - v_2}}{(s + v_2)} - \frac{\frac{v_2}{v_3 - v_2}}{(s + v_3)} \xrightarrow{\mathscr{L}^{-1}} P_{23}(t) = \frac{v_2}{v_3 - v_2}(e^{-v_2 t} - e^{-v_3 t}) \tag{70}$$

A.0.9 Calculation of $P_{13}(t)$

$$P_{13}^{'}(t) = q_{12}P_{23}(t) - v_1P_{13}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{13}(s) = v_1\tilde{P}_{23}(s) - v_1\tilde{P}_{13}(s) \xrightarrow{\tilde{P}_{23}(s)=\frac{\frac{v_2}{v_3-v_2}}{(s+v_2)}-\frac{\frac{v_2}{v_3-v_2}}{(s+v_3)}}$$

$$\tilde{P}_{13}(s) = \frac{v_1v_2}{v_3-v_2}(\underbrace{\frac{1}{(s+v_1)(s+v_2)}}_{\text{Term I}} - \underbrace{\frac{1}{(s+v_1)(s+v_3)}}_{\text{Term II}})$$

$$\text{Term I} = \frac{a}{(s+v_2)} + \frac{b}{(s+v_1)} = \frac{s(a+b)+av_1+bv_2}{(s+v_1)(s+v_2)} \longrightarrow \begin{cases} a = \frac{1}{v_1-v_2} \\ b = \frac{-1}{v_1-v_2} \end{cases}$$

$$\text{Term II} = \frac{a^{'}}{(s+v_3)} + \frac{b^{'}}{(s+v_1)} = \frac{s(a^{'}+b^{'})+a^{'}v_1+b^{'}v_3}{(s+v_1)(s+v_3)} \longrightarrow \begin{cases} a^{'} = \frac{1}{v_1-v_3} \\ b^{'} = \frac{-1}{v_1-v_3} \end{cases}$$

$$\tilde{P}_{13}(s) = \frac{v_1v_2}{v_3-v_2}\Big(\frac{\frac{1}{v_1-v_2}}{s+v_2} - \frac{\frac{1}{v_1-v_2}}{s+v_1} + \frac{\frac{1}{v_1-v_3}}{s+v_1} - \frac{\frac{1}{v_1-v_3}}{s+v_3}\Big) \xrightarrow{\mathscr{L}^{-1}}$$

$$P_{13}(t) = \frac{v_1v_2}{v_3-v_2}\Big[\frac{1}{v_1-v_2}\big(e^{-v_2t}-e^{-v_1t}\big) + \frac{1}{v_1-v_3}\big(e^{-v_1t}-e^{-v_3t}\big)\Big]$$

$$P_{13}(t) = \frac{v_1v_2}{(v_1-v_3)(v_1-v_2)}e^{-v_1t} + \frac{v_1v_2}{(v_1-v_2)(v_3-v_2)}e^{-v_2t} - \frac{v_1v_2}{(v_3-v_2)(v_1-v_3)}e^{-v_3t} \tag{71}$$

A.0.10 Calculation of $P_{03}(t)$

$$P_{03}^{'}(t) = q_{01}P_{13}(t) - v_0 P_{03}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{03}(s) = v_0\tilde{P}_{13}(s) - v_0\tilde{P}_{03}(s) \xrightarrow{\tilde{P}_{13}(s)}$$

$$\begin{aligned}\tilde{P}_{03}(s) &= \frac{v_0v_1v_2}{v_3-v_2}\Big[\frac{1}{v_1-v_2}\Big(\underbrace{\frac{1}{(s+v_0)(s+v_2)}}_{\text{Term I}} - \underbrace{\frac{1}{(s+v_0)(s+v_1)}}_{\text{Term II}}\Big) \\ &+ \frac{1}{v_1-v_3}\Big(\underbrace{\frac{1}{(s+v_0)(s+v_1)}}_{\text{Term III}} - \underbrace{\frac{1}{(s+v_0)(s+v_3)}}_{\text{Term IV}}\Big)\Big]\end{aligned}$$

$$\text{Term I} = \frac{a}{(s+v_0)} + \frac{b}{(s+v_2)} = \frac{s(a+b)+av_2+bv_0}{(s+v_0)(s+v_2)} \longrightarrow \begin{cases} a = \frac{1}{v_2-v_0} \\ b = \frac{-1}{v_2-v_0}\end{cases}$$

$$\text{Term II} = \text{Term III} = \frac{a^{'}}{(s+v_0)} + \frac{b^{'}}{(s+v_1)} = \frac{s(a^{'}+b^{'})+a^{'}v_1+b^{'}v_0}{(s+v_0)(s+v_1)} \longrightarrow \begin{cases} a^{'} = \frac{1}{v_1-v_0} \\ b^{'} = \frac{-1}{v_1-v_0}\end{cases}$$

$$\text{Term IV} = \frac{a^{''}}{(s+v_0)} + \frac{b^{''}}{(s+v_3)} = \frac{s(a^{''}+b^{''})+a^{''}v_3+b^{''}v_0}{(s+v_0)(s+v_3)} \longrightarrow \begin{cases} a^{''} = \frac{1}{v_3-v_0} \\ b^{''} = \frac{-1}{v_3-v_0}\end{cases}$$

$$\begin{aligned}\tilde{P}_{03}(s) &= \frac{v_0v_1v_2}{v_3-v_2}\Big[\frac{1}{v_1-v_2}\Big(\frac{\frac{1}{v_0-v_2}}{s+v_2} - \frac{\frac{1}{v_0-v_2}}{s+v_0} + \frac{\frac{1}{v_0-v_1}}{s+v_0} - \frac{\frac{1}{v_0-v_1}}{s+v_1}\Big) \\ &+ \frac{1}{v_1-v_3}\Big(\frac{\frac{1}{v_0-v_1}}{s+v_1} - \frac{\frac{1}{v_0-v_1}}{s+v_0} + \frac{\frac{1}{v_0-v_3}}{s+v_0} - \frac{\frac{1}{v_0-v_3}}{s+v_3}\Big)\Big] \xrightarrow{\mathscr{L}^{-1}}\end{aligned}$$

$$P_{03}(t) = \frac{v_0 v_1 v_2}{v_3 - v_2}\Big[\frac{1}{(v_1 - v_2)(v_0 - v_2)}\Big(e^{-v_2 t} - e^{-v_0 t}\Big) + \frac{1}{(v_1 - v_2)(v_1 - v_0)}\Big(e^{-v_1 t} - e^{-v_0 t}\Big)$$
$$+ \frac{1}{(v_1 - v_3)(v_1 - v_0)}\Big(e^{-v_0 t} - e^{-v_1 t}\Big) + \frac{1}{(v_1 - v_3)(v_0 - v_3)}\Big(e^{-v_0 t} - e^{-v_3 t}\Big)\Big]$$

$$P_{03}(t) = \frac{v_0 v_1 v_2}{v_3 - v_2}\Big[\frac{v_2 - v_3}{(v_0 - v_1)(v_0 - v_2)(v_0 - v_3)}e^{-v_0 t} - \frac{v_2 - v_3}{(v_0 - v_1)(v_1 - v_2)(v_1 - v_3)}e^{-v_1 t}$$
$$+ \frac{1}{(v_1 - v_2)(v_0 - v_2)}e^{-v_2 t} - \frac{1}{(v_1 - v_3)(v_0 - v_3)}e^{-v_3 t}\Big] \tag{72}$$

A.0.11 Calculation of $P_{44}(t)$

$$P'_{44}(t) = 1 \tag{73}$$

A.0.12 Calculation of $P_{34}(t)$

$$P'_{34}(t) = q_{34} \cancel{P_{44}(t)} \to 1 - v_3 P_{34}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{34}(s) = \frac{v_3}{s} - v_3 \tilde{P}_{34}(s)$$

$$\tilde{P}_{34}(s) = \frac{v_3}{s(s+v_3)} = \frac{a}{s} + \frac{b}{(s+v_3)} = \frac{s(a+b)+av_3}{s(s+v_3)} \longrightarrow \begin{cases} a = 1 \\ b = -1 \end{cases}$$

$$\tilde{P}_{34}(s) = \frac{1}{s} - \frac{1}{(s+v_3)} \xrightarrow{\mathscr{L}^{-1}} P_{34}(t) = 1 - e^{-v_3 t} \tag{74}$$

A.0.13 Calculation of $P_{24}(t)$

$$P'_{24}(t) = q_{23} P_{34}(t) - v_2 P_{24}(t) \xrightarrow{\mathscr{L}} S\tilde{P}_{24}(s) = v_2 \tilde{P}_{34}(s) - v_2 \tilde{P}_{34}(s) \xrightarrow{\frac{1}{s} - \frac{1}{(s+v_3)}}$$

$$\tilde{P}_{24}(s) = \underbrace{\frac{v_2}{s(s+v_2)}}_{\text{Term I}} - \underbrace{\frac{v_2}{(s+v_2)(s+v_3)}}_{\text{Term II}}$$

$$\text{Term I} = \frac{a}{s} + \frac{b}{(s+v_2)} = \frac{s(a+b)+av_2}{s(s+v_2)} \longrightarrow \begin{cases} a = 1 \\ b = -1 \end{cases}$$

$$\text{Term II} = \frac{a^{'}}{(s+v_3)} + \frac{b^{'}}{(s+v_2)} = \frac{s(a^{'}+b^{'}) + a^{'}v_2 + b^{'}v_3}{(s+v_2)(s+v_3)} \longrightarrow \begin{cases} a^{'} = \frac{v_2}{v_2-v_3} \\ b^{'} = \frac{-v_2}{v_2-v_3} \end{cases}$$

$$\tilde{P}_{24}(s) = v_2\Big(\frac{\frac{1}{v_2-v_3}}{s+v_2} - \frac{\frac{1}{v_2-v_3}}{s+v_3} + \frac{\frac{1}{v_2}}{s} - \frac{\frac{1}{v_2}}{s+v_2}\Big)$$

$$P_{24}(t) = \frac{v_2}{v_2-v_3}\big(e^{-v_2t} - e^{-v_3t}\big) + \big(1 - e^{-v_2t}\big)$$

$$P_{24}(t) = 1 + \frac{v_3}{v_2-v_3}e^{-v_2t} - \frac{v_2}{v_2-v_3}e^{-v_3t} \tag{75}$$

A.0.14 Calculation of $P_{14}(t)$

$$P_{14}^{'}(t) = q_{12}P_{24}(t) - v_1P_{14}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{14}(s) = v_1\tilde{P}_{24}(s) - v_1\tilde{P}_{14}(s) \xrightarrow{\tilde{P}_{24}(s)}$$

$$\tilde{P}_{14}(s) = \frac{v_1v_2}{v_2-v_3}\Big(\underbrace{\frac{1}{(s+v_1)(s+v_2)}}_{\text{Term I}} - \underbrace{\frac{1}{(s+v_1)(s+v_3)}}_{\text{Term II}}\Big) + v_1\Big(\underbrace{\frac{1}{s(s+v_1)}}_{\text{Term III}} - \underbrace{\frac{1}{(s+v_1)(s+v_2)}}_{\text{Term IV}}\Big)$$

$$\text{Term I} = \text{Term IV} = \frac{a}{(s+v_2)} + \frac{b}{(s+v_1)} = \frac{s(a+b)+av_1+bv_2}{(s+v_1)(s+v_2)} \longrightarrow \begin{cases} a = \frac{1}{v_1-v_2} \\ b = \frac{-1}{v_1-v_2} \end{cases}$$

$$\text{Term II} = \frac{a^{'}}{(s+v_3)} + \frac{b^{'}}{(s+v_1)} = \frac{s(a^{'}+b^{'})+a^{'}v_1+b^{'}v_3}{(s+v_3)(s+v_1)} \longrightarrow \begin{cases} a^{'} = \frac{1}{v_1-v_3} \\ b^{'} = \frac{-1}{v_1-v_3} \end{cases}$$

$$\text{Term III} = \frac{a^{''}}{s} + \frac{b^{''}}{(s+v_1)} = \frac{s(a^{''}+b^{''})+a^{''}v_1}{s(s+v_1)} \longrightarrow \begin{cases} a^{''} = \frac{1}{v_1} \\ b^{''} = -\frac{1}{v_1} \end{cases}$$

$$\begin{aligned} \tilde{P}_{14}(s) &= \frac{v_1v_2}{v_2-v_3}\Big[\frac{1}{v_1-v_2}\Big(\frac{1}{s+v_2}-\frac{1}{s+v_1}\Big)+\frac{1}{v_1-v_3}\Big(\frac{1}{s+v_1}-\frac{1}{s+v_3}\Big)\Big] \\ &+ v_1\Big[\frac{1}{v_1}\Big(\frac{1}{s}-\frac{1}{s+v_1}\Big)+\frac{1}{v_1-v_2}\Big(\frac{1}{s+v_1}-\frac{1}{s+v_2}\Big)\Big] \end{aligned}$$

$$\begin{aligned} P_{14}(t) &= \frac{v_1v_2}{v_2-v_3}\Big[\frac{1}{v_1-v_2}\Big(e^{-v_2t}-e^{-v_1t}\Big)+\frac{1}{v_1-v_3}\Big(e^{-v_1t}-e^{-v_3t}\Big)\Big] \\ &+ v_1\Big[\frac{1}{v_1}\Big(1-e^{-v_1t}\Big)+\frac{1}{v_1-v_2}\Big(e^{-v_1t}-e^{-v_2t}\Big)\Big] \end{aligned}$$

$$\begin{aligned} P_{14}(t) &= \frac{-v_2v_3}{(v_1-v_2)(v_1-v_3)}e^{-v_1t}+\frac{v_1v_3}{(v_1-v_2)(v_2-v_3)}e^{-v_2t} \\ &- \frac{v_1v_2}{(v_2-v_3)(v_1-v_3)}e^{-v_3t}+1 \end{aligned} \tag{76}$$

A.0.15 Calculation of $P_{04}(t)$

$$P_{04}^{'}(t) = q_{01}P_{14}(t)-v_0P_{04}(t) \xrightarrow{\mathscr{L}} s\tilde{P}_{04}(s) = v_0\tilde{P}_{14}(s)-v_0\tilde{P}_{04}(s) \xrightarrow{\tilde{P}_{14}(s)}$$

$$\tilde{P}_{04}(s) = v_0\Big[\Big(\frac{-v_2v_3}{(v_1-v_2)(v_1-v_3)} \times \underbrace{\frac{1}{(s+v_0)(s+v_1)}}_{\text{Term I}}\Big) + \Big(\frac{v_1v_3}{(v_1-v_2)(v_2-v_3)} \times \underbrace{\frac{1}{(s+v_0)(s+v_2)}}_{\text{Term II}}\Big)$$
$$+ \Big(\frac{-v_1v_2}{(v_2-v_3)(v_1-v_3)} \times \underbrace{\frac{1}{(s+v_0)(s+v_3)}}_{\text{Term III}}\Big) + \Big(\underbrace{\frac{1}{s(s+v_0)}}_{\text{Term IV}}\Big)\Big] \quad (77)$$

$$\text{Term I} = \frac{a}{(s+v_0)} + \frac{b}{(s+v_1)} = \frac{s(a+b)+av_1+bv_0}{(s+v_0)(s+v_1)} \longrightarrow \begin{cases} a = \frac{1}{v_1-v_0} \\ b = \frac{-1}{v_1-v_0} \end{cases}$$

$$\text{Term II} = \frac{a^{'}}{(s+v_0)} + \frac{b^{'}}{(s+v_2)} = \frac{s(a^{'}+b^{'})+a^{'}v_2+b^{'}v_0}{(s+v_0)(s+v_2)} \longrightarrow \begin{cases} a^{'} = \frac{1}{v_2-v_0} \\ b^{'} = \frac{-1}{v_2-v_0} \end{cases}$$

$$\text{Term III} = \frac{a^{''}}{(s+v_0)} + \frac{b^{''}}{(s+v_3)} = \frac{s(a^{''}+b^{''})+a^{''}v_3+b^{''}v_0}{(s+v_0)(s+v_3)} \longrightarrow \begin{cases} a^{''} = \frac{1}{v_3-v_0} \\ b^{''} = \frac{-1}{v_3-v_0} \end{cases}$$

$$\text{Term IV} = \frac{c}{s} + \frac{d}{(s+v_0)} = \frac{s(c+d)+cv_0}{s(s+v_0)} \longrightarrow \begin{cases} c = \frac{1}{v_0} \\ d = -\frac{1}{v_0} \end{cases}$$

$$\tilde{P}_{04}(s) = \frac{v_0v_2v_3}{(v_1-v_2)(v_1-v_3)} \times \Big(\frac{\frac{1}{v_1-v_0}}{s+v_1} - \frac{\frac{1}{v_1-v_0}}{s+v_0}\Big) + \frac{v_0v_1v_3}{(v_1-v_2)(v_2-v_3)} \times \Big(\frac{\frac{1}{v_2-v_0}}{s+v_0} - \frac{\frac{1}{v_2-v_0}}{s+v_2}\Big)$$
$$+ \frac{-v_0v_1v_2}{(v_2-v_3)(v_1-v_3)} \times \Big(\frac{\frac{1}{v_3-v_0}}{s+v_3} - \frac{\frac{1}{v_3-v_0}}{s+v_0}\Big) + v_0\Big(\frac{\frac{1}{v_0}}{s} - \frac{\frac{1}{v_0}}{s+v_0}\Big) \xrightarrow{\mathscr{L}^{-1}}$$

$$
\begin{aligned}
P_{04}(t) &= \frac{v_0 v_2 v_3}{(v_1-v_0)(v_1-v_2)(v_1-v_3)} \times \left(e^{-v_1 t} - e^{-v_0 t}\right) \\
&+ \frac{v_0 v_1 v_3}{(v_2-v_0)(v_1-v_2)(v_2-v_3)} \times \left(e^{-v_0 t} - e^{-v_2 t}\right) \\
&+ \frac{v_0 v_1 v_2}{(v_3-v_0)(v_2-v_3)(v_1-v_3)} \times \left(e^{-v_3 t} - e^{-v_0 t}\right) \\
&+ 1 - e^{-v_0 t}
\end{aligned}
$$

$$
\begin{aligned}
P_{04}(t) &= \left(1 + \frac{v_0^3 - v_0^2 v_1 - v_0^2 v_2 - v_3^2 v_2 + v_0 v_1 v_2 + v_0 v_1 v_3 + v_0 v_2 v_3}{(v_0-v_1)(v_1-v_2)(v_1-v_3)}\right) \times e^{-v_0 t} \\
&+ \frac{v_0 v_2 v_3}{(v_1-v_0)(v_1-v_2)(v_1-v_3)} \times e^{-v_1 t} \\
&- \frac{v_0 v_1 v_3}{(v_0-v_2)(v_1-v_2)(v_2-v_3)} \times e^{-v_2 t} \\
&+ \frac{v_0 v_1 v_2}{(v_0-v_3)(v_2-v_3)(v_1-v_3)} \times e^{-v_3 t} \\
&+ 1
\end{aligned} \tag{78}
$$

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