College level students’ understanding of functions in the context of modelling dynamic situations

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Abstract

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Concordia University, 2021

One of the critical tasks in early mathematical education is teaching the concept of function and its different representations through diverse examples and dynamic tasks. This task is meant to enable students to develop meaningful and powerful constructions between functional concepts and procedures, to conceptualize a function as a generalized process that accepts input and produces output and, therefore, to build essential conceptual and analytical thinking of functions that are necessary to form a covariational conception of function. This is particularly useful for College level students enrolled in Calculus courses who need a strong covariational conception of function to further succeed in mathematics courses.

My thesis reveals that, from a sample group of college level students enrolled in Calculus courses at Concordia University, many participants do not have a strong conceptual or/and analytical thinking of functions that is essential in meaningfully modeling dynamic situations. Majority of students, who have obstacle(s) or/and pseudo-thought(s) or/and misconception(s) due to them having weak analytical/conceptual understanding of functions, do not move flexibly between different representations of functions and fail to view a function as a process that maps values of one variable to values of another variable. Consequently, these students generally do not reason dynamic functional situations covariationally at a higher level. This outcome questions the readiness of Calculus students for subsequent mathematics courses, as well as the effectiveness of prior school curriculum in mathematics education.

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1 INTRODUCTION

Along the history of humankind, mathematics has been a leading actor in the development of society and the notion of function has been playing a critical role as a unifying concept, not only in different branches of mathematics, but also in other sciences such as physics, chemistry or biology, or everyday life, where it modelled real-life situations. For example, the narrative '*my height grows with age*' represents a function in which height is a dependent variable while time is a universal independent variable. How do college level students reason this basic situation? Do they notice that this dynamic phenomenon represents a function? Do they realize that height is an output variable while time is an input variable? Do students think how height changes continuously (increasing, generally until the age of 20s when it stops increasing) while time also changing continuously (always increasing), and these two variables changing in tandem? In other words, do college level students reason the given dynamic functional situation covariationally?

In this thesis, I will describe how students enrolled in college level Calculus courses at Concordia University understand functions when they are confronted with this notion in the context of problems about dynamic phenomena. The idea for the thesis was inspired by the work of Marilyn P. Carlson and her collaborators on students' understanding of Calculus, especially of the notion of function (Carlson, 1998; Carlson et al., 2002; Thompson & Carlson, 2017). The notion of function is fundamental in all domains of mathematics (under different names, e.g., mapping, transformation) and functional thinking is indispensable in applications.

As a motivation of my thesis, it is not only important to investigate whether college level Calculus students reason dynamic functional situations covariationally, but it is equally important to observe at what level they generally reason dynamic events covariationally in order to assess the effectiveness of current mathematical instruction in students' covariational thinking and the readiness of Calculus students for more advanced mathematics courses. Carlson et al. (2002) introduced five levels regarding students' covariational reasoning. For instance, in the example given above, if a student only makes a coordination between the two variables, height and time, then he/she reasons the given event covariationally at Level 1. A student's reasoning unable to move a higher covariational reasoning level (e.g. Level 2) may be due to her/his lack of strong conceptual or/and analytical thinking (or understanding) of functions when modelling functional relationships of dynamic situations. Thus, in my endeavour, I proceeded to examine the following research questions related to students' covariational reasoning:

How do college level Calculus students reason functional situations that rely on variables varying dynamically? Particularly, at what level do Calculus students generally reason dynamic functional situations covariationally?

What can we, as educators, conclude about Calculus students' conceptual and analytical thinking of functions in the context of modelling dynamic situations? How college level Calculus students being unable to reason dynamic function situations covariationally is related to Calculus students having a weak conceptual or/and analytical thinking of functions?

Students in traditional Calculus courses often hold a very limited idea of function, mostly as a formula of the form $y =$ [an algebraic expression formed of letters, numbers, names of elementary functions and arithmetic operations] (Giovaniello, 2017). They are generally used to having the formula given to them in a problem, than to having to model a relationship between varying quantities with a function. In this direction, a revealing comment on the students and Calculus courses is made by Carlson in (Carlson, 1998) and (Carlson et al., 2002):

> *The successful college algebra students in this study had limited understanding of many of the components of the function concept. Their narrow view of functions was demonstrated by the fact that they thought any function could be defined by a single formula and that all functions must be continuous. They did not understand the function notation and had difficulty understanding the role of the independent and dependent variable given a functional relationship. (Carlson, 1998, p.137)*

Some studies suggest that students have enormous difficulties with representations of functions (Vinner & Dreyfus, 1989) and so in moving flexibly and fluidly between different representations of a given functional situation (Oehrtman, Carlson & Thompson, 2008) due to highly procedural orientation in mathematical education (Kaldrimidou & Ikonomou, 1998), which may contribute to students being unable to reason dynamic events covariationally. When dealing with functions, students may use meaningless conceptual associations or/and rely on their uncontrolled memorized algebraic expressions (formed by school curriculum), and so be in pseudo-analytical or/and pseudo-conceptual modes of thinking (Vinner, 1997) which may also prevent them reasoning dynamic functional situations covariationally. In addition to having pseudo- thoughts when working with functions, students also may have some obstacles or/and misconceptions. Some of these obstacles can be classified as epistemological, didactical and cognitive obstacles (Bachelard, 1938/1983; Brousseau, 1997; Herscovics, 1989; Sierpinska, 1992; Sierpinska, 2019). Thus, I am also interested to examine the following questions;

What are common epistemological, didactical and cognitive obstacles, pseudo-thoughts and misconceptions in preventing college level Calculus students to successfully complete a dynamic function task?

Is there a certain correlation between Calculus students having obstacle(s) or/and pseudo-thought(s) or/and misconception(s), Calculus students having weak conceptual or/and analytical understanding of functions and Calculus students being unable to reason dynamic events covariationally at a higher level?

With my research project, I attempt to obtain and analyze data that would allow me to get some answers and formulate certain conjectures to these questions.

This is mostly a qualitative research study, although some descriptive statistics will be provided. The emphasis will be on evaluating and studying various ways students are thinking about functions in the context of modeling a physical phenomenon. Students enrolled in Concordia's course MATH 205 – Differential and Integral Calculus II, in Winter 2019 were given weekly assignments as extra-credit homework. During the beginning of the semester, students were given a first homework called Assignment 1 consisting of Problem 1 for which twenty-four students provided responses, and a second homework called Assignment 2 composed of Problem 2. While eighteen students provided solutions for Problem 2, I analyzed the solutions of fifteen participants who provided responses for both Problems 1 and 2 since I did not enough data for the remaining three students. Students were given more assignments in the following weeks, but their participation rate went down dramatically each week and some of students who provided solutions to Problem 1 and 2 did not provide solutions for the other problems given in the following weeks. Due to the inconsistency of participation, time constraints and other factors, with the advice of my supervisors, a decision was made to analyze the students' responses to Problems 1 and 2 only.

In these two assignments, I have asked students enrolled in MATH 205 two modified versions of the same problem that Carlson et al. used in (Carlson et al., 2002). – the "bottle filling" problem. The formulation of the bottle problem was modified to add more questions in order to compensate for the fact that I did not interview the students, as Carlson et al. did. Hence, in my research, I produced two problems associated with the bottle problem; one has a conceptual basis (Problem 1), while the other has an analytical basis (Problem 2). I adopted Carlson et al.'s theoretical construct. However, differently from their construct, I divided behaviors associated with covariational reasoning in three sections; graphical, verbal and algebraic representations of function while Carlson et al. considered only graphical and verbal representations of function in their study. I also made some changes in the description of behaviors of verbal and graphical representations of function. More substantial changes are the addition of 'representing relative magnitude(s) of the output variable on the y-axis while picturing relative magnitude(s) of the input on the x-axis' in MA3 (since it is important to show relative magnitude(s) symbolically/numerically on the axes at this level) and modification of description of behavior as 'constructing a line, curve, with definite directions' in MA2 for graphical representation of function (since an individual may not image only a straight line in her/him mind). All these additions and modifications are discussed in the theoretical framework.

Another substantial change, added a posteriori, is that if a student's behavior is supportive of MA1, MA2, MA3 and MA5, but not necessarily of MA4, we can say that her/his covariational reasoning has reached Level 5. As mentioned later in the theoretical framework, the mental actions MA1 through MA5 are supported by level 5 (L5) images and the instantaneous rate of change is conceptualized as resulting from smaller and smaller refinements of the average rate of change. However, it is likely that an individual has made this conceptualization of the average rates of change leading to the instantaneous rate of change sometime in the past and, later, when faced with a specific task, the individual does not repeat this conceptualization again because it is considered perhaps routine or simply an intermediate step already mastered. Therefore, the individual may not exhibit an awareness of the average rate of change of the output (thus no signs of MA4), but by coordinating the amount of change of the output (signs of MA3) and exhibiting a correct reasoning about the instantaneous rate of change of the output while picturing continuous changes in the input (signs of MA5), we allow the benefit of the doubt to prevail and pass to MA5.

In the next chapter, I offer a review of previous existing research on students' understanding of functions as recognized by specialists in mathematics education. The review does not have the ambition of being comprehensive and, instead, focuses on research into the covariational thinking about functions. In Chapter 3, I describe the theoretical framework that I used in analyzing students' solutions. Chapter 4 presents the methodology – the context and tools of how the research was conducted. Chapter 5 presents the results of my research organized into three sections: analysis of the students' responses for Problem 1, analysis of the students' responses for Problem 2 and comparative analysis of the results obtained for Problems 1 and 2. The final chapter, Chapter 6, offers some conclusions, recommendations and directions of future possible research.

2 LITERATURE REVIEW: RESEARCH ON STUDENTS' UNDERSTANDING OF FUNCTIONS

There are two perspectives in research on students' understanding of functions: 1) the function is viewed mainly as a formally defined mathematical concept; 2) the function is viewed mainly as a modeling tool for relationships between varying quantities. The first perspective is related to a set-theoretic conception of function which is represented today by the Dirichlet-Bourbaki definition, that is, "*a function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set*" (Vinner & Dreyfus, 1989, p.360). Students have difficulties understanding this formal definition of function, confuse the defining condition (or univalence condition) with the condition of one-to-one function and have trouble with representations of functions. The second perspective is associated with the dependence relation between quantities (or Euler's definition) that represents a covariational conception of function (Carlson et al., 2002, Thompson & Carlson, 2017), that is, "*a function is a dependence relation between two variables*" (Vinner & Dreyfus, 1989, p.360). In either context, students' understanding of modelling dynamic functional tasks can be obstructed by many factors while, in itself, the modelling could be quite a complex process.

More than four decades, researchers have been investigating students' and prospect teachers' understanding of the function concept (Carlson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Carlson & Oehrtman, 2004; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2010; Carlson, Madison & West, 2015; Even, 1992; Even & Bruckheimer, 1998; Herscovics, 1989; Kaldrimidou & Ikonomou, 1998; Malik, 1980; Markovits et al., 1986; Monk, 1992; Oehrtman, Carlson, & Thompson, 2008; Saldanha & Thompson, 1998; Sajka, 2003; Sierpinska, 1992; Sierpinska, 2019; Thompson, 1994; Thompson, 1994b; Thompson & Carlson, 2017; Vinner, 1983; Vinner & Dreyfus, 1989). The common result in these studies is that students (and even prospect teachers) have enormous difficulties in understanding the notion of function. These difficulties are related to having sometimes pseudo-thoughts (Vinner 1997), epistemological obstacles (Sajka, 2003; Sierpinska, 1992; Sierpinska, 2019), cognitive obstacles (Herscovics, 1989), didactical obstacles (Bachelard, 1938/1983; Brousseau, 1997) or misconceptions (Giovanniello, 2017). So, the question is: What actions students need to take in order to understand the function concept and overcome these difficulties? The answer to this question is well argumented by Sierpinska (1992).

When dealing with functions, a student first needs to recognize the existence of '*World of changes'*, then identify '*What changes*' in this world, and then look for some regularities in relationships between varying quantities (Sierpinska, 1992). In her work, Sierpinska provided the first two conditions of understanding functions as 'identification of changes observed in our world' and 'identification of regularities in relationships between changes': "*U(f)-1: Identification of changes observed in the surrounding world as a practical problem to solve*" and "*U(f)-2: Identification of regularities in relationships between changes as a way to deal with the changes*" (Sierpinska, 1992, p.31). Another important act of understanding identified by Sierpinska is "*U(f)-3: Identification of the subjects of change in studying changes*" in order to overcome the following epistemological obstacle "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes"* (Sierpinska, 1992, p.36). When an individual envisions '*World of changes*' and correctly identify '*What changes*' in this world, then he/she coordinates between variables, in other words, he/she starts to reason the situation covariationally at Level 1 (at least) (Carlson et al., 2002). Once he/she makes this basic coordination between variables, then he/she shall think of regularities in relationship between variables in order to reason the event covariationally at a higher level (e.g. Level 3). Ideally, school curriculum including textbooks developers would design mathematical instruction that encourage students to observe the existence of '*World of changes*', identify '*What changes*' in this world, and conceptualize '*how quantities change together in tandem*' in this world of changes.

Over the next sections, I first present the historical development of the function concept up to the middle of the 20th century (section 2.1) and then discuss the epistemology of the function concept in the end of the 20th century and the beginning of the 21th century (section 2.2). The latter section is divided into two sub-sections: past research on understanding the function concept (sub-section 2.2.1) and epistemology of a covariational conception of function (sub-section 2.2.2).

2.1 Historical development of the function concept up to the middle of the 20th century

The general idea of function started to appear in the history of mathematics well before 2000 BC when Babylonians first created tabulated functions. It evolved with the works of Greek mathematicians, such as Ptolemy and Aristotle, and of 14th century European mathematicians, such as Oresme. Its evolution reached completion with the discoveries of 17th century mathematicians such as Descartes, Newton and Leibniz, of 18th century mathematicians such as Bernoulli and Euler, and of 19th century mathematicians such as Cauchy, Fourrier, Bolzano, Weierstrass and Dirichlet.

According to Boyer (1946), there are four main eras in the development of the function concept: proportion, equation, and function, the latter splitting into two periods. In the era of proportion, relationships between quantities were represented schematically and describing motion began to attract more attention in the mathematical community (e.g. Oresme in the 14th century). In the era of equation, relationships between quantities with constrained variation were represented algebraically after the Viète's introduction (in the 16th century) of new symbolic algebra, and geometrically on the coordinate system after Descartes' discovery of Cartesian Geometry (in the 17th century). In the first era of function, a relationship between two continuously varying quantities, with values of one variable determined by values of other variable, was explicitly represented by a formula or a graph (in the 17th – 18th centuries, e.g. Newton, Leibniz, Euler). In the second era of function, which started with Dirichlet's introduction (in the 19th century) of the formal definition of function that lasted through present day, a relationship between two continuously varying quantities is represented clearly by a distinct law of correspondence in which values of one variable determined uniquely by values of another variable.

Simultaneously, according to Youschkevitch (1976/77), there are three main periods of the development of the function concept up to the middle of the 19th century:

1) The antiquity in which mathematicians worked on certain cases of functional dependence between two quantities which consist of general notions of functions and of variable quantities.

2) The Middle Ages (during the 14th century, especially in Europe) in which geometric and mechanical dependences were represented.

3) The Modern Period (after the 16th century) in which analytical representations of functions were introduced and a group of analytic functions were formulated by sums of infinite power series (e.g. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,\ -\infty < x < \infty$).

In the Modern Period, an analytical representation of functions led to a more efficient method of solving problems and that changed mathematics dramatically. In the middle of the 18th century, Euler provided a general definition of function since having only analytical forms of functions was not satisfactory in mathematical analysis. Then, in the 19th century, Dirichlet introduced the formal definition of function that opened a new era in the development of the theory of functions. However, past studies suggest that students have difficulties in understanding of the set-theoretic definition of function (Even & Bruckheimer, 1998; Herscovics, 1989; Malik, 1980; Markovits et al., 1986; Sierpinska, 1992; Vinner, 1983; Vinner & Dreyfus, 1989).

Note: A more detailed description of the historical development of the function concept up to the middle of the 20^{th} century can be found in Appendix A.

2.2 Epistemology of the function concept in the end of the 20^{TH} century and the beginning of the 21th century

Euler (in 1744) first defined a function of a variable quantity as an analytic expression formed, in any manner, by the variable quantity and constants. In 1755, he changed his 'analytic-expression' definition of function to a more global definition of function, by considering a function as a dependence relation between quantities, which represents a covariational conception of function. By introducing a more global definition of function, Euler overcame an important epistemological obstacle in the history of mathematics, that is, defining a function as an analytic expression. However, the existence of 'analytic expression' obstacle in educational system throughout the history is undeniable: 'a function is an algebraic expression in one real variable' (Carlson M. P., 1998; Giovanniello, 2017; Sierpinska, 2019).

There are consequences of considering a function only as an analytic expression. For instance, based on Euler's definition, the Dirichlet function (representing the relation from $\mathbb{R}$ to $\mathbb{R}$ composed of pairs (x, y), such that y = 1 if $x \in \mathbb{Q}$, y = 0 if $x \notin \mathbb{Q}$) would not be a function since it is defined by the two different expressions for the two different intervals of the domain. On the other hand, a given situation may represent a relationship between quantities without having any rule (e.g. my height has been growing in time). In addition to these, an algebraic expression, such as $f(x) = \sqrt{x}$, without defining its domain would be considered as a function.

After the introduction of the Dirichlet's definition of a singled-valued function and of the Bourbaki's set-theoretic definition of function, the formal definition of function is called the Dirichlet-Bourbaki (or only Dirichlet) definition that is widely accepted and used in the educational system since 1960s. The definition of function given by teachers at the college or university level is the set-theoretic definition of function, involving a Cartesian product of two sets and a deterministic relation (or kind of correspondence) between these two sets, that is:

> *A Function from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$. (Thomas, 2008, p.2)*

In school curriculum or textbooks in which a function is usually defined by a law, the domain of the function is presumed to be the largest set of real numbers for which the analytic expression produces real output values, in case of the domain is not being mentioned explicitly. This is called the 'natural domain'.

In his book of '*Didactical Phenomenology of Mathematical Structures'*, Freudenthal (1983) noted that there are two essential aspects of the function concept; '*arbitrariness'* (being implicit) referring to the arbitrary nature of functions, and '*univalence'* (being explicit) referring to the rule which assigns to each element in the first set unique element in the second set. Euler's definition of function represents multi-valued functions while the Dirichlet-Bourbaki definition which demands '*univalence*' argument represents single-valued functions. Despite the importance of '*univalence*' condition, most of students and many teachers do not know why functions have to be univalent (Even, 1992; Even and Bruckheimer, 1998).

2.2.1 Past Research On Understanding The Function Concept

Researchers revealed that many students have enourmous difficulties in understanding of the set-theoretic definition of function (Even & Bruckheimer, 1998; Herscovics, 1989; Malik, 1980; Markovits et al., 1986; Sierpinska, 1992; Vinner, 1983; Vinner & Dreyfus, 1989). These studies suggest that the set-theoretic approach by itself is not very successful in educational system.

In 1989, Vinner and Dreyfus conducted a research to investigate college level students' images and definitions for the concept of function. They observed that most of students did not use the set-theoretic definition of function and the students' responses were formed by their previous learning experiences with functions from science courses. The results showed that most of college level students used a scientific interpretation of function (e.g. as a dependence relation, a rule or a formula) in their responses by avoiding a formal interpretation of function associated with the set-theoretic definition. Consequently, Vinner and Dreyfus (1989) questioned the necessity of teaching the set-theoretic definition in early calculus courses:

> *…at least a doubt should be raised whether the Dirichlet-Bourbaki approach to the function should be taught in courses where it is not intensively needed. If discontinuous functions, functions with split domains, functions with exceptional points, strange functions are needed, we think that they should be introduced as cases extending the students' previous experience. The formal definition should be a conclusion of the various examples introduced to the students. (p.365)*

Malik (1980) also questioned the need of teaching the Dirichlet-Bourbaki definition:

…the necessity of teaching the modern definition of function at school level is not at all obvious and most of the instructors feel that pedagogical considerations were ignored while designing the course content and the mode of presentation. (p.490)

He then suggested that the set-theoretic definition should be introduced later in topology courses since Euler's definition of function is sufficient for pre-calculus or calculus courses:

We note that the definition of function as an expression or formula representing a relation between variables is for calculus or a pre-calculus course; is a rule of correspondence between reals for analysis; and a set theoretic definition with domain and range is required in the study of topology. Since only a small percentage of school students eventually study analysis and topology, the set theoretic definition could be postponed to the beginning of these courses and a simple and easily understandable definition should be taught at the elementary level. (Malik, 1980, p.492)

Even and Bruckheimer (1998) similarly stated that there may be didactical advantages to focus on Euler's definition and delay the introduction of Dirichlet-Bourbaki definition of function until it is more beneficial. They recommended that students shall engage in mathematical activities in which teachers represent both single-valued and multi-valued functions without putting any emphasis on the univalence condition that should be introduced when it is needed. Students shall be encouraged to focus on relationships between quantities and different representations of functions. This may contribute to the students' learning of the function concept.

..a number of studies indicate that students have difficulties in understanding and using the function concept. Consequently, it seems more reasonable to focus student attention on essential aspects of functions, such as how they describe relationships between variables, their power to mathematise situations, how representing a function in different ways can expand understanding about their behavior and how approaching them in different ways (e.g. object vs. process, point-wise vs. globally) may be useful in different situations… (Even & Bruckheimer, 1998, p.32)

Students begin to learn the formal approach of the function concept interfering with the univalent restriction generally in high school or in the first year of their college. After the introduction of the Dirichlet-Bourbaki concept of function, students start to learn discriminating functions from non-functions. However, differences between functions and non-functions and reasons why functions need to be univalent are not sufficiently represented in textbooks and verbalized by teachers in mathematical activities of today's school curriculum (Even and Bruckheimer, 1998). Even though the Dirichlet-Bourbaki definition is generally introduced in the textbooks and school curriculum, the examples used by teachers and the exercises provided in the textbooks are mostly related to functions whose rule of correspondence is given by a formula. This may cause students to summon functions as mental pictures of a formula.

Last twenty years, researchers reported studies which revealed some fundamental understandings and reasoning abilities that students need to have in order to succeed in early calculus courses (Carlson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Carlson & Oehrtman, 2004; Carlson, Madison & West, 2010; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2015; Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017). Some of these essential understandings involve the meaning of the function concept, function evaluation, function composition, function inversion, being necessary to learn key concepts of pre-calculus and calculus courses such as rate-of-change of function, accumulation and continuous covariation. On the other hand, some of crucial reasoning abilities students need to have are '*a process conception (or across-time view) of function'* and '*a covariational conception of function*'.

The '*APOS theory*' introduced in the 1990s involves the construction of mental actions, processes and objects (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky & McDonald, 2001). An '*Action*' is a static transformation of objects through some repeatable manipulations in order to obtain other objects. When an individual has an action conception of function which is also called as a '*point-wise conception of function*' (Monk, 1992), she/he tends to rely on her/his computational reasoning abilities and to perform operations by step-by-step procedure (e.g. ability to plug numbers into an algebraic expression and compute). A '*Process*' conception of function (or across-time conception of function, Monk, 1992) interferes with a dynamic transformation of objects through some repeatable actions result in producing the same transformed object. A function is viewed as a generalized process which accepts input and produces output. An individual may start the mathematical activity with some objects, operate these objects and obtain new objects as a result of those repeatable actions. In this level, he/she shall be able to combine the process with other processes and reverse it- in general, operate the processes in multiple directions (Breidenbach et al., 1992).

Some research on students' understanding of the function concept remarked that pre-calculus and calculus students often have a static (or action) conception of function, and that students need to have a dynamic (or process) conception of function in order to reason a given functional situation covariationally (Carlson, 1998; Carlson & Oehrtman, 2004; Carlson, Madison & West, 2010; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2015; Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017).

Breidenbach, Dubinsky, Hawks and Nichols (1992) investigated difficulties that college level students have for the function concept. In the first task, they asked students '*What is a function?'* before any instruction. The results revealed that only 14% of college level students have had a process conception of function while 24% have had an action conception of function. Students often conceptualize functions in terms of algebraic manipulations and standard procedures without thinking of function as a general mapping from a set of input values to a set of output values. In order to develop a dynamic (or process) conception of function, one needs to view a function as a generalized process which accepts input and produces output.

Carlson (1998) similarly examined college level and undergraduate students' understandings of the function concept. She found that many pre-calculus students did not have a process conception of function since they thought about functions in terms of algebraic manipulations, nor did these students have had necessary conceptual knowledge for modelling dynamic functional situations. Carlson concluded that this outcome is a result of limited mathematical instruction in educational system:

> *…curriculum developers underestimate the complexity of acquiring an understanding of the essential components of the function concept… and that current curricula provide little opportunity for developing the ability to: interpret and represent covariant aspects of functions, understand and interpret the language of functions, interpret information from dynamic functional events. (Carlson, 1998, p. 142)*

Even (1992) investigated pre-service mathematics teachers' knowledge and understanding of the inverse function. In the given tasks, Even asked 152 prospective teachers the following question:

> *Given* $f(x) = 2x - 10$ *and* $f^{-1}(x) = \frac{x+10}{2}$. *Find* $(f^{-1} \circ f)(512.5)$. *Explain.*
>
> (Even, 1992, p.558)

She expected teachers to use the idea of 'undoing' and find the answer as 512.5 by knowing the inverse function undoes what the function does. Almost half of (43%) of teachers provided a correct answer by using the reasoning of 'undoing'. However, half of these respondents who used the inverse property did some extra, redundant computations in order to find or prove that the correct answer is 512.5. On the other hand, more than half (62%) of the participants did not use the inverse property of function, instead, they tried to do all computations in order to find an answer. One half of these participants completed correct calculations as:

$$f(512.5) = 2(512.5) - 10 = 1015$$

$$f^{-1}(f(512.5)) = f^{-1}(1015) = \frac{1015 + 10}{2} = \frac{1025}{2} = 512.5$$

It is not clear whether these respondents understood the inverse property, but did not remember it or use it, or whether their thinking which might be formed by '*rote learning*' (e.g. reacting based on their memorized procedural associations) was purely mechanical so they focused on showing their analytical knowledge instead of thinking the conceptual aspects of the given question. It is also not clear whether these participants have strong conceptual knowledge since they approached the given problem analytically instead of recalling the conceptual meaning of inverse function, or whether their thinking was based on naïve conception of composition of functions. On the other hand, it seems that the rest of the respondents who failed in reaching the correct answer did not consider the idea of inverse function at all. Even remarked that both procedural and conceptual knowledge, and the relationship between them, interfere in understanding an idea. Thus, it is important to focus on the relationship between conceptual and procedural knowledge. When an individual fails in making appropriate connections between concepts and procedures, he/she may produce some answers and even correct ones sometimes by coincidence, but he/she may not understand what he/she is doing (Even, 1992, p.561).

In 1992, Monk conducted a study involving dynamic function events to investigate calculus level students' difficulties in understanding the function concept by using Across-time questions that ask students to describe changes in the output while picturing changes in the input. One across-time problem he asked was a model of a moving ladder standing against a wall and then the top of the ladder moving down while the bottom of the ladder pulled away from the wall. Monk aimed to find out whether students have a dynamic conception of function when solving problems involve dynamic functional situations. Students were expected to explain changes in positions of the top and of the bottom of the ladder, and to sketch a graph of the given functional event represented in the ladder model. He observed that many students did not have an across-time (or process) conception of function since they were unable to describe patterns in the relationship between the two continuously changing quantities. These students often calculated particular values of the height of the top of the ladder from particular values of the distance between the bottom of the ladder and the wall, being suggestive of them having a point-wise (or action) conception of function. Monk (1992) found that calculus students' main difficulties were associated with '*Iconic translation*' of a given situation, that is, '*correlating the shape of a graph with the physical characteristic of a given situation*' being considered as a cognitive obstacle. There are other cognitive obstacles involved in '*transforming pairs of numbers from an equation into pairs of coordinates*' or '*transforming an algebraic expression of a function to its graphical one*' being well reported by Herscovics (1989).

When examined university mathematics students' and high school teachers' difficulties with graphical representation of functions, Kaldrimidou & Ikonomou (1998) found that students' and teachers' difficulties are associated with particular characteristics of graphs (cognitive nature), the effectiveness of procedures (metacognitive nature), and common misconceptions about the nature of mathematics (epistemological nature). The investigators concluded that majority of participants who paid little attention to geometrical construction of functions had a strong tendency to use standard algebraic procedures related to functions instead of focussing on conceptual aspects of the given mathematical tasks. They think that this is a result of narrow or faulty mathematical instruction in educational system that pays too much attention to algebraic expressions of functions, creating a didactical obstacle (Brousseau, 1997). Regarding the educational system focusing on algebraic procedures rather than graphical representation of functions, Kaldrimidou & Ikonomou made the following comments:

> *The subject of mathematics, as taught in school, gives rise to epistemological and metacognitive considerations that favor the algebraic rather than the graphic representation of problems. This means the creation of an implicit context of communication that reinforces the algebraic rather than the graphic representation of problems. This produces a vicious circle, in the sense that the solution of a problem by means of an algebraic representation results in a reinforcement of this representation with respect to its effectiveness (metacognitive characteristic). The continuous repetition, and its reinforcement in school, strengthens this kind of representation and makes it the prominent and most appropriate procedure in mathematics (epistemological characteristics). (Kaldrimidou & Ikonomou, 1998, p.286)*

In her investigation of a secondary student's understanding of the function concept, Sajka (2003) analyzed a dialogue held with an average secondary school student, Kasia. Sajka's analysis revealed that Kasia's main difficulties in understanding the function concept are related to the nature of a mathematical notation (epistemological nature), the limited choice of mathematical tasks and narrow contexts in school curriculum (didactical nature), and also her misinterpretation of a given functional situation (cognitive nature). During the dialogue, Sajka noticed that the student's attention was drawn only to numerical equations and unknowns, so Kasia failed in moving to the level of thinking about meaning of a function. Sajka (2003) pointed out that: '*function is for her not an encapsulated concept, but it is still in the process of construction'* (p.250). She described the Kasia's difficulty with one of the epistemological obstacles being identified as '*EO(f)-4: (Unconscious scheme of thought) Thinking in terms of equations and unknowns to be extracted from them*' by Sierpinska (1992) *On Understanding The Notion of function*. In order to overcome this obstacle, one needs to exhibit a behaviour that supports the act of understanding identified as '*U(f)-4: Discrimination between two modes of mathematical thought: one in terms of known and unknown quantities, the other-in terms of variable and constant quantities*' (Sierpinska, 1992, p.37-38). Sajka also observed in the dialogue that Kasia often considered the idea of function as the same as the idea of the formula of a function. This indicates that she was maybe under an influence of memorized associations formed by limited mathematical tasks. Regarding individuals thinking functions as formulas, Sierpinska identified the following epistemological obstacle: '*EO(f)-11: (A conception of function) Only relationships describable by analytic formulae are worthy of being given the name of function*' (Sierpinska, 1992, p.46). In order to overcome this obstacle, Sierpinska (1992) suggests that one needs to exhibit a behaviour that supports the act of understanding identified as '*U(f)-9: Discrimination between a function and the analytical tools sometimes used to describe its law'* (p.46).

It seems that promoting the set-theoretic definition of function in early calculus courses is not very successful. Promoting typical examples and standard algebraic procedures in mathematics education does not also lead to a successful understanding of functions in students. Because educational system encourages students to use typical procedures and to manipulate standard algebraic equations, students often fail in developing more meaningful constructions or interpretations between functional concepts and procedures due to them lacking strong conceptual thinking of functions and having some common obstacles, such as cognitive, epistemological and didactical. These students often fail in viewing a function as a generalized process which accepts input and produces output, and consequently, they fail in thinking how continuously varying variables changes in tandem.

2.2.2 Epistemology of a Covariational Conception of Function

The idea of variables' values varying continuously has been foundational in the development of calculus and so the function concept. Newton, Leibniz and Euler (17^{th}- 18^{th} centuries) discussed about changes in one continuously varying quantity causing changes in another continuously varying quantity. However, due to the emergence of functional analysis and of abstract algebra, Dirichlet's conception of function attracted more attention and so became the foundation of the establishment of the modern set-theoretic definition of function while the covariational approach of function did not attract much attention.

Although the static set-theoretic definition of function is given in school curriculum at the college level, teachers often use dynamic expressions when describing the concepts of limit, continuity and derivative such as:*"as x grows, y takes on smaller and smaller values" or "the limit of a function f(x) as x approaches x_o"* (Sierpinska, 2019). The terms of 'correspondence' or 'Cartesian product' are often not mentioned in these dynamic expressions, causing students not practicing enough with these concepts. When given tasks for completion, students may not remember the static set-theoretic definition of function and only react based on their experiences or memorized associations related to dynamic expressions. This may result in students confusing and being unable to correlate a covariational conception of function with a set-theoretic conception of function, creating so an epistemological obstacle. On the other hand, if a teacher only focuses on the theoretical language of sets in beginning analysis courses and avoids the covariational approach, then this will produce a didactic obstacle (Sierpinska, 2019). In order to understand functions, it is important to study functions not only based on a set-theoretic conception of function, but also based on a covariational conception of function, and to harmonize between these two fundamental conceptions.

When looking at the epistemology of covariational reasoning, we can say that the notion of covariation first emerged in the works of Jere Confrey, Patrick Thompson and Marilyn Carlson during 1990s. Confrey's conception of covariation is related to coordinating values of two quantities as they change in relation to each other while Thompson's conception of covariation concern visualizing values of each quantity as varying continuously and then picturing two or more quantities as varying continuously and simultaneously.

Confrey and Smith described a covariation conception of function as requiring a coordination of an operational movement from y_n to y_{n+1} with corresponding movement from x_n to x_{n+1} (Confrey and Smith, 1994, p.137). They pointed out that: '*in a covariation approach, a function is understood as the* *juxtaposition of two sequences, each of which is generated independently through a pattern of data values*' (Confrey and Smith, 1995, p.67) and '*the elements and structure of the domain and range are cogenerated through simultaneous but independent actions, creating a covariational model of function*' (Smith and Confrey, 1994, p.337). Confrey and Smith (1994) stated that a covariation approach is fundamental in conceptualizing the rate of change of function (p. 153-154).

Thompson described meanings of quantity, quantification, quantitative structure, quantitative relationship and quantitative reasoning, and focused on which ways students think of varying quantities, quantitative relationships between variables, and also rates of changes of variables, in his works (1990, 1994, 1994a, 1994b, 2011). According to Thompson, a quantity is defined as an individual's conceptualization of a quality of an object involving a measurement process:

> *Quantities are conceptual entities. They exist in people's conceptions of situations. A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability. (Thompson, 1994a, p.187)*

Quantification is described as a process of an individual's conceptualization of assigning numerical values to qualities:

> *Quantification is a process by which one assigns numerical values to qualities. That is, quantification is a process of direct or indirect measurement. (Thompson, 1990, p. 5)*

While he defined a quantitative structure as '*a network of quantitative relationships*', he described a quantitative relationship as '*the conception of three quantities, two of which determine the third by a quantitative operation…For example, suppose an average speed is conceived by a multiplicative comparison of a distance and an interval of time*' (Thompson, 1990, p. 13).

According to Thompson, quantitative reasoning is defined as an individual's conceptualization of a mathematical situation quantitatively - in terms of quantities and quantitative relationships between these quantities:

> *Quantitative reasoning is the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships. (Thompson, 1990, p. 13)*

In Patrick Thompson's works, variation and covariation became a main target to explain the reasoning of individuals who pictured a mathematical situation quantitatively and also envisioned it dynamically (e.g.

imaging quantities whose values varied simultaneously). Regarding the idea of variation, Thompson (2011) noted that '*the mathematics of variation involves imagining a quantity whose value varies*' (p.46).

Saldanha and Thompson (1998) described the idea of covariation as an individual having in mind a sustained image of two quantities' values changing simultaneously, and concluded that images of covariation are developmental:

> *Thinking of covariation as the coordination of sequences fits well with employing tables to present successive states of a variation….In this regard, our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit and persistent realization that, at every moment, the other quantity also has a value. In early development one coordinates two quantities' values—think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values. (Saldanha and Thompson, 1998, p. 299)*

In 2017, Thompson and Carlson provided a more general definition of covariational conception of function being correlated with set-theoretic conception of function:

> *A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (Thompson and Carlson, 2017, p.444)*

By the term "invariant relationship", the researchers mean that the same relationship can be used to determine a value of the dependent variable from any value of the independent variable.

According to the Thompson & Carlson's theory of covariational reasoning, if an individual visualizes values of two quantities as varying continuously and images these variables varying simultaneously in relation to each other, then he/she reasons a given functional situation covariationally (Thompson & Carlson, 2017, p.425). For example, in a situation of a car moving on a straight line and always having a distance from an initial point; an individual is supposed to conceptualize a car's location in terms of a measure of the distance from an initial point, to visualize varying distance and also to envision elapsed time being measured as the car moving. If the individual conceptualizes the car's measured distance as varying, visualizes the elapsed time being measured as varying and coordinates these two variables as they vary continuously and simultaneously, then we say that the individual reasons the given mathematical situation covariationally. By referring to Carlson's work (1998), Thompson and Carlson (2017) remarked that covariational reasoning is fundamental for representing given dynamic functional situations both algebraically and graphically (p.452). However, many pre-calculus and calculus students are unable to reason dynamic events covariationally.

Carlson (1998) investigated college algebra, calculus and graduate mathematics students' understanding of the function concept according to their answers to certain tasks. In one of these tasks, students were asked to sketch a graph to represent the relationship between the values of two continuously changing quantities in a dynamic function event. In the bottle problem, students were supposed to construct a graph of the height of water as a function of the amount of water that's in the bottle (Figure 1):

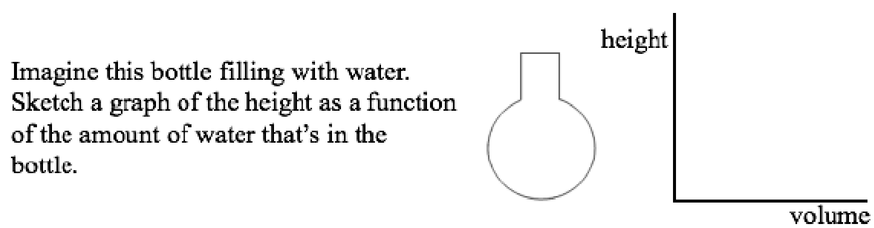


Figure 1. The bottle problem used in the Carlson's study

The results indicate that majority (74%) of high-performing second-semester calculus students failed in constructing a suitable graph of the height of water as a function of the amount of water being in the bottle. During the interviews, some of these students also failed in describing how changes in the amount of water causing changes in the height of water (Carlson, 1998, p.124-126). Carlson concluded that high-performing calculus 2 students had difficulty defining piecewise functions, and interpreting dynamic graphical information related to these functions (Carlson, 1998, p.138-139).

Carlson, Jacobs, Coe, Larsen & Hsu (2002) conducted a study to investigate second semester calculus students' ability to reason given dynamic functional situations covariationally. In their study, they defined *covariational reasoning* as the collection of '*cognitive activities involved in coordinating two* *varying quantities while attending to the ways in which they change in relation to each other*' (Carlson et al., 2002, page 355). In the theoretical framework, they classified images of covariation as developmental with five levels. These five developmental levels (from Level 1 to Level 5) are defined in terms of mental actions represented from MA1-coordination between values of two quantities varying continuously and simultaneously to MA5-conceptualization of refinements of a function's average rate of change on smaller and smaller intervals (detailed information given in the theoretical framework of my study). Differently from Saldanha and Thompson's work (1998), Carlson et al. (2002) added coordination of the average and instantaneous rates of change of one quantity with respect to another quantity in their investigation since they aimed to examine students' understanding of the rate of change of a function on successive intervals of the function's domain. They asked students the same bottle problem which was used by Carlson (1998). The researchers documented that only the quarter (25%) of high-performing second semester calculus students were able to construct an acceptable graph while most (70%) of participants sketched an increasing concave up or an increasing concave down graph in their responses to the bottle problem (Carlson et al., 2002, p.369). Similarly, in students' responses to the temperature problem with the given rate of change graph of temperature over an eight-hour time-period, they reported that only one fifth (20%) of high-performing second semester calculus students were able to draw an acceptable temperature graph while the quarter (25%) of these respondents sketched a temperature graph being similar to the given rate of change graph of temperature in the problem. Besides these, almost one third (30%) of participants were unable to mark the concavity changes when drawing their graphs (Carlson et al., 2002, p.377).

Carlson et al. (2002) concluded that most of students were able to coordinate the direction of the change in the output while thinking changes in the input (MA2), and many of these students were also able to coordinate the amount of change in the output while picturing changes in the input (MA3). However, these students had difficulty in interpreting and representing the rate of change information of a dynamic functional situation (MA4). They also had difficulty in conceptualizing the instantaneous rate of change of the output since they were unable to visualize smaller and smaller refinements of the average rate of change of the output with respect to the input (MA5). Even though a few students were able to exhibit some behaviors which support MA5, these students, during the interviews, were unable to describe how the instantaneous rate of change of the output with respect to the input was found. The investigators concluded that these students were not able to apply consistently Level 4 covariational reasoning and had difficulty in applying Level 5 covariational reasoning (Carlson et al., 2002, p.382-383).

In their work, Carlson & Oehrtman (2004) and Oehrtman, Carlson & Thompson (2008) reached to a similar conclusion that pre-calculus students exhibit behaviors that support only MA1 and MA2. When students were asked the same bottle problem, the examiners observed that many pre-calculus level students were unable to conceptualize the amount of change of the height of water while thinking changes in the volume of water (MA3). It is noticed that the students often constructed a concave up graph in their responses to the bottle problem and provided common explanation for their concave up graph, such as '*as the water is poured in it gets higher and higher on the bottle (MA2)"* (Carlson & Oehrtman, 2004, p.36; Oehrtman, Carlson and Thompson, 2008, p.163-164). Although some students mentioned about the rate of change of the output for a particular interval of the domain, these students were not able to describe how the rate changes over the domain of the function (MA4). Moreover, students who exhibited some behaviors that support MA5 stated that they used memorized procedures in order to reach their graphical constructions of the dynamic functional situation.

In a different direction, when faced with a simple function such as $f(x) = 3x$, many high performing pre-calculus students were unable to notice that the parentheses () stand as a marker for the independent variable, $f(x)$ represents the values of the dependent variable, f stands for the name of the function and $3x$ indicates how the independent variable x is mapped to the dependent variable $f(x)$. Oehrtman, Carlson & Thompson (2008) concluded that students' poor understandings of the functional notation and relying on their procedural memorized associations seem to contribute to students' inability of moving between different representations of a functional situation. They remarked that "*developing an understanding of function in such real-world situations that model dynamic change is an important bridge for success in advanced mathematics"* (Oehrtman, Carlson & Thompson, 2008, p. 154). The researchers suggest that mathematical instruction in educational system shall focus on conceptual orientation in teaching functions in order to encourage students to view a function as a process which accepts input and generates output, and so, to promote students to think how continuously varying quantities changes in tandem:

> *A mature function understanding that is revealed by students' using functions fluidly, flexibly, and powerfully is typically associated with strong conceptual underpinnings. Promoting this conceptual structure in students' understanding may be achieved through both curriculum and instruction including tasks, prompts, and projects that promote and assess the development of these "ways of thinking" in students. We advocate for greater emphasis on developing students' ability to speak about functions*

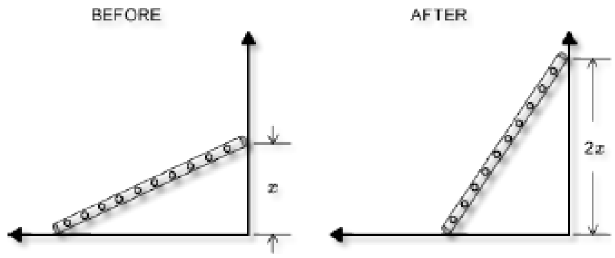
as entities that accept input and produce output, a more conceptual orientation to teaching function inverse and composition, the inclusion of tasks requiring simultaneous judgments about entire intervals of input or output values, and the development of students' ability to mentally run through a continuum of input values while imagining the changes in the output values…. (Carlson & Oehrtman, 2004, p.39; Oehrtman, Carlson and Thompson, 2008, p.167)

Carlson, Madison & West (2010, 2015) and Carlson, Oehrtman & Engelke (2010) conducted studies to investigate pre-calculus or beginning calculus students' reasoning abilities and understandings of some fundamental concepts related to functions. Foundational understanding and reasoning abilities, being essential for learning key ideas of pre-calculus and calculus courses, have been reported in these studies. Some of these foundational understandings are listed as: function composition, function inversion and the rate of change of function while fundamental reasoning abilities are described as: a process conception of function and a covariational conception of function.

In their investigation, Carlson, Oehrtman & Engelke (2010) used the spherical bottle problem (in which cylindrical section removed from the original bottle problem) to investigate college algebra and pre-calculus students' ability in constructing a graph of a dynamic function event, particularly, whether students are able to reason dynamic events covariationally. After conducting interviews with 47 students, they found that only nine students were able to coordinate the instantaneous rate of change of the height of water while imagining continuous changes in the volume (MA5) (Carlson, Oehrtman & Engelke, 2010, p.127). Testing 550 college algebra and 902 pre-calculus students with a final assessment, Carlson, Oehrtman & Engelke (2010) observed that most of these students relied on their procedural associations, failed in conceptualizing a function as a generalized process which accepts input and produces output and so were unable to reason the dynamic functional situations covariationally (p.139). The investigators remarked that a process conception of function plays a critical role in covariational reasoning: "*we view covariational reasoning as a refinement and extension of a process view of function and elaborate here why we claim that a process view of function is essential for imagining and describing how two quantities covary*" (Carlson, Oehrtman & Engelke, 2010, p. 117).

Carlson, Madison & West (2010, 2015) conducted similar studies to assess beginning calculus students' readiness for calculus courses, and the effectiveness of pre-calculus courses in preparing students to learn key concepts of calculus. They asked students the following ladder question (Figure 2):

A ladder that is leaning against a wall is adjusted so that the distance of the top of the ladder from the floor is twice as high as it was before it was adjusted.



The slope of the adjusted ladder is:

a. Less than twice what it was
b. Exactly twice what it was
c. More than twice what it was
d. The same as what it was before
e. There is not enough information to determine if any of a through d is correct.

Figure 2. The ladder problem used in the Carlson, Madison & West's works (2010, 2015)

The results suggest that only one third (27%) of 631 beginning calculus students chose the correct answer for this question. It seems that these students visualized how the slope of the ladder changes, and noticed that doubling the value of the numerator and decreasing the value of the denominator would generate a slope being more than twice what it was before. Almost half of students (48%) chose the answer b), exactly twice what it was. After conducting interviews with some of these students, the examiners concluded that these students were unable to visualize the slope of the straight line as a ratio of two quantities, only focussed on the amount of change of the top of the ladder, and failed in conceptualizing how the increase in the distance of the top of ladder from the floor by a factor 2 and the decrease in the distance of the base of the ladder from the wall changed the slope of the ladder (Carlson, Madison & West, 2015, p.227). After analyzing 631 students' responses to the 25-item multiple-choice instrument, the researchers found that vast majority of beginning calculus students had difficulty in understanding fundamental concepts of pre-calculus, such as the function concept, function composition, the rate of change of a function, were unable to view a function as a generalized process and so they had weaknesses in reasoning dynamic events covariationally. They concluded that beginning calculus students enrol in calculus courses without learning essential knowledge of pre-calculus and so they are not prepared to understand key ideas of calculus.

Calculus is based on the two theories: the theory of integration and the theory of differentiation. In order to understand these theories and other fundamental concepts of calculus, students need to conceptualize variables of a dynamic functional situation as varying smoothly and continuously in smaller and smaller bits. Covariational reasoning plays a vital role in the development of fundamental mathematical notions related to functions. Students and teachers need to understand the ideas of continuous variation and continuous covariation in order to develop more meaningful conceptions of mathematical notions associated with functions. In their study, Thompson and Carlson emphasized the importance of teaching ideas of continuous variation and continuous covariation earlier in mathematics education:

> *Students are unlikely to succeed in calculus if they meet these ways of thinking for the first time in calculus. Meaningful learning in calculus relies on students being able to inject meanings they have built in school mathematics into representations of them in calculus while at the same time creating a scheme that unites them symbolically. They must therefore begin building ideas of smooth continuous variation and covariation, constant rate of change, and process conceptions of rules of assignment in school mathematics. (Thompson & Carlson, 2017, p.453)*

Thompson and Carlson (2017) also emphasized the supporting role of quantitative reasoning in covariational reasoning. Like quantitative reasoning, covariational reasoning is also about understanding mathematical situations. When an individual is able to keep track of quantities' values changing simultaneously, he/she most strongly reasons a dynamic functional situation covariationally (Thompson and Carlson, 2017, p.446).

When faced with mathematical situations, students often rely on their memorized procedural associations formed by limited selection of standard school tasks, lack essential conceptual understanding and are unable to develop powerful constructions between functional concepts and procedures. Since students often fail in conceptualizing a function as a process that maps values of one variable to values of another variable, they are generally unable to reason dynamic functional situations covariationally. Consequently, many students enrol in calculus courses without learning fundamental function concepts and so are not ready to understand advanced key concepts of higher mathematics courses.

3 THE THEORETICAL FRAMEWORK

The theoretical framework of my research is based on three main sources. One part of it is the theory of understanding functions called “covariational framework”, developed by Marilyn P. Carlson and her collaborators (Carlson et al., 2002). The second part consists of concepts of conceptual, pseudo-conceptual, analytical and pseudo-analytical modes of thinking coined by Shlomo Vinner (Vinner, 1997) with the aim of capturing the origins of difficulty specific to learning mathematics at school. The last part composed of concepts of epistemological, didactical and cognitive obstacles, defined by Sierpinska (1992, 2019), Herscovics (1989), Bachelard (1938/1983) and Brousseau (1997), with the aim of revealing college level Calculus students’ common obstacles.

3.1 COVARIATIONAL UNDERSTANDING OF FUNCTIONS

In my work, I adopted the Covariation Framework rooted in Carlson et al.’s paper (2002) as main theoretical framework for investigating college level Calculus students’ abilities in covariational reasoning. First, I present the original framework followed by its adaptation for the purpose of this research project.

3.1.1 Carlson et al.’s “Covariational Framework”

The covariational reasoning is defined in Carlson et al. (2002) to be the collection of ‘*cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each othe*r’ (page 355). According to the original Framework, the development of covariational reasoning progresses through five levels. Each level is defined by the range “mental actions” (or thoughts about varying quantities) that it supports. The framework distinguishes five mental actions, describing each of them by its conceptual contents and by externally observable behaviors through which these thoughts are assumed to express themselves. The five mental actions and related behaviors are given in the Table 1 on page 360 in the cited article. I reproduce this table in Figure 3.

Table 1
Mental Actions of the Covariation Framework

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 3. Carlson et al.'s original model of mental actions of covariational reasoning

The concept of [mental] image [of a mathematical concept] is used to explain the levels of covariational reasoning. The description of the meaning of the word 'image' is taken from Thompson (1996) where he says:

> *By "image" I mean much more than a mental picture. Rather, I have in mind an image as being constituted by experiential fragments from kinesthesis, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images can entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining. (Thompson P. , Imagery and the devlopment of mathematical reasoning, 1996, p. 276)*

Thompson states that his understanding of "mental image" is different from (but not inconsistent with) Tall & Vinner's notion of "concept image" (Tall & Vinner, 1981), A student can develop a "concept image" of a mathematical concept, in Tall & Vinner's sense, only after having heard or seen its name and its definition; a concept image is an association in the learner's mind with a name, a word, For example, a person's concept image of the concept of function is made of all the associations he or she makes with the word "function" in the context of mathematics. Thompson summarizes Tall & Vinner's notion of "concept image" as follows: "*a concept image comprises the visual representations, mental pictures, experiences and impressions evoked by the concept name*" (Thompson, Imagery and the devlopment of mathematical reasoning, 1996, pp. 15-16 in online version). On the other hand, the mental images proposed by Thompson may be developed through experiencing and dealing with mathematical situations, problems, before or independently of being introduced to a concept's name and definition, and the focus is on what the person *does* with the image and/or elements of it (in other words, how the person *operates* with it): "*the notion of image I've attempted to develop focuses on the dynamics of mental operations*." (ibid.). A person may develop an image of two quantities co-varying in different ways without necessarily associating it with the name "function" or a well-defined mathematical concept (as Newton and Leibniz did).

Carlson et al.'s (2002) descriptions of the levels are reproduced below in Figure 4. The descriptions follow the same format. The authors considers the image of covariation of an individual to be at level n if it supports mental actions MA1 through MAn summarizing briefly the actions used in their earlier description illustrated in Figure 3 for levels 1 to 3, and slightly enriched, for levels 4 and 5.

According to Carlson et al., if the students' behaviors engage in a mental action related with a given level, and in the mental actions related with all lower levels, then the student's ability in covariational reasoning has reached that level. Our goal is that students fully participate in all mental actions presented in the covariation framework.

At Level 1, the image of covariation a person holds allows that person to see two variables changing simultaneously and coordinate changes in one with changes in the other. This level is labelled the "coordination level". So, according to the Covariational Framework, if we observe behaviors characteristic of mental action MA1 in a student, then we can say that the development of covariational reasoning in this student has attained at least Level 1.

At Level 2, the image of covariation additionally allows a person to become aware of the direction of the changes (increase, decrease or neither) of the variable assumed to be "dependent" with respect to the variable assumed to be “independent”. This level is called the “direction level”. So, if a student exhibits behaviors which are supportive of mental actions MA1 and MA2, we can claim that the development of covariational reasoning of this student has reached at least Level 2.

At Level 3, called the “quantitative coordination level”, the image of covariation additionally allows a person to estimate the amount of change of the dependent variable, also called output variable function, while considering changes in the independent variable, also called input variable. Thus, if a student exhibits behaviors supportive of mental actions MA1, MA2 and MA3, then we can say that this student’s covariational reasoning ability has reached at least Level 3 and passed levels 1 and 2.

At Level 4, called the “average rate level”, the image of covariation additionally allows a person to quantitatively coordinate the *average rate of change* of the output with respect to uniform changes in the input variable. If a student exhibits behaviors that are suggestive of mental actions MA1, MA2, MA3, and MA4, we can say that the development of covariational reasoning in this student has attained at least Level 4.

At Level 5, called “the instantaneous rate level”, the image additionally allows coordinating the instantaneous rate of change of the output with continuous changes in the input variable and also observes changes in the rate of change from increasing to decreasing or vice versa over the entire domain of the input (i.e., identify points of inflection). Since, in the theory, instantaneous rate of change is conceptualized as the limit of the average rate of change over smaller and smaller intervals of the input variable, Carlson et al. (2002) require that a person should exhibit behaviors symptomatic of MA4 (on top of MA1, MA2, MA3 and MA5) to be declared at Level 5.

Table 2
Levels of the Covariation Framework

Covariational Reasoning Levels

The covariation framework describes five levels of development of images of covariation. These images of covariation are presented in terms of the mental actions supported by each image.

Level 1 (L1). ***Coordination***

At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).

Level 2 (L2). ***Direction***

At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are *both* supported by L2 images.

Level 3 (L3). *Quantitative Coordination*

At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.

Level 4 (L4). *Average Rate*

At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.

Level 5 (L5). *Instantaneous Rate*

At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

Figure 4. Carlson et al.'s (2002) original model of levels of covariational reasoning.

3.1.2 Adaptation of Carlson et al.'s Covariational Framework to analyze the data

Adaptation of Carlson et al.'s covariational framework was used to explore and report the covariational reasoning of college level Calculus students in dynamic function events. I note that Carlson et al.'s covariational framework evaluates graphical/schematic representation of a function and verbal/contextual representation of a function. Since I am interested to observe whether Calculus students are able to represent a given functional situation, verbally, graphically or algebraically, I added a new section as algebraic/symbolic representation of a function including the descriptions of behaviors associated with covariational reasoning in my adaptation and made some changes in the descriptions of behaviors listed in Figure 3. These changes will be explained shortly.

At the coordination level (Level 1) which supports the Mental Action 1 (MA1), Carlson et al. expected that students will label the axes of the graph with verbal indications of coordinating two variables. On the one hand, labeling and speaking can occur at the same time in an interview. On the other hand, all labeling, verbalizing and symbolizing consist in writing, so they cannot be done simultaneously. We want to observe traces of three behaviors when analyzing written responses. Thus, I split the description of the behavior symptomatic of Level 1 in three parts; 1) labeling axes with names of variables, preferably putting the name of the variable implicitly assumed (in a problem) to be the input (or independent) variable on the horizontal axis and the name of the output (dependent) variable on the vertical axis, 2) verbalizing an awareness of coordinating these variables, 3) expressing name of the input and output variables algebraically/symbolically by letters/symbols (e.g. $f(x)$). These changes are visible in the new table identified as Table 1 below.

Table 1. Adaptation of Carlson's mental actions descriptions to my research

Name of mental action	**Description of mental actions**	**Behaviors associated with covariational reasoning (more substantial adaptations to the conditions of my research: no oral interviews, more detailed questions to be answered in writing)**		
		<u>Graphical/Schematic Representation</u>	<u>Verbal/Contextual Representation</u>	<u>Algebraic/Symbolic Representation</u>
Mental action 1 (MA1)	Coordinating the value of one variable with changes in the other.	Labeling axes with names of variables, preferably putting the names of the variable implicitly assumed to be the input (or independent) variable on the horizontal axis and the name of the output (dependent) variable on the vertical axis	Verbalizing an awareness of coordinating the two variables, the input and output	Expressing names of the input and output variables algebraically/symbolically – determination of functional notation. Names of the input and output variables are expressed explicitly by letters/symbols (e.g. y(x))

Mental action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing a line, curve, (or sequence of lines) with definite direction(s) (e.g. line rising from the left to right)	Verbalizing an awareness of the direction of change of the output while considering changes in the input	Expressing an awareness of the direction of change of the output while considering changes in the input, symbolically/algebraically. The direction of change of the output (with respect to changes in the input) is represented by symbols and/or numbers connected with inequality/equality signs (e.g. $0 \leq y_1(x_1) \leq y_2(x_2) \leq y(x)$ for the input $0 \leq x_1 \leq x_2 \leq x$)
Mental action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/ representing relative magnitude(s) (numerically or symbolically) of the output variable on the y-axis while picturing relative magnitude(s) of the input on the x-axis. Possibly constructing a secant line or more	Verbalizing an awareness of the amount of change of the output while considering successive changes in the input	Expressing an awareness of the amount of change of the output while considering successive changes in the input, symbolically/algebraically. A rule (or rules) of correspondence between covarying quantities is represented by algebraic expression(s), involving letters/ symbols/ numbers combined by operations, connected to functional notation with an equal sign. The domain and range of the function are expressed explicitly (e.g. $y(x) = x + 3$, the domain and range are all real numbers)
Mental action 4 (MA4)	Coordinating the average rate of change of the function with uniform increments of	Constructing contiguous secant lines for the domain	Verbalizing an awareness of the rate of change of the output (with respect to the input) while	Expressing an awareness of the average rate of change of the output while considering uniform increments of the input, symbolically/algebraically. The average rate of change of the

	change in the input variable		considering uniform increments of the input	function is considered as change in the output (Δy) divided by change in the input (Δx). Being aware of that the slope of secant line passing through two specific points, such as (x_1, y_1) and (x_2, y_2), represents the average rate of change of the function that is: $$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$
Mental action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes (direction of concavities and inflection points are correct)	Verbalizing an awareness of the instantaneous changes in the rate of change of the output as the input continuously takes values in its entire domain	Expressing an awareness of the instantaneous rate of change of the output with continuous changes in the input, symbolically/algebraically. The instantaneous rate of change of the output is expressed as the derivative of the dependent variable, being aware that the slope of the tangent line at a specific point, such as $(x_0, y(x_0))$, represents the instantaneous rate of change of the function at the point x_0 via the limit definition: $$y'(x_0) = \lim_{\mathrm{d}x_0 \to 0} \frac{\mathrm{d}y}{\mathrm{d}x_0} = \lim_{\mathrm{d}x_0 \to 0} \frac{y(x_0 + \mathrm{d}x_0) - y(x_0)}{\mathrm{d}x_0}$$ The inflection point is found as the point where the second derivative changes sign

At the direction level (Level 2) which supports the Mental Action 2 (MA2), the focus is on the coordinating the direction (e.g., increasing, decreasing or neither) of the changes in the output variable with respect to changes in the input variable. One of the associated behaviors for Mental Action 2 that is written in the original Carlson et al.'s table (Figure 3) is *'constructing an increasing straight line'.* However, if the framework is to be applicable to a wide range of problems, we cannot assume that the only model students will think of will be an increasing linear function. Thus, I modified the description of the first behavior to reflect a greater generality and applicability, as "constructing a line, curve, (or sequence of lines) with definite direction(s) (e.g. line rising from the left to right)" under the section of graphical representation in Table 1 above. The second behavior symptomatic of MA2 – verbally coordinating the direction of changes in the two variables – remains the same under the section of verbal representation. I introduce the third behavior symptomatic of MA2 written in the section of algebraic representation as: "Expressing an awareness of the direction of change of the output while considering changes in the input, symbolically/algebraically. The direction of change of the output (with respect to changes in the input) is represented by symbols and/or numbers connected with inequality/equality signs (e.g. $0 \le y_1(x_1) \le y_2(x_2) \le y(x)$ for the input $0 \le x_1 \le x_2 \le x$)".

At the quantitative coordination level (Level 3) which supports Mental Action 3 (MA3), plotting the points on the graph, verbalizing an awareness of the amount of change of the output while considering changes in the input and "expressing an awareness of the amount of change of the output while considering successive changes in the input, symbolically/algebraically" are the key behaviors. At this level, a rule (or rules) of correspondence between changing quantities is represented by algebraic expressions, involving letters/symbols/numbers combined by operations, connected to functional notation with an equal sign, and also the domain and range of the function need to be expressed explicitly (e.g. $y(x) = x + 3$, the domain and range are all real numbers). The word "successive" was added in the part of contextual representation in the table since changes in the input should be considered successively at Level 3. I added the following condition "representing relative magnitude(s) (numerically or symbolically) of the output variable on the y-axis while picturing relative magnitude(s) of the input on the x-axis, possibly constructing a secant line or more" in the section of schematic representation while I removed the part of *"constructing secant lines*" since I think that constructing secant lines should not be required at this level while it is necessary at Level 4. **Special case:** In a case that there exists a symmetry property between two functions (such as a case of bijectivity), if an individual verbally recognizes and uses the symmetry property between two bijective functions to coordinate the amount of change of the output (with respect to input), then this is considered as verbal evidence for MA3.

Regarding the average rate level (Level 4) which supports the Mental Action 4 (MA4), I kept first and second behaviors the same under the sections of graphical and verbal representations. I represent the third behavior symptomatic of MA4 which can be seen in the section of symbolic representation as: "expressing an awareness of the average rate of change of the output while considering uniform increments of the input, symbolically/algebraically. The average rate of change of the function is considered as change in the output (Δy) divided by change in the input (Δx). Being aware of that the slope of secant line passing through two specific points, such as (x_1, y_1) and (x_2, y_2), represents the average rate of change of the function that is: $M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$."

At the instantaneous rate level (Level 5) which supports the Mental Action 4 (MA5), while I made small changes and some organization under the sections of graphical and verbal representations, I introduce the third behavior symptomatic of MA5 as: "expressing an awareness of the instantaneous rate of change of the output with continuous changes in the input, symbolically/algebraically" under the section of algebraic representation. At this level, the instantaneous rate of change of the output is expressed as the derivative of the dependent variable. An individual needs to be aware of that the slope of the tangent line at a specific point, such as $(x_0, y(x_0))$, represents the instantaneous rate of change of the function at the point x_0 via the limit definition:

$$y'(x_0) = \lim_{\mathrm{d}x_0 \to 0} \frac{\mathrm{d}y}{\mathrm{d}x_0} = \lim_{\mathrm{d}x_0 \to 0} \frac{y(x_0 + \mathrm{d}x_0) - y(x_0)}{\mathrm{d}x_0}$$

The inflection point needs to be found and represented as the point where the second derivative changes sign by an individual at this level.

I made a more substantial change in defining Level 5. The original definition required "awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change" and, presumably, behaviors symptomatic of this awareness, that is, observation of MA4, since the authors say, "The mental actions identified as *MA1 through MA5* are supported by L5 images." While I agree with the assumption of awareness, I do not require that it is observable through behaviors characteristic of MA4. In theory, instantaneous rate of change is conceptualized as resulting from smaller and smaller refinements of the average rate of change. But, in a particular person, this conceptualization may have been done in a general case sometime in the past and, when, later, faced with a particular problem, the person may not redo the conceptualization. So the person may not exhibit behaviors characteristic of MA4 that is show an awareness of the average rate of change and then pass to the limit and show thinking about the instantaneous rate of change. Thus, if we observe behaviors which are suggestive of mental actions MA1, MA2, MA3 and MA5, but not necessarily of MA4, we can claim that the development of covariational reasoning in this student has reached Level 5.

Moreover, I identify a new level as '*Level 0. Unsatisfactory Covariational Reasoning'* that corresponds to behavior not supporting MA1 which I will denote, and use further, as MA0. This means that the development of a student's covariational reasoning has not even reached Level 1. At this level, when presented with a situation involving two changing quantities, the student does not show that he or she is coordinating their changes.

3.2 DIFFERENT MODES OF THINKING ACCORDING TO VINNER

A second set of concepts that I use in my theoretical foundation consist of the conceptual, pseudo-conceptual, analytical and pseudo-analytical modes of thinking defined by Shlomo Vinner (1997) in a paper aiming to analyze and report students' behaviors in learning or problem-solving situations. These concepts are also used by Carlson et al. in analyzing students' responses (Carlson et al., 2002).

For students to be engaged in a mathematical activity, it is expected that they are cognitively committed (meaning that students are intellectually involved with the subject) in a given learning or problem-solving situation. Vinner calls this state 'cognitive commitment' and he remarks the following about true learning or problem-solving situations: '*the student has to be cognitively committed (being in a learning mode) to the external stimuli that the student is going to absorb*' (page 99). Sometimes, classroom situations, which are learning or problem-solving situations for a teacher, become for some students pseudo-learning or pseudo-problem-solving situations. A student who is in a pseudo situation acts spontaneously without being cognitively involved with the topic, and tries to find the result (generally based on some memorized associations from past mathematical activities) without validating her/his answer (to see, for example, if the procedure is appropriate in this particular instance). The student is not concerned with whether his or her answer is valid but only with the fact whether the answer will be accepted by the teacher. Therefore he or she does not see the point of checking their answer – the control stage is missing in their thinking. This is the idea behind the notions of pseudo-conceptual and pseudo-analytical behaviors and underlying modes of thinking:

> *Pseudo-conceptual or pseudo-analytical thought processes are based on the belief that that a certain act will lead to an answer which will be accepted by society or will impress society (mathematics teachers in our case). In other words, I am dealing with the*

difference between the belief that statement X is true and the belief that statement X will be credited by person Y who is supposed to evaluate it. (Vinner, 1997, p. 115)

Pseudo-conceptual and pseudo-analytical thought processes share important common features – e.g., lack of control procedures – yet Vinner insists on keeping both these concepts (Vinner, 1997, p. 116). The reason being that they occur in classroom situations with different goals. When we evaluate students' responses in a discussion about the meanings of mathematical concepts, Vinner proposes to speak about conceptual and pseudo-conceptual behaviors and thinking processes. When the main goal is to solve a problem– he proposes to speak about analytic and pseudo-analytic behaviors and thinking processes. Sometimes, reflection on the meaning of a concept takes place within a problem-solving situation. In this case, Vinner proposes to keep evaluating students' behaviors in terms of analytic and pseudo-analytic behaviors:

There are contexts which are essentially problem-solving contexts. In these contexts the main activity is to analyze the situation and to find the suitable solution procedure. It is true that part of analysis may be conceptual. Especially the control procedures may involve conceptual elements. However, if the main process is a problem-solving process, I will speak about analytical of pseudo-analytical thought processes and behavior. (Vinner, 1997, p. 116)

Vinner has trouble, however, to keep to this convention in his "additional examples" (section 6, p. 118-119). So I will also depart from this convention and use the "conceptual" terms whenever the student is referring to the meaning of a concept, even if this occurs within a situation whose main goal is to solve a problem.

3.2.1 Conceptual and Pseudo-conceptual thinking

One common classroom activity has students and teachers participate in a discussion about the meanings of certain mathematical concepts. A student who is in a "*conceptual mode of thinking*" thinks about ideas, the meanings of mathematical notions and their logical connections.

Concepts are involved. Students are expected to think about concepts, their meaning and their interrelations. If they really do it, they are in a conceptual mode of thinking. If they do not, but, yet, succeed in producing answers which seem to be conceptual, then this will be described, in my terminology, as being in a pseudo-conceptual mode of thinking. (Vinner, 1997, p. 99)

Therefore, if a student is in some thought processes in which ideas, concepts and their mental connections are involved; this type of thinking is identified as *conceptual thinking* being based on conceptual understanding. The behavior that is produced by these conceptual thought processes is called a *conceptual behavior.* It is introduced as a behavior that…

> *…is based on meaningful learning and conceptual understanding. It is a result of thought processes in which concepts were considered, as well as relations between the concepts, ideas in which the concepts are involved, logical connections, and so on. (Vinner, 1997, p. 100)*

If a student fails to think about ideas, concepts associated with the ideas and their logical connections, this type of thinking is called as *pseudo-conceptual thinking*. The behavior that is produced by these pseudo-conceptual thought processes is called *pseudo-conceptual behavior*. Vinner describes pseudo-conceptual thoughts as …

> *… formed in a spontaneous way. They are not necessarily taught by teachers or the agents. Sometimes they are the natural cognitive reactions to certain cognitive stimuli. The students use them without going through any reflective procedure, control procedure or analysis of any kind. (Vinner, 1997, p. 101)*

He points out the following about the *pseudo-conceptual behavior*:

> *… a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior. (Vinner, 1997, p. 100)*

According to Vinner, if a student's behavior indicates that the control stage is missing when discussing the meaning of mathematical concepts and her/his answer does not make sense in the context, then we identify the student's behavior as *pseudo-conceptual behavior* which is a result of the student's *pseudo-conceptual mode of thinking:*

> *A dominant feature of the pseudo-conceptual thought processes is the uncontrolled associations which fail to become a meaningful framework for further thought processes. Most of us have some experience with situations in which we combine certain words or symbols and express them without knowing exactly what they mean. (Vinner, 1997, p. 103)*

In the following example, Vinner identifies the Student C's reaction as a typical pseudo-conceptual behavior (page 107-108):

> *Teacher: What is the distance between two points?*
>
> *Student A: The slope.*
>
> *Student B: A straight line.*
>
> *Student C: A segment.*
>
> *Teacher: …the distance between two points is the length of the segment connecting the two points.*
>
> *Student C: But this is exactly what I have said.*

We can determine from the Student C's responses that he/she had a lack of control mechanism when responding to the question. He/she first claimed that the distance between two points is a segment, and then thought that the segment which represents a geometrical entity is identical with the length of the segment which represents a number. It is obvious that there are no clear distinctions between two different entities in the student's mind which is an evidence of pseudo-conceptual behavior.

In my work, I use the notions of conceptual and pseudo-conceptual thinking to investigate and report the students' behaviors (true or pseudo) produced by some mental processes in moments of thinking about the meaning of concepts in concrete problem-solving situations. When a student is engaged in discussing the meanings of mathematical concepts, and is in a pseudo-conceptual mode, the student will try to use memorized associations which come from common rituals of answering standard mathematical questions. The student will probably not test, or fail to test, these associations in order to see if her/his answer is correctly derived or not.

3.2.2 Analytical and Pseudo-analytical thinking

Another common classroom activity in which the students and teachers participate is solving routine mathematical problems. The terms "analytical thinking" and "pseudo-analytical mode of thinking" are introduced by Vinner (1997) to examine and report certain specific students' behaviors in routine problem-solving situations. According to Vinner, when a student is in a routine problem-solving event, the student is supposed to have mental processes that are "analytical" in the sense which he explains by way of a diagrammatic model in his paper (Vinner, 1997, p. 111). In summary, the model assumes that analytical thinking process (being based on analytical understanding) about a routine mathematical problem consists in analyzing the problem to determine the type and structure of the problem, choosing a suitable solution procedure from a ‘pool of procedures’ (that student remembers from previous mathematical events) for this particular type of the problem, applying the selected solution procedure to the given problem in order to produce an answer, and verifying if the answer makes sense.

Vinner says that his paper is not considering situations of solving non-routine problems. The problems given to the participants in my study were non-routine. Yet some students interpreted them as routine, or behaved as if they were given a routine problem to solve. That is why Vinner’s framework could still be applied to analyzing their solutions.

When engaging routine or non-routine problem solving situations, most of students in the educational system (as some of the students who participated in my study) probably will try to use a memorized solution procedure that they remember from the past mathematical activities. In a problem-solving situation, it is vital to select a correct solution procedure. However, it is not sufficient to choose the right solution procedure in this process since the student also needs to apply the selected procedure accurately in her/his solution in order to find correct answer(s) by the end of the activity.

As Vinner explained in his paper, it is essential to mention the importance of reading comprehension skill in a problem-solving situation. When a student does not understand, or misunderstands what he/she reads in a given problem, he/she will end up making enormous errors in his/her responses. In addition to that, it is also critical to have a control mechanism in any problem-solving situation and activate such mechanism when responding to a question in order to see if the answer makes sense. Therefore, these two abilities, reading comprehension and control mechanism, are fundamental in solving both routine and non-routine mathematical problems.

In problem-solving situations, the students are supposed to be in the true analytical mode of thinking in order to exhibit true analytical behaviors. However, the students may have thought of processes that are ‘*pseudo-analytical’* instead of true analytical. In pseudo-analytical mode of thinking, when a mathematical problem is posed to a student, the particular student will have mental processes to determine the similarity of a given mathematical problem to one of problems that is chosen from a pool of typical problems (including their solution procedures), and apply the selected solution procedure to the given problem in order to produce an answer. The produced answer can even be the correct answer. Vinner presents the following necessary conditions for pseudo-analytical mental processes to occur:

> *A student who solves problems using the pseudo-analytical mode has to have the following:*
>
> *(A') A pool of typical questions and their solution procedures*
>
> *(B') Mental schemes by means of which a similarity of a given question to one of the questions in A' can be determined. (Vinner, 1997, p. 114)*

In pseudo-analytical thought processes, a student will match the given problem with one typical question and its solution procedure **based on superficial similarities and not on deeper analysis of the problem's type and structure**. The student will then **imitate** remembered solution procedure to obtain an answer. The student will not reflect on the plausibility of the answer – he/she will **not engage the control mechanisms**.

Vinner points out that,

> *the most characteristic feature of the pseudo-analytical behavior is the lack of control procedures. The person is responding to his or her spontaneous associations without a conscious attempt to examine them. The moment a result is obtained there are no additional procedures which are supposed to check the correctness of the answer. (Vinner, 1997, p. 114)*

Therefore, *'uncontrolled associations'* are the key feature not only in identifying the pseudo-conceptual behaviors, but also in identifying the pseudo-analytical behaviors.

3.2.3 Distinction between Pseudo-conceptual thinking and Misconceptions

It is important to distinguish between the pseudo-conceptual thoughts and misconceptions. Vinner identifies misconception as the concept that *'it is based on a belief about a certain mathematical situation'* while pseudo-conceptual thoughts *'are based on the belief that a certain act will lead to an answer which will be accepted by society'* (Vinner, 1997, p. 115). In addition to that, as Vinner mentions in his paper (page 121), the difference between pseudo-conceptual thoughts and misconceptions is that a student is cognitively involved in misconceptions while there is no 'cognitive involvement' in pseudo-conceptual mode of thinking. The lack of cognitive involvement is an important foundation for further pseudo-conceptual behavior. More precisely, Vinner points out the difference between the two modes as follows:

The difference between the pseudo-conceptual mode and misconceptions is in the cognitive dimension. The way I see it, misconceptions occur within cognitive frameworks. The pseudo-conceptual mode is outside the cognitive frameworks. A misconception is a result of cognitive involvement. It is a result of cognitive efforts. These efforts led to a wrong idea. On the other hand, when a pseudo-conceptual behavior occurs, there is no cognitive involvement. The person is looking for a satisfactory reaction to a certain stimulus while cognitive issues do not play a role (at least not a major role). The thought process is guided, in addition to other things by uncontrolled associations and superficial similarities. (Vinner, 1997, p. 121)

3.3 Epistemological, cognitive and didactical obstacles in the context of functions

Another set of concepts that I use in my study is formed by epistemological, cognitive and didactical obstacles as appear in work of Bachelard (1938/1983), Brousseau (1997), Herscovics (1989) and Sierpinska (1992, 2019) in aiming to reveal common obstacles college level Calculus students encounter when solving dynamic tasks. As discussed in the section 3.2, in a problem solving situation, a student is expected to be in a learning mode. However, the particular student may not be cognitively committed with the subject or the situation, so he/she may have some obstacles (besides maybe having some pseudo-thoughts) when solving the problem. These obstacles may be epistemological, cognitive or/and didactical.

Brousseau (1997) defined an obstacle as "*a previous piece of knowledge which was once interesting and successful but which is now revealed as false or simply unadopted*" (p.82) while Bachelard (1938/1983) represented the concept of '*epistemological obstacle'* relating to the progress of scientific knowledge:

When one looks for the psychological conditions of scientific progress, one is soon convinced that it is in terms of obstacles that the problem of scientific knowledge must be raised. The question here is not that of considering external obstacles, such as the complexity and transience of phenomena, or to incriminate the weakness of the senses and of the human spirit; it is in the very act of knowing, intimately, that sluggishness and confusion occur by a kind of functional necessity. It is there that we will point out causes of stagnation and even regression; it is there that we will reveal causes of inertia which we will call epistemological obstacles. (Quoted from Herscovics, 1989, p.61)

Herscovics (1989) used the term '*epistemology*' in the context of historical and critical study of science, specifically, and the growth of knowledge in general. The development of scientific knowledge encounters some periods of slow progress with some 'jumps' from old ways of knowing to new ways of knowing. This means that the process of growth of scientific knowledge (e.g. the process of learning mathematics) is discontinuous. Jumping from an old way of knowing to a new way of knowing requires some changes related to the knowledge in a person's mind. However, the individual may hold some '*blind beliefs*' and '*unconscious thoughts'* in her/his cognition, preventing her/him from knowing in new ways. If these '*blind beliefs*' and '*unconscious schemes of thought'* are widespread or have been widespread among human beings in some culture or society, then these obstacles are called '*epistemological obstacles'*. When we take a distance from our '*blind beliefs*' and '*unconscious thoughts'*, realize their consequences and consider other perspectives, we may overcome these epistemological obstacles (Sierpinska, 1992). Then, the jump can be described in terms of the new ways of knowing. Related to overcoming these obstacles, Sierpinska made the following comment:

> *When, eventually, we start seeing and doing things differently, the unquestioned beliefs turn into hypotheses or assumptions, the unconscious ways of thinking turn into explicit techniques or methods, and the obstacle is overcome (Sierpinska, 2019)*

She mentioned that the geocentric view of the universe, believing that the Earth is the center of the universe and planets orbit around it (from Ptolemy's Almagest), might well have been an epistemological obstacle in the history of astronomy. She also provided an example of an epistemological obstacle in mathematics being still present in the today's school curriculum, namely '*thinking of a function as an algebraic expression made of variables and constants'.* The "analytic-expression" obstacle creates other obstacles such as only focusing on operational (or procedural) aspects of functions and not considering functions as representing relationships between changing quantities.

On the one hand, according to Euler's first definition, an analytic expression is a set of a variable quantity and constant quantities combined with certain algebraic operations, representing a rule of a function. On the other hand, 'analytic expression' today stands for one type representation of function and not for function itself. Therefore, to be able to overcome this obstacle, it is vital to distinguish functions themselves from their representations as Sierpinska (1992) stated in her work: "*U(f)-15: Discrimination between different means of representing functions and the functions themselves*" (p.53). If a student is not able to discriminate between a function and analytical tools used to represent the function, then he/she may have not overcome the epistemological obstacle: "*EO(f)-11: Only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46), being used in my study. This epistemological obstacle may be very common among students due to highly procedural orientation being dominant in our world's educational system.

Another vital sign of understanding functions identified by Sierpinska is "*U(f)-3: Identification of the subjects of change in studying changes*". If an individual fails in identifying "*What changes*" in the *"World of changes",* then he/she may have the following epistemological obstacle also being used in my study; "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes"* (Sierpinska, 1992, p.36). Sierpinska identified other epistemological obstacles that I use in my study as follows:

> *EO(f)-5: Regarding the order of variables as irrelevant. (Sierpinska, 1992, p.38)*
>
> *EO(f)-7: A Pythagorean philosophy of number: everything is number. (Sierpinska, 1992, p.41)*
>
> *EO(f)-9: Proportion is a privileged kind of relationship. (Sierpinska, 1992, p.43)*
>
> *EO(f)-16: The changes of a variable are changes in time. (Sierpinska, 1992, p.55)*

Both historical and individual development of scientific knowledge causes difficulties that students experience in learning of mathematics. While historical development of scientific knowledge is associated with '*epistemological obstacles'*, individual development of scientific knowledge is related to '*cognitive obstacles'*. Herscovics distinguished between these two types of obstacles by the statements such as:

> *…just as the development of science is strewn with epistemological obstacles, the acquisition of new conceptual schemata by the learner is strewn with cognitive obstacles. And, just as epistemological obstacles are considered normal and inherent to the development of science, so should cognitive obstacles be considered normal and inherent to the learner's construction of knowledge. (Herscovics, 1989, p.61)*

In a learning situation, the critical point is cognitive balancing of new knowledge with existing old knowledge. Piaget called this the balancing factor '*Equilibration*' (Cohen, LeoNora M. & Kim, Younghee M., 1999). According to this equilibration theory, when an individual is learning a new knowledge, he/she is supposed to maintain a balance between assimilation, which involves integration of the new knowledge to the individual's existing cognitive structure, and accommodation, which involves changes formed in the individual's mental structures. But, if the individual fails to accommodate his/her existing cognitive structures in the process of construction of the new knowledge, then his/her existing cognitive structures may turn into '*cognitive obstacles*' (e.g. being unable to transform a pair of numbers from an equation into pairs of coordinates, or, correlating the shape of a graph with the physical features of a mathematical situation represented by the graph of a function) as Herscovics pointed out:

> *…some new ideas are involved which cannot easily be assimilated into the learner's existing cognition. New cognitive structures must be constructed and this process of accommodation may necessarily confront the student with major cognitive obstacles. (Herscovics, 1989, p.61)*

As noted in Section 3.1.2, I intend to observe whether Calculus students are able to represent a dynamic situation, verbally, graphically or algebraically. However, students may have difficulty in representing functions and also in moving from one type representation of the function to another type, creating a cognitive obstacle. I aim to study the cognitive obstacles involved in students being unable to move between different representations of a function, thus, I identify the following cognitive obstacles being used in my research:

1) CO-GTV: Cognitive obstacle of being unable to transform a graphical representation of function to its verbal one;

2) CO-GTA: Cognitive obstacle of being unable to transform a graphical representation of function to its algebraic one;

3) CO-VTG: Cognitive obstacle of being unable to transform a verbal representation of function to its graphical one;

4) CO-VTA: Cognitive obstacle of being unable to transform a verbal representation of function to its algebraic one;

5) CO-ATG: Cognitive obstacle of being unable to transform an algebraic representation of function to its graphical one.

I note that we can also identify another cognitive obstacle of being unable to transform an algebraic representation of function to its verbal one (CO-ATV). But, I did not use this identification in my research due to low participation of students for verbal representation of the function in Problem 2 since it seems that students did not find necessary to address the verbalization of the situation once they worked on the analytical side of the problem.

On the other hand, there are obstacles of didactical origin which are the result of narrow or faulty mathematical instruction (Brousseau, 1997). For instance, focusing on the set-theoretic definition of function at school or paying too much attention on analytic expressions of functions along the educational system, considered as didactical obstacles. Providing standard examples and routine tasks in school curriculum is another didactical obstacle. Associated with typical school examples of functions using almost exclusively x and y, I present the following didactical obstacle that I use in my investigation; DO-XY: Didactical obstacle involved in representing variables by the letters 'x' and 'y'. These obstacles caused by the type of instruction are the avoidable ones since they depend on choices made by the educational system. However, epistemological obstacles which are associated with the development of scientific knowledge and cognitive obstacles related to individual's process of accommodation are difficult ones to avoid due to the nature of new knowledge and to the learner's natural thinking processes (Herscovics, 1989).

4 METHODOLOGY

In my study, I have investigated the understanding of functions of college level Calculus students asked to model dynamic situations. I have posed two problems to two sections of Math 205 at Concordia University, class in which college level Calculus students are enrolled to come up to speed on pre-university level Calculus required by their program of study. The class enrollment of a typical section varies in between 75 and 100 students, and there are, in average, five such sections of the class in a given term. Their participation was voluntary, but presented as an incentive to obtain up to 2 points per problem extra credit on their regular homework assignment for the class. The 2 points amounted to up to 2% of their homework grade which on its turns counts for 10% of their final grade for the class. The collection of data was done with the help and cooperation of two instructors teaching the corresponding sections of the course, and with the approval of the course examiner.

The problems were developed from ‘the bottle problem’ used in their study by Carlson et al. (2002). I have modified the formulation of the bottle problem and generated multiple parts under Problems 1 and 2 to ask students more questions. This was motivated in part by the fact that our research study did not incorporate any interviews with the students.

4.1 DEVELOPING THE RESEARCH INSTRUMENT: QUESTIONS (MATHEMATICAL PROBLEMS) FOR STUDENTS

As research instrument, I have designed two non-routine tasks with the goal of having Calculus students to model dynamic situations and so to represent situations: verbally, graphically or/and algebraically. The students’ behaviors, exhibited in the process of solving such non-routine tasks, would provide information on college level Calculus students’ understanding of functions in the context of modeling changes between varying quantities. These two non-standard tasks were divided into Problem 1, representing a conceptual version of the bottle problem, and of Problem 2, representing an analytical version of the bottle problem. In this section, I will first present each problem exactly as it was given to the students, and then I will describe what I consider to be a Level 5 response. I note here that I have used a computer based program, called *Wolfram Mathematica*, to facilitate complex processes, such as plotting graphs or doing analytical calculations of polynomial functions with higher degree, in solving the dynamic tasks.

4.1.1 Problem 1 [The bottle problem – A conceptual version]

This part consists of the formulation and the expected solution for Problem 1 having a conceptual base. I present the formulation of Problem 1 which contains four questions: 1a, 1b, 1c and 1d.

4.1.1.1 Problem formulation

Suppose an evaporating flask is filled with water.

(a) Sketch a graph of the height of water as a function of the amount of water that is in the bottle.

(b) Explain why your graph represents this relationship.

(c) Does the graph of the function have a point of inflection? Justify your answer and, if yes, indicate clearly the point of inflection on the graph.

(d) Is there an interval where the height grows linearly with respect of the amount of the water? If yes, mark clearly the interval on your graph.

4.1.1.2 Expected solution

I have aimed for students to draw a continuously increasing graph while thinking about the instantaneous rates of change of the height of water in the bottle for amounts of water ranging from 0 to the volume of the whole bottle that was made of a spherical segment and a cylinder. In other words, a "correct" solution would show evidence of thinking about the relationship between the changes in amount of water and the height of water in the bottle at Level 5, i.e., "coordinating the instantaneous rate of change of the output with continuous changes in the input" (questions 1a, 1b, 1d) and also "observing changes in the rate of change from increasing to decreasing or vice versa", that is, identify point(s) of inflection (question 1c).

For a solution to be evaluated at Level 5 in the development of covariational reasoning, in response to question 1a students would have produced a concave down graph in the beginning until the amount of water reached the middle of the spherical part of the bottle when the graph becomes concave up, followed by a straight line starting from the moment when the amount of water reaches the end of the spherical part until the height reaches the end of the bottle. The axes would have been labelled correctly: the horizontal axis – "amount of water", or "volume of water", or "V", etc., and the vertical axis – "height of water" or "h", etc. Additionally, the relative magnitudes of the output and input variables should be represented symbolically on the axes. For instance, the graph (even though without secant lines) in Figure 5 is an example of a Level 5 solution for graphical representation of function (representing graphical signs of MA1, MA2, MA3 and MA5).

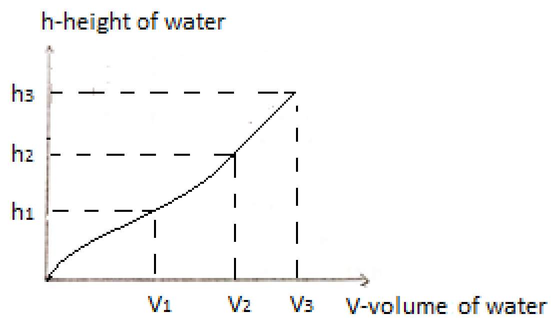


Figure 5. Expected kind of graph in response to question 1a

In response to question 1b, I expected students to explain in words the changes in the height of water in the bottle as the amount of water increased, in a manner consistent with the graph and referring to the situation of filling a bottle with water. I expected that their answer to question 1b may already contain elements of answers to questions 1c and 1d, and that students will add annotations to their graphs. I present an annotated graph in Figure 6 below.

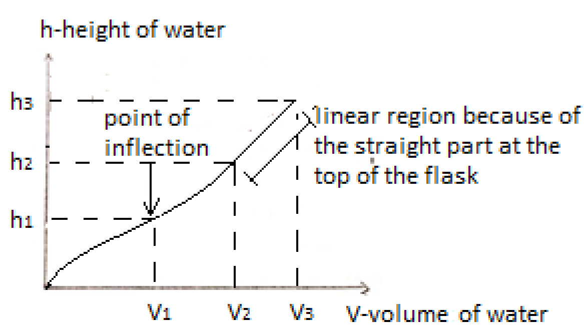


Figure 6. Annotated graph in the same solution as shown in previous figure.

I now introduce a student's response of a level 5 solution for verbal representation of function (his graph is similar to the one given in Figure 6), involving verbal signs of MA1, MA2, MA3, MA4 and MA5 to question 1b as an example:

b) The bottom and top parts of the flask have a similar structure. Therefore, the curvature of the graph will be similar. At the beginning the bottom part is narrower compared to the middle part. Thus, the slope will be shown with an increasing curve. However, the slope is higher and it will decrease towards to the middle/wide part. After the middle (the widest part) which I pointed out with the inflection point, after the inflection area, the graph will start to have similar pattern to the bottom part. At the end due to the straight part of the flask the graph will have a linear increase.

In response to question 1c, I expected students at level 5 to indicate that the inflection point will occur when water reached the middle of the spherical part of the bottle and explain it by reference to change of concavity or to change in the direction of the curvature (verbal sign of MA5). But I also accepted less accurate explanations such as the one below as level 5, where "change of direction of the graph" is mentioned, because the student referred to "curvature" in his answer to question 1b (written above):

c) Yes, towards the middle of the flask where the radius is the biggest the graph changes its direction.

In response to question 1d, I expected the interval corresponding to the linear part of the graph to be marked on the horizontal axis (graphical sign of MA5), but accepted it being marked on the curve, as in Figure 6. In the following example of response to question 1d, the student also provided a valid justification of the linearity, which was not required in this question but complements his answer to question 1b:

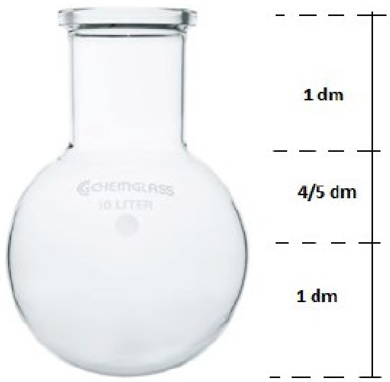
d) Yes, the neck of the flask is cylindrical; this will cause the graph a linear behavior. The height will be directly proportional to the amount of water.

4.1.2 Problem 2 [The bottle problem – An analytical version]

This part consists of the formulation and the expected solution for Problem 2 representing an analytical version of the bottle problem. I will present first the formulation of Problem 2 that contains multiple questions denoted 2a, 2b, 2c and 2d.

4.1.2.1 Problem formulation

This problem is about the evaporating flask being filled with water, as last week, but this time you are given the dimensions of the flask. We are assuming the radius of the sphere that forms the round part of the bottle is 1 unit (e.g., 1 dm = 10 cm) and the neck starts at 1 and 4/5 units from the bottom of the bottle; we also assume that the neck is 1 unit high.



a) Find a formula for the volume V of water as a function of the height h of the water: V(h) = ?. What is the domain of this function? What is the range?

b) Sketch a graph of the height as a function of the volume of the water, i.e., of the function h(V). Explain how you did it and what makes you sure you are right. What is the domain of this function? What is its range?

c) What is the height of the water if there are, approximately,

(i) 2 litres

(ii) 4 litres

(iii) 5 litres

of water in the flask? Note: 1 litre = 1 dm^3

d) Does the graph of the function h(V) have a point of inflection? If yes, what are its coordinates? Justify your response.

4.1.2.2 Expected solution

In the analytical version of the bottle problem, we assume that the radius of the spherical part of the flask and the neck are 1 unit while the distance from neck to the bottom of the flask is 1 and 4/5 units. In question 2a, I expected students to find a formula for the volume of water as a function of the height of water with its domain and range. I aimed for students to determine the volume of water in the spherical cap of the bottle visualizing it first by approximation as the sum of volumes of many thin circular cylinders whose number goes to infinity while the thickness of the cylinders goes to zero. In order to reach the formula of the volume of water in the spherical part of the flask, I expected an integral formula 'summing up' the areas of the circular discs would be obtained from the collapse of the thin cylinders by the students. I present an example of Level 5 solution for the algebraic representation of function below.

Considering the values (the height of the spherical part of the bottle and the radius of the bottle) that are given in the problem, the students would have found the value of the radius of cylindrical part of bottle as $\frac{3}{5}$ by using the Pythagorean Theorem as shown in Figure 7 below.

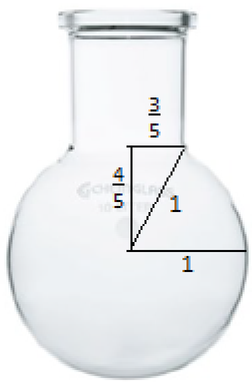


Figure 7. Finding the radius of cylindrical section of the flask.

In response to question 2a, the students would have imaged the flask on vertical and horizontal lines by labeling the horizontal axis as “x”, and the vertical axis as “y” like the one shown in Figure 8.

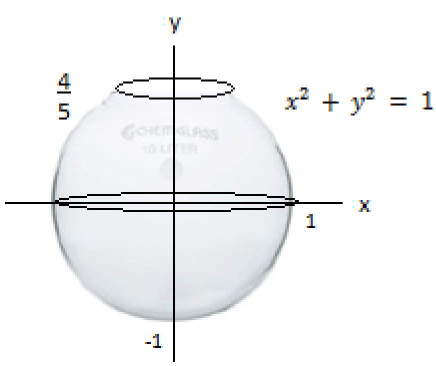


Figure 8. Representing the spherical section of the bottle on the x-y axes with an equation.

The students would have noticed that they have a half of the circle with center at (0, 0) and radius 1 for the first part of the spherical section of the bottle. Therefore, the equation representing the circle is: $x^2 + y^2 = 1$. This means that the radius of the disc (when the water reaches height h) is: $x = \sqrt{1 - y^2}$.

Since the area of the disc A equals to πx^2 $(or\ \pi r^2)$, so $A = \pi\,(1 - y^2)$. By summing up these areas, that is, integrating from -1 to h - 1, students would have had the formula for the volume of water in the spherical section of the flask as a function of the height h of water (a solution involving algebraic signs of MA1, MA2 and MA3):

$$V(h) = \int_{-1}^{0} A dy + \int_{0}^{h-1} A dy = \int_{-1}^{h-1} \pi (1 - y^2) dy = \pi(h-1) - \pi(h-1)^3 \frac{1}{3} - \pi(-1) + \pi(-1)^3 \frac{1}{3} = \pi h - \pi - \pi(h^3 - 3h^2 + 3h - 1)\frac{1}{3} + \pi - \pi\frac{1}{3} = \pi h^2 - \pi h^3 \frac{1}{3} \textit{ for } -1 \le h - 1 \le \frac{4}{5} \, dm \textit{ (or } 0 \le h \le \frac{9}{5}\textit{).}$$

Alternatively, the students would have pictured the spherical section of the flask by rotating it on horizontal line as seen in Figure 9:

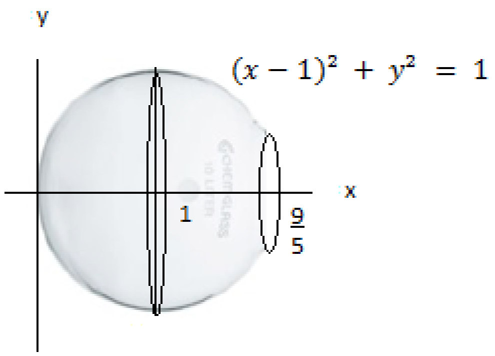


Figure 9. Rotating the flask on horizontal line.

Therefore, the students would have realized that the area under a curve which is half of the circle with center at (1, 0) and radius 1 would have represented with the following equation:

$(x-1)^2 + y^2 = 1$; e.g., the area under the graph of the function $y = \sqrt{1-(x-1)^2}$.

The volume of the spherical cap can be visualized as the sum of infinitely thin circular discs, perpendicular to the plane of the Figure 9. When the water reaches height x, and $0 \le x \le \frac{9}{5}$, the radius of this disc is equal to $r = \sqrt{1-(x-1)^2} = \sqrt{2x - x^2} = y$.

Hence, the area of the disc is: $A = \pi r^2 = \pi (2x - x^2)$. By summing up these areas, that is, integrating from 0 to h, students would have found the formula for the volume of water in the spherical part of the flask as a function of the height h of water (a solution containing algebraic signs of MA1, MA2 and MA3):

$$V(h) = \int_0^h \pi (2x - x^2) dx = \pi(h^2 - \frac{h^3}{3}) - (0 - 0) = \pi h^2 - \pi h^3 \frac{1}{3} \textit{ for } 0 \le h \le \frac{9}{5} \, dm.$$

The neck of the flask is a cylinder with radius $r = \sqrt{1^2 - (\frac{4}{5})^2} = \frac{3}{5}$. When the water in the neck reaches height H, the volume of water in it can be calculated using the formula for the volume of a cylinder: $V(h) = \pi r^2 H = \frac{9}{25}\pi H$. When the water reaches to the neck, the height H will be equal to $h - \frac{9}{5}$. Therefore, the total volume of water in the flask can be computed using the formula:

$V(h) = \frac{9}{25}\pi(h - \frac{9}{5}) +$ the volume of water in the spherical cap.

The latter volume of water (in the spherical cap) can be computed using the previously obtained volume formula ($\pi h^2 - \pi h^3 \frac{1}{3}$) for $0 \leq h \leq \frac{9}{5}$, taking $h = \frac{9}{5}$. Hence, the students would have found the formula for the total volume of water in the flask as a function of the height h of water (a solution interfering algebraic signs of MA1, MA2 and MA3):

$$V(h) = \frac{9}{25}\pi(h - \frac{9}{5}) + (\pi(\frac{9}{5})^2 - \pi(\frac{9}{5})^3 \frac{1}{3}) = \frac{9}{25}\pi(h - \frac{9}{5}) + \frac{162}{125}\pi \ \mathit{for} \ \frac{9}{5} < h \leq \frac{14}{5} \ dm.$$

In conclusion, the volume of water as a function of height of water can be evaluated using the following formula, representing a piecewise function:

$$V(h) = \begin{cases} \pi h^2 - \pi h^3 \frac{1}{3} & for \quad 0 \leq h \leq \frac{9}{5} \ dm \\ \frac{9}{25}\pi(h - \frac{9}{5}) + \frac{162}{125}\pi & for \quad \frac{9}{5} < h \leq \frac{14}{5} \ dm \end{cases}$$

It is not difficult to notice that the domain of this volume function is the interval formed by the minimum and maximum values of the height of water in the bottle. The range of the volume function can be found by just plugging the minimum and maximum values of the height of water into the two obtained volume formulas above. So, the students would have found the domain and the range of the volume function as follows (a response representing algebraic signs of MA3):

The domain of the volume function, $D_v = [0, \frac{14}{5}]$

The range of the volume function, $R_v = [0, V(\frac{14}{5})] = [0, \frac{207}{125}\pi] \sim [0, 5.2]$.

In question 2b, the students were expected to sketch a graph of the height of water as a function of the volume of the water. The graph of the volume V(h) of water as a function of the height of water (in which $\pi = 3.141559265359$) can be plotted in *Wolfram Mathematica* as shown below (I note that the students were advised, by their instructors, to use computer based program to sketch their graphs or to do complex analytical calculations):

$V[h\_] = Piecewise[\{\{3.14159265359 * (h\text{^}2)/3 * (3 - h), 0 < h \leq 9/5\}, \{(9/25) * 3.14159265359 * (h - 9/5) + (162/125) * 3.14159265359, 9/5 < h \leq 14/5\}\}];$

$Plot[V[h], \{h, 0, 14/5\}]$

Thus, the students would have obtained the following graph of the volume of water V(h) as a function of the height h of water (shown in Figure 10):

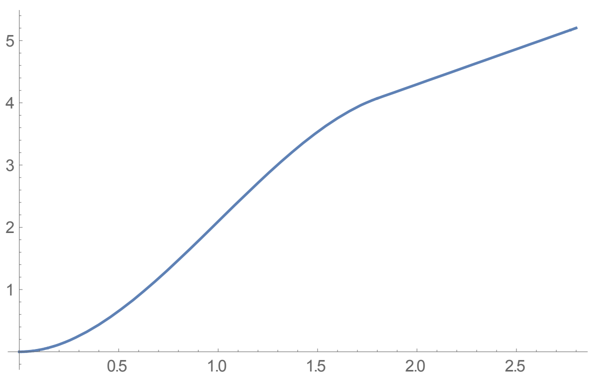


Figure 10. The graph of V(h) obtained in Wolfram Mathematica

By using 'InverseFunction' command on *Mathematica* (e.g. $Plot[InverseFunction[V][h], \{h, 0,5\}]$), they would have plotted the graph of the height of water h(V) being a reflection of the graph of the volume of water V(h) in the line V = h as seen in Figure 11 (representing graphical signs of MA1, MA2, MA3 and MA5):

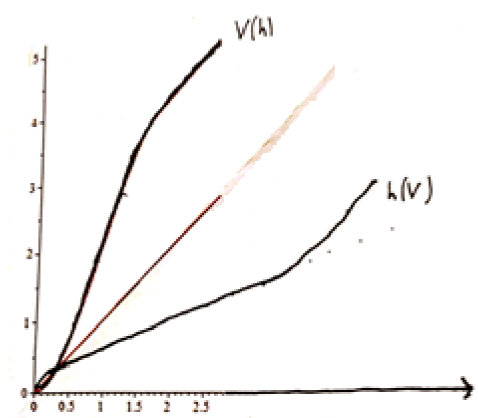


Figure 11. The graph of V(h) and its reflection h(V)

In question 2b, I also expected students to explain their graphs of the height of water as a function of the volume of water and to write the domain and range of the height function. I present a student's explanation for her graph as an example of a level 5 solution for verbal representation of the height function (involving verbal signs of MA1, MA2, MA3, MA4 and MA5):

> *As the volume starts increasing from zero, the height increases at a rapid rate as the radius is at its smallest. As the radius of the flask begins to increase with height, the rate of increase of the height slows. At* $h = 1$*, the rate of increasing of h is the lowest, and as volume begins to increase, the rate of increase of height increases until it reaches* $h = \frac{9}{5}$ *at which the height increases linearly as volume increases as the radius is constant.*

In this question, I expected that students would describe the function of the volume of water V(h) as bijective, so its inverse h(V), being the function of the height of the water, exists (being considered as verbal evidence of MA3). This means that the domain of the volume function V(h) actually equals the range of the height function h(V), while the range of the volume function V(h) equals the domain of the height function h(V) since the height function h(V) is a reflection of the volume function V(h) in the line V = h (verbal sign of MA3). Thus, the expectation was that students would find the domain and the range of the height function as follows (answers containing algebraic evidence of MA3):

The domain of the height function, $D_h = [0, \frac{207}{125}\pi] \sim [0, 5.2]$

The range of the height function, $R_h = [0, \frac{14}{5}]$.

In question 2c, I have aimed for students to find the values of the height of the water in the bottle for some specific given values of the volume of water. The values of the height of water can be found by plugging the given values of the volume of water into the piecewise volume function obtained in question 2a and solving these equations.

In part i) of question 2c in which the value of the volume of water is given as 2 litres, since the height of water is less than $\frac{9}{5}\ dm$ (by looking at the graph of the volume of water as a function of the height of the water in Figure 10), the students would have used the first formula of the piecewise volume function found in question 2a and so they would have had a cubic equation as the following (algebraic sign of MA3):

$$2 = \pi h^2 - \pi h^3 \frac{1}{3} \Rightarrow \pi h^3 - 3\pi h^2 + 6 = 0 \Rightarrow h^3 - 3h^2 + \frac{6}{\pi} = 0.$$

By using 'Solve' command in *Wolfram Mathematica* (e.g., Solve [πh^3-3πh^2+6==0, h]), the students would have obtained the following results:

$$h \approx -0.71683\ dm\ ,\ h \approx 0.96994\ dm,\ h \approx 2.7469\ dm.$$

Since the first result is negative and the last result represents the value of the height of water as more than $\frac{9}{5}\ dm$, the students would have determined that the second result is the correct value of the height of the water, that is $h \approx 0.96994\ dm$ (evidence of algebraic level MA3) is the suitable solution for the question posed in part i).

In part ii) of question 2c with the given value of the volume of water as 4 litres, the students would have noticed that the first formula of the piecewise volume function found in question 2a again should be used since the height of water is less than $\frac{9}{5}\ dm$ in this question too. So, they would have had the following cubic equation (algebraic evidence of MA3):

$$4 = \pi h^2 - \pi h^3 \frac{1}{3} \ \ => \pi h^3 - 3\pi h^2 + 12\ = 0\ \ =>\ h^3 - 3h^2 + \frac{12}{\pi}\ = 0.$$

By using 'Solve' command in *Wolfram Mathematica* (e.g., Solve [πh^3-3πh^2+12==0, h]), the students would have found the following outcomes:

$$h \approx -0.97969\ dm\ ,\ h \approx 1.7437\ dm,\ h \approx 2.2360\ dm.$$

Since the first outcome for the height of water is negative and the last outcome equals more than $\frac{9}{5}\ dm$, the students would have selected the second outcome of the value of the height of the water, that is $h \approx 1.7437\ dm$ (involving algebraic sign of MA3), as the most appropriate solution for the question asked in part ii).

In part iii) of question 2c with the given value of the volume of water as 5 litres, the students would have realized that they would need to use the second formula of the piecewise volume function obtained in question 2a since the height of water is more than $\frac{9}{5}\ dm$ (by looking at the graph in Figure 10). Hence, the following equation would be found (algebraic evidence of MA3):

$$5 = \frac{9}{25}\pi(h - \frac{9}{5})\ + \frac{162}{125}\pi$$

They would have observed that the equation being a linear at this time has only one solution (algebraic evidence of MA3):

$$h = \frac{625 - 81\pi}{45\pi} \approx 2.621\ dm$$

In question 2d, I expected students to take the second derivative of the volume function V(h) and find its root/s. By using the obtained root/s, they would have found the inflection point of the graph of the volume function. Then, they would have switched the coordinates of the obtained inflection point of the graph of the volume function in order to find the inflection point of the graph of the height function h(V).

By taking the second derivative of the function of the volume of water, that is $V(h) \ = \pi h^2 - \pi h^3 \frac{1}{3}$, the students would have found the following equation (algebraic sign of MA5):

$$V''(h) \ = 2\pi - 2\pi h.$$

Then, by making this equation equal to zero, they would have noticed that the function has one root as $h = 1$. By plugging this value of the height of the water into the volume function, they would have obtained the following value of the volume of water:

$$V(1) \ = \pi 1^2 - \pi 1^3 \frac{1}{3} \ = \frac{2}{3}\ \pi.$$

The students would have then concluded that the inflection point of the graph of the volume function V(h) is $(1, \frac{2}{3}\ \pi)$, since the height function h(V) is the inverse of the volume function V(h), hence the inflection point of the graph of the height function h(V) is $(\frac{2}{3}\ \pi, 1)$. Furthermore, they would have justified an inflection point as a point where the graph changes its shape from concave down to concave up (or from concave up to concave down), that is the point where second derivative of the volume function changes sign (algebraic sign of MA5).

4.2 DATA COLLECTION

The data for this research consists of written responses obtained from college level Calculus students who enrolled in Concordia University's course MATH 205 – Differential and Integral Calculus II in Winter 2019. The course is required for students in different programs to follow on more advanced courses and runs in several parallel sections. These written assignments were given to students enrolled in two sections by two different instructors, who voluntarily distributed the weekly assignments containing problems to Calculus students. These students would either have completed MATH 203 – Different and Integral Calculus I course or have had an equivalent course to MATH 203 prior to taking MATH 205 course. Hence, they were expected to have essential conceptual and analytical thinking been necessary for modeling dynamic events and so to represent functions verbally, graphically and algebraically since they are already familiar with the ideas of function, limit, derivative, accumulation, continuous variation and continuous covariation, have worked with different types of function, such as linear, quadratic and polynomial and have practiced with various representations of function; such as verbal, tabular, graphical and algebraic, for a minimum of two years. College level students, who were not interviewed but allowed to use a computer based program to sketch graphs or/and do complex calculations, voluntarily participated to complete their assignments at home, so they were not in the presence of a teacher while solving dynamic tasks. They were given extra bonus mark for every correct solution they provided in the weekly assignments.

In the beginning of the semester, students from two classes of MATH 205 course were given Assignment 1 consisting Problem 1 as a first homework, and Assignment 2 consisting Problem 2 as a second homework. Students were given more assignments in the following weeks, however, their participation in the assignments decreased dramatically each week and many students who provided solutions to Problem 1 did not provide solutions to Problem 2 or to other problems given in the following weeks. For instance, the twenty four students provided responses for Problem 1 while eighteen students provided solutions for Problem 2. But, the three participants out of the eighteen who responded to Problem 2 did not provide response to Problem 1. So while, for Problem 1, I have analyzed the responses of all twenty four participants, I chose to analyze, for Problem 2, the solutions of the fifteen respondents who provided responses for both Problems 1 and 2 as a result of having insufficient data for the other three participants. Because of weak participation in the assignments for the following weeks and due to time constraints, with the advice of my supervisors, a decision was made to analyze the students' responses to Problems 1 and 2 only.

4.3 Data analysis

Even though some explanatory statistics are provided, this is mainly a qualitative research study which interferes detailed investigation of behaviors/beliefs/views of small group of individuals for some period of time through in-depth data collection, organization and interpretation (Hammarberg, Kirkman & Lacey, 2016). The focus in my study is on studying and evaluating ways students are thinking about functions in the context of modeling dynamic situations. As a main theoretical construct which I discuss in Chapter 3, adaption of Carlson et al.'s (2002) Covariational Framework was used to analyze the collected data. For the analysis of the results, I also used the theoretical concepts of conceptual, analytical, pseudo-conceptual and pseudo-analytical modes of thinking (including misconceptions) defined by Vinner (1997), and of epistemological, cognitive and didactical obstacles which are rooted in the works of Bachelard (1938/1983), Brousseau (1997), Herscovics (1989) and Sierpinska (1992, 2019).

The data analysis of the examination started with carefully reading and comprehending the students' solutions to both problems in order to determine the general nature of the solutions and categorize student's behaviors on each task. Students' written responses to each problem were analyzed from the point of view of correctness, of level of covariational reasoning by using mental actions described in my theoretical framework and of common pseudo-thoughts, obstacles and misconceptions, which I discuss in Chapter 5 containing also a comparative analysis of the results obtained for both Problems 1 and 2. My data analysis consists of three stages. I note that, in the second stage, I analyzed the solutions of students who provided responses to Problem 1.

In the first stage, after evaluating each student' s solution based on correctness, I identified the evidences of MA1, MA2, MA3, MA4 and MA5 for graphical and verbal representations of the dynamic situation in Problem 1 in order to determine the covariational reasoning level reached by each student and evaluate each student's conceptual thinking of the function. When carefully analyzing students' written responses to Problem 1, I also identified common epistemological, cognitive and didactical obstacles, pseudo-thoughts, and misconceptions among college level Calculus students.

In the second stage, I repeated the same procedure for Problem 2. Differently from the process of detailed analysis of each student's solution to Problem 1, I collected the evidences of MA1, MA2, MA3, MA4 and MA5 for algebraic, graphical and verbal representations of the function. I note that I focused more on analyzing the data for algebraic and graphical representations of the dynamic event due to students' low participation in representing the dynamic situation verbally in Problem 2.

In the last stage, I comparatively analyzed the results that I obtained for Problems 1 and 2 in order to reach some general conclusion about level at which Calculus students often reason dynamic events covariationally, Calculus students' conceptual and analytical thinking of functions and common obstacles (epistemological, cognitive and didactical), pseudo-thoughts and misconceptions in preventing college level students successfully to complete dynamic tasks. Finally, I tried to find some correlation between Calculus students being unable to reason dynamic events covariationally and Calculus students having obstacles, pseudo-thoughts, misconceptions, weak conceptual and analytical understanding of functions.

After analyzing the students' solutions in each stage as described above, my analysis was sent to my supervisors for further correction, confirmation, disconfirmation or/and recommendation which have been very supportive not only in revealing findings and missing points of my research, but also in systematically organizing, planning and interpreting the textual data in every stage. This validation process which helped insure that details including errors were not missed during the in-depth data analysis continued back and forth with my supervisors for more than two years until we reached final conclusion regarding students' covariational reasoning levels (including students' conceptual and analytical thinking of functions), students' common obstacles, pseudo-thoughts and misconceptions and correlations between these aspects.

5 RESULTS

This chapter, aiming to present my answers to the research questions raised in the introduction, consists of three sections: analysis of responses collected for Problem 1 (section 5.1), analysis of the responses collected for Problem 2 (section 5.2) and a side by side analysis of the results obtained for Problems 1 and 2 (section 5.3). I intend to analyze college level Calculus students' solutions to both problems in order to conclude levels of students' covariational reasoning, including an assessment of the students' conceptual and analytical thinking of functions and what type of common pseudo-thoughts, obstacles and misconceptions students have when modeling dynamic functional situations. I am also interested in investigating correlations between students having obstacles, pseudo-thoughts, misconceptions, weak conceptual/analytical understanding of functions and what makes students unable to reason functional relationships of dynamic situations covariationally.

5.1 ANALYSIS OF STUDENTS' RESPONSES TO THE FIRST PROBLEM ABOUT BOTTLE FILLING: VERBALIZING OR/AND GRAPHING THE PROCESS

In this section, I analyze college level Calculus students' responses to Problem 1 (problem having a conceptual basis) from the following perspectives: correctness (sub-section 5.1.1), level of students' covariational reasoning (including students' conceptual thinking of the function) (sub-section 5.1.2) and students' common pseudo-thoughts, obstacles and misconceptions (sub-section 5.1.3).

5.1.1 Analysis from the point of view of correctness

From the twenty-four students who participated in the resolution of Problem 1, only four students (numbered 1, 2, 3 and 4), that is 17% of total number of students, produced correct answers for all questions posed in this problem (Table 2):

Table 2. Distribution of correct answers to Problem 1 and their percentages

Question	Number of correct answers to question(s)	% of correct answers among 24 students
Q1a	11	46
Q1b	9	38
Q1c	6	25
Q1d	18	75
All of them	**4**	**17**

Note: I am interested to observe whether students are able to represent a given functional situation graphically or verbally in Problem 1.

5.1.1.1 Question 1a

For question a) of the bottle problem, where students were asked to "sketch a graph of the height of water as a function of the amount of water that is in the bottle", eleven students (numbered 1, 2, 3, 4, 5, 7, 8, 9, 12, 14 and 15) out of twenty-four, which makes 46% of all students, constructed an acceptable graph of the height of the water as a function of the amount (or volume) of the water (including labeling axes correctly). Student 6 constructed a valid graph of the volume of the water as a function of the amount of the height of the water; this is not considered an entirely acceptable solution in our study, but partially so, based on the relation between the graph of an invertible function and that of its inverse.

5.1.1.2 Question 1b

For question 1b) which asked students to "explain why your graph represents this relationship", nine students (numbered 1, 2, 3, 4, 5, 6, 8, 10 and 11) out of twenty four, amounting for 38% of all students, provided a suitable or somewhat suitable explanation for their graphs. For example, Student 1's explanation was considered satisfactory:

> *When the flask is just starting to get filled in the beginning, the height is increasing at a faster rate compared to at the center of the spherical part of the flask because at the bottom the radius is smaller than at the middle. Once water is above the middle of the spherical section the water might increase more rapidly as the radius decreases. Once water height is above the sphere portion, the height increases linearly due to the cylindrical shape with a constant radius. (Student 1, response to Question 1b)*

while Student 9's explanation was not entirely satisfactory:

> *So as the height of the water rises, but depending on where in the flask sometimes like the height is small while the volume is larger. (Student 9, response to Question 1b)*

Two students (numbered 14 and 15) among those who sketched acceptable graphs were not able to write a valid justification of their graphs. Another student (numbered 12) attempted to explain his graph analytically by performing some calculations while in pseudo-analytical mode of thinking. On the other hand, two students out of the first group (numbered 10 and 11) divided their graphs into four intervals with an incorrect smooth curve for the spherical portion of the flask though they provided a suitable explanation of the behavior of the height with respect to the amount of water.

5.1.1.3 Question 1c

For question c), namely "Does the graph have a point of inflection? Justify your answer and, if yes, indicate clearly the point of inflection on the graph", only six students (numbered 1, 2, 3, 4, 6 and 9) out of twenty-four, which makes 25% of all students, plotted the inflection point correctly and provided valid justification. Meanwhile, seven students (numbered 5, 7, 8, 12, 13, 14 and 15) out of twenty-four identified the inflection point correctly without providing satisfactory justification. In addition to that, Student 17 also plotted the inflection point (without satisfactory justification) but labeled the axes incorrectly (height vs. time), while Students 18 and 23 plotted the inflection point without labeling the axes at all and their explanations are not satisfactory either. Two students (numbered 10 and 11) explained correctly the reasons why a point on a graph is an inflection point, but failed to identify the inflection point correctly on their graphs. In fact, each one of them has plotted two or three inflection points. Students 16, 20 and 24 claimed that 'there is no inflection point', Students 19 and 22 thought that the starting point of the cylindrical section of the flask corresponds to an inflection point, and Student 21 imagined the point on the graph corresponding to the starting point of filling the flask as an inflection point.

5.1.1.4 Question 1d

For question d), "Is there an interval where the height of the water rises linearly with respect to the amount of water? If yes, mark the interval on your graph", eighteen students out of the twenty-four (numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19 and 22) marked the linear part of the graph correctly on their graphs. Eight students (numbered 1, 2, 3, 4, 6, 7, 10 and 13) out of these eighteen came up with satisfactory justification for the linearly increasing part of the graph in their answers to the parts b) and d) although it was not specifically required.

5.1.2 Analysis from the point of view of level of students' covariational thinking

5.1.2.1 An overview

I start by presenting a table listing the assessments of students' solutions from the point of view of "correctness" (or "acceptable answers") and levels of covariational thinking (Table 3).

Table 3. Distribution of correct answers to Problem 1 and level of covariational reasoning for each student

Student's code	Q1a graph	Q1b explan.	Q1c infl.	Q1c justif.	Q1d linear	Number of correct answers per student	Level of covariational reasoning
S1 – S24	1/0[1]	1/0[2]	1/0[3]	1/0[4]	1/0[5]	sum[6]	0 – 5
S1	**1**	**1**	**1**	**1**	**1**	**5**	**5**
S2	**1**	**1**	**1**	**1**	**1**	**5**	**5**
S3	**1**	**1**	**1**	**1**	**1**	**5**	**5**
S4	**1**	**1**	**1**	**1**	**1**	**5**	**5**
S5	**1**	**1**	**1**	**0**	**1**	**4**	**5**
S6	**0**	**1**	**1**	**1**	**1**	**4**	**3**
S7	**1**	**0**	**1**	**0**	**1**	**3**	**4**
S8	**1**	**1**	**1**	**0**	**1**	**4**	**3**
S9	**1**	**0**	**1**	**1**	**1**	**4**	**3**
S10	**0**	**1**	**0**	**1**	**1**	**3**	**3**
S11	**0**	**1**	**0**	**1**	**1**	**3**	**3**
S12	1	0	1	0	1	3	2
S13	**0**	**0**	**1**	**0**	**1**	**2**	**3**
S14	1	0	1	0	1	3	2
S15	1	0	1	0	1	3	2
S16	0	0	0	0	1	1	1
S17	0	0	0	0	0	0	0
S18	0	0	0	0	0	0	0
S19	0	0	0	0	1	1	2
S20	0	0	0	0	0	0	2
S21	0	0	0	0	0	0	0
S22	0	0	0	0	1	1	2
S23	0	0	0	0	0	0	0
S24	0	0	0	0	0	0	0
# of correct ans. to question	11	9	13	8	18	59	Level 5 – 5 students, L.4 – 1 st. L.3 – 6 st. L.2 – 6 st, L.1 – 1 st. L.0 – 5 students
% of correct answers among 24 students	46	38	54	33	75	49	

Legend:

1) graph acceptable = 1, otherwise = 0

2) explanation satisfactory = 1, otherwise = 0

3) inflection point marked correctly = 1, otherwise = 0

4) justification of inflection point satisfactory = 1, otherwise = 0

5) linear part marked correctly = 1, otherwise = 0

6) sum of numbers in this row and columns 2 to 6.

I also present a table listing the assessments of students' solutions from the point of view of level of mental actions for graphical and verbal representations of the given functional situation (Table 4).

Table 4. Distribution of each student's engaged mental actions for graphical and verbal representations of the function

Student’s code	Student’s engaged mental Action for graphical representation	Student’s engaged mental Action for verbal representation
S1 – S24	**MA1 – MA5**	**MA1 – MA5**
S1	MA2	MA5
S2	MA2	MA5
S3	MA2	MA5
S4	MA2	MA5
S5	MA5	MA4
S6	**MA3**	**MA3**
S7	MA2	MA4
S8	MA2	MA3
S9	MA2	MA3
S10	MA2	MA3
S11	**MA3**	**MA3**
S12	MA2	MA1
S13	MA3	MA2
S14	**MA2**	**MA2**
S15	MA2	MA1
S16	**MA1**	**MA1**
S17	**MA0**	**MA0**
S18	**MA0**	**MA0**
S19	MA2	MA0
S20	MA2	MA0
S21	**MA0**	**MA0**
S22	MA2	MA0
S23	**MA0**	**MA0**
S24	**MA0**	**MA0**

Recall: In the theoretical framework, we have mentioned that: “*if we observe behaviors which are suggestive of mental actions MA1, MA2, MA3 and MA5, but not necessarily of MA4, we can claim that the development of covariational reasoning in this student has reached Level 5*”.

Based on the analysis of the students’ understanding of functions for the given dynamic situation represented in Problem 1, it is observed that five students (numbered 1, 2, 3, 4 and 5) provided a solution which was evaluated as representing Level 5 of covariational reasoning, showing evidence of students having strong conceptual thinking of functions when answering Problem 1 since they were able to make logical connections related to the ideas and concepts given in Problem 1 (Vinner, 1997). In other words, only five students exhibited behaviors that support Mental Action 5 for verbal or graphical representation of the function by coordinating the instantaneous rate of change of the values of the function with respect to continuous changes in the independent variable for the entire domain. Four of these students (numbered 1, 2, 3 and 4) were able to sketch a valid curve (which is concave down, then concave up, then linear) but without contiguous secant lines and relative magnitudes of the variables (graphical sign of MA2 but not of MA3) while Student 5 constructed a valid smooth curve with the relative magnitudes and the correct directions but without contiguous secant lines (being considered as graphical sign of MA5 even though secant lines do not exist). Beside this, they all indicated the inflection point correctly in their responses. On the other hand, Students 1, 2, 3 and 4 verbalized an awareness of the instantaneous changes in the rate of change of the height while the amount of water rises in the bottle (verbal sign of MA5) while Student 5 expressed an awareness of the average rate of change of the height with respect to volume (verbal sign of MA4). Thus, these five students' covariational reasoning has reached Level 5 since they exhibited behaviors being suggestive of MA5 for either verbal or graphical representation of the given functional situation.

Six students (numbered 6, 8, 9, 10, 11 and 13) provided a solution which was evaluated as representing Level 3 of covariational reasoning since their behaviors are supportive of MA3 for verbal or/and graphical representation of the function while Student 7's response seems to represent Level 4 of covariational reasoning since his behavior is indicative of MA4 for verbal representation of the given dynamic event while only MA2 for graphical representation of the situation. It seems that Students 6, 7, 8, 9, 10 and 11 have good or moderate conceptual thinking of the function (but not as good as the previous five students have) while Student 13 has weak conceptual thinking of the situation due to him being unable to develop meaningful verbal and geometrical constructions related to the conceptual elements of the dynamic situation given in Problem 1 (Vinner, 1997). I note that Student 13 was unable to neither coordinate the amount of change of the output (with respect to the input) verbally nor construct a correct graph of the height function or the volume function.

While Students 6 and 11 were able to coordinate the amount of change of the output while thinking changes in the input both verbally and graphically (verbal and graphical signs of MA3), Students 8, 9 and 10 verbalized an awareness of the amount of change of the output with respect to changes in the input (verbal sign of MA3) but failed in representing relative magnitude(s) of the output variable on the y-axis so no graphical evidence for MA3 appears. Oppositely, Although his graph of V(h) was incorrect, Student 13 represented relative magnitudes of the output with respect to corresponding magnitudes of the input including plotted points (graphical sign of MA3) but was unable to express an awareness of the amount of change of the output with respect to changes in the input thus no verbal evidence for MA3 seems to exist.

Moreover, the covariational reasoning of other six students (numbered 12, 14, 15, 19, 20 and 22) seem to have remained at Level 2 due to them being unable to coordinate the amount of change of the output with respect to changes in the input graphically or verbally while Student 16's covariationl reasoning has reached only Level 1 since he was unable to coordinate the direction of change of the output while thinking changes in the input both verbally and graphically. Hence, we can say that these seven students have had a weak conceptual thinking of the function when solving Problem 1 since they were not able to make more meaningful connections associated with the ideas represented in Problem 1 (Vinner, 1997). On the other hand, the covariational reasoning of remaining students (numbered 17, 18, 21, 23 and 24) attained at Level 0 because neither did these students verbalize an awareness of coordinating the two variables correctly nor did they labeled x-y axes, such as height and volume, correspondingly. These students seemed not to have an essential conceptual understanding of functions due to them failing to develop meaningful verbal and graphical representations of the situation related to the given concepts in Problem 1. The data reveals that majority of these students whose covariational reasoning have not reached Level 3 had obstacles (epistemological, didactical and cognitive), pseudo-thoughts, or/and some misconceptions which I discuss in the section 5.1.3.

In summary, we have observed that only 21% of all students provided a solution representing Level 5 of covariational reasoning and 25% of participants provided a solution being representative of Level 3 of covariational reasoning while other 25% of students provided a response being indicative of Level 2 of covariational reasoning and other 21% of respondents provided an answer representing Level 0 of covariational reasoning.

The results indicate that most of students (79%) failed in coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (no Level 5 of covariational reasoning) and 75% of the respondents were unable to coordinate the average rate of change of the output while thinking uniform increments of the input (no Level 4 of covariational reasoning).It seems that half of students were not able to coordinate the amount of change of the output while picturing changes in the input (no Level 3 of covariational reasoning) and almost quarter of students (21%) were unable to make a correct coordination between the two variables (no Level 1 of covariational reasoning). On the other hand, 83% of participants were unable to represent relative magnitudes of the output on the axes even though some of them plotted points correctly on their graphs, thus there is no sufficient evidence of MA3 for graphical representation of the function among these students' responses to Problem 1. These students did not construct secant lines either, so no signs of MA4 for geometrical representation of the function exist.

Eight students (numbered 17, 18, 19, 20, 21, 22, 23 and 24), which makes 33% of all students, were unable to coordinate between the two variables verbally. This is suggestive of them having very poor conceptual thinking of functions when answering Problem 1 (Vinner, 1997). Five (numbered 17, 18, 21, 23 and 24) of these students did not exhibit behaviors that support MA1 neither for graphical representation nor for verbal representation of the given event. This means that they failed in coordinating between the two variables both graphically and verbally. This suggests that these five students do not have essential conceptual thinking of functions when solving Problem 1. On the other hand, I have found that eleven students (numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11), amounting for 46% of respondents, have strong or good or moderate conceptual thinking of functions in the context of modeling dynamic situations.

5.1.2.2 Example of a response representing Level 5 of covariational reasoning

There were five students whose solutions were assessed as representing Level 5 of covariational reasoning, Students 1 to 5. I present an analysis of Student 1's and Student 5's solutions.

5.1.2.2.1 Student 1: a Level 5 solution with a strong verbal representation

Student 1 provided a response that was evaluated as representative of Level 5 of covariational reasoning since she verbally coordinated the instantaneous rate of change of the height while thinking continuous changes in the volume for the entire domain of the function. I reproduce below her solution to questions 1a, 1b, 1c and 1d.

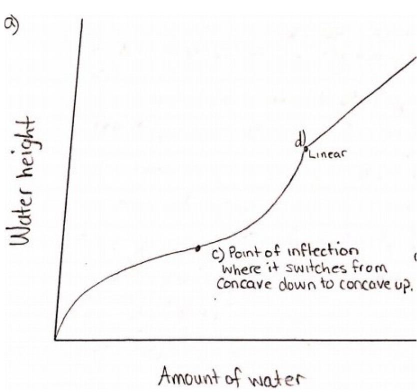


Figure 12. Image of Student 1's graph

> *(b) When the flask is just starting to get filled in the beginning, the height is increasing at a faster rate compared to at the center of the spherical part of the flask*

because at the bottom the radius is smaller than in the middle. Once water is above the middle of the spherical section,, the water height increases more rapidly as the radius decreases. (c) Point of inflection where it switches from concave down to concave up (marked on the graph). (d) Once water height is above the sphere portion, the height increases linearly due to the cylindrical shape with a constant radius. (Student 1)

When examining Student 1's responses, we see that the student constructed a valid graph (graphical evidences for both MA1 and MA2) but without representing magnitude(s) of the dependent and independent variables and contiguous secant lines (no graphical signs neither for MA3 nor for MA4), and verbalized an awareness of the instantaneous changes in the height with respect to the changes in the amount of water (verbal sign of MA5). In her answers, she considered three varying quantities as 'changes in height of water depend on changes in amount of water which depend on the radius of the top surface of the water at the particular height: h(V(r))'. She is aware that the changes in height are decreasing toward the middle of the spherical part of the bottle, while the volume of water increases, since she noted that 'when the flask is just starting to get filled in the beginning, the height is increasing at a faster rate compared to at the center of the spherical part of the flask because at the bottom the radius is smaller than in the middle' in her answer to question 1b (evidence of having strong conceptual thinking of the dynamic event). This is evidence of her being aware of instantaneous rate of change of the height with respect to the amount of water (verbal evidence for MA5). In addition, the student's remark of 'once water is above the middle of the spherical section, the water height increases more rapidly as the radius decreases' is also an indication of the student's awareness of the instantaneous changes in height while imagining the amount of water increases continuously (verbal evidence for MA5). In her response to question 1c, she identified the inflection point correctly by justifying it as 'point of inflection where it [the graph] switches from concave down to concave up' which is also an evidence of MA5 for verbalization of the functional relationship of the dynamic situation.

5.1.2.2.2 Student 5: A Level 5 solution due to satisfactory geometrical representation

Student 5's solution also appeared to represent Level 5 of covariational reasoning since he graphically coordinated the instantaneous rate of change of the height while thinking continuous changes in the volume for the entire domain of the function but without constructing secant lines on his graph. I reproduce his solution to questions 1a, 1b, 1c and 1d below.

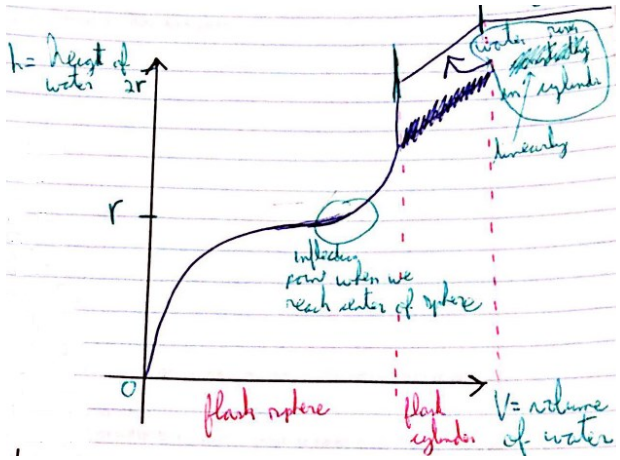


Figure 13. Image of Student 5's graph

> *(b) At first, the height raises quickly because the width is smaller and the width w=2r, where r is the radius. (c) The curve then reaches an inflection point and the other half of the sphere fills. (d) The height raises linearly when the water reaches the cylinder until the cylinder is full. Note that the sphere is not perfect: It has a square top for the cylinder, so the height of the water in the sphere is not exactly 2r. (Student 5)*

This student verbally and graphically considered the covariation between the volume of water as the independent variable and the height as the dependent variable (verbal and graphical signs of MA1). He measured the volume by the width of the flask at the particular height which depends on the radius of the surface of the water: h(V(w(r))). His answers to questions 1b and 1c imply that he considered the direction (verbal sign of MA2) and amount (verbal sign of MA3) of change of the height with respect to volume of water. Moreover the student expressed awareness of the rate of change of the height while picturing (approximately uniform) increments of the volume of water, from V=0 to V=volume of half a sphere, then from that volume to V=volume of the entire spherical part of the flask, and lastly from that volume to the volume of water in the whole flask (sign of having good or strong conceptual thinking of the situation). This can be regarded as verbal evidence for MA4 but not for MA5 since he did not consider what happens with height when volume V grows by smaller and smaller increments.

He plotted points on his graph and represented relative magnitudes of the height of water while considering changes in the volume of water (graphical sign of MA3). Even though he did not construct contiguous secant lines, his smooth curve with correct direction of concavities including plotted points and relative magnitudes is considered as supportive of MA5 for graphical representation of the function. Thus, I estimate that his covariational reasoning has reached Level 5.

5.1.2.3 Example of a response representing Level 4 of covariational reasoning

There was only one student whose solution appeared to represent Level 4 of covariational reasoning.

5.1.2.3.1 Student 7: Considering average rates of change of the height of water with respect to uniform increments of change in volume

Student 7 provided a response being evaluated as representative Level 4 of covariational reasoning since he verbally coordinated the average rate of change of the height while thinking changes in the volume of liquid. His solution to questions 1a, 1b, 1c and 1d were:

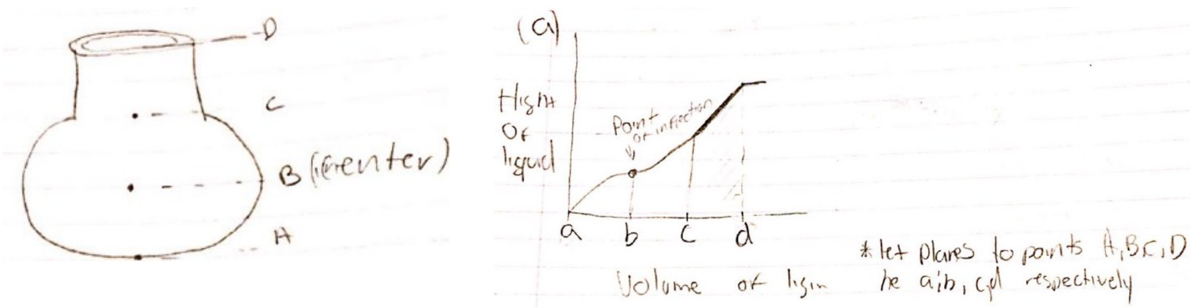


Figure 14. Image of Student 7's graph

> *(b) From a [to] b rate of increase in height is decreasing. From b to c rate of increase in height is increasing. From c to d rate of increase in height is constant therefore the graph is linear. (c) The graph has an inflection point at point B which is the middle of the spherical portion since concavity changes at this point.*
>
> *(d) From c to d the graph is linear. (Student 7)*

Student 7 correctly labeled the x-axis as "Volume of liquid" and the vertical axis as "Height of liquid", so condition 1) of MA1 is satisfied. But the points a, b, c, d marked on the x-axis do not refer to values of the volume, which would be numbers, but to planes, a different category of mathematical entities. So, one has doubts if he identified the covarying quantities correctly. Perhaps he was thinking of a, b, c, and d as representing the volumes of liquid in the bottle up to those planes, and not, literally, of the planes, and used the word "planes" as a shortcut. We can give him the benefit of the doubt and assume that he experienced the MA1 and meant the volume of liquid by writing the letters a, b, c and d in his responses. Then, we can claim that we have graphical evidences for MA1 and MA2 but not for MA3 since he failed in representing relative magnitude(s) of the height of liquid on his graph. We can also claim that we have verbal evidences for both MA3 and MA4 (in addition to signs of MA1 and MA2) since he expressed some awareness about the amount of change of the height and also how the rate of change of the height behaves in each interval by stating as 'from a to b rate of increase in height is decreasing, from b to c rate of increase in height is increasing' (evidence of having good or moderate conceptual thinking of the functional event). Therefore, I estimate that his covariational reasoning has reached Level 4. We see some evidence of his awareness of the fact that the inflection point is where the rate of change of the height changes from decreasing to increasing in his answers to questions 1b and 1c. However, this is not sufficient to claim that his covariational reasoning reached Level 5 since he did not conceptualize what happens with the height when the volume of liquid increases by smaller and smaller increments.

5.1.2.4 Examples of responses representing Level 3 of covariational reasoning

5.1.2.4.1 Student 6: Graphing V(h) instead of h(V)

In Student 6's solution, it is observed that he sketched the graph of the inverse function by treating the volume of water as a function of the height of water.

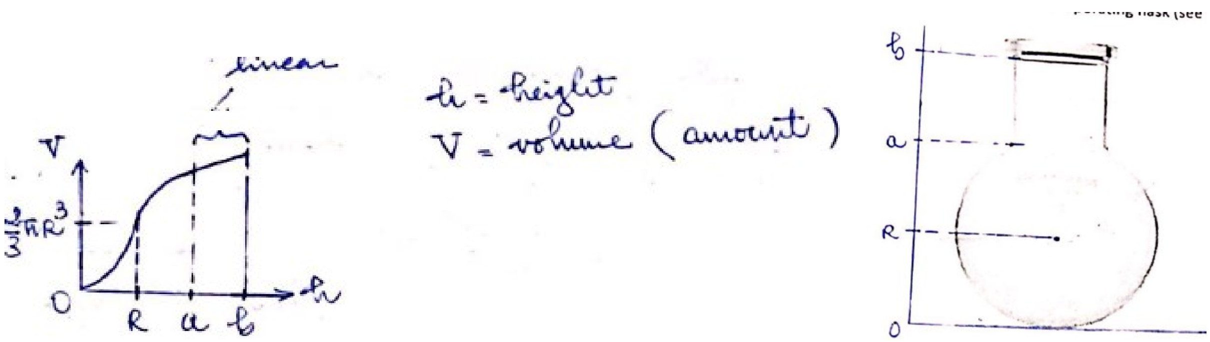


Figure 15. Image of Student 6's graph

Student 6 drew a correct graph of the function V(h) [instead of h(V)], labeling the axes with variables h and V. While he symbolically represented the three relative magnitudes of the independent variable (R, a, and b) on the horizontal line of his graph, he represented only one relative magnitude of the dependent variable on the vertical line (considered as graphical signs of MA1, MA2 and MA3).

His responses to questions 1b and 1c follow:

> *(b) 0→R while h increases, V increases more quickly as the flask gets bigger in this range, R→a The flask starts to get smaller in volume which makes V increase more slowly as h increases, a→b The cross-sectional area is the same at all points, hence the rate is constant. (c)Yes [there is an inflection point], 0→R: large change in V with small change in h; increasing slope ⇒concave up, R→a: small change in V with large change in h; decreasing slope ⇒concave down, The inflection point is h=R. (Student 6)*

The explanations are consistent with his graph. In these explanations, the student verbalized an awareness of:

- coordinating the variables he labeled the axes with, that is, height or h, and volume or V (verbal sign of MA1);

- coordinating the direction of change of the output (V in his case) while considering changes in the input (h in his case), since he speaks of the volume increasing while the height increases in his answer to question 1b (verbal sign of MA2);

- coordinating the amount of change of the volume of water while imagining changes in the height, since he speaks of V increasing more quickly while h increases from 0 to R in his response to question 1b and of decreasing and increasing slopes in question 1c (verbal sign of MA3) (evidence of having moderate conceptual thinking of the dynamic situation).

However, he did not draw contiguous secant lines nor did he appear to consider the average rate of change of the volume with respect to uniform increments of change in the height of water (so no signs of MA4). He also does not seem to have *instantaneous* changes of the rate of change in mind even though he speaks of “increasing” and “decreasing slope”, because the way he understands, for example, “increasing slope” is “large change in V with small change in h” (no verbal sign of MA5). While his graph represents the direction of concavities for the volume function correctly, it does not have a correct direction of concavities for the height function, thus violating the coordination of the instantaneous rate of change of the function (no graphical sign of MA5). Therefore, we conclude that his covariational reasoning has remained at Level 3.

5.1.2.4.2 Student 8: “Proportional growth”

Student 8 provided a solution also seemed to represent Level 3 of covariational reasoning since he verbally coordinated the amount of change of the height while thinking changes in the amount of water. I reproduce his responses to questions 1a, 1b, 1c and 1d below.

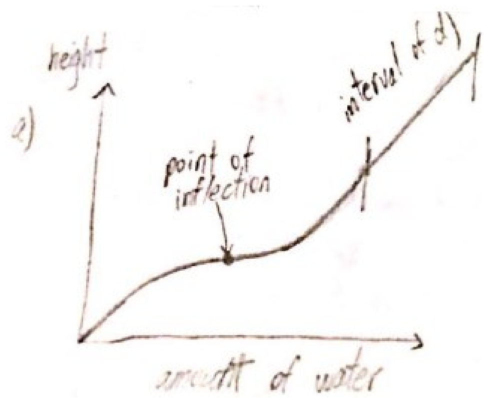


Figure 16. Image of Student 8's graph

> *(b) Height is always increasing proportionally to the amount of water, however, we should observe the round section of the flask, where it will require progressively increasing quantities of liquid to increase the height of water inside the flask. (c) At the widest part of the spherical section of the flask, the radius of cross-sectional circles gradually decrease, making the required amount of water needed to rise in the height lower. This is identically opposite to the lower half, which means the exact half of the sphere would also be the point of inflection in our graph. (d) After the round part is completely filled up, the height becomes a standard cylinder, where height increases linearly with the amount of water. (Student 8)*

Student 8 labeled the axes correctly as 'height' on the vertical axis and 'amount of water' on the horizontal axis and constructed a valid smooth curve but without relative magnitudes of the variables and secant lines on his graph (graphical signs for only MA1 and MA2). He verbalized an awareness of coordinating these variables (verbal sign of MA1) and the direction of change of the output while considering changes in the input since he wrote as 'height is always increasing proportionally to the amount of water' (verbal sign of MA2). By stating 'however, we should observe the round section of the flask where it will require progressively increasing quantities of liquid to increase the height of the water inside the flask' in his answer to question 1b (sign of having moderate conceptual thinking of the dynamic event), he verbalized an awareness of how the height changes while picturing changes in the amount of water (verbal sign of MA3). Hence, I estimate that the covariational reasoning of this student reached Level 3.

In his response to question 1c, he wrote that 'At the widest part of the spherical section of the flask, the radius of cross-sectional circles gradually decrease [that may indicate a misconception as right at that point, the radius has a maximum, until that point, the radius increases, after that it decreases], making the required amount of water needed to rise in height lower. This is identically opposite to the lower half, which means the exact half of the sphere would also be the point of inflection in our graph'. In his answer, the student pictured inputs to evaluate equal output, rather than estimating the output based on equal input. Besides that, he considered three varying quantities as 'changes in height of water depend on changes in amount of water which depend on the radius of the surface of the water at the particular height: h(V(r))'.

He did not appear to consider average rates of change of the height with respect to uniform increments of change in the amount of water (thus no sign of MA4). He also did not seem to think what happens with the height when the amount of water increases by smaller and smaller increments and, thus, failed to image instantaneous changes in the rate of change of the height with respect to the amount of water for the entire domain. Although he identified the point of inflection correctly, he was unable to explain why the point he marked on the graph is an inflection point since there is no mention of concavity changes (no signs of MA5).

In addition, we can determine (from his expressions of 'increasing proportionally' and 'progressively increasing') that he may not have overcome yet the epistemological obstacle of 'proportion privileged' which is identified as "*EO(f)-9: Proportion is a privileged kind of relationship*" (Sierpinska, 1992, p.43).

5.1.2.4.3 Student 10: Three points of inflection

Student 10 provided a response that was also evaluated as representing Level 3 of covariational reasoning. The student's solution to questions 1a, 1b, 1c and 1d were:

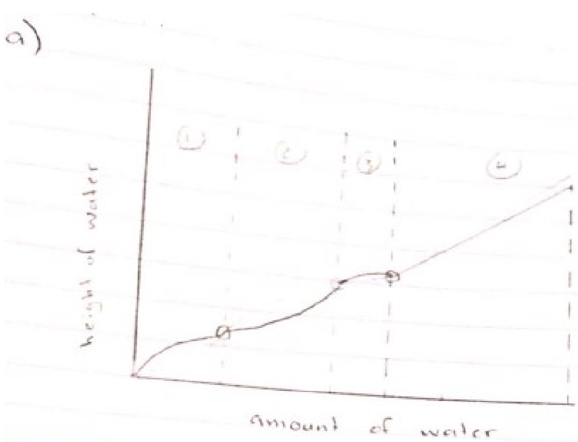


Figure 17. Image of Student 10's graph

(b) My graph is split up into 4 sections in order to represent the relationship between amount of water and height of water. Sections 1 and 3 of the graph represent the beginning and end parts of the spherical part of the flask. Because these parts are tapered or rounded, the height of the water will increase faster than the amount of water. Section 2 represents the main part of the spherical part of the flask, because it is the widest part, the amount of water will increase quicker than the height of the water. Part 4 of the graph represents the cylindrical part of the flask, the height and volume will increase at a constant amount. (c) The graph has three points of inflection marked by the circles on the graph. These points of inflection occur due to the change in shape of the flask. As the flask either tapers or gets wider we experience points of inflection. (d) Section 4 of the graph represents a linear relationship between height of water and amount of water. The linear relationship is due to the cylindrical shape of the flask. (Student 10)

Student 10 labeled the axes correctly as 'height of water' on the y-axis and 'amount of water' on the x-axis (graphical sign of MA1) and the direction of his graph is correct as well (graphical sign of MA2) but he was unable to provide an acceptable graph with relative magnitudes of the variables (thus no graphical sign of MA3). The immediate noticeable fact is that the student constructed a smooth curve with incorrect direction of concavities, identified three inflection points (in violation of coordinating the instantaneous changes in the height of water with respect to the amount of water) and divided his graph into four intervals as sections 1, 2, 3 and 4 (in violation of coordinating the average rate of change of the height of water with respect to the amount of water). On the other hand, he verbalized an awareness of both the direction of change of the height of water while considering changes in the amount of water (verbal sign of MA2) and the amount of change in the height of water while picturing changes in the amount of water (verbal sign of MA3) by statements such as 'because these parts are tapered or rounded (referring to Sections 1 and 3), the height of the water will increase faster than the amount of water' or '[In Section 4], the height and volume will increase at a constant amount' in his solution to question 1b (sign of having moderate conceptual thinking of the functional situation). In his response, he also wrote that 'Section 2 represents the main part of the spherical part of the flask, because it is widest part, the amount of water will increase quicker than the height of the water' in which he was not able to consider changes in the output with respect to changes in the input. In his solution to question 1c, he expressed the reason why the points marked on his graph are inflection points as 'these points of inflection occur due to the change in shape of the flask. As the flask either tapers or gets wider we experience points of inflection'. But, he failed in coordinating the average rate of change of the height of water with respect to uniform increments of change in the amount of water (no signs of MA4) and violated the coordination of instantaneous changes in the rate of change of the height of water with respect to smaller increments of change in the amount of water (so no signs of MA5), therefore, I conclude that his covariational reasoning has not reached neither Level 5 nor Level 4. Since he was unable to transform his verbal representation of the dynamic situation to an acceptable graphical construction, he may not have overcome yet the cognitive obstacle labeled as CO-VTG: Cognitive obstacle of being unable to transform a verbal representation of function to its graphical one, like Student 11 who also sketched a similar graph of the height function with four intervals of the domain.

5.1.2.4.4 Student 13: Assuming water is poured at a constant speed

Student 13 provided the following responses to questions 1a, 1b, 1c and 1d:

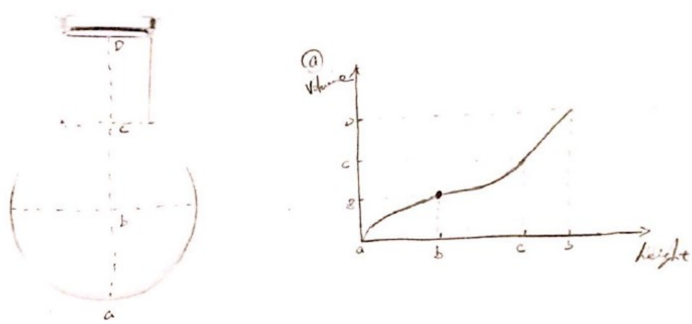


Figure 18. Image of Student 13's graph

> *(b) from a-b and b-c they have same of volume. if pour water at same speed, a-b and b-c will have same speed, but opposite, a-b from fast to slow, b-c from slow to fast. Since D-C is cylinder, if pour the water is same speed, they will have constant speed rise. (c) point b will be inflection point, there are increasing to decreasing swift. (d) from C-D is linearly increasing. (Student 13)*

Student 13's solution was also assessed at Level 3 of covariational reasoning. The shape of the curve looks somewhat like the graph of V(h), however the axes are labeled inversely. So we have graphical signs of MA1, MA2 and MA3 since he also represented relative magnitudes of the input and output variables on his graph including plotted points but without constructing secant lines. Although he verbalized an awareness of the directions of change in volume while considering changes in the height of water (verbal sign of MA2), the student who seems to have weak conceptual thinking of the situation did not verbalize an awareness of the amount of change of the volume with respect to the height of water (no verbal sign of MA3) and the average rate of change of the volume while considering uniform increments of the height of water (no verbal sign of MA4). When he mentions about 'speed' in his response to question 1b, it refers to pouring water at constant speed. Probably, the student imagined the variables as functions of time and, thus, overall three varying quantities, volume, height and time. It is possible that the student has not overcome yet the common epistemological obstacle "*EO(f)-16: The changes of a variable are changes in time*" (Sierpinska, 1992, p.55). Moreover, the student plotted the inflection point correctly on his graph but failed to explain why the inflection point is at h = b by explicit reference to concavity changes. Instead, the student refers to changes in something he calls "swift": "increasing to decreasing swift". It is not clear if he meant 'switch' or 'speed' by the word 'swift'. I estimate that the student did not exhibit behavior supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables*" and may not have overcome yet the corresponding epistemological obstacle labeled as "*EO(f)-5: Regarding the order of variables as irrelevant*" (Sierpinska, 1992, p.38).

5.1.2.5 Examples of responses representing Level 2 of covariational reasoning

5.1.2.5.1 Student 12: Analytic responses that do not explain the behavior of the function (a case of pseudo-analytical thought); covariational reasoning concealed

In his response to question 1a, the student produced an acceptable graph but without representing relative magnitudes of the variables and constructing secant lines on his graph.

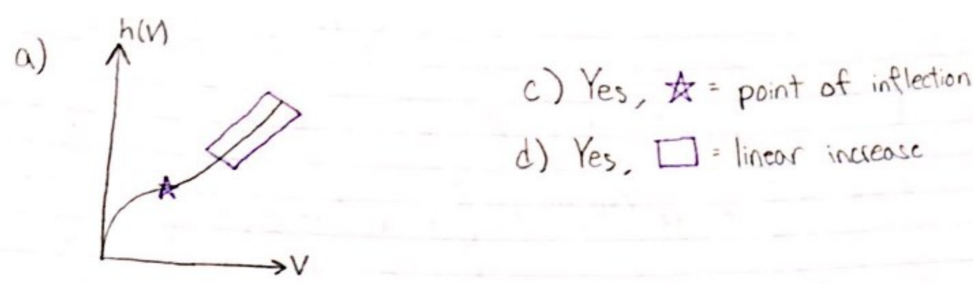


Figure 19. Image of Student 12's graph

The response to question 1b is very long. Student 12 proceeded to justify his graph in question 1a analytically. He started by deriving a formula for volume as a function of height. He first rotated the flask 90 degrees to the right on the x-axis (which is denoted by h), identified the values as h_0, h_1, h_2 and h_3 on the horizontal axis and r_1 on the vertical axis (which is denoted by $r(h)$) and conceptualized the circle with center at $(\frac{h_2-h_0}{2}, 0)$ and radius $\frac{h_2-h_0}{2}$ as: $(x - \frac{h_2-h_0}{2})^2 + f(h)^2 = (\frac{h_2-h_0}{2})^2$ in which $f(h)$ represents the radius of the disc that is ‘r’. He used the formula $V = \int \pi\, r^2 \mathrm{dr}$ in order to find the formulas for volume of water both in the spherical cap and in the cylindrical part of the flask.

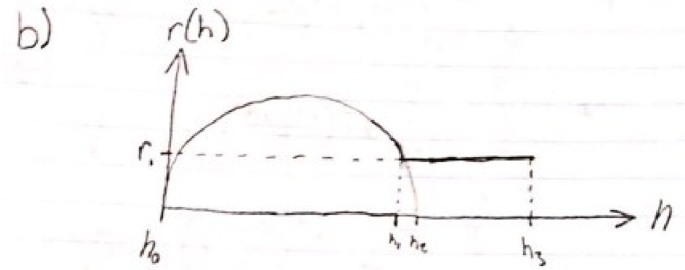


Figure 20. Image of the beginning of Student 12’s response to question 1b

> *b) We can make the cross section into a piecewise function from the two functions:*
>
> $f(h) = \sqrt{(\frac{h_2 - h_0}{2})^2 - (x - \frac{h_2 - h_0}{2})^2}$ *from* $[h_0, h_1]$
>
> $f(h) = r_1$ *from* $[h_1, h_3]$
>
> *And using a volume formula:*
>
> $V_2 = \int \pi\, r^2 dr$
>
> *Next we can plug in our two functions and their bounds, because volume is the same as the amount of water*
>
> $$V = \pi \int_{h_0}^{h_1} (\sqrt{(\frac{h_2 - h_0}{2})^2 - (x - \frac{h_2 - h_0}{2})^2})^2 dx + \pi \int_{h_1}^{h_3} {r_1}^2\, dx$$
>
> *(Student 12’s next response to question 1b)*

I note the notational inconsistency in the formula for the function f: the independent variable is called “h” in the name of the function (and on the graph) and “x” in the rule of the function, and the name of the independent variable is omitted altogether in specifying the intervals; the student is saying, for example, “from $[h_1, h_3]$” instead of “for $h \in [h_1, h_3]$”. On the graph in Figure 20, this function is called “$r(h)$”, not “$f(h)$”.

Moreover, the last formula in his response is the formula for the total volume of water in the whole flask, and writing it is questionable in a problem that studies the process of filling the flask with water. The student should have continued to elaborate on the function $f(h)$ and calculated the integrals with variable upper bound:

$V(\mathrm{h}) = \pi \int_{h_0}^{h} (\sqrt{(\frac{h_2 - h_0}{2})^2 - (x - \frac{h_2 - h_0}{2})^2})^2 dx$ for $\mathrm{h} \in [h_0, h_1]$ and

$V(\mathrm{h}) = \pi \int_{h_1}^{h} {r_1}^2\, dx$ for $\mathrm{h} \in [h_1, h_3]$

Instead, the student manipulates the formula for the total volume in a half-page of "calculations". He does not, however, calculate the value of the integral:

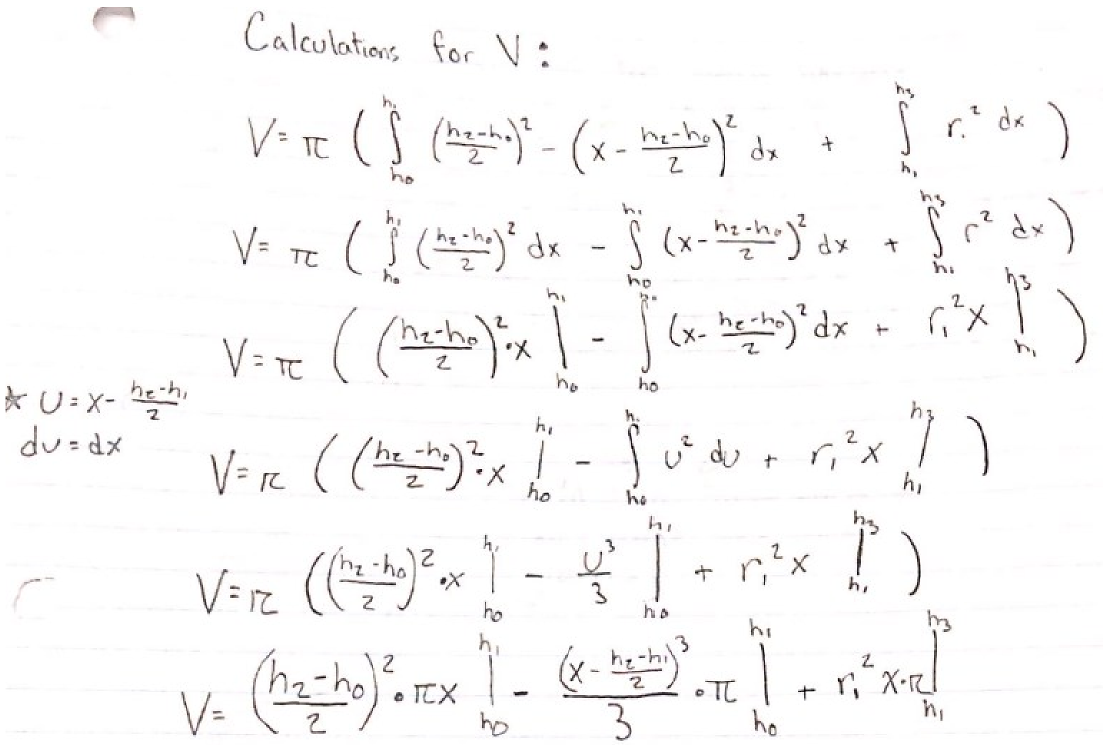


Figure 21. Student 12's manipulations of the formula for total volume of the flask

The last line in his calculations leaves the integrals unevaluated at the bounds, and this allows him to obtain a formula for volume of water as a function of height as a piecewise function:

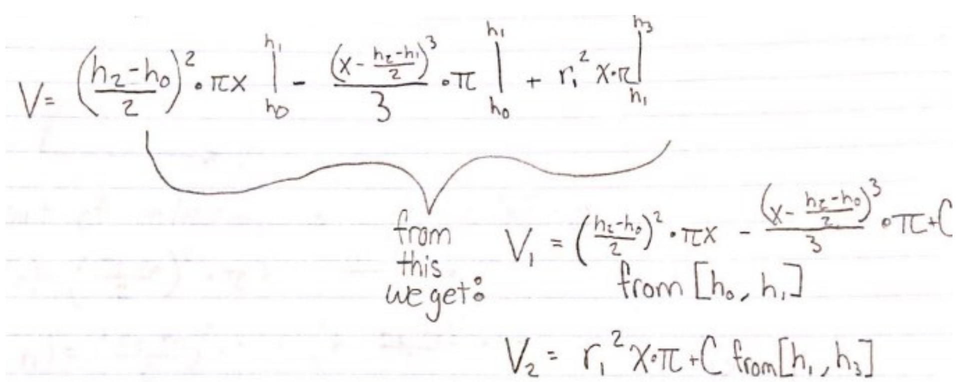


Figure 22. Image of the formula for volume as function of height that Student 12 obtained from calculating the integrals in Figure 21

There are similar imprecisions of notation in Figure 22. The name of the independent variable is not mentioned in the name of the function, nor in the intervals. Moreover, the constant C is not specified although it is essential, at least for the linear piece of the function (the "V_2"), which does not start at (0, 0) but it is shifted both horizontally and vertically.

The student then sketches a rough graph of the function $V(h)$ and says "However, this is the graph of amount of water as function of height and we need to find height as function of amount". This verbalization is clear evidence of awareness of what the independent and the dependent variables and of "coordinating them consistently with the labeling of the axes" (verbal sign of MA1).

He then proceeds to obtain the inverse function $h(V)$ (algebraic sign of MA1). In Figure 22, he had used the letter "x" for height and the letter "V" for volume; now he renamed V as "y". Then he switched those letters in his formulas for V_1 and V_2 obtaining two equations of the form $x = f(y)$. He did not solve these equations for y. Thus, we can say that this student may not have overcome yet the common didactical obstacle labeled as DO – XY: Didactical obstacle involved in representing variables with the letters 'x' and 'y'. With no further explanation, he then states that his graph in question 1a can be obtained from these equations (Student 12's following response to question 1b:

Next we must find the inverse functions, so if:

$V_1 = (\frac{h_2-h_0}{2})^2\pi h - \frac{(h-\frac{h_2-h_0}{2})^3}{3}\pi + C \qquad V_2 = {r_1}^2 h\pi + C$

$y = (\frac{h_2-h_0}{2})^2\pi x - \frac{(x-\frac{h_2-h_0}{2})^3}{3}\pi + C \qquad y = {r_1}^2 x\pi + C$

we must switch x and y:

$x = (\frac{h_2-h_0}{2})^2\pi y - \frac{(y-\frac{h_2-h_0}{2})^3}{3}\pi + C \qquad x = {r_1}^2 y\pi + C$ => *And from these equations, we make our final graph:*

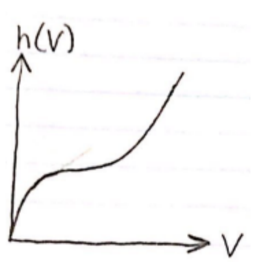


Figure 23. Image of the final part of Student 12's response to question 1b

In response to question 1c, about the inflection point, Student 12 marks the point correctly on his graph in Figure 19, but offers only the necessary analytic condition for the existence of an inflection point – the second derivative equal to 0 – and applies it to the function $V(h)$ not the function $h(V)$ for which he has no explicit formula:

c) point of inflection is when $V''(h) = 0$

$$V(h) = (\frac{h_2 - h_0}{2})^2 \pi h - \frac{(h - \frac{h_2 - h_0}{2})^3}{3} \pi$$

$$V'(h) = (\frac{h_2 - h_0}{2})^2 \pi - (h - \frac{h_2 - h_0}{2})^2 \pi$$

$$V''(h) = 1 - 2(h - \frac{h_2 - h_0}{2})\pi = 0$$

(Student 12's response to question 1c)

The student made a mistake in calculating the second derivative $V''(h)$ – there should be a zero and not "1" – so the solution of the equation $V''(h) = 0$ is not $h = \frac{h_2 - h_0}{2}$ (nor the radius of the sphere) as his graph in Figure 19 suggests. Obviously, the student had not used this equation to find the inflection point, and most likely had not used the equations for the functions $V(h)$ or $h(V)$ to sketch their graphs. His behaviors are suggestive of him having common pseudo-analytical thoughts when responding to Problem 1 (Vinner, 1997). One reason why he had these thoughts may be that the student has not overcome yet the most common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46). Whatever were his reasons, however, the result is that we do not know how he was really reasoning, how he figured out the particular shape of the graph. While he may have been reasoning covariationally at levels 3, 4, and 5, there is no written evidence for that. Since he was unable to transform his graphical construction of the functional situation to a verbal representation, I estimate that he may not have overcome yet the common cognitive obstacle labeled as CO – GTV: Cognitive obstacle of being unable to transform a graphical representation of function to its verbal one.

Student 12 sketched an acceptable graph with axes labeled correctly (graphical signs of MA1 and MA2) and his analytical solution indicates an awareness of the direction of change of the height with respect to the volume (algebraic signs of MA1 and MA2). Therefore, we can say that the covariational reasoning of this specific student has reached Level 2. The student who appeared to have weak conceptual thinking of the function did not represent relative magnitudes of the variables on the axes, nor did he display an awareness of the amount of change in height of water while picturing changes in volume (no signs of MA3). He did not draw secant lines and speak about the rate of change of the height of water slowing down and speeding up while envisioning changes of the volume of water in the spherical part of the bottle nor did he mention about the rate of change of the height of water being constant while picturing changes of the volume of water in the cylindrical part of the flask in his responses (no signs of MA4).

5.1.2.5.2 Student 14: Unsatisfactory verbalization of the functional relationship of the dynamic situation

Student 14's solution to Problem 1 follows:

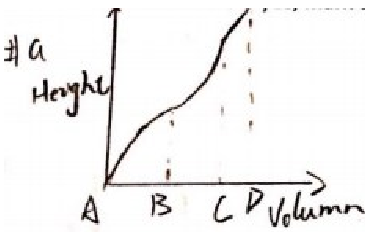


Figure 24. Image of student 14's graph

> *(b) A -> B the growth rate is decreasing with the increase of height, B -> C the growth rate is increasing with the increase of height, C -> D the growth rate is constant with the increase of height. (c) point B. (d) From C to D. (Student 14)*

Student 14 provided a valid graph with axes labeled correctly and expressed an awareness of changes in the height of water while imaging increase in volume, which are signs of MA1. His rising graph from left to right is also suggestive of him graphically coordinating the direction of change of the output while picturing changes in the input (graphical sign of MA2). While he symbolically represented four magnitudes (A, B, C and D) of the volume, he did not represent corresponding magnitudes of the height of water on his graph (no graphical sign of MA3). In his response to question 1b, he stated as 'A -› B, the growth rate is decreasing with the increase of height. B -› C, the growth rate is increasing with the increase of height. C -› D, the growth rate is constant with the increase of height' (evidence of having weak conceptual thinking of the function). It seems that he considered three varying quantities as 'changes in the rate of change of the height (e.g. 'the growth rate is decreasing') depend on changes in height of water (e.g. 'with the increase of height') which depend on the particular interval of the volume (e.g. 'A -> B' being magnitudes of the volume): R(h(v))'. In his responses, he doesn't explicitly verbalize an awareness of the amount of change of height with respect to changes in volume (no verbal sign of MA3); instead, he appears to verbalize the average rate of change of the height with respect to changes in height (in relation to the volume of water). Thus, I evaluate that the student was not able to coordinate the average rate of change of the height of water with respect to volume of water. He also failed in constructing secant lines on his graph (so no signs of MA4). He plotted the inflection point correctly, but without a justification. He may not have overcome yet the common cognitive obstacle of being unable to transform a graphical representation of a function to its verbal one labeled as CO – GTV since he provided a valid geometrical construction of the function, but his verbalization of the situation was insufficient.

5.1.2.5.3 Student 20: Constructing a linear graph of V(h) and having pseudo-thoughts

In her response to question 1a, Student 20 had an image of the following expressions in her mind:

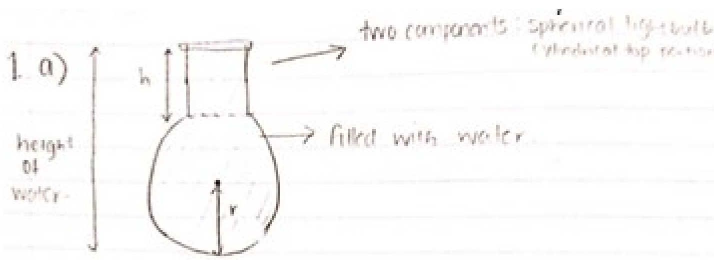


Figure 25. Student 20's image of the bottle

$$a)\begin{cases} Sphere: V = \frac{4}{3}\pi r^3 \\ Cylinder: V = \pi r^2 h \end{cases} => V = \frac{4}{3}\pi r^3 + \pi r^2 h$$

(Student 20's first response to question 1a)

The student used her memorized volume formulas of sphere and cylinder as $\mathrm{V} = \frac{4}{3}\pi \mathrm{r}^3$ for the volume of water in the spherical cap of the flask, $\mathrm{V} = \pi \mathrm{r}^2\mathrm{h}$ for the volume of water in the cylindrical part of the bottle and $\mathrm{V} = \frac{4}{3}\pi \mathrm{r}^3 + \pi \mathrm{r}^2\mathrm{h}$ (algebraic sign of MA1) for the total volume of the water in the bottle. I note that she used the same letter symbol 'r' in both her formulas.

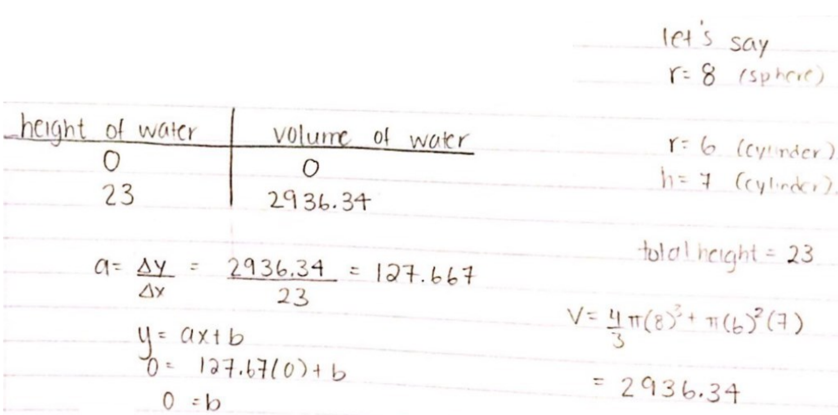


Figure 26. Image of Student 20's further response to question 1a

By imagining some numerical values, she wrote the value of the radius of sphere as eight, the value of the radius of cylinder as six and the value of the height of cylinder as seven, plugged these values into the total volume formula, that is $V = \frac{4}{3}\pi r^3 + \pi r^2 h$, in order to find the value of the total volume of the water in the bottle as $V = 2936.34$ (evidence of pseudo-analytical behavior). However, she then completely avoided the volume formulas of sphere and cylinder and used two other memorized expressions $y = ax + b$ and $a = \frac{\Delta y}{\Delta x}$ (but this time the independent variable is represented by the letter 'x' being the height of water while the dependent variable is represented the letter 'y' being the volume of water), which are commonly used in todays' mathematical activities, in order to find the value of the average rate of change of the function a as 127.667 and the value of the constant quantity b as zero (strong evidence of her having both pseudo-conceptual and pseudo-analytical thoughts). This student may not have overcome yet the common didactical obstacle labeled as DO – XY: Didactical obstacle involved in representing variables with the letters 'x' and 'y'.

She then constructed the following linear graph of the volume of water as a function of the height of water with its corresponding table:

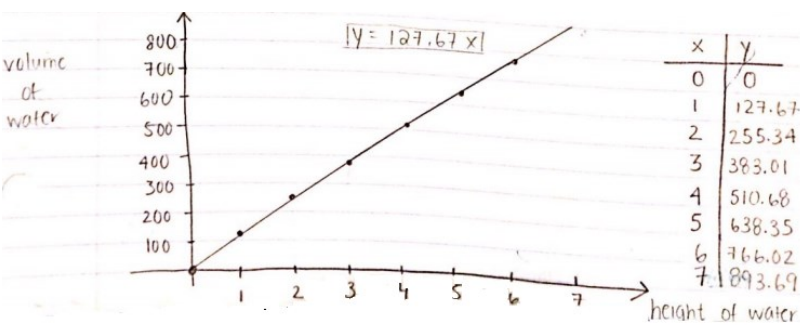


Figure 27. Image of Student 20's graph and table in question 1a

By using her linear function $y = 127.67x$, she produced a table containing the imagined numerical values of the height of water from 0 to 7 with the corresponding values of the volume of water and constructed a linear graph of the volume of water as function of the height of water with the associated numerical values (graphical and algebraic signs for MA1 and MA2). Very likely, this student has not overcome yet the following epistemological obstacle "*EO(f)-7: A Pythagorean philosophy of number: everything is number*" (Sierpinska, 1992, p.41). Based on her various algebraic equations including the table with imagined numerical values, we can also claim that she has not overcome yet the most common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46).

Another evidence of the student missing the control stage and providing incorrect comments lies on her explanations for questions 1b, 1c and 1d which were:

> *(b) The above graph represents the relationship between the height of the water as a function of the amount of the water in the bottle because the radius and height of both the cylindrical and spherical part of the light bulb remain the same. So, initializing those values, I obtain a total volume for the light bulb (V = 2936.34 u^3). Now, the total height would be the height of the cylinder and 2r of the sphere (total height of bulb = 23 units). At 0 height, volume is 0. Using the formula y = 127.67x, the graph between the height of bulb and volume of water is found. (c) there is no point of inflection because the curve(line) does not change in any other direction. (d) the height of water increases linearly with respect to the amount of water all throughout. (Student 20)*

In her solution to question 1b, she first claimed that 'the above graph represents the relationship between the height of the water as a function of the amount of water that is in the bottle' which is not inconsistent with her graph since she constructed a graph of the volume of the water as a function of the height of the water (strong evidence of pseudo-conceptual behavior). Thus, it appears that there is no sign of MA1 for verbal representation. She was not able to transform neither her graphical representation of the function to its verbal one (defined as the cognitive obstacle, CO - GTV) nor her verbal expression of the function to its graphical one (defined as the cognitive obstacle, CO - VTG) since her verbalization of the situation in question 1b does not correspond to her graphical construction in question 1a. She attempted to justify her linear graph by writing the value of the total volume of the water as $V = 2936.34$, the value of the total height of the flask as h = 23 and her last volume formula as $y = 127.67x$ (evidence of pseudo-analytical behavior). Like Student 12, she considered two letters, x and h, for the height of the water and two letters, y and V, for the volume of the water. She then wrote that she got her final graph 'between the height of bulb and volume of water' without mentioning which one is the function of the other.

It is evident that this student lacked essential conceptual thinking necessary for solving the dynamic task, used multiple memorized algebraic formulas to represent the volume function, failed in making logical connections between those expressions and conceptual aspects of the dynamic situation, lacked of control mechanism in her solutions, and so she was not able to develop meaningful construction of the given mathematical situation. This is a case of both pseudo-conceptual and pseudo-analytical modes of thinking.

> *…there are some contexts in which both the analytical and conceptual (or the pseudo-analytical and the pseudo-conceptual) modes are involved. (Vinner, 1997, p.116)*

5.1.2.6 Examples of responses representing Level 1 of covariational reasoning

5.1.2.6.1 Student 16: Also having both pseudo-conceptual and pseudo-analytical thoughts

Student 16 constructed the following graph of the volume of the water as a function of the height of the water in his response to question 1a:

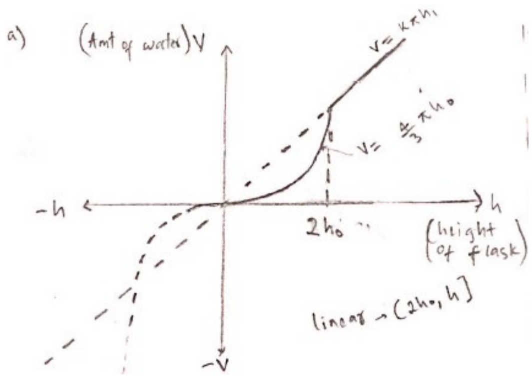


Figure 28. Image of Student 16's graph

In his solution to question 1b, the student divided the bottle into two parts as sphere and cylinder and into three intervals with four connecting points of 0, h_0, $2h_0$ and h (where h_0 is the radius of the sphere and h is the height of the flask). Then, he provided the volume formulas for the sphere and cylinder as $V = \frac{4}{3}\pi {h_0}^3 and\ V = \pi {\gamma_1}^2 h_1$ (where γ_1 is the radius of the cylinder and $h_1 = h - 2h_0$), respectively, and tried to take the derivative of the volume formula of the sphere with respect to its radius.

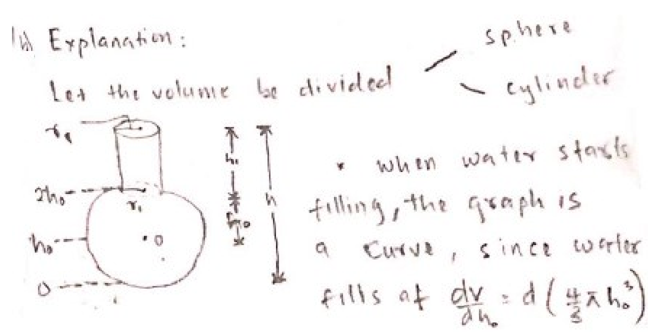


Figure 29. Image of beginning of Student 16's answer to question 1b

Student 16 then continued to explain his graph with his following response to question 1b:

> *(b)The curve part of the graph shows how water keeps filling as it is a sphere $(V = \frac{4}{3}\pi {h_0}^3)$.This continuous until a point $2h_{0,}$ where what starts filling in as a cylinder. Here $V = \pi {\gamma_1}^2 h_1$ and γ_1 is the radius of the cylinder and it is constant so, let at be K, then $V = \pi K h_1$ => this is a linear function which becomes the new volume from h = $2h_0$ distance. (Student 16)*

Student 16's answers to questions 1c and 1d were:

> *(c) The graph has an inflection point at h = 0, but since w_0 only consider the positive x, y of this graph, then it has no inflection on the positive x, y axis. An inflection point occurs when graph changes from increasing to decreasing or vice versa, but this graph keeps increasing always.*
>
> *(d) Yes, there is an interval where the volume increase linearly and that is from $2h_0$ to h, thus [$2h_0$, h]. (Student 16)*

Student 16 labeled the horizontal axis by "h" and the vertical axis by "V" (graphical sign of MA1). But "h" means the "height of flask" for him, which is tacitly assumed to be constant in the problem, not the variable height of the water. His geometrical construction represents the graph of the volume of a sphere as a function of the radius: $V(h_0) = \frac{4}{3} \cdot \pi \cdot h_0^3$, where h_0 is the radius of a sphere. The equation he wrote in Figure 29 $\frac{dV}{dh_0} = d(\frac{4}{3} \cdot \pi \cdot h_0^3)$ also represents h_0 as an independent variable. He seems to think that the formula of the volume of a sphere as a function of its radius is an equation of a sphere filling with water *radially*. He is aware that some quantities co-vary in the problem, but he fails to identify them correctly. It is therefore with some reservations that his solution is evaluated as representing Level 1 of covariational reasoning because some sentences in his responses to questions 1b and 1c are supportive of him being aware of that changes of height and volume are coordinated (verbal sign of MA1).

Based on his responses to Problem 1, we can claim that the student did not have necessary conceptual thinking when modeling the dynamic event since he failed in considering the conceptual aspects of the given task correctly. He relied on his uncontrolled memorized associations, failed in making logical connections between his procedural associations and conceptual elements of the situation and so was unable to develop meaningful interpretation of the given functional situation. This is also an example of common pseudo-conceptual and pseudo-analytical modes of thinking as in the case of student 20. Based on his responses, we conclude that this student, like Student 20, has not overcome yet the most common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46).

5.1.2.7 Examples of responses representing Level 0

5.1.2.7.1 Student 17: Height vs time

The student labeled the horizontal axis as "time" and the vertical axis as "height" (no graphical sign of MA1):

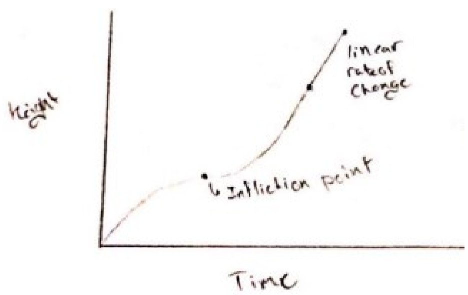


Figure 30. Image of Student 17's graph

His answer to question 1b starts with sentences stating the relationship between the direction of change in the rate of change and concavity (memorized rules).

> *avg rate of change increase –concave up*
> *avg rate of change decreases-concave down*
> *(b) my graph represent this relationship by showing a slow rate of change to fill up the bottom, as it approaches the midpoint the rate of change starts to increase and even more so at the neck. (Student 17)*

Student 17 provided the following responses to questions 1c and 1d:

> *(c) Yes, because there is a change in the rates.*
> *(d) Yes it would and it would happen in the neck area. (Student 17)*

When analyzing Student 17's solution to Problem 1, it is observed that he has constructed an acceptable shape of the graph with a correct inflection point, but did not consider 'volume (or amount of water)' as the independent variable in his graph and, instead, he pictured 'time' as the input. Possibly, the student has not overcome yet the common epistemological obstacle labeled as "*EO(f)-16: The changes of a* *variable are changes in time*" (Sierpinska, 1992, p.55). Therefore, it is observed that the covariational reasoning of this specific student did not even reach Level 1 since he did not produce a solution which supports MA1 (evidence of not having necessary conceptual thinking of the function). From the student's consideration of the two variables as 'height' and time', we can claim that this student also, like Student 7, may not have gone past experiencing the act of understanding labeled as "*U(f)-3: Identification of the subjects of change in studying changes*" and overcome yet the corresponding common epistemological obstacle of 'ignoring what changes' which is identified as "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes*" (Sierpinska, 1992, p.36). His answer to question 1b suggests that the student is aware that the rate of change of the height decreases until [when the water reaches] the middle of the spherical part of the flask and then increases until the beginning of the cylindrical part of the flask, except that the reference to time remains implicit, so that it sounds as if the independent variable was "the bottle" and its shape. He seems to rely, in sketching the graph, on a memorized rule (e.g. 'avg rate of change increase →concave up, avg rate of change decrease →concave down') which he probably remembered from previous mathematical events of answering similar questions rather than on a thorough analysis of the conceptual elements of the given dynamic task. By using this procedural association, he tried to explain the functional relationship that his graph represents by such statement; 'showing a slow rate of change to fill up the bottom, as it approaches the midpoint the rate of change starts to increase and even more so at the neck' in his response to question 1b, and the reason why the point he marked on his graph is the inflection point with such words; 'because there is a change in the rates' in his answer to question 1c. His statement of that 'the rate of change increases even more at the neck' is not accurate since the rate of change of the height of water is constant in the cylindrical section of the bottle. Based on the student's responses, we can say that he was also in both pseudo-conceptual and pseudo-analytical modes of thinking like students 16, 19 and 20 when responding to Problem 1 since he failed in thinking the concepts and ideas of the given functional situation correctly, used uncontrolled memorized procedures and failed in making meaningful connections between his procedural and conceptual associations.

5.1.3 Analysis from the point of view of students' common pseudo-thoughts, obstacles and misconceptions

I start with the data tabulated in Table 5 which lists the assessments of students' solutions from the point of view of common pseudo-thoughts, obstacles (epistemological, didactical and cognitive) and misconceptions.

Table 5. Distribution of each student's common pseudo-thoughts, obstacles, misconceptions

Student's code	Student's pseudo-thoughts	Student's epistemological obstacles	Student's didactical obstacles	Student's cognitive obstacles	Student's misconceptions
S1	---	---	---	---	---
S2	---	---	---	---	---
S3	---	---	---	---	---
S4	---	---	---	---	---
S5	---	---	---	---	---
S6	---	---	---	---	---
S7	---	---	---	---	---
S8	---	EO(f)-9	---	---	---
S9	---	---	---	---	---
S10	---	---	---	CO-VTG	---
S11	---	---	---	CO-VTG	---
S12	PSA	EO(f)-11	DO-XY	CO-GTV	---
S13	---	EO(f)-5, EO(f)-16	---	---	---
S14	---	---	---	CO-GTV	---
S15	---	EO(f)-11	---	CO-GTV	---
S16	PSA and PSC	EO(f)-11	---	---	---
S17	PSA and PSC	EO(f)-3, EO(f)-16	---	---	---
S18	---	EO(f)-3	---	---	---
S19	PSA and PSC	EO(f)-11	---	---	---
S20	PSA and PSC	EO(f)-7, EO(f)-11	DO-XY	CO-GTV and CO-VTG	---
S21	---	EO(f)-3, EO(f)-7, EO(f)-16	---	---	MC-IP
S22	---	EO(f)-3, EO(f)-11, EO(f)-16	---	CO-GTV	MC-IP
S23	---	EO(f)-3, EO(f)-16	---	---	---
S24	---	EO(f)-3	---	---	MC-IP

Abbreviation:

PSA: Pseudo-analytical thought,

PSC: Pseudo-conceptual thought,

MC-IP: Misconception about the inflection point

DO-XY: Didactical obstacle involved in representing variables with the letters 'x' and 'y'

CO-GTV: Cognitive obstacle of being unable to transform a graphical representation of function to its verbal one

CO-VTG: Cognitive obstacle of being unable to transform a verbal representation of function to its graphical one

"EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes" (Sierpinska, 1992, p.36)

"EO(f)-5: Regarding the order of variables as irrelevant" (Sierpinska, 1992, p.38)

"EO(f)-7: A Pythagorean philosophy of number: everything is number" (Sierpinska, 1992, p.41)

"EO(f)-9: Proportion is a privileged kind of relationship" (Sierpinska, 1992, p.43)
"EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions" (Sierpinska, 1992, p.46)
"EO(f)-16: The changes of a variable are changes in time" (Sierpinska, 1992, p.55)

The data suggests that seven students (numbered 12, 15, 16, 17, 19, 20 and 22), making 29% of the participants, relied on their procedural associations instead of focusing on the conceptual elements of the dynamic task. Consequently, Students 16, 17, 19 and 20 were in both pseudo-analytical and pseudo-conceptual modes of thinking (Vinner, 1997), while Student 12 had pseudo-analytical thoughts when responding Problem 1. I believe that the reason why Students 12, 16, 19 and 20 had common pseudo-analytical thoughts is that they have not overcome yet the common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46). Similarly, it seems that Students 15 and 22 have also not overcome yet this epistemological obstacle since they provided unnecessary analytic formulas in their responses to the dynamic task which has a conceptual basis.

Furthermore, Students 17, 18, 21, 22, 23 and 24, amounting for 25% of the respondents, may not have overcome yet the corresponding common epistemological obstacle of 'ignoring what changes' which is identified as "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes*" (Sierpinska, 1992, p.36) since they were unable to identify the name of the variables while Student 13 may not have overcome yet the epistemological obstacle labeled as "*EO(f)-5: Regarding the order of variables as irrelevant*" since she was not able to discriminate between the dependent and independent variables (Sierpinska, 1992, p.38). In addition, Students 13, 17, 21, 22 and 23 may not have overcome yet the epistemological obstacle "*EO(f)-16: The changes of a variable are changes in time*" (Sierpinska, 1992, p.55) due to them considering the independent variable as time. In summary, it looks like "*EO(f)-11, EO(f)-3 and EO(f)-16"* are the most common epistemological obstacles among college level Calculus students.

Meanwhile, Students 20 and 21 may not have overcome yet the epistemological obstacle labeled as "*EO(f)-7: A Pythagorean philosophy of number: everything is number*" (Sierpinska, 1992, p.41), and Student 8 has likely not overcome yet the epistemological obstacle of 'proportion privileged' which is identified as "*EO(f)-9: Proportion is a privileged kind of relationship*" (Sierpinska, 1992, p.43).

On the one hand, Students 12, 14, 15, 20 and 22 were not able to make a 'transformation from a graphical representation of a function to its verbal one', which is the common cognitive obstacle labeled as CO - GTV. On the other hand, Students 10 and 11 (and also student 20 since her verbal expression in question 1b does not correspond to her constructed graph in question 1a) seemed to be unable to 'transform a verbal representation of a function to its graphical one', which is the cognitive obstacle labeled as CO - VTG. Hence, based on the students' written responses to Problem 1, we can say that these seven students, amounting for 29% of the respondents, may not have overcome yet cognitive obstacles involved in moving between different representations of functions. In addition, Students 12 and 20 preferred to use the letters 'x' or/and 'y' in their analytical expressions. It is likely that this is due to students, probably these two as well, frequently studying with typical examples in mathematical instruction and those often involve the letters 'x' and 'y'. This is considered as an evitable common didactical obstacle (labeled as DO – XY: Didactical obstacle involved in representing variables with the letters 'x' and 'y') being generated by the narrow educational system (Herscovics, 1989).

Lastly, three students have showed a deep misconception about the concept of inflection point: Student 21 pictured an inflection point as appearing when the graph of a function touches the x-axis; Student 22 believed that an inflection point happens when the *shape* of graph changes (without providing valid explanation of what that shape may be); and Student 24 thought that an inflection point occurs when the first derivative changes sign.

5.2 Analysis of students' responses to the second problem about bottle filling: Thinking analytically about the process and then graphing and verbalizing the process

This section presenting the analysis of the students responses to Problem 2 consists of four subsections: analysis from the point of view of correctness (5.2.1), analysis from the point of view of level of students' covariational reasoning (including students' analytical and conceptual thinking of the dynamic event) (5.2.2) and analysis from the point of view of students' common pseudo-thoughts, obstacles and misconceptions (5.2.3). I aim to analyze whether students are able to represent a given functional situation, as exemplified here by Problem 2, in each of the following forms: algebraic, graphical and verbal. I note that in Problem 1, the analysis could only have been implemented for the graphical and, respectively, the verbal representations of a function. The type of mathematical questions posed in Problem 2 makes possible extending the analysis to the algebraic representation, hence a more comprehensive analysis.

5.2.1 Analysis from the point of view of correctness

Fifteen students (Students 1, 2, 4, 5, 6, 7, 9, 11, 12, 13, 14, 16, 20, 21 and 24) participated in the resolution of Problem 2. Unfortunately, no student out of fifteen provided correct responses for all questions asked in the second problem (Table 6).

Table 6. Distribution of correct answers to Problem 2 and their percentages.

Question	Number of correct answers to each question	% of correct answers among 15 students
Q2a	2	13
Q2b	1	6.66
Q2c	3	20
Q2d	1	6.66
All of them	0	0

5.2.1.1 Question 2a

In question 2a, with the given value of radius of the spherical part of the flask as 1 unit, and the neck 1 unit high, while the distance from neck to the bottom of the flask is 1 and 4/5 units, students were asked the following sub questions: "find a formula for the volume V of water as a function of the height h of the water: V(h)=? What is the domain of this function? What is the range?" Only two students (numbered 6 and 21) out of fifteen, which amounts for 13% of students, found a correct formula for the volume of water as a function of the height of the water and correct domain and range of this function. Student 11 obtained a correct formula but without calculating the integrals and referring to the domain and range of the function in question 2a. Student 13 provided a correct formula including the correct domain but with an incorrect range (although she provided a correct range in the beginning of her notes that she later changed to a wrong one). Additionally, Student 12 used the letter 'x' instead of the letter 'h' for the independent variable when providing the formula for the volume of water in the cylindrical part of the bottle although the domain and range he has given for the function are correct.

5.2.1.2 Question 2b

In question 2b, students were asked the following sub questions: "sketch a graph of the height as a function of the amount of the water, i.e., of the function h(V). Explain how you did it and what makes you sure you are right. What is the domain of this function? What is its range?" Only one student (student 6) out of fifteen, which makes only 6.66%, constructed an acceptable graph of the height of the water as a function of the amount of the water with correct relative magnitudes of both independent and dependent variables, provided a suitable explanation for his graph and wrote the correct domain and range of the height function. Student 1 also provided an acceptable graph, with satisfactory explanation, but without written relative magnitudes of the independent variable on her graph and without any mention of the domain and range of the height function with respect to volume. Students 4 and 7 also provided somewhat suitable explanations, but without acceptable graphs, and without writing the domain and range of the height function. Although he wrote that ‘y = x’ instead of ‘h = V’, student 4’s explanation was considered as satisfactory:

> *Firstly sketch the V(h) graph and reverse its graph about y = x to get h(V) (Student 4’s answer to question 2b)*

Student 7’s satisfactory explanation follows even though his graphs do not seem to be correctly constructed:

> *I sketched V(h) first and then I just mirror it (took inverse) to create h(V) (Student 7’s answer to question 2b)*

On the other hand, Student 11 has given the correct domain and range of the height function but with unacceptable graph and no explanation for her graph.

5.2.1.3 Question 2c

In question 2c, students were asked to “find the height of the water if there are, approximately, i) 2 litres, ii) 4 litres, iii) 5 litres of the water in the flask (with 1 litre =1dm^3)”. Only three students (numbered 5, 11 and 12), amounting for 20% of students, provided somewhat acceptable solutions of the height of the water for all three different values of the volume given in question 2c (although Student 12 did not provide any process about how he obtained correct answers for all three cases). It is important to note that these students did not provide a correct formula (including the integrals being evaluated in their formulas) for the entire domain of the function in their solutions to question 2a and correct responses to question 2b, but found correct answers of the three values of the height of the water for the given three different values of the volume of water in question 2c. Student 21 correctly found the solutions of the height of the water for the two values of the volume of the water given in the parts i) and ii), but he made a mistake in his calculations when trying to find the value of the height of water for the value of the volume of water given as 5 litres in the part iii), so he obtained an incorrect answer of the value of the height of water as 2.02 dm instead of 2.62 dm. Student 1 gave acceptable solutions of the height of the water for the two values of the volume of the water given in the parts ii) and iii) while she could not find the value of the height of water when the volume of water is 2 litres. On the other hand, student 14 obtained a correct answer for only the question asked in the part i) while student 6 gave a correct response for only the question asked in the part iii).

5.2.1.4 Question 2d

In question 2d, the following questions asked: "Does the graph of the function h(V) have a point of inflection? If yes, what are its coordinates? Justify your answer". Only one student (numbered 6) provided the coordinates of the inflection point of the graph of the function h(V) correctly with valid explanation as the second derivative changes sign at this point. Student 11 also provided the correct coordinates of the inflection point of the graph of the function h(V) but without satisfactory justification while Students 1, 7, 12 and 13 provided somewhat satisfactory justification for the inflection point of the graph of the function but without providing its full coordinates.

5.2.2 Analysis from the point of view of level of covariational thinking

5.2.2.1 An overview

I present a table summarizing the assessments of students' solutions from the point of view of "correctness" (or "acceptable answers") and levels of covariational thinking (Table 7).

Table 7. Distribution of correct answers to Problem 2 and level of covariational reasoning for each student

Student's code	Q2a	Q2b	Q2c	Q2d Coor.	Q2d Justf.	Number of correct answers per student	Level of covariational reasoning
S1 – S24	1/0[1]	1/0[2]	1/0[3]	1/0[4]	1/0[5]	sum[6]	0 – 5
S1	**0.25**	**0.50**	**0.66**	**0**	**1**	**2.41**	**5**
S2	0.25	0	0.33	0	0	0.58	2
S4	0.50	0.25	0	0	0	0.75	2
S5	0	0	1	0	0	1	2
S6	**1**	**1**	**0.33**	**1**	**1**	**4.33**	**5**
S7	0	0.25	0	0	1	1.25	2
S9	0.50	0	0	0	0	0.50	2
S11	**0.50**	**0.50**	**1**	**1**	**0**	**3**	**3**
S12	0.50	0	1	0	1	2.5	2
S13	**0.75**	**0**	**0**	**0**	**1**	**1.75**	**3**
S14	0.50	0	0.34	0	0	0.84	2
S16	0	0	0	0	0	0	1
S20	0	0	0	0	0	0	2
S21	**1**	**0.25**	**0.67**	**0**	**0**	**1.92**	**3**
S24	0	0	0	0	0	0	0
# of total correct answers	5.75	2.75	5.33	2	5	20.83	Level 5 – 2 students, L.4 – 0 st. L.3 – 3 st. L.2 – 8 st, L.1 – 1 st. L.0 – 1 st.
% of correct	38	18	36	13	33	28	

answers among 15 students							

Legend:

1) formula acceptable = 0.50, domain acceptable = 0.25, range acceptable = 0.25, otherwise = 0

2) graph acceptable = 0.25, explanation satisfactory = 0.25, domain acceptable = 0.25, range acceptable = 0.25, otherwise = 0

3) the value of the height in part i) acceptable = 0.34, the value of the height in part ii) acceptable = 0.33, the value of the height in part iii) acceptable = 0.33, otherwise = 0

4) coordinates of the inflection point acceptable = 1, otherwise = 0

5) justification of the inflection point satisfactory = 1, otherwise = 0

6) sum of numbers in this row and columns 2 to 6.

I now present a table listing the assessments of students' solutions from the point of view of level of mental actions for algebraic, graphical and verbal representations of the given dynamic situation (Table 8)

Table 8. Distribution of each student's engaged mental actions for graphical, verbal and algebraic representations of the function

Student's code	**Student's engaged mental Action for algebraic representation**	**Student's engaged mental Action for graphical representation**	**Student's engaged mental Action for verbal representation**
S1 – S24	**MA1 – MA5**	**MA1 – MA5**	**MA1 – MA5**
S1	**MA2**	**MA2**	MA5
S2	MA2	MA0	MA1
S4	**MA2**	**MA2**	**MA2**
S5	**MA2**	**MA2**	MA0
S6	**MA5**	**MA5**	MA3
S7	**MA2**	**MA2**	**MA2**
S9	MA2	MA0	MA0
S11	**MA3**	**MA3**	MA1
S12	**MA2**	**MA2**	MA1
S13	**MA3**	**MA3**	MA1
S14	**MA2**	**MA2**	MA1
S16	**MA1**	**MA1**	**MA1**
S20	**MA2**	**MA2**	MA0
S21	MA3	MA2	MA0
S24	**MA0**	**MA0**	**MA0**

Two students (numbered 1 and 6), which amounts for 13% of the students, provided a response being evaluated as representing Level 5 of covariational reasoning. This means that only two students exhibited behaviors that support Mental Action 5 for algebraic or graphical or verbal representation of the dynamic situation by coordinating the instantaneous rate of change of the values of the output variable with respect to continuous changes in the input variable for the entire domain (Student 1 did this coordination verbally while Student 6 did it algebraically and graphically). Student 1 verbalized an awareness of the instantaneous rate of change of the function with respect to continuous changes in the input variable (verbal sign of MA5) in her response to question 2b. It appears that Student 1 has a strong conceptual thinking of functions since she was able to develop meaningful verbal and geometrical constructions of the dynamic situation given in Problem 2 (Vinner, 1997). Recall that I also had the same conclusion when analyzing the student's solution to Problem 1. However, she constructed a valid graph without representing relative magnitudes of the independent variable (no graphical sign of MA3) and was unable provide a correct formula for the volume of water as a function of the height of the water (no algebraic sign of MA3). It seems that that the student has a weak analytical thinking of functions because she was unable to develop a meaningful analytical expression of the dynamic event when solving Problem 2 (Vinner, 1997). Contrary, Student 6's solution is indicative of him, algebraically and graphically, coordinating the instantaneous rate of change of the dependent variable with continuous changes in the independent variable (algebraic and graphical signs of MA5) while his response to question 2b is also suggestive of him verbally coordinating the amount of change of the output while thinking changes in the input (verbal sign of MA3), showing evidences of him not only having strong analytical thinking, but also having strong or good conceptual thinking of functions identifiable as he was making logical connections between the functional concepts and procedures related to the given dynamic situation without missing control mechanism (Vinner, 1997). I note that the student provided a correct graph of the height function in Problem 2 while he provided a correct graph of the volume function, instead of the height function which was asked in Problem 1. Recall that I concluded that he has a good or moderate conceptual thinking when analyzing his answers to Problem 1.

Three students (numbered 11, 13 and 21), that is 20% of participants, exhibited behaviors supporting MA3 for algebraic representation of the function, so their covariational reasoning have reached Level 3 since they were able to coordinate the amount of change of the output with respect to changes in the input algebraically. It appears that these students have a good or moderate analytical thinking since they were able to develop a suitable analytical expression of the dynamic situation (Vinner, 1997). Students 11 and 13 both algebraically and graphically coordinated the amount of change of the output while considering changes in the input (algebraic and graphical signs of MA3) in their solutions. Student 21 also algebraically coordinated the amount of change of the dependent variable while thinking changes in the independent variable since he provided a correct formula, including the correct domain and range of the volume function, in his response to question 2a and obtained some correct values of the height of water in his answers to question 2c (algebraic signs of MA3). However, he failed in representing relative magnitudes of the dependent and independent variables when constructing his graph in question 2b, so no graphical evidence of MA3 appears while the direction of change of the output with respect to input is visible on his graph (graphical sign of MA2). On the other hand, the three students did not show a strong conceptual understanding of the function since they were unable to develop a meaningful verbal construction of the given dynamic situation in Problem 2 (they engaged in MA1 or MA0 for verbal representation of the function). Recall: Student 11 engaged in MA3 for verbal representation of the function while Student 13 engaged in MA2 and Student 21 engaged in MA0 when answering Problem 1. Hence, we can say that Students 13 and 21 have a weak conceptual thinking of functions because they failed in developing meaningful constructions related to the functional concepts in both problems and also missing control stage in their solutions (Vinner, 1997).

Furthermore, eight students (numbered 2, 4, 5, 7, 9, 12, 14 and 20), amounting for 53% of respondents, provided a solution being representative of Level 2 of covariational reasoning since these students were able to coordinate the direction of change of the output with respect to input algebraically or graphically or sometimes verbally. Six of them (Students 4, 5, 7, 12, 14 and 20) provided responses supporting MA2 for both algebraic and graphical representations of the given functional situation. It seems that these six students do not have a strong analytical thinking or understanding of functions because they were unable to develop an acceptable algebraic expression of the volume function with its correct domain and range and also a suitable geometrical construction of the height function in Problem 2. Students 4 and 7 (whose solutions are similar) were able to coordinate the direction of change of the output while considering changes in the input all together algebraically, graphically and verbally (signs of MA2) while Students 5, 12, 14 and 20 exhibited behaviors supporting MA2 for algebraic and graphical representations of the situation and Students 2 and 9 produced behaviors being supportive of MA2 for only algebraic representation of the given dynamic situation, being indicative of them having a poor analytical understanding of functions. On the other hand, Students 12 and 14 engaged in MA1 for verbalization of the situation in Problem 2 while they engaged in either MA1 or MA2 for verbal representation of the function in Problem 1, being supportive of them having a weak conceptual understanding of functions. Also, Student 20 engaged in MA0 for verbalization of the situations in both problems 1 and 2, being indicative of him lacking essential conceptual understanding of functions.

While Student 16's covariational reasoning has reached only Level 1 since he made a correct coordination between the two variables algebraically, graphically and verbally, Student 24's covariational reasoning remained at Level 0 since he failed in making this coordination. It appears that these two students have very poor (or do not have an) analytical and conceptual thinking of functions due to them being unable to develop meaningful constructions between the functional concepts and procedures associated with the dynamic situation, and also missing control stage in their solutions (Vinner, 1997). These outcomes are consistent with the results found in the analysis of the students' responses to Problem 1. Recall: we had similar conclusion as Students 16 having weak conceptual thinking while Student 24 not having essential conceptual thinking when analyzing the students' responses to Problem 1 (Student 16 engaged in MA1 for graphical and verbal representations of the function and that Student 24 engaged in MA0 for both representations of the function in Problem 1). Three students (numbered 2, 9 and 24) were unable to make coordination between the variables graphically (no graphical sign of MA1). Especially, two of them (Students 9 and 24) did not exhibit behaviors that support MA1 for graphical representation nor for verbal representation of the given dynamic situation (no verbal and graphical signs of MA1).

The data shows that the majority of students (86.66%) failed in coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function graphically or verbally or algebraically (no signs of MA5). More than half of students (66.66%), algebraically or graphically or verbally, did not coordinate the amount of change of the output while picturing changes in the input (no signs of MA3). Almost three quarter of students (73.33%) failed in coordinating the direction of change of the output with respect to changes in the input verbally (no verbal sign of MA2), and five of these students (numbered 5, 9, 20, 21 and 24), which makes 33% of all students, did not coordinate the two variables verbally (no verbal sign of MA1). I have observed the same outcome in the analysis of the Students 20's, 21's and 24's responses to Problem 1. Therefore, we can claim that these three students do not have an essential conceptual thinking of functions being necessary for modeling dynamic situations. On the contrary, it seems that Students 1 and 6 have a strong or good conceptual thinking of functions.

Most of respondents (79%), except Students 6, 11, 13 and 21 who seemed to have had strong or good or moderate analytical thinking of the function, were not able to coordinate the amount of change of the output while picturing changes in the input algebraically (no algebraic signs of MA3). Because of that, they were not able to develop a suitable analytical construction of the dynamic situation; it seems that they have had a weak analytical thinking of the function when responding Problem 2 (Vinner, 1997). On the other hand, I can determine that at least 47% of participants (Students 12, 13, 14, 16, 20, 21 and 24) have weak conceptual thinking of functions based on the consideration of the students' responses to both Problems 1 and 2 which I discuss further in the section 5.3.

5.2.2.2 Example of a response representing Level 5 of covariational reasoning

There are two students whose solutions were assessed as representing Level 5 of covariational reasoning, Students 1 and 6. I present an analysis of Student 6's solution.

5.2.2.2.1 Student 6: A Level 5 solution with strong graphical, algebraic and verbal representations of the dynamic situation

The student 6's solution is long and detailed, so I will present it partially considering the parts that are most relevant for our assessment. In question 2a with the given values of radius of the spherical part of the flask as 1 unit and the distance from neck to the bottom of the flask as 1 and 4/5 units, student 6 first calculated the value of radius of the cylindrical part of the bottle by using the Pythagorean Theorem as it is shown in Figure 31.

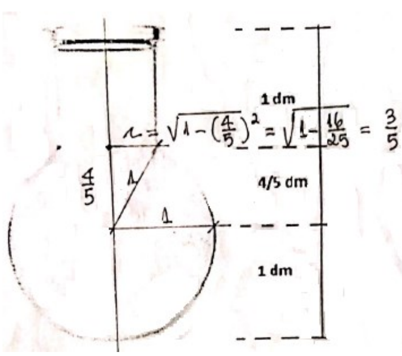


Figure 31. Student 6's use of Pythagorean Theorem

He then pictured the spherical part of the bottle on the Cartesian coordinate system as seen in Figure 32.

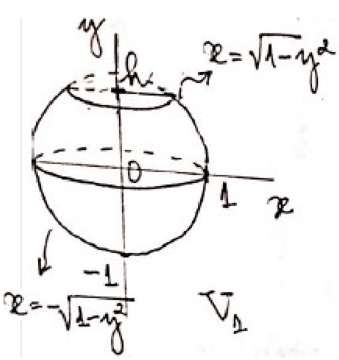


Figure 32. Student 6's image of spherical section of the flask

He noticed that the area under a curve which is half of the circle with center at (0, 0) and radius 1 can be represented with the equation: $x^2 + y^2 = 1$. He found the radius of the horizontal disc at level y to be: $x = \sqrt{1-y^2}$ when the water reaches height h, and realized that $A = \pi\,(1-y^2)$ by knowing the fact that the area A of the disc of radius R equals to πR^2. He then integrated this area function from -1 to h as follows:

$$V_1 = \int_{-1}^{h} \pi\,(\sqrt{1-y^2})^2\,dy = \pi h - \pi h^3\frac{1}{3} + \pi - \frac{1}{3}\pi = -\pi h^3\frac{1}{3} + \pi h + \frac{2}{3}\pi$$

(beginning of Student 6's response to question 2a)

(algebraic signs of MA1 and MA2)

After integration, he wrote as *"shift 1 unit to the right"* for the height of water (here, it seems that he visualized the height on the horizontal axis since it is independent variable) and obtained the following formula for the volume of water in the spherical section of the flask as a function of height h:

$$V_1(h-1) = -\pi(h-1)^3\frac{1}{3} + \pi(h-1) + \frac{2}{3}\pi$$

$$= -\pi(h^3 - 3h^2 + 3h - 1)\frac{1}{3} + \pi(h-1) + \frac{2}{3}\pi$$

$$V_1(h) = -\pi h^3\frac{1}{3} + \pi h^2$$

(Student 6's volume formula for the spherical cap written in question 2a)

(algebraic sign of MA3)

He then visualized the cylindrical section of the flask on Cartesian coordinate system (Figure 33):

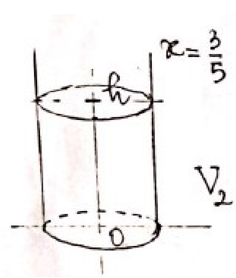


Figure 33. Student 6's image of cylindrical section of the flask

He first found the radius of the cylindrical part of the flask as $r = \sqrt{1^2 - (\frac{4}{5})^2} = \frac{3}{5}$ when the water in the neck reaches height h, plugged this value of the radius of the neck into the formula of the area of the disc (for the base of the cylinder), $A = \pi r^2 = \pi (\frac{3}{5})^2$, and then integrated the area function from 0 to h in order to find the formula for the volume of water in the cylinder as:

$$V_2 = \int_0^h \pi (\frac{3}{5})^2 \ dy = \frac{9}{25}\pi h$$

(Student 6's next response to question 2a)

After writing *"shift $V_1(\frac{9}{5})$ units up"*, he then calculated the volume of water in the spherical cap when the water reaches height $h = \frac{9}{5}$ as:

$$V_1(\frac{9}{5}) = -\frac{\pi}{3}(\frac{9}{5})^3 + \pi(\frac{9}{5})^2 = \frac{162}{125}\pi$$

(Student 6's following answer to question 2a)

(algebraic signs of MA1 and MA2)

After writing *"shift $\frac{9}{5}$ units to the right"* (again here, it seems that he imagined the height of water on the horizontal axis), he found the formula for the total volume of water in the flask by adding the volume of the spherical cap at height $h = \frac{9}{5}$ to the volume of the cylindrical section when the water reaches height $h - \frac{9}{5}$:

$$V_2(h - \frac{9}{5}) + \frac{162}{125}\pi = \frac{9}{25}\pi(h - \frac{9}{5}) + \frac{162}{125}\pi$$

$$V_2(h) = \frac{9}{25}\pi h + \frac{81}{125}\pi$$

(Student 6's volume formula for the cylindrical part written in question 2a)

(algebraic sign of MA3)

He then wrote the formula for the volume V of water as a function of the height h of the water as follows (algebraic signs of MA3, evidence of having strong analytical thinking of the function)

$$V(h) = \begin{cases} -\pi h^3 \frac{1}{3} + \pi h^2 \ , & 0 \le h \le \frac{9}{5} \\ \frac{9}{25}\pi h + \frac{81}{125}\pi \ , & \frac{9}{5} \le h \le \frac{14}{5} \end{cases}$$

(Student 6's final volume formula written in question 2a)

It seems that he made a typing error by making $h = \frac{9}{5}$ for both intervals of the domain which is however not invalidating the definition of the function as $V(h)$ is continuous in h and both branches give the same value of the function at $h = \frac{9}{5}$. He wrote correctly the domain and range of the volume function as:

$$Domain: D_{\text{v}} = [0, \frac{14}{5}],\ Range: V(h) = [0, V(\frac{14}{5})] = [0, \frac{207}{125}\pi]$$

(Student 6's final response to question 2a)

(algebraic sign of MA3).

For question 2b, student 6 correctly constructed the graph of the height of water as a function of the volume of water with clear indications of the values of the dependent variable with respect to the values of the independent variable as seen in Figure 34:

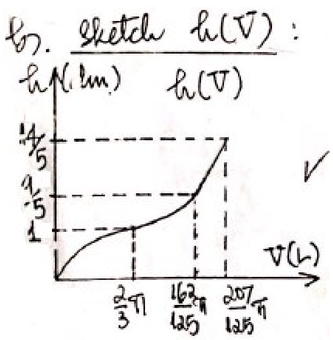


Figure 34. Image of Student 6's graph in question 2b

(graphical signs of MA1, MA2 , MA3 and MA5, but not of MA4 since no secant lines exist)

For the explanation on how he sketched his graph, the student mentioned clearly that the graph of h(V) is a reflection of the graph of V(h) in the line V = h. To further justify his action on the construction of his graph, he used the fact that the volume function V(h) is bijective, so its inverse h(V) exists (verbal sign of MA3, and also sign of having strong or good conceptual thinking of the dynamic situation):

how; reflect the graph of V(h) about the line V = h, what makes the graph correct; the function V(h) is bijective => There exists its inverse h(V) by reflecting V(h) about V = h

(Student 6's response for the explanation of his graph in question 2b)

By using the fact that the graph of h(V) is a reflection of the graph of V(h) in the line V = h, he then determined the domain and range of the height function h(V) as:

$V : D_V \rightarrow D_h$

$h : D_h \rightarrow D_V$

$Domain: D_h = [0, \frac{207}{125}\pi], Range: h(V) = [0, \frac{14}{5}]$

(Student 6's final response to question 2b)

(algebraic sign of MA3)

Since he is aware of the fact that the volume function V(h) is bijective, and its inverse h(V) whose graph is reflecting the graph of V(h) in the line V = h, we can say that this student view function as a process which accepts input and produces output. In addition to that the algebraic evidences of MA1, MA2 and MA3 that we have found in question 2a, we also found graphical evidences of MA1, MA2, MA3 and MA5 in question 2b. It is noteworthy to mention that this student constructed the graph of V(h) when he was asked to sketch a graph of the height of water as a function of the volume of water in Problem 1 while he did not repeat the same error in question 2b and correctly constructed the graph of h(V), being suggestive of him possession of a strong control mechanism vital for absence of pseudo-thoughts.

For question 2c, he wrote the following two results for the parts i) and ii) without showing any calculation or claiming to have used computing software like 'Wolfram Mathematica':

c) height of the water:

$i)\ V \approx 2L: h \approx 1dm,$

$ii)\ V \approx 4L: h \approx \frac{9}{5}dm,$

(Student 6's answers for parts i) and ii) of question 2c)

It is not clear how he calculated the heights of water when the volume is 2 and 4 litres. But, it is clear that when the volume is 2 litres, the height of water must be less than 1 dm ($h \approx 0.98\ dm$ and when the volume is 4 litres, the height of water must be less than 1.8 dm ($h \approx 1.74\ dm$). Therefore, these answers are not considered as correct (these are his only answers which were not accepted because they lack explanation).

On the other hand, he provided a correct answer for the part iii) in question 2c when the volume of water equals to 5 litres. He concluded that *"in the interval $\frac{162}{125}\pi \leq V \leq \frac{207}{125}\pi$, the graph is linear => the slope is constant"* and then made the following calculations by using the fact that the slope is constant in order to find the height of water as $h \approx 2.62\ dm$ when the volume is 5 litres:

$h(\frac{162}{125}\pi) = \frac{9}{5}, h(\frac{207}{125}\pi) = \frac{14}{5}, h(5) = h$

$$\frac{\frac{14}{5} - h}{\frac{207}{125}\pi - 5} = \frac{\frac{14}{5} - \frac{9}{5}}{\frac{207}{125}\pi - \frac{162}{125}\pi} <=> h = -\frac{\frac{207}{125}\pi - 5}{\frac{9}{25}\pi} + \frac{14}{5} \approx 2.62\ dm$$

(Student 6's response for part iii) of question 2c)

(algebraic signs of MA4 and MA5 for the linear part of the bottle)

In his response to the part iii) of question 2c, the student exhibited some behaviors supporting MA4 and MA5 since he explicitly mentioned about the rate of change of the volume of water for the cylindrical section of the bottle in his solution (e.g. 'the slope is constant') and also considered its value (e.g. '$\frac{9}{25}\pi$') in his calculations.

In his response to question 2d, he first wrote as "Yes, the inflection point is at $(\frac{2}{3}\pi, 1)$". He took the first and second derivative of the volume function for the spherical cap of the flask and set up the second derivative of the volume function equal to zero in order to find the value of the height of water as h = 1 at the inflection point:

$$V(h) = -\pi h^3 \frac{1}{3} + \pi h^2 \quad , \quad 0 \leq h \leq \frac{9}{5}$$

$$V'(h) = -\pi h^2 + 2\pi h$$

$$V''(h) = -2\pi h + 2\pi = 0 => \ h = 1$$

(Student 6's next response to question 2d)

(algebraic sign of MA5 for the spherical part of the bottle)

He identified the inflection point of V(h) as $(1, \frac{2}{3}\pi)$ where the second derivative of the volume function changes sign (algebraic sign of MA5):

$$\left\{ V''(h) \ => \ \underset{0}{\vdots} \ + \ \underset{1}{\vdots} \ - \ \underset{\frac{9}{5}}{\vdots} \ => \ (1, \frac{2}{3}\pi) \right.$$ *is the inflection point of V(h)*

(Student 6's following answer to question 2d)

By using the fact that h(V) is the inverse of V(h), he concluded that:

with h(V) being the inverse of V(h), the inflection point is then interval, -i.e. $(\frac{2}{3}\pi, 1)$ *and hence is the inflection point of h(V) (Student 6's final response to question 2d).*

In his response to question 2b, this student already recognized that the volume function V(h) is bijective and its inverse is h(V) whose graph is reflecting the graph of V(h) in the line V = h. In his response to question 2d, he again used the inverse relation between the two functions V(h) and h(V) and determined the inflection point of h(V) as the point, $(\frac{2}{3}\pi, 1)$ where its second derivative changes sign. Thus, this is renewed evidence of the student having strong analytical and conceptual thinking of functions (Vinner, 1997).

Based on his responses to the questions asked in Problem 2, we have observed that the student graphically, verbally and algebraically coordinated the two variables (MA1), the direction of change of the output while considering the input (MA2) and the amount of change of the output with respect to changes in the input (MA3). These results are consistent with the results that we found when analyzed the student's responses to Problem 1 (recall: he graphically and verbally coordinated the amount of change of the output while thinking changes in the input in the first problem). In Problem 2, he also exhibited behaviors that support MA4 and even MA5 for both graphical and algebraic representations of the function. Thus, we can claim that his covariational reasoning has reached Level 5 in Problem 2 (while it was reached Level 3 in Problem 1).

5.2.2.3 Example of responses representing Level 3 of covariational reasoning

There were three students whose solutions were assessed as representing Level 3 of covariational reasoning, students 11, 13 to 21. I first present an analysis of student 11's solution.

5.2.2.3.1 Student 11: a Level 3 solution; leaving the integrals unevaluated, dividing the graph into four intervals

In question 2a, like Student 6, Student 11 also pictured the flask on the Cartesian coordinate system as seen in Figure 35.

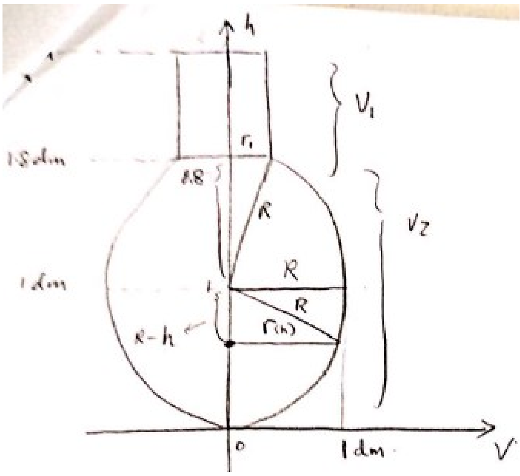


Figure 35. Student 11's image of the flask

She then used the Pythagorean theorem in order to find the value of the radius of the cylindrical section of the flask which is r_1 and the radius of the horizontal disc representing the area under the graph of the function, that is r(h), when the water reaches height h as:

$$r = \sqrt{R^2 - 0.8^2} = \sqrt{1^2 - 0.64} = \sqrt{0.36} = 0.6\ dm$$

$$r(h) = \sqrt{R^2 - (R-h)^2} = \sqrt{1^2 - (1-h)^2} = \sqrt{2h - h^2}$$

(beginning of Student 11's response to question 2a)

By considering the area A of the disc of radius *r(h)* as $A = \pi r(h)^2 = \pi\ (2h - h^2)$, she integrated this area function from 0 to h (but without evaluating the integral) in order to find the formula for the volume of water in the spherical part of the flask when the water reaches height h, and $0 \leq h \leq 1.8\ dm$, as follows:

When water level lower than 1.8 dm, $0 \leq h \leq 1.8\ dm$

$$V_2 = \int_0^h \pi\, r^2 = \int_0^h \pi\ (2h - h^2)\ dh$$

(Student 11's next response to question 2a)

(algebraic signs of MA1, MA2 and MA3)

She used the same procedure for the cylindrical section of the flask. She plugged the value of the radius of the neck into the area formula of the disc with radius r_1, such as $A = \pi\, {r_1}^2 = \pi\ (0.6)^2$, and then integrated this area function from 1.8 to h (but without evaluating the integral again) in order to find the formula for the volume of water in the cylindrical part of the flask when the water reaches height h, and $1.8dm < h \leq 2.8\ dm$, as follows:

When water level higher than 1.8 dm, $1.8dm < h \leq 2.8\ dm$

$$V_1 = \int_{1.8}^{h} \pi\, {r_1}^2 = \int_{1.8}^{h} \pi\, (0.6)^2\, dh = \int_{1.8}^{h} 0.36\pi\, dh$$

(Student 11's following answer to question 2a)

(algebraic signs of MA1, MA2 and MA3)

She then calculated the value of the volume of water in the spherical cap when the water reaches height $h = 1.8$ (at this time, she evaluated the integral), and added this value as $V_2 = 1.296\pi$ to the formula for the volume of water in the cylindrical part of the flask when the water reaches height $h - 1.8$ in order to find the formula for the total volume of water in the flask (again without evaluating the integration in the formula for the volume of water in the cylinder) as:

$$V_2 = \int_{0}^{1.8} \pi\, (2h - h^2)\, dh = \pi(1.8^2 - \frac{1.8^3}{3}) - 0 = \pi(3.24 - 1.944) = 1.296\pi$$

$$V = V_1 + V_2 = \int_{1.8}^{h} 0.36\pi\, dh + 1.296\pi$$

(Student 11's next answer to question 2a)

(algebraic signs of MA1, MA2 and MA3)

As conclusion, she wrote the following formula for the volume V of water as a function of the height h of the water (student 11's final response to question 2a) (evidence of having good or moderate analytical thinking of the dynamic situation):

$$V = \begin{cases} 1.296\pi + \int_{1.8}^{h} 0.36\pi\, dh \ , & 1.8 \leq h \leq 2.8 \\ \int_{0}^{h} \pi\, (2h - h^2)\, dh \ , & 0 \leq h \leq 1.8 \end{cases}$$

Like Student 6, she also set up height $h = 1.8$ for both intervals of the domain. Most importantly, the student did not evaluate the integrals in the formulas for the volume of water when the water reaches height h with $0 \leq h \leq 1.8$, and height h with $1.8 \leq h \leq 2.8$ even though she calculated the integral in the formula for the volume of water in the spherical part of the flask when water reaches height $h = 1.8$. In addition to that, she did not write the domain and range of the volume function.

In question 2b, the student constructed a similar graph to the one she drew when answering Problem 1, the graph being divided into four intervals. The graph represents the height of water as a function of the volume of water with the values of the dependent variable (as 1dm, 1.8dm and 2.8dm) but without all relative magnitudes of the independent variable (she represented the points of the input variable with letters as a, b, c and d on her graph) as seen in Figure 36:

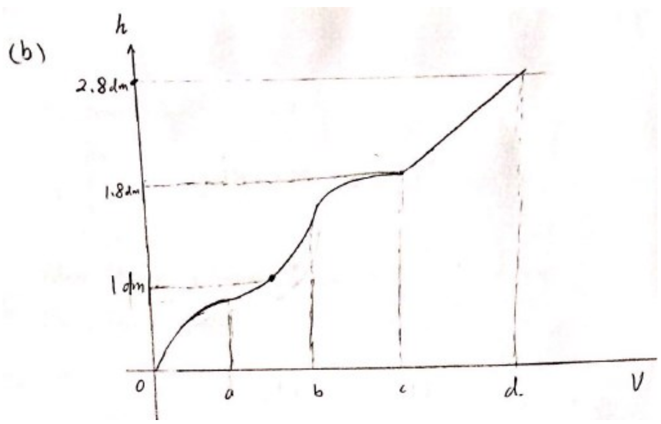


Figure 36. Image of Student 11's graph

(graphical signs of MA1, MA2 and MA3).

Without justification about the construction of the graph, the student calculated the value of the volume of water when the water reaches height $h = 0 \; and \; h = 2.8$ in order to find the domain and range of the height function (sign of having good analytical thinking of the function):

> *b) when* $h = 0$ *volume = 0*
>
> *when* $h = 2.8$*, volume* $= 1.296\pi + \int_{1.8}^{2.8} 0.36\pi \, dh$
>
> $V = 1.296\pi + 0.36\pi(2.8 - 1.8)$
>
> $V = 1.656\pi \; dm^3$
>
> *So, the domain of the function is* $[0, 1.656\pi]$
>
> *the range of the function is* $[0, 2.8 \; dm]$
>
> *(Student 11's final response to question 2b)*

(algebraic signs of MA3)

As observed in the student's response above, she analytically conceptualized the amount of change of the height of water while considering changes in the volume of water and found the correct domain and range of the height function (even though she did not provide the domain and range of the volume function in question 2a), considered as evidence of MA3 for algebraic representation of the function. On the other hand, she did not explicitly mention about the inverse relationship between the height function and the volume function. Since the student was unable to transform her algebraic expression of the dynamic situation to an acceptable graphical construction, she may not have overcome yet the cognitive obstacle labeled as CO-ATG: Cognitive obstacle of being unable to transform an algebraic representation of function to its graphical one.

When constructing the graph of h(V), the student labeled the axes with the correct names of variables (graphical sign of MA1) and constructed a smooth curve rising from the left to the right as being in the correct direction (graphical sign of MA2) although her smooth curve is incorrect. Besides these, she plotted three essential points representing the relative magnitudes of the output variable as h = 1 dm, h = 1.8 dm and h = 2.8 dm on the vertical axis (graphical sign of MA3), but while picturing four symbolic magnitudes of the input variable represented by the letters a, b, c and d. It is not clear what the values of the letters a and b are, however, we can notice that $c = 1.296\pi$ and $d = 1.656\pi$ from her algebraic expressions.

In question 2c, the student found the value of the volume of water in the spherical cap when the water reaches height $h = 1.8$ and concluded that "*so, when if the volume of water is less than 4.06944 dm^3 the height is lower than 1.8dm*" (verbal sign of MA1):

c) when height is 1.8 dm

$$V = \int_0^{1.8} \pi\,(2h - h^2)\,dh = 1.296\pi \approx 4.06944\ dm^3$$

(beginning of Student 11's answer to question 2c)

When responding to part i) of question 2c, she used the formula for the volume of water in the spherical part of the flask and her procedural knowledge in order to find the height of water, as $h \approx 0.98\ dm$, when the volume of water equals to 2 litres:

i) 2 litres

$$V = \int_0^{h} \pi\,(2h - h^2)\,dh = 2$$

$$h^2 - \frac{h^3}{3} = \frac{2}{3.14}$$

$$3.14h^3 - 9.42h^2 + 6 = 0 => h \approx 0.98\ dm$$

(Student 11's solution to part i) of question 2c)

(algebraic signs of MA3)

By using the same procedure, she found the height of water, as $h \approx 1.78\ dm$, when the volume of water reaches to 4 litres, in part ii) of question 2c:

ii) 4 litres

$$V = \int_0^h \pi\,(2h - h^2)\,dh = 4$$

$$h^2 - \frac{h^3}{3} = \frac{4}{3.14}$$

$$3.14h^3 - 9.42h^2 + 12 = 0 => h \approx 1.78\ dm$$

(Student 11's solution to part ii) of question 2c)

(algebraic signs of MA3)

In part iii) of question 2c, she noticed that she needed to use the formula for the total volume of water in the flask that she found in question 2a, that is the value of the volume of water ($V_2 = 1.296\pi = 4.06944\ dm^3$) in the spherical cap when the water reaches height $h = 1.8$ added to the formula for the volume of water in the cylindrical part of the flask when the water reaches height $h - 1.8$. Consequently, the student found the height of water as $h \approx 2.623\ dm$ when the volume of water equals to 5 litres by performing the following computations:

iii) 5 litres

$$V = \int_{1.8}^h 0.36\pi\,dh + 1.296\pi = \int_{1.8}^h 0.36\pi\,dh + 4.06944 = 5$$

$$\pi\,(0.36h - 0.36(1.8)) + 4.06944 = 5$$

$$0.36h - 0.648 = (5 - 4.06944)/3.14$$

$$=> 0.36h \approx 0.296 + 0.648 => \ h \approx 2.623\ dm$$

(Student 11's solution to part iii) of question 2c)

(algebraic signs of MA3)

When analyzing the student's responses to parts i), ii) and iii) of question 2c, we emphasize that the student evaluated at this time the integrals in the formulas for the total volume of water, as opposed to finalizing these integrals in her response to question 2a. Therefore, by evaluating the integrals in the formulas, not only she overcame what could have been classified as a cognitive obstacle (Herscovics, 1989) but she also provided satisfactory evidence of algebraic representation of the function for MA1, MA2 and MA3.

On the other hand, as we have observed of the student picturing the function as a generalized process in her responses to questions 2a and 2b (even though her smooth curve is incorrect), we make the same observation in her responses to question 2c, which suggests of her having a good analytical thinking of the function (Vinner, 1997).

In her response to question 2d, the student first answered '*Yes. The graph of the function h(V) have a point of inflection*". After that, she wrote the formula for the volume of water in the spherical cap without evaluating the integral in the formula, took the first derivative and the second derivative of the volume function, and then set up the second derivative of the function equal to zero to find the value of the height of water h = 1 and justify the inflection point as follows:

$$V = \int_0^h \pi\, r^2 dh = \int_0^h \pi\, (2h - h^2)\, dh$$

$$V' = \pi\, (2h - h^2) = 2h\pi\, - \pi h^2$$

$$V'' = 2\pi\, - 2\pi h = 2\pi\, (1 - h) = 0 => h = 1$$

(Student 11's answer to question 2d)

She plugged the value of the height of water, h = 1, into the formula for the volume of water in spherical part of the flask in order to find the value of the volume of water at the inflection point:

$$V(1) = \int_0^1 \pi\, (2h - h^2)\, dh = \pi(1 - \frac{1}{3}) - 0 = \frac{2}{3}\,\pi$$

(Student 11's following answer to question 2d)

The student then concluded that "*so the coordinates of the inflection point is* $(\frac{2}{3}\,\pi, 1)$". Her response to question 2d suggests that she has good understanding of the fundamental theorem of calculus. However, neither she mentioned the inverse relationship between the volume function V(h) and the height function h(V), nor she described the inflection point as a point where the second derivative of the volume function changes sign. Hence, there is no sufficient algebraic evidence for MA5 although we see some signs which support mental action 5 such as representing the instantaneous rate of change of the volume function for the spherical part of the flask as the first derivative of the function (e.g. $V'(h) = 2(h + h^2)\pi$).

In addition to graphical evidences of MA1, MA2 and MA3 being obtained from her responses to question 2b, we also found satisfactory algebraic evidences of MA1, MA2 and MA3 in her solutions to questions 2a, 2b, 2c and 2d. These results are consistent with the results that we found when analyzing the student's responses to Problem 1 involving the verbal and graphical representations of the height function Recall that the student's covariational reasoning was assessed to have reached Level 3 in her solution of Problem 1 when she, verbally and graphically, coordinated the amount of change of the output while picturing changes in the input. Therefore, considering all evidence, we can say that Student 11 reasons covariationally at Level 3.

5.2.2.3.2 Student 13's responses to Problem 2: A Level 3 solution, incorrect graph of V(h)

In problem 2a, Student 13 first claimed that "*there will have 3 functions for this height*", found the radius of the cylindrical section of the flask as 0.6 dm by using Pythagorean Theorem, and visualized the flask as seen in Figure 37.

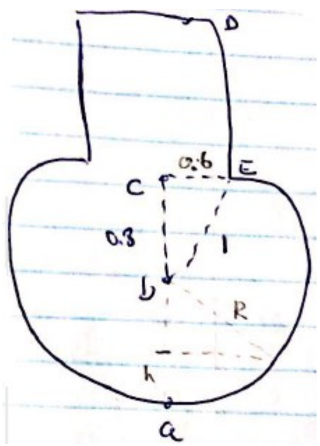


Figure 37. Student 13's image of the flask

For the first expression of the function, the student wrote as follows:

$$h = [0, 1] \quad range = [0, \tfrac{2}{3}\pi]$$

$$V(h) = \int_0^1 \pi\,(R^2 - (h - R)^2)dh + \tfrac{2}{3}\pi = \pi(h^2 - \tfrac{h^3}{3})$$

(Student 13's first answer to question 2a)

(algebraic signs of MA1, MA2 and MA3)

The student considered the domain of the function as $0 \leq h \leq 1$ and the range as $0 \leq V \leq \frac{2\pi}{3}$, and then integrated the area of the disc with radius $\sqrt{R^2 - (R-h)^2}$, that is $A = \pi\,(R^2 - (R-h)^2)$, from 0 to 1 (instead of integrating it from 0 to h) in order to find the formula for the volume of water in the first part of the spherical section of the bottle when the water reaches height h, and $0 \leq h \leq 1$. Despite the fact that the student did not consider the upper bound of the independent variable correctly in the integral formula for the volume of water in the first part of the spherical part of the flask, he found correctly the formula for the volume of water as $V(h) = \pi(h^2 - \frac{h^3}{3})$.

Continuing on the second expression of the function, the student's solution is:

$$h = [1, 1.8] \quad \textit{range} = [\tfrac{2}{3}\pi, 1.296\pi]$$

$$V(h) = \int_1^{1.8} \pi\,(R^2 - (h-R)^2)dh + \frac{2}{3}\pi = \pi(h^2 - \frac{h^3}{3}) - \pi(1 - \frac{1}{3}) + \frac{2}{3}\pi$$

$$= \pi(h^2 - \frac{h^3}{3})$$ *(Student 13's next answer to question 2a)*

(algebraic signs of MA1, MA2 and MA3)

The student stated the domain of the function as $1 \leq h \leq 1.8$ and the range as $\frac{2\pi}{3} \leq V \leq 1.296\pi$, integrated the area of the disc with radius $\sqrt{R^2 - (h-R)^2}$ from 1 to 1.8 (instead of integrating it from 1 to h) and added the result to the value of the volume of water when the water is at height $h = 1$, that is $\frac{2\pi}{3}$, to be able to obtain the formula for the total volume of water in the spherical section of the bottle when the water reaches height h, and $0 \leq h \leq 1.8$. Although the student found the correct volume formula, he again did not consider the upper bound of the independent variable correctly on the sign of integral of the formula for the volume of water in the second part of the spherical section of the flask (repeating the same mistake as in the first part of the solution).

For the third expression of the function, the student stated:

$$h = [1.8, 2.8] \quad \textit{range} = [1.296\pi, 1.65\pi]$$

$$V(h) = (h - 1.8)\pi\,(0.6)^2 + 1.296\pi$$

(Student 13's last response to question 2a)

(algebraic evidences of MA1, MA2 and MA3)

The student stated the domain of the volume function for the cylindrical part of the bottle as $1.8 \leq h \leq 2.8$ and the range as $1.296\pi \leq V \leq 1.65\pi$ and found the value of the volume of water in the spherical cap as 1.296π when the water reaches height $h = 1.8$. He then added this value to the formula for the volume of water in the cylindrical part of the flask when the water reaches height $h - 1.8$, that is $V(h) = (h - 1.8)\,\pi\,(0.6)^2$, to be able to obtain the formula for the total volume of water in the flask as shown above.

The student algebraically expressed the name of the input and output variables by letters h as the height of water and V as the volume of water (algebraic sign of MA1) and coordinated the direction of change of the output while considering changes in the input (algebraic sign of MA2). Although the student did not provide explicitly the domain and the range of the volume function, instead, he wrote the domain and range of the function for each section of the flask (for the first part of the spherical section, the second part of the spherical section and cylindrical section), he obtained the correct formula for the total volume of water as a function of the height of water (algebraic sign of MA3), supportive evidence for concluding that he has good or moderate analytical thinking of the situation. It seems that the student reasons the given functional situation covariationally at Level 3.

In question 2b, the student wrote the correct domain and an incorrect range of the volume function (instead of considering the height function as posed in the question) as "*Domain* $= [0, 2.8]$ *Range* $= [0, 2.32\pi]$". It is not clear how the student found the value of volume of water as 2.32π when the water reaches height $h = 2.8$ even though he wrote the correct value as 1.65π when responding question 2a. He then constructed an incorrect graph of the volume of water as a function of the height of water without any explanation (he drew the same graph when answering Problem 1) as shown in Figure 38 below.

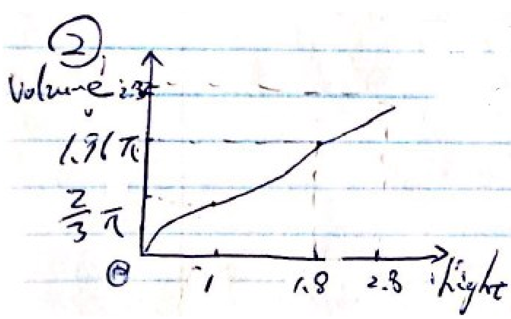


Figure 38. Image of Student 13's graph

(graphical signs of MA1, MA2 and MA3)

When constructing his graph of V(h) in question 2b, the student labeled the axes with the correct names of variables (graphical sign of MA1) and constructed a smooth curve rising from the left to the right as being in the correct direction (graphical sign of MA2) although his graph is not correct. In addition, he also graphically coordinated the amount of change of the output with respect to changes in the input by representing some relative magnitudes of the dependent and independent variables (graphical sign of MA3) although he incorrectly wrote the value of volume of water as 2.32π, instead of 1.65π, when the water reaches height $h = 2.8$. In addition to algebraic signs of MA1, MA2 and MA3 that we observed in question 2a, we also observed graphical signs of MA1, MA2 and MA3 in question 2b. It seems that his covariational reasoning has remained at Level 3 in Problem 2. These results are in line with the results that we obtained when analyzing his responses to Problem 1 (Recall: he graphically coordinated the amount of change of the output while considering changes in the input in Problem 1).

However, since he was unable to discriminate between the input and output variables, the student did not exhibit behavior supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables"* and has not overcome yet the corresponding epistemological obstacle labeled *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992) when responding question 2b (we had the same conclusion in the analysis of his responses to Problem 1), although he identified the dependent and independent variables correctly when answering question 2a. Since the student was not able to transform his algebraic formula representing the functional relationship of the dynamic situation to a suitable geometrical construction, he may not have overcome yet the cognitive obstacle labeled as CO-ATG: Cognitive obstacle of being unable to transform an algebraic representation of function to its graphical one.

For the parts i), ii) and iii) of question 2c, the student gave the following answers without showing any computation:

> *i) since* $\frac{2\pi}{3} = 2.1\ litres$*, height will be 1 dm, ii) since* $1.296\pi = 4.07\ litres$*, height will be 1.8 dm, iii) since* $1.65\pi = 5.18\ litres$*, height will be 2.8 dm (Student 13's response to question 2c)*

In his response to question 2c, the student did not exactly calculate the three values of the height of water for the three different values of the volume of water that were given in the question. He just projected or guessed some values of the height of water in his mind. These incorrect numerical values were not considered as evidence of covariational reasoning.

In question 2d, the student responded as follows:

> *d)* $V(h) = \pi(h^2 - \frac{h^3}{3}) \quad [0, 1.8]$
>
> $V'(h) = 2\pi h - \pi h^2 \quad V''(h) = 2\pi - 2\pi h = 2\pi(1 - h)$*, when* $V''(h) = 0$*,* $h = 1$
>
> *(Student 13's solution to question 2d)*

The student first wrote the formula for the volume of water (including its domain as [0, 1.8] without the mention of the name of the independent variable) in the spherical cap of the flask, took the first and second derivative of this volume function, set up the second derivative of the volume function equal to zero in order to obtain the value of the height of water as h = 1 at the inflection point, and concluded that *"So, h = 1 will be inflection point*" without considering the coordinates of the inflection point of h(V) which was asked in the question.

The student showed that the second derivative of the volume function changes sign at height h = 1 as:

d) --------0.5--------1---------1.5---------

$V''(h) > 0$ $V''(h) < 0$

Increasing *decreasing*

(Student 13's last response to question 2d)

When answering question 2d, he failed to write the coordinates of the inflection point of the height function. Like Student 12, Student 13 mistakes the "coordinates" with the "point" in the domain (possibly, these two students may have a misconception about coordinates of a point). In addition, the student did not mention the inverse relationship between the volume function V(h) and the height function h(V). In his solution, he exhibited some behaviors supporting Mental Action 5 such as representing the instantaneous rate of change of the volume function for the spherical section of the bottle as the first derivative of the function (e.g. $V'(h) = -\pi h^2 + 2\pi h$) and stating that the inflection point is the point where the second derivative of the function changes sign. However, these evidences are insufficient to assess that his covariational reasoning has reached level 5 (and even MA4) since he did not coordinate the instantaneous rate of change of the output (with respect to continuous changes in the input) for the entire domain.

5.2.2.4 Examples of students' responses representing Level 2 of covariational reasoning

These are eight students (numbered 2, 4, 5, 7, 9, 12, 14 and 20) whose solutions were assessed to be representative for Level 2 of covariational reasoning.

5.2.2.4.1 Student 12's responses to Problem 2: A Level 2 solution, using the letter 'x' in his formula

In question 2, student 12 visualized the bottle on the Cartesian coordinate system by rotating 90 degree to the right as seen in the (Figure 39).

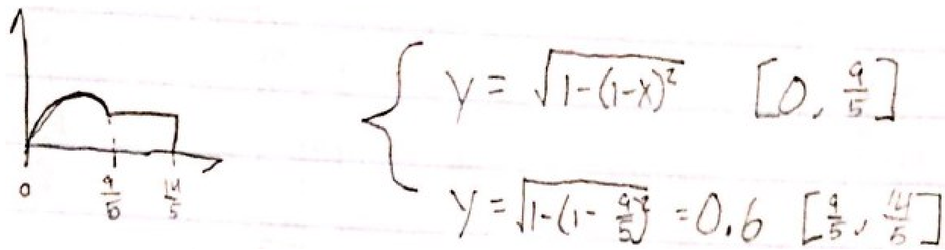


Figure 39. Student 12's image of the bottle

In his response above, the student found the radius of the cylindrical section of the flask which is $y_2 = 0.6$ (dm that is not written) when the water reaches height x, and $\frac{9}{5} \leq x \leq \frac{14}{5}$, and the radius of the horizontal disc representing the area under the graph of the function that is $y_1 = \sqrt{1-(1-x)^2}$ when the water reaches height x, and $0 \leq x \leq \frac{9}{5}$.

In question 2a, the student pictured the area of the disc with radius *0.6 dm* as $A_2 = \pi\, {y_2}^2 = \pi\,(0.6)^2$and the area of the disc with radius y_1 as $A_1 = \pi\, {y_1}^2 = \pi\,(1-(1-h)^2)$. He then integrated these two area formulas from 0 to h to be able to obtain the formula for the volume of water in the spherical section of the bottle, and the formula for the volume of water in the cylindrical section of the bottle as reproduced below:

> *a) V(h) must be piecewise function:*
>
> $V(h) = \int_0^h \pi\,(1-(1-h)^2)\, dh \quad [0,\frac{9}{5}]$
>
> $V(h) = \int_0^h \pi\, 0.6^2\, dh + C \quad [\frac{9}{5},\frac{14}{5}]$
>
> *(Student 12's first answer to question 2a)*

In his response above, the student did not mention the name of the independent variable when specifying the intervals; he wrote $[0, \frac{9}{5}]$ rather than $h \in [0, \frac{9}{5}]$, and $[\frac{9}{5}, \frac{14}{5}]$ instead of $h \in [\frac{9}{5}, \frac{14}{5}]$. Although the student considered the interval of the independent variable to be $[\frac{9}{5}, \frac{14}{5}]$, he was unable to consider the lower bound of the independent variable correctly in the integral of the formula for the volume of water in the cylindrical part of the flask; he wrote $\pi \int_0^h 0.6^2\, dh + C$ instead of $\int_{1.8}^h 0.36\pi \cdot dh$. In addition, the student used the name of the variable of integration as the same as the upper bound of the input (like student 11) using a constant C for a definite integral.

Without showing the computation, the student provided the following formula for the volume of water as a function of the height of the water:

> *Therefore:* $V(h) = \pi(h^2 - \frac{h^3}{3})$ $[0, \frac{14}{5}]$ $V(h) = \pi(0.36x + \frac{81}{125})$ $[\frac{9}{5}, \frac{14}{5}]$
>
> *(Student 12's next response to question 2a)*

We note in his expression above that the student used the letter "x" in the rule of the function (in the right side of the equation of the second formula) while he used the letter "h" for the name of the independent variable in the left side of the equation. We also observed this type of notational inconsistencies when we analyzed the student's responses to Problem 1 (e.g. he also used the letter "x" for the independent variable in his formula when answering the first problem).

Without further calculation once again, the student wrote the domain and range of the volume function and called the name of the independent variable as "*height of flask*" and the name of the dependent variable as "*volume for flask*":

> *Domain:* $[0, \frac{14}{5}]$ *(limited by height of flask), Range:* $[0, \frac{207\pi}{125}]$ *(limited by max volume for flask) (Student 12's final answer to question 2a)*

The student expressed algebraically the name of the input and output variables by the letters h (and also "x") for the height of water representing the input and V as the volume of water being the output (algebraic sign of MA1). He also algebraically coordinated the direction of change of the dependent variable while considering changes in the independent variable (algebraic sign of MA2), but he wrote the incorrect lower bound of the independent variable on the integration of the formula for the volume of water in the cylindrical part of the flask and used the letter "x" instead of "h" in the rule of the volume function which resulted in providing the wrong formula in terms of "x" even though he provided the correct domain and range of the volume function (no sufficient algebraic signs of MA3). As such, we see have supporting evidences of algebraic representation of the function for mental actions 1 and 2, thus, we can say that this student's covariational reasoning has reached Level 2. Recall that the student's covariational reasoning was also assessed at Level 2 on the results of Problem 1.

In question 2b, the student did not provide the domain and range of the height function. He constructed an incorrect graph of the volume of water (Figure 40) as a function of the height of water without justification (while he drew a correct graph of the height of water as a function of the volume of water when answering Problem 1).

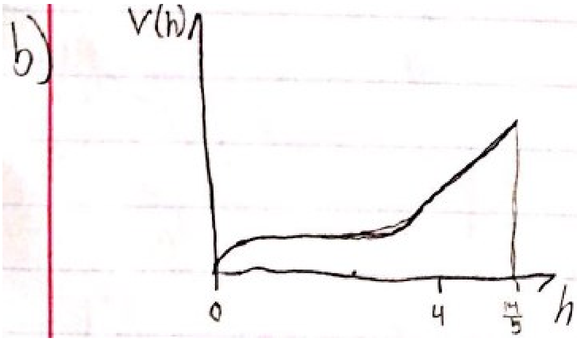


Figure 40. Image of Student 12's graph

(graphical signs of MA1 and MA2).

Constructing his graph of V(h) in question 2b, the student labeled the axes with the correct names of variables (graphical sign of MA1) and constructed a smooth curve rising from the left to the right as being in the correct direction (graphical sign of MA2) although his graph is not correct. He was unable to coordinate the amount of change of the output while considering changes in the input, graphically and verbally. Thus, his covariational reasoning remained at Level 2.

The student did not exhibit behavior supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables"* and has not overcome yet the corresponding epistemological obstacle labeled *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992). He failed in expressing the values of the dependent variable on the graph and wrote the value of the independent variable as 4 (representing the point being near to the starting point of the neck on his graph) positioned earlier than the value $\frac{14}{5}$ of the independent variable representing upper bound of the input. It is not clear why the student represented the number 4 (and how he found this number) as less than the number $\frac{14}{5}$ on his graph. The student may not have overcome yet the common cognitive obstacle of being unable to transform a algebraic representation of a function to its graphical one labeled as CO – ATG since he was not able to transform his algebraic expression (being in terms of 'x') to a graphical construction.

For the parts i), ii) and iii) of question 2c, the student wrote the three corresponding values of the height of water for the three different values of the volume of water without the mention of the names of the variables and without any calculation which shows how he obtained these values:

c) i) 2L = 0.97dm, ii) 4L = 1.744dm, iii) 5L = 2.621dm

(Student 12's response to question 2c)

To question 2d, the student responded as follows:

d) Yes, inflection point = $\frac{2}{3}\pi$*. This is mathematically the point where h'' (V)= 0, and where concavity changes. From a real-world perspective, this is how much water is in the flask when it is most thick. (Student 12's response to question 2d)*

When the student wrote as "***inflection point =*** $\frac{2}{3}\pi$***"***, he did not specify the name of the variable (height or volume) which represents the value $\frac{2}{3}\pi$, and did not provide any computation about how he found this value. Although the coordinates of the inflection point of h(V) were asked in the question, the student wrote the inflection point of V(h) and ignored the 'coordinates' of the inflection point of the height function completely (having misconception about coordinates of a point like student 13). This is renewed evidence of that the student was unable to discriminate between the dependent and independent variables, so he has not overcome yet the epistemological obstacle ***"EO(f)-5: Regarding the order of variables as irrelevant"*** (Sierpinska, 1992).

As the student's solution does not show any calculation of first and second derivative of the function, or any other procedure, his mention of the inflection point as "*the point where h(V)'' = 0, and where concavity changes"* in his response to question 2d suggests memorized associations and may be an example of rote learning. The student who seems to have weak analytical and conceptual thinking of the functional situation did not talk either about the inverse relationship between the height function and the volume function. This concludes the lack of higher MAs values in our assessment of the student's covariational reasoning.

5.2.2.4.2 Student 20's responses to Problem 2: A Level 2 solution, pseudo thoughts, linear graph of V(h)

In her response to question 2a, she first wrote as "***total height = 10 cm + 8 cm + 10 cm = 28 cm***", then added the volume formulas of sphere and of cylinder, being as $V = \frac{4}{3}\pi r^3 + \pi r^2 h$ and found the volume of water as $V = 5598.3$ by assigning the value of radius r (in both formulas of sphere and cylinder) as 9 cm:

$$a)\ V = \frac{4}{3}\pi r^3 + \ \pi r^2 h = \frac{4}{3}\pi(9)^3 + \ \pi(9)^2 10 = 5598.3\ u^3$$

(Student 20's first solution to question 2a)

She then claimed that "*Domain; (0, 28), Range; (0, 5598.3)*". It seems that the student added 10 cm to the 8 cm and divided it by two in order to find the value of radius as 9 cm. But, she used this value of radius in both formulas of sphere and cylinder. Her uncontrolled memorized associations are suggestive of her being in both pseudo-conceptual and pseudo-analytical thoughts like Student 16.

While she provided her memorized volume formulas of sphere and cylinder in terms of 'h' and 'V' in her first response above, she provided an another formula being in terms of 'x' and 'y' in her following response to question 2a (Figure 41):

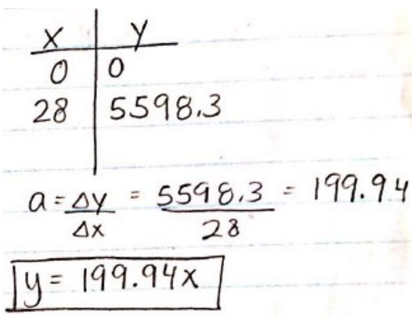


Figure 41. Image of Student 20's second solution to question 2a

I note that she provided a similar response to question 1a in Problem 1 as well, being suggestive of the student having weak analytical and conceptual thinking of functions.

From her answers, we can say that she imaged the independent variable as the height of water being in terms of the letter 'x' since it equals to 28 and the dependent variable as the volume of water being in terms of the letter 'y' since it equals to 5598.3. We can also say that she pictured the volume function as linear, so divided the value of volume of water to the value of height of water and found the second formula for the volume function as "$y = 199.94x$". Her uncontrolled memorized associations are renewed evidence of her being in both pseudo-conceptual and pseudo-analytical thoughts when responding to both Problems 1 and 2.

From her responses to question 2a, we see some algebraic evidence of her coordinating between the two variables, the height of water as h or x and the volume of water as V or y (algebraic sign of MA1). We also see some evidence of her coordinating the direction of change of the output while picturing changes in the input algebraically (algebraic sign of MA2).

In question 2b, she used her last obtained formula being as "$y = 199.94x$" in order to tabulate the values of the independent and dependent variables and to construct a graph of the volume function instead of the height function (Figure 42):

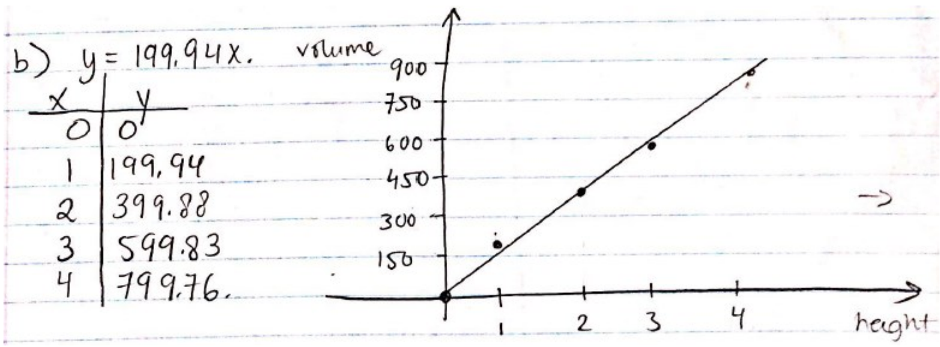


Figure 42. Image of Student 20's graph and table

She then again claimed that "*Domain; (0, 28), Range; (0, 5598.3)*" as she did in her response to question 2a. In her answers above, we see more evidence of the student being in both pseudo-analytical and pseudo-conceptual modes of thinking. In her graph of V(h), she labeled the axes with the correct names of variables (graphical sign of MA1) and constructed a straight line rising from the left to the right (graphical sign of MA2). We see also evidence of her symbolically/numerically coordinating the direction of change of the output while thinking changes in the input on the table she constructed (algebraic sign of MA2). Like Student 16, this student also did not exhibit behavior supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables"* and has not overcome yet the corresponding epistemological obstacle labeled *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992) when responding question 2b.

For parts i), ii) and iii) of question 2c, she found the following answers by using her formula of $y = 199.94x$ in which the letter 'y' being the volume of water and the letter 'x' being the height of water:

> *c) 2 litres = 2000 cm^3 ; x = 10 cm, 4L = 4000 cm^3 ; x = 20 cm, 5L = 5000 cm^3 ; x = 25 cm*
>
> *(Student 20's response to question 2c)*

From her responses to question 2c, we see more algebraic evidence of her coordinating the direction of change of the output while thinking changes in the input (algebraic sign of MA2). Thus, based on her responses to questions 2a, 2b and 2c, we can claim that her covariational reasoning attained at Level 2.

In her response to question 2d, she wrote that "*no, there are no point of inflection*".

5.2.2.5 Examples of a response representing Level 1 of covariational reasoning

There is only one student (numbered 16) whose solution was assessed to be representative for Level 1 of covariational reasoning.

5.2.2.5.1 Student 16's responses to Problem 2: A Level 1 solution, pseudo thoughts

For question 2a, Student 16 who also seems to have weak analytical and conceptual thinking of functions provided the following responses without supporting calculation or justification:

a) $V = \begin{cases} \frac{4}{3}\pi h^3 \text{ , } 0 \leq h \leq 18\ cm \\ \pi r^2 h \text{ , } 18 < h \leq 28\ cm \end{cases}$ *, Domain: [1, 28] for both, Range: [4.2, 9.952]*

(Student 16's response to question 2a)

The first thing we notice in his response is that the student used uncontrolled memorized volume formulas of the sphere and cylinder to represent the volume formulas in the spherical cap and the cylindrical part of the flask (strong evidence of pseudo-analytical behavior). We also notice that he used the height h instead of radius r in his volume formula of sphere. Beside this, he did not plug the value of the radius of the spherical cap of the flask (given as 1 dm = 10 cm) into his volume formula. While he used the radius r in his volume formula of cylinder, he did not attempt to find the value of this radius r nor did he try to find the value of the volume of water when the height reaches 18 cm and add this value to his second volume formula. Like Students 1 and 5, he totally ignored the value of the volume of water, in his second volume formula, when the height of water reaches 18 cm. Furthermore, it is not clear how he found the incorrect domain and the range of the volume function in his answer. It is also not clear if the domain of the function is height or volume in his response since he did not name the domain of the function. Based on his response to question 2a, we see some algebraic evidence of him coordinating between two variables, the height h and the volume V (algebraic sign of MA1).

For question 2b, he constructed a similar graph to the one he sketched when answering question 1a, and labeled the vertical axis as V being volume and the horizontal axis as h being height (which should be the opposite) without any explanation (only graphical sign of MA1 since the direction of change of the volume with respect to height on his graph is not clear). He wrote the same volume formula that he provided in his response to question 2a, and then claimed that "*the Domain –possible inputs of h ranges from 1cm to 27cm, and Range; output from the domain [4.2, 51592]*". Neither his graph including the identifications of dependent and independent variables nor domain and range are correct. In his response to question 2b, the student did not exhibit behavior supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables"* and has not overcome yet the corresponding epistemological obstacle labeled *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992).

In question 2c, the student used the volume formulas he obtained in question 2a, but incorrectly, so he obtained uncontrolled illogical answers. For the part i) of question 2c, he used his volume formula of cylinder, that is $V = \pi r^2 h$, instead of using his volume formula of sphere (evidence of pseudo-analytical and pseudo- conceptual behaviors). He then claimed that r (being the radius of cylinder) approximately equals to h and found the height of water as $h = 25cm$ when there is 2 litres water in the flask. In the part ii) of question 2c, he used his volume formula of sphere, that is $V = \frac{4}{3}\pi h^3$, and found the height of water as $h = 9.8\ cm$ when there is 4 litres water in the flask. By looking into his two responses to the parts i) and ii) of question 2c, we observe that the student actually claims incorrectly that the height of water is lower when there is more water in the flask. This is a strong evidence of the student being not only in pseudo-analytical mode of thinking but also in pseudo-conceptual mode of thinking. For part iii) of question 2c, he used his volume formula of sphere again instead of using the volume formula of cylinder (renewed evidence of pseudo-analytical and pseudo- conceptual behaviors), and found the height of water as $h = 10.6\ cm$ when there is 5 litres water in the flask. From his response to part iii), we again have the same observation of that the student continues claiming the height of water is lower when there is even more water in the flask (more evidence of pseudo- conceptual behavior).

Based on his responses to question 2a, 2b and 2c, we can say that the student was unable to coordinate the direction of change of the output while thinking changes in the input algebraically graphically and verbally (no algebraic, graphical and verbal signs of MA2). Hence, we can say that his covariational reasoning remained at Level 1 (Recall: his covariational reasoning has also reached Level 1 when answering Problem 1).

In his response to question 2d, he wrote the following answer being similar to his response to question 1c;

> *d) h(V) has no inflection point, at always increases always and there's no change, like that. So no. However, if the entire graph is considered, its inflection point would be at (0,0) (Student 16's response to question 2d)*

From his answers to question 2d, we can claim that the student has a misconception about the point of inflection.

5.2.2.6 Examples of a response representing Level 0 of covariational reasoning

There is one student (numbered 24) whose solution was assessed to be representative for Level 0 of covariational reasoning.

5.2.2.6.1 Student 24's responses to Problem 2: A Level 0 solution

While we see no response to other questions, student 24 provided the following answer to question 2a (Figure 43):

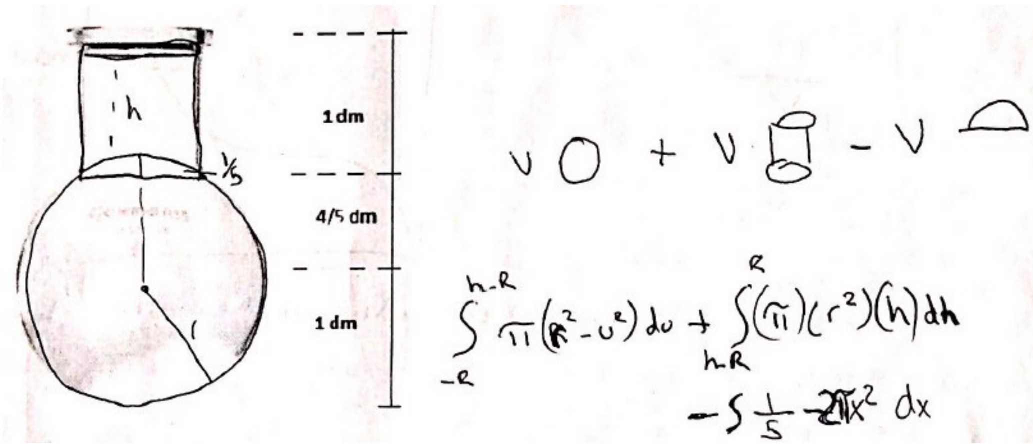


Figure 43. Image of Student 24's response to question 2a

We note that Student 24 did not identify the name of the dependent and independent variables, we thus conclude that his covariational reasoning has not reached Level 1.

5.2.3 Analysis from the point of view of students' common pseudo-thoughts, obstacles and misconceptions

I start with the data tabulated in Table 9 which shows the assessments of students' solutions from the point of view of common pseudo-thoughts, obstacles (epistemological, didactical and cognitive) and misconceptions.

Table 9. Distribution of each student's engaged common pseudo-thoughts, obstacles, misconceptions

Student's code	Student's engaged pseudo-thoughts	Student's engaged epistemological obstacles	Student's engaged didactical obstacles	Student's engaged cognitive obstacles	Student's engaged misconceptions
S1	---	---	---	CO-GTA, CO-VTA	MC-CP
S2	---	EO(f)-3, EO(f)-5	DO-XY	---	---
S4	---	---	---	---	---
S5	---	EO(f)-5	---	---	---
S6	---	---	---	---	---
S7	---	---	---	---	---
S9	---	---	---	---	---
S11	---	---	---	CO-ATG	---
S12	---	EO(f)-5	DO-XY	CO-ATG	MC-CP
S13	---	EO(f)-5	---	CO-ATG	MC-CP
S14	---	---	---	CO-ATG	---
S16	PSA and PSC	EO(f)-5, EO(f)-11	---	---	---

S20	PSA and PSC	EO(f)-5, EO(f)-7, EO(f)-11	DO-XY	---	MC-IP
S21	---	---	---	---	---
S24	---	EO(f)-3	---	---	---

Abbreviation:

PSA: Pseudo-analytical thought,

PSC: Pseudo-conceptual thought,

MC-IP: Misconception about the inflection point

MC-CP: Misconception about ‘coordinates’ of a point

DO-XY: Didactical obstacle involved in representing variables with the letters ‘x’ and ‘y’

CO-ATG: Cognitive obstacle of being unable to transform an algebraic representation of function to its graphical one

CO-GTA: Cognitive obstacle of being unable to transform a graphical representation of function to its algebraic one

CO-VTA: Cognitive obstacle of being unable to transform a verbal representation of function to its algebraic one

“EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes” (Sierpinska, 1992, p.36)

“EO(f)-5: Regarding the order of variables as irrelevant” (Sierpinska, 1992, p.38)

“EO(f)-7: A Pythagorean philosophy of number: everything is number” (Sierpinska, 1992, p.41)

“EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions” (Sierpinska, 1992, p.46)

I now present the data tabulated in Table 10 which lists the assessments of students’ solutions from the point of view of common pseudo-thoughts, obstacles (epistemological, didactical and cognitive) and misconceptions for both Problems 1 and 2.

Table 10. Distribution of each student’s engaged common pseudo-thoughts, obstacles, misconceptions

Student’s code	**Student’s engaged pseudo-thoughts**		**Student’s engaged epistemological obstacles**		**Student’s engaged didactical obstacles**		**Student’s engaged cognitive obstacles**		**Student’s engaged misconceptions**	
	Prob.1	Prob.2	Prob.1	Prob.2	Prob.1	Prob.2	Prob.1	Prob.2	Prob.1	Prob.2
S1	---	---	---	---	---	---	---	CO-GTA and CO-VTA	---	MC-CP
S2	---	---	---	EO(f)-3 EO(f)-5	---	DO-XY	---	---	---	---
S4	---	---	---	---	---	---	---	---	---	---
S5	---	---	---	EO(f)-5	---	---	---	---	---	---
S6	---	---	---	---	---	---	---	---	---	---
S7	---	---	---	---	---	---	---	---	---	---
S9	---	---	---	---	---	---	---	---	---	---

S11	---	---	---	---	---	---	**CO-VTG**	**CO-ATG**	---	---
S12	**PSA**	**---**	**EO(f)-11**	**EO(f)-5**	**DO-XY**	**DO-XY**	**CO-GTV**	**CO-ATG**	**---**	**MC-CP**
S13	**---**	**---**	**EO(f)-5 EO(f)-16**	**EO(f)-5**	**---**	**---**	**---**	**CO-ATG**	**---**	**MC-CP**
S14	---	---	---	---	---	---	**CO-GTV**	**CO-ATG**	---	---
S16	**PSA and PSC**	**PSA and PSC**	**EO(f)-11**	**EO(f)-5 EO(f)-11**	---	---	---	---	---	---
S20	**PSA and PSC**	**PSA and PSC**	**EO(f)-7 EO(f)-11**	**EO(f)-5 EO(f)-7 EO(f)-11**	**DO-XY**	**DO-XY**	**CO-GTV and CO-VTG**	**---**	**---**	**MC-IP**
S21	---	---	EO(f)-3 EO(f)-7 EO(f)-16	---	---	---	---	---	MC-IP	---
S24	---	---	**EO(f)-3**	**EO(f)-3**	---	---	---	---	MC-IP	---

Abbreviation (other abbreviations are already given above under Table 9):

CO-GTV: Cognitive obstacle of being unable to transform a graphical representation of function to its verbal one

CO-VTG: Cognitive obstacle of being unable to transform a verbal representation of function to its graphical one

"EO(f)-16: The changes of a variable are changes in time" (Sierpinska, 1992, p.55)

I have observed that Students 16 and 20 had common pseudo-conceptual and pseudo-analytical thoughts (Vinner, 1997) when answering Problem 2 since they may not have overcome the most common epistemological obstacle "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46). We concluded the existence of the same pseudo-thoughts and epistemological obstacle in these students while analyzing of the students' responses to Problem 1.

Moreover, six students (numbered 2, 5, 12, 13, 16 and 20), making 40% of participants, did not exhibit behaviors supporting the act of understanding labeled "*U(f)-5: Discrimination between the dependent and independent variables"*. Thus, they may not have overcome yet the corresponding common epistemological obstacle labeled *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992), since they were unable to discriminate between the output and input variables (Recall: Student 13 had this epistemological obstacle when answering both Problems 1 and 2). Students 2 and 24 may not have overcome the common epistemological obstacle of 'ignoring what changes' which is identified as "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes*"

(Sierpinska, 1992, p.36) since Student 24 totally ignored what changes (in both Problems 1 and 2) while Student 2 was unable to identify the name of the variables in his graphical representation of the height function. Students 20 have not overcome yet the epistemological obstacle "*EO(f)-7: A Pythagorean philosophy of number: everything is number*" (Sierpinska, 1992, p.41) since she used some imagined numerical values in her solutions (to both Problems 1 and 2). It seems that "*EO(f)-11, EO(f)-5 and EO(f)-3"* are the most common epistemological obstacles among college level Calculus students, observation consistent with the results obtained for Problem 1 (Recall: we have found that "*EO(f)-11, EO(f)-3 and EO(f)-16"* are the most common epistemological obstacles when analyzing the students' responses to Problem 1).

It appears that four students (numbered 11, 12, 13 and 14) which makes 27% of students were unable to make a 'transformation from an algebraic expression of a function to its graphical one', which is the common cognitive obstacle labeled as CO – ATG (Recall: the three students (numbered 11, 12, and 14) had a cognitive obstacle of being unable to transform either a graphical representation of the function to its verbal one or a verbal representation of the function to its graphical one). On the other hand, it seems that Student 1 was not able to make a 'transformation from graphical and verbal representations of a function to its algebraic one', which are the cognitive obstacles labeled as CO – GTA and CO – VTA since she was unable to find a correct formula although her correct graph is well explained by words in her solution to Problem 2. Hence, I can say that these five students, amounting for 33% of respondents, may not have overcome yet cognitive obstacles involved in moving different representations of functions. Since we estimate that students paid little attention to verbal representation of the function asked in Problem 2 and so most of them did not provide sufficient verbalization in their responses to question 2b, it will be realistic not to consider whether students were able to transform from an algebraic or a graphical representation of the function to its verbal one in Problem 2. Students 2, 12 and 20 used the letters 'x' or/and 'y' in their algebraic expressions when responding Problem 2. Recall: Students 12 and 20 also used these letters in their responses to Problem 1. This is potentially due to students practicing too often standard examples while engaged in mathematical activities that are in terms of 'x' and 'y', creating thus inadvertently a common didactical obstacle (labeled as DO – XY) which Students 12 and 20 displayed frequently.

Lastly, I have observed that students 1, 12 and 13 have a misconception about the 'coordinates' of a point, while Student 20 has a misconception about what constitutes a point of inflection of a graph of a function.

5.3 Comparative analysis of the results obtained for Problems 1 and 2

I note that, in this section, I considered the results of fifteen students who participated in both Problems 1 and 2. I try to answer the research questions summarized under the sub-titles below.

5.3.1 How do college level Calculus students reason functional situations that rely on variables varying dynamically? Particularly, at what level do Calculus students generally reason dynamic functional situations covariationally?

I first present a table showing level(s) of each student's covariational reasoning reached in Problems 1 and 2 (Table 11).

Table 11. Distribution of level(s) of covariational reasoning, for each student, reached in Problem 1 and 2

Student's code	**Student's covariational reasoning reached in Problem 1**	**Student's covariational reasoning reached in Problem 2**
S1 – S24	**Level 0 – 5**	**Level 0 – 5**
S1	**5**	**5**
S2	5	2
S4	5	2
S5	5	2
S6	**3**	**5**
S7	4	2
S9	3	2
S11	**3**	**3**
S12	**2**	**2**
S13	**3**	**3**
S14	**2**	**2**
S16	**1**	**1**
S20	**2**	**2**
S21	0	3
S24	**0**	**0**

As seen in Table 11, Students 11 and 13 seemed to have reasoned the given functional situations covariationally at Level 3 when answering both problems. Student 1's behaviors are supportive of that she reasoned the given events covariationally at Level 5 in both Problems 1 and 2, but this outcome is based on only verbal signs of MA5 since she even did not exhibit behaviors which support graphical or algebraic signs for MA3 in both problems. So, one doubts about her covariational reasoning reached level 5. On the other hand Student 6 whose covariational reasoning always attained at Level 3 when answering Problem 1 provided remarkable solution in Problem 2 which suggests he reasoned the dynamic situations covariationally at Level 5. Moreover, Students 12, 14 and 20 exhibited behaviors that are indicative of them reasoning the events covariationally at Level 2. The table also indicates that Student 16's responses are supportive of him reasoning the situations covariationally at Level 1 in both problems while Student 24's answers suggest that he was even unable to reason the events covariationally at Level 1. On the other hand, Students 2, 4 and 5, whose covariational reasoning seemed to have reached Level 5 in Problem 1, provided responses representing only Level 2 of covariational reasoning in Problem 2. It seems that Student 7 whose covariational reasoning appeared to have reached Level 4 in Problem 1 reasoned the given event covariationally only at Level 2 when responding Problem 2 while Student 9 whose covariational reasoning reached Level 3 when answering Problem 1 provided responses, in Problem 2, which represent only Level 2 of covariational reasoning. Student 21 whose covariational reasoning remained at Level 0 in Problem 1 exhibited behaviors being supportive of him reasoning the dynamic situation at Level 3 when responding Problem 2. Based on these results, we conclude that only four students (numbered 1, 6, 11 and 13), making 27% of the fifteen students, are consistently able to reason dynamic situations covariationally at Level 3 or Level 5 while the remaining are not able. Five students (numbered 12, 14, 16, 20 and 24) from the remaining ones, amounting for 33% of the fifteen, are consistently unable to reason dynamic events covariationally at Level 3.

Most of the twenty-four Calculus students (79%) were unable to coordinate the instantaneous rate of change of the function with continuous changes in the input variable for the entire of the function graphically or verbally (no Level 5 reasoning), most of these respondents (75% of all students) were not able to coordinate the average rate of change of the output while picturing uniform increments of the input (no Level 4 reasoning) and half of all participants were unable to coordinate the amount of change of the output while imagining changes in the input variable (no Level 3 reasoning) when responding to Problem 1. The one third of participants was even not able to coordinate between the two variables verbally while almost quarter (21%) of students were unable to make coordination between the two variables neither verbally nor graphically (no Level 1 reasoning).

We see more dramatic results in the analysis of the fifteen students' solutions to Problem 2. Majority of the fifteen Calculus students (87%) were unable to coordinate the instantaneous rate of change (or the average rate of change) of the dependent variable with continuous changes in the independent variable algebraically or graphically or verbally while two third (67%) of participants were not able to coordinate the amount of change of the output while thinking changes in the input when solving Problem 2.

We have observed that most of Calculus students are unable to reason dynamic function situations covariationally at Level 5. This indicates that Calculus students fail in conceptualizing how two varying quantities change together in tandem since they are not able to keep track of the quantities' values varying simultaneously and so to observe some patterns in the changing nature of the instantaneous rate. This outcome is consistent with what Carlson and her colleagues found in their works (Carlson et al., 2002, Carlson & Oehrtman, 2004; Carlson, Madison & West, 2010; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2015; Oehrtman, Carlson, & Thompson, 2008). We have also observed that many Calculus students are unable to develop not only Level 5 or Level 4 but also Level 3 covariational reasoning since they are not able to recognize some regularities in relationship between the values of two continuously varying variables in a dynamic function situation. Thus, we can say that Calculus students generally reason dynamic events covariationally at Level 2 (the direction level) and sometimes at Level 3 (the quantitative coordination level) but not at Level 5 (the instantaneous rate level) or Level 4 (the average rate level).

5.3.1.1 What can we, as educators, conclude about Calculus students' conceptual and analytical thinking of functions in the context of modelling dynamic situations?

I now present a table listing each student's engaged mental actions for graphical, verbal and algebraic representations of the situations being associated with Problems 1 and 2 (Table 12).

Table 12. Distribution of each student's engaged mental actions for graphical, verbal or algebraic representations of the situation for each problem

Student's code	**Student's engaged mental action for graphical representation**	**Student's engaged mental action for verbal representation**	**Student's engaged mental action for algebraic representation**	**Student's engaged mental action for graphical representation**	**Student's engaged mental action for verbal representation**
S1 – S24	**PROBLEM 1**		**PROBLEM 2**		
S1	**MA2**	**MA5**	**MA2**	**MA2**	**MA5**
S2	MA2	MA5	MA2	MA0	MA1
S4	**MA2**	**MA5**	**MA2**	**MA2**	**MA2**
S5	MA5	MA4	MA2	MA2	MA0
S6	**MA3**	**MA3**	**MA5**	**MA5**	**MA3**
S7	**MA2**	**MA4**	**MA2**	**MA2**	**MA2**
S9	MA2	MA3	MA2	MA0	MA0
S11	**MA3**	**MA3**	**MA3**	**MA3**	MA1
S12	**MA2**	**MA1**	**MA2**	**MA2**	**MA1**
S13	**MA3**	**MA2**	**MA3**	**MA3**	**MA1**
S14	**MA2**	**MA2**	**MA2**	**MA2**	MA1
S16	**MA1**	**MA1**	**MA1**	**MA1**	**MA1**
S20	**MA2**	**MA0**	**MA2**	**MA2**	**MA0**
S21	MA0	MA0	MA3	MA2	MA0

S24	**MA0**	**MA0**	**MA0**	**MA0**	**MA0**

When looking Table 12, we see couple of regularities in some students' engaged mental actions for both problems. For instance, one common pattern belongs to Student 6 who exhibited behaviors which always suggest MA3 in both problems and even of MA5 in problem 2. We can say that this student has the strongest conceptual and analytical thinking of functions among the fifteen students. The second participant who seems to have good or moderate conceptual and analytical thinking is Student 11 who always engaged in MA3 except the verbal representation case in problem 2 which was evaluated as MA1 (This is may be due to her putting less attention to verbalization of the situation). It appears that Student 13's behaviors are suggestive of also MA3 for both graphical and algebraic representations but MA2 or MA1 for verbal representations of the given events, which is indicative of him having a weak conceptual thinking while a good analytical thinking of functions. Contrary, Student 1's engaged mental actions often support MA2 for both graphical and algebraic representation of the situations while MA5 for verbal representation in both problems. Probably, this student does not have a strong procedural thinking while she has a necessary conceptual understanding of functions.

Student 4's behaviors are supportive of MA2 for almost all cases except the verbal representation case in problem 1 which was considered as MA5. Student 7 also engaged in mental actions generally being indicative of MA2 except for verbal representation of the situation given in problem 1 being suggestive of MA4. It looks like that both Students 4 and 7 have good or strong conceptual thinking while weak analytical thinking of functions.

Student 12 provided responses indicate MA2 for both graphical and algebraic representations while MA1 for verbal representations of the situations in both problems. Student 14's solutions suggest him having MA2 for almost all cases except the verbal representation case in the second problem which is evaluated as MA1 and Student 20's answers support MA2 for both graphical and algebraic representations while no mental action is observed for verbal representations of the events in both problems. These outcomes are suggestive of them having weak both conceptual and analytical understanding of functions. Student 16 always engaged in MA1 for all cases while Student 24 engaged in no mental action for all cases, being strongly indicative of them lacking both conceptual and analytical understanding of functions.

Student 2 exhibited behaviors which support MA5 for only verbal representation of the function in Problem 1 while MA2 or MA2 for either algebraic or graphical representation of the situations. This student may have a good conceptual thinking while a weak analytical thinking of functions. Moreover, Student 5 may also have a good conceptual understanding since he provided responses which suggest MA4 or MA5 for either verbal or graphical representation of the function in Problem 1 while he may have a weak analytical understanding since his behaviors are suggestive of MA2 for algebraic representation of the situation in Problem 2. Similarly, it seems that Student 9 has a good or moderate conceptual thinking since he engaged in MA3 for verbalization of the situation in Problem 1 while a weak analytical thinking due to him engaging MA2 for algebraic expression of the situation in Problem 2. Contrary, Student 21 appeared to have a good or moderate analytical thinking because of he engaged in MA3 for analytical representation in Problem 2 while he has a weak conceptual thinking to due him engaging MA2 or MA0 for either verbal or graphical representations of the situations when solving the dynamic tasks.

In summary, we have found that 54% of twenty-four participants (numbered 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24) appeared to have a weak conceptual thinking of functions when answering Problem 1 while at least 47% of fifteen students (numbered 12, 13, 14, 16, 20, 21, and 24) appeared to have a poor conceptual understanding of functions when responding both Problems 1 and 2. This result is in line with the previous research results (Even, 1992; Monk, 1992). On the other hand, 79% of fifteen respondents (except Students 6, 11, 13 and 21 who seemed to have a strong or moderate analytical understanding) seemed to have poor analytical thinking of functions when solving Problem 2. I note that many Calculus students having weak conceptual and analytical thinking of functions fail in conceptualizing how two varying variables change together in tandem.

I present a table listing each student's engaged mental actions for verbal representations of the functional situations in Problems 1 and 2 (Table 13).

Table 13. Distribution of each student's engaged mental actions for verbal representations of the situations in both problems

Student's code	**Student's engaged mental Action for verbal representation**	**Student's engaged mental Action for verbal representation**
S1 – S24	**PROBLEM 1**	**PROBLEM 2**
S1	**MA5**	**MA5**
S2	MA5	MA1
S4	MA5	MA2
S5	MA4	MA0
S6	**MA3**	**MA3**
S7	MA4	MA2
S9	MA3	MA0
S11	MA3	MA1

S12	**MA1**	**MA1**
S13	MA2	MA1
S14	MA2	MA1
S16	**MA1**	**MA1**
S20	**MA0**	**MA0**
S21	**MA0**	**MA0**
S24	**MA0**	**MA0**

From Table 13, while we see consistent results for seven students' mental actions (*marked in bold*), we observe some inconsistencies for the remaining eight students' engaged mental actions. All these eight students whose behaviors were supporting higher MAs levels for verbal representation of the function in Problem 1 provided responses which are suggestive them engaging lower MAs levels for verbal representation of the dynamic event in Problem 2. This may be suggestive of them putting less attention on verbal representation of function in the second problem since it was mostly analytical.

It is noticed in the table that Student 1's behaviors support MA5 and Student 6's answers indicate MA3 for verbal representation of the events in both problems. These two students seemed to have essential conceptual understanding of functions in the context of modeling dynamic events. In addition, while Students 12 and 16 exhibited behaviors being supportive of MA1 for verbalization of the functional situation in both problems, Students 20, 21 and 24 did not provide responses being indicative of even MA1. These five students appeared to lack essential conceptual understanding of functions being required for modeling dynamic situations. It seems that Students 13 and 14 also do not have strong or good conceptual understanding of functions due to them being unable to develop more meaningful verbal constructions of the given situations in both problems. In the cases of the six students (2, 4, 5, 7, 9 and 11), we observe a dramatic decrease on students' engaged MA level for verbal representation of function (e.g. Student 2's engaged mental action is MA5 in Problem 1 while MA2 in Porblem2). Probably, these students have a strong or good conceptual understanding of functions but they do not show their conceptual understanding every time, especially when the task is mostly analytical. However, it is clear that seven students (numbered 12, 13, 14, 16, 20, 21 and 24), which makes 47% of fifteen respondents, do not have a strong or good conceptual thinking (or understanding) of functions being necessary for modeling dynamic situations.

I present a table listing each student's engaged mental actions for graphical representations of the functional events in Problems 1 and 2 (Table 14).

Table 14. Distribution of each student's engaged mental actions for graphical representations of the situations in both problems

Student's code	Student's engaged mental Action for graphical representation	Student's engaged mental Action for graphical representation
S1 – S24	**PROBLEM 1**	**PROBLEM 2**
S6	**MA3**	**MA5**
S11	**MA3**	**MA3**
S13	**MA3**	**MA3**
S1	**MA2**	**MA2**
S4	**MA2**	**MA2**
S7	**MA2**	**MA2**
S12	**MA2**	**MA2**
S14	**MA2**	**MA2**
S20	**MA2**	**MA2**
S16	**MA1**	**MA1**
S24	**MA0**	**MA0**
S5	MA5	MA2
S2	MA2	MA0
S9	MA2	MA0
S21	MA0	MA2

When we look at the data for graphical representations of the situations, being differently from verbal representations, we realize that there are more stable results but still with low levels of students' engaged mental actions. Ten students (1, 4, 7, 11, 12, 13, 14, 16, 20 and 24) engaged in same mental action in both problems while the remaining five students (2, 5, 6, 9 and 21) engaged in different mental actions. We observe that some of these five students' mental action levels increased while the other students' mental actions levels decreased when responding Problem 2. For instance, we see an increase from MA3 to MA5 for Student 6's engaged mental action. This is due to him constructing an inverse graph of the height function in Problem 1, which was considered as violation of MA4 and MA5 for graphical representation of the function. On the other hand, he corrected his mistake by applying a control mechanism and constructed a valid graph of the height function in problem 2. We thus conclude that this student has necessary conceptual and analytical understanding of functions. While we make an observation of Student 6 correcting his mistake in problem 2, oppositely, Student 5 who correctly sketched a graph of the height function in problem 1 makes the same mistake in problem 2 by constructing the graph of the volume function, so while he engaged in MA5 when answering Problem 1, he engaged in only MA2 when responding Problem 2. Even though Students 11 and 13 engaged in MA3 for the graphical representation of function, they constructed incorrect graphs of the functions in both Problems 1 and 2.

Moreover, six students (1, 4, 7, 12, 14 and 20) engaged in MA2 and Student 16 engaged in MA1 while Student 24 did not even engage in MA1 when solving both problems. At least half of these students (numbered 12, 14 and 20) do not have necessary conceptual and algebraic understanding of functions. On the other hand, Students 2 and 9 who seemed not to have essential analytical understanding engaged in MA2 when responding Problem 1 while engaged in MA0 when answering Problem 2. Contrary, Student 21 who appeared to have weak conceptual understanding engaged in MA0 when answering Problem 1 while engaged in MA2 when solving Problem 2.

I present a table listing each student's engaged mental actions for algebraic representation of the functional situation given in Problem 2 (Table 15).

Table 15. Distribution of each student's engaged mental actions for algebraic representation of the situation in problem 2

Student's code	**Student's engaged mental action for algebraic representation**
S1 – S24	**PROBLEM 2**
S6	**MA5**
S11	**MA3**
S13	**MA3**
S21	**MA3**
S1	MA2
S2	MA2
S4	MA2
S5	MA2
S7	MA2
S9	MA2
S12	MA2
S14	MA2
S16	MA1
S20	MA2
S24	MA0

For the data of algebraic representation of the function, we observe that only Student 6 engaged in MA5 while Students 11, 13 and 21 engaged in MA3. These four students appeared to have had a strong or good or moderate analytical understanding of function due them being able to develop a suitable analytical construction of the given dynamic situation in Problem 2. While Student 16 engaged in MA1 and Student 24 engaged in MA0, the remaining nine students exhibited behaviors which suggest MA2 for algebraic representation of function. It seems that these eleven students, amounting for 79% of participants, do not have the strong or good analytical thinking (or understanding) of functions being required for modeling dynamic situations.

I present a table listing each student's engaged mental actions for verbal representation of the functional situation in Problem 1 and for algebraic representation of the functional event in Problem 2 (Table 16).

Table 16. Distribution of each student's engaged mental actions for verbal representation of the function in Problem 1 and for algebraic representation of the function in Problem 2

Student's code	**Student's engaged mental Action for verbal representation**	**Student's engaged mental action for algebraic representation**
S1 – S24	**PROBLEM 1**	**PROBLEM 2**
S1	**MA5**	**MA2**
S2	**MA5**	**MA2**
S4	**MA5**	**MA2**
S5	**MA4**	**MA2**
S6	**MA3**	**MA5**
S7	**MA4**	**MA2**
S9	**MA3**	**MA2**
S11	**MA3**	**MA3**
S12	MA1	MA2
S13	MA2	MA3
S14	MA2	MA2
S16	MA1	MA1
S20	MA0	MA2
S21	MA0	MA3
S24	MA0	MA0

As seen in the table, it seems that only two students (numbered 6 and 11) whose solutions are supportive of at least MA3 for both representations of the functions have necessary conceptual and analytical understanding of functions for modeling dynamic situations. It looks like Student 13 is having necessary analytical thinking while not having a strong conceptual thinking of functions. Six students (numbered 1, 2, 4, 5, 7 and 9) appeared not to have strong algebraic understanding of functions (since they were unable to develop a valid algebraic construction) but may have good or moderate conceptual understanding (since they were able to provide a sufficient verbalization of the given situation in Problem 1). Another five students (numbered 12, 14, 16, 20 and 24) who did not engage in MA3 for both representations of the situations seemed to have neither essential conceptual understanding nor analytical understanding of functions. On the other hand, Student 21 who appeared to lack an essential conceptual thinking when responding Problem 1 seemed to have a necessary analytical thinking of functions when answering Problem 2.

As having powerful data, I now present a table showing the correlation between students' engaged mental actions for graphical representations of the situations in Problem 1 and 2, and algebraic representation of the situation in Problem 2 (Table 17).

Table 17. Distribution of each student's engaged mental actions for graphical representations of the situations in both problems and algebraic representation of the situation in Problem 2

Student's code	Student's engaged mental Action for graphical representation	Student's engaged mental Action for graphical representation	Student's engaged mental action for algebraic representation
S1 – S24	**PROBLEM 1**	**PROBLEM 2**	**PROBLEM 2**
S6	**MA3**	**MA5**	**MA5**
S11	**MA3**	**MA3**	**MA3**
S13	**MA3**	**MA3**	**MA3**
S1	**MA2**	**MA2**	**MA2**
S4	**MA2**	**MA2**	**MA2**
S7	**MA2**	**MA2**	**MA2**
S12	**MA2**	**MA2**	**MA2**
S14	**MA2**	**MA2**	**MA2**
S20	**MA2**	**MA2**	**MA2**
S16	**MA1**	**MA1**	**MA1**
S24	**MA0**	**MA0**	**MA0**
S5	MA5	MA2	MA2
S2	MA2	MA0	MA2
S9	MA2	MA0	MA2
S21	MA0	MA2	MA3

As noticed on the table above, the data for eleven students (*marked in bold*) are consistent and we observe significant correlation between students' engaged mental actions for graphical representations of the situations in both problems and algebraic representation of the function in the second problem. Only three students (numbered 6, 11 and 13) engaged in at least MA3 and six students (numbered 1, 4, 7, 12, 14 and 20) always engaged in only MA2 while Student 16 regularly engaged in MA1 and Student 24 engaged in MA0 for all three cases represented on the table.

I present a table showing the correlation between students' engaged mental actions for graphical representations of the situations in Problem 1 and 2, algebraic representation of the situation in Problem 2 and verbal representation of the situation in Problem 1 (Table 18).

Table 18. Distribution of each student's engaged mental actions for graphical representations of the situations in both problems, algebraic representation of the situation in Problem 2 and verbal representation of the situation in Problem 1

Student's code	Student's engaged mental Action for graphical	Student's engaged mental Action for graphical	Student's engaged mental action for algebraic	Student's engaged mental Action for verbal

	representation	representation	representation	representation
S1 – S24	**PROBLEM 1**	**PROBLEM 2**	**PROBLEM 2**	**PROBLEM 1**
S6	**MA3**	**MA5**	**MA5**	**MA3**
S11	**MA3**	**MA3**	**MA3**	**MA3**
S13	**MA3**	**MA3**	**MA3**	MA2
S1	**MA2**	**MA2**	**MA2**	MA5
S4	**MA2**	**MA2**	**MA2**	MA5
S7	**MA2**	**MA2**	**MA2**	MA4
S12	**MA2**	**MA2**	**MA2**	MA1
S14	**MA2**	**MA2**	**MA2**	**MA2**
S20	**MA2**	**MA2**	**MA2**	MA0
S16	**MA1**	**MA1**	**MA1**	**MA1**
S24	**MA0**	**MA0**	**MA0**	**MA0**
S5	MA5	MA2	MA2	MA4
S2	MA2	MA0	MA2	MA5
S9	MA2	MA0	MA2	MA3
S21	MA0	MA2	MA3	MA0

I note that the results of verbal representation on the function of Problem 2 is not considered in the table above due to very low MA levels which may be the result of students not focusing on verbalization in question 2b. The students may have considered that they have addressed the verbalization in Problem 1, or that it is redundant given their algebraic work shown. In Table 18, while we notice frequent consistencies between graphical and algebraic representations of the situations, we observe much less consistencies between these two representations and verbal representation of the situation.

We observe that only two students (numbered 6 and 11) who have necessary conceptual and analytical thinking of functions engaged in at least MA3 for all cases of representations while Student 14 engaged in MA2, Student 16 engaged in MA1 and Student 24 engaged in MA0 for all four cases. Student 13 also frequently engaged in MA3 for all cases except for verbal representation (in Problem) 1 being assessed as MA2. Students 1, 4 and 7 engaged in MA2 for all cases except for verbalization of the function (in Problem 1) being assessed either MA5 or MA4. On the other hand, Students 12 and 20 also frequently engaged in MA2 except for verbal representation being assessed as MA1 or MA0. For the remaining four students (numbered 2, 5, 9 and 21) we have very inconsistent outcomes changing from MA5 to MA0.

5.3.1.2 How college level Calculus students being unable to reason dynamic function situations covariationally is related to Calculus students having a weak conceptual or/and analytical thinking of functions?

I now present a table listing each student's engaged minimum and maximum covariational reasoning (CV) levels based on consideration of the 'and' and 'or' conjunctions (without considering verbal representation of the situation in Problem 2) and my conclusions about student' conceptual and analytical thinking of the functions (Table 19).

Table 19. Distribution of each student's engaged minimum and maximum CV levels and my conclusions about students' conceptual and analytical thinking of functions

Student's code	Max CV level reached (or conjunction) in Problem 1	Max CV level reached (or conjunction) in Problem 2	Min CV level reached (and conjunction) in Problem 1	Min CV level reached (and conjunction) in Problem 2 (without considering verbal representation)	My conclusions about student' conceptual and analytical thinking of functions
S1	5	5	2	2	Good conceptual and weak analytical
S2	5	2	2	0	Good conceptual and weak analytical
S4	5	2	2	2	Good conceptual and weak analytical
S5	5	2	4	2	Good conceptual and weak analytical
S6	3	5	3	5	Good conceptual and good analytical
S7	4	2	2	2	Good conceptual and weak analytical
S9	3	2	2	0	Good conceptual and weak analytical
S11	3	3	3	3	Good conceptual and good analytical
S12	**2**	**2**	**1**	**2**	**Weak conceptual and weak analytical**
S13	3	3	2	3	Weak conceptual and good analytical
S14	**2**	**2**	**2**	**2**	**Weak conceptual and weak analytical**
S16	**1**	**1**	**1**	**1**	**Weak conceptual and weak analytical**
S20	**2**	**2**	**0**	**2**	**Weak conceptual and weak analytical**
S21	**0**	**3**	**0**	**2**	**Weak conceptual and good analytical**
S24	**0**	**0**	**0**	**0**	**Weak conceptual and weak analytical**

As seen on Table 19, we observe a certain correlation between students' engaged maximum/minimum CV levels and students' conceptual and analytical thinking of the functions. The first clear observation we make is that Student 6 (being the only student) who seemed to have good conceptual and analytical thinking of functions were able to reason the dynamic situations covariationally at higher levels, such as Level 3 or Level 5. Student 5 who seemed to have a good conceptual thinking of functions was able reason the events covariationally sometimes at level 5 while sometimes at level 2 due to him having weak analytical understanding of functions. Two students (numbered 11 and 13) who seemed to have a good analytical thinking were also often able to reason the situations covariationally at higher level, such as Level 3. On the other hand, five students (numbered 1, 2, 4, 7 and 9) who appeared to have a weak analytical thinking of functions were able reason the events covariationally sometimes at higher levels, such as level 5 or level 3, while generally at lower levels, such as level 2.

Four students (numbered 12, 16, 20 and 24) who seemed to have weak conceptual and analytical thinking of functions were generally reason dynamic events covariationally at lower levels, such as Level 2, 1 or 0. Student 14 who also seemed to have weak conceptual and analytical thinking of functions was able to reason the situations covariationally at Level 2. Student 21 who seemed to have a weak conceptual thinking were inconsistently reasoning the events covariationally at different levels, such as level 0, 2 or 3. These results indicate that Calculus students who have a weak conceptual or/and analytical thinking of functions tend to reason dynamic functional situations covariationally at lower levels.

5.3.2 What are common epistemological, didactical and cognitive obstacles, pseudo-thoughts and misconceptions in preventing college level Calculus students to successfully complete a dynamic function task?

We have observed that almost one third (29%) of twenty-four students relied on their procedural associations instead of focusing on the conceptual elements of the dynamic task given in Problem 1 while more than half of these respondents were in pseudo-analytical mode (and sometimes also in pseudo-conceptual mode) of thinking since they have not overcome yet the common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46). Some of these students (numbered 16 and 20) continued having both pseudo-conceptual and pseudo-analytical thoughts when responding Problem 2. The quarter of participants may not have overcome yet another common epistemological obstacle labeled as "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes*" (Sierpinska, 1992, p.36) due to them being unable to identify the name of the variables when responding Problem 1 while one fifth of students may not have overcome yet the epistemological obstacle "*EO(f)-16: The changes of a variable are changes in time*" (Sierpinska, 1992, p.55) since they considered the independent variable as time. Almost half (40%) of fifteen students may not have overcome yet the epistemological obstacle labeled as *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992, p.38) when answering Problem 2 due to them being unable to discriminate between the output and input variables. Hence, we conclude that "*EO(f)-3, EO(f)-5, EO(f)-11 and EO(f)-16"* are the common epistemological obstacles among college level Calculus students.

One fifth of twenty-four participants had a common cognitive obstacle labeled as CO – GTV since they were not able to transform a graphical representation of the function to its verbal one in Problem 1 while some students had a cognitive obstacle labeled as CO – VTG since they were unable to transform a verbal representation of the function to its graphical one. Hence, we conclude that almost one third of twenty-four students may not have overcome yet cognitive obstacles involved in moving between different representations of functions (in Problem 1). More than quarter (27%) of fifteen respondents had a common cognitive obstacle labeled as CO – ATG due to them being unable to transform an algebraic expression of the functional situation to its graphical one while one student had the cognitive obstacles labeled as CO – GTA and CO – VTA due to her being unable to obtain a correct formula although her valid graph is well explained by words in her solution to Problem 2. Thus, we conclude that one third of fifteen students may not have overcome yet cognitive obstacles involved in moving between different representations of functions (in Problem 2), being consistent with our previous conclusion. We also conclude that the cognitive obstacles labeled as CO – GTV and CO – ATG are the common cognitive obstacles among Calculus students.

We have observed that some Calculus students may not have overcome yet a common didactical obstacle labeled as DO – XY: Didactical obstacle involved in representing variables with the letters 'x' and 'y' due to them frequently practicing typical examples, in mathematical activities, that are often in terms of 'x' and 'y'. Finally, we have observed that some Calculus students have a common misconception about the concept of inflection point while others have a common misconception about the coordinates of a point.

5.3.2.1 Is there a certain correlation between Calculus students having obstacle(s) or/and pseudo-thought(s) or/and misconception(s), Calculus students having weak conceptual or/and analytical understanding of functions, and Calculus students being unable to reason dynamic events covariationally at a higher level?

I present the data displaying students' pseudo-thought(s) or/and misconception, students' obstacle(s) (epistemological, didactical and cognitive), students' engaged minimum/maximum covariational reasoning (CV) levels (without considering verbal representation of the function in Problem 2) and students' conceptual and analytical thinking of functions (Table 20):

Table 20. Distribution of students' pseudo-thought(s) or/and misconception, students' obstacle(s), students' engaged minimum/maximum covariational reasoning (CV) levels and students' conceptual and analytical thinking of functions

Student's code	**Student's engaged pseudo-thoughts and misconceptions**		**Student's engaged obstacles (epistemological, didactical and cognitive)**		**Student's engaged minimum /maximum CV levels from 0 to 5 (without considering verbal representation in Problem 2)**		**Student's conceptual and analytical thinking of functions**
	Problem 1	**Problem2**	**Problem 1**	**Problem 2**	**Problem 1**	**Problem 2**	
S1	---	MC-CP	---	CO-GTA CO-VTA	2/5	2/5	Good conceptual and weak analytical
S2	**---**	**---**	**---**	**EO(f)-3 EO(f)-5 DO-XY**	**2/5**	**0/2**	**Good conceptual and weak analytical**
S4	---	---	---	---	2/5	2/2	Good conceptual and weak analytical
S5	---	---	---	EO(f)-5	4/5	2/2	Good conceptual and weak analytical
S6	---	---	---	---	3/3	5/5	Good conceptual and good analytical
S7	---	---	---	---	2/4	2/2	Good conceptual and weak analytical
S9	---	---	---	---	2/3	0/2	Good conceptual and weak analytical
S11	---	---	CO-VTG	CO-ATG	3/3	3/3	Good conceptual and good analytical
S12	**PSA**	**MC-CP**	**EO(f)-11 DO-XY CO-GTV**	**EO(f)-5 DO-XY CO-ATG**	**1/2**	**2/2**	**Weak conceptual and weak analytical**
S13	---	MC-CP	EO(f)-5 EO(f)-16	EO(f)-5 CO-ATG	2/3	3/3	Weak conceptual and good analytical
S14	**---**	**---**	**CO-GTV**	**CO-ATG**	**2/2**	**2/2**	**Weak conceptual and weak analytical**
S16	**PSA, PSC**	**PSA, PSC**	**EO(f)-11**	**EO(f)-5 EO(f)-11**	**1/1**	**1/1**	**Weak conceptual and weak analytical**
S20	**PSA, PSC**	**PSA, PSC, MC-IP**	**EO(f)-7 EO(f)-11 DO-XY CO-GTV CO-VTG**	**EO(f)-5 EO(f)-7 EO(f)-11 DO-XY**	**0/2**	**2/2**	**Weak conceptual and weak analytical**
S21	**MC-IP**	**---**	**EO(f)-3 EO(f)-7 EO(f)-16**	**---**	**0/0**	**2/3**	**Weak conceptual and good analytical**
S24	**MC-IP**	**---**	**EO(f)-3**	**EO(f)-3**	**0/0**	**0/0**	**Weak conceptual and weak analytical**

Abbreviations for obstacles, pseudo-thoughts and misconceptions are given in the sections 5.1.3 and 5.2.3.

According to the data represented on the table above, we can say that there are some visible correlations between Calculus students having obstacle(s), pseudo-thought(s), or/and a misconception or/and a weak conceptual thinking or/and a weak analytical thinking of functions and students being unable to reason dynamic function situations covariationally at higher levels.

First of all, I would like to note that Student 6 (being the only student in my study), who has had good conceptual and analytical understanding of functions and seemed not to have any obstacle, pseudo-thought or misconception, and so consequently, he was able to reason the dynamic situations covariationally at higher levels. Student 11, who seemed to have only two cognitive obstacles was consistently able to reason the dynamic situations covariationally at Level 3 due to her having good conceptual and analytical thinking of functions. Student 13 had couple of obstacles and a misconception due to him having a weak conceptual understanding of functions but he generally was able to reason the events covariationally at Level 3 since he has had a good analytical thinking like Student 11. Student 1 who has had a weak analytical thinking and two cognitive obstacles (and a misconception) was still able to reason the events covariationally at higher level due to her having a good conceptual thinking of functions.

Students 4, 7 and 9 who seemed not to have had any obstacle but had a weak analytical thinking were often able to reason the dynamic situations covariationally at Level 2. Student 5 who appeared to have one obstacle sometimes reasoned the dynamic event covariationally at Level 2 due to him having a weak analytical thinking. Student 2 who seemed to have had couple of obstacles and a weak analytical understanding tend to reason dynamic events covariationally at Level 2 (while sometimes at Level 5 due to him having a good conceptual thinking).

The remaining six students (numbered 12, 14, 16, 20, 21 and 24) who had couple of obstacles or/and pseudo thought(s) or/and a misconception and weak conceptual and analytical thinking of functions were often unable to reason the dynamic situations covariationally at a higher level (except Student 21 whose covariational reasoning sometimes has reached Level 3 due to him having a good analytical thinking of functions).

The results suggest that majority of Calculus students, who have obstacle(s) or/and pseudo-thought(s) or/and misconception(s) due to them having weak analytical or/and conceptual understanding of functions, consequently tend to reason dynamic functional situations covariationally at a lower level.

6 CONCLUSIONS, RECOMMENDATIONS AND DIRECTIONS OF FUTURE POSSIBLE RESEARCH

This chapter consists of the sections of conclusions, recommendations and directions of future possible research.

6.1 CONCLUSIONS

As mentioned in the literature review, there are some concerns about the necessity of teaching the set-theoretic definition in early calculus courses since many students have enourmous difficulties in understanding of the set-theoretic definition of function (Even & Bruckheimer, 1998; Herscovics, 1989; Malik, 1980; Markovits et al., 1986; Sierpinska, 1992; Vinner, 1983; Vinner & Dreyfus, 1989). On the other hand, some researchers suggest that students need to have foundational understandings, such as, rate of change, accumulation and continuous covariation, and reasoning abilities, such as a process conception of function and a covariational conception of function in order to succeed in Pre-calculus and Calculus courses (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Carlson & Oehrtman, 2004; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2010; Carlson, Madison & West, 2015; Monk, 1992; Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017). Unfortunately, investigations revealed that students enroll in Pre-Calculus or Calculus courses without learning key concepts of functions (Madison & West, 2010; Carlson, Oehrtman & Engelke, 2010; Carlson, Carlson, Madison & West, 2015), having strong conceptual thinking or understanding of functions (Even, 1992; Monk, 1992) due to highy procedural orientation in mathematics education (Kaldrimidou & Ikonomou, 1998; Oehrtman, Carlson & Thompson; 2008) moving flexibly, fluidly and powerfully between different types of representations of functions (Carlson, 1998; Carlson & Oehrtman, 2004; Oehrtman, Carlson, & Thompson, 2008), having a process conception of function (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Monk, 1992) and so developing a strong covariational conception of function (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Thompson & Carlson, 2017), being necessary for Pre-Calculus and Calculus students to succeed in more advanced mathematics courses. These students often have common obstacles, such as epistemological (Sajka, 2003; Sierpinska, 1992; Sierpinska, 2019), cognitive (Herscovics, 1989) and didactical (Bachelard, 1938/1983; Brousseau, 1997), or/and pseudo-thoughts (Vinner, 1997) or/and misconceptions (Carlson 1998**;** Giovaniello, 2017). My conclusions, which I describe below, are similar with these conclusions.

Covariational reasoning plays an important role in the development of fundamental mathematical notions related to functions. To be able to understand key concepts of calculus, a student needs to grasp the idea of continuous covariation and thus conceptualize variables of a dynamic situation as varying smoothly and continuously in smaller and smaller bits. In our study, we have observed that Calculus students are unable to reason dynamic function situations covariationally at Level 5. This means that Calculus students fail in conceptualizing how two varying variables change together in tandem since they are not able to keep track of the variables' values changing simultaneously and so to recognize some patterns in the changing nature of the instantaneous rate. This result is consistent with other results found by Carlson and her colleagues in their works (Carlson et al., 2002, Carlson & Oehrtman, 2004; Carlson, Madison & West, 2010; Carlson, Oehrtman & Engelke, 2010; Carlson, Madison & West, 2015; Oehrtman, Carlson, & Thompson, 2008).

My study revealed that majority of the twenty-four Calculus students (79%) were unable to coordinate the instantaneous rate of change of the function with continuous changes in the input variable for the entire domain of the function graphically or verbally (no Level 5 reasoning), most of these students (75% of all students) were not able to coordinate the average rate of change of the output while envisioning uniform increments of the input (no Level 4 reasoning) and half of all respondents were unable to coordinate the amount of change of the output while thinking changes in the input variable (no Level 3 reasoning) when responding to Problem 1. The one third of participants was even unable to coordinate between the two variables verbally while almost quarter (21%) of students were not able to make coordination between the two variables neither verbally nor graphically (no Level 1 reasoning). We have observed more dramatic results in the analysis of the fifteen students' solutions to Problem 2. Majority of these fifteen Calculus students (87%) were not able to coordinate the instantaneous rate of change (or the average rate of change) of the dependent variable with continuous changes in the independent variable algebraically or graphically or verbally while two third (67%) of respondents were unable to coordinate the amount of change of the output while picturing changes in the input when solving Problem 2.

Based on these results, I conclude that many College level Calculus students are unable to develop not only Level 5 or Level 4 but also Level 3 covariational reasoning, coordinating the amount of change of the output with respect to changes in the input variable. This means that Calculus students fail in thinking how two varying quantities changes simultaneously since they are unable to observe some regularities in relationship between the values of two continuously changing quantities in a dynamic function event. Then, one might ask the following question: Why Calculus students are not able to reason dynamic events covariationally? My results may shine a light on this question as well.

I have found that more than half (54%) of twenty-four participants have a weak conceptual thinking of functions when answering Problem 1 (and at least 47% of fifteen students seemed to have a poor conceptual thinking of functions when responding both Problem 1 and 2). These results are consistent with previous research results (Even, 1992; Monk, 1992). On the other hand, more than three quarter (79%) of fifteen respondents appeared to have a weak analytical thinking of functions when solving Problem 2. I conclude that many Calculus students having weak conceptual and analytical understanding of functions contribute them failing to think how two varying quantities change together in tandem.

I have found that more than quarter (29%) of twenty-four participants relied on their memorized procedural associations formed in mathematical activities of school curriculum instead of focusing on the conceptual aspects of the given dynamic situation, consequently, were unable to move easily between the two representations of the function when responding Problem 1 and so to think about how the two varying quantities, height and volume, change together in tandem. More than half of these students had pseudo-analytical thoughts (and sometimes also pseudo-conceptual thoughts) since they have not overcome yet the common epistemological obstacle labeled as "*EO(f)-11: only relationships describable by analytic formulae are worthy of being given the name of functions*" (Sierpinska, 1992, p.46). Some of these respondents (numbered 16 and 20) also had both pseudo-conceptual and pseudo-analytical thoughts when responding Problem 2. Besides this, a quarter of participants may not have overcome yet another common epistemological obstacle labeled as "*EO(f)-3: Regarding changes as phenomena; focusing on how things change, ignoring what changes*" (Sierpinska, 1992, p.36) while almost quarter of students (21%) may not have overcome yet the epistemological obstacle "*EO(f)-16: The changes of a variable are changes in time*" (Sierpinska, 1992, p.55) when answering Problem 1. In addition, Almost half (40%) of fifteen participants may not have overcome yet another common epistemological obstacle labeled as *"EO(f)-5: Regarding the order of variables as irrelevant"* (Sierpinska, 1992, p.38) while some students (numbered 2 and 24) have not overcome yet the epistemological obstacle "*EO(f)-3"* when answering Problem 2. Thus, we conclude that "*EO(f)-3, EO(f)-5, EO(f)-11 and EO(f)-16"* are the common epistemological obstacles among college level Calculus students. These Calculus students who had pseudo-thought(s) or/and epistemological obstacle(s) were often unable to recognize some patterns in the relationship between the two continuously varying quantities and picture how these two quantities changes simultaneously, being indicative of some correlation between students having epistemological obstacle(s) or/and pseudo-thought(s) and them being unable to reason dynamic events covariationally. Consequently, these students were unable to move flexibly, fluidly and powerfully between different representations of the dynamic functional situations due to them having not only epistemological obstacle(s) but also cognitive and didactical obstacles.

The study revealed further that almost one third (29%) of twenty-four students may not have overcome yet cognitive obstacles involved in moving different representations of a function (particularly CO – GTV or CO – VTG) when responding to Problem 1. On the other hand, one third of fifteen participants may not have overcome yet cognitive obstacles materializing in translating one type representation of a function to another type when responding to Problem 2 (particularly CO – ATG or CO – GTA or CO – VTA). The data suggests that the cognitive obstacles labeled as CO – GTV and CO – ATG are the common cognitive obstacles among Calculus students. Many of Calculus students, except Students 1, 11 and 13, who had cognitive obstacles were unable to reason the given dynamic events covariationally at Level 3 while almost all these students, except Student 1, were not able to reason the situations covariationally at Level 5, being suggestive a certain correlation between students having cognitive obstacles and being unable to reason dynamic events covariationally at higher level.

Moreover, when answering Problem 2, a fifth (20%) of fifteen students used the letters 'x' and 'y' in their algebraic expressions so they may not have overcome yet a common didactical obstacle involved in representing variables in terms of 'x' and 'y' (labeled as DO – XY). Most of these students (except Student 2) had this didactical obstacle also when responding Problem 1. These students also had multiple epistemological obstacles, pseudo-thought(s) and a misconception and, consequently, were unable to reason the dynamic situations covariationally at Level 3. On the other hand, some students had a misconception about the point of inflection in Problem 1 while more than quarter of respondents had a misconception about either coordinates of a point or the inflection point when answering Problem 2. At least half of these students also seemed to be unable to reason the given dynamic events covariationally at Level 3.

Regarding the '*APOS theory*' mentioned in the literature review, I can conclude that at least half of twenty-four students seemed not to conceptualize a function as a generalized process which accepts input and produces output when responding Problem 1 while more than three quarter (80%) of fifteen respondents (except Students 6, 11 and 13) appeared not to have a process conception of function when responding Problem 2. Most of these students had weak conceptual understanding or/and analytical understanding of functions, were unable to move flexibly between different representations of the functional situations and had common obstacles, such as epistemological, cognitive, didactical, or/and pseudo-thought(s), or/and a misconception when solving the two dynamic tasks. Consequently, these students were unable to reason the given dynamic situations covariationally at higher levels such as Level 5 or Level 4 or even Level 3.

In my study only one student (numbered 6), who had essential conceptual and analytical thinking of functions and did not have any obstacle, pseudo-thought or misconception, was able to move flexibly, fluidly and powerfully between different representations of the functional situations, had a strong process conception of function and so was able to reason the two dynamic events covariationally at higher levels such as Level 5 or Level 3.

All twelve respondents, except Students 5 and 6, whose covariational reasoning has reached at least Level 3 were not able to move flexibly and fluidly between graphical and verbal representations of the function when answering Problem 1 and almost half of these respondents have not had overcome yet at least one obstacle, such as cognitive or epistemological or didactical. The remaining twelve students, except Students 14 and 24, whose covariational reasoning has not reached Level 3 had at least couple of obstacles (epistemological, cognitive or/and didactical), pseudo-thought(s) or/and a misconception due to them having a weak conceptual understanding of functions. These students who often do not have a process conception of function were also unable to move flexibly and fluidly between graphical and verbal representations of the function when responding to Problem 1. Although I identified only one obstacle (epistemological) and a misconception for Student 24 due to him not providing sufficient solutions to both Problems 1 and 2, he is the only student whose covariational reasoning has reached Level 0 in both problems and he is not able to move between different representations of functions at all, showing that he lacks not only essential conceptual and analytical thinking of functions but also a process conception of function.

On the other hand, four students (numbered 1, 11, 13 and 21) whose covariational reasoning has reached at least Level 3 were not able to move flexibly and fluidly between various representations of the function situation when answering Problem 2 and have not overcome yet at least one obstacle, such as cognitive or epistemological. The remaining ten students whose covariational reasoning has not reached even Level 3 did not have a process conception of function and were generally unable to move flexibly and fluidly between different representations of the situation when responding Problem 2. One third of fifteen students, who had multiple obstacles or/and pseudo thought(s) or/and a misconception due to them having weak conceptual and analytical understanding of functions, were able to move poorly or very poorly between different representations of the situations and so they were generally unable to reason the dynamic situations covariationally at higher levels

The data indicates that most of college level Calculus students, who have obstacle(s) or/and pseudo-thought(s) or/and misconception(s) due to them having weak analytical understanding or/and weak conceptual understanding of functions, are often unable to move flexibly, fluidly and powerfully between different representations of functions and so they fail to conceptualize a function as a process which maps values of one variable to values of another variable. Consequently, these students generally do not reason dynamic functional situations covariationally at higher levels.

6.2 Recommendations

The results question the readiness of Calculus students for subsequent mathematics courses, as well as the effectiveness of prior school curriculum in mathematics education on the topic of functions and related concepts. It is important to teach the key concepts related to functions, such as rate of change, accumulation and continuous covariation, through non-standard examples and dynamic events earlier in school curriculum in order to promote students to build strong conceptual and analytical thinking of functions before they enroll Calculus courses. It is preferable that educational system focuses on both conceptual and procedural orientations equally and encourages students to work with different representations of functions through dynamic tasks in mathematical education in order to help them to develop meaningful constructions between functional concepts and procedures and so to build a strong covariational conception of function.

My other recommendation is consistent with what Dr. Sierpinska said in her 1992's paper. On the one hand, functional situations first appear as typical examples of functions in school curriculum, and students discover dynamic models of relationships between variables much later by experience. On the other hand, functions first appeared as models of relationships in the history of mathematics in order to describe and predict functional situations. Therefore, it may be better if, in teaching, functions appear as models of relationships between variables before introducing some standard examples of functions and definitions related to functions (Sierpinska, 1992, p. 32 and 57). Building on this, I would like to recommend that students should also be encouraged to work with non-standard examples and non-routine tasks in order to overcome the didactical obstacle involved in promoting typical examples in educational system.

6.3 DIRECTIONS OF FUTURE POSSIBLE RESEARCH

In the literature review, I have mentioned about students lacking conceptual thinking or understanding of functions due to highy procedural orientation being dominant in mathematics education (Kaldrimidou & Ikonomou, 1998; Oehrtman, Carlson & Thompson; 2008). I would like to say a few words about Quebec's educational system. There have been efforts in the Quebec's educational program (at the pre-university level) to engage students more in using functions as a modeling tool. This is to be achieved, among others, by means of "situational problems" as of March 2021[1].

According to the first document in footnote, Quebec's mathematics education in secondary cycle requires students to get involved in 'solving a situational problem', 'using mathematical reasoning' and 'communicating by using mathematical language'. In the page 18 of the first document, it is written as follows, regarding 'solving a situational problem':

> *Solving a situational problem involves using a heuristic or discovery approach. In mathematics, this means being able to find a coherent solution to a situational problem under one of the following conditions:*
>
> *– The situation has not been previously presented in the learning process.*
>
> *– Finding a satisfactory solution involves using a new combination of rules or principles that the student may or may not have previously learned.*
>
> *– The solution or the way in which it is to be presented has not been encountered before.*
>
> *Solving a situational problem involves discernment, research and the development of strategies entailing the mobilization of knowledge. It also requires the students to carry out a series of actions such as: decoding the elements that can be processed mathematically, representing the situational problem by using a mathematical model, working out a mathematical solution, validating this solution and sharing the information related to the situational problem and the proposed solution……………In arithmetic, the students use their number and operation sense as well as the*

[1] (http://www.education.gouv.qc.ca/fileadmin/site_web/documents/PFEQ/chapter61.pdf and http://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/PFEQ_mathematique-deuxieme-cycle-secondaire_EN.pdf and also https://ca.ixl.com/standards/quebec/math, see grades from 8 to 12, emphasizing word problems, for instance, please see sections 7-8 AR an 7-8 AL in grade 8 which represents Quebec's secondary cycle 1 year 1 mathematical program).

relationships between these operations. They manipulate numerical expressions related to different sets of numbers, using processes for mental or written computation or using technology. They validate and interpret the numerical results in light of the context. In algebra, the students use different types of representations. They construct algebraic expressions, tables and graphs in order to generalize, interpret and solve the situational problem. They identify the unknown, solve equations to discover its value(s) and interpret them in light of the context.

(p.18 of the first document in the footnote)

It appears that Quebec's mathematical program encourages students to work with 'situational problems'. These problems involve 'decoding elements being processed mathematically', 'representing the problem by a mathematical model', 'working out a mathematical solution' and 'validating the solution'. Students are encouraged to use different types of representations to construct algebraic expressions and graphical constructions of situations, and to find the value(s) of algebraic equations when solving situational problems.

This being said about Quebec's secondary school mathematical program, in my research, I concluded that around half of Calculus students seemed to have a strong or good or moderate conceptual understanding of functions while only quarter of Calculus students seemed to have a strong or good or moderate analytical understanding of functions. This may be a result of college level students feeling intimidated by 'situational problems'. On the other hand, almost one third of twenty-four students relied on their procedural associations instead of focusing on the conceptual aspects of the given dynamic task in Problem 1. This may be a result of them feeling intimidated by 'typical algebraic procedures/formulas' formed in their earlier (or previous) mathematical education. However, there was no information about college level students' previous mathematical experiences (and I have just glanced at Quebec's secondary school mathematics program). Thus, a future research, aiming to study a correlation between Quebec's pre-university treatment of functions and its effect on college level students enrolled in calculus courses, may look into college level Calculus students' mathematical background and also Quebec's earlier mathematics education in order to reach more general conclusion about the impact of Quebec's mathematical program on college level students' learning.

I have observed that although Students 11 and 13 constructed incorrect graphs when responding both problems, I have determined that they exhibited behaviors that support MA3 for graphical representation of the dynamic situations since they correctly plotted points and represented relative magnitudes of the input and output variables on their graphs. While I adopted Carlson et al.'s theoretical construct in which behaviors are listed as '*plotting points/constructed secant lines*' for graphical representation of a function, I have changed the description of these behaviors for geometrical construction as 'plotting points/representing relative magnitude(s) of input and output variables/possibly constructing secant line(s)' in my adaptation. I note that no respondent constructed secant lines (graphical sign of MA4) when responding both problems. On the other hand, I think that 'constructing a smooth curve' may also be a condition for graphical representation of a function at Level 3 (instead of at Level 5) since we are interested in finding an algebraic expression representing a rule(s) of correspondence between changing quantities which should be represented by a correct smooth curve (not like the Student 11's and 13's curves) at this level. It may be also more appropriate to add a condition of 'constructing the slope(s) of the tangent line(s)' in the description of behaviors for graphical representation of a function at Level 5 since we are interested in finding instantaneous changes in the rate of change of the output (with respect to continuous changes in the input) which needs to be represented by the slope(s) of the line(s) tangent to the graph of the function. These suggestions for modifying the theoretical framework, as well as others, could be considered in future investigations when studying advanced mathematics students' understanding of functions.

Another future investigation may attempt to answer the following question: At what level Calculus students often move between different representations of a function? A future research project could rely on my theoretical construct and consider also each level of covariational reasoning as a level of ability to move between representations of a function. For instance, the examiner may describe Level 1 of ability to move between representations of a function as follows; if an individual exhibits behaviors which are suggestive of her/him able to move between graphical, verbal and algebraic representations of a function at Level 1 which will require the individual to coordinate the two variables all graphically, verbally and (*instead of 'or'*) algebraically, then his/her ability to move between the three representations of the function has reached Level 1.

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APPENDIX A

SUMMARY OF HISTORICAL DEVELOPMENT OF THE FUNCTION CONCEPT UP TO THE MIDDLE OF THE 20TH CENTURY

Antiquity Period

In Babylonian mathematics (2000 BC), mathematicians created tables of squares, cubes, reciprocals, square/cube roots (all based on natural numbers), worked on tabulated functions by using sexagesimal numeral system and developed algebraic methods for solving equations. According to O'Connor & Robertson (2005), although the tables produced by Babylonian mathematicians characterize functions, the function concept was not present at that time since they did not think about the terms related to functions.

In Greek mathematics dated back to 6th century BC, Pythagoreans discovered quantitative relationship between music and mathematics. They conceived the idea of function by measuring physical quantities like the length of a string. With the discovery of *Pythagorean Theorem,* Pythagoras also contributed to the development of mathematical analysis in antique times. The symptoms for conic sections, the first and second degree equations related to the conic sections were represented during the Ancient Greek period. These equations were introduced by equalities of areas of some rectangles or triangles.

Youschkevitch states that "*Until about the third century AD….No algebraic formula, no kind of literal algorithm, no analytical expression was ever introduced*" (Youschkevitch, 1976/1977, p.41). Around AD 200-300, Greek mathematician Diophantus of Alexandria who worked on algebraic equations and provided numerical solutions first introduced algebraic symbols like signs for the first six power of the unknown quantity and sign of equality.

In Ptolemy's *'Almagest'* (around AD 150), a mathematical model related to astronomy presented as the Plotemaic System. In this geocentric model where the Earth is at the center of the universe, Ptolemy calculated the positions of sun, moon and planets continuously changing in time by using some procedures and tabulated functions with numerical examples and sometimes in a general manner by verbally or graphically. Ptolemy who used geometric theorems in his *'Almagest'* produced a trigonometric table, by computing chords of a circle, which is equivalent to the table of the sine function's values. He noticed that a chord chosen in a given circle is related to an arc corresponding to the length of the chord. Although his discovery is directly related to a function, there is a doubt about his understanding of the notion of function (O'Connor & Robertson, 2005).

Archimedes who lived between 287-212 BC worked on quadratures and cubatures, applied concepts of infinitesimals in his works in a way which is similar to the modern integral calculus and used the method of exhaustion in order to prove geometrical theorems like the area of a circle or volume of a spheroid. His famous works are the '*Quadrature of the Parabola'*, '*Measurement of a Circle'* and '*On the Sphere and Cylinder*'. In his *'Measurement of a Circle'* he actually calculated the area of a circle while he calculated the surface area of a sphere in his '*On the Sphere and Cylinder'*. Even if he came up with the same integral $\int_a^b x^2 dx$ in his works, he failed in representing a general concept of a definite integral.

Aristotle (around 350 BC) used the term 'motion' in sense of 'change' and categorized it in three main forms as; change with respect to quality (qualitative change or alteration), change with respect to quantity (quantitative change like increase-decrease) and local motion (change in place) which consists of both uniform motion in which equal distances corresponds to equal intervals of time and non-uniform (difform) motion in which the velocity is changing. However, despite considerable progress, none of the above considerations led to a general concept of a function present in antiquity.

Middle Ages

In the 14th century, European mathematicians among whom Oresme were interested in changes of quantities and local motion, representing ideas of instantaneous velocity, acceleration and variable quantity. In his '*Latitude of forms'*, Oresme (1342-1382) who thought about measurable quantities been represented by points, lines and surfaces introduced the idea of a function explicitly on the x and y coordinates by distinguishing 'latitude *(latitudo)*' which represents the intensity of a quality (*intensio*) being along a vertical line from 'longitude *(longitudo)*' which represents extension of a quality (*extensio*) being along a horizontal line. He considered the intensity of qualities continuously changing in time. He applied this idea to his analysis of local motion in which the intensity was introduced as the velocity, the extension as the time and the area of the shape as the distance traveled.

Oresme considered three main kinds of qualities: 1) *Uniform quality* in which a body is moving with a constant velocity and the line of intensity is parallel to the line of longitudes. The related figure is a rectangle. 2) *Uniformly difform quality* in which the velocity of a moving body is changing and the line of intensity is a straight line which represents the hypotenuse of a corresponding triangle. 3) *Difformly difform qualities* in which the acceleration of a body is changing, thus all other cases which are not associated with uniform and uniformly difform qualities. He proved the mean speed rule known as the '*Merton Theorem*' (by finding the area of trapezoid) which was introduced by Heytesbury, Swineshead and Dumbleton in the 14th century. The theorem states that: '*uniformly difform motion is equivalent to a uniform motion with a velocity equal to the velocity of the accelerated movement at the middle moment of time'* (Youschkevitch, 1976/1977, p 48). Even though Oresme considered 'latitudes' and 'longitudes' of a quality as dependent variable and independent variable quantities in a general form and developed functional relationship between these quantities both geometrically and mechanically, he too failed in providing general definitions of these concepts. Three centuries later, Galileo (1600s-until 1642) who also proved the mean speed theorem by method of indivisibles began to conceive the idea of function. In his work of motion, he introduced quantities and relations between variables by means of geometrical expressions and of straight line segments. By comparing two concentric circles in which one's diameter is twice that of the smaller one, Galileo started to understand the idea of mapping between sets (O'Connor & Robertson, 2005).

In sum, European mathematicians in the Middle Ages considered variables continuously changing in time while conceptualized geometrical and mechanical forms of functions. They also considered infinitesimals when solving problems related to the areas of figures, infinite geometric series and instantaneous velocities. However, no explicit definitions of the concepts of function and of changing quantities were present in the mathematics of the Middle Ages.

Modern Period

Before the modern era, tabulated functions were introduced verbally, geometrically and kinematically, but, analytical method of representing functions by means of algebraic equations and formulas was absent. Until the beginning of the 17th century, mathematicians already studied real and complex numbers, thought about the concept of function as a relation between sets of numbers, advanced in trigonometry and represented signs of mathematical operations and relations. At the end of the 16th century, Viète's introduction of a new symbolic algebra revolutionized the development of the function concept and, after decades of works in rectilinear motions, curvilinear motions and the forces causing the motion became an object of study in the 17th century.

Viète, Fermat and Descartes Eras

Viète (1591) who is known as a founder of the new algebra used letters as symbols for both known (consonants) and unknown (vowels) quantities in his algebraic equations which were later used in the infinitesimal calculus by his successors like Descartes, Newton and Leibniz. His literal systematic algebra was significant convention in symbolizing algebraic equations in a general form.

Fermat who is known, along with Descartes, as the inventor of analytic geometry used (in his 1637 study, published in 1679) analytical methods of representing functions by the application of the new symbolic algebra to the study of geometry. He described unknown quantities in algebraic equations as continuous line segments in geometrical representations and developed methods for finding local maxima, local minima and tangent of differentiable functions by associating algebraic equations with curves and surfaces.

At the time of Fermat writing his convention, Descartes who is famous for his legacy in the development of Cartesian Geometry (or Coordinate Geometry) also conceived the idea of representing functions analytically. His leading goal was to minimize the solution of algebraic equations to standard procedures and produce their real roots in terms of known quantities. In his volume 'La *Geometrie*' (1637), the letters near the end of the alphabet (e.g. x, y and z) represent unknown quantities in equations while letters near the beginning of the alphabet (e.g. a, b and c) represent known quantities. Descartes invented what is now known as a Cartesian coordinate system which created a link between algebra and geometry. Therefore, he was able to describe geometric figures algebraically by using the coordinates of the points which produce the figures, and solve geometric problems by using algebra or vice versa. In a Cartesian system, he associated a geometric curve in a plane with an equation, called x the variable on the horizontal line and y the variable on the vertical line, and considered the numerical coordinates in order to determine the unique position of points.

Descartes pointed out that:

"*Prenant successivement infinies diverses grandeurs pour la ligne y, on en trouvera aussi infinies pour la ligne x, et ainsi on aura une infinité de divers points tels que celui qui est marqué C, par le moyen desquels on décrit la ligne courbe démandée."* (quoted from Youschkevitch, 1976/1977, p.52)

He means that if we take successively an infinite number of different values for the line y, we will find an infinite number of values for the line x and so an infinite number of unique points C by means of which the required curve shall be obtained. By taking unique values for y and obtaining the associated values for x, he actually considered a dependence relation between changing quantities and was able to compute the values of one quantity with respect to a given values of another quantity.

Moreover, he distinguished between algebraic (geometrical) curves which can be defined by using algebraic equations and non-geometrical (mechanical) curves. He classified geometric curves in kinds; the first kind curves include conic sections (circle, parabola, ellipse and hyperbola) which are expressed by second degree equations, the second kind curves described by third and fourth degree equations, the third kind curves represented by fifth and sixth degree equations and so on. The main characteristic of geometric curves is that all the points of a geometric curve can be obtained by construction from lower order curves (e.g. parabola obtained by lines which represent sides of a rectangle). On the other hand, only some of points can be obtained in mechanical curves which do not involve precise and exact measurement (e.g. spirals).

European mathematicians conceived the general notion of function during the 17^{th} century as a relation between 'quantities' rather than 'sets of numbers' (which were considered by earlier mathematicians). By using analytical representations of functions by means of equations and formulas with strict rules, the 17^{th} century mathematicians opened a new era in mathematical analysis. Application of the new symbolic algebra to geometry inspired other mathematicians like Newton and Leibniz and made a significant revolution in the development of infinitesimal calculus.

Newton and Leibniz Eras

Viète' literal systematic algebra made it possible to build up a general theory of equations while Fermat's and Descartes' analytic geometry made it possible to apply this new algebra to geometry. Newton (1664-1670), independently, also represented functional relationships between quantities analytically. In addition, like Oresme and Galileo, he expressed basic concepts of kinematics like time and motion and a geometrical relationship between these concepts. As his predecessors, he imaged velocity continuously changing in time when studying on motions.

In his *Method of fluxions and infinite series* (1670, published in 1736) which has been a leading treatise in the development of infinitesimal calculus, Newton had written out his theory of fluxions in which he called a changing quantity a '*fluent*' and its rate of change (or its derivative) a '*fluxion*' referring to its velocity. He introduced the independent variable as a correlated quantity and the dependent variable as a related quantity. Time was always considered as the universal independent variable by him while his focus was on velocities of variable quantities.

In his work, he verbalized the two main problems of the calculus which are inverse of each other; the first problem (the problem of tangents), with a given distance, is subject to finding the velocity of motion, the second problem (the problem of quadrature), with a given velocity, is about determining the distance of motion. The first problem refers to differentiation of functions while the second problem refers to integration of functions. Newton's analytical representations of functions in finite expression or by sums of infinite power series played central role in the development of infinitesimal analysis.

At the time of Newton was writing his discovery of differentiation and integration, independently, Leibniz developed the fundamental notations of infinitesimal calculus. In 1684, he expressed a new method for calculating maxima, minima and tangents. He introduced the differential of the ordinate y as dy and the differential of the abscissa x as dx which is an arbitrary segment in a curve, then he concluded that the ratio $\frac{dy}{dx}$ representing the slope of the tangent line of the curve is the same as the ratio of the ordinate to the subtangent.

We see the word 'function' first, in the statement of '…*other kinds of lines which, in a given figure, perform some function*…' (O'Connor & Robertson, 2005, p.2), in his work of '*The inverse method of tangents, or about functions*' (1673) in which he explained line segments associated with a curve for both geometrical and transcendental curves. He introduced, like Newton, the inverse problem of finding ordinates with a given property of a curve's tangent. In the same work, he defined functions as line segments formed by infinitely many straight lines associated with a fixed point and points of a given curve. In his explanation, he described these line segments as abscissas, ordinates and segments of tangents, normals, subtangents and subnormals (published in 1692 and 1694). Leibniz developed many notations that are still used in calculus today. He classified functions and curves as 'algebraic' ones expressed by an equation of a specific order and 'transcendental' ones expressed by equations of an infinite order (while Descartes categorized curves as 'geometrical' and 'mechanical').

While Leibniz and Newton, independently, made all these outstanding discoveries which are fundamentals for both differential and integral calculus, a general definition of function was still absent at that period of time.

Bernoulli and Euler Eras

In 1694, J. Bernoulli send Leibniz a letter describing his discovery that the lengths, or areas under some curves represented by integrals could be expressed by means of infinite series which Leibniz already knew. In this letter, Bernoulli expressed a function as: "…*a quantity somehow formed from indeterminate and constant quantities*" (quoted from O'Connor & Robertson, 2005, p.2). In a letter from 1696, he stated as "*diverse quantities* $\overline{X^1}, \overline{X^2}$ *are given by an indeterminate x and by constants… algebraically or transcendentally.*" (quoted from Youschkevitch, 1976/1977, p.57)

Even though he provided a general idea of function by means of analytical representations, a general definition of function was not present in his letters or articles. After exchanging their ideas through letters, Bernoulli and Leibniz discussed about the most suitable notation for a function of one or more than one variable, and discriminated functions by means of indices (e.g. $\overline{X^1}, \overline{X^2}$).

In 1718, Bernoulli published the article in which he provided the first explicit definition of a function:

"On appelle fonction d'une grandeur variable une quantité composée de quelque manière que ce soit de cette grandeur variable et de constants." (quoted from Youschkevitch, 1976/1977, p.60)

It basically means that we call function of a variable magnitude a quantity composed, in a certain manner, by that variable magnitude and constants. He conceived the concept of analytically represented functions, however, he did not explain how he obtained a function from the independent variable nor did he discriminate between single valued and multivalued functions.

Leonhard Euler significantly contributed in the development of the notion of function with his '*Introductio in analysin infinitorum*' (1744, published in 1748) in which he started to introduce the definitions of basic concepts related to functions. Euler first defined a function of a variable quantity as an analytic expression formed, in any manner, by the variable quantity and constants:

"*A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. Thus any analytic expression which, besides the variable z, contains also constant quantities is a function of that z; thus* $a + 3z$; $az - 4z^2$; $az + b\sqrt{a^2 - z^2}$; cz, *etc. are functions of z*". (Euler, 1988, p.3)

Euler first worked on standard algebraic operations in his analytical expressions, and then continued with transcendent operations and reached exponential-logarithmic functions and many other functions, resulting into infinite series, by integrating differential equations. He divided functions into two types, algebraic (divided into rational and irrational) and transcendental, depending on the nature of the analytic expression. He described the operations, such as addition, subtraction, multiplication, division, raising to a power and extraction of roots, involved in combining the variable quantity and constant quantities. He stated that any function can be expressed with an infinite series. This also involves certain algebraic operations by which variable and constant quantities can be arranged in a law of function. He spoke about multiples of functions and powers of functions, arising from a single operation, discriminated by the name of function. *Euler did not consider constant functions as functions. He called constant functions as constant quantities which are determined. Changing the value of variable quantity* *does not change the output value resulting in a constant function holding the same value. On the other hand, he claimed that 'a function itself of a variable quantity will be a variable quantity'. But in reality, a function of a variable quantity could be a constant quantity.*

In his *Introductio*, Euler provided a most general form of analytically expressed function by means of infinite power series containing sum of infinite products as $y = A + Bz + Cz^2 + Dz^3 + \cdots$. In his representation, he considered powers of z to be any numbers by commenting as "*to render this explanation broader, not only positive integral powers of z should be admitted, but any power. Thus there will be no doubt that any function of z could be transmuted into as infinite expression of the type* $Az^{\varphi} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} + \cdots$ *the exponents ϕ, β, γ, δ etc. denoting any numbers"* (quoted from Youschkevitch, 1976/1977, p.62).

He understood the idea of that functions analytically expressible could be developed into infinite power series whose terms may contain not only positive exponents but also fractional or negative ones. Since a function y could be expressed by some series whose terms contain powers of z, therefore z could be represented in terms of y by inverting the series. Defining function by means of analytical expressions whose form is infinite power series played a major role in the development of geometry, mechanics and physics during the entire 18th century.

Euler only considered analytic functions in Volume 1 of his *Introductio* while he mentioned about other type of functions in Volume 2 of his compendium. For Euler, the analytical law of a functional relation remains unchanged in a continuous function which could be introduced by just one expression while discontinuous functions (or mixed) could be represented by different laws for different intervals of the function's domain. By defining continuous function with an analytic law or equation representing the relation between coordinates of points of a curve, Euler emphasized the importance of the singleness of an analytical law. He then described discontinuous curves as "*all curves not determined by any definite equation, of the kind won't to be traced by a free stroke of the hand."* (quoted from Youschkevitch, 1976/1977, p.67)

In his *'Institutiones calculi differentialis'* (published in 1755), changed his definition of function provided in the Volume 1 of his '*Introductio'* to the more global definition of function in which he considered a function as a dependence relation between quantities;

> *"If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter."*

(Euler, 1755, 'Institutiones Calculi Differentialis', quoted from Youschkevitch, 1976/1977, p.70)

Introducing the more general definition of function as a dependence relation between variables and investigating relations between main properties of different classes of functions of one variable were important contributions in the development of the 18^{th} century's mathematics. Euler's discoveries inspired many other mathematicians like Cauchy, Fourrier, Dirichlet, Bolzano and Weierstrass who contributed enormously to the general theory of analytic functions during the 19^{th} century.

Cauchy, Fourrier, Dirichlet, Bolzano, Weierstrass and Bourbaki Eras

While only a single analytical expression of function was considered for almost two centuries until the end of 18^{th} century, mathematicians started to think and study on different kinds of functions (differentiable or piecewise discontinuous), which might not be analytically expressible or might be represented by more than one analytic law.

In his *Fragment sur les fonctions discontinues* (1780), Charles worked on examples involving discontinuous functions that were represented by different analytic expressions for different intervals could be expressed by one equation.

In his '*Memoire sur les fonctions continues'* (published in 1844), Cauchy provided a discontinuous function (in Euler's sense) of $f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$ expressible by a single equation $y = \sqrt{x^2}$ for every $-\infty < x < +\infty$ which represents a continuous function. Cauchy also showed that an infinitely differentiable function at a given point could not be analytic at that point and gave the following example; $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$.

In his *Cours d'analyse* (1821), Cauchy wrote the following statement in which he considered a dependence relation between quantities like Euler:

"If variable quantities are so joined between themselves that, the value of one of these being given, one can conclude the values of all the others, one ordinarily conceives these diverse quantities expressed by means of the one of them, which then takes the name independent variable; and the other quantities expressed by means of the independent variable are those which one calls functions of this variable." *(*quoted from O'Connor, J., & Robertson, E., 2005, p.5)

On the one hand, Euler claimed that mixed functions are not representable by trigonometric series, on the other hand, Fourier (1805) rejected this idea and stated that even discontinuous arbitrary functions could be representable by sinus and cosines of multiple arcs. However, he failed in doing sufficient analysis for the problem of expressing mixed functions with such trigonometric series. Representing functions by means of trigonometric series in which the coefficients of Fourier series of a given function $f(x)$ is equal to the integrals of the products $f(x)\cos nx \ \ and\ f(x)\sin nx$ motivated other mathematicians who started to produce general definitions of integral.

In his *Théorie analytique de la chaleur* (published in 1821-1822), Fourier described a function a sequence of values or ordinates, each of which is arbitrary, in his definition of a function:

"In general, the function f(*x*) *represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given of the abscissa x, there are an equal number of ordinates f*(*x*)*. All have actual numerical values, either positive or negative or nul. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity." (*quoted from O'Connor, J., & Robertson, E., 2005, p.5)

Dirichlet (1829-1837) found that a bounded function which is piecewise continuous and piecewise monotone over a given interval could be expressed by Fourier series converges to that function. Any arbitrary (discontinuous in Euler's sense) curve being hand-drawn over a given interval could be introduced by a single analytic equation, so becomes a continuous curve. Like Lobathchevsky, Dirichlet (1835) also used the word 'gradually' meaning continuously in Cauchy's sense in his more detailed definition of a continuous function, containing geometrical explanation, in which he imaged the different parts of a curve sharing a common law or different laws or no law:

"*Imagine a and b to be two fixed values and x a variable, which is supposed to assume one after the other all values between a and b. If to each x there corresponds a unique finite y in such a manner that while x runs continuously through the interval from a to b,* $y = f(x)$ *varies gradually also, then y is a continuous function of x for this interval. It is not at all necessary that y depends on x in this whole interval by the same law, and it is not even necessary to imagine a dependency expressible by mathematical operations…This definition does not prescribe a common law to the different parts of the curve; it can be thought of as being composed of parts of the most different kinds or completely without law.*" (quoted from Youschkevitch, 1976/1977, p.78)

Dirichlet introduced his famous function called '*Dirichlet function*' which is discontinuous at each point of the interval $0 \leq x \leq 1$:

$$f(x) = \begin{cases} 0, & for\ rational\ values\ of\ x \\ 1, & for\ irrational\ values\ of\ x \end{cases}.$$

Even though Lobathchevsky and Dirichlet considered discontinuous functions in their examples, it is not clear why their definitions are restricted to continuous functions.

Bolzano (in a treaty from 1817 but published much later) first introduced the epsilon-delta method when defined the notion of limit. He also provided the modern definition of derivative. The definition of derivative is known today as:

"Given a function $g(x)$, the derivative $g'(x)$ is understood to be the slope of the graph of g at each point x in the domain…Let $g: A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A." (Understanding Analysis, Stephen Abbott, p.145-148)

In his '*Cours d'analyse'* (1821), Cauchy discussed infinitely small quantities and provided a verbal definition of limit based on these quantities without using the epsilon-delta method:

"*When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.*" (quoted from Kline, 1990, p.951)

By using the limits as basis, he then introduced the concepts of convergence, continuity and derivative. He defined continuity of functions by considering the fact that an infinitesimal change in the input variable produces an infinitesimal change in the output variable:

"*the function* f(x) *will remain continuous with respect to* x *between the given limits, if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.*" (quoted from Kline, 1990, p.951)

While Cauchy focused on the continuity of the sum of a convergent series of continuous functions without distinguishing between pointwise continuity and uniform continuity, Weierstrass (in 1842-1844) formalized the concept of uniform convergence, by standard notions, written today as:

"Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) converges uniformly on A to a limit function f defined on A if, for every number $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ and $x \in A$".(Understanding Analysis, Stephen Abbott, p.177)

In 1854, Weierstrass provided the modern form of ε- δ definition of limit as:

"Let $f(x)$ to be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write $lim_{x \to x_0} f(x) = L$, if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that, for all x, $0 < |x - x_0| < \delta => |f(x) - L| < \varepsilon$".
(Thomas and Finney, Calculus and Analytic Geometry, p.70)

Weierstrass proved that a function being continuous over a closed interval could be expressed in that interval by a sum of convergent series of polynomials. The definitions of pointwise convergence and uniform convergence of sum of infinite series of functions were also provided during this period of time.

It is evident that Euler's general definition of a function was recognized and developed by his successors such as Cauchy, Lobathchevsky and Dirichlet who produced more detailed definition of a continuous function. Moreover, in 1939, Bourbaki provided the set-theoretic definition of function as a set of ordered pairs in which each element in the first set assigns to exactly one element in the second set:

"Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x. We give the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation x; y is said to be the value of the function at the element x, and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the same function". (quoted from Kleiner, 1989, p.299)

Many correspondences, like discontinuous functions, functions with split domains and functions with exceptional points, were defined as functions after the introduction of the Dirichlet-Bourbaki definition. Mathematicians, until the end of the 18th century, avoided the idea of representing a function with multiple analytic expressions or without any expression by basically focusing on the idea of representing function with a single analytic law. The explicit definitions of fundamental concepts of calculus, such as the limit concept, pointwise/uniform convergence, continuity, derivative and the function concept, were not yet present since mathematicians prior to the 19th century did not much focus on the ideas of limit and infinite series of functions. From beginning of the 19th century, mathematicians started to work with different types of functions, such as piecewise discontinuous functions. Cauchy, Bolzano, Dirichlet and Weierstrass focussed on fundamental concepts of calculus and significantly contributed in the introduction of the limit concept including the formal ε - δ definition and the definitions of the concepts of uniform convergence, continuity and derivative. After the introduction of these foundational definitions, studying infinite sums of functions and working with examples of continuous but nowhere differentiable functions became a target in mathematical analysis.

APPENDIX B

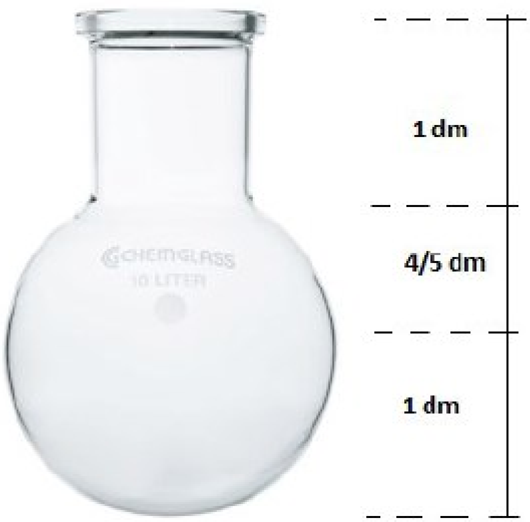
THE STATEMENT OF PROBLEM 1

Suppose an evaporating flask (see picture below) is filling with water.

a) Sketch a graph of the height of the water as a function of the amount of the water that is in the bottle.
b) Explain why your graph represents this relationship.
c) Does the graph of the function have a point of inflection? Justify your answer and, if yes, indicate clearly the inflection point on the graph.
d) Is there an interval where the height of the water increases linearly with respect to the amount of the water? If yes, mark the interval on your graph.

THE STATEMENT OF PROBLEM 1

This problem is about the evaporating flask being filled with water, as last week, but this time you are given the dimensions of the flask. We are assuming the radius of the sphere that forms the round part of the bottle is 1 unit (e.g., 1 dm = 10 cm) and the neck starts at 1 and 4/5 units from the bottom of the bottle; we also assume that the neck is 1 unit high.



a) Find a formula for the volume V of water as a function of the height h of the water: V(h) = ?. What is the domain of this function? What is the range?
b) Sketch a graph of the height as a function of the volume of the water, i.e., of the function h(V). Explain how you did it and what makes you sure you are right. What is the domain of this function? What is its range?
c) What is the height of the water if there are, approximately,

   (i) 2 litres

   (ii) 4 litres

   (iii) 5 litres

   of water in the flask? Note: 1 litre = 1 dm^3

d) Does the graph of the function h(V) have a point of inflection? If yes, what are its coordinates? Justify your response.