Computability of Function Spaces from Harmonic Analysis

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ABSTRACT

Computability of Function Spaces from Harmonic Analysis

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In computer science and mathematics, a computable function is one for which a computer program exists and can give its values in finite time. We explore notions of computability for function spaces such as the Hardy space $H^1(\mathbb{R})$ and the Besov space $B^p(\mathbb{T})$ from harmonic analysis. After a comprehensive introduction to computable analysis and studying several function spaces, we establish some original results. Namely, that there exists a dense computability structure on the Besov space.

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Chapter 1

Introduction

The theory of computation (or recursion theory) deals with the study of computable functions. Broadly speaking, in computer science and mathematics, a computable function is one for which a computer program exists and can give its values in finite time. From another point of view, such objects are said to be computable if they can be approximated by other objects of the same nature which are already known to be computable. The existence of an approximation to a function, for example, can be seen as the mathematical analogue of what computer scientists refer to as the existence of a Turing Machine for that function. Thanks to Alan Turing, the theory of computability has been very popular within the field of theoretical computer science. However, computability within the context of mathematical analysis is not as common. In fact, one can see that mathematicians such as Banach and Mazur in [1] and Grzegorczyk in [10] were studying such concepts as early as 1937. Recent work in the field of computability involves more well-known concepts in mathematics, such as Julia sets in [4], harmonic measure in [2], or even the Blaschke product in [15].

The interaction between computability and analysis can lead to surprising results. For instance, it was found in [18] that there exist solutions of the form $u(x_1, ..., x_n, t)$ to the wave equation which are computable functions at time $t = 0$, but not at time $t = 1$. Moreover, mathematicians have developed several theories and definitions in order to incorporate the notion of computability in familiar settings such as Banach spaces, and metric spaces [3]. Despite this, the theory of computability within analysis is still not uniform and there exist many seemingly different schools of thought within the field. As such, the intersection between analysis and computability remains wide open.

Our goal is to explore notions of computability for function spaces such as the Hardy space $H^1(\mathbb{R})$ and the Besov space $B^p(\mathbb{T})$ from harmonic analysis. Chapter 2 gives a comprehensive overview of many notions from computable analysis (the analysis of computable objects in mathematics), which range from the most basic notion of a computable function $f : \mathbb{N} \to \mathbb{N}$ to more advanced notions such as computable metric spaces and computability structures. Next, Chapter 3 gives some background on the harmonic analysis side and is presented in a way that highlights the progression of ideas and spaces for which we wish to establish notions of computability. It begins with the Hardy space and ends with the Besov space, the latter begin the space which our results are concerned with. Finally, Chapter 4 presents original results, namely that there exists a dense computability structure on the Besov space.

To the best of our knowledge, notions involving the computability of function spaces found in harmonic analysis have not previously been studied. As such, we end by noting that our work is only the beginning of a long list of possible results concerning the intersection between the two fields and state a couple of open questions which would be natural to address given our results.

Chapter 2

Background in Computability

In order to introduce concepts of computability in the context of real function spaces, one needs to define what it is for a function from $\mathbb{N} \to \mathbb{R}$ to be computable and the reason for this will be clear in our work to follow. To do this, we build up to such a definition, starting from the naturals and working our way up to the reals. This chapter is meant to be a self contained introduction to computability, as it is still a new branch of mathematical logic. Most of the notions presented in this chapter can be found in [12, 18, 17, 23, 19] and [6]. First, we begin with functions mapping to the natural numbers.

2.1 Computability on the Naturals

We denote the set $\mathbb{N} \setminus \{0\}$ by $\mathbb{N}_+$.

Definition 2.1.1. Let $n \in \mathbb{N}_+$. For $i \in \{1, \ldots, n\}$, we define the functions

- $I_i^n : \mathbb{N}^k \to \mathbb{N}, \quad I_i^n(x_1, \ldots, x_n) = x_i$, the projection function onto the i-th coordinate of an n tuple of integers,
- $s : \mathbb{N} \to \mathbb{N}, \quad s(x) = x + 1$, the successor function,
- $z : \mathbb{N} \to \mathbb{N}, \quad z(x) = 0$, the zero function.

We say that I_i^n, s, and z are initial functions.

These functions are called initial functions since, as we shall see shortly, they are at the base of all computable functions mapping to the naturals.

Definition 2.1.2. Let $n, k \in \mathbb{N}_+$. Let $g : \mathbb{N}^n \to \mathbb{N}$ and $f_1, f_2, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$. Now, consider the function $h : \mathbb{N}^k \to \mathbb{N}$,

$$h(x_1, \ldots, x_k) = g\big(f_1(x_1, \ldots, x_k), ..., f_n(x_1, \ldots, x_k)\big).$$

We say that h is obtained by composition of the functions g and $f_1, ..., f_n$.

Definition 2.1.3. Suppose $f : \mathbb{N}^n \to \mathbb{N}$ and $g : \mathbb{N}^{n+2} \to \mathbb{N}$. For $h : \mathbb{N}^{n+1} \to \mathbb{N}$, if

$$h(0, y_1, \ldots, y_n) = f(y_1, \ldots, y_n)$$
$$h(x + 1, y_1, \ldots, y_n) = g\big(h(x, y_1, \ldots, y_n), x, y_1, \ldots, y_n\big),$$

then h is said to be obtained by primitive recursion from f and g.

Let $n \in \mathbb{N}_+$ and let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be a function such that for any $(x_1, \ldots, x_n) \in \mathbb{N}^n$, there exists a $y \in \mathbb{N}$ such that $g(x_1, \ldots, x_n, y) = 0$. Now consider the function $f : \mathbb{N}^n \to \mathbb{N}$ be defined as:

$$f(x_1, \ldots, x_n) = \min\{y \in \mathbb{N} : g(x_1, \ldots, x_n, y) = 0\}.$$

The process of obtaining the smallest such integer is one which will be important later on. For this reason, we define it as the μ - operator.

Definition 2.1.4. For $n \in \mathbb{N}_+$ and g defined as above, we say that f is obtained from g using the μ-operator if

$$\mu^y\big(g(x_1, \ldots, x_n, y) = 0\big) = \min\{y \in \mathbb{N} : g(x_1, \ldots, x_n, y) = 0\} = f(x_1, \ldots, x_n),$$

We have now defined the most basic operations and functions with range $\mathbb{N}$ and are ready to introduce the concept of computability of a function.

Definition 2.1.5. For $k \in \mathbb{N}_+$, we say a function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if it can be obtained in finitely many steps from initial functions using either composition, primitive recursion or the μ-operator.

In the literature, the terms computable and recursive are often used interchangeably.

Each initial function is trivially computable. Because of this, there are many ways of constructing a computable function from simpler functions which are known to be computable. This is in fact how one can show that a given function is computable and we illustrate this idea below.

Example 2.1.1. *Let $k \in \mathbb{N}_+$. For $a \in \mathbb{N}$ let $c_a : \mathbb{N}^k \to \mathbb{N}$*

$$c_a(x) = a \quad \forall x \in \mathbb{N}^k.$$

Then c_a is computable for any $a \in \mathbb{N}$. This can be seen by induction on a. First, we have that $c_0 : \mathbb{N}^k \to \mathbb{N}$

$$c_0(x) = 0, \quad \forall x \in \mathbb{N}^k.$$

Since

$$c_0(x) = z(I_1^k(x)),$$

where z is the zero function. From this, it follows that c_0 is computable. Assume now that c_a is computable for some $a \in \mathbb{N}$. We have that

$$c_{a+1}(x) = a + 1 = s(a) = s(c_a(x)),$$

where s is the successor function. So, as c_{a+1} is a composition of computable functions, it must be computable. Thus, c_a is computable for all $a \in \mathbb{N}$ by induction.

Suppose we wish to show that a function on $\mathbb{N}$ is computable. To do so, we would need to use Definition 2.1.3 with a suitable function whose domain is one dimension less than that of the set $\mathbb{N}$, which does not make sense. Hence, in order to show that a function on $\mathbb{N}$ is computable, the following proposition and its corollary can be used instead.

Proposition 2.1.6. *Let $a \in \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ a computable function. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by*

$$h(0) = a$$
$$h(x+1) = g(h(x), x), \quad \forall x \in \mathbb{N}.$$

Then h is computable.

Proof. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be defined by

$$H(x, y) = h(x).$$

Since for any $x \in \mathbb{N}_+$, $h(x) = H(x, 0)$, it suffices to show that H is computable. To that end, we see that

$$H(0, y) = h(0) = c_a(y),$$
$$H(x + 1, y) = g(H(x, y), x).$$

Letting now

$$F : \mathbb{N} \to \mathbb{N}, f(y) = c_a(y)$$
$$G : \mathbb{N}^3 \to \mathbb{N}, G(a, b, c) = g(I_1^3(a, b, c), I_2^3(a, b, c)),$$

which are both computable by definition or by composition of computable functions, we get that H is computable, since it can be obtained by primitive recursion of F and G. □

Corollary 2.1.7. *Let $a \in \mathbb{N}$ and let $g : \mathbb{N} \to \mathbb{N}$ be a computable function. Now consider $h : \mathbb{N} \to \mathbb{N}$ defined by*

$$h(0) = a$$
$$h(x + 1) = g(h(x)).$$

Then h is computable.

Proof. Define the function $g' : \mathbb{N}^2 \to \mathbb{N}$ by $g'(x, y) = g(x)$. Then g' is computable since it is the composition of g and the projection function onto the first coordinate. Moreover, we have

$$h(0) = a$$
$$h(x + 1) = g'(h(x), x).$$

Hence, by Proposition 2.1.6, it follows that h is computable. □

In fact, the result which is more useful in showing that a function mapping $\mathbb{N}$ to $\mathbb{N}$ is computable is Corollary 2.1.7, simply because it is simpler to apply. Now we show some examples of computable functions which are obtained from other computable functions.

Example 2.1.2. *Let* $f : \mathbb{N}^2 \to \mathbb{N}$ *be a computable function and let* $g : \mathbb{N}^2 \to \mathbb{N}$ *be defined by*

$$g(x, y) = f(y, x).$$

Then g *is computable, since it can be obtained by composition of projection functions and* f*. Namely,*

$$g(x, y) = f(I_2^2(x, y), I_1^2(x, y)).$$

Consider the functions $\text{sgn}, \overline{\text{sgn}} : \mathbb{N} \to \mathbb{N}$ defined by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{\text{sgn}}(x) = \begin{cases} 0 & \text{if } x \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

These two functions will be very useful in proofs to come, so, for the sake of exposition, we show that they are computable.

Proposition 2.1.8. sgn *and* $\overline{\text{sgn}}$ *are computable functions.*

Proof. Define $g, g' : \mathbb{N}^2 \to \mathbb{N}$ by $g(x, y) = 1$ and $g'(x, y) = 0 \quad \forall (x, y) \in \mathbb{N}^2$. We first note that g and g' are computable by Example 2.1.1, since they are just constant functions. Moreover, we have

$$\text{sgn}(0) = 0$$
$$\text{sgn}(x + 1) = g(\text{sgn}(x), x).$$

and

$$\overline{\text{sgn}}(0) = 1$$
$$\overline{\text{sgn}}(x + 1) = g'(\overline{\text{sgn}}(x), x).$$

Hence, by Proposition 2.1.6, sgn and $\overline{\text{sgn}}$ are computable. □

Example 2.1.3. *The function* $\overline{s} : \mathbb{N} \to \mathbb{N}$

$$\overline{s}(x) = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

is computable. This can be seen by considering the function $g : \mathbb{N}^2 \to \mathbb{N}, \quad g(x, y) = I_2^2(x, y)$, *which is computable since it is an initial function. Then we have that*

$$\overline{s}(0) = 0$$
$$\overline{s}(x + 1) = g(\overline{s}(x), x),$$

Hence, by Proposition 2.1.6, $\overline{s}$ *is computable.*

Since, until now, we have only considered functions whose range is $\mathbb{N}$, it is natural to define a modified subtraction. That is, we require a function that, gives the difference of two natural numbers when it is nonnegative, and 0 when the difference would be negative.

Definition 2.1.9. For $x, y \in \mathbb{N}$ we define the modified subtraction by

$$x \;\dot{-}\; y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.1.10. *Let* $f : \mathbb{N}^2 \to \mathbb{N}, \quad f(x, y) = x \;\dot{-}\; y$. *Then* f *is computable.*

Proof. Define $g : \mathbb{N}^2 \to \mathbb{N}$ by

$$g(x, y) = f(y, x).$$

By Example 2.1.2, in order to show that f is computable, it is enough to show that g is computable. To do this, we must find computable functions $F : \mathbb{N} \to \mathbb{N}$ and $G : \mathbb{N}^3 \to \mathbb{N}$ such that g can be obtained by primitive recursion using F and G, as in Definition 2.1.3. To that end, we can see that

$$g(0, y) = y \;\dot{-}\; 0 = I_1^1(y)$$
$$g(x + 1, y) = y \;\dot{-}\; (x + 1) = \overline{s}(I_1^3(g(x, y), x, y)),$$

where $\overline{s}$ is the computable function from Example 2.1.3. So letting $F(a) = I_1^1(a)$ and $G(a,b,c) = \overline{s}(I_1^3(a,b,c))$, we have that these two functions are computable by composition and are such that

$$g(0,y) = F(y)$$
$$g(x+1,y) = G(g(x,y),x,y).$$

□

The proposition below will be assumed to be true, without proof, but it can easily be shown using primitive recursion. It will be useful in proving the statements to come.

Proposition 2.1.11. *Suppose $f, g : \mathbb{N}^k \to \mathbb{N}$ are computable functions. Then, $f+g$ and fg are computable as functions from $\mathbb{N}^k$ to $\mathbb{N}$.*

Corollary 2.1.12. *Let $f_1, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$ be computable functions. Then $f_1 + \cdots + f_n$ and $f_1 \cdot \cdots \cdot f_n$: are computable as functions from $\mathbb{N}^k$ to $\mathbb{N}$.*

Another common operation is that of taking the absolute value of the difference between two natural numbers. Hence, we consider the example below, along with sketch of the proof as to why this function is computable.

Example 2.1.4. *Let $A : \mathbb{N}^2 \to \mathbb{N}, \quad A(x,y) = |x-y|$. Then A is computable, since for all $x, y \in \mathbb{N}$, we have*

$$|x-y| = (x \ \dot{-}\ y) + (y \ \dot{-}\ x).$$

Indeed, this can be seen by checking that the left and right hand sides of the equation are equal in two cases: when $x \leq y$ and when $y \leq x$. Let $f, g : \mathbb{N}^2 \to \mathbb{N}$ be the computable functions from Proposition 2.1.10 and the proof of this same proposition, respectively. Then,

$$A(x,y) = f(x,y) + g(x,y).$$

So by Corollary 2.1.12, it follows that A is computable.

We continue our exposition by considering the some more common operations between natural numbers, such as the floor and remainder of a division.

Example 2.1.5. *Let $f : \mathbb{N}^2 \to \mathbb{N}$ be defined by*

$$f(x,y) = \begin{cases} \left\lfloor \frac{x}{y} \right\rfloor & \text{if } y \geq 1 \\ 0 & \text{if } y = 0. \end{cases}$$

The fact that f is computable can be seen as follows: let $x, y \in \mathbb{N}, \quad y \geq 1$ and let $k = \left\lfloor \frac{x}{y} \right\rfloor$. Then

$$k \leq \frac{x}{y} < k+1 \implies ky \leq x < (k+1)y.$$

Thus,

$$k = \min\{z \in \mathbb{N} : x < (z+1)y\}.$$

We now note that for $a, b \in \mathbb{N}$

$$a < b \iff b \dot{-} a > 0 \iff \overline{\text{sgn}}(b \dot{-} a) = 0.$$

By this fact, we have that for $x, y, z \in \mathbb{N}$

$$x < (z+1)y \iff \overline{\text{sgn}}((z+1)y \dot{-} x) = 0. \qquad (*)$$

Now define $g : \mathbb{N}^3 \to \mathbb{N}$ by $g(x,y,z) = \overline{\text{sgn}}((z+1)y \dot{-} x)\,\text{sgn}(y)$. We can see that g is computable by considering the functions $t, u, v : \mathbb{N}^3 \to \mathbb{N}$

$$t(x,y,z) = (z+1)y = (s \circ I_3^3) I_2^3(x,y,z)$$
$$u(x,y,z) = t(x,y,z) \dot{-} x$$
$$v(x,y,z) = \text{sgn}(y)$$

which are all computable functions (since they are compositions of functions which were previously shown to be computable). Then, g is computable since it is the product of two computable functions, namely $g(x,y,z) = u(x,y,z) \cdot v(x,y,z)$. Moreover, it follows from $(*)$ that

$$f(x,y) = \mu^z(g(x,y,z) = 0) \qquad \forall x, y, z \in \mathbb{N}.$$

So f can be obtained using the μ-operator with a computable function, meaning that f is computable.

Example 2.1.6. *Let f be defined as in Example 2.1.5. Let $g : \mathbb{N}^2 \to \mathbb{N}$ be defined by*

$$g(x, y) = x \dot{-} f(x, y) \cdot y.$$

This function, which in fact turns out to be the remainder of the division of x by y (if $y > 0$), is computable. We first note that the we know there exist $u, v \in \mathbb{N}$ such that $x = yu + v$, and $0 \leq v < y$. Hence, dividing by y on both sides yields

$$\frac{x}{y} = u + \frac{v}{y} \quad \text{and} \quad 0 \leq \frac{v}{y} < 1.$$

It can therefore be seen that $\left\lfloor \frac{x}{y} \right\rfloor = u$. Thus,

$$x = y \cdot \left\lfloor \frac{x}{y} \right\rfloor + v \implies v = x - y \cdot \left\lfloor \frac{x}{y} \right\rfloor.$$

So, this means that $g(x, y) = v$ and hence g is computable, since the modified subtraction is computable.

Following the work we presented, one can consider other functions with range being $\mathbb{N}$ and show that they are computable. In other words, the possibilities are endless. However, we now move on to other notions of computability within the context of the natural numbers. For example, computability notions can also be defined when subsets of $\mathbb{N}^k$ are involved.

Definition 2.1.13. Let $S \subseteq \mathbb{N}^k$. We way that S is a computable set in $\mathbb{N}^k$ if the function $\chi_S : \mathbb{N}^k \to \mathbb{N}$ is computable.

Proposition 2.1.14. *Let S and T be computable sets in $\mathbb{N}^k$. Then $S \cup T$, $S \cap T$ and S^c are computable.*

Proof. Since S and T are computable, we know that the functions χ_S and χ_T are computable. Moreover, for each $x \in \mathbb{N}^k$, we have

$$\chi_{S \cup T}(x) = \operatorname{sgn}(\chi_S(x) + \chi_T(x)), \quad \chi_{S \cap T}(x) = \chi_S(x) \cdot \chi_T(x), \quad \chi_{S^c}(x) = 1 \dot{-} \chi_S(x).$$

These are computable functions by either composition of functions which we know are computable (for the first and last) or Corollary 2.1.13 (the second). □

Proposition 2.1.15. *Suppose $f : \mathbb{N}^k \to \mathbb{N}$ is computable. Then, the set*

$$S = \left\{x \in \mathbb{N}^k : f(x) = 0\right\}$$

is computable.

Proof. We can see that $\chi_S(x) = \overline{\text{sgn}}(f(x))$, and that this function is computable by arguments similar to those used before. □

Proposition 2.1.16. *Suppose $f, g : \mathbb{N}^k \to \mathbb{N}$ are computable. Then, the set*

$$S = \left\{x \in \mathbb{N}^k : f(x) = g(x)\right\}$$

is computable.

Proof. We can write

$$S = \left\{x \in \mathbb{N}^k : A(f(x), g(x)) = 0\right\},$$

where $A(x, y) = |x - y|$ is as defined in Example 2.1.4. It then follows that

$$\chi_S(x) = \overline{\text{sgn}}(A(f(x), g(x))),$$

which is computable. □

Example 2.1.7. *The set $\Delta = \{(x, y) : y|x\}$ is computable. This can be seen by letting g be the function from Example 2.1.6 and $x, y \in \mathbb{N}$. We have*

$$y|x \iff g(x, y) = 0.$$

So,

$$\chi_\Delta(x, y) = \overline{\text{sgn}}(g(x, y)) \qquad \forall x, y \in \mathbb{N}.$$

And so since g and $\overline{\text{sgn}}$ are computable functions, it follows that χ_Δ is computable by composition and hence that Δ is a computable set.

Example 2.1.8. *The set $2\mathbb{N} = \{2x : x \in \mathbb{N}\}$ is computable. We can let Δ be the set from Example 2.1.7 and therefore have*

$$\chi_{2\mathbb{N}}(x) = \chi_\Delta(x, c_2(x)),$$

which is a computable function by composition.

Our brief exposition of computable subsets of $\mathbb{N}^k$ now allows us consider functions which can be defined by cases and show that they can be computable under the right assumptions. This is in fact quite powerful, as it widens the set of functions for which computability can be established.

Proposition 2.1.17. *Let $k, n \in \mathbb{N}_+$. Let $f_1, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$ be computable functions and let $S_1, \ldots, S_n \subseteq \mathbb{N}^k$ be computable sets. Assume that for each $x \in \mathbb{N}^k$ there exists a unique $i \in \{1, \ldots, n\}$ such that $x \in S_i$. Then, the function $F : \mathbb{N}^k \to \mathbb{N}$ defined by*

$$F(x) = \begin{cases} f_1(x) & \text{if } x \in S_1 \\ & \vdots \\ f_n(x) & \text{if } x \in S_n \end{cases}$$

is computable.

Proof. For each $x \in \mathbb{N}^k$ we have

$$F(x) = f_1(x)\chi_{S_1}(x) + \cdots + f_n(x)\chi_{S_n}(x).$$

Since all of the sets in question are computable, so are their characteristic functions. It follows from Corollary 2.1.12 that F is a computable function. $\square$

We now turn to results involving functions who have higher dimensional ranges.

Definition 2.1.18. Let $n \in \mathbb{N}_+$ and $f : \mathbb{N}^k \to \mathbb{N}^n$. Then there exist unique functions $f_1, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$ such that for any $x \in \mathbb{N}^k$,

$$f(x) = (f_1(x), \ldots, f_n(x)).$$

We say that $f_1, \ldots, f_n$ are the component functions of f.

Definition 2.1.19. A function $f : \mathbb{N}^k \to \mathbb{N}^n$ is computable if and only if each of its component functions are computable.

Proposition 2.1.20. *Let $n \in \mathbb{N}_+$, $f : \mathbb{N}^k \to \mathbb{N}^n$ and $g : \mathbb{N}^n \to \mathbb{N}$ be computable functions. Then, $g \circ f : \mathbb{N}^k \to \mathbb{N}$ is a computable function.*

Proof. Let $f_1, \dots, f_n : \mathbb{N}^k \to \mathbb{N}$ be the component functions of f and $x \in \mathbb{N}^k$. We have

$$g \circ f(x) = g(f(x)) = g(f_1(x), \dots, f_n(x)).$$

Since g and the component functions of f are computable, it follows that $g \circ f$ is computable by composition. $\square$

Proposition 2.1.21. *Let $n, l \in \mathbb{N}_+$, $f : \mathbb{N}^k \to \mathbb{N}^n$ and $g : \mathbb{N}^n \to \mathbb{N}^l$ be computable functions. Then, $g \circ f : \mathbb{N}^k \to \mathbb{N}^l$ is a computable function.*

Proof. Let $x \in \mathbb{N}^k$. We have

$$g \circ f(x) = g(f(x)) = (g_1(f(x)), \dots, g_l(f(x))) = (g_1 \circ f(x), \dots, g_l \circ f(x)).$$

By Proposition 2.1.20, we have that $g_i \circ f$ is computable for each $i \in \{1, \dots, l\}$. Hence, the component functions of $g \circ f$ are computable and so $g \circ f$ is a computable function. $\square$

The definitions above only deal with functions whose range is the natural numbers. Because we wish to see what it means for a function ranging on the reals to be computable, we must therefore work our way up to such a definition. In other words, we must go from the naturals, to the rationals, to the reals. This takes us to the next step, which is computability on the integers.

2.2 Computability on the Integers

Here we now consider functions from $\mathbb{N}^k$ to $\mathbb{Z}$. Since to any integer z, one can associate a pair of natural numbers r, p via the expression $z = (-1)^r p$, the definition of computable functions with range $\mathbb{Z}$ follows quite nicely. In this subsection, we let $k \in \mathbb{N}_+$, unless stated otherwise.

Definition 2.2.1. We say that $f : \mathbb{N}^k \to \mathbb{Z}$ is computable if there exist computable functions $p, r : \mathbb{N}^k \to \mathbb{N}$ such that for any $x \in \mathbb{N}^k$

$$f(x) = (-1)^{r(x)} p(x).$$

It is also known that an integer can be written as the difference between two natural numbers. This yields an equivalent characterization of computable functions with range $\mathbb{Z}$.

Proposition 2.2.2. *Let $f, g : \mathbb{N}^k \to \mathbb{N}$ be computable functions. Let $h : \mathbb{N}^k \to \mathbb{Z}$ be defined by*

$$h(x) = f(x) - g(x).$$

Then h is computable in the sense of Definition 2.2.1.

Proof. We let $p : \mathbb{N}^k \to \mathbb{N}$ be

$$p(x) = |f(x) - g(x)| = A(f(x), g(x)),$$

where $A : \mathbb{N}^2 \to \mathbb{N}$ is the function from Example 2.1.4. So, p is computable by composition, since A, f, g are computable. Moreover, we can consider the set $S = \{x \in \mathbb{N}^k : f(x) < g(x)\}$ and define the function $r : \mathbb{N}^k \to \mathbb{N}$ by

$$r(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}.$$

We can see that S is a computable set since we can express its characteristic function as $\chi_S(x) = \operatorname{sgn}(g(x) \dot{-} f(x))$, which is a computable function. Thus, r is a computable function and since we have that for any $x \in \mathbb{N}^k$

$$h(x) = (-1)^{r(x)} p(x),$$

it follows that h is computable. □

Proposition 2.2.3. *Suppose $f : \mathbb{N}^k \to \mathbb{Z}$ is computable. Then there exist computable functions $u, v : \mathbb{N}^k \to \mathbb{N}$ such that*

$$f(x) = u(x) - v(x) \quad \forall x \in \mathbb{N}^k.$$

Proof. Suppose f is computable. Then, there exist computable functions $p, r : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = (-1)^{r(x)} p(x).$$

We notice now that we can write

$$f(x) = \begin{cases} c_0(x) - p(x) & \text{if } r(x) \text{ is odd} \\ p(x) - c_0(x) & \text{if } r(x) \text{ is even} \end{cases}.$$

Consider the functions

$$u(x) = \begin{cases} c_0(x) & \text{if } x \in S^c \\ p(x) & \text{if otherwise} \end{cases}.$$

and

$$v(x) = \begin{cases} p(x) & \text{if } x \in S^c \\ c_0(x) & \text{otherwise} \end{cases}.$$

where $S = \{x \in \mathbb{N}^k : r(x) \text{ is even}\}$. Since S is a computable set ($\chi_S(x) = \chi_{2\mathbb{N}}(r(x))$), it follows from Proposition 2.1.17 that u, v are computable functions. Finally, we have that $f = u - v$. □

The two previous prepositions therefore suggest that a function $f : \mathbb{N}^k \to \mathbb{Z}$ is computable if and only if it can be written as the difference of tow computable functions from $\mathbb{N}^k$ to $\mathbb{N}$.

In a similar fashion to the previous section, we now show some useful results about computability of this wider class of functions.

Proposition 2.2.4. *Let $f, g : \mathbb{N}^k \to \mathbb{Z}$ be computable functions. Then the functions $-f, f + g, f \cdot g : \mathbb{N}^k \to \mathbb{Z}$ are computable.*

Proof. Since f and g are computable, there exist computable functions $u_f, u_g, v_f, v_g : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = u_f(x) - v_f(x), \text{ and } g(x) = u_g(x) - v_g(x).$$

We can now express the functions in question as follows:

$$-f(x) = -(u_f(x) - v_f(x)) = v_f(x) - u_f(x),$$

$$f + g(x) = u_f(x) - v_f(x) + u_g(x) - v_g(x) = (u_f + u_g)(x) - (v_f + v_g)(x).$$

Moreover, we note that since f, g are computable, by Definition 2.2.1, we also have that there exist computable functions $r_f, p_f, r_g, p_g : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = (-1)^{r_f(x)} p_f(x), \quad g(x) = (-1)^{r_g(x)} p_g(x).$$

Finally,

$$fg(x) = (-1)^{r_f(x)+r_g(x)} p_f(x) p_g(x).$$

From these expressions for the functions in question, it follows that all three are computable. $\square$

One can consider more functions with range $\mathbb{Z}$ and show they are computable by arguments which are similar to those presented above. We now again move on to computability notions involving subsets on the integers. The results presented may seem repetitive, and this is because their analogues were presented in the previous section. We only re-visit them for the sake of consistency.

Proposition 2.2.5. *Suppose $f : \mathbb{N}^k \to \mathbb{Z}$ be computable functions. Then the sets $S = \left\{x \in \mathbb{N}^k : f(x) = 0\right\}$ and $T = \left\{x \in \mathbb{N}^k : f(x) > 0\right\}$ are computable.*

Proof. Since f is computable, we have that for any $x \in \mathbb{N}^k$, $f(x) = (-1)^{r(x)} p(x)$, where $r, p : \mathbb{N}^k \to \mathbb{N}$ are computable functions. Now,

$$f(x) = 0 \iff p(x) = 0.$$

So, by Proposition 2.1.15, S is computable. Similarly,

$$x \in T \iff f(x) > 0 \iff r(x) \in 2\mathbb{N}, p(x) \neq 0.$$

To show T is computable, we define the sets

$$T_1 = \left\{x \in \mathbb{N}^k : r(x) \in 2\mathbb{N}\right\}, \quad T_2 = \left\{x \in \mathbb{N}^k : p(x) \neq 0\right\}.$$

T_1 is computable since its characteristic function is the composition of computable functions. Namely, $\chi_{T_1} = \chi_{2\mathbb{N}}(r(x))$. Next, T_2 is computable since it is the complement of the computable set from Proposition 2.1.15. Moreover, since $T = T_1 \cap T_2$, it follows that T is computable. $\square$

Corollary 2.2.6. *Let $f, g : \mathbb{N}^k \to \mathbb{Z}$ be computable. Then the sets*

$$S = \left\{x \in \mathbb{N}^k : f(x) = g(x)\right\}$$
$$T = \left\{x \in \mathbb{N}^k : f(x) < g(x)\right\}$$
$$V = \left\{x \in \mathbb{N}^k : f(x) \leq g(x)\right\}$$

are computable.

Proof. Let $h : \mathbb{N}^k \to \mathbb{Z}, \quad h(x) = g(x) - f(x)$. So, $h = g + (-f)$. Note that

$$h(x) = 0 \iff g(x) = f(x) \iff x \in S.$$

So, it follows that

$$S = \left\{x \in \mathbb{N}^k : h(x) = 0\right\},$$

which is computable by Proposition 2.2.5. The proof for the sets T and V is similar and is therefore omitted. □

We are now ready to move on to computability of functions whose range is the rationals.

2.3 Computability on the Rationals

The contents of this section are again presented in a similar way to the previous sections. We begin with the key definition of a computable function from $\mathbb{N}^k$ to $\mathbb{Q}$. Since any rational number z can be associated with a triple of natural numbers via the expression $z = (-1)^r p/q$, this will yield the following intuitive definition. In this section $k \in \mathbb{N}_+$, unless stated otherwise.

Definition 2.3.1. We say that $f : \mathbb{N}^k \to \mathbb{Q}$ is computable if there exist computable functions $p, q, r : \mathbb{N}^k \to \mathbb{N}$ such that $q(x) \neq 0 \quad \forall x \in \mathbb{N}^k$ and

$$f(x) = (-1)^{r(x)} \frac{p(x)}{q(x)}, \quad \forall x \in \mathbb{N}^k.$$

Now, we also know that any rational number can be written as the quotient of two integers. This yields the following proposition, which we state without proof.

Proposition 2.3.2. *A function $f : \mathbb{N}^k \to \mathbb{Q}$ is computable if and only if there exist computable functions $u : \mathbb{N}^k \to \mathbb{Z}$ and $v : \mathbb{N}^k \to \mathbb{N}$ such that*

$$f(x) = \frac{u(x)}{v(x)}, \quad \forall x \in \mathbb{N}^k.$$

So far, we have seen different definitions of computable functions depending on their range. Since, for example, $\mathbb{Z} \subset \mathbb{Q}$, a natural question to ask would be, if a function is computable as a function from $\mathbb{N}^k$ to $\mathbb{Z}$, will is still satisfy the definition of a computable function from $\mathbb{N}^k$ to $\mathbb{Q}$? We address this question in the following proposition.

Proposition 2.3.3. *If $f : \mathbb{N}^k \to \mathbb{Z}$ is computable, then f is computable as a function from $\mathbb{N}^k$ to $\mathbb{Q}$. Conversely, if $f : \mathbb{N}^k \to \mathbb{Z}$ is computable as a function from $\mathbb{N}^k$ to $\mathbb{Q}$, then f is computable as a function $\mathbb{N}^k \to \mathbb{Z}$.*

Proof. Suppose f is computable as a function from $\mathbb{N}^k \to \mathbb{Q}$. Then, by Definition 2.3.1, we have

$$f(x) = (-1)^{r(x)} \frac{p(x)}{g(x)},$$

where $p, q, r : \mathbb{N}^k \to \mathbb{N}$ are computable. Moreover,

$$\begin{aligned} f(x) \in \mathbb{Z} &\implies \frac{p(x)}{q(x)} \in \mathbb{N} \\ &\implies \frac{p(x)}{q(x)} = \left\lfloor \frac{p(x)}{q(x)} \right\rfloor. \end{aligned}$$

We now consider the function $h : \mathbb{N}^2 \to \mathbb{N}$ defined by $h(x, y) = \left\lfloor \frac{x}{y} \right\rfloor$, which is computable by Example 2.1.5 and can conclude the following series of implications:

$$\begin{aligned} &\implies \frac{p(x)}{q(x)} = h(p(x), q(x)) \\ &\implies f(x) = (-1)^{r(x)} h(p(x), q(x)). \end{aligned}$$

Hence, since p, q are computable functions from $\mathbb{N}^k$ to $\mathbb{N}$ and h is a computable function from $\mathbb{N}^2$ to $\mathbb{N}$, it follows that $h \circ (p, q)$ is a computable function from $\mathbb{N}^k$ to $\mathbb{N}$ by composition. So, f satisfies Definition 2.2.1, from which it follows that f is computable as a function from $\mathbb{N}^k$ to $\mathbb{Z}$. □

We now again have results involving the sum and multiplication of computable functions mapping to the rational numbers.

Proposition 2.3.4. *Let $f, g : \mathbb{N}^k \to \mathbb{Q}$ be computable. Then, the functions $-f, f + g, fg : \mathbb{N}^k \to \mathbb{Q}$ are computable. Moreover, if $f(x) \neq 0 \quad \forall x \in \mathbb{N}^k$, then the function $\frac{1}{f} : \mathbb{N}^k \to \mathbb{Q}$ is computable.*

Proof. Since $f, g : \mathbb{N}^k \to \mathbb{Q}$ are computable, we have computable functions $u_f, u_g : \mathbb{N}^k \to \mathbb{Z}$ and $v_f, v_g : \mathbb{N}^k \to \mathbb{N}$ such that for any $x \in \mathbb{N}^k$,

$$f(x) = \frac{u_f(x)}{v_f(x)}, \quad g(x) = \frac{u_g(x)}{v_g(x)}.$$

Claim 1. *$-f$ is a computable function.*

Proof of claim: For any $x \in \mathbb{N}^k$, we can write

$$(-f)(x) = \frac{-u_f(x)}{v_f(x)}.$$

Letting $u_{-f} = -u_f$, this is a computable function from $\mathbb{N}^k$ to $\mathbb{Z}$ and $v_{-f} = v_f$ also a computable function from $\mathbb{N}^k$ to $\mathbb{N}$, we get that $-f$ is computable.

Claim 2. *$f + g$ is a computable function.*

Proof of claim: For any $x \in \mathbb{N}^k$, we can write

$$(f+g)(x) = \frac{u_f(x)}{v_f(x)} + \frac{u_g(x)}{v_g(x)} = \frac{u_f v_g(x) + u_g v_f(x)}{v_f v_g(x)}.$$

Letting $u_{f+g} = u_f v_g + u_g v_f$, this is a computable from $\mathbb{N}^k$ to $\mathbb{Z}$ (since the u's are computable from $\mathbb{N}^k$ to $\mathbb{Z}$ and the v's are computable from $\mathbb{N}^k$ to $\mathbb{N}$ and so to $\mathbb{Z}$) and $v_{f+g} = v_f v_g$ is also a computable function from $\mathbb{N}^k$ to $\mathbb{N}$. From this, we get that $f + g$ is computable.

Claim 3. *fg is a computable function.*

Proof of claim: For any $x \in \mathbb{N}^k$, we can write

$$(fg)(x) = \frac{u_f(x)}{v_f(x)} \cdot \frac{u_g(x)}{v_g(x)} = \frac{u_f u_g(x)}{v_f v_g(x)}.$$

Letting $u_{fg} = u_f u_g$, this is a computable function from $\mathbb{N}^k$ to $\mathbb{Z}$ (since it is the product of computable functions) and $v_{fg} = v_f v_g$ is also a computable function from $\mathbb{N}^k$ to $\mathbb{N}$ (again, because it is the product of computable functions). From this, we get that fg is computable.

Claim 4. *$\frac{1}{f}$ is a computable function.*

Proof of claim: Since f is computable, we also know that there exist functions r_f, p_f, q_f : $\mathbb{N}^k \to \mathbb{N}$ computable such that for any $x \in \mathbb{N}^k$ for which $f(x) \neq 0$, we can write

$$f(x) = (-1)^{r_f(x)} \frac{p_f(x)}{q_f(x)}.$$

Then,

$$\left(\frac{1}{f}\right)(x) = (-1)^{r_f(x)} \frac{q_f(x)}{p_f(x)}.$$

Letting $r_{\frac{1}{f}} = r_f$, $p_{\frac{1}{f}} = q_f$ and $q_{\frac{1}{f}} = p_f$ all computable functions by definition, we get that $\frac{1}{f}$ is computable. $\square$

In addition to results involving the computability of functions, we can also look at analytic properties of computable functions. Namely, that any computable function from $\mathbb{N}^k$ to $\mathbb{Q}$ is in some sense computably bounded.

Proposition 2.3.5. *Let $f : \mathbb{N}^k \to \mathbb{Q}$ be computable. Then there exists a computable function $M : \mathbb{N}^k \to \mathbb{N}$ such that $|f(x)| < M(x)$ for any $x \in \mathbb{N}^k$.*

Proof. Since f is computable as a function from $\mathbb{N}^k \to \mathbb{Q}$, we can write

$$f(x) = (-1)^{r(x)} \frac{p(x)}{q(x)},$$

where $p, q, r : \mathbb{N}^k \to \mathbb{N}$ are computable functions. Then, we have that

$$|f(x)| = \frac{p(x)}{q(x)} \leq p(x) < p(x) + 1.$$

Letting $M(x) = u(x) + 1$, which is a computable function from $\mathbb{N}^k \to \mathbb{N}$ yields the desired result. □

Finally, we can state the following result which ties back to our discussion of functions with higher dimensional ranges in the first section of this chapter.

Proposition 2.3.6. *Suppose $f : \mathbb{N}^k \to \mathbb{Q}$ and $g : \mathbb{N}^n \to \mathbb{N}^k$ are computable. Then the function $f \circ g : \mathbb{N}^n \to \mathbb{Q}$ is computable.*

Proof. There exist computable functions $r, p, g : \mathbb{N}^k \to \mathbb{N}$ such that

$$f(x) = (-1)^{r(x)} \frac{p(x)}{q(x)}, \quad \forall x \in \mathbb{N}^k.$$

Hence,

$$f(g(x)) = (-1)^{r(g(x))} \frac{p(g(x))}{q(g(x))}, \quad \forall x \in \mathbb{N}^n.$$

From this and Proposition 2.1.20, it follows that $f \circ g$ is computable. □

Now, since any real number can be approximated by a sequence of rational numbers, we will use this fact in order to continue our exposition to functions from $\mathbb{N}$ to $\mathbb{R}$, therefore building our way up to finally study real-valued functions.

2.4 Computability on the Reals

In this section we again let $k \in \mathbb{N}_+$, unless otherwise stated. The exposition in this section is of particular importance for our original results in Chapter 4.

Definition 2.4.1. A real-valued function $f : \mathbb{N}^k \to \mathbb{R}$ is computable if there exists a computable $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x,i)| < 2^{-i}, \quad \forall x \in \mathbb{N}^k, \quad \forall i \in \mathbb{N}.$$

We say that F is a computable approximation of f.

Hence, a computable real-valued function is one whose values can be uniformly approximated by some computable rational valued function.

Proposition 2.4.2. *Let $f : \mathbb{N}^k \to \mathbb{R}$ be computable. Then there exists a computable function $M : \mathbb{N}^k \to \mathbb{N}$ such that $|f(x)| < M(x)$ for any $x \in \mathbb{N}^k$.*

Proof. Since f is computable, we have that there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x,i)| < 2^{-i}, \quad \forall x \in \mathbb{N}^k, \quad \forall i \in \mathbb{N}.$$

Thus, for any $i \in \mathbb{N}$, by the reverse triangle inequality and Proposition 2.3.5 (which gives us an upper bound on the function F) it follows that

$$|f(x)| - M_F(x,i) < |f(x)| - |F(x,i)| < 2^{-i}.$$

In particular, for $i = 0$, we have

$$|f(x)| < 1 + M_F(x,0).$$

Letting $M(x) = M_F(x,0) + 1$, which is a computable function from $\mathbb{N}^k \to \mathbb{N}$, yields the desired result. □

Proposition 2.4.3. *Suppose $f : \mathbb{N}^k \to \mathbb{R}$ is computable and let $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ be its computable approximation. Then there exists a computable $M : \mathbb{N}^k \to \mathbb{N}$ such that $|F(x,i)| < M(x)$ for any $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$.*

Proof. By definition of computable approximation, for each $i \in \mathbb{N}$, we have

$$|F(x,i)| - |f(x)| \leq |f(x) - F(x,i)| < 2^{-i}.$$

By Proposition 2.4.2, we have that we can bound f above by the computable function $M_f : \mathbb{N}^k \to \mathbb{N}$. Thus, we get

$$|F(x,i)| \leq |f(x)| + 2^{-i} < |f(x)| + 1 < M_f(x) + 1.$$

Letting $M(x) = M_f(x) + 1$, which is a computable function from $\mathbb{N}^k \to \mathbb{N}$ yields the desired result. $\square$

Lemma 2.4.4. *Suppose $f : \mathbb{N}^k \to \mathbb{R}$ is any function (non necessarily computable) and $F : \mathbb{N}^{k+1} \to \mathbb{Q}$, $M : \mathbb{N}^k \to \mathbb{N}$ are computable functions such that*

$$|f(x) - F(x,i)| < M(x) \cdot 2^{-i}, \quad \forall x \in \mathbb{N}^k, \quad \forall i \in \mathbb{N},$$

then f is computable.

Proof. Since $M : \mathbb{N}^k \to \mathbb{N}$, it follows that for any $x \in \mathbb{N}^k$, $M(x) < 2^{M(x)}$. Hence, for any x,

$$M(x)2^{-M(x)} < 1 \implies M(x)2^{-M(x)}2^{-i} < 2^{-i}.$$

Thus,

$$|f(x) - F(x, i + M(x))| < M(x)2^{-(i+M(x))} < 2^{-i}.$$

Defining $F' : \mathbb{N}^{k+1} \to \mathbb{Q}$ as

$$F'(x,i) = F(x, i + M(x)),$$

it follows that F' is computable (by Proposition 2.3.6) and so f is computable, since there exists a computable function $F' : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that for any x and i,

$$|f(x) - F'(x,i)| < 2^{-i}.$$

$\square$

We now turn to results involving sums and products of computable real-valued functions, as we have done in the past. The proofs of these results are significantly more involved than those of simpler ranges such as $\mathbb{Z}$ and $\mathbb{Q}$.

Proposition 2.4.5. *Let $f, g : \mathbb{N}^k \to \mathbb{R}$ be computable functions. Then $-f, |f|, f+g : \mathbb{N}^k \to \mathbb{R}$ are computable.*

Proof. Since $f, g : \mathbb{N}^k \to \mathbb{R}$ are computable, there exist $F, G : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that $\forall x \in \mathbb{N}^k, i \in \mathbb{N}$

$$|f(x) - F(x,i)| < 2^{-i}, \quad \text{and} \quad |g(x) - G(x,i)| < 2^{-i}.$$

Claim 1. *$-f$ is a computable function.*

Proof of claim: This follows from the fact that

$$\begin{aligned} |-f(x) - (-F(x,i))| &= |(-1)(f(x) - F(x,i)| \\ &= |f(x) - F(x,i)| \\ &< 2^{-i}. \end{aligned}$$

So, since $-F$ is computable, this implies that $-f$ is computable.

Claim 2. *$|f|$ is a computable function.*

Proof of claim: We have that

$$||f(x)| - |F(x,i)|| \leq |f(x) - F(x,i)| < 2^{-i}.$$

Since F is computable, so is $|F|$ by the fact that

$$|F(x,i)| = (-1)^{c_0(x,i)} \frac{u(x,i)}{v(x,i)}.$$

It now follows immediately that $|f|$ is computable.

Claim 3. *$f + g$ is a computable function.*

Proof of claim: We have that for any $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$,

$$|f(x) + g(x) - (F(x,i) + G(x,i))| \leq |f(x) - F(x,i)| + |g(x) - G(x,i)| < 2 \cdot 2^{-i} = 2^{-i+1}.$$

In particular, we have

$$|(f+g)(x) - (F+G)(x,i+1)| < 2^{-i}.$$

The function $(F+G)(x,i+1)$ is computable since it is the composition of computable functions. From this, it follows that $f+g$ is computable. $\square$

Proposition 2.4.6. *Let $f, g : \mathbb{N}^k \to \mathbb{R}$ be computable. Then, $fg : \mathbb{N}^k \to \mathbb{R}$ is computable.*

Proof. Since f and g are computable, there exist computable functions $F, G : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that for any $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$,

$$|f(x) - F(x,i)| < 2^{-i}, \quad |g(x) - G(x,i)| < 2^{-i}.$$

So it follows that

$$\begin{aligned}
|fg(x) - FG(x,i)| &= |f(x)g(x) - f(x)G(x,i) + f(x)G(x,i) - F(x,i)G(x,i)| \\
&\leq |f(x)g(x) - f(x)G(x,i)| + |f(x)G(x,i) - F(x,i)G(x,i)| \\
&= |f(x)||g(x) - G(x,i)| + |G(x,i)||f(x) - F(x,i)| \\
&\leq M_f(x)2^{-i} + M_G(x)2^{-i} \qquad (*) \\
&= 2^{-i}(M_f(x) + M_G(x)).
\end{aligned}$$

Where $(*)$ follows by Propositions 2.4.2 and 2.4.3. Now, since M_f and M_G are computable functions, so is their sum. Thus, by Lemma 2.4.4, fg is computable. □

Proposition 2.4.7. *Let $f : \mathbb{N}^k \to \mathbb{R}$ be a computable function such that $f(x) \neq 0 \quad \forall x \in \mathbb{N}^k$. Then, $\frac{1}{f} : \mathbb{N}^k \to \mathbb{R}$ is computable.*

Proof. Since f is computable, we have that

$$|f(x) - F(x,i)| < 2^{-i}, \quad \forall x \in \mathbb{N}^k, i \in \mathbb{N},$$

where $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ is a computable function. Now, we wish to bound the following expression

$$\left|\frac{1}{f(x)} - \frac{1}{F(x,i)}\right| = \frac{|f(x) - F(x,i)|}{|f(x)F(x,i)|}.$$

To do so, we must bound $|f(x)|$ and $|F(x,i)|$ from below. To this end, we first show the following claims:

Claim 1. $\forall x \in \mathbb{N}^k \quad \exists i_x \in N$ *such that* $3 \cdot 2^{-i_x} < |F(x, i_x)|$.

<u>Proof of claim:</u> Let $x \in \mathbb{N}^k$, then

$$\begin{aligned}
f(x) \neq 0 &\implies |f(x)| > 0 \\
&\implies \exists i_x \quad 4 \cdot 2^{-i_x} < |f(x)| \qquad (*)
\end{aligned}$$

For this i_x, by the reverse triangle inequality we also have that

$$|f(x)| - |F(x, i_x)| < 2^{-i_x} \implies |f(x)| < 2^{-i_x} + |F(x, i_x)|. \qquad (**)$$

Now, by $(*)$ and $(**)$, it follows that

$$4 \cdot 2^{-i_x} < 2^{-i_x} + |F(x, i_x)| \implies 3 \cdot 2^{-i_x} < |F(x, i_x)|.$$

Claim 2. *If $x \in \mathbb{N}^k$ and $i_x \in \mathbb{N}$ are such that $3 \cdot 2^{-i_x} < |F(x, i_x)|$, then*

(a) $2 \cdot 2^{-i_x} < |f(x)|$

(b) $2^{-i_x} < |F(x, i_x)| \quad \forall i \geq i_x$

Proof of claim: Let $x \in \mathbb{N}^k$ and $i_x \in \mathbb{N}$ be such that $3 \cdot 2^{-i_x} < |F(x, i_x)|$.
(a) By the reverse triangle inequality and the assumption, we have that

$$3 \cdot 2^{-i_x} - |f(x)| < |F(x, i_x)| - |f(x)| < 2^{-i_x}.$$

This implies that

$$|f(x)| > 2^{-i_x} - 3 \cdot 2^{-i_x} - 2^{-i_x} = 2 \cdot 2^{-i_x}.$$

(b) By the reverse triangle inequality and (a), we have that for any i,

$$\begin{aligned} |F(x, i)| - |f(x)| < 2^{-i} &\implies |f(x)| - 2^{-i} < |F(x, i)| \\ &\implies 2 \cdot 2^{-i_x} - 2^{-i} < |f(x)| - 2^{-i} < |F(x, i)|. \end{aligned}$$

Then, it follows that for any $i \geq i_x$, we have

$$2^{-i_x} = 2 \cdot 2^{-i_x} - 2^{-i_x} < 2 \cdot 2^{-i_x} - 2^{-i} < |f(x)| - 2^{-i} < |F(x, i)|.$$

Claim 3. *Let $S = \{(x, y) \in \mathbb{N}^{k+1} : 3 \cdot 2^{-y} < |F(x, y)|\}$. Then, there exists a computable function $\phi : \mathbb{N}^k \to \mathbb{N}$ such that $(x, \phi(x)) \in S \quad \forall x \in \mathbb{N}^k$.*

Proof of claim: By a corollary analogous to Corollary 2.2.6, we have that S is a computable set. Moreover, for any $x \in \mathbb{N}^k$, there is a $y \in \mathbb{N}$ such that $(x, y) \in S$ (by our first claim). We now construct that desired ϕ by letting

$$\phi(x) = \mu^y \left(\chi_S(x, y) \dot{-} 1 = 0\right).$$

Indeed, this function is computable since S is a computable set, so $\chi_S : \mathbb{N}^{k+1} \to \mathbb{N}$ is a computable function and so is the modified subtraction. This concludes the proof of the claim.

Now, letting ϕ be the function from our last claim, we can see that for any $x \in \mathbb{N}^k$,

$$3 \cdot 2^{-\phi(x)} < |F(x, \phi(x))|.$$

Moreover, by our second claim, we get the desired lower bounds. Namely,

(a) $\implies 2 \cdot 2^{-\phi(x)} < |f(x)|$

(b) $\implies 2^{-\phi(x)} < |F(x,i)| \quad \forall i \geq \phi(x) \implies 2^{-\phi(x)} < |F(x, i+\phi(x))| \quad \forall i \in \mathbb{N}$.

Now, in light of the previously obtained bounds, we are ready to proceed to the proof of the statement of this proposition. To that end, let $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$. We have:

$$\begin{aligned}
\left| \frac{1}{f(x)} - \frac{1}{F(x, i+\phi(x))} \right| &= \frac{|f(x) - F(x, i+\phi(x))|}{|f(x)F(x, i+\phi(x))|} \\
&< \frac{|f(x) - F(x, i+\phi(x))|}{2 \cdot 2^{-\phi(x)} \cdot 2^{-\phi(x)}} \\
&< \frac{2^{-(i+\phi(x))}}{2^{-2\phi(x)+1}} \\
&= 2^{-i-1+\phi(x))} \\
&\leq 2^{-i} M(x),
\end{aligned}$$

where $M(x) = 2^{\phi(x)}$, which is a computable function from $\mathbb{N}^k \to \mathbb{N}$. Thus, by Lemma 2.4.4, it follows that $\frac{1}{f}$ is computable. □

In addition to computable real-valued functions, we can also consider the definition of computable real numbers. This concept is purely for expository purposes.

Definition 2.4.8. Let $x \in \mathbb{R}$. We say that x is a computable number if there exists a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that

$$|x - f(k)| < 2^{-k} \quad \forall k \in \mathbb{N}.$$

It is a known fact that if f is a computable real-valued function, then for any x in the domain of f, $f(x)$ is a computable real number. Moreover, we can see that there certainly exist non computable real numbers because the number of computable functions from $\mathbb{N}$ to

$\mathbb{R}$ is countable, so the number of computable real numbers must also be countable. Hence, since the set of real numbers is uncountable, one can deduce that the set of computable real numbers is but a small subset of the set of reals. The following results will be important for our original work at the end of this thesis, however, they are presented without proof. The complete proofs can be found in [12].

Proposition 2.4.9. *Let $x, y \in \mathbb{R}$ be computable numbers. Then $-x, x+y, xy$ are computable numbers. Moreover, if $x \neq 0$, then $\frac{1}{x}$ is a computable number.*

Theorem 2.4.10. *If x, y are computable real numbers and $x > 0$, then x^y is a computable real number.*

Since we are interested in function spaces in harmonic analysis, the circle is often a domain which will be used. As such, it is useful to know that π is a computable number. This can be intuitively explained by the fact that there exists an algorithm which can compute the i-th digit of π.

Theorem 2.4.11. *π is a computable real number.*

2.5 Effective Enumerations

Having established the fundamentals of computability theory for functions and numbers, we now turn our attention to notions of computability within different settings. For example given a set, one may wish enumerate the elements of this set using a computable function. This is called an effective enumeration. We first establish to basic results.

Proposition 2.5.1. *There exists a computable surjection g from $\mathbb{N}$ to $\mathbb{N}^2$.*

Proof. It suffices to show that we can find $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ computable functions such that $S = \{(\tau_1(i), \tau_2(i)) : i \in \mathbb{N}\} = \mathbb{N}^2$. To that end, we can define

$$\tau_1(i) = (i)_1, \tau_2(i) = (i)_2,$$

where the notation $(i)_k$ denotes the power of the $k-$th prime in the prime power factorization of the integer i. That is, given an integer i, we can write $i = \prod_{j=0}^{\infty} p_j^{(i)_j}$ and $p_0 = 2, p_1 =$

$3, p_2 = 5$ and so on. The function

$$(i,k) \to (i)_k = \begin{cases} \max\{j : p_k^j \text{ divides } i\} & i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

is computable (we state this without proof), so τ_1 and τ_2 are also computable.

One side of the inclusion follows immediately, since $\tau_1(i), \tau_2(i) \in \mathbb{N}$. For the other direction, let $(a,b) \in \mathbb{N}^2$ and $i = 3^a 5^b$. Then, $(i)_1 = a$ and $(i)_2 = b$. So

$$(a,b) = (\tau_1(i), \tau_2(i)) \implies (a,b) \in S.$$

Letting $g(i) = (\tau_1(i), \tau_2(i))$, this completes the proof. □

Proposition 2.5.2. *There exists a computable surjection from* $\mathbb{N}$ *to* $\mathbb{Q}$.

Proof. Consider the function $r : \mathbb{N} \to \mathbb{Q}$ defined by

$$r(i) = (-1)^{(i)_2} \frac{(i)_0}{(i)_1 + 1},$$

where the notation $(i)_k$ denotes the power of the $k-$th prime in the prime power factorization of the integer i. Since we know that $(i)_k$ is computable, then it is true that r is computable. Now, to show r is surjective, we must show that $r(\mathbb{N}) \subseteq \mathbb{Q}$ and $\mathbb{Q} \subseteq r(\mathbb{N})$. The first inclusion follows immediately. For the second inclusion, let $q \in \mathbb{Q}$. Then we can find $a, b, c \in \mathbb{N}$ such that

$$q = (-1)^c \frac{a}{b+1}.$$

Letting now $i = 2^a 3^b 5^c$, then $(i)_0 = a, (i)_1 = b$ and $(i)_2 = c$. So, $q \in r(\mathbb{N})$. □

A set for which an effective enumeration would be desirable is the set of finite sequences of natural numbers. For this, we can turn to the following result.

Proposition 2.5.3. *Let S be the set of all finite sequences of natural numbers. There exist computable functions* $\sigma : \mathbb{N}^2 \to \mathbb{N}$ *and* $\eta : \mathbb{N} \to \mathbb{N}$ *such that*

$$\{(\sigma(i,0), \ldots, \sigma(i, \eta(i)) : i \in \mathbb{N}\} = S.$$

Proof. Consider the functions

$$\sigma(i,k) = (i)_k \dot{-} 1, \quad \eta(i) = \begin{cases} \max\{k : p_k \text{ divides } i\} & \text{if } i \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Claim 1. $\{(\sigma(i,0), \ldots, \sigma(i,\eta(i)) : i \in \mathbb{N}\} \subseteq S$.

Proof of claim: For a fixed $i \in \mathbb{N}$, the sequence $(\sigma(i,0), \ldots, \sigma(i,\eta(i))$ is clearly a finite sequence of natural numbers by the definitions of σ and η. Hence $(\sigma(i,0), \ldots, \sigma(i,\eta(i)) \in S$.

Claim 2. $S \subseteq \{(\sigma(i,0), \ldots, \sigma(i,\eta(i)) : i \in \mathbb{N}\}$.

Proof of claim: Let $(m_0, \ldots, m_n) \in S$. It is enough to find and i such that $(m_0, \ldots, m_n) = (\sigma(i,0), \ldots, \sigma(i,\eta(i))$. To that end, consider $i = p_0^{m_0+1} + \cdots + p_n^{m_n+1}$. Then, we'll have that

$$\eta(i) = n, \quad \sigma(i,\eta(i)) = m_n, \text{ and } \forall j \in \{0, \ldots, n\}\, \sigma(i,j) = (i)_j - 1 = m_j + 1 - 1 = m_j.$$

Thus,

$$(m_0, \ldots, m_n) \in \{(\sigma(i,0), \ldots, \sigma(i,\eta(i)) : i \in \mathbb{N}\}.$$

□

This proposition in fact shows that we can enumerate the set of all finite sequences of natural numbers in a computable way. The same holds for the set of all finite sequences of rational numbers, as we shall see next.

Proposition 2.5.4. *Let $r : \mathbb{N} \to \mathbb{Q}$ be a fixed computable surjection (the one from Proposition 2.5.2, for example) and let S be the set of all finite sequences of rational numbers. There exist computable functions $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ such that*

$$\{\big(r(\sigma(i,0)), \ldots, r(\sigma(i,\eta(i)))\big) : i \in \mathbb{N}\} = S.$$

Proof. Consider the computable functions σ and η as in the proof of Proposition 2.5.3.

Claim 1. $\left\{\big(r(\sigma(i,0)), \ldots, r(\sigma(i,\eta(i)))\big) : i \in \mathbb{N}\right\} \subseteq S$.

Proof of claim: For a fixed $i \in \mathbb{N}$, the sequence $\big(r(\sigma(i,0)), \ldots, r(\sigma(i,\eta(i)))\big)$ is clearly a finite sequence of rational numbers by the definitions of the functions σ, η and r. Hence $\big(r(\sigma(i,0)), \ldots, r(\sigma(i,\eta(i)))\big) \in S$.

Claim 2. $S \subseteq \{\big(r(\sigma(i,0)),\ldots,r(\sigma(i,\eta(i)))\big) : i \in \mathbb{N}\}$.

Proof of claim: Let $(m_0,\ldots,m_n) \in S$. So, since r is a surjective map to $\mathbb{Q}$ and each $m_j \in \mathbb{Q}$, we have that $\forall j, \quad \exists b_j \in \mathbb{N}$ such that $m_j = r(b_j)$. Thus,

$$(m_0,\ldots,m_n) = (r(b_0),\ldots,r(b_n)).$$

Now, $(b_0,\ldots,b_n)$ is a finite sequence of natural numbers. So by Proposition 2.5.3 we can write

$$(b_0,\ldots,b_n) = (\sigma(i,0),\ldots,\sigma(i,\eta(i)))$$

for some $i \in \mathbb{N}$. I.e., $\forall j < n$, $b_j = \sigma(i,j)$ and $b_n = \sigma(i,\eta(i))$. Thus,

$$(m_0,\ldots,m_n) = (r(\sigma(i,0)),\ldots,r(\sigma(i,\eta(i)))) \in \{\big(r(\sigma(i,0)),\ldots,r(\sigma(i,\eta(i)))\big) : i \in \mathbb{N}\}.$$

□

We can also show such a result for the set of all intervals with rational endpoints, which we will call rational intervals.

Proposition 2.5.5. *Let S be the set of all rational intervals,*

$$\{(a,b) : a,b \in \mathbb{Q}, a < b\}.$$

There exist computable functions $s, w : \mathbb{N} \to \mathbb{Q}$ such that

$$\{(s(i),w(i)) : i \in \mathbb{N}\} = S.$$

Proof. First, let r, be some fixed computable surjection $r : \mathbb{N} \to \mathbb{Q}$ (we can let r be the enumeration of all rational numbers from Proposition 2.5.2). Since r is actually a sequence, we will write $r(i) = r_i$.

Consider

$$s(i) = r_{\tau_1(i)}, w_i = r_{\tau_1(i)} + f((\tau_2(i)),$$

where τ_1, τ_2 are the computable functions from the proof of Proposition 2.5.1, and $f : \mathbb{N} \to \mathbb{Q}_+, f(i) = \frac{(i)_0+1}{(i)_1+1}$ is an enumeration of all positive rational numbers (this can be shown with a proof similar to that of Proposition 2.5.2).

Let $A = \{(s(i), w(i)) : i \in \mathbb{N}\}, B = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$. We have $A \subseteq B$ by the fact that for any fixed i,

$$s(i), w(i) \in \mathbb{Q} \text{ and } f(i) \in \mathbb{Q}_+ \implies s(i) < w(i) \implies (s(i), w(i)) \in B.$$

For the inclusion $B \subseteq A$, we take any $(a, b) \in B$ and define $c = b - a \in \mathbb{Q}_+$. Since a, c are both rational numbers, by the enumeration of rational numbers, we can therefore find $u, v \in \mathbb{N}$, such that $a = r_u$ and $c = f(v)$. Then, $(u, v) \in \mathbb{N}^2$, so by Proposition 2.5.1, there exists and i such that $(u, v) = (\tau_1(i), \tau_2(i))$. Then for this i,

$$s(i) = r_{\tau_1(i)} = r_u = a, \quad w(i) = r_{\tau_1(i)} + f(\tau_2(i)) = a + f(v) = a + c = b.$$

Hence, $(a, b) \in A$.

Letting $I(i) = (s(i), w(i))$, this completes the proof. □

2.6 Notions of Computability in Metric Spaces

When establishing computability within the context of metric spaces, one often desires to establish a certain structure to the space. For example, one can define a certain subset of the space to be computable; if an element of the space can be approximated by elements of this subset, we can then say that this element is also computable. In the literature, a common term is that of a computable metric space. More on computability structures can be found in [5]. This section is of particular interest to us, as the results and definitions established here will be incorporated in our original results in Chapter 4.

Definition 2.6.1. A tuple (X, d, α) is a computable metric space if (X, d) is a metric space and

- $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X with respect to the metric d,
- the function $\mathbb{N}^2 \to \mathbb{R}, (n, m) \to d(\alpha(n), \alpha(m))$ is computable.

Since a metric is a real-valued functions, one can see how our exposition of computable real-valued functions plays an important role here. An example of a computable metric space is $(C[0,1], d_\infty, \alpha)$, where α is an enumeration of particular subset of the set of piecewise linear functions. This is shown in [13].

Another way to impose computability on metric spaces is via computability structures. We start with a few preliminary definitions.

Definition 2.6.2. If (x_i) and (y_j) are sequences in X, we say that $((x_i), (y_j))$ is an effective pair in (X, d) if the function $f : \mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto d(x_i, y_j)$ is computable.

By $(x_i) \diamond (y_i)$, we denote the statement that $((x_i), (y_i))$ is an effective pair in (X, d) .

Definition 2.6.3. Let (X, d) be a metric space and (x_i) be a sequence in X such that $(x_i) \diamond (x_i)$. Then, we say that (x_i) is an effective sequence in (X, d).

Although it may seem counter-intuitive, it is not always true that $(x_i)\diamond(x_i)$. To illustrate this, consider the sequence $(x_i) = (a, b, a, a, \dots)$. Then the function $\mathbb{N}^2 \to \mathbb{N}, (i, j) \mapsto d(x_i, x_j)$ is not computable if $d(a, b)$ is an incomputable number.

We can relate this notation to computable metric spaces via the following definition.

Definition 2.6.4. Suppose α is a sequence in X which is dense in (X, d). Then, (X, d, α) is a computable metric space if and only if $\alpha \diamond \alpha$. In this case, we call α an effective separating sequence for (X, d).

Definition 2.6.5. Suppose (x_i) and (y_i) are sequences in X. We say that (x_i) is computable with respect to (y_i) and write $(x_i) \preceq (y_i)$ if there exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d(x_i, y_{f(i,k)}) < 2^{-k}, \quad \forall i, k \in \mathbb{N}.$$

Definition 2.6.6. Let (X, d) be a metric space and let $\mathcal{S}$ be a nonempty set of sequences in X. Suppose the following holds:

1. $(x_i), (y_i) \in S \implies (x_i) \diamond (y_i)$,

2. $(y_j) \in S$, and (x_i) is any sequence in X such that $(x_i) \preceq (y_j) \implies (x_i) \in S$.

Then, we say that $\mathcal{S}$ is a computability structure on (X, d).

The significance of a computability structure on a metric space is, as stated before, it allows us to in a way fix the elements in the space which are computable and in turn deduce computability properties about the rest of the elements in the space.

Definition 2.6.7. For a computable metric space (X, d, α) we denote the set of all sequences in X which are computable with respect to α as

$$\mathcal{S}_\alpha = \{(x_i) \in X^{\mathbb{N}} : (x_i) \preceq \alpha\}.$$

This is an example of a computability structure, since it can be shown that $\mathcal{S}_\alpha$ is a computability structure on (X, d). We now give another basic example of a computability structure.

Example 2.6.1. *For a fixed $a \in X$, the set $\mathcal{S} = \{(x_i)\}$, where $x_i = a$, is a computability structure on the metric space (X, d). In order to see this, we first can see that the map $(i, j) \mapsto d(x_i, x_j)$ is in fact the map $(i, j) \mapsto 0$, meaning it is computable. So, it is true that $(x_i) \diamond (x_i)$. Next, suppose we have $(y_j) \preceq (x_i)$. This means that there exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that*

$$d(y_j, a) = d(y_j, x_{f(j,k)}) < 2^{-k}, \quad \forall j, k \in \mathbb{N}.$$

Since the distance is nonnegative, this implies that it must be true that $y_j = a, \forall j$. Thus, $(y_j) = (x_i) \in \mathcal{S}$.

This example illustrates the fact that any metric space has a computability structure. Moreover, we can have many computability structures on any given metric space (X, d).

Definition 2.6.8. Suppose $\mathcal{S}$ is a computability structure on (X, d). Let (x_i) be a sequence in X. We say that (x_i) is computable with respect to $\mathcal{S}$ if $(x_i) \in \mathcal{S}$.

Definition 2.6.9. Let $x \in X$. We say that x is a computable point in (X, d) with respect to $\mathcal{S}$ if the sequence $(x, x, x, \ldots, x) \in \mathcal{S}$.

Definition 2.6.10. For a computability structure $\mathcal{S}$ on a metric space (X, d), we denote the set of all computable points in X with respect to $\mathcal{S}$ by $\mathcal{S}^0$.

Definition 2.6.11. We say that a computability structure on (X, d) is dense if $\mathcal{S}^0$ is dense in (X, d).

So, $\mathcal{S}$ will be a dense computability structure on X if and only if $\mathcal{S}^0$ is dense in X. Note that not every metric space has a dense computability structure. Take for example $X = \{0, \gamma\}$, where γ is an incomputable real number, with the Euclidean metric.

Theorem 2.6.12. *Let (X, d) be a computable metric space, $\mathcal{S}$ be a computability structure on it and $x \in X$. Then the following are equivalent:*

1. *x is a computable point with respect to $\mathcal{S}$,*
2. *there exist $(y_j) \in \mathcal{S}$ and $j \in \mathbb{N}$ such that $x = y_j$*
3. *there exist $(y_j) \in \mathcal{S}$ and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(x, y_{f(k)}) < 2^{-k}, \quad \forall k \in \mathbb{N}$.*

Proposition 2.6.13. *Let $\mathcal{S}$ be a computability structure on (X, d) and $a, b \in \mathcal{S}^0$. Then $d(a, b)$ is a computable number.*

Proof. Since $a, b \in \mathcal{S}^0$ we have that the sequences $(a, a, a, \dots)$ and $(b, b, b, \dots)$ are in $\mathcal{S}$. So, $(a, a, a, \dots) \diamond (b, b, b, \dots)$ and the function $F : \mathbb{N}^2 \to \mathbb{R}$ $F(i, j) = d(a, b)$ is computable. Hence, $d(a, b)$ is a computable number. □

Proposition 2.6.14. *Let (X, d) be a metric space. Suppose D is a non-empty subset of X such that $d(x, y)$ is a computable number for all $x, y \in D$. Then, there exists a computability structure, call it $\mathcal{S}$ on (X, d) such that $D = \mathcal{S}^0$.*

Proof. First, define

$$\mathcal{S} = \{(a, a, a, ...) : a \in D\}.$$

$\mathcal{S}^0 = D$ holds trivially. Now, we must check that both conditions of a computability structure are satisfied.

Claim 1. *For any $a, b \in D$, $(a, a, a, \dots) \diamond (b, b, b, \dots)$.*

Proof of claim: Let $F : \mathbb{N}^2 \to \mathbb{R}$, $F(i, j) = d(a, b)$. By the assumption, we know that $d(a, b)$ is a computable number. Therefore, F is a computable function, since it is just the constant function taking on $d(a, b)$ as its only values. Hence, $(a, a, a, \dots) \diamond (b, b, b, \dots)$.

Claim 2. *If (x_i) is a sequence in X and $a \in D$ is such that $(x_i) \preceq (a, a, a, \dots)$, then $(x_i) \in \mathcal{S}$.*

Proof of claim: It is enough to show that there exists an element in D, call it c such that $\forall i$, $x_i = c$. This follows by applying a reasoning identical to that found in Example 2.6.1. $\square$

If D is dense in X, then that means $\mathcal{S}^0$ is a dense set in X. Hence, this directly implies that $\mathcal{S}$ is a dense computability structure on (X, d).

Definition 2.6.15. A computability structure $\mathcal{S}$ on (X, d) is said to be separable if there exists a sequence α such that (X, d, α) is a computable metric space and $\mathcal{S} = \mathcal{S}_\alpha$.

From this, we can see that each separable computability structure is a dense one. Having finished the exposition of computability and notions of computability within the context of metric spaces, we now move on to the second part of this project, which is to gain a solid background in spaces found in harmonic analysis. We first begin with the Hardy space, which is an example of a metric space.

Chapter 3

Background in Analysis

Since our work involves two very different areas of mathematics, the notation in the following section is completely independent to that of the previous section (it restarts).

3.1 The Hardy Space

3.1.1 Complex Hardy Space on the Disk

We begin with some short background on the Hardy space. First, the complex $\mathcal{H}^p(\mathbb{D})$. For a reference on this material see [14], [20] and [22].

Definition 3.1.1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in $\mathbb{C}$ and $\mathcal{H}(\mathbb{D})$ denote the collection of holomorphic functions on $\mathbb{D}$. Given $F \in \mathcal{H}(\mathbb{D})$, the average $M_p(F, r)$ of F on a circle radius $r < 1$ centred at the origin is given by

$$M_p(F,r) = \begin{cases} \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(re^{i\theta})| d\theta\}, & \text{if } p = 0 \\ \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & \text{if } 0 < p < \infty \\ \sup\limits_{\theta} |F(re^{i\theta})|, & \text{if } p = \infty. \end{cases}$$

For a fixed p and $F \in \mathcal{H}(\mathbb{D})$, M_p is a monotonically increasing function in r. In fact, this is the main result in [11], where the author (Hardy) discusses these averages in more detail and shows that they are log convex functions of r. It is likely for this reason that these spaces are called Hardy spaces.

Definition 3.1.2. For $0 < p \leq \infty$, we define the complex Hardy space $\mathcal{H}^p(\mathbb{D})$ as

$$\mathcal{H}^p(\mathbb{D}) = \{F \in \mathcal{H}(\mathbb{D}) : \|F\|_{\mathcal{H}^p} = \sup_{r<1} M_p(F, r) < \infty\}.$$

When $p = 0$, we define the Nevanlinna class as

$$\mathcal{N}(\mathbb{D}) = \{F \in \mathcal{H}(\mathbb{D}) : \|F\|_{\mathcal{N}} = \sup_{r<1} M_0(F, r) < \infty\}.$$

Moreover, if $0 < s < p < \infty$, $\mathcal{H}^\infty(\mathbb{D}) \subset \mathcal{H}^p(\mathbb{D}) \subset \mathcal{H}^s(\mathbb{D}) \subset \mathcal{N}(\mathbb{D})$ by Jensen's inequality.

3.1.2 Hardy Spaces on the Circle

For a given $F \in \mathcal{H}^p(\mathbb{D}), 0 < p < \infty$, there exists an $F^* \in L^p(\mathbb{T})$ such that $\lim_{r\to 1} \left\|F^*(e^{i\theta}) - F(re^{i\theta})\right\|_p = 0$. We call F^* the boundary values of F.

Definition 3.1.3. We define the complex Hardy space on the circle as

$$\mathcal{H}^p(\mathbb{T}) = \{F^* : F \in \mathcal{H}^p(\mathbb{D})\}.$$

Definition 3.1.4. For $1 \leq p \leq \infty$, we define the real Hardy space on the circle as

$$H^p(\mathbb{T}) = \{Re(F) : F \in \mathcal{H}^p(\mathbb{T})\}.$$

In order to obtain the full real Hardy space, one can then look at these functions and allow multiplication by complex scalars and addition to get a vector space over the complex numbers. Note that, for $p < 1$, the boundary values have to be taken in the sense of distributions.

3.1.3 Real Hardy Spaces on the Line

Following the definition found in [8], we define the Schwartz space as

$$\mathscr{S}(\mathbb{R}) = \left\{f \in C^\infty(\mathbb{R}) : \|f\|_{\mathscr{S}}^N = \sup_{x\in\mathbb{R}} (1 + |x|)^N |(\partial f(x)| < \infty, \forall N \in \mathbb{N}\right\}.$$

This space is also commonly called the space of rapidly decreasing functions, since any function in it will itself, together with all its derivatives, vanish at infinity faster than any power of $|x|$. Moreover, we denote the dual of the Schwartz space, i.e. the space of tempered distributions, by $\mathscr{S}'(\mathbb{R}^n)$. That is, it is the space of all functionals from $\mathscr{S}$ to $\mathbb{R}$ that are linear and continuous.

For $\Phi \in \mathscr{S}(\mathbb{R})$ and $t > 0$ we define its dilation Φ_t as $\Phi_t(x) = t^{-1}\Phi(x/t)$. Also, we define the maximal function $M_\Phi f$ as $M_\Phi f(x) = \sup_{t>0} |f * \Phi_t(x)|$. Note that $M_\Phi f$ is well defined when f is a tempered distribution since the convolution of a Schwartz function and a tempered distribution is a C^∞ function.

Definition 3.1.5. For $0 < p \leq \infty$ and a fixed $\Phi \in \mathscr{S}(\mathbb{R})$ with $\int \Phi \, dx = 1$, we define the real Hardy space on the real line as

$$H^p(\mathbb{R}) = \{f \in \mathscr{S}'(\mathbb{R}) : M_\Phi f \in L^p(\mathbb{R})\}.$$

We define

$$\|f\|_{H^p} := \|M_\Phi f\|_{L^p}.$$

This is a norm for $p \geq 1$. Moreover, for $1 < p \leq \infty$, it turns out that $H^p(\mathbb{R})$ is equivalent to $L^p(\mathbb{R})$ and for $p = 1$, $H^1(\mathbb{R}) \subsetneq L^1(\mathbb{R})$.

Another way of defining the real Hardy space $H^p(\mathbb{R})$ is via a construction which is analogous to that of the real Hardy space $H^p(\mathbb{T})$. Namely, for a function $F(x, y)$ on $\mathbb{R}^2_+$ say that $F \in H^p(\mathbb{R})$ if $\sup_{y>0} \|F_y\|_{L^p} < \infty$, where $F_y(x) = F(x, y)$ for a fixed $y \in \mathbb{R}_+$.

A third way of defining the real Hardy space is based off functions that capture the inherent cancellation and increased integrability properties of Hardy space functions. We call such functions atoms, since they represent the most basic building blocks for the space $H^1_{at}(\mathbb{R})$, which will be defined below. Note that the following definition is restricted to $p = 1$, although there exists a definition for atoms corresponding to other values of $p < 1$.

Definition 3.1.6. For $p = 1$, we say a function $a : \mathbb{R} \to \mathbb{R}$ is an atom if there exists a finite interval $I \subset \mathbb{R}$ such that

1. $\operatorname{supp}(a) \subset I$
2. $|a| \leq \frac{1}{\ell(I)}$
3. $\int_I a(x) \, dx = 0$,

where $\ell(I)$ denotes the length of I.

We say that $f \in \mathscr{S}'$ admits an atomic decomposition if there exists a sequence of atoms $\{a_k\}$ and a sequence of scalars $\{\lambda_k\} \in \ell^1(\mathbb{N})$ such that

$$f = \sum_k \lambda_k a_k, \tag{3.1.1}$$

where the convergence is not only in the sense of distributions, but also in the L^1 norm.

Definition 3.1.7. For $p = 1$, we define the atomic Hardy space on the real line as

$$H^1_{at}(\mathbb{R}) = \left\{ f \in \mathscr{S}' : \|f\|_{H^1_{at}} = \inf \sum_k |\lambda_k| < \infty \right\},$$

where the infimum is taken over all atomic decompositions (3.1.1) of f.

As explained in [22], H^1_{at} coincides with the real Hardy space H^1 as defined above. Hence, any function in the real Hardy space H^1 will enjoy many properties inherited by the elementary building blocks (the atoms).

Since atoms are quite easy to characterize and their linear combinations are dense in H^1, a space based on them seemed like a good setting in order to incorporate computability notions in the context of Hardy spaces. However, we quickly realized that these atoms are in fact too general for this goal. In other words, we required a set of atoms which have an explicit formula. This lead us to consider systems of Haar wavelets, which are closely related to H^1 atoms and for which there exists an explicit formula. The choice to study wavelets also seemed rather intuitive, as this mathematical tool is widely used in signal and image processing, so computability should hold for such functions.

3.2 Haar Wavelets

We note that in general, wavelets (and in particular the Haar wavelets) are normalized in L^2, whereas the atoms from Definition 3.1.6 are normalized in L^1. More on wavelets and the Haar function can be found in [16]. Nonetheless, what was attractive about the Haar system is that Haar functions, when appropriately normalized, satisfy all the conditions of the H^1 atoms while having an explicit formula and shape which is easily adaptable in a computability setting. Moreover, there are many results concerning convergence of linear combinations of Haar wavelets, such as that they form a basis for C^1 functions.

First, we note that families of wavelets are given by identifying two base functions (called mother and father wavelets), then the subsequent functions (called children) in the family will be a linear combination of the previous functions in the generation right before them.

Definition 3.2.1. For $1 \leq k$, $1 \leq j \leq 2^k$, the dyadic interval $I_{k,j}$, followed by its two children $I_{k+1,2j-1}$ and $I_{k+1,2j}$ is given by

$$I_{k,j} = \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right),$$

$$I_{k+1,2j-1} = \left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right) \qquad I_{k+1,2j} = \left[\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right).$$

Definition 3.2.2. For $1 \leq k$, $1 \leq j \leq 2^k$, the family of Haar wavelets on the unit interval $[0,1)$ is given by

$$\text{Father wavelet} : h_{00} = \chi_{[0,1)},$$

$$\text{Mother wavelet} : h_{01} = \chi_{I_{1,1}} - \chi_{I_{1,2}},$$

$$\text{Subsequent Haar functions} : h_{kj} = 2^{k/2}\left(\chi_{I_{k+1,2j-1}} - \chi_{I_{k+1,2j}}\right).$$

It is clear that Haar wavelets inherit many nice properties from the fact that they are based on dyadic intervals. As such, since we know that dyadic numbers are rational (and hence computable), a natural conclusion would be that they are a good candidate for computability notions. However, linear combinations of Haar wavelets are not dense in all of H^1_{at}. As a result, this inspired us to consider what DeSouza in his paper [7] calls the special atoms space.

3.3 The Special Atoms Space

Since an atom from Definition 3.1.6 is a rather general term, it turns out that when requirements are added to the atoms which are allowed to be used as the building blocks, the space generated is no longer the whole Hardy space H^1, but instead a subset of it, which we shall introduce later. For the purpose of adapting our work to the computability setting, we restrict our definitions to one dimension, but these spaces can be studied in higher dimensions. We begin with some basic definitions of atoms and special atoms. Our work in this section is based on DeSouza's work in his paper [7].

Notation 1. *We think of the circle* $\mathbb{T}$ *as the interval* $[-\pi, \pi)$.

Definition 3.3.1. For $1/2 < p < \infty$ and for each finite interval I in $\mathbb{T}$, the special atom $a_I : \mathbb{T} \to \mathbb{R}$ is defined as

$$a_I(x) = \frac{1}{\ell(I)^{1/p}}\chi_L(x) - \frac{1}{\ell(I)^{1/p}}\chi_R(x),$$

where L is the left half of the interval I, R is the right half and $\ell(I)$ denotes the length of the interval.

We can see that special atoms are atoms, but with an added symmetry condition.

Consider the constant function $b(x) = 1/2\pi$. Despite the fact that b is not a special atom, we shall consider b as being a special atom. This is because it must be included in the collection of special atoms in order to generate the special atoms space. This just makes things easier in terms of nomenclature.

We say that $f : \mathbb{T} \to \mathbb{R}$ admits a special atomic decomposition if there exist a sequence of special atoms $\{a_k\}$ and a sequence of scalars $\{\lambda_k\} \in \ell^1(\mathbb{N})$ such that

$$f = \sum_k \lambda_k a_k, \tag{3.3.1}$$

where a_j is either a_{I_j} or b and the convergence is in the sense of distributions. When $p \geq 1$, the convergence is also in the $L^p(\mathbb{T})$ norm.

Definition 3.3.2. For $1/2 < p < \infty$, we define the special atoms space as the space of all limits of linear combinations of special atoms. That is,

$$B^p(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{R} : \|f\|_{B^p} = \inf \sum_{j=0}^{\infty} |\lambda_j| < \infty \right\},$$

where the infimum is taken over all special atomic decompositions (3.3.1) of f.

The special atoms space therefore inherits many interesting attributes which come as a consequence of the added symmetry property of the special atoms. Namely, the fact that its building blocks are described by an explicit formula. So, B^p now seems like a good candidate for our goal of establishing computability notions within the context of a function space in harmonic analysis. However, the issue now comes with the fact that the norm is not at all computable, because of the presence of the infimum. Intuitively the fact that one would want to avoid such a norm makes sense, since calculating it involves checking all possible representations of a given f in terms of special atoms, which can go on forever. This lead us to search for alternative norms on the special atoms space. In the section to follow, we study two more function spaces with their own norms and see how they relate to the special atoms space of DeSouza in [7].

We begin by considering a space which is also generated by special atoms, called the Besov space. This follows the work on Frazier, Jawerth and Weiss in [9]. We adapt the results in higher dimension, although the only case of interest is when $n = 1$. Note that the exposition to follow deals with functions on $\mathbb{R}^n$, as opposed to on the circle.

We say a norm on a Banach space B on $\mathbb{R}^n$ is translation invariant if for any $h \in \mathbb{R}^n$ and $f \in B$,

$$\|f(\cdot - h)\|_B = \|f\|_B .$$

Similarly, we a say the norm is dilation invariant if for any $t > 0$ and $f \in B$,

$$\|f_t\|_b = \|f\|_B ,$$

where the dilation f_t is defined as $f_t(x) = t^{-n} f(x/t)$.

Now, let us consider the following minimality problems, called problems P and P_0:

- P: finding the minimal (smallest) Banach space B such that $\mathscr{S}(\mathbb{R}^n) \subset B \subset \mathscr{S}'(\mathbb{R}^n)$ and $\|\cdot\|_B$ is translation and dilation invariant.

- P_0: finding the minimal (smallest) Banach space B^0 such that $\mathscr{S}_0(\mathbb{R}^n) \subset B^0 \subset \mathscr{S}_0'(\mathbb{R}^n)$ and $\|\cdot\|_{B^0}$ is translation and dilation invariant.

Here $\mathscr{S}(\mathbb{R}^n)$ is the Schwartz space and $\mathscr{S}_0(\mathbb{R}^n) = \{f \in \mathscr{S}(\mathbb{R}^n) : \int f = 0\}$.

The work in [9] (chapter 3) proves the following theorem in great detail. For our purposes, we only state the main result.

Theorem 3.3.3. *Suppose* $(B, \|\cdot\|)$ *is a Banach space continuously contained in the space of tempered distributions. If the norm of* B *is translation and dilation invariant and* $\mathscr{S}(\mathbb{R}^n) \subset B$*, then* $L^1(\mathbb{R}^n) \subset B$ *and*

$$\|f\|_B \leq \|\chi_{Q_0}\|_B \|f\|_{L^1}.$$

Here χ_{Q_0} *is the characteristic function of the unit cube* $Q_0 = \{x = (x_1, x_2, x_3, \ldots, x_n) : 0 \leq x_j \leq 1, j = 1, \ldots, n\}$ *in* $\mathbb{R}^n$*.*

Theorem 3.3.3 states that the solution to problem P is $L^1(\mathbb{R}^n)$. However we know that L^1 is not a good space, since many operations often encountered in analysis (the Hardy-Littlewood maximal operator, for example) are not bounded on L^1. For this reason, the authors then considered problem P_0, which is a modification of the problem P. We shall see later that the solution to problem P_0 is the space B_ψ, which will be defined shortly.

For $f, g \in L^1(\mathbb{R}^n)$, we define the convolution on $\mathbb{R}^n$ as

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Convolutions have many properties. Namely, for $f, g, h \in L^1(\mathbb{R}^n)$, we have that

a) $f * (g + h) = (f * g) + (f * h)$,

b) $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{R}$,

c) $f * g = g * f$,

d) $(f * g) * h = f * (g * h)$,

e) $f * g$ is continuous,

f) $\widehat{f * g}(x) = \hat{f}(x)\hat{g}(x)$,

where $\hat{f}$ denotes the Fourier transform on the real line of f, that is, for a $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx.$$

Definition 3.3.4. For a fixed $N \in \mathbb{Z}_+$, we let ψ be a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that:

1. $\mathrm{supp}(\psi) \subset B_1(0)$,

2. ψ is radial;

3. $\psi \in C^\infty(\mathbb{R}^n)$,

4. $\int_{\mathbb{R}^n} x^\gamma \psi(x)dx = 0$ if $|\gamma| \leq N$, where $\gamma \in \mathbb{Z}_+^n, x^\gamma = x^{\gamma_1} x^{\gamma_2} \ldots x^{\gamma_n}$ and $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$,

5. $\int_0^\infty [\hat{\psi}(t\xi)]^2 \frac{dt}{t} = 1$ if $\xi \in \mathbb{R}^n \backslash \{0\}$.

Definition 3.3.5. For a fixed ψ as in Definition 3.3.4, we define the space B_ψ as

$$B_\psi = \left\{ f \in \mathscr{S}_0'(\mathbb{R}^n) : \|f\|_{B_\psi} = \int_0^\infty \|\psi_t * f\|_{L^1} \frac{dt}{t} < \infty \right\}.$$

The spaces B_ψ obtained using various ψ will all coincide. This means that the definition above is actually defining one unique space which is independent of the choice of ψ.

Because of the nature of its norm, the space B_ψ will inherit many of its properties from the Calderón reproducing formula, which gives an integral representation of the identity operator. We give two versions of the formula below.

Theorem 3.3.6. *Let $\psi \in L^1(\mathbb{R}^n)$ be a real-valued, function satisfying conditions 2 and 5 in Definition 3.3.4. For $f \in L^2(\mathbb{R}^n)$, we can write*

$$f(x) = \int_0^\infty (\psi_t * \psi_t * f)(x) \frac{dt}{t}.$$

The equality above is to be interpreted in the sense that the right hand side converges to the left hand side in the L^2 norm as the lower bound of the integral goes to 0 and the upper bound goes to infinity.

For the convergence of the Calderón Reproducing Formula in $L^p, 1 < p < \infty$, see [24]. Some more results involving the Calderón reproducing formula are given below. Namely, the Calderón reproducing formula for $L^1(\mathbb{R})$ functions.

Theorem 3.3.7. *Let $\psi \in L^1(\mathbb{R}^n)$ be a real-valued function with integral zero satisfying conditions 2 and 5 in Definition 3.3.4. For $f \in L^1(\mathbb{R}^n)$ with $\hat{f} \in L^1(\mathbb{R}^n)$, we can write*

$$\lim_{\epsilon \to 0, \delta \to \infty} f_{\epsilon,\delta}(x) = \lim_{\epsilon \to 0, \delta \to \infty} \int_\epsilon^\delta (\phi_t * \phi_t * f)(x) \frac{dt}{t} = f(x)$$

for all $x \in \mathbb{R}^n$ provided f is the continuous representative of the equivalence class determined by f in $L^1(\mathbb{R}^n)$.

The proofs of Theorem 3.3.7 uses Fubini's theorem and Young's inequality to show that $f_{\epsilon,\delta} \in L^1(\mathbb{R}^n)$ by finding a bound for its L^1 norm and can be found in [9]. We will now derive a known distributional version of the Calderón reproducing formula, which is needed for the space B_ψ, as it is a subset of the tempered distributions. However, some preliminary notions as first required.

Definition 3.3.8. For $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $u \in \mathscr{S}'(\mathbb{R}^n)$, we define the refection $\tilde{\phi}$ and the translation $\tau_x u$ as

$$\tilde{\phi}(y) = \phi(-y), y \in \mathbb{R}^n,$$

$$(\tau_x u)(y) = u(y - x), y \in \mathbb{R}^n.$$

Hence, from [20], we can say that the convolution of a distribution u with a Schwartz function ϕ is given by

$$(u * \phi)(x) = u(\tau_x \tilde{\phi}).$$

This also illustrates the fact that the convolution between a distribution and a Schwartz function is a function, as stated when studying the real Hardy spaces on the line in section 3.1.3.

Now, we must define the convolution of a distribution with an L^1 function. From [21], we know that if f is a bounded distribution and $h \in L^1(\mathbb{R}^n)$, then the convolution $f * h$ is a distribution acting on a Schwartz function $\phi \in \mathscr{S}$ as follows:

$$f * h(\phi) = \int_{\mathbb{R}^n} (f * \tilde{\phi})(x)\tilde{h}(x)dx. \tag{3.3.2}$$

Moreover, from [20], we again have many properties for such a convolution. Namely, for $\phi \in \mathscr{S}_n$ and u a tempered distribution,

a) $u * \phi \in C^\infty(\mathbb{R}^n)$

b) $u * \phi$ has polynomial growth, hence is a tempered distribution

c) $\widehat{(u * \phi)} = \hat{\phi}\hat{u}$

d) $(u * \phi) * \psi = u * (\phi * \psi)$, for every $\psi \in \mathscr{S}_n$

e) $\hat{u} * \hat{\phi} = \widehat{(\phi u)}$.

Now we establish the Calderón reproducing formula in the sense of distributions. Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$. Moreover, let ψ be as in Definition 3.3.4, that is, it is smooth with compact support. For $t > 0$ we have that $f*(\psi_t*\psi_t)$ is the convolution of an L^1 function (since ψ_t is smooth with compact support, meaning it is in L^1 and so is its reflection, and so is its convolution with itself) with a tempered distribution ($f \in \mathscr{S}'$). Hence, using (3.3.2) we write

$$f * (\psi_t * \psi_t)(\phi) = \int_{\mathbb{R}^n} (f * \tilde{\phi})(x)\widetilde{\psi_t * \psi_t}(x)dx. \tag{3.3.3}$$

The term $f * \tilde{\phi}$ is the convolution between a tempered distribution and a Schwartz function, which is well defined as a function, as mentioned before. Moreover, the integrand on right hand side of (3.3.3) is in fact a smooth function of compact support.

We therefore have established that (3.3.3) is the integral of a product of a smooth function with a function of compact support, and is therefore convergent, so we can use the properties of distributions and the Fourier transforms to write, in the formal sense,

$$\begin{aligned}\int_{\mathbb{R}^n} (f * \tilde{\phi})(x)\widetilde{\psi_t * \psi_t}(x)dx &= \int_{\mathbb{R}^n} \widehat{(f * \tilde{\phi})}(\xi)\widehat{\psi_t * \psi_t}(\xi)d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi)(\hat{\psi}_t(\xi))^2 d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi)(\hat{\psi}(t\xi))^2 d\xi.\end{aligned}$$

Hence,

$$f * (\psi_t * \psi_t)(\phi) = \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi)(\hat{\psi}(t\xi))^2 d\xi.$$

Using the expression obtained above, we take the integral over $\mathbb{R}_+$ in order to obtain the desired Calderón formula for distributions. Namely,

$$\begin{aligned}
\int_0^\infty f * (\psi_t * \psi_t)(\phi)\frac{dt}{t} &= \int_0^\infty \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi)(\hat{\psi}(t\xi))^2 d\xi \frac{dt}{t} \\
&= \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi)\left(\int_0^\infty (\hat{\psi}(t\xi))^2 \frac{dt}{t}\right) d\xi \\
&= \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\tilde{\phi}}(\xi) d\xi \\
&= \int_{\mathbb{R}^n} f(x)\tilde{\phi}(x) dx \\
&= f(\phi).
\end{aligned}$$

To summarize, we have shown that the following equation holds in the sense of distributions

$$\int_0^\infty f * (\psi_t * \psi_t)\frac{dt}{t} = f.$$

As it turns out, the space B_ψ is a function space only consisting of L^1 functions. To show this, we must show that the distributions contained in the space are also L^1 functions.

Proposition 3.3.9. *For any $f \in B_\psi$, $f \in L^1$.*

Proof. For a fixed and suitable ψ, let $f \in B_\psi$. By the definition of this space, we know that f must be a tempered distribution (it belongs to $\mathscr{S}_0'$) and that

$$\|f\|_{B_\psi} = \int_0^\infty \|\psi_t * f\|_1 \frac{dt}{t} < \infty.(\dagger)$$

Now consider

$$\frac{(\psi_t * \psi_t * f)(x)}{t}.$$

Since $\psi \in \mathscr{S}$, by the first property of convolutions, it follows that the expression above is a smooth function, call it $g(x,t)$. We now show that this function is in L^1, then use the Calderón reproducing formula in order to conclude that f is an L^1 function. To that end, fix a $t > 0$,

$$\begin{aligned}
\int_{\mathbb{R}} |g(x,t)|\, dx &= \frac{1}{t}\int_{\mathbb{R}} |\psi_t * \psi_t * f(x)|\, dx \\
&= \frac{1}{t}\|\psi_t * \psi_t * f\|_{L^1} \\
&\leq \frac{1}{t}\|\psi_t\|_{L^1}\|\psi_t * f\|_{L^1} \\
&= \frac{C_\psi}{t}\|\psi_t * f\|_{L^1}.
\end{aligned}$$

Recalling the fact that $f \in B_\psi$, and by †, we can take integrals on both sides and get

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |g(x,t)|\, dxdt &\leq \int_{\mathbb{R}} \frac{C_\psi}{t} \left\|\psi_t * f\right\|_{L^1} dt \\ &= C_\psi \left\|f\right\|_{B_\psi} \\ &< \infty.\end{aligned}$$

Therefore, we conclude that $g \in L^1(dx \times dt)$. Moreover, by Fubini's theorem, we can also conclude that for almost every $x \in \mathbb{R}$, $\int_{\mathbb{R}} g(x,t)dx \in L^1(dt)$ along with for almost every $t \in \mathbb{R}$, $\int_{\mathbb{R}} g(x,t)dt \in L^1(dx)$. So, this gives us that

$$\int_{\mathbb{R}} \psi_t * \psi_t * f(x)\frac{dt}{t} \in L^1(dx).$$

By the Calderón reproducing formula for distributions which we established, this L^1 function acts as a distribution identically to f. From this, it follows that $f \in L^1(dx)$. To summarize, we have shown that, for $f \in B_\psi$ there exists a positive constant C_ψ such that

$$\left\|f\right\|_{L^1} \leq C_\psi \left\|f\right\|_{B_\psi}.$$

□

In [9], it is stated that for any ψ, the space B_ψ is a Banach space which is continuously imbedded in $\mathscr{S}_0'(\mathbb{R}^n)$, whose norm is translation and dilation invariant and that $\mathscr{S}_0(\mathbb{R}^n)$ is contained inside it. These claims were checked, but are omitted in this exposition. The important thing to notice from this is that for any suitable ψ, the space B_ψ is the solution to problem P_0. This therefore suggests that the space in Definition 3.3.5 is independent of the choice of ψ. Indeed, for this reason, the authors of [9] call the solution to problem P_0 the Besov space $\dot{B}_1^{0,1}(\mathbb{R}^n)$.

When $p = 1$, a special atom as defined in Definition 3.3.1 is an H^1 atom, as in Definition 3.1.6. From the atomic decomposition (3.1.1) and since we saw above that the convergence in B_ψ gives convergence in L^1, the inclusion $\dot{B}_1^{0,1}(\mathbb{R}) \subset H^1(\mathbb{R})$ follows. The work of Wilson and Uchiyama in [25] gives an example as to why this inclusion is strict. Namely, the example they give in this paper turns out to be an H^1 atom that is not in the space B_ψ. They construct this example by building a function b in H^1 and an appropriate ψ so that the large frequencies of this ψ match up with those of b when convolved with each other. The function they construct is

$$b(x) = -\sum_{k=1}^{\infty} k^{-1+\epsilon_2} sin(2^k \pi x) \chi_{[-2,-1]}(x).$$

There exist other, more common, definitions of the Besov space in terms of the Fourier transform or interpolation, but we do not include them here and instead given an equivalent definition below. When $n = 1$, there is a very convenient decomposition of the Besov space $\dot{B}_1^{0,1}(\mathbb{R})$ via our familiar special atoms. Here, the special atoms are the same as those used by DeSouza (refer to Definition 3.3.1), with $p = 1$.

Theorem 3.3.10. *The special atoms space B^1 coincides with $\dot{B}_1^{0,1}(\mathbb{R})$ and there exist constants M, N such that for any $f \in \dot{B}_1^{0,1}(\mathbb{R})$,*

$$N \|f\|_{B^1} \leq M \|f\|_{\dot{B}_1^{0,1}} \leq M \|f\|_{B^1}.$$

The proof of Theorem 3.3.10 can be found in [9]. This theorem therefore allows us to consider the norm on the Besov space (i.e. B_ψ, for any suitable ψ) as a norm on the special atoms space. However, this norm is again not readily adaptable in the computability setting without an explicit formula for a suitable ψ. Moreover, in order to discuss computability, it is simpler to work on a bounded domain.

3.4 Besov Spaces on the Circle

Following DeSouza's result in [7], we instead turned to the Besov-Bergman-Lipschitz spaces and their relationship with the special atoms space, stated below. Note that we have now returned to the setting of functions on the circle.

Definition 3.4.1. For $0 < \alpha < 1, 1 \leq r, s \leq \infty$, we define the Besov-Bergman-Lipschitz spaces $\Lambda(\alpha, r, s)$ as

$$\Lambda(\alpha, r, s) = \left\{ f : \mathbb{T} \to \mathbb{R}; \quad \|f\|_{\Lambda(\alpha,r,s)} < \infty \right\},$$

where

$$\|f\|_{\Lambda(\alpha,r,s)} = \|f\|_{L^r(dx)} + \left(\int_{\mathbb{T}} \frac{(\|f(x+t) - f(x)\|_{L^r(dx)})^s dt}{|t|^{1+\alpha s}} \right)^{\frac{1}{s}}.$$

When $s = 1$, we'll have a norm on this space. However, for other values of s, we'll only have a quasi norm (because the triangle inequality will only be satisfied with a constant). Moreover, this space is non homogeneous because of the presence of the L^r norm of f in the beginning.

The main result from [7] is what will allow us to have a suitable norm on the special atoms space. In this paper, the author begins by showing the following intermediate result.

Theorem 3.4.2. *For $1 < p < \infty$, if $f \in B^p, 1 < p < \infty$, then $f \in \Lambda(1-1/p, 1, 1)$. Moreover,*

$$\|f\|_{\Lambda(1-1/p,1,1)} \leq C_p \|f\|_{B^p},$$

where C_p is an absolute constant depending only on p.

This was proved by finding an upper bound for the norm of an atom. A similar computation will be repeated in Chapter 4. This result then leads to the main result of [7].

Theorem 3.4.3. *For $1 < p < \infty$, we have that $f \in B^p$ if and only if $f \in \Lambda(1 - 1/p, 1, 1)$. Moreover, there are absolute constants M and N such that*

$$N \|f\|_{B^p} \leq \|f\|_{\Lambda(1-1/p,1,1)} \leq M \|f\|_{B^p}.$$

Theorem 3.4.3 will allow us to have a suitable norm on the special atoms space. In order to be able to tie together all of the spaces for which a decomposition in terms of special atoms exists, that is relate $\dot{B}_1^{0,1}(\mathbb{R})$ to B^p and to $\Lambda(\alpha, r, s)$, we must let $p = 1$ in B^p. By Theorem 3.4.3, the equivalent Besov-Bergman-Lipschitz space would then be $\Lambda(0, 1, 1)$. However, this result is inconclusive in the literature in the case $p = 1$. Therefore, the best function space setting for computability notions to be explored is in the special atoms space B^p with $p > 1$ and with norm $\|\cdot\|_{\Lambda(1-1/p,1,1)}$. Thankfully, as stated previously, this is a true norm, so it induces a metric.

Chapter 4

Computability of the Besov Space

The goal of this chapter is to try to relate concepts from computability in the context of the Besov space.

4.1 Exploration

Definition 4.1.1. We consider distances on the circle modulo 2π. That is, for $x, y \in \mathbb{T}$, their distance on the circle is denoted as $|x - y|$.

This distance takes into account the periodicity which comes from identifying the endpoints of the circle with each other.

Definition 4.1.2. We define the distance between two subintervals I and J of the circle as

$$d(I, J) = |s_I - s_J| + |w_I - w_J|,$$

where s_I, s_J are the left endpoints of I and J respectively, and similarly w_I, w_J are the right endpoints.

Definition 4.1.3. If $p > 1$ is a computable real number, we define a computable special atom to be either a special atom $a_I : \mathbb{T} \to \mathbb{R}, a_I(x) = \frac{1}{\ell(I)^{1/p}}\chi_L(x) - \frac{1}{\ell(I)^{1/p}}\chi_R(x)$, where $I \subseteq \mathbb{T}$ has computable endpoints, or the function $b : \mathbb{T} \to \mathbb{R}, b(x) = 1/2\pi$.

For any given interval I with rational endpoints, the special atom a_I is a computable special atom, since rational numbers are computable. Moreover, the constant function $b = 1/2\pi$ is also a computable special atom since π is a computable number.

Definition 4.1.4. We define a rational special atom as a special atom whose support is an interval with rational endpoints.

Let $p > 1$ and a_{I_1} and a_{I_2} be two special atoms supported in $I_1 = (s_1, w_1), I_2 = (s_2, w_2)$, respectively and let $m_i = \frac{s_i+w_i}{2}, i = 1, 2$ be the midpoints of each interval. We note that one of (or both) the special atoms may be of the exceptional form $b(x) = \frac{1}{2\pi}$. However, in what follows we will disregard this case since it actually makes the calculations much simpler and is therefore not so insightful. So, we have

$$a_{I_1}(x) = -c_1\chi_R(x) + c_1\chi_L(x)$$
$$a_{I_2}(x) = -c_2\chi_R(x) + c_2\chi_L(x),$$

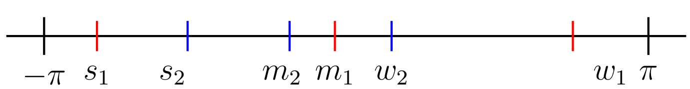
where $c_1 = \frac{1}{\ell(I_1)^{1/p}}$ and $c_2 = \frac{1}{\ell(I_2)^{1/p}}$. We have 5 possible cases for the configuration of the intervals:

1. $I_1 \cap I_2 = I_2$ or $I_1 \cap I_2 = I_1$

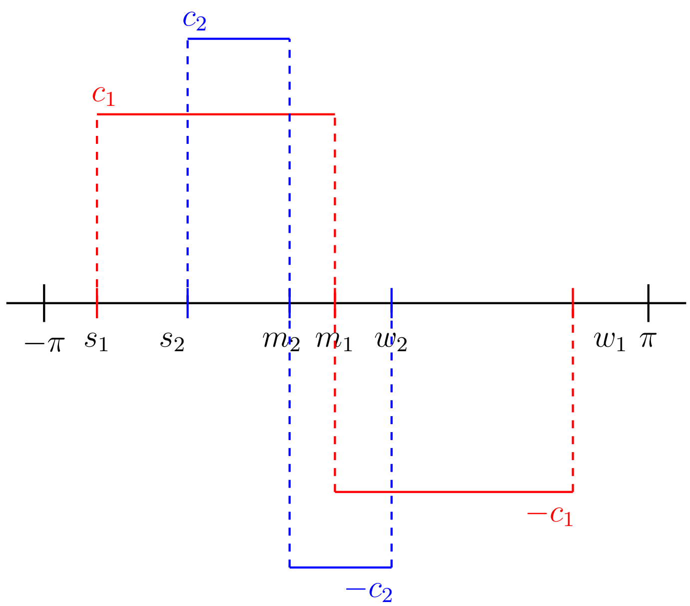
2. $I_1 \cap I_2 = (s_2, w_1)$ or $I_1 \cap I_2 = (s_1, w_2)$

3. $I_1 \cap I_2 = \emptyset$

Since this is is just an exploration we will only consider the case 1, with an added assumption on their midpoints. This is illustrated below:



So, the special atoms look like:



Let now

$$d(x) = a_{I_1}(x) - a_{I_2}(x),$$

and let:

$$A_1 = (-\pi, s_1), A_2 = (s_1, s_2), A_3 = (s_2, m_2),$$
$$A_4 = (m_2, m_1), A_5 = (m_1, w_2), A_6 = (w_2, w_1), A_7 = (w_1, \pi).$$

Then,

$$d(x) = \begin{cases} 0 & \text{if } x \in A_1 \\ c_1 & \text{if } x \in A_2 \\ c_1 - c_2 & \text{if } x \in A_3 \\ c_1 + c_2 & \text{if } x \in A_4 \\ -c_1 + c_2 & \text{if } x \in A_5 \\ -c_1 & \text{if } x \in A_6 \\ 0 & \text{if } x \in A_7 \end{cases}$$

In order to simplify calculations to come, we let, for $i \in \{1, \ldots, 7\}$

$$A_i = (l_i, r_i), B_i = |d(x)| \text{ for } x \in A_i$$

and for $x \in A_i$ and $y \in A_j$

$$C_{i,j} = |d(x) - d(y)|.$$

Now, by DeSouza's result and a change of variables, we have that

$$\begin{aligned}
\|a_{I_1} - a_{I_2}\|_{B^p} &= \|d\|_{\Lambda(1-1/p,1,1)} \\
&= \|d\|_{L^1} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|d(x+t) - d(x)|}{|t|^{2-1/p}} dxdt \\
&= \|d\|_{L^1} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|d(x) - d(y)|}{|x-y|^{2-1/p}} dxdy.
\end{aligned}$$

We begin by calculating the L^1 norm. Recalling that $c_1 \le c_2$ and that both are positive constants, we may perform the following computations

$$\begin{aligned}
\|d\|_{L^1} &= \sum_{2\le i\le 6} \int_{A_i} B_i dx \\
&= \sum_{2\le i\le 6} B_i \ell(A_i) \\
&= c_1\ell(A_2) + (c_2 - c_1)\ell(A_3) + (c_1 + c_2)\ell(A_4) + (c_2 - c_1)\ell(A_5) + c_1\ell(A_6) \\
&= c_1(s_2 - s_1) + (c_1 - c_2)(m_2 - s_2) + (c_1 + c_2)(m_1 - m_2) + (c_2 - c_1)(w_2 - m_1) + c_1(w_1 - w_2).
\end{aligned}$$

Now for the second part of the Besov norm:

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|d(x) - d(y)|}{|x-y|^{2-1/p}} dxdy.$$

In order to evaluate this integral, we look at different values of x and y in the domain of integration and see that the calculation of the double integral can be generalized as below. If $(x, y) \in A_i \times A_j$, without loss of generality we can assume that $j < i$ (since the integrand is symmetric in x and y and moreover, if $i = j$ then $C_{i,j} = 0$). Let us also assume that $l_i - r_j \le \pi$ so that $|x - y| = x - y$. Setting $\alpha = 2 - 1/p$, we have $\int_{A_j}\int_{A_i} \frac{|d(x)-d(y)|}{|x-y|^{2-1/p}} dxdy =$

$$\begin{aligned}
&= C_{i,j} \int_{l_j}^{r_j}\int_{l_i}^{r_i} \frac{1}{(x-y)^\alpha} dxdy \\
&= \frac{C_{i,j}}{1-\alpha} \int_{l_j}^{r_j} (r_i - y)^{1-\alpha} - (l_i - y)^{1-\alpha} dy \\
&= \frac{C_{i,j}}{(1-\alpha)(2-\alpha)} \left((r_i - l_j)^{2-\alpha} - (r_i - r_j)^{2-\alpha} + (l_i - r_j)^{2-\alpha} - (l_i - l_j)^{2-\alpha}\right)
\end{aligned}$$

$$= \frac{C_{i,j}}{(1-1/p)(1/p)} \left((r_i - r_j)^{1/p} - (r_i - l_j)^{1/p} - (l_i - r_j)^{1/p} + (l_i - l_j)^{1/p} \right).$$

Note that because of our simplifying assumption, we do not have to deal with the periodicity. In a more general case, the expression above would be similar and will involve the endpoints of the intervals along with π.

Finally, putting this together with the first part of the Besov norm, we get that:

$$\|a_{I_1} - a_{I_2}\|_{B^p} = \sum_{2 \le i \le 6} B_i \ell(A_i) + \frac{1}{(1-1/p)(1/p)} \sum_{1 \le i,j \le 7} C_{i,j} D_{i,j}.$$

Recall that this expression for the Besov norm of the difference of any two special atoms a_{I_1} and a_{I_2} was obtained under the assumption that the intervals I_1 and I_2 are configured as in Case 1. The other two cases remain to be covered, however we can easily adapt the explicit formula for the norm to the other configurations of the intervals. In any case, we'll get an expression which is similar to the one above (by similar, we mean that the terms depending on i, j will still be linear combinations of endpoints of the partition and of π, so it changes nothing as far as computability is concerned). This analysis will be particularly useful when proving the computability of the norm in Section 4.3.

In fact, this exploration allows us conclude something more general regarding the form of the Besov norm of step functions.

Lemma 4.1.5. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is a step function of the form*

$$f(x) = \sum_{i=1}^{n} c_i \chi_{I_i}(x),$$

where $\{I_i\}$ is a partition of the interval $\mathbb{T}$. Then

$$\|f\|_{B^p} = \sum_{i=1}^{n} |c_i| \ell(I_i) + \frac{1}{(1-1/p)(1/p)} \sum_{1 \le i,j \le n} |c_i - c_j| D_{i,j},$$

where $D_{i,j}$ will be some linear combination of π and terms depending on the endpoints of the intervals I_i and I_j, with each element of the combination being raised to the power $1/p$ (see $D_{i,j}$ above for an example).

We continue the analysis of Case 1 by looking at how the norm of the difference of two special atoms behaves when both special atoms are very close.

Let $I = (s, w)$ and $I_k = (s_k, w_k)$ be such that $I_k \to I$ as $k \to \infty$, and assume that they are configured as in case 1 (from above). By the convergence of the intervals, we have that

$$|s - s_k| \to 0, |w - w_k| \to 0. \quad (1).$$

If we consider the midpoints of the intervals, call them m and m_k, it also follows immediately that

$$|m - m_k| \to 0. \quad (2).$$

Moreover, letting $c = \frac{1}{\ell(I)^{1/p}}$ and $c_k = \frac{1}{\ell(I_k)^{1/p}}$, we can see that

$$|I_k| \to |I| \implies c_k \to c \implies |c - c_k| \to 0 \quad (3).$$

Now, we must show that both parts of the Besov norm converge to 0. To this end, let us first consider the L^1 norm:

$$\|a_I - a_{I_k}\|_{L^1} = c(s_k - s) + (c - c_k)(m_k - s_k) + (c + c_k)(m - m_k) + (c_k - c)(w_k - m) + c(w - w_k).$$

The elements of the addition above each go to zero, by facts (1), (2) and (3). Hence, this shows that $\|a_I - a_{I_k}\|_{L^1} \to 0$. It remains to show that the second part of the Besov norm converges to zero. To that end, we will analyze

$$\int_{A_j} \int_{A_i} \frac{C_{i,j}}{|x - y|^{2-1/p}} dx dy,$$

which is an equivalent expression for the second part of the Besov norm, for different combinations of i and j. We will see that the convergence of the second part of the Besov norm to zero will be due to one of two reasons: either the measure of the domain of integration will go to zero, or the integrand goes to zero. In fact, this will require a separate analysis of the three following cases:

1. $i = j$ (trivial case)

2. one of (or both) i and j is even

3. both i and j are odd.

Before studying all three cases, we first note that the function

$$f(x,y) := \frac{C_{i,j}}{|x-y|^{2-1/p}} \in L^1(\mathbb{T}^2),$$

for any pair of i and j, f will vanish on squares $A_i \times A_i$ covering the diagonal (see case i below). So, the absolute continuity of the integral gives us that

$$\iint_S f \to 0 \text{ as } |S| \to 0.$$

Now, in:

(Case i): $i = j$.

In this case, since $x \in A_i$ and $y \in A_j$, we'll have that $b(x) = b(y)$. So, $C_{i,j} = 0$, meaning that the integrand is zero on $A_i \times A_i$ so

$$\int_{A_j}\int_{A_i} f(x,y)dxdy = 0.$$

(Case ii): one of (or both) i and j is even

This is in fact that case where the convergence is due to the measure of the domain of integration going to zero. By observing the A_i's above, we can see that the even indexed subintervals are those whose lengths go to zero as $I_k \to I$. Hence, whenever one of (or both) i and j is even, we'll have that

$$|A_i \times A_j| = \ell(A_i)\ell(A_j) \to 0.$$

It therefore follows immediately from the absolute continuity of the integral mentioned above that

$$\int_{A_j}\int_{A_i} f(x,y)\, dxdy \to 0.$$

(case iii): both i and j are odd.

In this case, it can be seen that $C_{i,j} = |c - c_k|, |c_k - c|, 2|c - c_k|$ or $2|c_k - c|$. However, all four values will go to zero as $k \to \infty$. Hence, once again, it follows that

$$\int_{A_j}\int_{A_i} \frac{C_{i,j}}{|x-y|^{2-1/p}}\, dxdy \to 0.$$

We have therefore illustrated that, in case (i), as the supports of two special atoms get very close, the Besov norm of the difference of both functions will consist of the sum of the elements which are themselves going to zero. In fact, the following proposition shows that this result holds in general (i.e. without any assumptions as to how the intervals are configured).

Proposition 4.1.6. *For a fixed $p > 1$, if I is a subinterval of $\mathbb{T}$ and $\{I_k\}_{k=0}^{\infty}$ is a sequence of subintervals of $\mathbb{T}$ converging to I, then $\|a_I - a_{I_k}\|_{B^p} \to 0$.*

Proof. We prove the proposition without any assumption on the configuration of the intervals I and any given I_k. Let $I = (s, w)$ and $I_k = (s_k, w_k)$ (with midpoints m and m_k) such that $I_k \to I$. That is, we have that

$$d(I, I_k) \to 0 \text{ as } k \to \infty.$$

Claim 1.

$$|s - s_k| \to 0, \quad |w - w_k| \to 0, \ \textit{and} \ |m - m_k| \to 0 \quad \textit{as} \quad j \to \infty.$$

<u>Proof of claim:</u> Since $d(I, I_k) = |s - s_k| + |w - w_k|$, it follows immediately that

$$|s - s_k| \to 0, |w - w_k| \to 0.$$

Now, since the midpoints are given by

$$m = \frac{s + w}{2}, m_k = \frac{s_k + w_k}{2},$$

we get

$$|m - m_k| = \left|\frac{s + w - s_k - w_k}{2}\right| \leq \left|\frac{s - s_k}{2}\right| + \left|\frac{w - w_k}{2}\right|,$$

from which the convergence is clear.

Consider $c := \ell(I)^{-1/p}$ and $c_k := \ell(I_k)^{-1/p}$.

Claim 2. *$c_k \to c$ as $k \to \infty$.*

The proof of this claim is omitted, as it is very similar to that of the previous claim.

Claim 3. *$a_{I_k} \to a$ pointwise for a.e. $x \in \mathbb{T}$ as $k \to \infty$.*

Proof of claim:

This claim follows immediately by applying the result from the previous claim. Namely, and assuming $c_k \to c$, we'll have for a.e. $x \in \mathbb{T}$

$$c_k\chi_{(s_k,m_k)}(x) \to c\chi_{(s,m)},$$

and

$$-c_k\chi_{(m_k,w_k)}(x) \to -c\chi_{(m,w)}.$$

So, this gives

$$a_k(x) = c_k\chi_{(s_k,m_k)}(x) - c_k\chi_{(m_k,w_k)}(x) \to c\chi_{(s,m)} - c\chi_{(m,w)} = a(x).$$

Note that this convergence in fact cannot be uniform, and this is in fact observed in our explicit from before.

Now, we let

$$d_k(x) = a_k(x) - a(x), \quad f_k(x,y) = \frac{|d_k(x) - d_k(y)|}{|x-y|^{2-1/p}}.$$

By DeSouza's result, we have that

$$\|a_{I_k} - a_I\|_{B^p} = \|d_k\|_{L^1} + \int_{\mathbb{T}}\int_{\mathbb{T}} f_k(x,y)\,dxdy.$$

Claim 4. *$\|d_k\|_{L^1} \to 0$ as $k \to \infty$.*

Proof of claim:

The convergence to zero of the first part of the norm follows from the Generalized Lebesgue Dominated Convergence Theorem, combined with Claim 2. Firstly, the sequence $\{d_k\}$ is a sequence of $L^1(\mathbb{T})$ functions and $|d_k| \to 0$ pointwise a.e. by Claim 3. Now can dominate each $|d_k|$ as

$$|d_k(x)| = |(a_{I_k} - a_I)(x)| \leq c_k + c \text{ pw a.e..}$$

Letting $g(x) = c_k + c$, then $g \in L^1(\mathbb{T})$ and $c_k + c \to 4c$ by Claim 2. Moreover,

$$\int_{\mathbb{T}} c_k + c \to \int_{\mathbb{T}} 2c = 4\pi c < \infty.$$

Hence, by the Generalized Lebesgue Dominated Convergence Theorem, it follows that

$$\int_{\mathbb{T}} |a_{I_k} - a_I| \to 0, \text{ as } k \to 0.$$

In other words

$$\|d_k\|_{L^1} \to 0.$$

Claim 5. $\int_{\mathbb{T}}\int_{\mathbb{T}} f_k(x,y)dxdy \to 0$ *as* $k \to \infty$.

Proof of claim:

For a fixed k, in order to explicitly calculate this double integral, we must see how I_k and I are positioned with respect to each other. Depending on this, we partition the circle accordingly, say into N_k different subintervals where N_k is a finite number. So, $\cup_{i=1}^{N_k} A_i = \mathbb{T}$, where $A_i = (l_i, r_i)$ and if $i < j$ then $r_i < l_j$. Then,

$$\int_{\mathbb{T}}\int_{\mathbb{T}} f_k(x,y)\,dxdy = \sum_{1\le i,j\le N_k} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy.$$

In fact, since a_{I_k} and a_I each define three points (left, right and midpoint), this means that the circle will be divided into 7 intervals. So we can claim that the number of intervals is bounded above by $N = 10$. Namely,

$$\sum_{1\le i,j\le N_k} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy \le \sum_{1\le i,j\le N} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy.$$

On each rectangle $A_j \times A_j$, f_k is a constant, call it C_{ij} divided by either $(x-y)^\alpha$ or $(y-x)^\alpha$, where $\alpha = 2 - 1/p$. In fact, we can divide the integral as

$$\sum_{i=j} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy + \sum_{i<j} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy + \sum_{i>j} \int_{A_i}\int_{A_j} f_k(x,y)\,dxdy.$$

For the first summand: since $i = j$, we have that $C_{ij} = 0$ so $f_k = 0$ and the integral goes to zero.

For the other two summands: $f_k = \frac{C_{ij}}{(y-x)^\alpha}$ or $f_k = \frac{C_{ij}}{(x-y)^\alpha}$. A direct calculation of the integral will give an expression which goes to zero because of one of two reasons. First, for some combinations of i and j, we'll have that $\ell(A_i)$ or $\ell(A_j)$ will go to zero which results in the integral going to zero. The second reason is because, if the length of the rectangle $A_i \times A_j$ does not go to zero, then we'll have that C_{ij} goes to zero. In this scenario, the integral will still go to zero as it can be written as C_{ij} times some expression which is bounded independently of k. One can refer to the exploration above in order to see how these two reasons come about. Hence, $\int_{\mathbb{T}}\int_{\mathbb{T}} f_k(x,y)dxdy$ can be written as the sum of three terms, all of which go to zero as $k \to \infty$.

Finally, our main result follows by Claims 4 and 5. □

This proposition is powerful in that it allows us to conclude that any special atom, with the exception of $b(x) = 1/2\pi$, can be approximated by a sequence of special atoms living in rational intervals in the Besov norm. Namely, that the space consisting of finite linear combinations of special atoms with support in rational intervals and the function $b(x) = 1/2\pi$ is dense in $B^p, p > 1$.

4.2 Establishing Computability Notions on the Besov Space

For $p > 1$, consider the set

$$D = \left\{ \sum_{j=0}^{n} \lambda_j g_j : \lambda_j \in \mathbb{Q}, g_j \text{ is a rational special atom or } g_j = b, n \in \mathbb{N} \right\}.$$

From before, we know that D is dense in B^p. Now we can consider a fixed enumeration of all rational intervals (we know this exists by Proposition 2.5.5), call it $\{I_k\}_{k\in\mathbb{N}}$, we let

$$a_k = \begin{cases} a_{I_{k-1}} & \text{if } k \geq 1 \\ \frac{1}{2\pi} & \text{if } k = 0. \end{cases}$$

This notation illustrates the fact the enumeration of rational intervals allows us also to enumerate the set of rational special atoms, since to each rational interval $I_{k-1}, k \geq 1$, we may identify uniquely the rational special atom a_k. A natural progression would therefore be to enumerate the set D, in a manner similar to that found in Propositions 2.5.2 to 2.5.5.

Theorem 4.2.1. *Let D be defined as above. There exists a sequence $\alpha : \mathbb{N} \to D$ such that α is the composition of computable functions and*

$$E = \{\alpha(i) : i \in \mathbb{N}\} = D.$$

Proof. Recall that from Proposition 2.5.4, we know there exists an enumeration of all finite sequences of rational numbers. Namely,

$$\{\big(r(\sigma(i,0)),\ldots,r(\sigma(i,\eta(i)))\big) : i \in \mathbb{N}\},$$

where $r : \mathbb{N} \to \mathbb{Q}$ is a computable surjection and

$$\sigma(i,k) = (i)_k \;\dot{-}\; 1, \quad \eta(i) = \begin{cases} \max\{k : p_k \text{ divides } i\} & \text{if } i \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

In order to simplify notation, we denote $r(n) = r_n$. In other words, the $i-th$ finite sequence of rational numbers will be given by

$$\big(r_{\sigma(i,0)},\ldots,r_{\sigma(i,\eta(i))}\big).$$

Setting $\alpha : \mathbb{N} \to D$ as

$$\alpha(i) = \sum_{j=0}^{\eta(i)} r_{\sigma(i,j)} a_j$$

We must show that the inclusion between both sets holds in both directions. That is, we show the following claims.

Claim 1. $E \subseteq D$.

Proof of claim: Fix an $i \in \mathbb{N}$ and consider

$$\alpha(i) = \sum_{j=0}^{\eta(i)} r_{\sigma(i,j)} a_j \in E.$$

Clearly $\alpha(i)$ is a finite linear combination of the functions a_k,with rational coefficients, since the function r maps onto $\mathbb{Q}$. Hence, $\alpha(i) \in D$ and since i was arbitrary, this proves the claim.

Claim 2. $D \subseteq E$.

Proof of claim: Consider a function $f \in D$.

$$f = \sum_{j=0}^{n} \lambda_j g_j,$$

for some $n \in \mathbb{N}$ and where for each j, $\lambda_j \in \mathbb{Q}$ and g_j is a rational special atom or the special atom $b(x) = 1/2\pi$.

We must first identify each g_j by an a_{m_j} where we recall that the definition of a_k above. To that end, we must re-order the g_j by letting

$$g_j = a_{m_j}, m_j \in \mathbb{N}, 0 \leq j \leq n.$$

So, letting $N = \max\{m_0, \ldots, m_n\}$, we now have

$$f = \sum_{j=0}^{N} \beta_j a_j,$$

For some rational numbers $\beta_0, \ldots, \beta_N$. Now, since the sequence $(\beta_0, \ldots, \beta_N)$ is a finite sequence of rational numbers, by Proposition 2.5.4, we know there exists an $i \in \mathbb{N}$ such that

$$(\beta_0, \ldots, \beta_N) = (r_{\sigma(i,0)}, \ldots, r_{\sigma(i,\eta(i))}).$$

Thus, we can finally conclude that for some $i \in \mathbb{N}$, f can be written as

$$\alpha(i) = \sum_{j=0}^{N} r_{\sigma(i,j)} a_j = f.$$

Hence, $f \in E$

□

This proposition shows that $\alpha : \mathbb{N} \to B^p$ is a dense sequence in B^p. We continue our exploration of computability within the context of the special atoms space by looking at computability structures. In particular, we have the following result.

Theorem 4.2.2. *Let D be defined as above. For a fixed computable number $p > 1$, there exists a dense computability structure, $\mathcal{S}$, on the special atoms space B^p such that $D = \mathcal{S}^0$.*

Proof. Fix a computable real number $p > 1$. For $i, j \in \mathbb{N}$, we know that the two elements from the set D will look like

$$\alpha_i(x) = \sum_{k=1}^{n_i} \lambda_{i_k} a_k(x),\ \alpha_j(x) = \sum_{k=1}^{n_j} \lambda_{j_k} a_k(x),$$

where $\lambda_{i_k}, \lambda_{j_k} \in \mathbb{Q}$ (so they are computable), $n_i, n_j \in \mathbb{N}$ (so they are computable) and a_k is a computable special atom.

So, $\alpha_i - \alpha_j$ will clearly be a step function. Letting $c_k = \lambda_{i_k} - \lambda_{j_k}$ and $N = \max\{n_i, n_j\}$, by Lemma 4.1.5, we therefore know that there exists some partition of $\mathbb{T}$, call it $\{I_k\}$ such that

$$\|\alpha_i - \alpha_j\|_{B^p} = \sum_{k=1}^{N} |\lambda_{i_k} - \lambda j_k| \ell(I_k) + \frac{p^2}{p+1} \sum_{1 \le k,m \le N} |\lambda_{i_k} - \lambda_{j_k} - (\lambda_{i_m} - \lambda_{j_m})| D_{k,m}.$$

Under the assumption that all constants involved are rational numbers, for a fixed (i, j), $\|\alpha_i - \alpha_j\|_{B^p}$ will indeed be a computable number, since the explicit formula we give for it consists of finite sums of computable numbers. Hence, for the metric space (B^p, d), with d being the metric induced by the norm, we have that $d(f, g)$ is a computable number for all $f, g \in D$. Our result now follows immediately from Proposition 2.6.14. In other words, this shows that there exists a dense computability structure on B^p. □

Chapter 5

Open Questions

Since the bridge between computability and function spaces is still being built, there are many questions that remain open. In particular, we showed that for any two finite rational linear combinations of computable special atoms the number $\|a_{I_1} - a_{I_2}\|_{B^p}$ is computable. However, is it true that the norm of the difference of two finite linear combinations of atoms is computable uniformly as a function of i, j? Namely, is it true that

$$(i, j) \mapsto \|\alpha_i - \alpha_j\|_{B^p}$$

is a computable as a function $\mathbb{N}^2 \to \mathbb{R}$? If one succeeds in showing this, it would then be true that the for computable $p > 1$ the Besov space (B^p, d) with metric induced by the Besov-Bergamn-Lipschitz norm (with $\alpha = 1 - 1/p, s = r = 1$), is a computable metric space. One can also address this question for other values of p.

One can also try to look at establishing other computability structures on the Besov space. For example, since we showed that there exists a dense computability structure on B^p for $p > 1$, a natural question would be to ask if there exists a separable computability structure on this space. This question is equivalent to saying that there exists α such that (B^p, d, α) is a computable metric space.

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