Invariant Measures of Random Dynamical Systems with Constant Probabilities

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Abstract

Invariant Measures of Random Dynamical Systems with Constant Probabilities

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In this PhD thesis we are concerned with the existence of the invariant measures and the absolutely continuous invariant measures under one-dimensional transformations. The thesis consists of two articles [1], [21].

We establish the existence of invariant measures for random maps with constant probabilities and for nonautonomous random dynamical systems generalizing Krylov-Bogoliubov Theorem. We present results on the existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a, b]$ using the theory of bounded variation.

We study the dynamics of a new family of transformations. We defined a general formula for the density function for any transformation belonging to our family, and we find some special properties for this family. This allowed us to study the random maps with constant probabilities based on these maps and to prove that the density function f of random map which is constructed from our family maps $T = \{\tau_1, \tau_2, ..., \tau_n; p_1, p_2, ..., p_n\}$ is the combination $f(x) = p_1 f_1 + p_2 f_2 + ... + p_n f_n$, where $f_1, f_2, ..., f_n$ are the invariant density functions of $\tau_1, \tau_2, ..., \tau_n$ respectively. We defined another family of transformations, and we proved that the density functions for any transformations belonging to this family are $f(x) = 1$. We present an example to find the density function of the random maps by conjugations.

We created two classes of chaotic maps with desired invariant densities using two methods of solving the inverse Frobenius-Perron problem (IFPP). We studied the Lyapunov exponent and the autocorrelation properties for one of these classes.

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Dedications

This thesis is dedicated to the memory of my father, Mahmoud Akelh AlFarajat. I miss him everyday. To my beloved mother Khitam, my father-in-law Ahmad AlFdoul and my mother-in-law Sara. Finally, I must thank my wonderful wife, Yasmeen for her support.

Eyad AlFarajat

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Chapter 1

Introduction

Ergodic theory is the mathematical study of the long-term average behaviour of systems, concerned with study of dynamical systems from the point of view of orbits' statistical behaviour under a transformation. The basic ingredients are a measurable space, a measurable transformation acting on points in the measurable space and an invariant measure on the measurable space. Most specialists consider it is as a study of invariant measures of dynamical systems. Dynamical systems can have a large set of invariant measures. One of the most important problems in the ergodic theory of dynamical systems is the existence of absolutely continuous invariant measures.

In this thesis, we present results about the absolutely continuous invariant measures in dynamical systems of a simple single maps, in the random dynamical systems and in nonautonomous random dynamical systems.

The importance of absolutely continuous invariant measures follows from the fact that they are considered physically significant, for example, only these measures can be visualized using computers. The existence of acim, theory and examples of the maps of $[0, 1]$ into itself has a long history. See for instance, the works of Ulam and von Neumann ([52] 1947), Rényi ([44] 1957), Lasota and Yorke ([36] 1973) and Jabłoński, Góra and Boyarsky ([29] 1996).

There is a natural procedure for finding an absolutely continuous invariant measure. It is the iterating of the canonical measure μ. First construct the images of μ under the mapping $\mu_n = \mu \circ \tau^{-n}$, then take the averages $\nu_n = \sum_{k=0}^{n-1} \mu_k / n$ and take some *-weak accumulation point. Special properties of the mapping ($e.g.$ its uniform expansion) may be reflected in the properties of the limit measure (absolute continuity). An alternative way is to iterate the density function with the transfer operator, and use the properties of τ to prove a compactness property of a resulting sequence. The existence of an absolutely continuous invariant measure is not granted and is due in many cases to hyperbolic properties of the mapping, such as large derivatives on big sets of points. Once found, the absolutely continuous invariant measure serves via the ergodic theorem to pronounce statements about typical (with respect to the canonical measure) behaviour of the system [26].

The main tool we shall use throughout is the Frobenius-Perron Operator. The Frobenius-Perron operator describes the evolution of density functions in a dynamical system, the invariant density is a fixed point of P_τ, $P_\tau(f) = f$. The existence of invariant densities for a class of chaotic point transformations has been proved by Lasota and Yorke [36]. We can approximate the fixed point of the Frobenius-Perron operator P_τ by the fixed point of a matrix operator. Frobenius-Perron operator is an example of Markov operator [35], and possesses nice properties such as linearity, positivity, preservation of integrals. With the aid of this operator we will be able to find meaningful invariant measures, study their properties, and show why they are important in describing chaotic phenomena [11].

A random dynamical system of special interest is a random map where the process switches from one map to another according to fixed probabilities [42]. Random maps with constant probabilities are an important special case of skew products. Let $T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\}$, be a random map with constant probabilities, where $\{\tau_1, \tau_2, ..., \tau_k\}$ is a set of measurable transformations, and $\{p_1, p_2, ..., p_k\}$ is a set of constant probabilities, that is $p_i > 0$, $i = 1, 2, ..., k$, $\sum_{i=1}^{k} p_i = 1$. In the case that p_i are not constant functions, the random map is said to have a position dependent probabilities $p_i(x)$ and the random map is a position dependent random map. A measure μ is called invariant under the random maps with constant probabilities T if it satisfies the condition that $\mu(A) = \sum_{i=0}^{k} p_i \mu(\tau_i^{-1}(A))$, for each measurable set A. The existence and properties of invariant measures for random maps reflect their long-time behaviour and play an important role in understanding their chaotic nature. In 1984, Pelikan [42] proved sufficient conditions for the existence of acim for random maps with constant probabilities. For the theory of the existence and properties of invariant measures for random maps see [42, 11, 5, 50, 8, 30, 9]. Random dynamical systems provide a useful framework for modeling and analyzing various physical, social, finance and economic phenomena [50, 12, 48]. A random map as a model was introduced by mathematicians half a century ago [40]. Since the 1970s, the random map model has attracted the attention of physicists [32]. Such a dynamical system has recently found application in the study of fractals [7], in modeling interference effects in quantum mechanics [13], and in computing metric entropy [49].

Autonomous systems are rare in nature. A more realistic approach to modeling real life processes is to consider non-autonomous models. In this thesis we review the framework for studying a nonautonomous random dynamical systems. We consider the non-autonomous dynamical system $\{T_n\}$, where $T_n = \left\{\tau_{_{1(n)}}, \tau_{_{2(n)}}, ..., \tau_{_{k(n)}}; p_1, p_2, ..., p_k\right\}$. See the definition in Section 4.1. We study the existence of the invariant measures and the absolutely continuous invariant measures under one-dimensional non-autonomous random transformations. We present results on the existence of an absolutely continuous invariant measure (acim) on $[a, b]$ using the theory of bounded variation.

The general solution of the inverse Frobenius– Perron problem (IFPP), $i.e.$, constructing a chaotic dynamical system with given invariant density is obtained for the class of one-dimensional unimodal complete chaotic maps. There are different approaches to solving the IFPP. We presented two approaches from them. The first approach is matrix approach. Matrix method is outlined in the work of P. Góra and A. Boyarsky (1993 [22]), the work of Bollt (1999 [10]), the work of McDonald and Wyk (2017 [39]), the work of Rogers, Shorten and Naughton (2007 [45]) and the work of Nie and Coca (2016 [41]). The matrix method, gives us a simple relationship between the given density f and τ, where f is any piecewise constant density function. That is by expressing f in the form of the leading eigenvector, one we can determine the Ulam's matrix and hence the chaotic map τ. The column stochastic matrix can be treated as Ulam's transition matrix. The second approach is the conjugation approach, this approach was developed by Ulam (1960 [51]) Grossman and Thomae (1977 [24]), Gyorgyi and Szepfalusy (1984 [25]), Baranovsky and Daems (1995 [6]) and Jiang (1995 [31]). Conjugation function approach, makes use of the following equivalence relation between two mappings: The map $\tau : I \to I$ is conjugate to a piecewise linear map $\sigma : J \to J$,$i.e.$, there exists a one-to-one map $h : I \xrightarrow{\text{onto}} J$ such that $\tau = h \circ \sigma \circ h^{^{-1}}$, for a σ with a uniform invariant density, τ can then be found via the conjugating function (see Example 2.5.14).

The thesis is organized as follows.

In Chapter 2, we introduce some relevant concepts of random dynamical system including necessary theorems from measure theory and ergodic theory.

In Chapter 3, we proved the existence of invariant measure for random maps with constant probabilities, which is constructed from a continuous maps on compact space. These results are a generalization of Krylov-Bogoliubov Theorem.

In Chapter 4, we prove the existence of invariant measure for nonautonomous random dynamical systems. These results are a generalization of Krylov-Bogoliubov Theorem. We present the properties of the Frobenius-Perron operator with respect to $\widehat{T}_0^n$. We obtain a Lasota-Yorke inequality under an expanding on average condition. We present results on the existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a, b]$ using the theory of bounded variation.

In Chapter 5, We present results about a new class of families of piecewise linear transformations and about random maps with constant probabilities constructed from those transformations on the interval $[0, 1]$. We present the properties of these families. The main result is in Proposition 5.2.2. For another family of transformations we prove that the invariant density for any transformation of the family is $f = 1$. Finally, we present an example of finding the density function of the random maps by conjugations.

In Chapter 6, We present a particular class of Markov transformations.

In Chapter 7, We created two classes of chaotic maps with desired invariant densities using two methods of solving the inverse Frobenius-Perron problem (IFPP). We studied the Lyapunov exponent and the autocorrelation properties for one of these classes.

Chapter 2

Background

2.1 Review of dynamical systems and ergodic theory

Let us consider a probability space $(X, \mathfrak{B}, \mu)$ where X is a set, $\mathfrak{B}$ is a σ-algebra of subsets of X and μ is a measure such that $\mu(X) = 1$. The measurable transformation $\tau : X \to X$ is said to be μ-preserving transformation if $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$. Sometimes we say that μ is τ-invariant measure. The quadruple $(X, \mathfrak{B}, \mu, \tau)$ is called a dynamical system, while one refers to τ as the dynamics. It models a system with motion; being at an instance in state x_0, in the next instance the system is going to be in state $\tau(x_0)$. For a $x_0 \in X$ the elements of the set

$$\{x_0, \tau(x_0), \tau^2(x_0), ...\}$$

are called iterates of x_0 where $\tau^{n+1} = \tau^n \circ \tau = \tau \circ \tau^n$, the whole set is called the orbit starting in x_0 and the collection of all such orbits is called discrete dynamical system in X induced by τ. Here "discrete" refers to the fact that we may think of n as a discrete time parameter.

If τ is a (one to one) transformation from X to itself, then it said to be invertible and the condition for τ to be measure preserving in this case can be written as $\mu(\tau(A)) = \mu(A)$ for all $A \in \mathfrak{B}$.

A measurable set A is said to be invariant under τ (with respect to μ) provided

$$\mu(A \setminus \tau^{-1}(A)) = \mu(\tau^{-1}(A) \setminus A) = 0,$$

that is, modulo sets of measure 0, $\tau^{-1}(A) = A$. The dynamics on A is independent of $X \setminus A$ and $(X, \mathfrak{B}|_A, \mu|_A, \tau|_A)$ is a dynamical system as well.

If A is any set, the characteristic function of A, χ_A is the function on X defined by $\chi_A = 1$ if $x \in A$, and 0 otherwise. It is clear that the function χ_A is measurable if and only if the set A is measurable. For any measurable set A,

A is invariant under τ if and only if $\chi_A \circ \tau = \chi_A$ almost everywhere (abbreviated as a.e.) on X.

Another interesting behaviour is the accumulation of states around some subset of the phase space. We call a compact set $A \subset X$ an attractor if the iterates of every bounded set $B \subset X$ are uniformly tending to A. Sometimes not all states in X tend to A.

Ergodicity describe a dynamical system which has same behaviour averaged over time as averaged over the space of the system states, here is the formal definition.

Definition 2.1.1. A measure preserving transformation is ergodic if for every $A \in \mathfrak{B}$, such that $\tau^{-1}(A) = A$, $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Lemma 2.1.2. *[34] τ is ergodic if and only if every measurable invariant function is constant.*

The most fundamental idea in ergodic theory is the following fact proved by G.D. Birkhoff in 1931 .

Theorem 2.1.3. ***(Birkhoff's Ergodic Theorem)*** *[11]: Suppose $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is measure preserving, where $(X, \mathfrak{B}, \mu)$ is $\sigma-$finite, and $f \in L^1(X, \mathfrak{B}, \mu)$. Then there exists a function $f^* \in L^1(X, \mathfrak{B}, \mu)$ such that*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) \longrightarrow f^*,\ \mu - a.e. \tag{2.1.4}$$

Furthermore, $f^* \circ \tau = f^*$ $\mu - a.e.$ and if $\mu(X) < \infty$, then $\int_X f^* d\mu = \int_X f d\mu$. Moreover if τ is ergodic and $(X, \mathfrak{B}, \mu)$ is a normalized measure space, then f^* is constant and

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) \longrightarrow \int_X f d\mu \tag{2.1.5}$$

for almost every x.

The Birkhoff's Ergodic Theorem implies that if $\tau : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is ergodic and μ is τ-invariant and E is a measurable subset of X, then the orbit of almost every point of X visits the set E with asymptotic frequency $\mu(E)$.

Definition 2.1.6. Let μ and ν be two measures on the same measurable space. We say that ν is absolutely continuous with respect to μ, (and write $\nu \ll \mu$) if for every $A \in \mathfrak{B}$ for which $\mu(A) = 0$, we have $\nu(A) = 0$.

Definition 2.1.7. Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. Then $\tau : X \to X$ is said to be nonsingular if and only if $\tau_* \mu \ll \mu$, $i.e.$, if for any $A \in \mathfrak{B}$ such that $\mu(A) = 0$, we have $\tau_* \mu(A) = \mu(\tau^{-1}(A)) = 0$.

To test absolute continuity it is often useful to use the next theorem.

Theorem 2.1.8. *[16] $\nu \ll \mu$ if and only if for given $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\nu(A) < \epsilon$.*

If $\nu \ll \mu$, then it is possible to represent ν in terms of μ. This is the essence of the **Radon-Nikodym Theorem**:

Theorem 2.1.9. *[11] Let $(X, \mathfrak{B})$ be a space and let ν and μ be two normalized measures on $(X, \mathfrak{B})$. If $\nu \ll \mu$, then there exists a unique $f \in L^1(X, \mathfrak{B}, \mu)$ such that for every $A \in \mathfrak{B}$,*

$$\nu(A) = \int_A f d\mu \ll \mu. \tag{2.1.10}$$

f is called the Radon-Nikodym derivative and is denoted by $d\nu/d\mu$.

Example 2.1.11. μ is the length measure on X. ν assigns to each subset Y of X, twice the length of Y. Then, $\frac{d\nu}{d\mu} = 2$.

2.2 Frobenius-Perron Operator

Let $I = [a, b]$ and consider the measure space $(I, \mathfrak{B}, \lambda)$ where $\mathfrak{B}$ is a σ-algebra of subsets of I and λ is the normalized Lebesgue measure on I. Let $\tau : I \to I$ be a non-singular transformation,

$i.e.$, $\lambda(\tau^{-1}(A)) = 0$ whenever $\lambda(A) = 0$, and μ be a measure absolutely continuous with respect to λ ($\mu \ll \lambda$) where μ has a density f . Let us assume that τ is nonsingular, we define the **Frobenius-Perron operator** P_τ on L^1 corresponding to τ by

$$\int_A P_\tau f d\lambda = \mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f d\lambda \tag{2.2.1}$$

for all $A \in \mathfrak{B}$ and $f \in L^1$. Let $A = [a, x]$, differentiating both sides, we obtain,

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([a,x])} f d\lambda, \; a.e., \tag{2.2.2}$$

and the corresponding Frobenius-Perron operator P_τ can be expressed by

$$P_\tau f(x) = \sum_{w \in \tau^{-1}(x)} \frac{f(w)}{|\tau'(w)|}. \tag{2.2.3}$$

The existence and the uniqueness of P_τ, follows by the **Radon-Nikodym Theorem**. The operator P_τ transforms probability density functions into probability density functions under the transformation τ, where τ is assumed to be nonsingular. One of the most important properties of P_τ is that its fixed points are the densities of measures invariant under τ [11].

Next we will state some useful properties for Frobenius-Perron operator in general.

Proposition 2.2.4. *[11] Let $f, g \in L^1$, $h \in L^\infty$, and $\alpha, \beta \in R$. $P_\tau : L^1 \to L^1$ satisfies the following properties:*

- *(Linearity) $P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g, a.e.$*
- *(Positivity) If $f > 0$ then $P_\tau f > 0$.*
- *(Preservation of Integrals) $\int_I P_\tau f d\lambda = \int_I f d\lambda$.*
- *(Contraction property) $\|P_\tau\| \leq \|f\|$.*
- *(Composition property) If $\tau, \sigma : I \to I$ are nonsingular, then $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$. In particular, $P_{\tau^n} f = P_\tau^n f$.*

- *(Adjoint property)* $\int_I (P_\tau f) \cdot g d\lambda = \int_I f \cdot U_\tau g d\lambda$*, where* $U_\tau : L^\infty \to L^\infty$ *is called the* ***Koopman operator*** *and is defined by* $U_\tau g = g \circ \tau$*.*

The following proposition says that a density f^* is a fixed point of P_τ if and only if it is the density of a $\tau-$invariant measure μ, absolutely continuous with respect to a measure λ.

Proposition 2.2.5. *[11] Let* $\tau : X \to X$ *be nonsingular. Then* $P_\tau f^* = f^*$ *a.e., if and only if the measure* $\mu = f^* \cdot \lambda$*, defined by* $\mu(A) = \int_A f^* d\lambda$*, is* $\tau-$*invariant, i.e., if and only if* $\mu(\tau^{-1}A) = \mu(A)$ *for all measurable sets* A*, where* $f^* \geq 0$ *and* $\| f^* \|_1 = 1$*.*

Let

$$D = D(X, \mathfrak{B}, \mu) = \{f \in L^1(X, \mathfrak{B}, \mu) : f \geqslant 0 \text{ and } \|f\|_1 = 1\},$$

denote the space of probability density functions. A function $f \in D$ is called a density function or simply a density.

If $f \in D$, then

$$\nu(A) = \int_A f d\mu \ll \mu$$

is a measure and f is called the density of ν and is written as $d\nu/d\mu$.

2.3 Spaces of Functions and Measures

The results which are presented in this section are derived from the books of Góra and Boyarsky [11] (1997), Royden and Fitzpatrick [47] (2010), Kingman and Taylor [34] (1966), Roussas [46] (2014), Walkden [53] (2002).

We recall some fundamental ideas from measure theory.

Definition 2.3.1. Let $\mathbf{F}$ be a linear space. A function $\| \cdot \| : \mathbf{F} \to R^+$ is called a norm if it has the following properties for each $f, g \in \mathbf{F}$ and $\alpha \in R$,

(1) $\|f\| = 0 \Leftrightarrow f = 0$

(2) $\|\alpha f\| = |\alpha|\|f\|$

(3) $\|f+g\| \leq \|f\| + \|g\|$.

The space $\mathbf{F}$ endowed with a norm $\|\cdot\|$ is called a normed linear space.

Definition 2.3.2. A sequence $\{f_n\}$ in a normed linear space is said to converge to f in the metric space $\mathbf{F}$ provided

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Definition 2.3.3. A sequence $\{f_n\}$ in a normed linear space is a Cauchy sequence if for any $\epsilon > 0$, there exists an $N \geq 1$ such that for any $n, m \geq N$,

$$\|f_n - f_m\| < \epsilon.$$

Every convergent sequence is a Cauchy sequence.

Definition 2.3.4. A normed linear space $\mathbf{F}$ is complete if every Cauchy sequence converges, $i.e.$, if for each Cauchy sequence $\{f_n\}$ there exists $f \in \mathbf{F}$ such that $f_n \to f$. A complete normed space is called a Banach space.

Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. If a property is true except for a subset having measure zero, then we say this property is true almost everywhere.

Definition 2.3.5. Let $1 \leq p < \infty$. The family of real-valued measurable functions (or rather a.e.-equivalence classes of them) $f : X \to R$ satisfying

$$\int_X |f(x)|^p d\mu < \infty \tag{2.3.6}$$

is called the $L^p(X, \mathfrak{B}, \mu)$ space and is denoted by $L^p(\mu)$ when the underlying space is clearly known, and by L^p where both the space and the measure are known. The integral in 2.3.6 is assigned a special notation

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu\right)^{\frac{1}{p}},$$

and is called the L^p norm of f. L^p with the norm $\|\cdot\|_p$ is a complete normed space, $i.e.$, a Banach space. The space of almost everywhere bounded measurable functions on $(X, \mathfrak{B}, \mu)$ is denoted by L^∞. Functions that differ only on a set of $\mu-$measure 0 are considered to represent the same element of L^∞. The L^∞ norm is given by

$$\|f\|_\infty = ess\sup f(x) = \inf\{M : \mu\{x : |f(x)| > M\} = 0\}.$$

The space L^∞ with the norm $\|\cdot\|_\infty$ is a Banach space.

Definition 2.3.7. The space of bounded linear functionals on a normed space $\mathbf{F}$ is called the adjoint space of $\mathbf{F}$ and is denoted by $\mathbf{F}^*$. The weak convergence in $\mathbf{F}$ is defined as follows: a sequence $\{f_n\}_1^\infty \subset \mathbf{F}$ converges weakly to an $f \in \mathbf{F}$ if and only if for any $G \in \mathbf{F}^*$, $G(f_n) \to G(f)$ as $n \to \infty$. Similarly, a sequence of functionals $\{G_n\}_1^\infty \subset \mathbf{F}^*$ converges in the $^*-$weak topology to a functional $G \in \mathbf{F}^*$ if and only if for any $f \in \mathbf{F}$, $G_n(f) \to G(f)$ as $n \to \infty$.

Theorem 2.3.8. ***(Kakutani-Yosida Theorem)*** *[11]: Let $\mathbf{F}$ be a Banach space and let $T : \mathbf{F} \to \mathbf{F}$ be a bounded linear operator. Assume there exists $c > 0$ such that $\|T^n\| \leq c$, $n = 1, 2, \ldots$. Furthermore, if for any $f \in A \subset \mathbf{F}$, the sequence $\{f_n\}$, where*

$$f_n = \frac{1}{n}\sum_{k=1}^{n} T^k f,$$

contains a subsequence $\{f_{n_k}\}$ which converges weakly in $\mathbf{F}$, then for any $f \in \overline{A}$,

$$\frac{1}{n}\sum_{k=1}^{n} T^k f \to f^* \in \mathbf{F}$$

(norm convergence) and $T(f^*) = f^*$.

We now consider spaces of continuous and differentiable functions. Let X be a compact metric space.

Definition 2.3.9. Let f be a real valued function defined on a set A of real numbers. We say that f is continuous at the point x_0 in A provided that for each $\epsilon > 0$, there is a $\delta > 0$ for which if $x \in A$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$. The function f is said to be continuous (on A) provided it is continuous at each point in its domain A.

The following definitions and theorems are from [11].

Definition 2.3.10. $C^0(X) = C(X)$ is the space of all continuous real functions $f : X \to R$, with the norm

$$\|f\|_{C^0} = \sup_{x \in X} |f(x)|. \tag{2.3.11}$$

Definition 2.3.12. Let $r \geqslant 1$, $C^r(X)$ denotes the space of all $r-$times continuously differentiable real functions $f : X \to R$, with the norm

$$\|f\|_{C^r} = \max_{0 \leqslant k \leqslant r} \sup_{x \in X} |f^{(k)}(x)|. \tag{2.3.13}$$

where $f^{(k)}(x)$ is the $k-$th derivative of $f(x)$ and $f^{(0)}(x) = f(x)$.

Definition 2.3.14. $\mathcal{M}(X)$ denotes the spaces of all measures μ on $\mathfrak{B}(X)$. The norm, called the total variation norm on $\mathcal{M}(X)$, is defined by

$$\|\mu\| = \sup_{A_1 \bigcup ... \bigcup A_N = X} \{|\mu(A_1)| + ... + |\mu(A_N)|\}. \tag{2.3.15}$$

where the supremum is taken over all finite partitions of X.

Theorem 2.3.16. *Let X be a compact metric space. Then the adjoint space of $C(X), C^*(X)$, is equal to $\mathcal{M}(X)$.*

Theorem 2.3.17. *(Scheffé's Theorem) If $f_n \geq 0$, $\int f_n d\lambda = 1, n = 1, 2, ...$ and $f_n \longrightarrow f$ a.e. with $\int f d\lambda = 1$, then $f_n \longrightarrow f$ in $L^1 - norm$.*

Definition 2.3.18. Let X be a metric space and μ a measure defined on the σ- algebra of Borel sets. We say that μ is a Radon measure if $\mu(K) < \infty$ for all compact sets and

$$\mu(E) = \inf_{\substack{U \supset E \\ U-open}} \mu(U) = \sup_{\substack{K \subset E \\ K-compact}} \mu(K) \text{ , for all } E \subset \mathfrak{B}(X).$$

Theorem 2.3.19. *(Lusin's Theorem)[18] Suppose that μ is a Radon measure on metric space X and $f : X \to C$ is a measurable function that vanishes outside a set of finite measure. Then for any*

$\epsilon > 0$ there exists $g \in C^c(X)$ such that $g = f$ except on a set of measure $< \epsilon$. If f is bounded, g can be taken to satisfy

$$\sup_{x\in X} |g(x)| < \sup_{x\in X} |f(x)|.$$

Where $C^c(X)$ is the linear space of continuous real-valued functions on X.

Definition 2.3.20. The weak topology of measures is a topology of weak convergence on $\mathcal{M}(X)$. i.e.,

$$\mu_n \to \mu \Leftrightarrow \int g d\mu_n \to \int g d\mu$$

for all $g \in C(X)$.

In view of Theorem 2.3.16 this is sometimes referred to as the topology of $*-$weak convergence.

Theorem 2.3.21. *The weak topology of measures is metrizable and any bounded (in norm) subset of $\mathcal{M}(X)$ is compact in the weak topology of measures.*

We now present an important corollary of Theorem 2.3.16.

Corollary 2.3.22. *The set of probability measures is compact in the weak topology of measures.*

Definition 2.3.23. A function $f : X \to R$ is simple if it takes only a finite number of different values.

Note these values must be finite. Writing them as $a_i, 1 \le i \le N$, and letting $A_i = \{x \in X : f(x) = a_i\}$, we can write

$$f = \sum_{i=1}^{N} a_i \chi_{A_i},$$

where χ_A is the characteristic function of A.

Theorem 2.3.24. ***Simple Approximation Theorem*** *[47] Let $(X, \mathfrak{B}, \mu)$ be a measure space and f a measurable function on X. Then there is a sequence $\{\psi_n\}$ of simple functions on X that converges pointwise on X to f and has the property that*

$$|\psi_n| \le |f| \text{ on } X \text{ for all } n.$$

(1) If X is $\sigma-$finite, then we may choose the sequence $\{\psi_n\}$ so that each ψ_n vanishes outside a set of finite measure.

(2) If f is nonnegative, we may choose the sequence $\{\psi_n\}$ to be increasing and each $\psi_n \geq 0$ on X.

(3) if f is bounded on X. Then for each $\epsilon > 0$, there are simple functions ϕ_ϵ and ψ_ϵ on X such that $\phi_\epsilon \leq f \leq \psi_\epsilon$ and $0 \leq \psi_\epsilon - \phi_\epsilon \leq \epsilon$ on X.

2.4 Krylov-Bogolyubov Theorem

Theorem 2.4.1. *[11] Let $\tau : X \to X$ be a measurable transformation of $(X, \mathfrak{B}, \mu)$. Then τ is $\mu-$preserving if and only if*

$$\int f(x)d\mu = \int f(\tau(x))d\mu, \tag{2.4.2}$$

for any $f \in L^\infty$. If X is compact and (2.4.2) holds for any continuous function f, then τ is $\mu-$preserving.

Proof. Assume τ is $\mu-$preserving. Let $f \in L_\infty$. For a simple function $f = \Sigma_{k=1}^{n} c_k \chi_{A_k}$, since τ is measure preserving,

$$\begin{aligned} \int f \circ \tau d\mu &= \int \left[\sum_{k=1}^{n} c_k \cdot \chi_{A_k} \circ \tau\right] d\mu = \int \left[\sum_{k=1}^{n} c_k \cdot \chi_{\tau^{-1}(A_k)}\right] d\mu \\ &= \sum_{k=1}^{n} c_k \cdot \mu(A_k) = \int f d\mu \end{aligned} \tag{2.4.3}$$

Therefore, the Equation (2.4.2) holds for f simple, for this reason and according to the Simple Approximation Theorem 2.3.24, there is an increasing sequence $\{f_n\}$ of simple functions on X that converge pointwise on X to f. Hence $\{f_n \circ \tau\}$ is an increasing sequence of simple functions on X that converge pointwise on X to $f \circ \tau$. By using the Monotone Convergence Theorem twice, we have

$$\int f \circ \tau d\mu = \lim_{n\to\infty} \int f_n \circ \tau d\mu = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

Conversely, assume (2.4.2) holds. For $A \in \mathfrak{B}$, since $\mu(X) < \infty$, the function $f = \chi_A$ belongs to $L^1(X, \mu)$ and $f \circ \tau = \chi_{\tau^{-\iota}(A)}$.

$$\mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f d\mu = \int_A f \circ \tau d\mu = \int_A f d\mu = \mu(A).$$

□

Krylov-Bogolyubov Theorem says that for every a probability measure ν, every limit point of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} v \circ \tau^{-i}$$

is an invariant measure.

Theorem 2.4.4. ***(Krylov-Bogoliubov**) [11] Let X be a compact metric space and let $\tau : X \longrightarrow X$ be continuous. Then there exists a $\tau-$invariant normalized measure on X.*

Proof. Let ν be a normalized measure on X. Consider the sequence μ_n defined by

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \tau_*^i \nu, \tag{2.4.5}$$

where the operator $\tau_*^i \nu = v \circ \tau^{-i}$. The sequence $\{\mu_n\}_{k=1}^{\infty}$ contains a weakly convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$, since it is precompact in the weak topology of measures. Let μ be a limit point of this subsequence. We will prove that μ is $\tau-$invariant normalized measure on X. To this end it is enough to show that for any continuous function g the Equation (2.4.2) holds. We have

$$\begin{aligned}
|\mu(g) - \mu(g \circ \tau)| &= \lim_{k\to\infty} |\mu_{n_k}(g) - \mu_{n_k}(g \circ \tau)| \\
&= \lim_{k\to\infty} |\frac{1}{n_k}\sum_{i=0}^{n_k-1}(v \circ \tau^{-i})(g) - \frac{1}{n_k}\sum_{i=0}^{n_k-1}(v \circ \tau^{-i-1})(g)| \\
&= \lim_{k\to\infty} \frac{1}{n_k}|v(g) - v \circ \tau^{n_k}(g)| \\
&\le \lim_{k\to\infty} \frac{2\sup|g|}{n_k} = 0,
\end{aligned}$$

and Equation (2.4.2) is proved. Note that since τ is continuous $g \circ \tau$ is continuous for any continuous g and then the $*-$weak convergence of μ_{n_k} implies $\mu_{n_k}(g \circ \tau) \to \mu(g \circ \tau)$. □

2.5 Some theorems on the existence of acim's

In this section we will present some results about the existence of an absolutely continuous invariant measure for a piecewise differentiable mapping on an interval.

Let $\mathcal{P} = \{I_1, I_2, ..., I_n\}$, $I_i, = (x_{i-1}, x_i)$, $i = 1, 2, ..., n$ be a partition of I, $\tau : I \to I$ and $\tau_i = \tau_{|Ii}$. τ is a Markovian map for $\{I_i\}$ if it satisfies::

M1 (Piecewise smoothness) τ_i has a C^2-extension to the closure $\overline{I_i}$ of I_i,

M2 (Local invertibility) τ_i is strictly monotone,

M3 (Markov property) $\tau(\overline{I_i})$ is a union of some intervals $\overline{I_j}$.

If for each $i = 1, 2, ..., n$, if τ_i is linear, then τ is called a piecewise linear Markov transformation.

The class of piecewise linear Markov transformations is a simple class of piecewise monotonic transformations and the matrix representation of the corresponding Frobenius–Perron operator can be calculated easily. In fact, it is a matrix which follows from the following theorem.

Theorem 2.5.1. *[11] Let $\tau : (I, \mathfrak{B}, \lambda) \to (I, \mathfrak{B}, \lambda)$ be a piecewise linear Markov transformation with respect to the partition $\mathcal{P} = \{I_1, I_2, ..., I_n\}$. Then there exists a $n \times n$ matrix M_τ such that $P_\tau f = f M_\tau^T$ for every piecewise constant $f = (f_1, f_2, ..., f_n)$. The matrix $M_\tau = (m_{ij})$ is defined by*

$$m_{ij} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}$$

Where T denotes transpose.

Example 2.5.2. Let $\omega : [0, 1] \to [0, 1]$ be a piecewise linear Markov transformation on the partition $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, defined by

$$\omega(x) = \begin{cases} 1 - 4x, & \text{for } 0 \le x < \frac{1}{4} \\ 2(x - \frac{1}{4}), & \text{for } \frac{1}{4} \le x < \frac{1}{2} \\ \frac{1}{2} - 2(x - \frac{1}{2}), & \text{for } \frac{1}{2} \le x < \frac{3}{4} \\ 4x - 3, & \text{for } \frac{3}{4} \le x < 1 \end{cases}$$

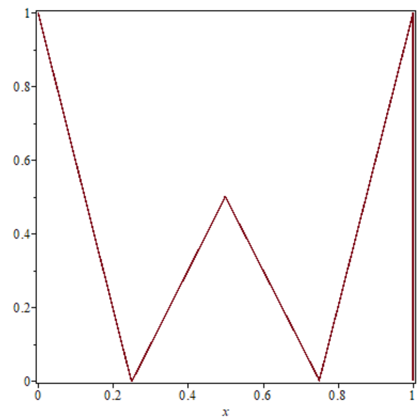

Figure 2.1: The map ω in Example 2.5.2.

ω-map is piecewise expanding and satisfy the conditions of Theorem 2.5.5. The matrix representation of P_ω is M_ω where

$$M_\omega = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Let $f = [x_1, x_2, x_3, x_4]$, where $x_i = f|I_i$, $I_i = \left[\frac{i-1}{4}, \frac{i}{4}\right]$, $i = 1, 2, 3, 4$. The normalized density of the map ω is the left eigenvector of M_ω with eigenvalue 1. Hence

$$f = [3, 3, 1, 1].$$

Therefore, the density of the invariant measure with respect to the Lebesgue measure is

$$g(x) = \frac{3}{2} \cdot \chi_{[0,\frac{1}{2}]} + \frac{1}{2} \cdot \chi_{[\frac{1}{2},1]}.$$

$\square$

Theorem 2.5.3. *(Folklore theorem)[11] Assume that (M1-M3) hold, and τ satisfy:*

F1 (Aperiodicity) there exists an integer n such that $\tau^n(\overline{I_i}) = X$ for all i.

F2 (Eventually expansive) there exist $n \in N$ and a constant $C > 1$ such that the derivative is *defined and* $|(\tau^n)'| \geq C$.

Then τ *has an ergodic invariant probability measure* μ *such that* $d\mu = \rho dx$*, where* ρ *is piecewise continuous and satisfies* $1/A \leq \rho(x) \leq A$ *for some* $A > 0$.

Theorem 2.5.4. *[43] The invariant measure is unique and has density bounded if it's satisfy:*

U1 τ *is a Markov transformation.*

U2 $|\tau''/(\tau')^2| \leq \theta < \infty$ *where* τ', τ'' *are defined,*

U3 $|\tau'(x)| \geq \lambda > 1$*, where* τ' *are defined.*

Let $\vee(.)$ be the standard one dimensional variation of a function and $BV(I)$ be the space of functions of bounded variations on I equipped with the norm $\|.\|_{BV} = \vee(.) + \|.\|_{L^1}$. Lasota and Yorke [36] proved the following important result for the existence of an acim for a single transformation using bounded variation methods:

Theorem 2.5.5. *[36] Let* $\tau : [0,1] \to [0,1]$ *be a piecewise* C^2 *transformation such that* $\inf |\tau'| > 1$. *Then for any* $f \in L^1[0,1]$ *the sequence* $\frac{1}{n}\sum_{k=1}^{n} P_\tau^k f$ *is convergent in norm to* $f^* \in L^1[0,1]$*. The limit function has the following properties:*

(1) $f > 0 \Rightarrow f^* > 0$.

(2) $\int_0^1 f d\lambda = \int_0^1 f^* d\lambda$.

(3) $P_\tau f^* = f^*$ *and consequently* $d\mu^* = f^* d\lambda$ *is invariant under* τ.

(4) $f^* \in BV[0,1]$*. Moreover there exists* c *independent to the choice of initial* f *such that* $\vee_{[0,1]} f^* \leq c \parallel f \parallel_1$.

Example 2.5.6. [11] Let $\tau : [0,1] \to [0,1]$ be defined for any $\alpha \in (0,1)$, $\tau(x) = \dfrac{x}{\alpha} \cdot \chi_{[0,\alpha]}(x) + \dfrac{1-x}{1-\alpha} \cdot \chi_{[\alpha,1]}(x)$, and let $f(x) = 1$. Then $f(x)$ is the invariant density of τ on $[0,1]$, $P_\tau f = f$.

$$P_\tau f(x) = (\alpha) + (1-\alpha) = 1$$

Example 2.5.7. Let $\tau : [0,1] \to [0,1]$ be defined by $\tau(x) = x + x^2 (mod 1)$, and let $h(x) = \frac{1}{x} + \frac{1}{1+x}$. Then $h(x)$ is the infinite invariant density of τ on $[0,1]$, $P_\tau h = h$.

$$\begin{aligned} P_\tau h(x) &= \frac{h(\tau_1^{-1}(x))}{\tau'(\tau_1^{-1}(x))} + \frac{h(\tau_2^{-1}(x))}{\tau'(\tau_2^{-1}(x))} \\ &= \frac{\frac{1}{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4x}} + \frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1+4x}}}{1 + (-1+\sqrt{1+4x})} + \frac{\frac{1}{-\frac{1}{2}+\frac{1}{2}\sqrt{5+4x}} + \frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{5+4x}}}{1 + (-1+\sqrt{5+4x})} \\ &= 2\left(\frac{2}{4x} + \frac{2}{4+4x}\right) = h(x) \end{aligned} \tag{2.5.8}$$

Definition 2.5.9. Two transformations $\tau : I \to I$ and $\sigma : J \to J$ on intervals I and J are called conjugate if there exists a bijective continuous map $h : I \longrightarrow J$ such that:

$$\sigma(x) = (h \circ \tau \circ h^{-1})(x).$$

The map h is called the conjugation.

Theorem 2.5.10. *[11] Let $\tau : I \to I$ be nonsingular and let $h : I \to I$ be a diffeomorphism. then we have:*

(1) $P_\tau f = f$ implies $P_\sigma g = g$, where $\sigma = h \circ \tau \circ h^{-1}$ and $g = (f \circ h^{-1}).|(h^{-1})'|$;

(2) if f is a $\tau-$invariant density, then g is a $\sigma-$invariant density.

Proof. (1) Let $P_\tau f = f$. Using the composition property for Frobenius-Perron operator we get

$$P_\sigma(P_h f) = P_h \circ P_\tau \circ P_{h^{-1}} \circ P_h f = P_h \circ P_\tau \circ P_{h^{-1} \circ h} f = P_h \circ P_\tau f = P_h f. \tag{2.5.11}$$

We have to show that $P_h f = g$. But that immediately follows from

$$P_h f(x) = \sum_{i=1}^{n} f \circ h_i^{-1} |(h_i^{-1})'| \chi_{[a_{i-1}, a_i]} = (f \circ h^{-1}) |(h^{-1})'| = g. \tag{2.5.12}$$

where h is monotonic $(n = 1)$, since it is a diffeomorphism. By using the Equation 2.5.11, we get

$$P_\sigma(g) = P_\sigma(P_h f) = P_h f = g$$

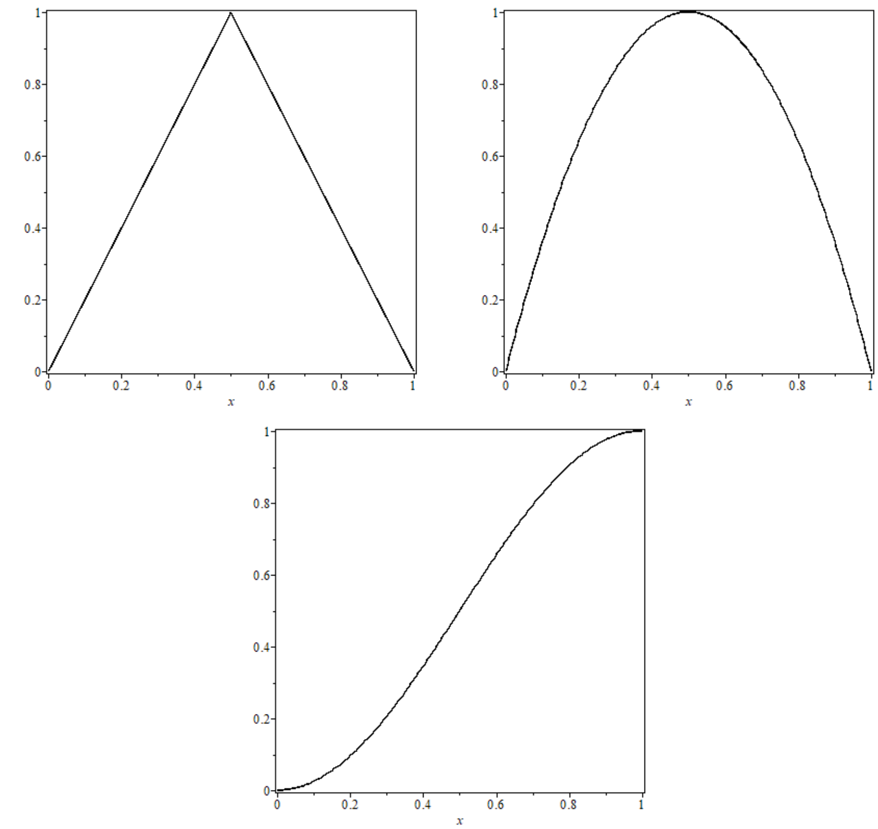

Figure 2.2: Top left graph is the map $\tau_0(x)$. Top right graph is the map $G_4(x)$. Bottom graph shows the conjugation map $h(x)$. For Example 2.5.14.

(2) Let $\int_I f d\lambda = 1$. Then $\int_I g d\lambda = \int_I P_h f d\lambda = \int_I f d\lambda = 1$. $\square$

Corollary 2.5.13. *If τ_1 is the tent map and τ_1 and τ_2 are conjugated by h ($\tau_2 = h \circ \tau_1 \circ h^{-1}$), then*

$$f_2 = |(h^{-1})'|.$$

From **corollary 2.5.13** we can find the relation,

$$h^{-1}(x) = \pm \int_0^x f_2(t)dt.$$

Example 2.5.14. Let $X = [0, 1]$. Consider the map $\tau_0 : [0, 1] \to [0, 1]$ defined by $\tau_0 = 1 - |2x - 1|$ (tent map) and $G_4 : [0, 1] \to [0, 1]$ be the logistic map that is defined by $G_4 = 4x(1 - x)$. Let, moreover, $h : [0,1] \to [0,1]$ be defined by $h(x) = \sin^2(\frac{\pi x}{2})$. τ_0 and G_4 are conjugated by h.

Notice that the probability density function of τ_0 is $g(x) = 1$. By applying Proposition 2.5.10, $g = (f \circ h^{-1}).|(h^{-1})'|$, where $h^{-1}(x) = \frac{2}{\pi}\arcsin(\sqrt{x})$, and then we have,

$$g = 1 \cdot \left(h^{-1}(x)\right)'.$$

Therefore, using the change of variables formula we obtain,

$$g = \frac{1}{\pi\sqrt{x(1-x)}},$$

and g is the invariant density of the logistic map.

Example 2.5.15. Let $X = [0,1]$, and let us consider the maps

$$\tau_1(x) = \begin{cases} \dfrac{2x}{1-x^2}, & \text{for } 0 \leq x < \sqrt{2}-1 \\ \dfrac{1-x^2}{2x}, & \text{for } \sqrt{2}-1 \leq x < 1 \end{cases},$$

$$\tau_2(x) = \begin{cases} \dfrac{50x}{25-16x^2}, & \text{for } 0 \leq x < \frac{5}{8}(\sqrt{29}-5) \\ \dfrac{100x}{75-48x^2} + \dfrac{1}{6}, & \text{for } \frac{5}{8}(\sqrt{29}-5) \leq x < \frac{1}{2} \\ \dfrac{25-150x}{108x^2-36x-72} + \dfrac{1}{6}, & \text{for } \frac{1}{2} \leq x < \frac{5}{6}\sqrt{2} - \frac{2}{3} \\ \dfrac{3x^2-x-2}{1-6x} + \dfrac{1}{6}, & \text{for } \frac{5}{6}\sqrt{2} - \frac{2}{3} \leq x < \frac{1}{6}(\sqrt{29}-1) \\ \dfrac{3x^2-x-2}{\frac{2}{3}-4x}, & \text{for } \frac{1}{6}(\sqrt{29}-1) \leq x < 1 \end{cases}.$$

The invariant densities for τ_1 and τ_2 are $f_1(x) = \dfrac{4}{\pi}\dfrac{1}{1+x^2}$ and

$$f_2(x) = \begin{cases} \dfrac{80}{\pi(25+16x^2)}, & \text{for } 0 \leq x < \frac{1}{2} \\ \dfrac{10}{\pi(3x^2-x+\frac{13}{6})}, & \text{for } \frac{1}{2} \leq x < 1, \end{cases}$$

respectively. τ_1 and τ_2 are conjugated by $h(x) = (\frac{5}{4}x)\chi_{[0,\frac{2}{5}]} + (\frac{5}{6}x + \frac{1}{6})\chi_{[\frac{2}{5},1]}$.

2.6 Random dynamical systems

Let $(X, \mathfrak{B}_X, \mu)$ be a probability space and let τ be an $\mu-$preserving measurable map on X. A **random dynamical system** f on the measurable space $(Y, \mathfrak{B}_Y)$ over $(X, \mathfrak{B}_X, \mu, \tau)$ is generated by mappings f_α , $\alpha \in X$, so that:

(1) the map $(\alpha, x) \to f_\alpha(x)$ is measurable, and

(2) it satisfies the cocycle property $f_\alpha^{n+m} = f_{\tau^m(\alpha)}^n \circ f_\alpha^m$ for all $n, m \in Z^+$, $\alpha \in X$.

The associated random orbits are $x_0, x_1, ...$, where $x_0 \in Y$ and $x_{n+1} = f_{\tau^n(\alpha)}(x_n)$. This random dynamical system is denoted by $(X, \mathfrak{B}_X, \mu, \tau, f)$.

Let X be a nonempty set and $\mathbb{T}$ be the set of two sided or one sided discrete or continuous times $(\mathbb{T} = R, R^+, R^-, Z, Z^+, Z^-)$.

Definition 2.6.1. [3](Random dynamical systems) Let $(X, \mathfrak{B}, \mu, \{\phi(t), t \in \mathbb{T}\})$ be a metric dynamical system and $(Y, \mathcal{F})$ be a measurable space. A random dynamical system on $(Y, \mathcal{F})$ over $(X, \mathfrak{B}, \mu, \{\phi(t), t \in \mathbb{T}\})$ is a mapping $\theta : \mathbb{T} \times X \times Y \to Y$, $(t, x, y) \mapsto \theta(t, x, y)$ satisfying the following condition: the mappings $\theta(t, x) : \theta(t, x, \cdot) : Y \to Y$ form a cocycle, that is, they satisfy (i) $\theta(0, x) = id_Y$, for all $x \in X$ (if $0 \in X$); (ii) $\theta(t + s, x) = \theta(t, \phi(s)x) \circ \theta(s, x)$ for all $t, s \in \mathbb{R}$, $x \in X$.

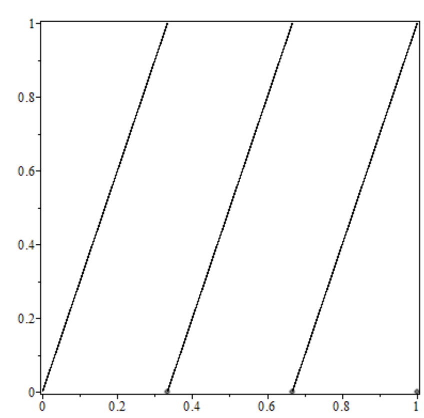

Figure 2.3: The map $\tau(x)$ in Example 2.6.2.

Example 2.6.2. $([0,1],\mathfrak{B},\lambda,\tau)$ is a metric dynamical system where $\tau:[0,1]\to[0,1]$ is defined by $\tau(x)=3x(mod1)$ and λ is the Lebesgue measure on [0,1].

Definitions 2.6.3. [3]

(1) A random dynamical system on $(Y,\mathcal{F})$ over $(X,\mathfrak{B},\mu,\{\phi(t),t\in\mathbb{T}\})$ is a measurable random dynamical system if $\mathfrak{B}(\mathbb{T})\times\mathfrak{B}\times\mathcal{F}$, $\mathcal{F}$ measurable.

(2) A measurable random dynamical system on $(Y,\mathcal{F})$ over $(X,\mathfrak{B},\mu,\{\phi(t),t\in\mathbb{T}\})$ is a continuous or topological random dynamical system if Y is a topological space and $\theta(.,x,.)$: $\mathbb{T}\times Y\to Y$ is continuous for every $x\in X$.

(3) A continuous or topological random dynamical system on $(Y,\mathcal{F})$ over $(X,\mathfrak{B},\mu,\{\phi(t),t\in\mathbb{T}\})$ is a smooth random dynamical system of class C^k if Y is a manifold and $\theta(t,x)=\theta(t,x,.)$: $Y\to Y$ is C^k, $1\leq k\leq\infty$ for every $(t,x)\in\mathbb{T}\times X$.

2.6.1 Skew product

Let $(\Omega,\mathcal{A},\sigma,\nu)$ be a dynamical system and let $(Y,\mathcal{B},\tau_w,\mu_w)_{w\in\Omega}$ be a family of dynamical systems such that the functions $\tau_w(x)$ are $\mathcal{A}\times\mathcal{B}$ measurable. A skew product of σ and $\{\tau_w\}_{w\in\Omega}$ is a transformation $S:\Omega\times Y\to\Omega\times Y$ defined by

$$S(w,x)=(\sigma(w),\tau_w(x)), \tag{2.6.4}$$

where $w\in\Omega$ and $x\in Y$.

In fact, An important application of a skew product construction is the random maps with constant probabilities. Let $(X,\mathcal{B},\lambda)$ be a measure space and $\Omega=\Sigma^+=\{1,2,3,...,k\}^{\{\mathbf{N}\cup 0\}}=\{w=\{w_i\}_{i=0}^{\infty}:w_i\in\{1,2,3,...,k\}\}$, be the set of set of all one sided infinite sequences. Let $\tau_j:X\to X$, $j=1,2,...,k$ be nonsingular piecewise one-to-one transformations and $p_1,p_2,...,p_k$ be constant probabilities such that $\Sigma_{j=1}^k p_j=1$. The topology on Ω is the product of the discrete topology on $\{1,2,3,...,n\}$ and the Borel probability measure μ_p on Ω is defined as

$$\mu_p\left(\{w:w_0=i_0,w_1=i_1,...,w_n=i_n\}\right)=p_{i_0},p_{i_1},...,p_{i_n}.$$

Let $\sigma : \Omega \to \Omega$ be the left shift. Now consider the skew product $S : \Omega \times X \to \Omega \times X$ defined by

$$S(w, x) = (\sigma(w), \tau_{w_0}(x)), \tag{2.6.5}$$

where $w \in \Omega$ and $x \in X$,

$$S^2(w, x) = (\sigma^2(w), \tau_{w_1} \circ \tau_{w_0}(x)), \tag{2.6.6}$$

and for any integer $N \geq 1$,

$$S^N(w, x) = (\sigma^N(w), \tau_{w_{N-1}} \circ \tau_{w_{N-2}} \circ ... \circ \tau_{w_1} \circ \tau_{w_0}(x)), \tag{2.6.7}$$

A random map with constant probabilities is

$$T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\},$$

with constant probabilities $p_i > 0$, $\sum_{i=1}^{k} p_i = 1$. The random map with constant probabilities T is defined by choosing τ_i with probability p_i, *i.e.*, for any $x \in X$, $T(x) = \tau_i(x)$ with probability p_i. For any integer $N \geq 0$, the iterates of the random map T are $T^N(x) = \tau_{i_N} \circ \tau_{i_{N-1}} \circ ... \circ \tau_{i_1}(x)$ with probability $\prod_{j=1}^{N} p_{i_j}$. $T^N(x)$ can be viewed as the second component of the S^N of the skew product S. Pelikan [42] defined a T-invariant measure m as follows:

Definition 2.6.8. Let T be a random map on X and μ be a measure on X. The measure μ is invariant under the random map T if

$$\mu(E) = \Sigma_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{2.6.9}$$

for any measurable set $E \in \mathcal{B}$.

2.7 The existence of absolutely continuous invariant measures for random maps

Pelikan [42] gives the following sufficient condition for the existence of absolutely continuous invariant measures for these random maps $T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\}$:

$$\sum_{j=1}^{k} \frac{p_j}{|\tau_j'|} \leq \alpha < 1, \tag{2.7.1}$$

for some constant α. the existence of acim, theory and examples of the maps of $[0, 1]$ into itself has a long history. See for instance, the works of Ulam and von Neumann ([52] 1947), Rényi ([44] 1957), Lasota and Yorke ([36] 1973) and Jabłoński, Góra and Boyarsky ([29] 1996).

The Frobenius-Perron operator P_T with respect to the random map with constant probabilities T is given by

$$P_T f = \sum_{i=1}^{k} p_i P_{\tau_i} f, \tag{2.7.2}$$

where P_{τ_i} is the Frobenius-Perron operator of the transformation τ_i. Operator P_T can be expressed by

$$P_T f(x) = \sum_{i=1}^{k} p_i \left(P_{\tau_i}(f)\right)(x) = \sum_{i=1}^{k} p_i \sum_{j=1}^{n_i} \frac{f(\tau_{i,j}^{-1}(x))}{|\tau'(\tau_{i,j}^{-1}(x))|} \chi_{\tau_i([x_{j-1}, x_j])}(x). \tag{2.7.3}$$

Where $\chi_{[x_{j-1}, x_j]}$ is the characteristic function of the interval $[x_{j-1}, x_j]$, *i.e.*, $\chi_{[x_{j-1}, x_j]}(x) = 1$ if $x \in [x_{j-1}, x_j]$, and 0 otherwise. The key for the indices in Equation (2.7.3):

$i =$ transformation numbers, $i = 1, 2, \cdots, k$.

$n_i =$ the total number of sub-transformation for each τ_i, $i = 1, 2, \cdots, k$.

$j =$ sub-transformation numbers $\tau_{i,j} = \tau_i|_{I_j}$, $j = 1, 2, \cdots, n_i$ and $i = 1, 2, \cdots, k$.

Measure μ is $T-$invariant measure if and only if $\mu(A) = \sum_{i=0}^{k} p_i \mu(\tau_i^{-1}(A))$ for all $A \in \mathfrak{B}$. $P_T f^* = f^*$ if and only if $\mu = f^* \lambda$ is $T-$invariant absolutely continuous measure.

Next we will state some useful properties for Frobenius-Perron operator with respect to the random map .

Proposition 2.7.4. *[50] Let α, β be constant. Then if $f, g \in L^1([0,1])$, $h \in L^\infty([0,1])$, and . $P_T : L^1([0,1]) \to L^1([0,1])$ enjoys the following properties:*

(1) (Linearity) $P_T(\alpha f + \beta g) = \alpha P_T f + \beta P_T g, a.e.$

(2) (Positivity) If $f > 0$ then $P_T f = \sum_{k=1}^{K} p_k P_{\tau_k} f > 0$.

(3) (Preservation of Integrals) $\int_{[0,1]} P_T f d\lambda = \left(\sum_{k=1}^{K} p_k\right) \int_{[0,1]} f d\lambda = \int_{[0,1]} f d\lambda$.

(4) (Contraction property) $\|P_T f\|_1 \leq \|f\|_1$.

(5) (Composition property) If $T, R : L^1([0,1]) \to L^1([0,1])$ are two random maps, then $P_{T \circ R} f = P_T \circ P_R f$. In particular, for any $n \geq 1$, $P_{T^n} f = P_T^n f$.

Lemma 2.7.5. *$P_T f^* = f^*$ if and only if $\mu = f^* \lambda$ is $T-$invariant.*

2.7.1 Random maps of piecewise linear Markov transformations and the Frobenius-Perron operator:

One of the important property for the piecewise linear Markov transformations is the invariant densities can be computed easily since the Frobenius-Perron operator can be represented by a finite-dimensional matrix (see Theorem 2.5.1). This property is inherited by random maps which are constructed from piecewise linear Markov transformations (see Section 3.4.6. in [50]).

Example 2.7.6. Consider the random map

$$T = \left\{ \tau_1, \tau_2, \tau_3; \frac{1}{2}, \frac{1}{8}, \frac{3}{8} \right\},$$

where τ_1, τ_2 and τ_3 are the piecewise linear Markov transformations shown in Figure 2.4.

Using, Theorem 2.5.1 and the Frobenius-Perron operator P_T, we have

$$P_T f = (\frac{1}{2} M_{\tau_1}^c + \frac{1}{8} M_{\tau_2}^c + \frac{3}{8} M_{\tau_3}^c) \pi^f.$$

Where c denotes transpose. It can be easily shown that the solution of the matrix equation $M_T^c \pi^f = \pi^f$ is

$$\pi^f = \left[\frac{239}{483}, \frac{2584}{3703}, \frac{24}{23}, 1 \right]^c$$

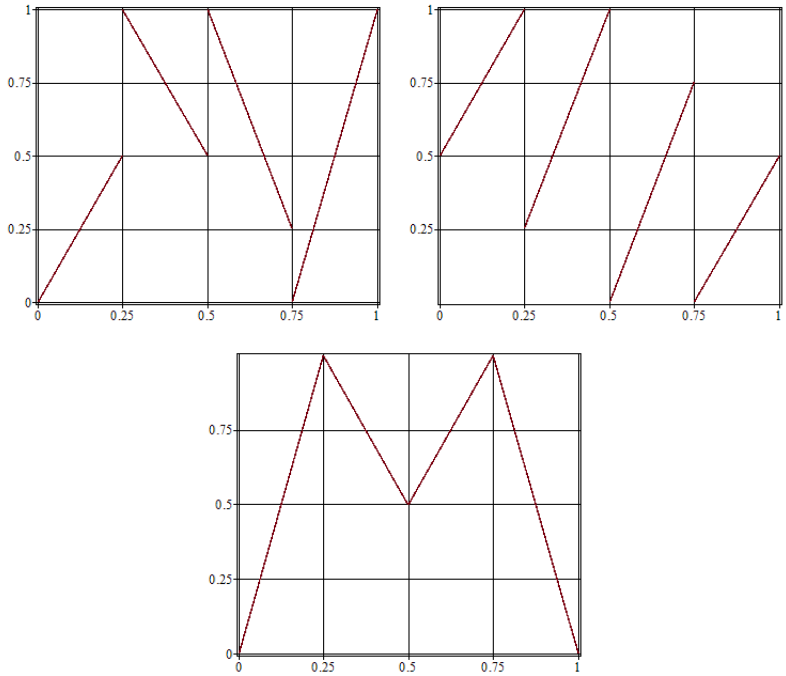

Figure 2.4: Top left: $\tau_1(x)$. Top right: $\tau_2(x)$. Bottom: $\tau_3(x)$, Example 2.7.6.

and the invariant density of T is

$$f(x) = \begin{cases} \dfrac{5497}{35950}, & \text{for } 0 \leq x < \frac{1}{4} \\ \dfrac{3876}{17975}, & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ \dfrac{5796}{17975}, & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\ \dfrac{11109}{35950}, & \text{for } \frac{3}{4} \leq x < 1 \end{cases}.$$

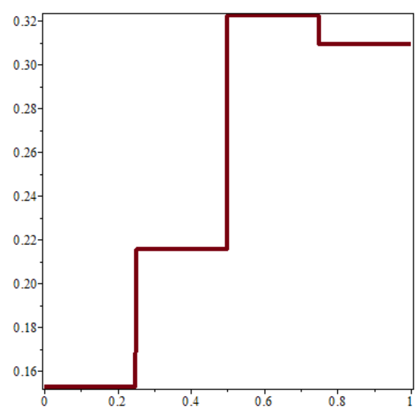

Figure 2.5: The invariant density of T, Example 2.7.6.

Chapter 3

Existence of invariant measures for continuous random maps.

In this section 3.2 we prove necessary and sufficient conditions for existence of an invariant measure for random maps. Before that in section 3.1, we mentioned some basics and discussed some rationale. The **Krylov-Bogoliubov Theorem** 2.4.4 is one of the theory which establishes the existence of invariant measures for continuous transformations (regardless it's expanding or nonexpanding transformations) on a compact space. In [3], For a Polish space, the author introduces a topology of weak convergence of measures which let him carry over the Krylov-Bogoliubov theorem and prove that each continuous random dynamical system on a compact space has at least one invariant measure and he generalized that to a random compact set.

3.1 Measurable transformation

Let X be a compact metric space and consider the measure space $(X, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is a σ-algebra of subsets of X and μ is the normalized measure on X. Let $\mathcal{M}(X)$ denotes the spaces of all measures on $\mathfrak{B}(X)$. A random map

$$T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\},$$

where $\tau_1, \tau_2, ..., \tau_k$ be a collection of measurable transformations from $(X, \mathfrak{B})$ to $(X, \mathfrak{B})$, $p_i > 0$, $\sum_{i=1}^{k} p_i = 1$. The random map with constant probabilities T is defined by choosing τ_i with constant probability p_i, in other words for any $x \in X$, $T(x) = \tau_i(x)$ with probability p_i. For an integer $N \geq 0$, the iterate of the random map T is performed as follows:

$$T^N(x) = \tau_{i_N} \circ \tau_{i_{N-1}} \circ ... \circ \tau_{i_1}(x),$$

with probability $\prod_{j=1}^{N} p_{i_j}$. The transition function of the Markov process of the random map T is the following

$$\mathbb{P}(x, A) = p_1 \chi_A(\tau_1(x)) + ... + p_k \chi_A(\tau_k(x)),$$

from a point $x \in X$ into a set $A \in \mathfrak{B}(X)$, and χ_A is the characteristic function of the set A on X that takes the value 1 on A and 0 on $X \setminus A$.

Given the random map with constant probabilities T and $g : X \to Y$ we can consider the composition $g(T) : X \to Y$ to generate a random map with constant probabilities $g(T)(x) = \{g(\tau_1(x)), g(\tau_2(x)), ..., g(\tau_k(x)); p_1, p_2, ..., p_k\}$,. In particular if $g : X \to R^*$ is an extended real-valued function on X (R^* is the extended real number system defined by adding two points $-\infty$ and $+\infty$ to the real numbers), then $g(T)$ defines an extended random map on X.

Lemma 3.1.1. *Let τ_i, $i = 1, 2, ..., k$, be a collection of measurable transformations from $(X, \mathfrak{B})$ to $(X, \mathfrak{B})$ and $g : X \to R^*$ a measurable as a function with extended real-values, then the composition $g(\tau_i)$ is measurable.*

Proof. For each i and for any Borel set in **B** in R^* we have

$$\begin{aligned}\{x : g(\tau_i)(x) \in \mathbf{B}\} &= \tau_i^{-1}\{y : g(y) \in \mathbf{B}\} \\ &= \tau_i^{-1}(\mathbf{A}), \text{for some } \mathbf{A} \in \mathfrak{B}\end{aligned}$$

. □

If μ is a measure on $(X, \mathfrak{B})$ and T is a random map with constant probabilities constructed by collection of measurable transformations from $(X, \mathfrak{B})$ to $(X, \mathfrak{B})$ then we can use T to define a measure v on $\mathfrak{B}$ by putting

$$v(\mathbf{A}) = \mu(T^{-1}(\mathbf{A})), \text{ for } \mathbf{A} \in \mathfrak{B}. \tag{3.1.2}$$

By this definition of v, $(X, \mathfrak{B}, v)$ is a measure space.

The following change of variable formula in an integral.

Theorem 3.1.3. *Let $g : X \to R^*$ be a measurable function. Then we have*

$$\int_X g(T)d\mu = \int_X gdv \tag{3.1.4}$$

Proof. Consider non-negative functions $g : X \to R^+$. It is enough to prove (3.1.4) when $g = \chi_A$ is the characteristic function of the set A (where χ_A is the characteristic function , and $g \circ T = \sum_{i=0}^{n} p_i g \circ \tau_i = \sum_{i=0}^{n} p_i \chi_{\tau_i^{-1}}$). Then

$$g(\tau_i)(x) = \begin{cases} 1, & \text{if } x \in \tau_i^{-1}(A) \\ 0, & \text{if } x \notin \tau_i^{-1}(A) \end{cases},$$

so that $g(\tau_i)$ is the characteristic function of $\tau_i^{-1}(A)$, a set in $\mathfrak{B}$. Thus, by 3.1.2

$$\begin{aligned}\int_X gdv &= v(A) = \mu(T^{-1}(A)) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(A)) \\ &= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k .1 d\mu = \int_X \sum_{k=1}^{K} p_k \chi_A(\tau_k(x)) d\mu \\ &= \sum_{k=1}^{K} p_k \int_X g(\tau_k(x)) d\mu = \int_X g(T(x)) d\mu.\end{aligned}$$

The proof is completed. □

Definition 3.1.5. μ is called $T-$invariant measure if and only if it satisfies the following condition:

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k d\mu(x) = \sum_{i=0}^{K} p_i \mu(\tau_i^{-1}(A)). \tag{3.1.6}$$

Lemma 3.1.7. *Let X be a compact metric space and let $(X, \mathfrak{B}, \mu)$ be a measure space with normalized measure μ and T be a random map on X. Then μ is $T-$invariant measure if and only if for any function $g \in C(X)$*

$$\int g d\mu = \int g \circ T d\mu, \tag{3.1.8}$$

where $g \circ T = \sum_{i=0}^{n} p_i g \circ \tau_i$.

Proof. Assume (3.1.8) holds. For the measurable function g, by Theorem 3.1.3 we have $\int_X g \circ T d\mu = \int_X g dv$. By Theorem 2.3.16, we infer from 3.1.8 that $\mu(T^{-1}(A)) = \mu(A)$ for every measurable set A.

Conversely, assume μ is $T-$invariant measure. X is a compact metric space, then the adjoint space of $C(X)$ is equal to $\mathcal{M}(X)$ (Theorem 2.3.16). First we have to prove that 3.1.8 holds when $f = \chi_A$ for the characteristic function .

$$\begin{aligned}
\int f \circ T d\mu &= \int \chi_{{}_A} \circ T d\mu = \int_A 1 \circ T d\mu \\
&= \sum_{k=1}^{n} p_k \int_A 1 \circ \tau_k d\mu = \sum_{k=1}^{n} p_k \int_{\tau_k^{-1}(A)} 1 d\mu \\
&= \sum_{k=1}^{n} p_k \mu(\tau_k^{-1}(A)) = \mu(A) = \int \chi_{{}_A} d\mu = \int_A f d\mu.
\end{aligned}$$

For a simple function $f = \Sigma_{k=1}^{n} c_k \chi_{A_k}$, since μ is $T-$invariant measure,

$$\int f \circ T d\mu = \int \left[\sum_{k=1}^{n} c_k . \chi_{A_k} \circ T \right] d\mu = \int \left[\sum_{k=1}^{n} c_k p_k . \chi_{\tau_k^{-1}(A_k)} \right] d\mu = \sum_{k=1}^{n} c_k . \mu(A_k) = \int f d\mu.$$

Thus,

$$\int f d\mu = \int f \circ T d\mu, \tag{3.1.9}$$

Therefore (3.1.8) holds for f simple. According to the the Simple Approximation Theorem (Theorem 2.3.24), there is an increasing sequence $\{f_n\}$ of simple functions on X that converge pointwise on X to the any continuous function g. Hence $\{f_n \circ T\}$ is an increasing sequence of simple functions on X that converge pointwise on X to $g \circ T$. By using the Monotone Convergence Theorem two times and applying the equality of (3.1.9) for simple functions, we have

$$\int g \circ T d\mu = \lim_{n \to \infty} \int f_n \circ T d\mu = \lim_{n \to \infty} \int f_n d\mu = \int g d\mu.$$

□

3.2 The Generalization of Krylov-Bogoliubov Theorem for random maps

Next, we generalize Krylov-Bogoliubov Theorem to random maps. An analogous result, for continuous time Random Dynamical Systems, is proven in Theorem 1.5.8. of Arnold's book [3]. For simplicity we consider $k = 2$.

Theorem 3.2.1. *(**The Generalization of Krylov-Bogoliubov Theorem**) Let X be a compact metric space and let $(X, \mathfrak{B}, \lambda)$ be a measure space with normalized measure λ and let $\tau_i : X \longrightarrow X$, $i = 1, 2$ be continuous transformations. Consider the random map $T = \{\tau_1, \tau_2, p_1, p_2\}$ with constant probabilities p_1, p_2. Then there exists a $T-$invariant normalized measure μ.*

Proof. Let ν in $\mathcal{M}(X)$ be a normalized measure (for example we can take a Dirac measure). Define the sequence $\mu_n \in \mathcal{M}(X)$ by

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} T_*^j \nu$$

Then for $B \in \mathfrak{B}$

$$\begin{aligned}
\mu_n(B) &= \frac{1}{n} \left(\nu + T_* \nu + ... + T_*^{n-1} \nu\right)(B) \\
&= \frac{1}{n} \left(\nu(B) + \nu(T^{-1}(B)) + ... + \nu(T^{-(n-1)}(B))\right) \\
&= \frac{1}{n} \left(\nu(B) + \sum_{k_1=1}^{2} p_{k_1} \nu(\tau_{k_1}^{-1}(B)) + ... + \sum_{k_1,k_2,..,k_n} p_{k_1} p_{k_2} ... p_{k_n} \nu(\tau_{k_1}^{-1} \tau_{k_2}^{-1} ... \tau_{k_n}^{-1}(B))\right)
\end{aligned}$$

where $\prod_{i=1}^{n-1} p_{k_i}$ is the probability of $T^{n-1}(x)$. For example,

$$\sum_{k_1,k_2} p_{k_1} p_{k_2} \nu(\tau_{k_1}^{-1} \tau_{k_2}^{-1}(B)) = p_1^2 \nu(\tau_1^{-2}(B)) + p_1 p_2 \nu(\tau_1^{-1} \tau_2^{-1}(B)) + p_1 p_2 \nu(\tau_2^{-1} \tau_1^{-1}(B)) + p_2^2 \nu(\tau_2^{-2}(B))$$

Since $\mathcal{M}(X)$ is weak* compact, some subsequence μ_{n_k} converges, as $k \longrightarrow \infty$, to a measure $\mu \in \mathcal{M}(X)$, We shall show that μ is $T-$invariant, and for that it is enough to show that for any function $f \in C(X)$.

$$\int f d\mu = \int f \circ T d\mu. \tag{3.2.2}$$

We have,

$$\begin{aligned}
|\mu(f) - \mu(f \circ T)| &= \lim_{n_k \longrightarrow +\infty} |\mu_{n_k}(f) - \mu_{n_k}(f \circ T)| \\
&= \lim_{n_k \longrightarrow +\infty} \left| \frac{1}{n_k} \left(\nu + ... + T_*^{n_k - 1}\nu\right)(f) - \frac{1}{n_k} \left(T_*\nu + ... + T_*^{n_k}\nu\right)(f) \right| \\
&= \lim_{n_k \longrightarrow +\infty} \frac{1}{n_k} |\nu(f) - T_*^{n_k}\nu(f)| \\
&\leq \lim_{n_k \longrightarrow +\infty} \frac{2 \sup |f|}{n_k} = 0.
\end{aligned}$$

Therefore, μ is $T-$invariant, as claimed. □

Chapter 4

Existence of an absolutely continuous invariant measure for nonautonomous random maps

We present results on the existence of invariant measures for nonautonomous random dynamical systems. We prove the existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a, b]$ using the theory of bounded variation.

In Section 4.1, we give the definitions and introduce the notation to the nonautonomous random dynamical systems. In Section 4.2, we prove the generalization of the Krylov-Bogoliubov Theorem to the nonautonomous random dynamical systems. In Section 4.3, we prove the existence of an absolutely continuous invariant measures for the limit random map T. In Section 4.4, we introduce the properties of Frobenius-Perron operator for nonautonomous random maps. Finally, in Section 4.5, we prove a form of the Lasota-Yorke inequality and we prove the existence of invariant BV densities for nonautonomous random maps.

4.1 Definitions and notations for nonautonomous random dynamical systems.

Let X be a compact metric space and let $(X, \mathfrak{B}, \lambda)$ be a measure space, $\mathfrak{B}$ is a σ-algebra of subsets of X with normalized measure λ. Let $\tau_{1(n)}, \tau_{2(n)}, ..., \tau_{k(n)}$ be a k sequences of a transformations with a continuous limits $\tau_1, \tau_2, ..., \tau_k$ respectively. Consider the random maps

$$T_n = \left\{\tau_{1(n)}, \tau_{2(n)}, ..., \tau_{k(n)}; p_1, p_2, ..., p_k\right\}$$

which converge uniformly to a continuous random map

$$T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\},$$

with constant probabilities $p_1, p_2, ..., p_k$, $p_i > 0$, $\sum_{i=1}^{k} p_i = 1$.

Definition 4.1.1. The nonautonomous random dynamical systems on the metric compact space X is defined by:

$$x_{m+1} = T_m(x_m), m = 0, 1, 2, ...$$

where $x_0 \in X$ and $T_0 = \left\{\tau_{1(0)}, \tau_{2(0)}, ..., \tau_{k(0)}; p_1, p_2, ..., p_k\right\}$, where $\tau_{i(0)}$ are the identity transformations for all i. This generates the discrete time process $\widehat{T}_m^n$, which is defined for all $x \in X$

$$\widehat{T}_m^n = T_n \circ T_{n-1} \circ ... \circ T_{m+1} \circ T_m, m \leq n.$$

In particular,

$$\widehat{T}_0^n = T_n \circ T_{n-1} \circ ... \circ T_1 \circ T_0.$$

4.2 Existence of invariant measures for continuous nonautonomous random maps.

In [23] Góra, Boyarsky and Keefe proved the generalization of the Krylov-Bogoliubov's Theorem to the nonautonomous setting : every limit point of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1}(\widehat{\tau}_0^i)_*\nu,$$

is a τ-invariant measure for every probability measure ν. We generalize this result to random maps.

Theorem 4.2.1. *Let $\widehat{T}_0^n$ be as defined as in Section 4.1. Let ν be normalized measure on X. Define the measures $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}(\widehat{T}_0^i)_*\nu$. Let μ be a $*-$weak limit point of the sequence $\{\mu_n\}_{n\geq 1}$. Then μ is a $T-$invariant normalized measure.*

Proof. We follow the proof of original Krylov-Bogoliubov theorem. We use $k = 2$. We have the random maps

$$T_n = \left\{\tau_{1(n)}, \tau_{2(n)}; p_1, p_2\right\}, n = 1, 2, ...$$

Which converge uniformly to the continuous random map

$$T = \{\tau_1, \tau_2; p_1, p_2\},$$

with constant probabilities p_1, p_2.

For $B \in \mathfrak{B}$, we have

$$\begin{aligned}\mu_n(B) &= \frac{1}{n}\Big(\nu(B) + (\widehat{T}_0^1)_*\nu(B) + ... + (\widehat{T}_0^{n-1})_*\nu(B)\Big)\\ &= \frac{1}{n}\left(\nu(B) + (T_1 \circ T_0)_*\nu(B) + ... + (T_{n-1} \circ ... \circ T_1 \circ T_0)_*\nu(B)\right)\\ &= \frac{1}{n}(\nu(B) + \sum_{j_1=1, j_2=1}^{2} p_{j_1}p_{j_2}\nu(\tau_{j_1(0)}^{-1} \circ \tau_{j_2(1)}^{-1}(B)) + ...\\ &+ \sum_{j_1=1, j_2=1, j_3=1, ..., j_{n-1}=1}^{2} p_{j_1}p_{j_2}p_{j_3}...p_{j_{n-1}}\nu(\tau_{j_1(0)}^{-1} \circ \tau_{j_2(1)}^{-1} \circ \tau_{j_3(2)}^{-1} \circ ... \circ \tau_{j_n(n-1)}^{-1}(B))\end{aligned}$$

By assumption, μ is a $*$-weak limit point of the sequence $\{\mu_n\}_{n\geq 1}$, some subsequence μ_{n_j} converges, as $j \longrightarrow \infty$, to a measure μ. We shall show that μ is T- invariant, and for that it is enough to show that for any function $g \in C(X)$, $\mu(g) = \mu(g \circ T)$.

We have,

$$
\begin{aligned}
|\mu(g) - \mu(g \circ T)| &= \lim \left|\mu_{n_j}(g) - \mu_{n_j}(g \circ T)\right| \\
&= \lim \left| \frac{1}{n_j} \sum_{i=0}^{n_j-1} \left((\widehat{T}_0^i)_* \nu\right)(g) - \frac{1}{n_j} \sum_{i=0}^{n_j-1} \left((\widehat{T}_0^i)_* \nu\right)(g \circ T) \right| \\
&= \lim \frac{1}{n_j} \left| \left(\nu(g \circ \widehat{T}_0^1) + ... + \nu(g \circ \widehat{T}_0^{n_j-1})\right) - \left(\nu(g \circ T \circ \widehat{T}_0^1) + ... + \nu(g \circ T \circ \widehat{T}_0^{n_j-1})\right) \right| \\
&= \lim \frac{1}{n_j} \left| \left(\nu(g \circ \widehat{T}_0^1) + \sum_{i=1}^{n_j-1} \left(\nu(g \circ \widehat{T}_0^i) - \nu(g \circ T \circ \widehat{T}_0^{i-1})\right) - \nu(g \circ T \circ \widehat{T}_0^{n_j-1}) \right) \right| \\
&= \lim \frac{1}{n_j} \left| \left(\nu(g \circ \widehat{T}_0^1) + \sum_{i=1}^{n_j-1} \left(\nu(g \circ T_i \circ \widehat{T}_0^{i-1}) - \nu(g \circ T \circ \widehat{T}_0^{i-1})\right) - \nu(g \circ T \circ \widehat{T}_0^{n_j-1}) \right) \right| \\
&= \lim \frac{1}{n_j} \left| \left(\nu(g \circ \widehat{T}_0^1) + \sum_{i=1}^{n_j-1} \left(\nu\left((g \circ T_i - g \circ T)(\widehat{T}_0^{i-1})\right)\right) - \nu(g \circ T \circ \widehat{T}_0^{n_j-1}) \right) \right|.
\end{aligned}
$$

We have,

$$
\begin{aligned}
\nu\left((g \circ T_i - g \circ T)(\widehat{T}_0^{i-1})\right) &\leq \sup|g \circ T_i - g \circ T| \leq \omega_g\left(\sup|T_i - T|\right) \\
&\leq \omega_g \sum_{j=1}^{2} p_j \sup|\tau_{j(i)} - \tau_j|,
\end{aligned}
$$

where ω_g is the modulus of continuity of g,

$$\omega_g = \sup_{\rho(x,y)<\delta} |g(x) - g(y)|.$$

By assumption, $\tau_{i(n)} \to \tau_i$ uniformly so we can find $N > 1$, such that for an arbitrary $\epsilon > 0$ there exist $\delta > 0$ when $\omega_g(\delta) < \epsilon$ and $\sup \rho(\tau_{j(i)} - \tau_j) < \delta$ for all $i > N$.

Therefore, for $n_j > N$, we have

$$|\mu(g) - \mu(g \circ T)| \leq \lim \frac{1}{n_j}((2N+2) \cdot \sup|g| + (n_j - N)\epsilon).$$

Thus, μ is $T-$invariant, as claimed. $\square$

4.3 Existence of an absolutely continuous invariant measure for the limit map.

The next definition is for Frobenius-Perron operator with respect to T.

Definition 4.3.1. Let $T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\}$ be a random map constructed from nonsingular transformations $\{\tau_i\}_{i=1}^k$ on I. For all $f \in L^1(I)$

$$P_T f = \sum_{i=1}^{k} p_i \cdot P_{\tau_i} f, \tag{4.3.2}$$

where Frobenius-Perron operator P_τ can be expressed by

$$P_\tau f(x) = \sum_{w \in \tau^{-1}(x)} \frac{f(w)}{|\tau'(w)|}. \tag{4.3.3}$$

In case $\tau = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1$ the Frobenius-Perron operator P_τ can be expressed by

$$P_\tau f(x) = \sum_{w \in \tau^{-1}(x)} \frac{f(w)}{|D_\tau(w)|}, \tag{4.3.4}$$

where D_τ is the first derivative of the composite function τ.

Theorem 4.3.5. *[50] Let $T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\}$ be a random map and P_T be its Frobenius-Perron operator. For every density f^*, $P_T f^* = f^*$, a.e., if and only if $\mu = f^* \lambda$ is $T-$invariant.*

We going to prove the following theorems.

Theorem 4.3.6. *Let T_n and T be as defined as in Section 4.1 and let $T_n \to T$ uniformly, f_n is the invariant density associated with T_n and $f_n \to f$ weakly in L^1. Then $P_T f = f$.*

Proof. To prove that $P_T f = f$, it is sufficient to prove that for any continuous function g on X,

$$|\int g(f - P_T f) d\lambda| = 0.$$

We have

$$
\begin{aligned}
|\int g(f-P_T f)d\lambda| \leq |\int g(f-f_n)d\lambda| + |\int g(f_n - P_{T_n} f_n)d\lambda| \\
+ |\int g(P_{T_n} f_n - P_T f_n)d\lambda| + |\int g(P_T f_n - P_T f)d\lambda|
\end{aligned}
\tag{4.3.7}
$$

Since $f_n \to f$ weakly in L^1 the first summand tends to 0. The second is equal to 0 since $P_{T_n} f_n = f_n$.

Let us set

$$I_1 = |\int g(P_{T_n} f_n - P_T f_n)d\lambda|$$

$$I_2 = |\int g(P_T f_n - P_T f)d\lambda|$$

We have,

$$
\begin{aligned}
I_1 &= |\int (g \circ T_n - g \circ T) f_n d\lambda| \\
&= |\int (\sum_i p_i (g \circ \tau_{i(n)}) - \sum_i p_i (g \circ \tau_i)) f_n d\lambda| \\
&\leq \sum_i (p_i \sup |g \circ \tau_{i(n)} - g \circ \tau_i|) \int |f_n| d\lambda \\
&\leq \sum_i \left(p_i \omega_g \left(\sup |\tau_{i(n)} - \tau_i| \right) \right) \int |f_n| d\lambda \to 0,
\end{aligned}
\tag{4.3.8}
$$

we reached this through the assumption, for every i, $\tau_{i(n)} \to \tau_i$ uniformly, $\int |f_n| d\lambda$ are uniformly bounded and ω_g is the modulus of continuity of g.

$$I_2 \leq \int (g \circ T)|f_n - f| d\lambda \to 0 \tag{4.3.9}$$

Since $f_n \to f$ weakly and $g \circ T$ is bounded. Hence, $P_T f = f$. □

4.4 Properties of the Frobenius-Perron operator with respect to $\widehat{T}_0^n$.

The Frobenius-Perron operator $P_{\widehat{T}_0^n}$ for the nonautonomous random map $\widehat{T}_0^n$ is coming from random maps composition property, and it is given by

$$P_{\widehat{T}_0^n} f = P_{T_n \circ T_{n-1} \circ \ldots \circ T_1 \circ T_0} f = P_{T_n} \circ P_{T_{n-1}} \circ \ldots \circ P_{T_1} \circ P_{T_0} f, \tag{4.4.1}$$

where P_{T_k} is the Frobenius-Perron operator of the random map T_k, $k \in \{0, 1, 2, ..., n\}$.

The properties of $P_{\widehat{T}_0^n}$ resemble the properties of the classical Frobenius-Perron operator of a single transformation see [11].

Proposition 4.4.2. *Let α, β be constant. Then if $f, g \in L^1([0,1])$. $P_{\widehat{T}_0^n} : L^1([0,1]) \to L^1([0,1])$, $P_{\widehat{T}_0^n}$ has the following properties:*

(1) (Linearity) $P_{\widehat{T}_0^n}(\alpha f + \beta g) = \alpha P_{\widehat{T}_0^n} f + \beta P_{\widehat{T}_0^n} g$.

(2) (Positivity) If $f > 0$ then $P_{\widehat{T}_0^n} f > 0$.

(3) (Preservation of Integrals) $\int_{[0,1]} P_{\widehat{T}_0^n} f d\lambda = \int_{[0,1]} f d\lambda$.

(4) (Contraction property) $\|P_{\widehat{T}_0^n} f\|_1 \le \|f\|_1$.

(5) (Composition property) If $\widehat{T}_0^n, \widehat{R}_0^n : L^1([0,1]) \to L^1([0,1])$ are two nonautonomous random maps, then $P_{\widehat{T}_0^n \circ \widehat{R}_0^n} f = P_{\widehat{T}_0^n} \circ P_{\widehat{R}_0^n} f$. In particular, for any $m \ge 1$, $P_{(\widehat{T}_0^n)^m} f = P^m_{\widehat{T}_0^n} f$, where

$$P^m_{\widehat{T}_0^n} f = P_{\widehat{T}_0^n} \circ P_{\widehat{T}_0^n} \circ \ldots \circ P_{\widehat{T}_0^n}, \text{ m times.} \tag{4.4.3}$$

Proof. It is enough to prove this properties for $\widehat{T}_{n-1}^n = T_n \circ T_{n-1}$, where $T_n = \left\{\tau_{1(n)}, \tau_{2(n)}; p_1, p_2\right\}$. Using the properties of the Frobenius-Perron operator with respect to τ and T.

(1) (Linearity) Let $F = \alpha f + \beta g$. Then,

$$\begin{aligned}
P_{\widehat{T}_{n-1}^n}(F) &= P_{T_n \circ T_{n-1}}(F) = P_{T_n} \circ P_{T_{n-1}}(F) \\
&= p_1 P_{\tau_{1(n)}} \left(P_{T_{n-1}}(F)\right) + p_2 P_{\tau_{2(n)}} \left(P_{T_{n-1}}(F)\right) \\
&= p_1^2 P_{\tau_{1(n)}} \circ P_{\tau_{1(n-1)}}(F) + p_1 p_2 P_{\tau_{1(n)}} \circ P_{\tau_{2(n-1)}}(F) \\
&+ p_1 p_2 P_{\tau_{2(n)}} \circ P_{\tau_{1(n-1)}}(F) + p_2^2 P_{\tau_{2(n)}} \circ P_{\tau_{2(n-1)}}(F).
\end{aligned}$$

Then by using the linearity of P_τ two times, we have

$$\begin{aligned}
P_{\widehat{T}^n_{n-1}}(\alpha f + \beta g) &= \alpha p_1^2 P_{\tau_{1(n)}} \circ P_{\tau_{1(n-1)}}(f) + \beta p_1^2 P_{\tau_{1(n)}} \circ P_{\tau_{1(n-1)}}(g) \\
&+ \alpha p_1 p_2 P_{\tau_{1(n)}} \circ P_{\tau_{2(n-1)}}(f) + \beta p_1 p_2 P_{\tau_{1(n)}} \circ P_{\tau_{2(n-1)}}(g) \\
&+ \alpha p_1 p_2 P_{\tau_{2(n)}} \circ P_{\tau_{1(n-1)}}(f) + \beta p_1 p_2 P_{\tau_{2(n)}} \circ P_{\tau_{1(n-1)}}(g) \\
&+ \alpha p_2^2 P_{\tau_{2(n)}} \circ P_{\tau_{2(n-1)}}(f) + \beta p_2^2 P_{\tau_{2(n)}} \circ P_{\tau_{2(n-1)}}(g) \\
&= \alpha P_{\widehat{T}^n_{n-1}} f + \beta P_{\widehat{T}^n_{n-1}} g.
\end{aligned}$$

(2) (Positivity) Let $f > 0$ by using the positivity of P_τ two times, then we have

$$\begin{aligned}
P_{\widehat{T}^n_{n-1}}(f) &= p_1^2 P_{\tau_{1(n)}} \circ P_{\tau_{1(n-1)}}(f) + p_1 p_2 P_{\tau_{1(n)}} \circ P_{\tau_{2(n-1)}}(f) \\
&+ p_1 p_2 P_{\tau_{2(n)}} \circ P_{\tau_{1(n-1)}}(f) + p_2^2 P_{\tau_{2(n)}} \circ P_{\tau_{2(n-1)}}(f) \\
&> 0.
\end{aligned}$$

(3) (Preservation of Integrals) By using the preservation of integrals of P_τ two times, then we have

$$\begin{aligned}
\int_{[0,1]} P_{\widehat{T}^n_{n-1}} f d\lambda &= \int_{[0,1]} p_1^2 P_{\tau_{1(n)} \circ \tau_{1(n-1)}}(f) + p_1 p_2 P_{\tau_{1(n)} \circ \tau_{2(n-1)}}(f) \\
&+ p_1 p_2 P_{\tau_{2(n)} \circ \tau_{1(n-1)}}(f) + p_2^2 P_{\tau_{2(n)} \circ \tau_{2(n-1)}}(f) d\lambda \\
&= \int_{\tau^{-1}_{1(n-1)}([0,1])} p_1^2 P_{\tau_{1(n)}}(f) d\lambda + \int_{\tau^{-1}_{2(n-1)}([0,1])} p_1 p_2 P_{\tau_{1(n)}}(f) d\lambda \\
&+ \int_{\tau^{-1}_{1(n-1)}([0,1])} p_1 p_2 P_{\tau_{2(n)}}(f) d\lambda + \int_{\tau^{-1}_{2(n-1)}([0,1])} p_2^2 P_{\tau_{2(n)}}(f) d\lambda \\
&= \int_{[0,1]} p_1^2 P_{\tau_{1(n)}}(f) d\lambda + \int_{[0,1]} p_1 p_2 P_{\tau_{1(n)}}(f) d\lambda \\
&+ \int_{[0,1]} p_1 p_2 P_{\tau_{2(n)}}(f) d\lambda + \int_{[0,1]} p_2^2 P_{\tau_{2(n)}}(f) d\lambda \\
&= p_1^2 \int_{\tau^{-1}_{1(n)}([0,1])} f d\lambda + p_1 p_2 \int_{\tau^{-1}_{1(n)}([0,1])} f d\lambda \\
&+ p_1 p_2 \int_{\tau^{-1}_{2(n)}([0,1])} f d\lambda + p_2^2 \int_{\tau^{-1}_{2(n)}([0,1])} f d\lambda \\
&= (p_1 + p_2)^2 \int_{[0,1]} f d\lambda = \int_{[0,1]} f d\lambda
\end{aligned}$$

(4) (Contraction property) Let $f \in L^1([0,1])$. Let $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$. Then $f^+, f^- \in L^1([0,1])$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. By the linearity of $P_{\widehat{T}^n_{n-1}}$ we have

$$P_{\widehat{T}^n_{n-1}} f = P_{\widehat{T}^n_{n-1}}(f^+ - f^-) = P_{\widehat{T}^n_{n-1}} f^+ - P_{\widehat{T}^n_{n-1}} f^-$$

and

$$\|P_{\widehat{T}^n_{n-1}} f\|_1 = \int_{[0,1]} \left|P_{\widehat{T}^n_{n-1}} f\right| d\lambda \leq \int_{[0,1]} P_{\widehat{T}^n_{n-1}} |f|\, d\lambda = \int_{[0,1]} |f|\, d\lambda = \|f\|_1.$$

(5) (Composition property) Let $\widehat{T}^n_{n-1}, \widehat{R}^n_{n-1} : L^1([0,1]) \to L^1([0,1])$ are two nonautonomous random maps, then by using the composition property of P_T two times we have

$$P_{\widehat{T}^n_{n-1} \circ \widehat{R}^n_{n-1}} f = P_{T_n \circ T_{n-1} \circ R_n \circ R_{n-1}} f = P_{T_n} \circ P_{T_{n-1}} \circ P_{R_n} \circ P_{R_{n-1}} f = P_{\widehat{T}^n_{n-1}} \circ P_{\widehat{R}^n_{n-1}} f.$$

□

4.5 Existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a,b]$.

In this section we present results on the existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a,b]$ using the theory of functions of bounded variation. The original result for maps is due to Lasota and Yorke 1973 [36], Since then, the bounded variation proof has been generalized in a number of directions [29, 4, 27, 23].

Let $I = [a,b]$ and (I, B, λ) be a measure space, where λ is normalized Lebesgue measure on I. Let $\widehat{T}^n_0$ be as defined as in Section 4.1. $\widehat{T}^n_0$ is a nonautonomous random maps for the random maps

$$T_n = \left\{\tau_{1(n)}, \tau_{2(n)}, \ldots, \tau_{k(n)}; p_1, p_2, \ldots, p_k\right\},$$

where $\tau_{1(n)}, \tau_{2(n)}, \ldots, \tau_{k(n)}$ are k sequences of a transformations of I into I. For all n they are piecewise one-to-one and differentiable, nonsingular transformations on a partition $\mathcal{P}$ of I, $\mathcal{P} =$

$\{I_1, I_2, ..., I_m\}$. Denote by $V(\cdot)$ the standard one dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on I equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_1$.

The following theorems come from [11].

Theorem 4.5.1. *Let f and g be of bounded variation on $I = [a, b]$. Then so are their sum, difference and product. Also, we have*

$$V_I(f \pm g) \leq V_I f + V_I g,$$

$$V_I(f \cdot g) \leq AV_I f + BV_I g,$$

where $A = \sup\{|g(x)| : x \in I\}, B = \sup\{|f(x)| : x \in I\}$.

Theorem 4.5.2. *Let $f : [a, b] \longrightarrow R$ have a continuous derivative f' on $[a, b]$. Then*

$$V_{[a,b]} f = \int_a^b |f'(x)| d\lambda \tag{4.5.3}$$

Theorem 4.5.4. *Let $A, B \subset I$ and $\lambda(A \cap B) = 0$. If $f \in BV(A \cup B)$, then,*

$$V_{A \cup B}(f) \geq V_A(f|_A) + V_B(f|_B).$$

Theorem 4.5.5. *(Yorke's Inequality) Let $f : I \longrightarrow R$ be of bounded variation. Let $A \subset I$ and let χ_A be the characteristic function of the interval A. Assume $l = l(A) > 0$. Then $f\chi_A$ is of bounded variation and*

$$V_I(f\chi_A) \leq 2V_A(f|_A) + \frac{2}{l} \int_A |f| d\lambda \tag{4.5.6}$$

Theorem 4.5.7. *(Helly's First Theorem) Let an infinite family of functions $F = \{f\}$ be defined on all interval $[a, b]$. If all functions of the family and the total variation of all functions of the family are bounded by a single number, $i.e.$,*

$$|f(x)| \leq K, V_{[a,b]} f \leq K, \text{for all } f \in F,$$

then there exists a sequence $\{f_n\} \subset F$ that converges at every point of $[a, b]$ to some function f^* of bounded variation, and $V_{[a,b]} f^* \leq K$.

Let $g_{i(n)}(x) = \frac{p_i}{|\tau'_{i(n)}|}$, $i = 1, ..., k$. We assume the following conditions for all n

C1 (Average expanding condition) $\sum_i g_{i(n)}(x) < \alpha < 1, x \in I$.

C2 $g_{i(n)} \in BV([0,1]), i = 1, 2, ..., k$.

We going to prove the following theorem.

Theorem 4.5.8. *Let $f \in L^1(I)$ and $\tau_i : I \longrightarrow I$ be a piecewise C^2 transformations on a partition $\mathcal{P}$ of I. Suppose $I_i \in \mathcal{P}$. For $m \geq 1$ let us define*

$$T = \tau_m \circ \tau_{m-1} \circ \cdots \circ \tau_2 \circ \tau_1, \tag{4.5.9}$$

$T_i = T|_{I_i}$ and $D_T(x) = \tau'_m(\tau_{m-1} \circ \cdots \circ \tau_1) \cdot \tau'_{m-1}(\tau_{m-2} \circ \cdots \circ \tau_1) \cdot \ldots \cdot \tau'_2(\tau_1) \cdot \tau'_1(x)$. If $f|_{I_i} \in BV(I_i)$. Then

$$V_{T_i(I_i)}\left(\left(f \cdot |D_{T_i}^{-1}|\right) \circ T_i^{-1}(x)\right) \leq \Gamma \cdot V_{I_i}(f|_{I_i}) + \Lambda \int_{I_i} |f| d\lambda, \tag{4.5.10}$$

where $\Gamma = \sup(D_{T_i}^{-1})$ and $\Lambda = \dfrac{\max |(D_{T_i}^{-1})'|}{\min |(D_{T_i}^{-1})|}$.

Proof. We have

$$\begin{aligned}
V_{T_i(I_i)}\left(f \cdot |D_{T_i}^{-1}| \circ T_i^{-1}\right) &= \int_{T_i(I_i)} \left|d\left(f \cdot |D_{T_i}^{-1}| \circ T_i^{-1}\right)\right|, \text{ by Theorem 4.5.2,} \\
&= \int_{I_i} \left|d\left(f \cdot |D_{T_i}^{-1}|\right)\right|, \text{ using the standard change of variables,} \\
&\leq \int_{I_i} |f' \cdot |D_{T_i}^{-1}|| d\lambda + \int_{I_i} |f \cdot |D_{T_i}^{-1}|'| d\lambda \\
&\leq \Gamma \cdot V_{I_i}(f|_{I_i}) + \Lambda \int_{I_i} |f| d\lambda
\end{aligned} \tag{4.5.11}$$

where $\Gamma = \sup(D_{T_i}^{-1})$ and $\Lambda = \dfrac{\max |(D_{T_i}^{-1})'|}{\min |(D_{T_i}^{-1})|}$. ∎

In [42], Pelikan proved a form of the Lasota-Yorke inequality for Random maps.

Lemma 4.5.12. *Let $T = \{\tau_1, \tau_2, ..., \tau_k; p_1, p_2, ..., p_k\}$ be a random map, where τ_i are piecewise monotonic transformations that satisfy the $C1$ and $C2$ conditions, then, for any $f \in BV(I)$,*

$$V_I(P_T(f(x))) \leq AV_I(f) + B\|f\|_1, \tag{4.5.13}$$

where

$$A = 3\left(\max_k \sum_{i=1}^{m} \frac{p_i}{\inf|\tau_i'(x)|}\right),$$

and

$$B = \sum_{i,k} \frac{\sup|\sigma_i^{j\prime}|}{\inf \sigma_i^j} + \sup \sigma_i^j,$$

with $\sigma_i^j = |\frac{d}{dx}\psi_i^j|$, $\psi_i^j = \left(\tau_i|_{I_j}\right)^{-1}$.

Theorem 4.5.14. *Let T_n be a sequence of random maps as defined as in Section 4.1, for the random map which is constructed from piecewise C^2 transformations that satisfying the Condition 4.5.13, with common constants $0 < A < 1$ and $B < \infty$. Then, for any density $f \in BV(I)$, the sequence $f_n = \dfrac{1}{n}\sum_{i=1}^{n} P_{\widehat{T}_1^i} f$ forms a pre-compact set in L^1 and any convergent subsequence converges to a density of an acim of the limit map T.*

Lemma 4.5.15. *Under the assumptions of Lemma 4.5.12, for any density $f \in BV(I)$*

$$V_I\left(P_{\widehat{T}_1^n}(f)\right) \le A^n V_I(f) + \frac{B}{1-A}, \tag{4.5.16}$$

where $0 < A < 1$, and $B > 0$.

Proof. The Frobenius–Perron operator $P_{\widehat{T}_1^n}$ preserves the integral of positive functions and since f is a density function, we have $\int |P_{\widehat{T}_1^n} f| d\lambda = \|f\|_1 = 1$ for all $n \ge 1$. Since $P_{\widehat{T}_1^n} f = P_{T_n} \circ P_{T_{n-1}} \circ \ldots \circ P_{T_1} f$, by applying the Lasota-Yorke inequality for Random maps 4.5.13, we have

$$V_I\left(P_{\widehat{T}_1^n}(f)\right) \le A^n V_I(f) + B\sum_{i=1}^{n} A^{i-1} \le A^n V_I(f) + \frac{B}{1-A}, n \ge 1.$$

□

Lemma 4.5.17. *Let T_n and T be as defined as in Section 4.1 and let $T_n \to T$ uniformly. Under the assumptions of Theorem 4.5.14, for any density function f, $P_{T_n} f \to P_T f$ weakly in L^1, as $n \to \infty$.*

Proof. We will prove that for any density function f, $P_{T_n} f \to P_T f$ weakly in L^1, as $n \to \infty$. Let $g \in L^\infty(I, \lambda)$ be an arbitrary bounded function, fix an $\epsilon > 0$. By Lusin's 2.3.19 for any $\delta > 0$ there exists an open set $U \subset I$, $\lambda(U) < \delta$, and a continuous function $G \in C^0(I)$ such that $g = G$

on $I \setminus U$ and $\sup|G| \leq \|g\|_\infty$. The Frobenius–Perron operator is a conjugate of the Koopman operator, that is for any $f \in L^1$ and any $g \in L^\infty$, we have $\int_I P_\tau f g d\lambda = \int_I f g \circ \tau d\lambda$. Therefore, we can write

$$
\begin{aligned}
|\int_I (P_{T_n} fg - P_T fg)\, d\lambda| &= |\int_I \left(\sum_{i=1}^{k} p_i \cdot P_{\tau_{i(n)}} fg - \sum_{i=1}^{k} p_i \cdot P_{\tau_i} fg \right) d\lambda| \\
&\leq \sum_{i=1}^{k} p_i |\int_I \left(P_{\tau_{i(n)}} f \cdot g - P_{\tau_i} f \cdot g \right) d\lambda| \\
&\leq \sum_{i=1}^{k} p_i \int_I f |g \circ \tau_{i(n)} - g \circ \tau_i| d\lambda \\
&= \sum_{i=1}^{k} p_i \int_I f |g \circ \tau_{i(n)} - G \circ \tau_{i(n)} + G \circ \tau_{i(n)} - G \circ \tau_i + G \circ \tau_i - g \circ \tau_i| d\lambda \\
&\leq \sum_{i=1}^{k} p_i (\int_I f |g \circ \tau_{i(n)} - G \circ \tau_{i(n)}| d\lambda + \int_I f |G \circ \tau_{i(n)} - G \circ \tau_i| d\lambda \\
&+ \int_I f |G \circ \tau_i - g \circ \tau_i| d\lambda)
\end{aligned}
$$

Let $\sup|G| \leq \|g\|_\infty = M_g$. Let $I_f(t) = \sup_{A:\lambda(A)<t} \int_A |f| d\lambda$. It is known that $I_f(t) \to 0$ as $t \to 0$. Let ω_G be the modulus of continuity of G *i.e.* $\omega_G(t) = \sup_{|x-y|\leq t} |G(x) - G(y)|$. We have, $\omega_G(t) \to 0$ as $t \to 0$. By Lemma (5) in [23] there exists a constant K such that for any interval J we have $\lambda(\tau_i^{-1}(J)) \leq K\lambda(J)$. Then we have

$$
\begin{aligned}
|\int_I (P_{T_n} fg - P_T fg)\, d\lambda| &\leq 2M_g I_f(K\delta) + \omega_G(\sup|\tau_{i(n)} - \tau_i|) + 2M_g I_f(K\delta) \\
&= \omega_G(\sup\|\tau_{i(n)} - \tau_i\|_\infty) + 4M_g I_f(K\delta)
\end{aligned} \tag{4.5.18}
$$

Let us fix an $\epsilon > 0$. Since $\|\tau_{i(n)} - \tau_i\|_\infty \to 0$, as $n \to 0$ we can find $N \geq 1$ such that for all $n \geq N$ we have $\omega_G(\sup\|\tau_{i(n)} - \tau_i\|_\infty) < \epsilon$. We can also find an $\delta > 0$ that $4M_g I_f(K\delta) < \epsilon$. $\square$

The proof of Theorem 4.5.14 follows the ideas of the proof of Theorem 4 in [23].

Proof of Theorem 4.5.14. By the assumption, let f be a density function, $f \in BV(I)$. Let $f_n =$

$\frac{1}{n}\sum_{i=1}^{n} P_{\widehat{T}_1^i} f$. Using the Equation 4.5.16, $P_{\widehat{T}_1^i} f$ and f_n, $i, n \geq 1$ have uniformly bounded variation. Hence, for a bounded variation density f, these functions are uniformly bounded and Helly's Theorem 4.5.7 implies the existence of a subsequence $\{f_{n_k}\}_{k\geq 1}$ convergent almost everywhere to a function f^* of bounded variation. Thus, by the Lebesgue dominated convergence theorem, $\int_I f^* d\lambda = 1$. Therefore, by Scheffé's Theorem 2.3.17, $f_{n_k} \to f^*$ in the L^1-norm. Thus, the sequence $f_n = \frac{1}{n}\sum_{i=1}^{n} P_{\widehat{T}_1^i} f$ forms a pre-compact set in L^1.

By Lemma 4.5.17, for any density function F, $P_{T_n}F \to P_T F$ weakly in L^1, as $n \to \infty$. Now, let $\{f_{n_k}\}_{k\geq 1}$ be a subsequence of $\{f_n\}_{n\geq 1}$ convergent in L^1 to f^*. We will show that f^* is the density of an acim of T. We have

$$P_T f^* = P_T\left(\lim_{n_k\to\infty} f_{n_k}\right) = \lim_{n_k\to\infty} P_T f_{n_k}$$

We will show that $P_T f_{n_k} - f_{n_k}$ converges weakly in L^1 to 0. Let $\phi_i = P_{\widehat{T}_1^i} f$, $i = 1, 2, ...$ Moreover, $\phi_{i+1} = P_{T_{i+1}}\phi_i$. Then, $f_n = \frac{1}{n}(\phi_1 + \phi_2 + ... + \phi_{n-1} + \phi_n)$, . We have

$$\begin{aligned}
P_T f_{n_k} - f_{n_k} &= \frac{1}{n_k}(P_T\phi_1 + P_T\phi_2 + \; + P_T\phi_{n-1} + P_T\phi_{n_k}) - \frac{1}{n_k}(\phi_1 + \phi_2 + \; + \phi_{n-1} + \phi_{n_k}) \\
&= \frac{1}{n_k}(P_T\phi_{n_k} - \phi_1) + \frac{1}{n_k}\sum_{i=1}^{n_k-1}(P_T\phi_i - \phi_{i+1}) \\
&= \frac{1}{n_k}(P_T\phi_{n_k} - \phi_1) + \frac{1}{n_k}\sum_{i=1}^{n_k-1}(P_T\phi_i - P_{T_{i+1}}\phi_i)
\end{aligned}$$

Let N and δ be chosen as in Lemma 4.5.17 . Let $n_k \geq N+2$. Then, using estimate 4.5.18, we have

$$\begin{aligned}
|\int_I (P_T f_{n_k} - f_{n_k}) g d\lambda| \leq & \frac{1}{n_k}\int_I |(P_T\phi_{n_k} - \phi_1)g| d\lambda \\
& + \frac{1}{n_k}\sum_{i=1}^{N}\int_I |(P_T\phi_i - P_{T_{i+1}}\phi_i)g| d\lambda + \frac{1}{n_k}\sum_{i=N+1}^{n_k-1}\int_I |(P_T\phi_i - P_{T_{i+1}}\phi_i)g| d\lambda \\
& \leq \frac{2}{n_k}M_g + \frac{2}{n_k}NM_g + \frac{n_k - 1 - N}{n_k}(2\epsilon).
\end{aligned}$$

The right hand side becomes smaller than (2ϵ) as $n_k \to \infty$. Since $\epsilon > 0$ is arbitrary this proves that $P_T f_{n_k} - f_{n_k}$ converges weakly in L^1 to 0 and $P_T f^* = f^*$. □

Chapter 5

On the absolutely continuous invariant measures for the random maps:

We study the dynamics of a new family of transformations. We find a general formula for the invariant density of any transformation in our family. The properties of the family allow us to prove that the invariant density function f of random map constructed from our family maps $T = \{\tau_1, \tau_2, ..., \tau_n; p_1, p_2, ..., p_n\}$ is the combination $f = p_1 f_1 + p_2 f_2 + ... + p_n f_n$, where $f_1, f_2, ..., f_n$ are the invariant density functions of $\tau_1, \tau_2, ..., \tau_n$, respectively. We also consider another family of transformations, and prove that the invariant density for any transformation of the family is $f = 1$. We present an application example to find the density function of the random maps by conjugations.

5.1 Introduction

A dynamical system, a space X with a mapping $\tau : X \to X$ or a family of mappings $\mathscr{T}$, may have a large number of invariant measures. There are invariant measures that are absolutely continuous with respect to some canonical measure on X. When X is an interval those which are playing the most important role are measures absolutely continuous with respect to the Lebesgue measure. The importance of absolutely continuous invariant measures (**acim**) is due to a heuristic belief that canonical measures are the ones that represent physical objects. Regarding previous studies of the absolutely continuous invariant measures, there are many examples of transformations on an interval and the absolutely continuous measures invariant under those transformations. Rényi (1957) [44] was the first one to define a class of transformations that have an acim. He introduced a family of one-dimensional maps $\tau_\beta(x) = \beta x(mod1)$, $\beta \in (1, \infty)$, $x \in [0, 1)$, where $\beta > 1$ is not necessarily an integer. Lyubich (2002) [38] proved that almost any real quadratic map $\tau_c : x \to x^2 + c, c \in [-2, 1/4]$, has either an attracting cycle or an acim. The literature on the existence and the properties of acim is very rich, see for example references in [20, 36, 11, 28].

In this chapter we will consider some families of piecewise linear maps. Let I be the interval $[0, 1]$. Let $\mathcal{P} = \{I_1, ..., I_N\}$ where $I = \bigcup I_i$ and $I_i^\circ \cap I_j^\circ = \emptyset$, $i \neq j$, $N \geq 2$ and $I_i = [x_{i-1}, x_i]$. We assume that N is even and $x_{\frac{N}{2}} = \frac{1}{2}$. There is no condition for the intervals to be of the same length. Let τ be a transformation of I onto itself, and $\tau_i = \tau_{|I_i}$ for each $i = 1, 2, ..., N$. We assume the following conditions:

$C1$. $\tau(x)$ is a piecewise linear map, and for every $I_i \subset \left[0, \frac{1}{2}\right]$ there exist $I_j \subset \left[\frac{1}{2}, 1\right]$ such that they have the same image and the same slope of the line, $i.e.$, $\tau(I_i) = \tau(I_j)$ with $|\tau_i'(x)| = |\tau_j'(y)|$, $x \in I_i, y \in I_j$. The I_j corresponding to different I_i's are different.

$C2$. $\tau(x_i) \in \left\{0, \frac{1}{2}, 1\right\}$ where x_i, $i = 1, ..., N$ are the partition points, $\tau(\overline{I_i}) \in \left\{[0, 1], [0, \frac{1}{2}], [\frac{1}{2}, 1]\right\}$ and $\tau(I) = [0, 1]$.

These conditions define the family of simple maps from the unit interval to itself. Let $\mathscr{T}$ be the family of all transformations that are satisfying the above conditions. The graphs in Figure 7.1 show some examples.

Maps of our class $\mathscr{T}$ find application in modelling and designing true random number generators (TRNG) with decreased voltage supply sensitivity in electronics [19].

In Section 5.2,we present the notation and summarize the results that we will need in the sequel. In Section 5.3, we study properties of $\mathscr{T}$ and give some examples. The main result is proved in Section 5.4. More results about the random map T with constant probabilities are in Section 5.5. In Section 5.6, we present examples of transformations and study the density functions of these transformations by the conjugations. In Section 5.7, we define another new family of transformations and we prove its properties.

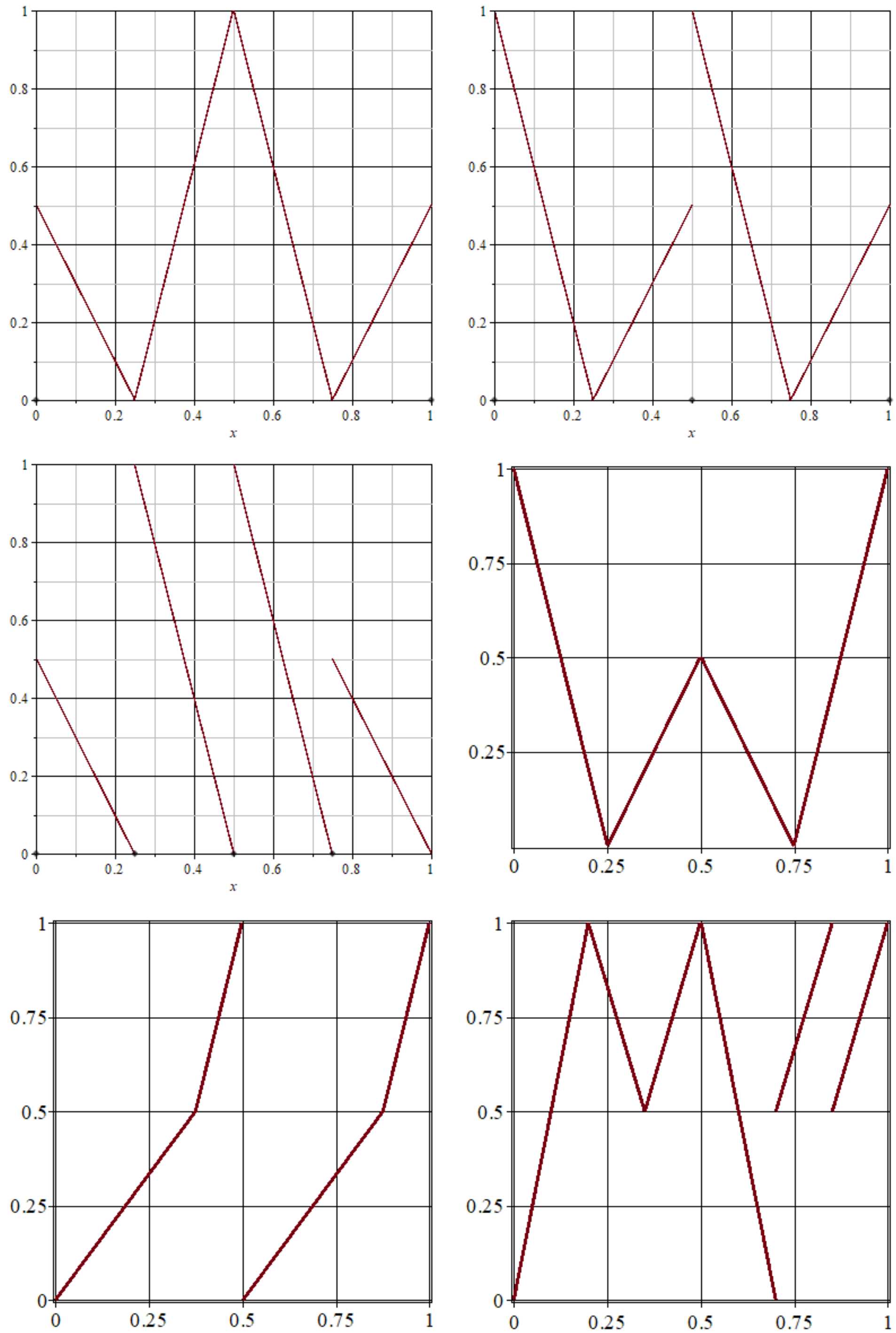

Figure 5.1: Examples of a transformations satisfying $C1$ and $C2$.

5.2 Some properties of $\mathscr{T}$ and the main result with motivating examples.

Let us consider the random map with constant probabilities $T = \{\tau_1, \tau_2; p_1, p_2\}$, where $\tau_i : [0,1] \longrightarrow [0,1]$; $k = 1,2$ are piecewise expanding maps. If τ_1 preserves a density f_1 and τ_2 preserves a density f_2, then consider the combination $f = p_1 f_1 + p_2 f_2$. We can ask if our random map T preserves the density f? In general, the answer is **no**. We have

$$\begin{aligned} P_T f &= P_T(p_1 f_1 + p_2 f_2) \\ &= p_1 P_T f_1 + p_2 P_T f_2 \\ &= p_1 \left[p_1 P_{\tau_1} f_1 + p_2 P_{\tau_2} f_1\right] + p_2 \left[p_1 P_{\tau_1} f_2 + p_2 P_{\tau_2} f_2\right] \\ &= p_1^2 f_1 + p_1 p_2 \left[P_{\tau_1} f_2 + P_{\tau_2} f_1\right] + p_2^2 f_2. \end{aligned}$$

in general different from f.

Example 5.2.1. Let $T = \left\{\tau, \omega; \frac{1}{2}, \frac{1}{2}\right\}$ and

$$\tau(x) = \begin{cases} 4x, & \text{for } 0 \le x < \frac{1}{4} \\ \frac{3}{2} - 2x, & \text{for } \frac{1}{4} \le x < \frac{3}{4}, \\ 2x - \frac{3}{2}, & \text{for } \frac{3}{4} \le x < 1 \end{cases} \qquad \omega(x) = \begin{cases} 1 - 4x, & \text{for } 0 \le x < \frac{1}{4} \\ 2x - \frac{1}{2}, & \text{for } \frac{1}{4} \le x < \frac{1}{2} \\ \frac{3}{2} - 2x, & \text{for } \frac{1}{2} \le x < \frac{3}{4} \\ 4x - 3, & \text{for } \frac{3}{4} \le x < 1 \end{cases}.$$

The invariant densities for τ and ω are $f_\tau(x) = \frac{8}{7}\chi_{[0,\frac{1}{2}]}(x) + \frac{6}{7}\chi_{[\frac{1}{2},1]}(x)$, $f_\omega(x) = \frac{3}{2}\chi_{[0,\frac{1}{2}]}(x) + \frac{1}{2}\chi_{[\frac{1}{2},1]}(x)$, respectively. Thus, the combination of the invariant density functions of τ and ω is $f = \frac{37}{28}\chi_{[0,\frac{1}{2}]}(x) + \frac{19}{28}\chi_{[\frac{1}{2},1]}(x)$ and $P_T f = \frac{281}{224}\chi_{[0,\frac{1}{2}]}(x) + \frac{167}{224}\chi_{[\frac{1}{2},1]}(x)$. Therefore, f is not the fixed point for P_T.

For transformations of family $\mathscr{T}$ this property holds. The following proposition presents some properties of $\mathscr{T}$.

Proposition 5.2.2. *Any $\tau \in \mathscr{T}$ has the following properties:*

(1.) τ is a piecewise monotonic expanding map.

(2.) The unique invariant density function of τ is piecewise constant on $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, i.e., $f(x) = (c)\chi_{[0,\frac{1}{2}]}(x) + (2-c)\chi_{[\frac{1}{2},1]}(x)$. The invariant density of τ can be also expressed as

$$f(x) = \sum_{i=1}^{n} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{\tau_i([x_{i-1},x_i])}(x).$$

(3.) Let $f_1, f_2, ..., f_n$ be the invariant densities of $\tau_1, \tau_2, ..., \tau_n$ respectively, $\tau_i \in \mathscr{T}$, $i = 1, 2, ..., n$. Then, for each j,

$$P_{\tau_j} f_i(x) = f_j(x),$$

for any $i = 1, 2, ..., n$. Moreover, the invariant density function of the random map $T = \{\tau_1, \tau_2, ..., \tau_n; p_1, p_2, ..., p_n\}$ is $f = p_1 f_1 + p_2 f_2 + ... + p_n f_n$.

We will prove the Proposition 5.2.2 in the next section. The following examples illustrate properties described in Proposition 5.2.2.

Example 5.2.3. Let $I = [0, 1]$, consider the transformations:

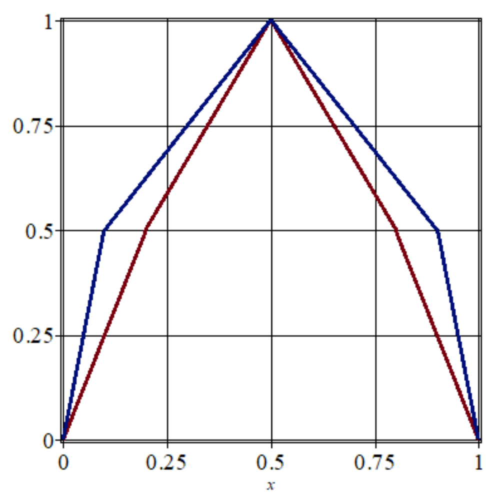

Figure 5.2: τ_1 and τ_2, for example 5.2.3.

$$\tau_1(x) = \begin{cases} 5x, & \text{for } 0 \le x < \frac{1}{10} \\ \frac{5}{4}x + \frac{3}{8}, & \text{for } \frac{1}{10} \le x < \frac{1}{2} \\ \frac{13}{8} - \frac{5}{4}x, & \text{for } \frac{1}{2} \le x < \frac{9}{10} \\ 5 - 5x, & \text{for } \frac{9}{10} \le x < 1 \end{cases}, \quad \tau_2(x) = \begin{cases} \frac{5}{2}x, & \text{for } 0 \le x < \frac{2}{10} \\ \frac{5}{3}x + \frac{1}{6}, & \text{for } \frac{2}{10} \le x < \frac{1}{2} \\ \frac{11}{6} - \frac{5}{3}x, & \text{for } \frac{1}{2} \le x < \frac{8}{10} \\ \frac{5}{2} - \frac{5}{2}x, & \text{for } \frac{8}{10} \le x < 1 \end{cases}.$$

The invariant densities for τ_1 and τ_2 are $f_1(x) = \frac{2}{5}\chi_{[0,\frac{1}{2}]}(x)+\frac{8}{5}\chi_{[\frac{1}{2},1]}(x)$ and $f_2(x) = \frac{4}{5}\chi_{[0,\frac{1}{2}]}(x)+\frac{6}{5}\chi_{[\frac{1}{2},1]}(x)$ respectively. Consider the random map $T_1 = \{\tau_1, \tau_2; p_1, p_2\}$, where $p_1, p_2 > 0$ and $p_1 + p_2 = 1$. Let $f = p_1 f_1 + p_2 f_2$. Then, we have

$$f = \left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[0,\frac{1}{2}]}(x) + \left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[\frac{1}{2},1]}(x).$$

We want to prove that T_1 preserves the density f. We need to find the Frobenius-Perron operator:

$$\begin{aligned}
P_{T_1}f(x) &= \sum_{k=1}^{2} p_k\left(P_{\tau_k}(f)\right)(x) = \sum_{k=1}^{2} p_k \sum_{i=1}^{2} \frac{f(\tau_{k,i}^{-1}(x))}{|\tau'(\tau_{k,i}^{-1}(x))|}\chi_{\tau(x_{i-1},x_i)}(x) \\
&= p_1\frac{1}{5}\left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[0,\frac{1}{2}]} + p_1\frac{4}{5}\left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[\frac{1}{2},1]} \\
&+ p_1\frac{4}{5}\left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[\frac{1}{2},1]} + p_1\frac{1}{5}\left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[0,\frac{1}{2}]} \\
&+ p_2\frac{2}{5}\left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[0,\frac{1}{2}]} + p_2\frac{3}{5}\left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[\frac{1}{2},1]} \\
&+ p_2\frac{3}{5}\left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[\frac{1}{2},1]} + p_2\frac{2}{5}\left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[0,\frac{1}{2}]} \\
&= [p_1^2\left(\frac{2}{25}\right) + p_1p_2\left(\frac{4}{25}\right) + p_1^2\left(\frac{8}{25}\right) + p_1p_2\left(\frac{6}{25}\right) + p_1p_2\left(\frac{4}{25}\right) \\
&+ p_2^2\left(\frac{8}{25}\right) + p_1p_2\left(\frac{16}{25}\right) + p_2^2\left(\frac{12}{25}\right)]\chi_{[0,\frac{1}{2}]} + [p_1^2\left(\frac{8}{25}\right) \\
&+ p_1p_2\left(\frac{16}{25}\right) + p_1^2\left(\frac{32}{25}\right) + p_1p_2\left(\frac{24}{25}\right) + p_1p_2\left(\frac{6}{25}\right) \\
&+ p_2^2\left(\frac{12}{25}\right) + p_1p_2\left(\frac{24}{25}\right) + p_2^2\left(\frac{18}{25}\right)]\chi_{[\frac{1}{2},1]}
\end{aligned}$$

$$
\begin{aligned}
P_{T_1} f(x) &= \left[p_1^2\left(\frac{10}{25}\right) + p_1p_2\left(\frac{10}{25}+\frac{20}{25}\right) + p_2^2\left(\frac{20}{25}\right)\right]\chi_{[0,\frac{1}{2}]} \\
&+ \left[p_1^2\left(\frac{40}{25}\right) + p_1p_2\left(\frac{40}{25}+\frac{30}{25}\right) + p_2^2\left(\frac{30}{25}\right)\right]\chi_{[\frac{1}{2},1]} \\
&= \left[p_1\left(\frac{10}{25}\right)(p_1+p_2) + p_2\left(\frac{20}{25}\right)(p_1+p_2)\right]\chi_{[0,\frac{1}{2}]} \\
&+ \left[p_1\left(\frac{40}{25}\right)(p_1+p_2) + p_2\left(\frac{30}{25}\right)(p_1+p_2)\right]\chi_{[\frac{1}{2},1]} \\
&= \left[p_1\left(\frac{2}{5}\right) + p_2\left(\frac{4}{5}\right)\right]\chi_{[0,\frac{1}{2}]}(x) + \left[p_1\left(\frac{8}{5}\right) + p_2\left(\frac{6}{5}\right)\right]\chi_{[\frac{1}{2},1]}(x).
\end{aligned}
$$

The proof is complete.

The following example of seven different piecewise linear transformations from family $\mathscr{T}$ illustrates property (3.) in Proposition 5.2.2.

Example 5.2.4. Consider the following seven transformations defined on different partitions of the interval $[0, 1]$. The first map:

$$
\tau_1(x) = \begin{cases}
\frac{1}{2\alpha_1}x, & \text{for } 0 \le x < \alpha_1 \\
\frac{1}{2(\alpha_2-\alpha_1)}(x-\alpha_1)+\frac{1}{2}, & \text{for } \alpha_1 \le x < \alpha_2 \\
\frac{1}{2(\alpha_2-\frac{1}{2})}(x-\alpha_2)+1, & \text{for } \alpha_2 \le x < \frac{1}{2} \\
\frac{-1}{2(\alpha_2-\frac{1}{2})}(x-\frac{1}{2})+\frac{1}{2}, & \text{for } \frac{1}{2} \le x < \alpha_4 \\
\frac{-1}{2(\alpha_2-\alpha_1)}(x-\alpha_4)+1, & \text{for } \alpha_4 \le x < \alpha_5 \\
\frac{-1}{2\alpha_1}(x-\alpha_5)+\frac{1}{2}, & \text{for } \alpha_5 \le x < 1
\end{cases}
$$

Where $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \alpha_3 = \frac{1}{2} < \alpha_4 < \alpha_5 < \alpha_6 = 1$, $\alpha_4 = 1-\alpha_2$ and $\alpha_5 = 1-\alpha_1$. The invariant density of $\tau_1(x)$ is $f_1(x) = (4\alpha_1)\chi_{[0,\frac{1}{2}]}(x) + (2-4\alpha_1)\chi_{[\frac{1}{2},1]}(x)$.

The second map:

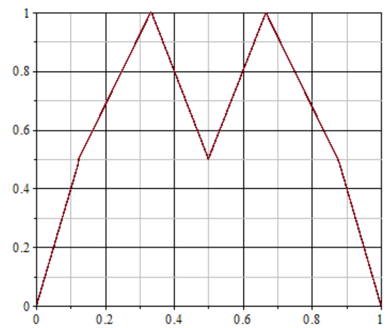

Figure 5.3: τ_1 with $\alpha_1 = \frac{1}{8}$ and $\alpha_2 = \frac{1}{3}$.

$$\tau_2(x) = \begin{cases} \frac{-1}{2\beta_1}x + \frac{1}{2}, & \text{for } 0 \le x < \beta_1 \\ \frac{1}{\frac{1}{2}-\beta_1}(x-\beta_1), & \text{for } \beta_1 \le x < \frac{1}{2} \\ \frac{-1}{\frac{1}{2}-\beta_1}(x-\frac{1}{2})+1, & \text{for } \frac{1}{2} \le x < \beta_3 \\ \frac{1}{2\beta_1}(x-1)+\frac{1}{2}, & \text{for } \beta_3 \le x < 1 \end{cases}$$

Where $\beta_0 = 0 < \beta_1 < \beta_2 = \frac{1}{2} < \beta_3 < \beta_4 = 1$ and $\beta_3 = 1 - \beta_1$. The invariant density of $\tau_2(x)$ is $f_2(x) = (2\beta_1 + 1)\chi_{[0,\frac{1}{2}]}(x) + (1 - 2\beta_1)\chi_{[\frac{1}{2},1]}(x)$.

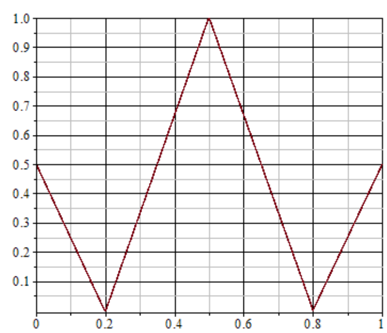

Figure 5.4: τ_2 with $\beta_1 = \frac{1}{5}$.

First we show that $P_{\tau_1} f_2 = f_1$:

$$P_{\tau_1} f_2(x) = \sum_{i=1}^{6} \frac{f_2(\tau_{1,i}^{-1}(x))}{|\tau_1'(\tau_{1,i}^{-1}(x))|} \chi_{\tau_1([x_{i-1},x_i])}(x)$$

$$= (2\beta_1 + 1)(2\alpha_1)\chi_{\left[0,\frac{1}{2}\right]} + (2\beta_1+1)(2\alpha_2 - 2\alpha_1)\chi_{\left[\frac{1}{2},1\right]} + (2\beta_1+1)(1-2\alpha_2)\chi_{\left[\frac{1}{2},1\right]}$$

$$+ (1-2\beta_1)(1-2\alpha_2)\chi_{\left[\frac{1}{2},1\right]} + (1-2\beta_1)(2\alpha_2-2\alpha_1)\chi_{\left[\frac{1}{2},1\right]} + (1-2\beta_1)(2\alpha_1)\chi_{\left[0,\frac{1}{2}\right]}$$

$$= (4\alpha_1\beta_1 + 2\alpha_1 + 2\alpha_1 - 4\alpha_1\beta_1)\chi_{\left[0,\frac{1}{2}\right]}$$

$$+ (4\alpha_2\beta_1 - 4\alpha_1\beta_1 + 2\alpha_2 - 2\alpha_1 - 4\alpha_2\beta_1 + 2\beta_1 - 2\alpha_2 + 1$$

$$- 2\alpha_2 + 1 + 4\alpha_2\beta_1 - 2\beta_2 + 2\alpha_2 - 2\alpha_1 - 4\alpha_2\beta_1 + 4\alpha_1\beta_1)\chi_{\left[\frac{1}{2},1\right]}$$

$$= (4\alpha_1)\chi_{\left[0,\frac{1}{2}\right]} + (2-4\alpha_1)\chi_{\left[\frac{1}{2},1\right]}$$

$$= f_1(x)$$

Now, $P_{\tau_2} f_1 = f_2$:

$$P_{\tau_2} f_1(x) = \sum_{i=1}^{4} \frac{f_1(\tau_{2,i}^{-1}(x))}{|\tau_2'(\tau_{2,i}^{-1}(x))|} \chi_{\tau_2([x_{i-1},x_i])}(x)$$

$$= (4\alpha_1)(2\beta_1)\chi_{\left[0,\frac{1}{2}\right]} + (4\alpha_1)\left(\frac{1}{2} - \beta_1\right)\chi_{[0,1]}$$

$$+ (2-4\alpha_1)\left(\frac{1}{2} - \beta_1\right)\chi_{[0,1]} + (2-4\alpha_1)(2\beta_1)\chi_{\left[0,\frac{1}{2}\right]}$$

$$= (8\alpha_1\beta_1 + 2\alpha_1 - 4\alpha_1\beta_1 + 1 - 2\beta_1 - 2\alpha_1 + 4\alpha_1\beta_1 + 4\beta_1 - 8\alpha_1\beta_1)\chi_{\left[0,\frac{1}{2}\right]}$$

$$+ (2\alpha_1 - 4\alpha_1\beta_1 + 1 - 2\alpha_1 - 2\beta_1 + 4\alpha_1\beta_1)\chi_{\left[\frac{1}{2},1\right]}$$

$$= (2\beta_1 + 1)\chi_{\left[0,\frac{1}{2}\right]} + (1-2\beta_1)\chi_{\left[\frac{1}{2},1\right]}$$

$$= f_2(x)$$

The third map:

$$\tau_3(x) = \begin{cases} \frac{-1}{\gamma_1}x + 1, & \text{for } 0 \le x < \gamma_1 \\ \frac{-1}{1-2\gamma_1}(x - \frac{1}{2}) + \frac{1}{2}, & \text{for } \gamma_1 \le x < \frac{1}{2} \\ \frac{1}{1-2\gamma_1}(x - \frac{1}{2}) + \frac{1}{2}, & \text{for } \frac{1}{2} \le x < \gamma_3 \\ \frac{1}{\gamma_1}(x-1) + 1, & \text{for } \gamma_3 \le x < 1 \end{cases}$$

Where $\gamma_0 = 0 < \gamma_1 < \gamma_2 = \frac{1}{2} < \gamma_3 < \gamma_4 = 1$ and $\gamma_3 = 1 - \gamma_1$. The invariant density of $\tau_3(x)$

is $f_3(x) = (2\gamma_1)\chi_{[0,\frac{1}{2}]}(x) + (2-2\gamma_1)\chi_{[\frac{1}{2},1]}(x)$.

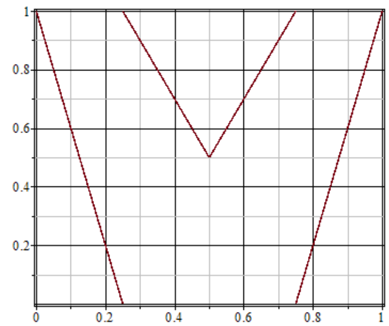

Figure 5.5: τ_3 with $\gamma_1 = \frac{1}{4}$.

Now, $P_{\tau_2} f_3 = f_2$:

$$
\begin{aligned}
P_{\tau_2} f_3(x) &= \sum_{i=1}^{4} \frac{f_3(\tau_{2,i}^{-1}(x))}{|\tau_2'(\tau_{2,i}^{-1}(x))|}\chi_{\tau_2([x_{i-1},x_i])}(x) \\
&= (2\gamma_1)(2\beta_1)\chi_{\left[0,\frac{1}{2}\right]} + (2\gamma_1)\left(\frac{1}{2}-\beta_1\right)\chi_{[0,1]} \\
&+ (2-2\gamma_1)\left(\frac{1}{2}-\beta_1\right)\chi_{[0,1]} + (2-2\gamma_1)(2\beta_1)\chi_{\left[0,\frac{1}{2}\right]} \\
&= (4\gamma_1\beta_1 + \gamma_1 - 2\gamma_1\beta_1 + 1 - 2\beta_1 - \gamma_1 + 2\gamma_1\beta_1 + 4\beta_1 - 4\gamma_1\beta_1)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (\gamma_1 - 2\gamma_1\beta_1 + 1 - \gamma_1 - 2\beta_1 + 2\gamma_1\beta_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= (2\beta_1+1)\chi_{\left[0,\frac{1}{2}\right]} + (1-2\beta_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= f_2(x)
\end{aligned}
$$

Now, $P_{\tau_3} f_2 = f_3$:

$$
\begin{aligned}
P_{\tau_3} f_2(x) &= \sum_{i=1}^{4} \frac{f_2(\tau_{3,i}^{-1}(x))}{|\tau_3'(\tau_{3,i}^{-1}(x))|} \chi_{\tau_3([x_{i-1},x_i])}(x) \\
&= (2\beta_1 + 1)\,(\gamma_1)\,\chi_{[0,1]} + (2\beta_1 + 1)\,(1 - 2\gamma_1)\,\chi_{\left[\frac{1}{2},1\right]} \\
&+ (1 - 2\beta_1)\,(1 - 2\gamma_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1 - 2\beta_1)\,(\gamma_1)\,\chi_{[0,1]} \\
&= (\gamma_1 + 2\gamma_1\beta_1 + \gamma_1 - 2\gamma_1\beta_1)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (\gamma_1 + 2\gamma_1\beta_1 + 1 - 2\gamma_1 + 2\beta_1 - 4\gamma_1\beta_1 + 1 - 2\gamma_1 + 2\beta_2 + 4\gamma_1\beta_1 + \gamma_1 - 2\gamma_1\beta_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= (2\gamma_1)\chi_{\left[0,\frac{1}{2}\right]} + (2 - 2\gamma_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= f_3(x)
\end{aligned}
$$

The fourth map:

$$
\tau_4(x) = \begin{cases}
\frac{-1}{2\delta_1}x + 1, & \text{for } 0 \le x < \delta_1 \\
\frac{1}{2(\delta_2-\delta_1)}(x - \delta_1) + \frac{1}{2}, & \text{for } \delta_1 \le x < \delta_2 \\
\frac{-1}{\frac{1}{2}-\delta_2}(x - \frac{1}{2}), & \text{for } \delta_2 \le x < \frac{1}{2} \\
\frac{-1}{2\delta_1}(x - \frac{1}{2}) + 1, & \text{for } \frac{1}{2} \le x < \delta_4 \\
\frac{1}{2(\delta_2-\delta_1)}(x - \delta_4) + \frac{1}{2}, & \text{for } \delta_4 \le x < \delta_5 \\
\frac{-1}{\frac{1}{2}-\delta_2}(x - 1), & \text{for } \delta_5 \le x < 1
\end{cases}
$$

Where $\delta_0 = 0 < \delta_1 < \delta_2 < \delta_3 = \frac{1}{2} < \delta_4 < \delta_5 < \delta_6 = 1$, $\delta_4 = \frac{1}{2} + \delta_1$ and $\delta_5 = \frac{1}{2} + \delta_2$. The invariant density of $\tau_4(x)$ is $f_4(x) = (1 - 2\delta_2)\chi_{[0,\frac{1}{2}]}(x) + (1 + 2\delta_2)\chi_{[\frac{1}{2},1]}(x)$.

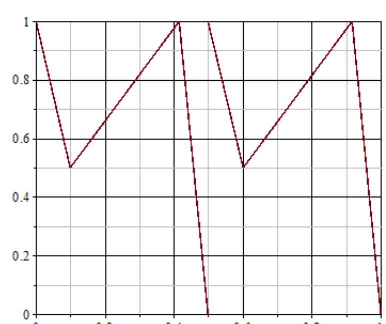

Figure 5.6: τ_4 with $\delta_1 = \frac{1}{10}$ and $\delta_2 = \frac{5}{12}$.

Now, $P_{\tau_4} f_3 = f_4$:

$$P_{\tau_4} f_3(x) = \sum_{i=1}^{6} \frac{f_3(\tau_{4,i}^{-1}(x))}{|\tau_4'(\tau_{4,i}^{-1}(x))|} \chi_{\tau_4([x_{i-1},x_i])}(x)$$

$$= (2\gamma_1)(2\delta_1)\chi_{[\frac{1}{2},1]} + (2\gamma_1)(2\delta_2 - 2\delta_1)\chi_{[\frac{1}{2},1]} + (2\gamma_1)\left(\frac{1}{2} - \delta_2\right)\chi_{[0,1]}$$

$$+ (2 - 2\gamma_1)(2\delta_1)\chi_{[\frac{1}{2},1]} + (2 - 2\gamma_1)(2\delta_2 - 2\delta_1)\chi_{[\frac{1}{2},1]} + (2 - 2\gamma_1)\left(\frac{1}{2} - \delta_2\right)\chi_{[0,1]}$$

$$= (\gamma_1 - 2\gamma_1\delta_2 + 1 - \gamma_1 - 2\delta_2 + 2\gamma_1\delta_2)\chi_{[0,\frac{1}{2}]}$$

$$+ (4\gamma_1\delta_1 + 4\gamma_1\delta_2 - 4\gamma_1\delta_1 + \gamma_1 - 2\gamma_1\delta_2 + 4\delta_1 - 4\gamma_1\delta_1$$

$$+ 4\delta_2 - 4\delta_1 - 4\gamma_1\delta_2 + 4\gamma_1\delta_1 + 1 - 2\delta_2 - \gamma_1 + 2\gamma_1\delta_2)\chi_{[\frac{1}{2},1]}$$

$$= (1 - 2\delta_2)\chi_{[0,\frac{1}{2}]} + (1 + 2\delta_2)\chi_{[\frac{1}{2},1]}$$

$$= f_4(x)$$

Now, $P_{\tau_3} f_4 = f_3$:

$$P_{\tau_3} f_4(x) = \sum_{i=1}^{4} \frac{f_4(\tau_{3,i}^{-1}(x))}{|\tau_3'(\tau_{3,i}^{-1}(x))|} \chi_{\tau_3([x_{i-1},x_i])}(x)$$

$$= (1 - 2\delta_2)(\gamma_1)\chi_{[0,1]} + (1 - 2\delta_2)(1 - 2\gamma_1)\chi_{[\frac{1}{2},1]}$$

$$+ (1 + 2\delta_2)(1 - 2\gamma_1)\chi_{[\frac{1}{2},1]} + (1 + 2\delta_2)(\gamma_1)\chi_{[0,1]}$$

$$= (\gamma_1 - 2\gamma_1\delta_2 + \gamma_1 + 2\gamma_1\delta_2)\chi_{[0,\frac{1}{2}]}$$

$$+ (\gamma_1 - 2\gamma_1\delta_2 + 1 - 2\gamma_1 - 2\delta_2 + 4\gamma_1\delta_2 + 1 - 2\gamma_1 + 2\delta_2 - 4\gamma_1\delta_2 + \gamma_1 + 2\gamma_1\delta_2)\chi_{[\frac{1}{2},1]}$$

$$= (2\gamma_1)\chi_{[0,\frac{1}{2}]} + (2 - 2\gamma_1)\chi_{[\frac{1}{2},1]}$$

$$= f_3(x)$$

The fifth map:

$$\tau_5(x) = \begin{cases} \frac{1}{\zeta_1}x, & \text{for } 0 \leq x < \zeta_1 \\ \frac{1}{2(\zeta_2-\zeta_1)}(x-\zeta_1), & \text{for } \zeta_1 \leq x < \zeta_2 \\ \frac{-1}{\frac{1}{2}-\zeta_2}(x-\frac{1}{2}), & \text{for } \zeta_2 \leq x < \frac{1}{2} \\ \frac{1}{\zeta_1}(x-\frac{1}{2}), & \text{for } \frac{1}{2} \leq x < \zeta_4 \\ \frac{1}{2(\zeta_2-\zeta_1)}(x-\frac{1}{2}-\zeta_1), & \text{for } \zeta_4 \leq x < \zeta_5 \\ \frac{-1}{\frac{1}{2}-\zeta_2}(x-1), & \text{for } \zeta_5 \leq x < 1 \end{cases}$$

Where $\zeta_0 = 0 < \zeta_1 < \zeta_2 < \zeta_3 = \frac{1}{2} < \zeta_4 < \zeta_5 < \zeta_6 = 1$, $\zeta_4 = \frac{1}{2} + \zeta_1$ and $\zeta_5 = \frac{1}{2} + \zeta_2$. The invariant density of $\tau_5(x)$ is $f_5(x) = (1 + 2(\zeta_2 - \zeta_1))\chi_{[0,\frac{1}{2}]}(x) + (1 - 2(\zeta_2 - \zeta_1))\chi_{[\frac{1}{2},1]}(x)$.

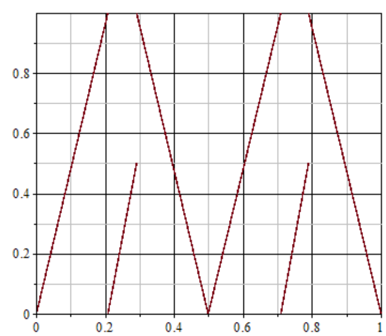

Figure 5.7: τ_5 with $\zeta_1 = \frac{5}{24}$ and $\zeta_2 = \frac{7}{24}$.

Now, $P_{\tau_5} f_4 = f_5$:

$$
\begin{aligned}
P_{\tau_5} f_4(x) &= \sum_{i=1}^{6} \frac{f_4(\tau_{5,i}^{-1}(x))}{|\tau_5'(\tau_{5,i}^{-1}(x))|} \chi_{\tau_5([x_{i-1},x_i])}(x) \\
&= (1-2\delta_2)\,(\zeta_1)\,\chi_{[0,1]} + (1-2\delta_2)\,(2\zeta_2-2\zeta_1)\,\chi_{\left[0,\frac{1}{2}\right]} + (1-2\delta_2)\left(\frac{1}{2}-\zeta_2\right)\chi_{[0,1]} \\
&+ (1+2\delta_2)\,(\zeta_1)\,\chi_{[0,1]} + (1+2\delta_2)\,(2\zeta_2-2\zeta_1)\,\chi_{\left[0,\frac{1}{2}\right]} + (1+2\delta_2)\left(\frac{1}{2}-\zeta_2\right)\chi_{[0,1]} \\
&= (\zeta_1 - 2\zeta_1\delta_2 + 2\zeta_2 - 2\zeta_1 - 4\zeta_2\delta_2 + 4\zeta_1\delta_2 + \frac{1}{2} - \zeta_2 - \delta_2 + 2\zeta_2\delta_2 + \zeta_1 + 2\zeta_1\delta_2 \\
&+ 2\zeta_2 - 2\zeta_1 + 4\zeta_2\delta_2 - 4\zeta_1\delta_2 + \frac{1}{2} - \zeta_2 + \delta_2 - 2\zeta_2\delta_2)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (\zeta_1 - 2\zeta_1\delta_2 + \frac{1}{2} - \zeta_2 - \delta_2 + 2\zeta_2\delta_2 + \zeta_1 + 2\zeta_1\delta_2 + \frac{1}{2} - \zeta_2 + \delta_2 - 2\zeta_2\delta_2)\chi_{\left[\frac{1}{2},1\right]} \\
&= (1+2(\zeta_2-\zeta_1))\chi_{[0,\frac{1}{2}]} + (1-2(\zeta_2-\zeta_1))\chi_{[\frac{1}{2},1]} \\
&= f_5(x)
\end{aligned}
$$

Now, $P_{\tau_4} f_5 = f_4$:

$$
\begin{aligned}
P_{\tau_4} f_5(x) &= \sum_{i=1}^{6} \frac{f_5(\tau_{4,i}^{-1}(x))}{|\tau_4'(\tau_{4,i}^{-1}(x))|} \chi_{\tau_4([x_{i-1},x_i])}(x) \\
&= (1+2(\zeta_2-\zeta_1))\,(2\delta_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1+2(\zeta_2-\zeta_1))\,(2\delta_2-2\delta_1)\,\chi_{\left[\frac{1}{2},1\right]} \\
&+ (1+2(\zeta_2-\zeta_1))\left(\frac{1}{2}-\delta_2\right)\chi_{[0,1]} + (1-2(\zeta_2-\zeta_1))\,(2\delta_1)\,\chi_{\left[\frac{1}{2},1\right]} \\
&+ (1-2(\zeta_2-\zeta_1))\,(2\delta_2-2\delta_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1-2(\zeta_2-\zeta_1))\left(\frac{1}{2}-\delta_2\right)\chi_{[0,1]} \\
&= (2\delta_1 + 4\zeta_2\delta_1 - 4\zeta_1\delta_1 + 2\delta_2 + 4\zeta_2\delta_2 - 4\zeta_1\delta_2 - 2\delta_1 - 4\zeta_2\delta_1 + 4\zeta_1\delta_1 \\
&+ \frac{1}{2} - \delta_2 + \zeta_2 - 2\zeta_2\delta_2 - \zeta_1 + 2\zeta_1\delta_2 + 2\delta_1 - 4\zeta_2\delta_1 + 4\zeta_1\delta_1 + 2\delta_2 - 4\zeta_2\delta_2 \\
&+ 4\zeta_1\delta_2 - 2\delta_1 + 4\zeta_2\delta_1 - 4\zeta_1\delta_1 + \frac{1}{2} - \delta_2 - \zeta_2 + 2\zeta_2\delta_2 + \zeta_1 - 2\zeta_1\delta_2)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (\frac{1}{2} - \delta_2 + \zeta_2 - 2\zeta_2\delta_2 - \zeta_1 - 2\zeta_1\delta_2 + \frac{1}{2} - \delta_2 - \zeta_2 + 2\zeta_2\delta_2 + \zeta_1 - 2\zeta_2\delta_2)\chi_{\left[\frac{1}{2},1\right]} \\
&= (1-2\delta_2)\chi_{\left[0,\frac{1}{2}\right]} + (1+2\delta_2)\chi_{\left[\frac{1}{2},1\right]} \\
&= f_4(x)
\end{aligned}
$$

The sixth map:

$$\tau_6(x) = \begin{cases} \frac{1}{2\eta_1}x + \frac{1}{2}, & \text{for } 0 \le x < \eta_1 \\ \frac{-1}{\frac{1}{2}-\eta_1}(x - \frac{1}{2}), & \text{for } \eta_1 \le x < \frac{1}{2} \\ \frac{-1}{\frac{1}{2}-\eta_1}(x - \eta_3), & \text{for } \frac{1}{2} \le x < \eta_3 \\ \frac{1}{2\eta_1}(x-1)+1, & \text{for } \eta_3 \le x < 1 \end{cases}$$

Where $\eta_0 = 0 < \eta_1 < \eta_2 = \frac{1}{2} < \eta_3 < \eta_4 = 1$, $\eta_3 = 1 - \eta_1$. The invariant density of $\tau_6(x)$ is $f_6(x) = (1 - 2\eta_1)\chi_{[0,\frac{1}{2}]}(x) + (1 + 2\eta_1)\chi_{[\frac{1}{2},1]}(x)$.

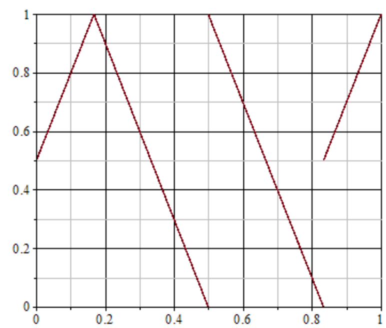

Figure 5.8: τ_6 with $\eta_1 = \frac{1}{6}$.

Now, $P_{\tau_6} f_5 = f_6$:

$$P_{\tau_6} f_5(x) = \sum_{i=1}^{4} \frac{f_5(\tau_{6,i}^{-1}(x))}{|\tau_6'(\tau_{6,i}^{-1}(x))|}\chi_{\tau_6([x_{i-1},x_i])}(x)$$

$$= (1 + 2(\zeta_2 - \zeta_1))\,(2\eta_1)\,\chi_{[\frac{1}{2},1]} + (1 + 2(\zeta_2 - \zeta_1))\left(\frac{1}{2} - \eta_1\right)\chi_{[0,1]}$$

$$+ (1 - 2(\zeta_2 - \zeta_1))\left(\frac{1}{2} - \eta_1\right)\chi_{[0,1]} + (1 - 2(\zeta_2 - \zeta_1))\,(2\eta_1)\,\chi_{[\frac{1}{2},1]}$$

$$= (\frac{1}{2} - \eta_1 + \zeta_2 - 2\zeta_2\eta_1 - \zeta_1 + 2\zeta_1\eta_1 + \frac{1}{2} - \eta_1 - \zeta_2 + 2\zeta_2\eta_1 + \zeta_1 - 2\zeta_1\eta_1)\chi_{[0,\frac{1}{2}]}$$

$$+ (2\eta_1 + 4\zeta_2\eta_1 - 4\zeta_1\eta_1 + \frac{1}{2} - \eta_1 + \zeta_2 - 2\zeta_2\eta_1 - \zeta_1 + 2\zeta_1\eta_1 + \frac{1}{2} - \eta_1 - \zeta_2$$

$$+ 2\zeta_2\eta_1 + \zeta_1 - 2\zeta_1\eta_1 2\eta_1 - 4\zeta_2\eta_1 + 4\zeta_1\eta_1)\chi_{[\frac{1}{2},1]}$$

$$= (1 - 2\eta_2)\chi_{[0,\frac{1}{2}]} + (1 + 2\eta_2)\chi_{[\frac{1}{2},1]}$$

$$= f_6(x)$$

We have $P_{\tau_5} f_6(x) = P_{\tau_5} f_4(x) = f_5$.

The seventh map:

$$\tau_7(x) = \begin{cases} \frac{1}{2\vartheta_1}x, & \text{for } 0 \le x < \vartheta_1 \\ \frac{-1}{2(\vartheta_2-\vartheta_1)}(x-\vartheta_1)+1, & \text{for } \vartheta_1 \le x < \vartheta_2 \\ \frac{-1}{\frac{1}{2}-\vartheta_2}(x-\frac{1}{2}), & \text{for } \vartheta_2 \le x < \frac{1}{2} \\ \frac{1}{2\vartheta_1}(x-\frac{1}{2}), & \text{for } \frac{1}{2} \le x < \vartheta_4 \\ \frac{1}{2(\vartheta_2-\vartheta_1)}(x-\vartheta_5)+1, & \text{for } \vartheta_4 \le x < \vartheta_5 \\ \frac{1}{\frac{1}{2}-\vartheta_2}(x-\vartheta_5), & \text{for } \vartheta_5 \le x < 1 \end{cases}$$

Where $\vartheta_0 = 0 < \vartheta_1 < \vartheta_2 < \vartheta_3 = \frac{1}{2} < \vartheta_4 < \vartheta_5 < \vartheta_6 = 1$, $\vartheta_4 = \frac{1}{2} + \vartheta_1$ and $\vartheta_5 = \frac{1}{2} + \vartheta_2$. The invariant density of $\tau_7(x)$ is $f_7(x) = (1-2(\vartheta_2-2\vartheta_1))\chi_{[0,\frac{1}{2}]}(x) + (1+2(\vartheta_2-2\vartheta_1))\chi_{[\frac{1}{2},1]}(x)$.

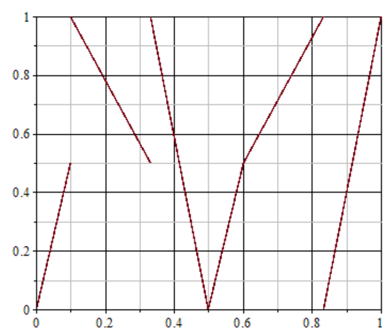

Figure 5.9: τ_7 with $\vartheta_1 = \frac{1}{10}$ and $\vartheta_2 = \frac{1}{3}$.

Now, $P_{\tau_7} f_6 = f_7$:

$$
\begin{aligned}
P_{\tau_7} f_6(x) &= \sum_{i=1}^{6} \frac{f_6(\tau_{7,i}^{-1}(x))}{|\tau_7'(\tau_{7,i}^{-1}(x))|}\chi_{\tau_7([x_{i-1},x_i])}(x) \\
&= (1-2\eta_1)\,(2\vartheta_1)\,\chi_{\left[0,\frac{1}{2}\right]} + (1-2\eta_1)\,(2\vartheta_2-2\vartheta_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1-2\eta_1)\left(\frac{1}{2}-\vartheta_2\right)\chi_{[0,1]} \\
&+ (1+2\eta_1)\,(2\vartheta_1)\,\chi_{\left[0,\frac{1}{2}\right]} + (1+2\eta_1)\,(2\vartheta_2-2\vartheta_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1+2\eta_1)\left(\frac{1}{2}-\vartheta_2\right)\chi_{[0,1]} \\
&= (2\vartheta_1 - 4\vartheta_1\eta_1 + \frac{1}{2} - \vartheta_2 - \eta_1 + 2\vartheta_2\eta_1 + 2\vartheta_1 + 4\vartheta_1\eta_1 + \frac{1}{2} - \vartheta_2 + \eta_1 - 2\vartheta_2\eta_1)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (2\vartheta_2 - 2\vartheta_1 - 4\vartheta_2\eta_1 + 4\vartheta_1\eta_1 + \frac{1}{2} - \vartheta_2 - \eta_1 + 2\vartheta_2\eta_1 + 2\vartheta_2 \\
&- 2\vartheta_1 + 4\vartheta_2\eta_1 - 4\vartheta_1\eta_1 + \frac{1}{2} - \vartheta_2 + \eta_1 - 2\vartheta_2\eta_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= (1-2(\vartheta_2-2\vartheta_1))\chi_{[0,\frac{1}{2}]} + (1+2(\vartheta_2-2\vartheta_1))\chi_{[\frac{1}{2},1]} \\
&= f_7(x)
\end{aligned}
$$

And, eventually, $P_{\tau_6} f_7 = f_6$:

$$
\begin{aligned}
P_{\tau_6} f_7(x) &= \sum_{i=1}^{4} \frac{f_7(\tau_{6,i}^{-1}(x))}{|\tau_6'(\tau_{6,i}^{-1}(x))|}\chi_{\tau_6([x_{i-1},x_i])}(x) \\
&= (1-2(\vartheta_2-2\vartheta_1))\,(2\eta_1)\,\chi_{\left[\frac{1}{2},1\right]} + (1-2(\vartheta_2-2\vartheta_1))\left(\frac{1}{2}-\eta_1\right)\chi_{[0,1]} \\
&+ (1+2(\vartheta_2-2\vartheta_1))\left(\frac{1}{2}-\eta_1\right)\chi_{[0,1]} + (1+2(\vartheta_2-2\vartheta_1))\,(2\eta_1)\,\chi_{\left[\frac{1}{2},1\right]} \\
&= (\frac{1}{2} - \eta_1 - \vartheta_2 + 2\vartheta_2\eta_1 + 2\vartheta_1 - 4\vartheta_1\eta_1 + \frac{1}{2} - \eta_1 + \vartheta_2 - 2\vartheta_2\eta_1 - 2\vartheta_1 + 4\vartheta_2\eta_1)\chi_{\left[0,\frac{1}{2}\right]} \\
&+ (2\eta_1 - 4\vartheta_2\eta_1 + 8\vartheta_1\eta_1 + \frac{1}{2} - \eta_1 - \vartheta_2 + 2\vartheta_2\eta_1 + 2\vartheta_1 - 4\vartheta_1\eta_1 + \frac{1}{2} \\
&- \eta_1 + \vartheta_2 - 2\vartheta_2\eta_1 - 2\vartheta_1 + 4\vartheta_2\eta_1 + 2\eta_1 + 4\vartheta_2\eta_1 - 8\vartheta_1\eta_1)\chi_{\left[\frac{1}{2},1\right]} \\
&= (1-2\eta_2)\chi_{\left[0,\frac{1}{2}\right]} + (1+2\eta_2)\chi_{\left[\frac{1}{2},1\right]} \\
&= f_6(x)
\end{aligned}
$$

5.3 The main result proof.

We can describe three types of transformations that belong to $\mathscr{T}$. **The first type**: the maps are symmetric around $x = \frac{1}{2}$. **The second type**: the graphs of the maps in $(0, \frac{1}{2})$ and in $(\frac{1}{2}, 1)$ are identical. **The third type**: is neither symmetric nor identical maps and satisfying the conditions $C1$ and $C2$.

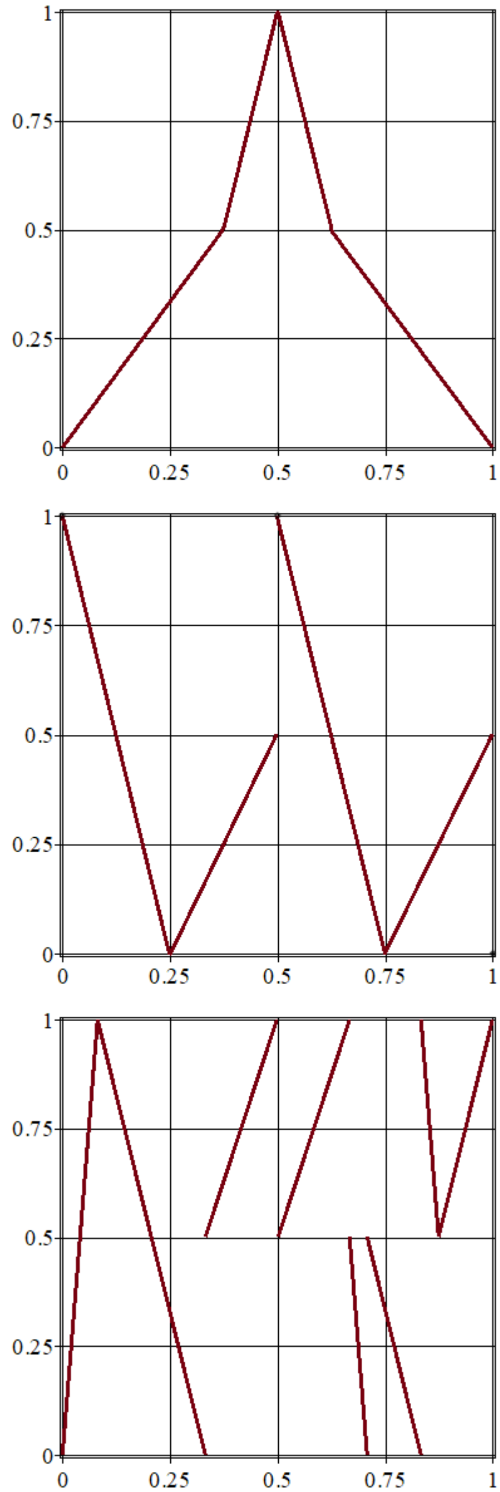

Figure 5.10: Examples of a $\mathscr{T}$ transformations types.

5.3.1 Proposition 5.2.2 proof.

(1.) $\tau(x)$ is a piecewise monotonic expanding map, since for every interval $I_i = (x_{i-1}, x_i)$ we have $|\tau(x_i) - \tau(x_{i-1})| \in \left\{\frac{1}{2}, 1\right\}$ and $|x_i - x_{i-1}| < \frac{1}{2}$. Thus, $|\tau'(x)| > 1$ for all $x \in I_i$.

(2.) In this proof we will start with the first type of maps in the family $\mathscr{T}$, when the transformations are symmetric. The other types are treated similarly.

Let τ be a piecewise linear symmetric transformation with N branches, $\tau \in \mathscr{T}$. N is even number, there are $\frac{N}{2}$ branches on each side of $x_{\frac{N}{2}} = \frac{1}{2}$. The interval $[0, \frac{1}{2}]$ is the domain of the branches $\tau_1, \tau_2, ..., \tau_{\frac{N}{2}}$ and the interval $[\frac{1}{2}, 1]$ is the domain of the branches $\tau_{\frac{N}{2}+1}, \tau_{\frac{N}{2}+2}, ..., \tau_N$. The pairs τ_i, τ_{N+1-i} where $i = 1...\frac{N}{2}$, have the same image and the same slope of the line. For more clarity, see Figure 5.11. We consider the density in the form $f(x) = (c)\chi_{[0,\frac{1}{2}]}(x) + (2-c)\chi_{[\frac{1}{2},1]}(x)$. We have

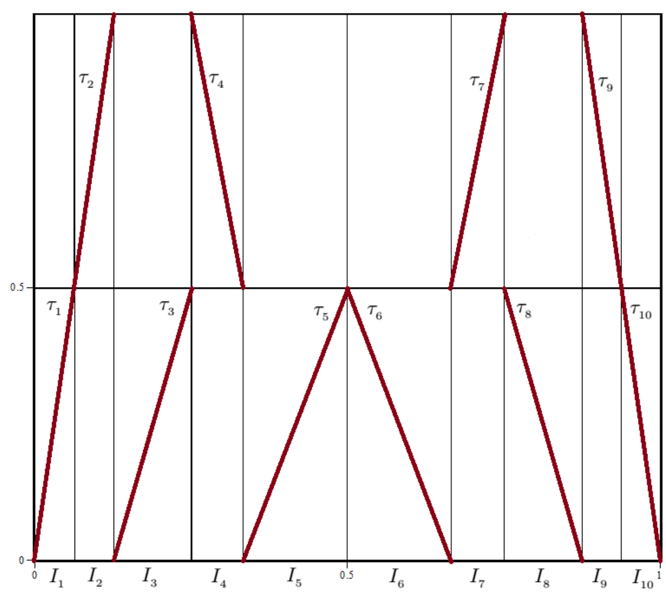

Figure 5.11: An example of a $\mathscr{T}$ transformation of **the first type** with $N = 10$.

$$
\begin{aligned}
P_\tau f(x) &= \sum_{i=1}^{N} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau([x_{i-1},x_i])}(x) \\
&= \frac{c}{|\tau_1'(\tau_1^{-1}(x))|}\chi_{\tau([x_0,x_1])} + \dots + \frac{c}{|\tau_{\frac{N}{2}}'(\tau_{\frac{N}{2}}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}-1},x_{\frac{N}{2}}\right]\right)} \\
&+ \frac{2-c}{|\tau_{\frac{N}{2}+1}'(\tau_{\frac{N}{2}+1}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}},x_{\frac{N}{2}+1}\right]\right)} + \dots + \frac{2-c}{|\tau_N'(\tau_N^{-1}(x))|}\chi_{\tau([x_{N-1},x_N])} \\
&= \frac{c+2-c}{|\tau_1'(\tau_1^{-1}(x))|}\chi_{\tau([x_0,x_1])} + \dots + \frac{c+2-c}{|\tau_{\frac{N}{2}}'(\tau_{\frac{N}{2}}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}-1},x_{\frac{N}{2}}\right]\right)} \\
&= \frac{2}{|\tau_1'(\tau_1^{-1}(x))|}\chi_{\tau([x_0,x_1])} + \dots + \frac{2}{|\tau_{\frac{N}{2}}'(\tau_{\frac{N}{2}}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}-1},x_{\frac{N}{2}}\right]\right)} \\
&= \left(\frac{1}{|\tau_1'(\tau_1^{-1}(x))|} + \frac{1}{|\tau_N'(\tau_N^{-1}(x))|}\right)\chi_{\tau([x_0,x_1])} + \dots \\
&\dots + \left(\frac{1}{|\tau_{\frac{N}{2}}'(\tau_{\frac{N}{2}}^{-1}(x))|} + \frac{1}{|\tau_{\frac{N}{2}+1}'(\tau_{\frac{N}{2}+1}^{-1}(x))|}\right)\chi_{\tau\left(\left[x_{\frac{N}{2}-1},x_{\frac{N}{2}}\right]\right)} \\
&= \frac{1}{|\tau_1'(\tau_1^{-1}(x))|}\chi_{\tau([x_0,x_1])} + \dots + \frac{1}{|\tau_{\frac{N}{2}}'(\tau_{\frac{N}{2}}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}-1},x_{\frac{N}{2}}\right]\right)} \\
&+ \frac{1}{|\tau_{\frac{N}{2}+1}'(\tau_{\frac{N}{2}+1}^{-1}(x))|}\chi_{\tau\left(\left[x_{\frac{N}{2}},x_{\frac{N}{2}+1}\right]\right)} + \dots + \frac{1}{|\tau_N'(\tau_N^{-1}(x))|}\chi_{\tau([x_{N-1},x_N])} \\
&= \sum_{i=1}^{N} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|}\chi_{\tau_i([x_{i-1},x_i])}(x).
\end{aligned}
$$

For each i, the interval $\tau_i([x_{i-1}, x_i])$ is equal to one of the intervals $[0, \frac{1}{2}], [\frac{1}{2}, 1]$ or $[0, 1]$. Let $J_1 = \{i : 1 \le i \le N, [0, \frac{1}{2}] \subset \tau([x_{i-1}, x_i])\}$ and $J_2 = \{i : 1 \le i \le N, [\frac{1}{2}, 1] \subset \tau([x_{i-1}, x_i])\}$. J_1 and J_2 are not necessarily disjoint. Then, we have

$$
P_\tau f(x) = \sum_{i=1}^{N} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|}\chi_{\tau_i([x_{i-1},x_i])}(x) = \sum_{i\in J_1} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|}\chi_{[0,\frac{1}{2}]}(x) + \sum_{i\in J_2} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|}\chi_{[\frac{1}{2},1]}(x)
$$

And by the linearity property of each of τ_i, first derivative of τ_i is constant. Therefore,

$$
f(x) = C_1\chi_{[0,\frac{1}{2}]}(x) + C_2\chi_{[\frac{1}{2},1]}(x).
$$

Since it is a density $C_2 = 2 - C_1$. If we set

$$c = \sum_{i \in J_1} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|},$$

we proved that $f(x) = (c)\chi_{[0,1/2]}(x) + (2-c)\chi_{[1/2,1]}(x)$ is P_τ invariant. It is easy to show that c ¡ 2 given conditions $C1$ and $C2$. The uniqueness of the invariant density, follows, for example, by the Folklore Theorem 6.1.1. of [11]. The proof is complete for the symmetric transformations type. If τ belongs to the second type or the third type of $\mathscr{T}$ then the construction of the proof in an analogous way.

(3.) Let $\tau_1, \tau_2 \in \mathscr{T}$ and f_1, f_2 be the density functions of τ_1, τ_2 respectively. First, we want to prove that, $P_{\tau_2} f_1(x) = f_2(x)$ and $P_{\tau_1} f_2(x) = f_1(x)$.

$$\begin{aligned}
P_{\tau_2} f_1(x) &= \sum_{i=1}^{n} \frac{f_1(\tau_{2,i}^{-1}(x))}{|\tau_2'(\tau_{2,i}^{-1}(x))|} \chi_{\tau_2([x_{i-1},x_i])}(x) \\
&= \frac{c_1}{|\tau_{2,1}'(x)|} \chi_{\tau_2([x_0,x_1])}(x) + \dots + \frac{c_1}{|\tau_{2,\frac{n}{2}}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}-1}, x_{\frac{n}{2}}\right]\right)}(x) \\
&+ \frac{2-c_1}{|\tau_{2,\frac{n}{2}+1}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}}, x_{\frac{n}{2}+1}\right]\right)}(x) + \dots + \frac{2-c_1}{|\tau_{2,n}'(x)|} \chi_{\tau_2([x_{n-1},x_n])}(x) \\
&= \frac{c_1+2-c_1}{|\tau_{2,1}'(x)|} \chi_{\tau_2([x_0,x_1])}(x) + \dots + \frac{c_1+2-c_1}{|\tau_{2,\frac{n}{2}}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}-1}, x_{\frac{n}{2}}\right]\right)}(x) \\
&= \frac{2}{|\tau_{2,1}'(x)|} \chi_{\tau_2([x_0,x_1])}(x) + \dots + \frac{2}{|\tau_{2,\frac{n}{2}}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}-1}, x_{\frac{n}{2}}\right]\right)}(x) \\
&= \frac{1}{|\tau_{2,1}'(x)|} \chi_{\tau_2([x_0,x_1])}(x) + \dots + \frac{1}{|\tau_{2,\frac{n}{2}}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}-1}, x_{\frac{n}{2}}\right]\right)}(x) \\
&+ \frac{1}{|\tau_{2,\frac{n}{2}+1}'(x)|} \chi_{\tau_2\left(\left[x_{\frac{n}{2}}, x_{\frac{n}{2}+1}\right]\right)}(x) + \dots + \frac{1}{|\tau_{2,n}'(x)|} \chi_{\tau_2([x_{n-1},x_n])}(x) \\
&= f_2(x).
\end{aligned}$$

Note that the linearity of τ_2 gives us $|\tau_2'(\tau_{2,i}^{-1}(x))| = |\tau_2'(x)|$. The proof of $P_{\tau_1} f_2 = f_1$ will be the same argument, and the same proof steps for any two transformations in $\mathscr{T}$.

Next, we claim that the invariant density function of the random map

$$T = \{\tau_1, \tau_2, ..., \tau_n; p_1, p_2, ..., p_n\},$$

is $f = p_1 f_1 + p_2 f_2 + ... + p_n f_n$. We have to prove that f is a fixed point for P_T. For this we will use the properties of Frobenius–Perron operator for random maps and the $\mathscr{T}$ property, $P_{\tau_j} f_i = f_j$. Therefore,

$$
\begin{aligned}
P_T f(x) &= P_T(p_1 f_1(x) + p_2 f_2(x) + \dots + p_n f_n(x)) \\
&= p_1 P_T f_1(x) + p_2 P_T f_2(x) + \dots + p_n P_T f_n(x) \\
&= p_1 \left[p_1 P_{\tau_1} f_1(x) + p_2 P_{\tau_2} f_1(x) + \dots + p_n P_{\tau_n} f_1(x)\right] + p_2 [p_1 P_{\tau_1} f_2(x) + p_2 P_{\tau_2} f_2(x) \\
&+ \dots + p_n P_{\tau_n} f_2(x)] + \dots + p_n \left[p_1 P_{\tau_1} f_n(x) + p_2 P_{\tau_2} f_n(x) + \dots + p_n P_{\tau_n} f_n(x)\right] \\
&= p_1 \left[p_1 f_1(x) + p_2 f_2(x) + \dots + p_n f_n(x)\right] + p_2 \left[p_1 f_1(x) + p_2 f_2(x) + \dots + p_n f_n(x)\right] \\
&+ \dots + p_n \left[p_1 f_1(x) + p_2 f_2(x) + \dots + p_n f_n(x)\right] \\
&= p_1 \left[f(x)\right] + p_2 \left[f(x)\right] + \dots + p_n \left[f(x)\right] \\
&= f(x)[p_1 + p_2 + \dots + p_n] = f(x).
\end{aligned}
$$

This completes the proof.

5.4 More results in random map with constant probabilities which is constructed from $\mathscr{T}$.

The result in Theorem 2.5.10 can be generalized to random maps.

Theorem 5.4.1. *Let $\tau_1, \tau_2 \in \mathscr{T}$, $T_1 = \{\tau_1, \tau_2; p_1, p_2\}$ be a random map with constant probabilities $p_1, p_2 > 0$, $p_1 + p_2 = 1$, and let $h : I \to I$ be a diffeomorphism function. Then we have:*

(1) $P_{T_1} f = f$ implies $P_{T_2} g = g$, where $T_2 = \{\tau_3, \tau_4; p_1, p_2\}$, $\tau_3 = h \circ \tau_1 \circ h^{-1}$, $\tau_4 = h \circ \tau_2 \circ h^{-1}$ and $g = (f \circ h^{-1}).|(h^{-1})'|$;

(2) if f is a T_1-invariant density, then g is a T_2-invariant density.

Proof. We follow the proof of the Theorem 2.5.10.

(1) Let $P_{T_1} f = f$ where $f = p_1 f_1 + p_2 f_2$. Using the composition property for Frobenius-Perron operator in random map and the properties of the family $\mathscr{T}$ we get

$$
\begin{aligned}
P_{T_2}(P_h f) &= p_1 P_{\tau_3}(P_h f) + p_2 P_{\tau_4}(P_h f) \\
&= p_1 P_h \circ P_{\tau_1} \circ P_{h^{-1}} \circ P_h f + p_2 P_h \circ P_{\tau_2} \circ P_{h^{-1}} \circ P_h f \\
&= p_1 P_h \circ P_{\tau_1} \circ P_{h^{-1} \circ h} f + p_2 P_h \circ P_{\tau_2} \circ P_{h^{-1} \circ h} f \\
&= p_1 P_h \circ P_{\tau_1} f + p_2 P_h \circ P_{\tau_2} f \\
&= p_1 P_h \circ P_{\tau_1} (p_1 f_1 + p_2 f_2) + p_2 P_h \circ P_{\tau_2} (p_1 f_1 + p_2 f_2) \\
&= p_1 P_h \left(p_1 P_{\tau_1} f_1 + p_2 P_{\tau_1} f_2\right) + p_2 P_h \left(p_1 P_{\tau_2} f_1 + p_2 P_{\tau_2} f_2\right) \\
&= p_1 P_h (p_1 f_1 + p_2 f_1) + p_2 P_h (p_1 f_2 + p_2 f_2) \\
&= p_1^2 P_h f_1 + p_1 p_2 P_h f_1 + p_1 p_2 P_h f_2 + p_2^2 P_h f_2 \\
&= \left(p_1^2 P_h f_1 + p_1 p_2 P_h f_2\right) + \left(p_1 p_2 P_h f_1 + p_2^2 P_h f_2\right) \\
&= p_1 P_h (p_1 f_1 + p_2 f_2) + p_2 P_h (p_1 f_1 + p_2 f_2) \\
&= p_1 P_h f + p_2 P_h f \\
&= P_h f (p_1 + p_2) \\
&= P_h f
\end{aligned}
$$

We have to show that $P_h f = g$. But that immediately follows from

$$
P_h f(x) = \sum_{i=1}^{n} f \circ h_i^{-1} |(h_i^{-1})'| \chi_{[a_{i-1}, a_i]}(x) = (f \circ h^{-1}) |(h^{-1})'| = g.
$$

where h is monotonic $(n = 1)$, since it is a diffeomorphism.

Therefore,

$$
P_{T_2}(g) = P_{T_2}(P_h f) = P_h f = g
$$

(2) Follow the proof of Proposition 2.5.10 property number 2. $\square$

Definition 5.4.2. Let $T_1 = \{\tau_1, \tau_2; p_1, p_2\}$ and $T_2 = \{\sigma_1, \sigma_2; p_3, p_4\}$ be a two random maps with constant probabilities $p_1, p_2 p_3, p_4 > 0$, $p_1 + p_2 = 1$ and $p_3 + p_4 = 1$. Where the transformations $\tau_i : I \to I$ and $\sigma_i : J \to J$ on intervals I and J. T_1 and T_2 are called conjugate if there exists a bijection continuous map $h : I \longrightarrow J$ such that

$$\sigma_i(x) = (h \circ \tau_i \circ h^{-1})(x).$$

The map h is called the conjugation transformation.

Corollary 5.4.3. *If*

$$T_1 = \left\{\sigma, \omega; \frac{1}{2}, \frac{1}{2}\right\}$$

and

$$\sigma(x) = \begin{cases} 4x, & \textit{for } 0 \le x < \frac{1}{4} \\ \frac{3}{2} - 2x, & \textit{for } \frac{1}{4} \le x < \frac{1}{2} \\ 4x - 2, & \textit{for } \frac{1}{2} \le x < \frac{3}{4} \\ \frac{5}{2} - 2x, & \textit{for } \frac{3}{4} \le x < 1 \end{cases}, \quad \omega(x) = \begin{cases} 1 - 4x, & \textit{for } 0 \le x < \frac{1}{4} \\ 2x - \frac{1}{2}, & \textit{for } \frac{1}{4} \le x < \frac{1}{2} \\ \frac{3}{2} - 2x, & \textit{for } \frac{1}{2} \le x < \frac{3}{4} \\ 4x - 3, & \textit{for } \frac{3}{4} \le x < 1 \end{cases},$$

The invariant densities are $f_\sigma(x) = \frac{1}{2}\chi_{[0,\frac{1}{2}]}(x) + \frac{3}{2}\chi_{[\frac{1}{2},1]}(x)$, $f_\omega(x) = \frac{3}{2}\chi_{[0,\frac{1}{2}]} + \frac{1}{2}\chi_{[\frac{1}{2},1]}$, *correspondingly. Thus, the invariant density of* T_1 *is* $f(x) = 1$. T_1 *and* T_2 *are conjugated by* h*, then invariant density of* T_2 *is*

$$g = |(h^{-1})'|.$$

From **corollary** 5.4.3 we can find the relation,

$$h^{-1}(x) = \pm \int_0^x f_2(t)dt.$$

5.5 Examples of the conjugation transformations.

In this section, we give three examples. The first two examples show application of the Theorem 2.5.10. The last example is application of Theorem 5.4.1.

Example 5.5.1. Let $h : [0, 1] \to [0, 1]$ be defined by $h(x) = \sin^2(\frac{\pi x}{2})$, and let

$$\Lambda_1(x) = \begin{cases} 3x, & \text{for } 0 \leq x < \frac{1}{6} \\ \frac{3}{2}x + \frac{1}{4}, & \text{for } \frac{1}{6} \leq x < \frac{1}{2} \\ -\frac{3}{2}x + \frac{7}{4}, & \text{for } \frac{1}{2} \leq x < \frac{5}{6} \\ 3 - 3x, & \text{for } \frac{5}{6} \leq x < 1 \end{cases}.$$

The map $\Lambda_1(x)$ belongs to family $\mathscr{T}$, with a density function $f_{\Lambda_1}(x) = \frac{2}{3}\chi_{[0,\frac{1}{2}]}(x) + \frac{4}{3}\chi_{[\frac{1}{2},1]}(x)$. Consider the map $\Lambda_2(x) = \left(h \circ \Lambda_1 \circ h^{-1}\right)(x)$. Using trigonometric identities, we obtain:

$$\Lambda_2(x) = \begin{cases} x(4x-3)^2, & \text{for } 0 \leq x < \frac{2-\sqrt{3}}{4} \\ \frac{1}{2} - (2)^{\frac{1}{2}}\left((1-x)^{3/2} + (x)^{3/2}\right) + \frac{3}{(2)^{\frac{3}{2}}}\left((1-x)^{1/2} + (x)^{1/2}\right), & \text{for } \frac{2-\sqrt{3}}{4} \leq x < \frac{2+\sqrt{3}}{4} \\ (1-x)(4(1-x)-3)^2, & \text{for } \frac{2+\sqrt{3}}{4} \leq x < 1 \end{cases}.$$

By Theorem 2.5.10, the density function of Λ_2 is $f_{\Lambda_2} = (f_{\Lambda_1} \circ h^{-1}).|(h^{-1})'|$, and we have

$$f_{\Lambda_2}(x) = \begin{cases} \frac{2}{3}\frac{1}{\pi\sqrt{x(1-x)}}, & \text{for } 0 \leq x < \frac{1}{2} \\ \frac{4}{3}\frac{1}{\pi\sqrt{x(1-x)}}, & \text{for } \frac{1}{2} \leq x < 1 \end{cases}.$$

Example 5.5.2. In this example, we show a sequence of transformations that are conjugate to the tent map ($\tau_0 = 1 - |2x - 1|$) through conjugating functions $h_n(x) = \sin^{2n}(\frac{\pi x}{2})$. Consider $\tau_n(x) = \left(h_n \circ \tau_0 \circ h_n^{-1}\right)(x)$, $h_n^{-1}(x) = \frac{2}{\pi}\arcsin(x^{\frac{1}{2n}})$. Then,

$$\tau_n(x) = 2^{2n}x(1 - x^{\frac{1}{n}})^n.$$

Note that, if $n = 1$ we get the logistic map $\tau_1(x) = 4x(1 - x)$. The invariant density of the tent map τ_0 is constant $f_0(x) = 1$. By applying Theorem 2.5.10, we have $f_n = (f_0 \circ h_n^{-1}).|(h_n^{-1})'|$.

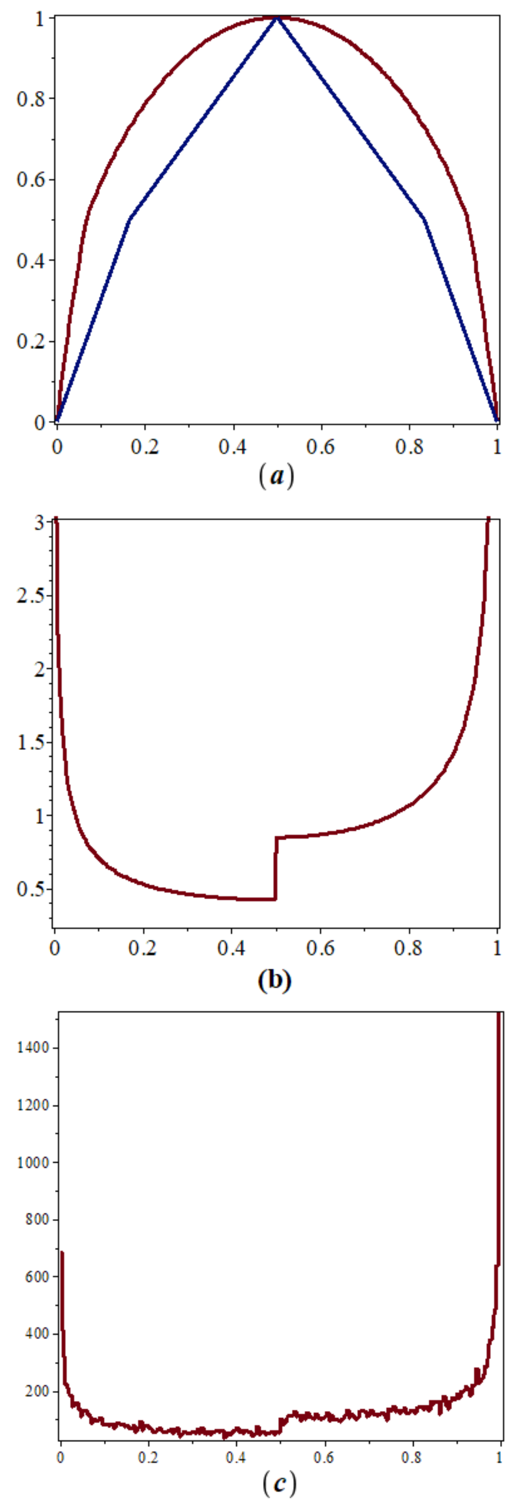

Figure 5.12: (a) Λ_1 and Λ_2 maps. (b) The graph of $f_{\Lambda_2}(x)$. (c) The histogram of 25000 iterations of Λ_2.

Therefore,

$$f_n(x) = 1 \cdot \left(h_n^{-1}(x)\right)' = \frac{1}{n\pi\sqrt{x^{2-\frac{1}{n}}(1-x^{\frac{1}{n}})}}$$

is the invariant density of the τ_n.

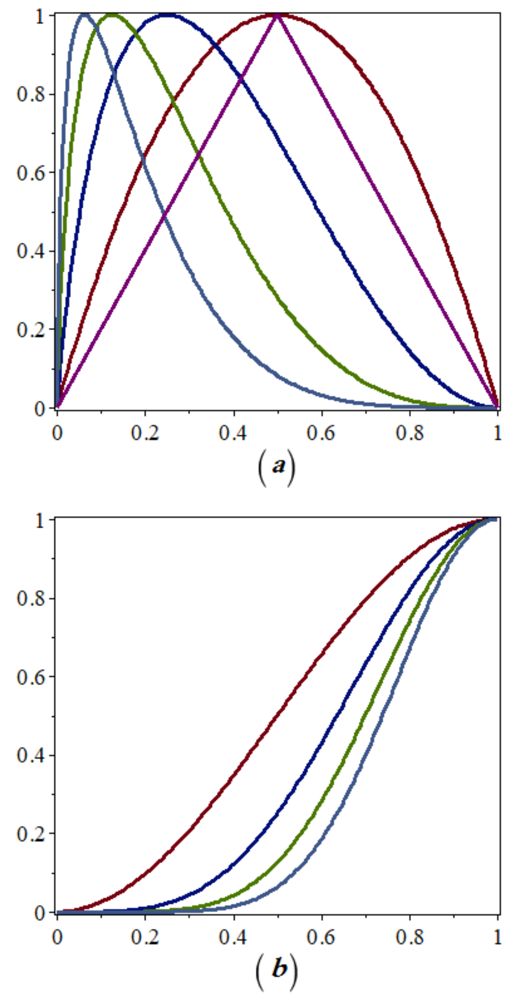

Figure 5.13: (a) $\tau_0, \tau_1, \tau_2, \tau_3$ and τ_4 maps. (b) h_1, h_2, h_3 and h_4.

Example 5.5.3. In this example, we use transformations from Example 5.5.1 and from Example 5.5.2. Let $T_1 = \left\{\tau_0, \Lambda_1; \frac{1}{2}, \frac{1}{2}\right\}$ be a random map with constant probabilities. By applying Proposition 5.2.2, the invariant density function of the random map T_1 is $f(x) = \frac{1}{2}f_0(x) + \frac{1}{2}f_{\Lambda_1}(x)$, $i.e.$,

$$f(x) = \frac{5}{6}\chi_{[0,\frac{1}{2}]}(x) + \frac{7}{6}\chi_{[\frac{1}{2},1]}(x).$$

Consider the random map $T_2 = \left\{\tau_1, \Lambda_2; \frac{1}{2}, \frac{1}{2}\right\}$. Applying Theorem 5.4.1, the invariant density function of the random map T_2 is $g = (f \circ h^{-1}).|(h^{-1})'|$. We have

$$g(x) = \begin{cases} \frac{5}{6}\frac{1}{\pi\sqrt{x(1-x)}}, & \text{for } 0 \leq x < \frac{1}{2} \\ \frac{7}{6}\frac{1}{\pi\sqrt{x(1-x)}}, & \text{for } \frac{1}{2} \leq x < 1 \end{cases}.$$

5.6 Absolutely continuous invariant measures for a large class of transformations

Let I be the unit interval, and let $\mathcal{P}$ be a finite partition of I into subintervals. More specifically, $\mathcal{P} = \{I_t, ..., I_N\}$, where $I = \bigcup I_i$, $I_i^\circ \cap I_j^\circ = \emptyset$, $i \neq j$, $N \geq 2$ and $I_i = [x_{i-1}, x_i]$. Let $\mathscr{T}^*$ denote the class of a piecewise linear transformations $\tau : I \to I$ and $\tau_i = \tau_{|Ii}$, $i = 1, 2, ..., N$. We assume the following conditions:

C^*1. The image of every subinterval I_i belonging to $\left[0, \frac{1}{2}\right]$, is equal to $\left[0, \frac{1}{2}\right]$ or equal to $\left[\frac{1}{2}, 1\right]$.

C^*2. For every subinterval I_i belonging to $\left[0, \frac{1}{2}\right]$, there exist $I_j \in \left[\frac{1}{2}, 1\right]$ such that $\overline{\tau(I_i)} = [0, 1] \setminus (\tau(I_j))^\circ$, and $|\tau_i'(x)| = |\tau_j'(x)|$.

The graphs in Figure 5.14 show some examples of maps in $\mathscr{T}^*$:

Lemma 5.6.1. *Let τ be a piecewise linear transformations $\tau : \left[0, \frac{1}{2}\right] \to [0, 1]$, defined on the partition $\mathscr{Q} = \{I_i\}$ and $\tau_i = \tau_{|I_i}$, $i = 1, 2, ..., K$. If the image of every subinterval $I_i \in \mathscr{Q}$, is equal to $\left[0, \frac{1}{2}\right]$ or equal to $\left[\frac{1}{2}, 1\right]$ then,*

$$\sum_{i=1}^{n} \frac{1}{\tan(\theta_i)} = 1, \tag{5.6.2}$$

where θ_i are the angles made by the graphs of τ_i with the lines $y = 0$ or $y = \frac{1}{2}$. An example is shown in the Figure 5.15.

Proof. Let $\mathscr{Q} = \left\{I_i = \left(x_{i-1}, x_i\right) : i = 1, 2, ..., K\right\}$ be a partition of $\left[0, \frac{1}{2}\right]$, where $0 = x_0 < x_1 < ... < x_{K-1} < x_K = \frac{1}{2}$, and it satisfies the lemma condition. Then we have $\tan(\theta_i) = \frac{1}{2(x_i - x_{i-1})}$

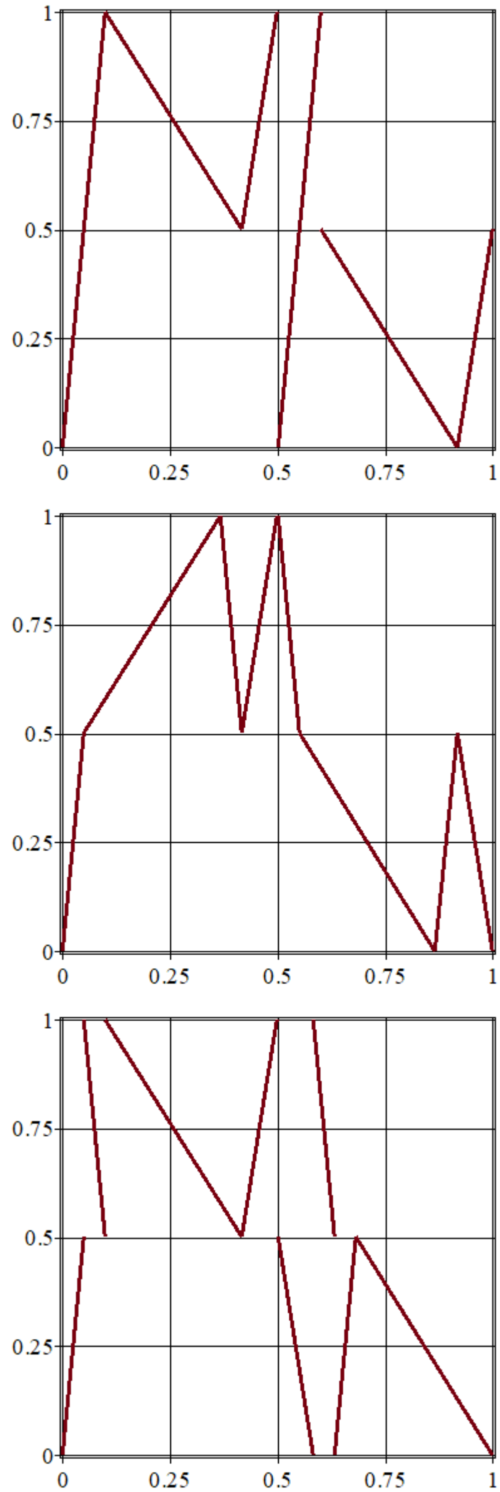

Figure 5.14: Examples of transformations satisfying $C^{*}1$ and $C^{*}2$.

for all i. Therefore,

$$\sum_{i=1}^{n} \frac{1}{\tan(\theta_i)} = 2(x_1 - 0) + 2(x_2 - x_1) + 2(x_3 - x_2) + ... + 2(\frac{1}{2} - x_{K-1}) = 1.$$

$\square$

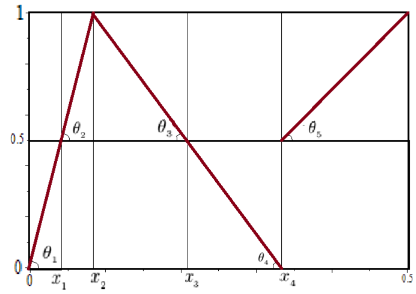

Figure 5.15: We have $\tan(\theta_1) = \frac{1}{2x_1}, \tan(\theta_2) = \frac{1}{2x_2 - 2x_1}, \tan(\theta_3) = \frac{1}{2x_3 - 2x_2}, \tan(\theta_4) = \frac{1}{2x_4 - 2x_3}$ and $\tan(\theta_5) = \frac{1}{1 - 2x_4}$.$\sum_{i=1}^{n} \frac{1}{\tan(\theta_i)} = 1$.

Proposition 5.6.3. *Any $\tau(x) \in \mathscr{T}^*$ enjoys the following properties:*

- *(1.) $\tau(x)$ is a piecewise monotonic expanding map.*
- *(2.) The $\sum_{i=1}^{n} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{\tau_i([x_{i-1}, x_i])}(x) = 1$.*
- *(3.) The invariant density of τ is $f(x) = 1$.*

Proof. We will prove (2.). Let $\mathscr{Q} = \{I_i = (x_{i-1}, x_i) : i = 1, 2, ..., K\}$ be a partition of $[0, 1]$, where $0 = x_0 < x_1 < ... < x_{\frac{N}{2}} = \frac{1}{2} < x_{\frac{N}{2}+1} < ... < x_{N-1} < x_N = 1$. Let the transformation $\tau \in \mathscr{T}^*$ be defined on the partition $\mathscr{Q}$ and $\tau_i = \tau_{|I_i}$, then τ_i satisfies the conditions C^*1 and C^*2 . Then we have:

$$\sum_{i=1}^{N} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{\tau([x_{i-1},x_i])}(x) =$$

$$\left(\sum_{\tau(I_i)=[0,\frac{1}{2}]} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{[0,\frac{1}{2}]}(x) + \sum_{\tau(I_i)=[0,\frac{1}{2}]} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{[\frac{1}{2},1]}(x) \right)$$

$$+ \left(\sum_{\tau(I_i)=[\frac{1}{2},1]} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{[0,\frac{1}{2}]}(x) + \sum_{\tau(I_i)=[\frac{1}{2},1]} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \chi_{[\frac{1}{2},1]}(x) \right)$$

$$= \left(\sum_{i=1}^{\frac{N}{2}} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \right) \chi_{[0,\frac{1}{2}]}(x) + \left(\sum_{i=1}^{\frac{N}{2}} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} \right) \chi_{[\frac{1}{2},1]}(x).$$

By Lemma 5.6.1, we have $\sum_{i=1}^{\frac{N}{2}} \frac{1}{|\tau_i'(\tau_i^{-1}(x))|} = 1$. The proof is complete. □

The following example gives a piecewise linear map in $\mathscr{T}^*$.

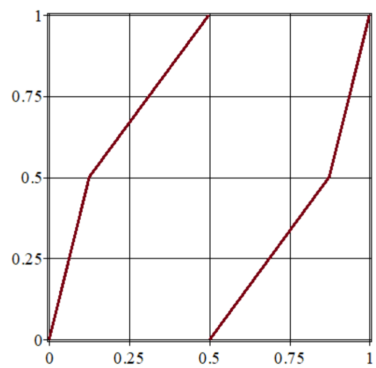

Figure 5.16: Map τ_8 of Example 5.6.4 ($x_1 = \frac{1}{8}$ and $x_2 = \frac{7}{8}$).

Example 5.6.4. Let $X = [0, 1]$, and let us consider the following map in the family $\mathscr{T}^*$.

$$\tau_8(x) = \begin{cases} \frac{1}{2x_1}x, & \text{for } 0 \le x < x_1 \\ \frac{1}{2(\frac{1}{2}-x_1)}(x - \frac{1}{2}) + 1, & \text{for } x_1 \le x < \frac{1}{2} \\ \frac{1}{2(\frac{1}{2}-x_1)}(x - \frac{1}{2}), & \text{for } \frac{1}{2} \le x < x_3 \\ \frac{1}{2x_1}(x - 1) + 1, & \text{for } x_3 \le x < 1 \end{cases}$$

Where $x_0 = 0 < x_1 < x_2 = \frac{1}{2} < x_3 < x_4 = 1$, $x_3 = 1 - x_1$. We claim that, the invariant density of $\tau_8(x)$ is $f(x) = 1$.

$$
\begin{aligned}
P_{\tau_8} f(x) &= \sum_{i=1}^{4} \frac{f(\tau_{8,i}^{-1}(x))}{|\tau_8'(\tau_{8,i}^{-1}(x))|} \chi_{\tau_8([x_{i-1},x_i])}(x) \\
&= (2x_1)\, \chi_{\left[0,\frac{1}{2}\right]}(x) + (1-2x_1)\, \chi_{\left[\frac{1}{2},1\right]}(x) + (1-2x_1)\, \chi_{\left[0,\frac{1}{2}\right]}(x) + (2x_1)\, \chi_{\left[\frac{1}{2},1\right]} \\
&= (1)\chi_{\left[0,\frac{1}{2}\right]}(x) + (1)\chi_{\left[\frac{1}{2},1\right]}(x) \\
&= 1 = f(x).
\end{aligned}
$$

Chapter 6

On a particular class of Markov transformations

In the this chapter we will present a class of transformations which are $\mathcal{P}$-semi-Markov transformations but after more detailed analysis turn out to be $\mathcal{P}$-Markov transformations as well.

6.1 Preliminaries

In this section, we going to present some results about Markov transformations and semi-Markov transformations and the matrix representation of the corresponding Frobenius–Perron operator.

Definition 6.1.1. Let $\mathcal{P} = \{I_1, I_2, ..., I_n\}$, $I_i, = (x_{i-1}, x_i)$, $i = 1, 2, ..., n$ be a partition of I, $\tau : I \to I$ and $\tau_i = \tau_{|Ii}$. If for each $i = 1, 2, ..., n$ τ_i is a homeomorphism from I_i to a connected union of intervals of $\mathcal{P}$, then τ is called a $\mathcal{P}$-Markov transformation.

The partition $\mathcal{P} = \{I_i\}_{i=1}^{n}$ is referred to as a Markov partition with respect to τ. Let us define the class of transformations $\mathcal{T}_{\mathcal{M}}$ as

$$\mathcal{T}_{\mathcal{M}} = \left\{\tau : |\tau'(x)| > 0, \text{on each } I_i\right\}.$$

If each τ_i is also linear on I_i we say τ is a piecewise linear $\mathcal{P}$-Markov transformation and denote this class of $\mathcal{P}$-Markov transformations by $\mathcal{L}_{\mathcal{M}} \subset \mathcal{T}_{\mathcal{M}}$. The class of piecewise linear $\mathcal{P}$-Markov transformations is a class of piecewise monotonic transformations and the matrix representation of the corresponding Frobenius–Perron operator can be calculated easily. In fact, it is a matrix defined in the following theorem.

Theorem 6.1.2. *[11] Let $\tau : (I, \mathfrak{B}, \lambda) \to (I, \mathfrak{B}, \lambda)$ where $\tau \in \mathcal{L}_{\mathcal{M}}$ with respect to the partition $\mathcal{P} = \{I_1, I_2, ..., I_n\}$. Then there exists a $n \times n$ matrix M_τ such that $P_\tau f = f M_\tau^T$ for every piecewise constant $f = (f_1, f_2, ..., f_n)$. The matrix $M_\tau = (m_{ij})_{1 \le i,j \le n}$ is defined by*

$$m_{ij} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}.$$

The following theorem is proved in [14].

Theorem 6.1.3. *If a transformation $\tau \in \mathcal{L}_{\mathcal{M}}$ and is piecewise expanding, then any τ-invariant density is constant on intervals of P.*

In [22], the authors introduce a new class of piecewise linear transformation called a $\mathcal{P}$-semi-Markov. They prove a number of theorems about these new maps, showing that given a piecewise constant density on intervals of a partition, it is always possible to find a $\mathcal{P}$-semi-Markov transformation that leaves the density invariant.

Definition 6.1.4. A transformation $\tau : I \to I$ is called $\mathcal{P}$**-semi-Markov transformation** if there exist disjoint intervals $Q_j^{(i)}$ such that for any $i = 1, ..., N$ we have $I_i = \cup_{j=1}^{k(i)} Q_j^{(i)}$, $\tau_{Q_j^{(i)}}$ is monotonic, and $\tau(Q_j^{(i)}) \in \mathcal{P}$.

Theorem 6.1.3 can be generalized to the semi-Markov case.

Theorem 6.1.5. *[22] Let τ be a $\mathcal{P}$-semi-Markov transformation, and $\tau_{Q_j^{(i)}}$ is linear with slope greater than 1 for $j = 1, ..., k(i)$, $i = 1, ..., N$. Then any τ-invariant density is constant on intervals of $\mathcal{P}$.*

Next theorem from [22] and [54] allows us to represent a Frobenius-Perron operator of a $\mathcal{P}$-semi-Markov map as a matrix.

Theorem 6.1.6. *Let $\tau : (I, \mathfrak{B}, \lambda) \to (I, \mathfrak{B}, \lambda)$ be a piecewise linear $\mathcal{P}$-semi-Markov transformations. The Frobenius-Perron matrix is $M_\tau = (a_{ij})_{1\le i,j\le n}$, where*

$$a_{ij} = \begin{cases} \sum_k |\left(\tau_k^{(i)}\right)'|^{-1}, & \text{if } \tau(Q_k^{(i)}) = I_j \\ 0, & \text{otherwise.} \end{cases}$$

Let $f = [f_1, f_2, .., f_n]$. If $f = fM_\tau$, f is a fixed point of the Frobenius-Perron operator P_τ of τ, $i.e.$, for all j,

$$\sum_{i=1}^{n}\sum_{k} \frac{f(\tau_k^{(i)}(x)^{-1})}{|\tau'(\tau_k^{(i)}(x)^{-1})|} . \chi_{\tau(Q_k^{(i)})}(x) = f_j, \tag{6.1.7}$$

where the second summation runs over all subintervals of I_i such that $\tau(Q_k^{(i)}) = I_j$. This equation can be simplified by noticing that $f(\tau_k^{(i)}(x)^{-1}) = f_j$, so

$$\sum_{i=1}^{n}\sum_{k} \frac{f_i}{|\tau_k^{(i)'}|} = f_j,$$

$$\sum_{i=1}^{n}\left(\sum_{k} \frac{1}{|\tau_k^{(i)'}|}\right) .f_i = f_j,$$

$$\sum_{i=1}^{n} a_{ij} .f_i = f_j.$$

Hence equation (6.1.7) is equivalent to $f = fM_\tau$.

6.2 A class of transformations belonging to the class of Markov transformations

It is easy to notice that any $\mathcal{P}$-Markov transformation is $\mathcal{P}$-semi-Markov and there are many $\mathcal{P}$-semi-Markov transformations that are not $\mathcal{P}$-Markov.

Let $\mathcal{Z}$ denote the set of integers. Define the class of transformations $\mathcal{T}_{\mathcal{Z}}$ as

$$\mathcal{T}_{\mathcal{Z}} = \left\{\tau : |\tau'(x)| = z_i > 0, \text{on each } I_i, z_i \in \mathcal{Z}\right\}.$$

If each τ_i is also linear on I_i we say τ is a piecewise linear transformation. We define the class $\mathcal{L}_{\mathcal{Z}} = \{\tau \in \mathcal{T}_{\mathcal{Z}}; \tau \text{ is a piecewise linear transformation}\}$.

Theorem 6.2.1. *If $\tau : (I, \mathfrak{B}, \lambda) \to (I, \mathfrak{B}, \lambda)$, where $\tau \in \mathcal{L}_{\mathcal{Z}}$ with respect to the partition $\mathcal{P} = \mathcal{P}\{x_1, x_2, ..., x_n\}$, where $x_1, x_2, ..., x_n$ are rational numbers. Then there is a finite partition of I with equal subintervals $\mathcal{P}^* = \mathcal{P}^*\{x_1^*, x_2^*, ..., x_N^*\}$ such that τ is a $\mathcal{P}^*$-Markov transformations.*

Proof. Let $\mathcal{P}^*$ be a finite partition for I with equal subintervals

$$\mathcal{P}^* = \mathcal{P}^*\{x_1^*, x_2^*, ..., x_N^*\},$$

and the number of the subintervals N is equal to the least common multiple (LCM) of the denominators of the interval limits of $\mathcal{P}$ and the denominators of the intercepts of the linear transformations. For example, if the the interval limits are $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{2}{3}$ and $x_3 = 1$, and the intercepts of the linear transformations are $\left\{2, \frac{1}{2}, \frac{7}{9}\right\}$ then the N =LCM$(4, 3, 2, 9) = 36$. Based on that, we have $\{x_1, x_2, ..., x_n\} \subseteq \{x_1^*, x_2^*, ..., x_N^*\}$, and for each i the image $\tau(x_i^*) \in \{x_1^*, x_2^*, ..., x_N^*\}$. Therefore, $\tau|_{(x_{i-1}^*, x_i^*)}$ is monotonic and $\tau|_{(x_{i-1}^*, x_i^*)}$ is a union of intervals of $\mathcal{P}^*$. □

Example 6.2.2. Let

$$\tau_1(x) = \begin{cases} 2x + \frac{1}{3}, & \text{for } 0 \le x < \frac{1}{3} \\ \frac{4}{3} - 3x, & \text{for } \frac{1}{3} \le x < \frac{4}{9} \\ \frac{11}{4} - 4x, & \text{for } \frac{4}{9} \le x < \frac{2}{3} \\ 3x - 2, & \text{for } \frac{2}{3} \le x < 1 \end{cases}$$

The map of $\tau_1(x)$ is a $\mathcal{P}^*$-Markov transformation with respect to the partition $\mathcal{P}^*$, see Figure 6.1. The number of the intervals for the partition $\mathcal{P}^*$ is LCM$(4, 3, 9) = 36$.

Example 6.2.3. Let τ be a piecewise linear $\mathcal{P}$-semi-Markov transformation with the condition that the value of (τ_i') belongs to the non-zero integer numbers, and we have a disjoint intervals $Q_j^{(i)}$

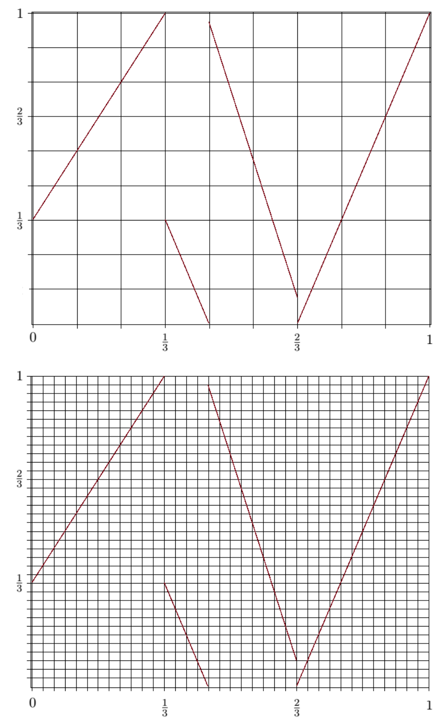

Figure 6.1: The map $\tau_1(x)$.

such that for any $i = 1, ..., N$ we have $I_i = \cup_{j=1}^{k(i)} Q_j^{(i)}$. Then there is a finite partition $\mathcal{P}^*$ with equal intervals and the number of the intervals is equal to the least common multiple (LCM) of the denominators of the interval limits, such that τ is $\mathcal{P}^*$-Markov transformations.

Example 6.2.4. Let

$$\tau_2(x) = \begin{cases} 4x + \frac{1}{3}, & \text{for } 0 \leq x < \frac{1}{6} \\ -8x + \frac{7}{3}, & \text{for } \frac{1}{6} \leq x < \frac{1}{4} \\ 4x - 1, & \text{for } \frac{1}{4} \leq x < \frac{1}{3} \\ 2x - \frac{2}{3}, & \text{for } \frac{1}{3} \leq x < \frac{1}{2} \\ -2x + 2, & \text{for } \frac{1}{2} \leq x < \frac{2}{3} \\ 4x - 2, & \text{for } \frac{2}{3} \leq x < \frac{3}{4} \\ 4x - \frac{8}{3}, & \text{for } \frac{3}{4} \leq x < \frac{5}{6} \\ -2x + 2, & \text{for } \frac{5}{6} \leq x < 1 \end{cases}$$

τ_2 is a piecewise linear $\mathcal{P}$-semi-Markov (see Figure (6.2)), with respect to the partition $\mathcal{P} = \{I_1, I_2, I_3\}$, where $I_1 = \left\{Q_1^{(1)}, Q_2^{(1)}, Q_3^{(1)}, Q_4^{(1)}, Q_5^{(1)}\right\}$, $I_2 = \left\{Q_1^{(2)}, Q_2^{(2)}\right\}$ and $I_3 = \left\{Q_1^{(3)}, Q_2^{(3)}, Q_3^{(3)}\right\}$. The map $\tau_2(x)$ is a Markov transformation with respect to the partition $\mathcal{P}^*$. The number of the intervals of the partition $\mathcal{P}^*$ is 24 (see Figure (6.3)). By using Definition 6.1.6 we have the transition matrix and the τ_2-invariant density function,

$$M_{\tau_2} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$f(x) = \begin{cases} \frac{6}{5}, & \text{for } 0 \leq x < \frac{1}{3} \\ \frac{18}{25}, & \text{for } \frac{1}{3} \leq x < \frac{2}{3} \\ \frac{27}{25}, & \text{for } \frac{2}{3} \leq x < 1 \end{cases}$$

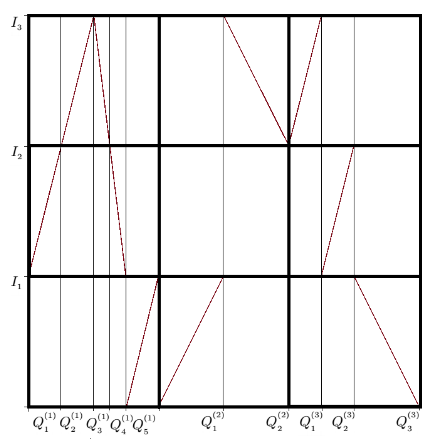

Figure 6.2: The map $\tau_2(x)$ is semi-Markov transformation.

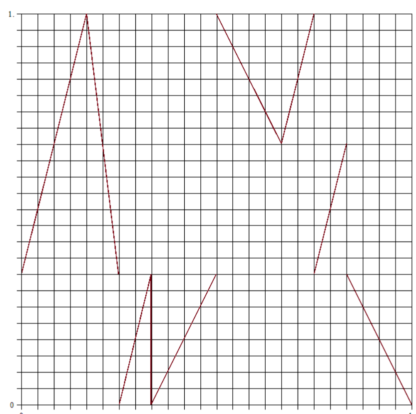

Figure 6.3: The map $\tau_2(x)$ is a Markov transformation with respect to the partition $\mathcal{P}^*$.

Chapter 7

Create chaotic maps.

In Chapter 5 we defined a family of transformations $\mathscr{T}$. This family is a class of chaotic maps for which we can find the matrix representation of the corresponding Frobenius–Perron operator. In this chapter, we present more properties for the transformations family $\mathscr{T}$. We created two classes of chaotic maps with desired invariant densities using two methods of solving the inverse Frobenius-Perron problem (IFPP). In the last section, we studied the Lyapunov exponent and the autocorrelation properties for one of these classes.

The Frobenius-Perron operator describes the evolution of density functions in a dynamical system. Finding the fixed points of this operator is referred to as the Frobenius-Perron problem ($i.e.$, $P_\tau f = f$, f is the invariant density under τ). Therefore, if we are given a density function f, the Inverse Frobenius–Perron Problem (IFPP) is to determine a point transformation τ such that the dynamical system $x_{i+1} = \tau(x_i)$ has f as its unique invariant probability density function. There are different approaches to solving the IFPP. The most popular of these branches matrix approach and conjugation approach.

7.1 Matrix approach.

Matrix method is outlined in the work of P. Góra and A. Boyarsky (1993 [22]), the work of Bollt (1999 [10]), the work of McDonald and Wyk (2017 [39]), the work of Rogers, Shorten and Naughton (2007 [45]) and the work of Nie and Coca (2016 [41]). The matrix method, gives us a relationship between the given density f and τ, where f is any piecewise constant density function. By expressing f in the form of the leading eigenvector, we can determine the Ulam's matrix and hence the chaotic map τ. The column stochastic matrix can be treated as Ulam's transition matrix.

7.1.1 The Ulam's matrix

We will start by choosing a natural number $\rho > 0$, then we build the eigenvector

$$\pi^f = \left[\frac{1}{\rho}, \frac{1}{\rho}, ..., \frac{1}{\rho}, \frac{1}{\rho}, 1, 1, ..., 1, 1\right]_{2(\rho+1)}. \tag{7.1.1}$$

The number of $\frac{1}{\rho}$'s is the same as number of 1's . Next we determined the Ulam's matrix A_ρ. The matrix A_ρ has the form:

$$\begin{bmatrix}
\frac{1}{\rho+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\rho+1} \\
\frac{1}{\rho+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\rho+1} \\
\frac{1}{\rho+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\rho+1} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\frac{1}{\rho+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\rho+1} \\
\frac{1}{\rho+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\rho+1} \\
0 & \frac{\rho}{\rho+1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\rho}{\rho+1} & 0 \\
0 & \frac{1}{\rho+1} & \frac{\rho-1}{\rho+1} & \ddots & 0 & 0 & 0 & 0 & \iddots & \frac{\rho-1}{\rho+1} & \frac{1}{\rho+1} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\
0 & 0 & 0 & \ddots & \frac{2}{\rho+1} & 0 & 0 & \frac{2}{\rho+1} & \iddots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \frac{\rho-1}{\rho+1} & \frac{1}{\rho+1} & \frac{1}{\rho+1} & \frac{\rho-1}{\rho+1} & \iddots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & \frac{\rho}{\rho+1} & \frac{\rho}{\rho+1} & 0 & \iddots & 0 & 0 & 0
\end{bmatrix} \tag{7.1.2}$$

The size of the matrix A_ρ is $2(\rho + 1) \times 2(\rho + 1)$. There are several interesting properties of matrix A, which we outline here:

1. Each column sums to 1 in the matrix A (column stochastic),

2. The matrix is a positive matrix,

3. The matrix has a single dominant eigenvalue of value 1, and the corresponding eigenvector π^f is

$$\left[\frac{1}{\rho}, \frac{1}{\rho}, ..., \frac{1}{\rho}, \frac{1}{\rho}, 1, 1, ..., 1, 1\right]_{2(\rho+1)}.$$

4. There is a unique invariant probability density function

$$f(x) = \left(\frac{2}{\rho+1}\right)\chi_{[0,\frac{1}{2}]}(x) + \left(\frac{2\rho}{\rho+1}\right)\chi_{[\frac{1}{2},1]}(x).$$

By FP-eigenvector of A we mean the eigenvector with eigenvalue 1.

Example 7.1.3. Let $\rho = 5$, then we have

$$\pi^f = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Therefore,

$$A_5 = \begin{bmatrix}
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
0 & \frac{5}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\
0 & \frac{1}{6} & \frac{4}{6} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & 0 \\
0 & 0 & \frac{2}{6} & \frac{3}{6} & 0 & 0 & 0 & 0 & \frac{3}{6} & \frac{2}{6} & 0 & 0 \\
0 & 0 & 0 & \frac{3}{6} & \frac{2}{6} & 0 & 0 & \frac{2}{6} & \frac{3}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

There is the unique invariant probability density function $f(x) = \left(\frac{1}{3}\right)\chi_{[0,\frac{1}{2}]}(x) + \left(\frac{5}{3}\right)\chi_{[\frac{1}{2},1]}(x)$. See Figure 7.1. $\square$

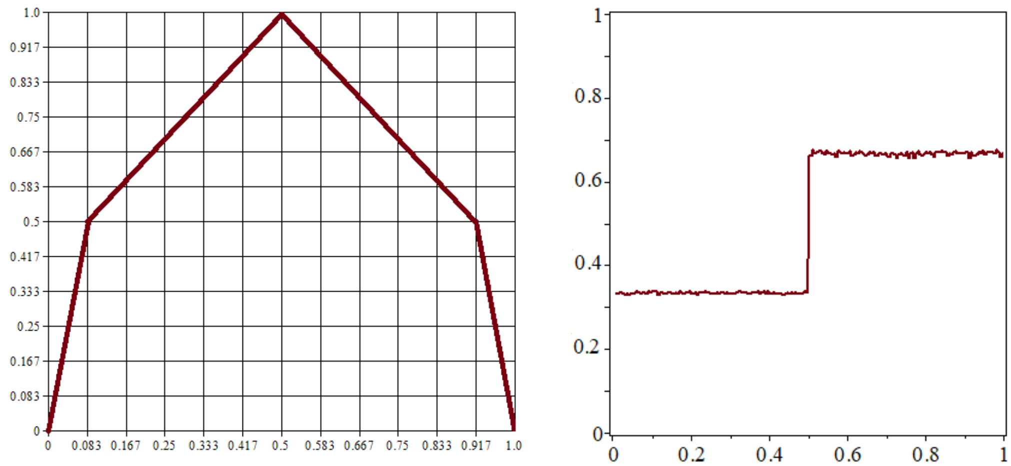

Figure 7.1: The one-dimensional map corresponding to matrix A_5 and the histogram of 10^6 iterations of τ (approximation to the invariant density).

7.1.2 The chaotic map

There are infinitely many of integer numbers $\rho > 0$, for each ρ we consider the eigenvector $\pi^f = \left[\frac{1}{\rho}, \frac{1}{\rho}, ..., \frac{1}{\rho}, \frac{1}{\rho}, 1, 1, ..., 1, 1\right]^T$ with respect to $(2(\rho+1))$ equal subintervals. This eigenvector is FP-eigenvector of A_ρ. To generate a chaotic map with a desired invariant density, we interpret the matrix A_ρ as a map of the unit interval to itself. Generally, τ_ρ is defined as:

$$\tau_\rho(x) = \begin{cases} (\rho+1)x, & \text{for } 0 \le x < \frac{1}{2(\rho+1)} \\ \frac{\rho+1}{\rho}(x-\frac{1}{2})+1, & \text{for } \frac{1}{2(\rho+1)} \le x < \frac{1}{2} \\ -\frac{\rho+1}{\rho}(x-\frac{1}{2})+1, & \text{for } \frac{1}{2} \le x < \frac{2\rho+1}{2(\rho+1)} \\ -(\rho+1)(x-1), & \text{for } \frac{2\rho+1}{2(\rho+1)} \le x < 1 \end{cases} \tag{7.1.4}$$

τ_ρ is $1-$parameter family of maps of the interval $[0,1]$ into itself, $\tau_\rho \in \mathscr{T}$. For each ρ, we have the invariant probability density function

$$f(x) = \left(\frac{2}{\rho+1}\right)\chi_{[0,\frac{1}{2}]}(x) + \left(\frac{2\rho}{\rho+1}\right)\chi_{[\frac{1}{2},1]}(x).$$

We can check τ_ρ preserves the density f, using the Frobenius-Perron operator P_τ. We have

$$
\begin{aligned}
P_{\tau_\rho} f(x) &= \sum_{i=1}^{4} \frac{f(\tau_{\rho i}^{-1}(x))}{|\tau_\rho'(\tau_{\rho i}^{-1}(x))|}\chi_{\tau(x_{i-1},x_i)}(x) \\
&= \frac{2}{\rho+1}\frac{1}{\rho+1}\chi_{[0,\frac{1}{2}]} + \frac{2}{\rho+1}\frac{\rho}{\rho+1}\chi_{[\frac{1}{2},1]} + \frac{2\rho}{\rho+1}\frac{\rho}{\rho+1}\chi_{[\frac{1}{2},1]} + \frac{2\rho}{\rho+1}\frac{1}{\rho+1}\chi_{[0,\frac{1}{2}]} \\
&= \left(\frac{2}{(\rho+1)^2} + \frac{2\rho}{(\rho+1)^2}\right)\chi_{[0,\frac{1}{2}]} + \left(\frac{2\rho}{(\rho+1)^2} + \frac{2\rho^2}{(\rho+1)^2}\right)\chi_{[\frac{1}{2},1]} \\
&= \left(\frac{2}{\rho+1}\right)\chi_{[0,\frac{1}{2}]} + \left(\frac{2\rho}{\rho+1}\right)\chi_{[\frac{1}{2},1]}.
\end{aligned}
$$

The invariant density f is a fixed point of P_{τ_ρ}. Therefore, f is the unique invariant probability density function under τ_ρ. The uniqueness follows, for example, by the Folklore Theorem 6.1.1. in [11]. A natural entry point to delve into the study of maps τ_ρ is to consider τ_ρ when $\rho = 1$, because it is one of the famous maps studied in the dynamical systems. It's called the tent map $\tau_1 = 1 - |2x - 1|$. The τ_ρ is easy to analyze because it is piecewise linear. The τ_ρ possesses rich dynamics and has several interesting properties. We will start by finding the fixed point of τ_ρ. The nature of the fixed points plays an important role in analyzing the dynamical behaviour of the map. The fixed points satisfy the relation $\tau_\rho(x) = x$. There are two fixed points of τ_ρ at $x = 0$ and at $x = \dfrac{3\rho+1}{4\rho+2}$ and they are unstable fixed points ($|\tau'(x)| > 1$). The plot of τ_ρ^n map, $n-$th iterates of τ_ρ are shown in the Figure 7.2. τ_ρ^n map has exactly 2^n periodic points of period n. The set of periodic points of the tent map is countable and it is dense in $[0, 1]$. Using the definition of τ_ρ, $\tau_\rho^2 = \tau_\rho \circ \tau_\rho$ can be explicitly written as follows:

$$
\tau_\rho^2(x) = \begin{cases}
(\rho+1)^2 x, & \text{for } 0 \le x < \frac{1}{2(\rho+1)^2} \\
\frac{(\rho+1)^2}{\rho}x + \left(\frac{\rho-1}{2\rho}\right), & \text{for } \frac{1}{2(\rho+1)^2} \le x < \frac{1}{2(\rho+1)} \\
-\left(\frac{\rho+1}{\rho}\right)^2 x + \frac{1}{2}\left(\frac{\rho+1}{\rho}\right)^2 + \left(\frac{\rho-1}{2\rho}\right), & \text{for } \frac{1}{2(\rho+1)} \le x < \frac{1}{2} - \frac{\rho}{2(\rho+1)^2} \\
-\frac{(\rho+1)^2}{\rho}x + \frac{(\rho+1)^2}{2\rho}, & \text{for } \frac{1}{2} - \frac{\rho}{2(\rho+1)^2} \le x < \frac{1}{2} \\
\frac{(\rho+1)^2}{\rho}x - \frac{(\rho+1)^2}{2\rho}, & \text{for } \frac{1}{2} \le x < \frac{1}{2} + \frac{\rho}{2(\rho+1)^2} \\
\left(\frac{\rho+1}{\rho}\right)^2 x - \frac{1}{2}\left(\frac{\rho+1}{\rho}\right)^2 + \left(\frac{\rho-1}{2\rho}\right), & \text{for } \frac{1}{2} + \frac{\rho}{2(\rho+1)^2} \le x < \frac{2\rho+1}{2(\rho+1)} \\
-\frac{(\rho+1)^2}{\rho}x + \frac{(\rho+1)^2}{\rho} + \left(\frac{\rho-1}{2\rho}\right), & \text{for } \frac{2\rho+1}{2(\rho+1)} \le x < 1 - \frac{1}{2(\rho+1)^2} \\
-(\rho+1)^2 x + (\rho+1)^2, & \text{for } 1 - \frac{1}{2(\rho+1)^2} \le x < 1
\end{cases} \tag{7.1.5}
$$

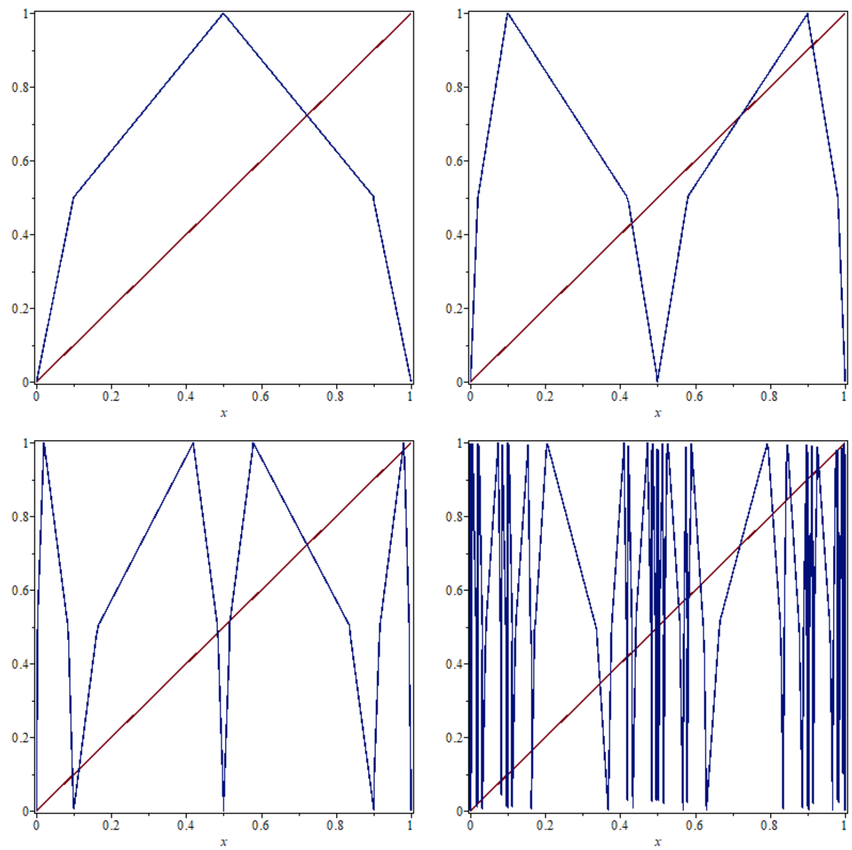

Figure 7.2: First, Second, Third and Sixth Iterate for the τ_ρ Map, $\rho = 4$.

7.2 Conjugation approach

Conjugation approach, developed by Ulam (1960 [51]) Grossman and Thomae (1977 [24]), Gyorgyi and Szepfalusy (1984 [25]), Baranovsky and Daems (1995 [6]) and Jiang (1995 [31]). Conjugation function approach, makes use of the following equivalence relation between two mappings: The map $\tau : I \to I$ is conjugate to a piecewise linear map $\sigma : J \to J$ if there exists a homeomorphism map $h : I \to J$ such that $\tau = h \circ \sigma \circ h^{-1}$. For a σ with a uniform invariant density, τ can then be found via the conjugating function (see Example 5.5.1 when $\rho = 2$). Conjugacy takes orbits of τ to orbits of σ. This follows since we have $h(\tau^n(x)) = \sigma^n(h(x))$ for all $x \in I$, so h takes the $n-$th point on the orbit of x under τ to the $n-$th point on the orbit of $h(x)$ under σ.

Similarly, h^{-1} takes orbits of σ to orbits of τ.

Definition 7.2.1. [25] Let $I = [0,1]$. A map $\tau : I \to I$ is said to be unimodal if there exists a turning point $x^* \in I$ such that the map τ can be expressed as $\tau(x) = \min\{\tau_l(x), \tau_r(x)\} = \tau_l(x) \cdot \chi_{[0,x^*]} + \tau_r(x) \cdot \chi_{[x^*,1]}$, where $\tau_l : [0,x^*] \to I$ and $\tau_r(x) : [x^*,1] \to I$ are continuous, differentiable except possibly at finite points, monotonically increasing and decreasing, respectively, and onto the unit-interval in the sense that $\tau_l(0) = \tau_r(1) = 0$ and $\tau_l(x^*) = \tau_r(x^*) = 1$.

A unimodal map $\tau : I \to I$ is said to be complete chaotic if it is chaotic in a probabilistic sense that it preserves an absolutely continuous invariant measure, $i.e.$, there exists such an absolutely continuous invariant measure, denoted as η, that the following identity is held for any measurable subset A of I:

$$\eta(A) = \eta(\tau^{-1}(A)).$$

By the previous definition, τ_ρ is a class of unimodal complete chaotic maps.

Now, to explain the conjugation function approach, we start with the τ_ρ, with a density function $f_{\tau_\rho}(x) = \frac{2}{\rho+1}\chi_{\left[0,\frac{1}{2}\right]}(x) + \frac{2\rho}{\rho+1}\chi_{\left[\frac{1}{2},1\right]}(x)$. Let $h : [0,1] \to [0,1]$ be defined by $h(x) = \frac{x}{2-x}$. Consider the map $\sigma_\rho(x) = \left(h \circ \tau_\rho \circ h^{-1}\right)(x)$. We have

$$\sigma_\rho(x) = \begin{cases} \frac{(\rho+1)x}{1-\rho x}, & \text{for } 0 \le x < \frac{1}{4\rho+3} \\ \frac{5\rho x+\rho+3x-1}{3\rho-\rho x-3x+1}, & \text{for } \frac{1}{4\rho+3} \le x < \frac{1}{3} \\ \frac{3\rho-\rho x-3x+1}{5\rho x+\rho+3x-1}, & \text{for } \frac{1}{3} \le x < \frac{2\rho+1}{2\rho+3} \\ \frac{-(\rho+1)(x-1)}{-\rho+\rho x+3x+1}, & \text{for } \frac{2\rho+1}{2\rho+3} \le x < 1 \end{cases} \tag{7.2.2}$$

By Theorem 2.5.10, the density function of σ_ρ is $f_{\sigma_\rho} = (f_{\tau_\rho} \circ h^{-1}).|(h^{-1})'|$, and we have

$$f_{\sigma_\rho}(x) = \begin{cases} \frac{2}{\rho+1}\frac{2}{(1+x)^2}, & \text{for } 0 \le x < \frac{1}{3} \\ \frac{2\rho}{\rho+1}\frac{2}{(1+x)^2}, & \text{for } \frac{1}{3} \le x < 1 \end{cases}.$$

See Figure 7.3.

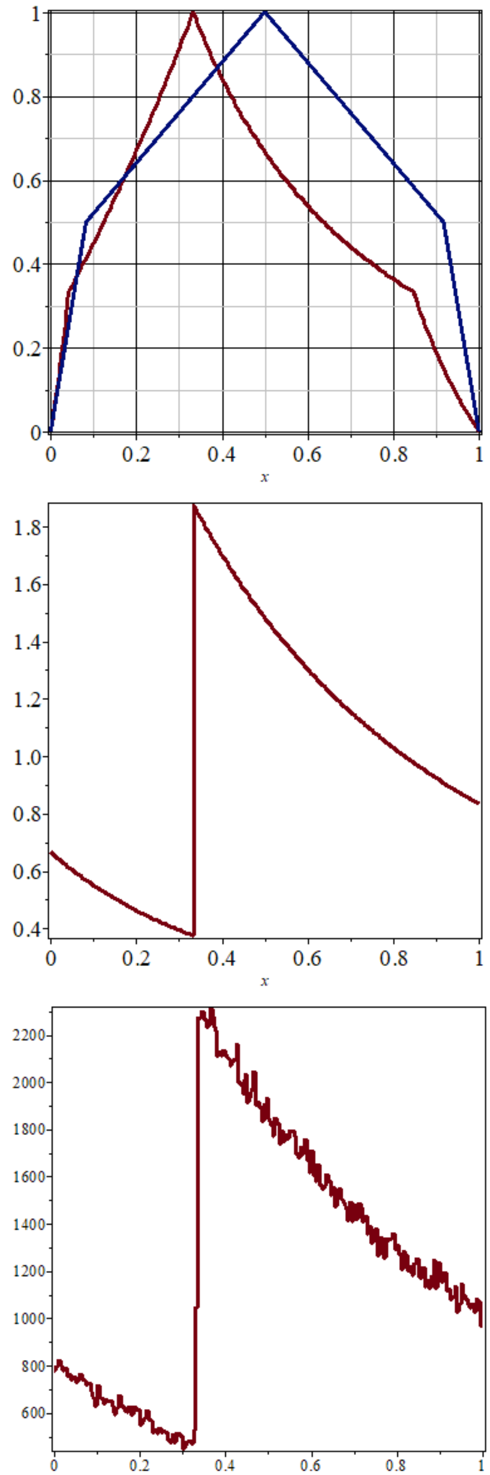

Figure 7.3: (Top) τ_5 and σ_5 maps. (Middle) The graph of $f_{\sigma_5}(x)$. (Bottom) The histogram of 250000 iterations of σ_5 (approximation to the invariant density).

7.3 Lyapunov exponent and autocorrelation properties

The theory of autocorrelation functions is given in most books on time series analysis, for examples [37], [17] and [33]. In this section, we explain the relationship between chaotic maps and the important concepts of Lyapunov exponent, mean functions, autocovariance functions, and autocorrelation functions.

For a $1-$dimensional map, the Lyapunov exponent gives the average rate of divergence of trajectories over the attractor. We used definition from [15].

Definition 7.3.1. For discrete system (one-dimensional maps or fixed point iterations) $x_n = \tau(x_{n-1})$ and for an orbit starting with x_0, the Lyapunov exponent can be defined as follows:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|\tau'(x_i)| \tag{7.3.2}$$

The Lyapunov exponent can be negative (stable fixed point), zero (bifurcation point), and positive (chaos). In [2] a chaotic orbit of a map τ is defined to be a bounded orbit with a positive Lyapunov exponent. If μ is an ergodic invariant measure for τ, then the right hand side in Equation 7.3.2 converges to $\int_0^1 \log|\tau'(x)|d\mu$ by the Birkhoff Ergodic Theorem. Thus $\int_0^1 \log|\tau'(x)|d\mu$ measures the exponent of the speed of the divergence.

Definition 7.3.3. The number $\int_0^1 \log|\tau'(x)|d\mu$ is called the Lyapunov exponent of τ.

It is straightforward to calculate the value of the Lyapunov exponent of τ_ρ because τ_ρ is a piecewise linear map and it is related to the slopes of the map segments. Then we have

$$\int_0^1 \log|\tau_\rho'(x)|d\mu = \log(\rho+1) - \frac{\rho}{\rho+1}\log(\rho). \tag{7.3.4}$$

The value of the Lyapunov exponent in Equation 7.3.4 is positive since

$$(\rho+1)\log(\rho+1) > (\rho)\log(\rho),$$

that is a signature of chaos. The Lyapunov exponent is dependent on the value of the ρ, when the value of ρ increases, the value of the Lyapunov exponent decreases.

$$\lim_{\rho \longrightarrow \infty} \left(\int_0^1 \log |\tau_\rho'(x)| d\mu \right) = \lim_{\rho \longrightarrow \infty} \left(\log(\rho+1) - \frac{\rho}{\rho+1} \log(\rho) \right) = 0 \tag{7.3.5}$$

The Figure 7.4 shows the values of the Lyapunov exponent (Equation 7.3.2) for τ_3 are getting arbitrarily close to the value 0.2442190501 which we calculated by Equation 7.3.4 and the values of the Lyapunov exponent for τ_{20} are getting arbitrarily close to the value 0.08314310874.

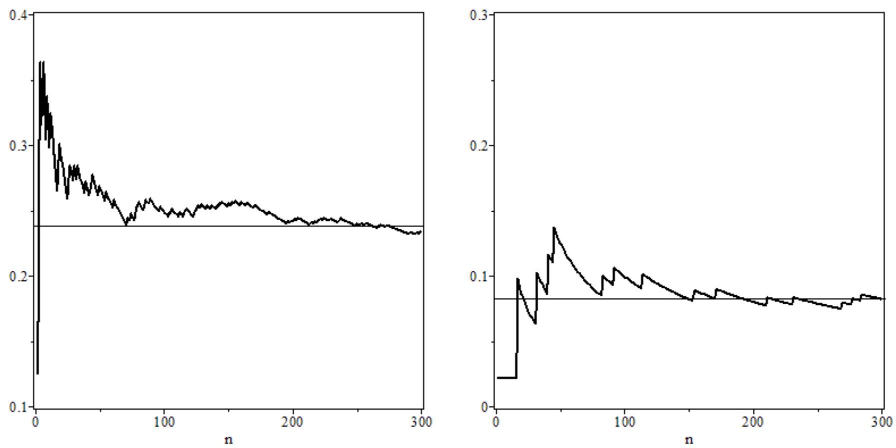

Figure 7.4: The Lyapunov exponent for τ_3 (left). The Lyapunov exponent for τ_{20} (right). For lags $1 - 300$.

Autocorrelation is another of the statistical properties of chaotic maps. We use the definition of autocorrelation in [37] to find how quickly two approaching trajectories diverge.

We calculate

$$\begin{aligned} C(k) &= \int_0^1 \left(\tau_\rho^{n+k}(x) - \overline{\tau_\rho(x)} \right) \left(\tau_\rho^{n}(x) - \overline{\tau_\rho(x)} \right) d\mu(x) \\ &= \int_0^1 \tau_\rho^{n+k}(x) \tau_\rho^{n}(x) d\mu(x) - \left(\overline{\tau_\rho(x)} \right)^2, \end{aligned} \tag{7.3.6}$$

and

$$C(0) = \overline{\tau_\rho^2(x)} - \left(\overline{\tau_\rho(x)} \right)^2, \tag{7.3.7}$$

where

$$\overline{\tau_\rho(x)} = \int_0^1 \tau_\rho^n(x) d\mu(x), \forall n. \tag{7.3.8}$$

$$\overline{\tau_\rho^2(x)} = \int_0^1 \left(\tau_\rho^n(x)\right)^2 d\mu(x). \tag{7.3.9}$$

Note that τ_ρ^n denotes $n-$times composition of the map τ_ρ. The mean function $\overline{\tau_\rho(x)}$ is independent of n since the integral is with respect to the invariant measure. By Equation 7.3.6, if $C(k) = 0$, then there is no correlation.

The definition of the autocorrelation coefficient of a stochastic process is

$$R(k) = \frac{C(k)}{C(0)}\,, R(0) = 1. \tag{7.3.10}$$

For $n = 1$, the following are calculated for τ_ρ

$$C(0) = \frac{\rho^2 + 14\rho + 1}{48(\rho+1)^2}, \tag{7.3.11}$$

$$\overline{\tau_\rho(x)} = \frac{3\rho+1}{4(\rho+1)}, \tag{7.3.12}$$

and

$$C(k) = \int_0^1 \tau_\rho^{1+k}(x)\tau_\rho(x)d\mu(x) - \left(\overline{\tau_\rho(x)}\right)^2. \tag{7.3.13}$$

τ_ρ	$\overline{\tau_\rho(x)}$	$C(0)$	$R(1)$	$R(2)$	$R(3)$	$R(4)$	$R(5)$
τ_2	0.583333	0.076388	-0.171717	0.057239	-0.019079	0.006359	-0.002119
τ_4	0.650000	0.060833	-0.336986	0.202191	-0.121315	0.072789	-0.043673
τ_7	0.687500	0.048177	-0.466216	0.349662	-0.262246	0.196684	-0.147513
τ_{15}	0.718750	0.035481	-0.634174	0.554902	-0.485539	0.424847	-0.371741
τ_{30}	0.733870	0.028637	-0.765524	0.716135	-0.669933	0.626712	-0.586279
τ_{99}	0.745000	0.023308	-0.910625	0.892413	-0.874564	0.857073	-0.839932

Table 7.1: Table for autocorrelation function values when $n = 1$, for lags $1 - 5$.

We conclude from the Table (7.1) and from the Figure (7.5), that the autocorrelation function $R(k)$ decays from $R(0) = 1$ down to zero. The autocorrelation function is dependent on the value of the ρ. When the value of ρ increases, the absolute value of the autocorrelation function increases. For large values of ρ the autocorrelation function decays slowly to zero. If ρ are close to 1 leading to a rapid decay of the autocorrelation function, and approaching trajectories diverge rapidly. τ_ρ is

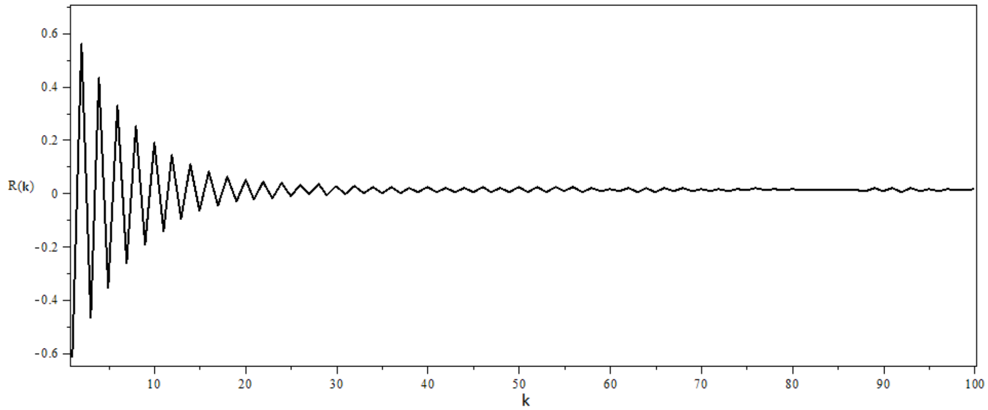

Figure 7.5: Autocorrelations $R(k)_{(n=1)}$ for $\tau_{15}(x)$ map, for lags $1 - 100$.

mixing since $C(k) \longrightarrow 0$ as $k \longrightarrow \infty$ for every ρ. Note that, mixing implies weak mixing and weak mixing implies ergodicity.

Note that,

$$\begin{aligned}\int_0^1 \left(\tau_\rho^{n+k}(x) - \tau_\rho^n(x)\right)^2 d\mu(x) = \int_0^1 \left(\tau_\rho^{n+k}(x)\right)^2 d\mu(x) - 2\int_0^1 \tau_\rho^{n+k}(x)\tau_\rho^n(x) d\mu(x) \\ + \int_0^1 \left(\tau_\rho^n(x)\right)^2 d\mu(x).\end{aligned} \tag{7.3.14}$$

We have

$$\begin{aligned}C(k) &= \overline{\tau_\rho^2(x)} - \left(\overline{\tau_\rho(x)}\right)^2 - \frac{1}{2}\int_0^1 \left(\tau_\rho^{n+k}(x) - \tau_\rho^n(x)\right)^2 d\mu(x) \\ &= C(0) - \frac{1}{2}\int_0^1 \left(\tau_\rho^{n+k}(x) - \tau_\rho^n(x)\right)^2 d\mu(x).\end{aligned} \tag{7.3.15}$$

Then, we calculate

$$2\left(C(0) - C(k)\right) = \int_0^1 \left(\tau_\rho^{n+k}(x) - \tau_\rho^n(x)\right)^2 d\mu(x). \tag{7.3.16}$$

Chapter 8

Conclusion

In this thesis, we used the Perron-Frobenius operator P_τ with respect to the single map τ and P_T with respect to the random map with constant probabilities T, to discuss the properties of the Frobenius-Perron operator with respect to the nonautonomous random map $\widehat{T}_0^n$ and to prove the existence of an absolutely continuous invariant measure for the nonautonomous random maps on $[a, b]$ using the theory of bounded variation and the Lasota-Yorke inequality from [42]. We extend the Lasota-Yorke inequality, in [36], into a form for the composition function that constructed from a piecewise C^2 transformations.

We present results on the existence of invariant measures for nonautonomous random dynamical systems, generalizing Krylov-Bogoliubov Theorem.

We discuss the dynamics of a new family of transformations. We find the invariant density for any transformation belonging to our family and the special properties for this family allowed us to get a unique acim under the random maps with constant probabilities T which is constructed from our family maps. We created two classes of chaotic maps with desired invariant densities by using one parameter ρ. We studied the Lyapunov exponent and the autocorrelation properties for one of these classes.

The multiplicity of the topics and the results in this thesis give us a great horizon for looking forward to the next steps. In chapters $3-4$, we present results on ergodic theory for the random maps with constant probabilities which are constructed from continuous maps on compact space, which build our ability to study the ergodic theory of multidimensional random maps on compact space. We also aspire to study ergodic theory for other spaces like Polish space (Polish space is a separable completely metrizable topological space). In chapters $5-7$, Our study focus on a type of transformations, which is a piecewise linear transformations. We will continue to study this type of transformations and try to find more results and more practical applications in the real life.

Bibliography

[1] AlFarajat, E. ; Góra, P. On the absolutely continuous invariant measures for the random maps. Submitted to, *International Journal of Bifurcation and Chaos IJBC-D-20-00505*. **Dec. 2020**.

[2] Alligood, K. T. ; Sauer, T. D. ; Yorke,J. A. *Chaos: An Introduction to Dynamical Systems*. *Springer-Verlag* **1997**, New York.

[3] Arnold, L. Random Dynamical Systems. *Springer* **1998**, Berlin.

[4] Bahsoun, W. Absolutely continuous invariant measures for random maps. Ph.D. Thesis, *Concordia University* **2004**.

[5] Bahsoun, W. ; Góra, P. Position dependent random maps in one and higher dimensions. *Stud. Math. 166* **2005**, 271.

[6] Baranovsky, A. ; Daems, D. Design of one dimensional chaotic maps with prescribed statistical properties. *Int. J. Bifurcation and Chaos*, *5*,**1995**, 1585–1598.

[7] Barnsley, M. F. Fractals Everywhere. *Academic. Press* **1988**, New York.

[8] Behnia, S. ; Akhshani, A. ; Ahadpour, S.; Akhavan, A. ; Mahmodi, H. Cryptography based on chaotic random maps with position dependent weighting probabilities. *Chaos Solitons Fractals 40* **2009**, 362.

[9] Behnia, S. ; Jafarizadeh,M. A.; Akhshani, A. Random Maps with Parameter-Dependent Probabilities. *Jou. of the Phy. Soc. of Japan Vol. 79, No. 12* **2010**, 124002.

[10] Bollt, E. M. Controlling chaos and the inverse Frobenius–Perron problem: Global stabilization of arbitrary invariant measures, *Int. J. Bifurcation and Chaos 10*, **2000** 1033–1050.

[11] Boyarsky, A. ; Góra, P. *Laws of Chaos*. Publisher: Birkhauser, **1997**.

[12] Boyarsky, A. ; Góra, P. *A dynamical model for interference effects and two slit experiment of quantum physics*. *Phys, Lett. A 168* **1992**, 103–112.

[13] Boyarsky, A. ; Góra, P. *A random map model for quantum interference*. *Commun. Nonlinear Sci. Numer. Simulation 15* **2010**, 1974–1979.

[14] Boyarsky, A. ; Haddad, G. All Absolutely Continuous Invariant Measures of piecewise linear Markov maps are piecewise constant. *Adv. in Appl. Math. 2*, **1981**, 284–289 .

[15] Choe, G. H. Computational Ergodic Theory. *Springer-Verlag Berlin Heidelberg* **2005**.

[16] Dunford, N. ; Schwartx, J. T. Linear Operators, Part I: General Theory. *Wiley Classics Library* Interscience **1988**, ISBN: 0471608483.

[17] Falconer, K. Fractal Geometry Mathematical. *Foundations and Applications*, Second Edition, Publisher: John Wiley and Sons, Ltd **2003**.

[18] Folland, G. B. Real Analysis: Modern Techniques and Their Applications *Pure and Applied Mathematics: A Wiley-Interscience Series* **1999**.

[19] Drutarovský, M. ; Galajda, P. A robust chaos-based true random number generator embedded in reconfigurable switched-capacitor hardware. *Radioengineering, Vol. 16, No. 3* **2007**, 120–127.

[20] Góra, P. Invariant densities for piecewise linear maps of the unit interval. *Ergod. Th. & Dynam. Sys. 29* **2009**, 1549–1583.

[21] Góra, P. ; AlFarajat, E. Existence of absolutely continuous invariant measure for the nonautonomous random maps. Unpublished manuscript, *Department of Mathematics and Statistics, Concordia University* **2021**, Montreal, Canada.

[22] Góra, P. ; Boyarsky, A. A matrix solution to the inverse Perron-Frobenius Problem. *Proceedings of the American Mathematics Society, Vol. 118, No.2* **1993**.

[23] Góra, P. ; Boyarsky, A. ; Keefe, C. Absolutely Continuous Invariant measures for Non-autonomous Dynamical Systems. *J. Math.Anal.Appl.470* **2019**, 159–168.

[24] Grossmann, S. ; Thomae, S. Invariant distributions and stationary correlation functions of one-dimensional discrete processes. *Zeitschrift für Naturforschung a* **1977**, *32*, 1353–1363.

[25] Gyögyi, G. ; Szépfalusy, P. Fully developed chaotic 1-d maps. *Z. Phys. B 55 no. 2,* **1984**, 179–186.

[26] Hazewinkel, M. (Ed.), Encyclopaedia of Mathematics, Supplement III, *Kluwer*, **2001** (ISBN 1-4020-0198-3).

[27] Hsieh, L.-Y. S. Ergodic theory of multidimensional random dynamical systems. *Ph.D. Thesis, University of Victoria* **2008**.

[28] Islam, S. Absolutely continuous invariant measures of linear interval maps. *Int. J. Pure Appl. Math. 27* **2006**, *4* , 449–464.

[29] Jabłoński, M. ; Góra, P. ; Boyarsky, A. A general existence theorem for absolutely continuous invariant measures on bounded and unbounded intervals. *Nonlinear World 3* **1996**, 183–200.

[30] Jafarizadeh M. A. ; Behnia, S. Hierarchy of random chaotic maps with an invariant measure, *J. Math. Phys. (N.Y.) 44* **2003**, 5386.

[31] Jiang,Y. On Ulam-von Neumann Transformations. *Commun. Math. Phys. 172* **1995**, 449–459.

[32] Kauffman, S. A. Metabolic stability and epigenesis in randomly constructed genetic nets. *Journal of Theoretical Biology, 22, Issue 3* **1969**, 437–467.

[33] Kirchgässner, G. ; Wolters, J. Introduction to Modern Time Series Analysis. *Springer Berlin Heidelberg* **2007**, New York.

[34] Kingman, J. ; Taylor, S. Introduction to Measure and Probability, *Cambridge University Press* **1966**.

[35] Lasota, A. ; Mackey, M. C. Chaos, fractals and noise. *Springer-Verlag*, AMS, volume 97, **1994**.

[36] Lasota, A. ; Yorke, J. A. On the Existence of Invariant Measures for Piecewise Monotonic Transformations. *Trans. Amer. Math. Soc. 186* **1973** 481–488.

[37] Lindsey, J. K. Statistical Analysis of Stochastic Processes in Time. *Cambridge University Press* **2004**.

[38] Lyubich, M. Almost Every Real Quadratic Map Is Either Regular or Stochastic. *Annals of Mathematics*, Second Series **2002** , 156, No. 1 ,1-78.

[39] McDonald, A. ; van Wyk, M. Solution of the Inverse Frobenius–Perron Problem for Semi–Markov Chaotic Maps via Recursive Markov State Disaggregation. *25th EUSIPCO* **2017** , ISBN 978-0-9928626-7-1 © EURASIP

[40] Metropolis, N. ; Ulam, S. A property of randomness of an arithmetical function. *American Mathematical Monthly 60* **1953**, 252–3.

[41] Nie, X. ; Coca, D. A matrix-based approach to solving the inverse Frobenius–Perron problem using sequences of density functions of stochastically perturb e d dynamical systems. *Commun Nonlinear Sci Numer Simulat 54* **2018**, 248–266.

[42] Pelikan, S. Invariant Densities for Random Maps of the Interval. *Trans. Amer.Math. Soc. 281* **1984**, 813–825.

[43] Pianigiani, G. First Return Map and Invariant Measures. *Israel J. Math. 35* **1980**, 32–48.

[44] Rényi, A. Representations for real numbers. *Acta Math. Acad. Sci. Hung. 8* **1957**, 477–493.

[45] Rogers ; Shorten ; Heffernan ; Naughton. Synthesis of piecewise-linear chaotic maps: Invariant densities, autocorrelation, and switching. *International Journal of Bifurcation and Chaos, Vol. 18, No. 8* **2008** 2169–2189.

[46] Roussas, G. G. An Introduction to Measure-Theoretic Probability, Second edition. *Academic Press is an imprint of Elsevier* **2014**.

[47] Royden, H. L. ; Fitzpatrick, P. M. Real Analysis, Fourth Edition Pearson Education. *Inc.,publishing as Prentice Hall* **2010**.

[48] Schenk-Hoppe, K. R. Random Dynamical Systems in Economics, working paper series. ISSN, *Institute of empirical research in economics, University of Zurich* **2000**, 1424–0459.

[49] Słomczyń ski, W. ; Kwapień, J. ; Życzkowski, K. A random map model for quantum interference. *Chaos: An Iterdis. Journal of Nonlinear Science 10* **2000**, 180.

[50] Swishchuk, A. ; Islam, S. Random Dynamical Systems in Finance. *Taylor and Francis Group* **2013**.

[51] Ulam, S. M. A Collection of Mathematical Problems. *Interscience* **1960**.

[52] Ulam, S. ; Von Neumann, J. On combination of stochastic and deterministic processes. *American Mathematical Society 53* **1947**, 1120.

[53] Walkden, C. Ergodic Theory *CUP* **2002**. (ISBN0521808413).

[54] Wei, N. Solutions of the Inverse Frobenius-Perron Problem. A Thesis for The Department of Mathematics and Statistics, Concordia University **2015**.