Simplicial volume and non-positive curvature

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Abstract

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The simplicial volume is a non-negative real valued homotopy invariant of closed connected manifolds measuring how efficient the fundamental class can be represented by real singular cycles. The problem of determining whether the simplicial volume of a given manifold is non-zero has been a challenge. It is known that the simplicial volume of negatively curved manifolds is positive [15]. Losing the negative bound on the sectional curvature, it has been shown that locally symmetric spaces of non-compact type have positive simplicial volume [21]. In their 2018 paper, C.Connell and S.Wang showed that the simplicial volume of n-manifolds with non-positive sectional curvature and negative $\lfloor \frac{n}{4} + 1 \rfloor$-Ricci curvature have positive simplicial volume, which confirms the Gromov's conjecture in special cases. The conjecture states that the simplicial volume of manifolds with non-positive sectional curvature and negative Ricci-curvature is positive. In this master thesis we will introduce required notions and preliminaries and present detailed proofs of the results mentioned above.

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Chapter 1

Introduction

For the first time, Gromov introduced the notion of the simplicial volume of a closed, connected and orientable manifold M in his 1982 paper [26]. This homotopy invariant is denoted by $||M|| \in [0, \infty)$, and measures how "efficiently" the fundamental class of M may be represented using real cycles.

Definition 1.0.1 (Simplicial Volume)**.** *Let M be an oriented closed connected n-manifold with fundamental class $[M] \in H_n(M, \mathbb{Z})$. The simplicial volume of M is defined as follows,*

$$||M|| := inf\{||\alpha||_1 : \partial\alpha = 0,\ [\alpha] = i_*([M])\}$$

where $i_* : H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})$ is the change of coefficient homomorphism, and for $\alpha = \Sigma_i a_i \sigma_i \in C_n(M, \mathbb{R})$ with $\sigma_i \in C^0(\Delta^n, M)$ we define,

$$||\alpha||_1 := \sum_i |a_i|$$

If M is not an orientable manifold then we take the oriented connected double cover of M denoted by $\overline{M}$, and define, $||M|| := \frac{1}{2}||\overline{M}||$.

To find a lower bound for the simplicial volume one may proceed as follows, for any arbitrary representative $\alpha = \sum_i a_i \sigma_i$ of the fundamental class find a special representative $\lambda(\alpha) = \sum_i b_i \tau_i$ with smaller $||.||_1$ so that the volume of all simplices τ_i are bounded by a universal constant only depending on the manifold, see the argument in Section 3.2. In 1980 H. Inoue and K. Yano proved that indeed in a negatively curved manifold geodesic simplicies have a universal bound on their volume, and for any simplex one can easily find a geodesic simplex that represents the same homology class. Therefore they proved the following theorem,

Theorem 1.0.2. *The simplicial volume of closed, connected Riemannian manifolds with negative sectional curvature is positive.*

We will explain the details of the proof of this theorem in Chapter 3. Moreover the simplicial volume of hyperbolic manifolds, i.e. manifolds with constant negative sectional curvature, has been explicitly calculated. Namely if M^n is a hyperbolic manifold, then $||M|| = vol(M)/\nu_n$, where ν_n is the maximum volume of all geodesic simplices in M. For instance the simplicial volume of closed connected surfaces with genus $g \geqslant 2$ is equal to

$4g - 4$, [26]. Now the question is what happens if we loose the negative upper bound for the sectional curvature, i.e. what can we say about the simplicial volume of non-positively curved manifolds. In the paper of Gromov([26]) the question was raised as whether the simplicial volume of closed locally symmetric space of non-compact type is positive. Note that a non-compact type manifold has non-positive sectional curvature by definition. This question was mentioned in variety of sources, [25], [34], [30] and [6] until 2005 when it was completely answered by J.F. Lafont and B. Schmidt, [21]. However there might be geodesic simplices with arbitrary large volume in non-positively curved manifolds but the idea of the proof in this case is to look for simplices with uniformly bounded volume by which the fundamental class can be represented. They follow Thurston's method, called the straightening method, see Chapter 4, to find these simplices. Note that the method in negative curvature case is also a straightening method, and Thurston's method is a refined version of the straightening used by H. Inoue and K.Yano. We will be explaining the details of the proof of J.F. Lafont and B. Schmidt for the following theorem in Chapter 5,

Theorem 1.0.3. *If M is a closed locally symmetric spaces of non-compact type, then $||M|| > 0$.*

Note that in the above theorem, "closed" shall mean compact and without boundary as usual. A different straightening procedure was developed by Savage to show the positivity of the simplicial volume for co-compact quotients of $SL_n(\mathbb{R})/SO_n(\mathbb{R})$, [30]. The following conjecture is attributed to Gromov.

Conjecture 1.0.4 (M.Gromov)**.** *If M is a closed manifold with non-positive curvature and negative Ricci-curvature, then $||M|| > 0$.*

In 2018 C. Connell and S. Wang showed the following theorem, which indeed confirms Conjecture 1.0.4 in some special cases,

Theorem 1.0.5. *The simplicial volume of closed n-manifolds with non-positive curvature and negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature is positive.*

The Ricci type curvature stated in the theorem above is a stronger assumption on the Ricci-curvature than negative Ricci-curvature, i.e. for $k = n$ we have $Ric_k < 0$ is equivalent to $Ricci < 0$, and for $k < n$, $Ric_k < 0$ implies $Ricci < 0$. For the definition of k-Ricci curvature see the Definition 6.4.3. We will also present a detailed proof of the Theorem 1.0.5. Non-positively curved Riemannian manifolds can be classified by their geometric rank, which is the minimum dimension of parallel stable Jacobi fields along geodesics, see Definition **??**, or in the case of symmetric spaces of non-compact type, the maximum k for which there is a complete totally geodesic submanifold isometric to $\mathbb{R}^k$. Higher rank manifolds turn out to have universal cover which are either metric products or symmetric spaces of non-compact type, [33], [17], and hence their simplicial volume is understood by theorems above. The remaining class of geometric rank one manifolds which includes manifolds with both vanishing and non-vanishing simplicial volume will be studied in Chapter 5 in a special case. The simplicial volume has been shown to vanish for several large classes of manifolds. Manifolds that admit a non-degenerate action of circle, or more generally a polarized F-structure [18], [10], [26], certain affine manifolds [24] and manifolds with amenable fundamental group [26]. Note that trivial, abelian, solvable and nilpotent groups are all amenable. In particular simply connected manifolds have zero simplicial volume. The simplicial volume of manifolds with free fundamental group also vanishes. Here are some other important results on the simplicial volume of manifolds some of which we will be using through the subsequent chapters.

Theorem 1.0.6 (Proportionality Principle)**.** *Let M, M' be two closed Riemannian manifolds with isometric universal cover, then*

$$\frac{||M||}{Vol(M)} = \frac{||M'||}{Vol(M')}$$

Theorem 1.0.7 ([29, Proposition 3.2.4])**.** *For a pair of closed oriented manifolds M, M' we have,*

$$C||M||.||M'|| \geqslant ||M \times M'|| \geqslant ||M|| \times ||M'||$$

where $C > 1$ is a constant that only depends on the dimension of $M \times M'$. If $dim(M) = m$ and $dim(M') = n$, the inequality holds for $C = \binom{m+n}{n}$.

Theorem 1.0.8. *For $n \geqslant 3$, the connected sum of a pair of n-manifolds M, M' satisfies*

$$||M\#M'|| = ||M|| + ||M'||$$

Theorem 1.0.9 ([31])**.** *Co-compact quotients of $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ have positive simplicial volume.*

Theorem 1.0.10 ([29, Proposition 3.2.5])**.** *Let M be an n-manifold. If $H^n(\pi_1(M)) = 0$ then $||M|| = 0$. In particular the simplicial volume any simply connected manifold vanishes.*

Theorem 1.0.11 ([29, Proposition 3.3.9])**.** *Let M be a locally symmetric n-manifold whose universal cover has a non-trivial compact factor, then $||M|| = 0$.*

Theorem 1.0.12 ([37])**.** *The simplicial volume of oriented closed connected smooth manifolds that admit a non-trivial smooth S^1-action vanishes.*

Theorem 1.0.13 ([26])**.** *Let M be a compact, connected and oriented Riemannian n-manifold with constant sectional curvature -1 on $int(M) = M \backslash \partial M$ and finite volume. Then,*

$$||M, \partial M|| = \frac{vol(M)}{\nu_n}$$

In particular, if M is closed then

$$||M|| = \frac{vol(M)}{\nu_n}$$

where ν_n is the volume of the regular ideal simplices in $\mathbb{H}^n$, i.e. $\nu_n = \sup\{vol(\sigma) : \sigma \in S_n(\mathbb{H}^n)\}$, where $S_n(M)$ is the set of all n-simplices in M.

Theorem 1.0.14 (No-gap Theorem, [16, Theorem A])**.** *Let $d \geqslant 4$ be an integer. For every $\epsilon > 0$ there is an orientable closed connected d-manifold such that $0 < ||M|| \leqslant \epsilon$. Hence the set of simplicial volumes of orientable closed connected d-manifolds is dense in $\mathbb{R}_{\geqslant 0}$.*

Theorem 1.0.15 (No-gap Theorem, [16, Theorem B])**.** *For every $q \in \mathbb{Q}_{\geqslant 0}$ there is an orientable closed connected 4-manifold M_q so that $||M_q|| = q$.*

Chapter 2

Preliminaries

2.1 Riemannian geometry

In this section we explain some basics on Riemannian geometry which is required to follow the rest of the notes.

Definition 2.1.1. *A Riemannian metric on a smooth manifold M is a smooth section of the 2-tensor bundle $T^*M \otimes T^*M$ so that for every point $x \in M$, $g(x) : T_xM \otimes T_xM \to \mathbb{R}$ is a symmetric positive-definite bi-linear form. A smooth manifold is called Riemannian if it is equipped with a Riemannian metric (Indeed every smooth manifold admits a Riemannian metric).*

Definition 2.1.2. *Let (M, g) be a Riemannian manifold. Let $\widetilde{M}$ be the universal cover of M with the covering map $p : \widetilde{M} \to M$. Since p is a local diffeomorphism one can pull back the metric g on $\widetilde{M}$, let us denote it by $\widetilde{g}$. The volume entropy of (M, g) is defined as follows,*

$$h((M, g)) := \lim_{R\to\infty} \frac{1}{R} vol(B_R(x))$$

where $x \in \widetilde{M}$ is a point and $B_R(x)$ is a ball around x with radius R with respect to the pull-back metric $\widetilde{g}$.

Remark 2.1.3. *The volume entropy defined above does not depend on the choice of the base point $x \in \widetilde{M}$. The volume entropy of the hyperbolic space with constant curvature -1 is equal to 1.*

The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric is a Riemannian manifold. The shortest path from one point to another in the Euclidean space is simply the straight line going through the points. To define the shortest path between two points p, q in an arbitrary Riemannian manifold (M, g) one might define the following distance function on the manifold,

$$d(p, q) := \inf_{\gamma} \int_I \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt$$

where $\gamma : I \to M$ varies on the set of piecewise differentiable curves in M from p to q. And define the curve that meets the infimum value above to be the shortest path from p to q.

But it is not clear why such a curve exist. A better way of approaching this generalization is the following, a straight line γ in the Euclidean space satisfies $\frac{d}{dt}(\dot{\gamma}) = \ddot{\gamma} = 0$, where γ is the velocity vector along γ which is indeed a vector field along the curve γ. It would be a good idea to define a "second derivative" of a curve, or in general derivative of a vector field on a Riemannian manifold.

Definition 2.1.4. *Let M be a smooth manifold. A connection on TM is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ satisfying the following properties, for every $X, Y, Z \in \Gamma(TM)$, $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$,*

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_Z(aX + bY) = a\nabla_Z X + b\nabla_Z Y$
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

The vector field $\nabla_X Y$ is called the covariant derivative of Y in the direction of X. If g is a fixed metric on M, then a connection ∇ is called Levi-Civita connection if it satisfies the following conditions,

- $\nabla_X Y - \nabla_Y X = [X, Y]$
- $\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

Remark 2.1.5. *For any metric g on a manifold M, there is a unique Levi-Civita connection.*

Definition 2.1.6. *Let M be a smooth manifold and $\gamma : I \to M$ be a peicewise smooth curve in M. Let ∇ be a connection on TM. There is a unique operator $D_t : \Gamma(TM_{|_\gamma}) \to \Gamma(TM_{|_\gamma})$ that satisfies the followings for every $X, Y, Z \in \Gamma(TM_{|_\gamma})$, $f \in C^\infty(I)$ and $a, b \in \mathbb{R}$,*

- $D_t(aX + bY) = aD_t X + bD_t Y$
- $D_t(fY) = f'Y + fD_t Y$
- *If V is an extendible vector field along γ and $\widetilde{V}$ is an extension of V then we have*

$$D_t V(t) = \nabla_{\dot{\gamma}} \widetilde{V}$$

From now on we shall always mean the Levi-Civita connection by ∇ and the corresponding covariant derivative along curve γ by D_γ.

Definition 2.1.7. *A curve γ in a Riemannian manifold M is called geodesic if $D_\gamma \dot{\gamma} = 0$.*

Definition 2.1.8. *Define a $(3,1)$-tensor field $R \in \Gamma(T^*(M)^{\otimes 3} \otimes TM)$ as follows,*

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

This $(3,1)$-tensor field is called curvature tensor. And for any two independent local vector fields X, Y on M the sectional curvature $sect(X, Y)$ is defined as follows,

$$sect(X, Y) := \frac{g(R(X, Y)Y, X)}{||X||^2||Y||^2 - g(X, Y)}$$

We may denote the sectional curvature by R too through the note. The Ricci-curvature of M is denoted by Ricci is a $(2,0)$-tensor field defined by $Ricci = tr(sect)$ where the trace *operator is defined by* $tr : \Gamma(T^*M^{\otimes k} \times TM^{\otimes l}) \to \Gamma(T^*M^{\otimes (k-1)} \times TM^{\otimes (l-1)})$ *by*

$$F \mapsto \Big((\omega_1, \ldots, \omega_{k-1}, X_1, \ldots, X_{l-1}) \mapsto$$

$$Tr\big(X \mapsto F(., \omega_1, \ldots, \omega_{k-1}, X, X_1, \ldots, X_{l-1})\big)\Big)$$

Definition 2.1.9. *A Riemannian manifold M is called negatively curved if $sect(X, Y) < -\delta < 0$ for some non-zero positive real number δ, every two independent tangent vectors $X, Y \in T_pM$ and every point $p \in M$. A Riemannian manifold is called non-positively curved if the sectional curvature is always non-positive and there is no universal negative upper bound for the sectional curvature. Positively curved and non-negatively curved are defined similarly.*

Definition 2.1.10. *Let M be a Riemannian manifold with Levi-Civita connection ∇. Let V be a vector field along a geodesic γ in M. We say that V is a Jacobi field along γ if it is a solution to the following equation,*

$$D_\gamma D_\gamma(-) + R(-, \dot\gamma)\dot\gamma = 0$$

If M is a complete non-positively curved Riemannian manifold, then a Jacobi field V along γ is called stable if there is a constant $C > 0$ so that $||V(t)|| < C$ for all $t \geqslant 0$.

Remark 2.1.11. *A Jacobi field J along a fixed geodesic γ is uniquely determined by the initial conditions $J(0) = v \in T_{\gamma(0)}M$ and $D_\gamma J(0) = w \in T_{\gamma(0)}M$. Furthermore for every $v, w \in T_{\gamma(0)}M$ there is a Jacobi field J with $J(0) = v$ and $D_\gamma J(0) = w$. Therefore the dimension of jacobi fields along a curve γ is $2n$ where $dim(M) = n$.*

Lemma 2.1.12. *(Uniqueness of stable Jacobi fields) Let M be a complete Riemannian manifold with non-positive curvatur, $\gamma : [0, \infty) \to M$ be a geodesic ray, and let $v \in T_pM$, $p = \gamma(0)$. There is a unique stable Jacobi field Y along γ with $Y(0) = v$.*

Proof. The uniqueness follows from the fact that in Hadamard spaces, complete non-positively curved manifolds, the length of a Jacobi field is a convex function. This is simply because,

$$\big(||Y(t)||^2\big)'' = \langle Y(t), Y(t)\rangle'' = 2\langle Y(t), Y'(t)\rangle'$$

$$= 2(||Y'(t)||^2 + \langle Y(t), Y''(t)\rangle)$$

$$= 2\big(||Y'(t)||^2 - \langle R(Y(t), \dot\gamma(t))\dot\gamma(t), Y(t)\rangle\big) \geqslant 0$$

Suppose Y is a stable Jacobi vector field along γ with $Y(0) = 0$. If we show that $Y \equiv 0$ then the uniqueness follows. Since $||Y(t)||^2$ is a convex function and bounded so we have, $(||Y(t)||^2)'$ is increasing and non-positive. But since $\frac{d}{dt}_{|t=0}||Y(t)||^2 = 2\langle Y(0), Y'(0)\rangle = 0$ so then we have $||Y(t)||^2$ is constant 0 and therefore $Y(t) = 0$. To prove the existence let Y_n be the unique Jacobi field along γ with $Y_n(0) = v$ and $Y_n(n) = 0$. Applying the Rauch's comparison theorem to $Y_n - Y_m$ we get,

$$||Y_n'(0) - Y_m'(0)|| \leqslant \frac{1}{t}||Y_n(t) - Y_m(t)||$$

in particular we have,

$$||Y'_n(0) - Y'_m(0)|| \leqslant \frac{1}{n}||Y_m(n)||$$

but since $||Y_m(t)||^2$ is convex and it reaches 0 at $t = m$ it must be monotone decreasing in the interval $[0, m]$, therefore,

$$||Y'_n(0) - Y'_m(0)|| \leqslant \frac{1}{n}||Y_m(n)|| \leqslant \frac{1}{n}||v||$$

Thus $||Y'_n(0)||$ is a Cauchy sequence with limit w, say. If Y_v is the unique Jacobi field along γ with $Y_v(0) = v$ and $Y'_v(0) = w$ then it follows immediately that Y_v, as the limit of Y_n, is a stable Jacobi field along γ. □

Theorem 2.1.13. *Let M be a non-positively curved complete and simply connected manifold. Then it is diffeomorphic to a Euclidean space. And M is a normal neighborhood of every point in the manifold, thus for any two points $p, q \in M$ there is a unique geodesic γ in M with $\gamma(0) = p$ and $\gamma(1) = q$.*

Proof. Let $p \in M$ be a point. Let $exp_p : U \to M$ be the exponential map that is defined on an open subset of $U \subset T_pM$ by $v \mapsto \gamma_{p,v}(1)$, where $\gamma_{p,v}$ is the unique geodesic passing through p with the velocity vector $\dot{\gamma}(0) = v$ that is defined on the interval $[0, 1]$. One can show that if the sectional curvature is non-positive then the exponential map defined above is a local diffeomorphism wherever it is defined. On the other hand it is known that a manifold is metric complete if and only if it is geodesically complete, that means all geodesics are defined on $\mathbb{R}$, i.e. $U = T_pM$. Putting all these together one shows that exp_p is a universal covering map, and since M is simply connected so exp_p is a global diffeomorphism. It is not hard to show that if $exp_p : U \to V$ is a diffeomorphism between a neighborhood U of $0 \in T_pM$ and a neighborhood $V \subset M$ of p then for every two points $p, q \in V$ there is a unique geodesic γ in V with $\gamma(0) = p$ and $\gamma(1) = q$. Thus the statement follows immediately. □

Theorem 2.1.14 (Gauss-Bonnet)**.** *Suppose γ is a curved polygon on an oriented Riemannian 2-manifold (M, g), and γ is positively oriented as the boundary of an open set Ω with compact closure. Then*

$$\int_\Omega K dA + \int_\gamma \kappa_N ds + \sum_i \epsilon_i = 2\pi$$

where K is the Gaussian curvature of M, dA is the volume form, $D_t\dot{\gamma}(t) = \kappa_N(t)N(t)$ and finally the $\epsilon'_i s$ are the rotation angles of γ at "sharp points", see Figure 2.1 and [22, pg.162].

Definition 2.1.15 (Geometric Rank)**.** *Let M be a symmetric manifold (of non-compact type). A k-flat submanifold of M is a complete totally geodesic submanifold of M that is isometric to $\mathbb{R}^k$. The rank of M is denoted by $rk(M)$ and is defined to be the maximum number k such that M has a k-flat submanifold.*

Manifolds with negative sectional curvature have rank equal to 1. In particular hyperbolic spaces have rank = 1. One can easily construct manifolds with rank one and arbitrary "large" parts with zero sectional curvature by taking two copies of a flat manifold and cutting off a ball and connecting two copies with a negatively curved cylinder along the holes. Here is also two large classes of manifolds with rank at least 2,

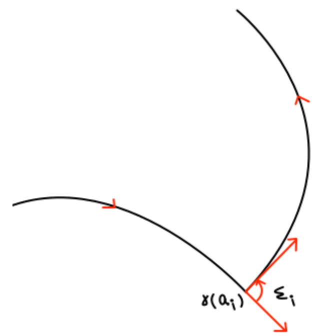

Figure 2.1:

Example 2.1.16. *Let G be a semi-simple Lie group with finite center and no compact factor. Consider the space $M = \Gamma\backslash G/K$ where $K \subset G$ is a maximal compact subgroup and $\Gamma \subset G$ is a uniform lattice. This manifold admits a metric with non-positive curvature and $rank(M) \geqslant 2$. See [33] for the details of why this manifold is in the desired class.*

Example 2.1.17 (Direct product)**.** *If N_1, N_2 are two non-positively curved manifolds then their product $M = N_1 \times N_2$ is a non-positively curved manifold with $rank(M) \geqslant 2$. This is easily seen by the fact that each geodesic in M lies in an immersed flat two-plane. Let $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ be a geodesic in M, if non of the γ^1, γ^2 is constant then γ is contained in the flat two plane spanned by $\gamma^1 \times \gamma^2 \subset M$. And if γ is parallel to one of the factors then each geodesic γ' parallel to the other factor determines a flat two plane containing γ.*

Definition 2.1.18 (Hessian)**.** *Let (M, g) be a Riemannian manifold and $f \in C^\infty(M, \mathbb{R})$ be a map. Then the Hassian of f is denoted by $\nabla^2 f$ and is defined as follows, for vector fields $X, Y \in \Gamma(TM)$,*

$$\nabla^2 f(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

2.2 Locally symmetric spaces

In this section we will discuss some properties of locally symmetric spaces that we will need in Chapter 5 to study positivity of the simplicial volume of locally symmetric spaces of non-compact type.

Definition 2.2.1. *A Riemannian manifold M is said to be a locally symmetric space if for every point $p \in M$ there exist a neighborhood U of p and an isometry $s : U \to U$ so that it leaves p fixed and $d_p s = -id_{T_pM}$.*

Remark 2.2.2. *Note that if M is an arbitrary Riemanian manifold and $p \in M$ any point on M, then there exist a neighborhood U of p and a unique diffeomorphism $s_{p,U} : U \to U$ such that $s(p) = p$ and $d_p s = -id_{T_pM}$. For any two neighborhoods U, V of p for which the maps $s_{p,U}$ and $s_{p,V}$ exist we have $s_{p,U\cap V}$ exist and $s_{p,U\cap V} = s_{p,U}|_{U\cap V} = s_{p,V}|_{U\cap V}$. Therefore there is a maximal neighborhood of p with a diffeomorphims $s_p : U \to U$ with $s_p(p) = p$ and $d_p s_p = -id_{T_pM}$. The map s_p is called geodesic symmetry centered at p.*

Definition 2.2.3. *A locally symmetric space is called a symmetric space if all the geodesic symmetry maps can be extended to the entire manifold.*

Lemma 2.2.4. *Universal cover of a complete locally symmetric space is a symmetric spaces.*

Proof. Without loss of generality, let M be a connected locally symmetric space and $\widetilde{M}$ be its universal cover with the induced metric. Thus $\widetilde{M}$ is a locally symmetric space as it is locally isometric to M. Now let $s_p : U \to U$ be a geodesic symmetry centered at p in $\widetilde{M}$. Now suppose $x \in \widetilde{M} \backslash U$. Let $\gamma : I \to \widetilde{M}$ be a smooth curve from p to x. Cover the image of γ by finitely many open neighborhoods $U_0, ..., U_k$ where $U_0 = U$, $U_i \cap U_j = \varnothing$ for $|i-j| \geqslant 2$, for each of which there exist a geodesic symmetry s_i centered at some point of U_i and also a chart $\phi_i : U_i \to \widetilde{U}_i \subset \mathbb{R}^n$. Now let $g_i : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry that extends the isometry map $\phi_i \circ s_i \circ \phi_{i-1}^{-1} : \phi_{i-1} \circ s_{i-1}^{-1}(U_i \cap U_i) \to \phi_i(U_{i-1} \cap U_i)$ to an isometry of $\mathbb{R}^n$. Now we define the extension of s_p at x to be $s_p \circ \phi_0^{-1} \circ g_1^{-1} \circ ... \circ g_{k-1}^{-1} \circ g_k^{-1} \circ \phi_i(x)$. One can check that this extension is an isometry and the definition does not depend on the choice of γ, using the fact that $\widetilde{M}$ is simply connected. for more details see [12]. □

Because of the proportionality principle, 1.0.6, to determine the positivity of the simplicial volume of a manifold M, it would be enough to show that there is a manifold with positive simplicial volume and universal cover isometric with that of M. Therefore let us talk about (simply connected complete) symmetric spaces and their properties.

Lemma 2.2.5 ([14, Theorem 3.2, Ch 2])**.** *Let G be a locally compact group with a countable basis. Suppose G is acting transitively and continuously on a locally compact Hausdorff space M. Let $p \in M$ be a point and H be the subgroup of G leaving p fixed. Then $H \subset G$ is a closed subgroup and the map $G/H \to M$ defined by $gH \mapsto g.p$ is a homeomorphism.*

Furthermore for a Riemannian manifold one can prove the following theorem,

Theorem 2.2.6 ([14, Theorem 3.3, Ch 4])**.** *The followings hold,*

- *Let M be a Riemannian manifold and $I(M)$ be the isometry group of M. Then the compact-open topology of $I(M)$, turns it into a locally compact group with countable basis, and $I(M)$ acts smoothly on M. Furthermore the subgroup $\widetilde{K} \subset I(M)$ that leaves a point $p \in M$ is a compact subgroup.*

- *If M is a globally symmetric space, then the identity connected component of isometry group $I(M)$, denoted by $G := I_0(M)$, admits a smooth structure compatible with the compact-open topology that turns it into a Lie group acting transitively and smoothly on M. If $K \subset G$ is the maximal subgroup of G fixing a point $p \in M$ then, K is a compact subgroup and $M \cong G/H$ (diffeomorphism).*

So in order to study the structure of symmetric spaces one might attempt to use the theory of Lie groups and Lie algebras. Note that for a semisimple Lie group G and a maximal compact subgroup $K \subset G$, the quotient group G/K can be equipped with a metric that is G-equivariant and turns the quotient into a globally symmetric Riemannian manifold. To see this, let $\mathfrak{g}, \mathfrak{k}$ be the Lie-algebra of G and K respectively. If G is a semisimple Lie group, i.e. by definition $\mathfrak{g}$ is a semisimple Lie algebra, then the Cartan Killing form is a non-degenerate bi-linear form on $\mathfrak{g}$ defined as follows,

$$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$$

$$(X, Y) \mapsto Tr(ad(X)ad(Y))$$

where $ad : \mathfrak{g} \to \mathfrak{gl}_n(\mathfrak{g})$ is the derivative of the adjoint representation of G at e, defined by $g \mapsto Ad(g) \in GL_n(\mathfrak{g})$, where $Ad(g)$ is the derivative of $c_g : G \to G$ at e, taking h to ghg^{-1}. Now since the killing form is a non-degenerate bi-linear form we can define the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, let us denote it by $\mathfrak{p}$. So then we have, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Now one can check that the differential of the projection map $G \to G/K$ is a linear map $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to T_K(G/K)$ with kernel equal to $\mathfrak{k}$ and it induces an isomorphism between $\mathfrak{p}$ and $T_K(G/K)$, where by index K we mean the coset $K \in G/K$. One can show that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. So the the killing form induces a norm on $\mathfrak{p}$, consequently on $T_K(G/K)$, denote this norm by g_K. now since G/K is a homogenuous space, i.e. for every point $x \in G/K$ there is a diffeomorphism $f_x : G/K \to G/K$ so that $f_x(K) = x$, we can transform the norm g_K to a norm g_x on the tangent space at the point x using f_x for any point $x \in G/K$. Therefore we get a G-equivariant metric g on G/K defined by $g(x) = g_x$.

Lemma 2.2.7 ([14, Theorem 3.5, Ch4]). *Let M be a symmetric space, G be the identity component of isometries of M and $K \subset G$ be the isotropy subgroup at the point $p \in M$. G is a semi-simple Lie group. If $\mathfrak{g}, \mathfrak{k}$ are the Lie algebras of G, K respectively, then we have $T_pM \cong \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k} \subset \mathfrak{g}$ with respect to the Killing form. Furthermore, we have $\mathfrak{p} = \{X \in \mathfrak{g} : d_ec_p(X) = -X\}$ and $\mathfrak{k} = \{X \in \mathfrak{g} : d_ec_p(X) = X\}$ where $c_p : G \to G$ is defined by $g \mapsto s_pgs_p$, where s_p is the geodesic symmetry at the point p. And finally M is isometric with G/K, where the metric on G/K is the one constructed in the paragraph right before the Lemma.*

Definition 2.2.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra that admits an involutive automorphism $s : \mathfrak{g} \to \mathfrak{g}$, i.e. $s \neq I$ and $s^2 = id$ such that the Lie subalgebra of fixed points of s, denoted by $\mathfrak{u}$, is compactly imbedded in $\mathfrak{g}$. Then we call the pair $(\mathfrak{g}, s)$ an orthogonal symmetric lie algebra. In addition if $\mathfrak{u} \cap \mathfrak{z} = 0$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$, then the pair is called an effectice orthogonal symmetric Lie algebra. Note that with the notation of Lemma 2.2.7, the pair $(\mathfrak{g}, d_ec_p)$ is an orthogonal symmetric Lie algebra.*

In the following definition by compact Lie algebra we shall mean Lie algebra of a compact group. And by Cartan decomposition we shall mean, there is a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, i.e. $B_\theta(X, Y) := -B(X, Y)$ is positive definite bi-linear form, and $\mathfrak{e}$ is the orthogonal complement of $\mathfrak{u}$ with respect to B_θ.

Definition 2.2.9. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra. Let $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ be the decomposition of $\mathfrak{g}$ into the eigenspaces of s corresponding to $+1, -1$ eigenvalues.*

- *If $\mathfrak{g}$ is a compact semi-simple Lie algebra, then $(\mathfrak{g}, s)$ is said to be of compact type.*
- *If $\mathfrak{g}$ is a non-compact semi-simple Lie algebra, and $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is a Cartan decomposition, then $(\mathfrak{g}, s)$ is said to be of non-compact type.*
- *If $\mathfrak{e}$ is an abelian ideal in $\mathfrak{g}$, then $(\mathfrak{g}, s)$ is said to be of Euclidean type.*

Theorem 2.2.10 ([14, Theorem 1.1, Ch V]). *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra. Then there exist ideals $\mathfrak{I}_0, \mathfrak{I}_-, \mathfrak{I}_+$ in $\mathfrak{g}$ with the following properties,*

- $\mathfrak{g} = \mathfrak{I}_0 \oplus \mathfrak{I}_- \oplus \mathfrak{I}_+$
- *$\mathfrak{I}_0, \mathfrak{I}_-, \mathfrak{I}_+$ are invariant under s and orthogonal with respect to the Killing form on $\mathfrak{g}$.*

- *Let s_0, s_-, s_+ be the restriction of s to $\mathfrak{I}_0, \mathfrak{I}_-, \mathfrak{I}_+$ respectively. Then the pairs $(\mathfrak{I}_0, s_0), (\mathfrak{I}_-, s_-), (\mathfrak{I}_+, s_+)$ are effective orthogonal symmetric Lie algebras of the Euclidean type, compact type and non-compact type respectively.*

Theorem 2.2.11 ([14, Theorem 3.1, ChV])**.** *Suppose that $(\mathfrak{g}, s)$ is an effective orthogonal symmetric Lie algebra and (G, K) is the corresponding pair of Lie groups. Here G is a Lie group with Lie algebra $\mathfrak{g}$ and K is the corresponding connected closed Lie subgroup of G to the Lie subalgebra $\mathfrak{k} := \{X \in \mathfrak{g} : s(X) = X\}$. Let g be an arbitrary G-equivariant Riemannian metric on G/K. Then the following hold,*

- *If R is the curvature tensor associated with the metric g, then $R(X,Y)Z = -[X,[Y,Z]]$. Thus the curvature tensor of the quotient G/K does not depend on the metric as long as the metric is G-equivariant.*
- *If $(\mathfrak{g}, s)$ is of compact type then the sectional curvature of G/K is non-negative.*
- *If $(\mathfrak{g}, s)$ is of non-compact type then the sectional curvature of G/K is non-positive*
- *If $(\mathfrak{g}, s)$ is of Euclidean type then the sectional curvature of G/K is vanishes.*

An immediate consequence of the Theorem 2.2.11 together with the Theorem 2.2.10 is the following theorem,

Theorem 2.2.12. *Every simply connected symmetric space M is isometrically decomposed to the Cartesian product of symmetric spaces M_0, M_-, M_+ where,*

1. *M_0 is isometric to a Euclidean space*
2. *M_- is a non-positively curved manifold that is not decomposed into Riemannian product of any Euclidean space with any other Riemannian manifold. These manifolds are called of non-compact type.*
3. *M_+ is a non-negatively curved manifold that is not decomposed into Riemannian product of any Euclidean space with any other Riemannian manifold. These manifolds are called of compact type. Furthermore, M_+ is compact.*

Now we state another decomposition theorem for complete simply connected symmetric spaces, known as de Rham decomposition theorem, which will reduce the study of classification problem of symmetric spaces to the classification of irreducible symmetric spaces. Before stating the theorem we introduce the notion of linear holonomy group and a lemma that will give an idea of the proof of the de Rham decomposition theorem.

Definition 2.2.13 (Linear Holonomy Group)**.** *Let M be a Riemannian manifold with Levi-Civita connection ∇. For any closed curve $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = x \in M$ we define,*

$$\widetilde{\gamma} : T_xM \longrightarrow T_xM$$
$$X \mapsto Par_{x,x}^{\nabla,\gamma}(X)$$

where $Par_{x,x}^{\nabla,\gamma}$ is the parallel transport along γ with respect to the connection ∇ from tangent space at x to itself. Now we can think of $\widetilde{\gamma}$ as an invertible linear transformation in $GL_n(\mathbb{R})$. Thus we define the linear holonomy group as the following,

$$\{\widetilde{\gamma} \in GL_n(\mathbb{R}) \big|\ \gamma : [0,1] \to M,\ \gamma(0) = \gamma(1) = x, \}$$

and denote it by Hol_x^{∇}.

Definition 2.2.14. *Let M be a Riemannian manifold with a connection ∇. Let Hol_x^{∇} be the linear holonomy group at a fixed point $x \in M$. We say that M is reducible if and only if T_x contains a non-trivial subspace T'_x that is stable under the holonomy group, i.e. for any $X' \in T'_x$ and $g \in Hol_x^{\nabla}$ we have, $g(X') \in T'_x$.*

Lemma 2.2.15 ([19, Proposition 6.1, Ch 4])**.** *Let M be a connected, simply connected and complete Riemannian manifold. Then M is reducible if and only if there are non-zero dimensional Riemannian manifolds M', M'' so that $M \cong M' \times M''$.*

Proof. Let $p \in M$ be a point such that $T_pM = T'_pM \bigoplus T''_pM$ where the subspaces T'_p, T''_pM are stable under the linear holonomy group action. Now one can show that the subspaces T'_p, T''_p are involutive. Let T', T'' be the distributions obtained by parallel transporting the spaces T'_pM, T''_pM along geodesics to other points of M. Let M', M'' be the the maximal integral submanifolds of M associated with T', T'' respectively. The goal is to prove that $M \cong M' \times M''$. The isometry $f : M \to M' \times M''$ is defined as follows, let $q \in M$ be an arbitrary point in M, and let γ be a geodesic in M with $\gamma(0) = p$ and $\gamma(1) = q$. If $\dot{\gamma}(t) = \dot{\gamma}'(t) + \dot{\gamma}''(t)$ is the corresponding decomposition of $\dot{\gamma}(t)$, then define the following curves in T'_pM and T''_pM respectively, $Par_{t,0}^{\gamma}(\dot{\gamma}'(t))$ and $Par_{t,0}^{\gamma}(\dot{\gamma}''(t))$. Now let γ_1 and γ_2 be two curves in M such that $Par_{t,0}^{\gamma}(\dot{\gamma}_1(t)) = Par_{t,0}^{\gamma}(\dot{\gamma}'(t))$ and similarly, $Par_{t,0}^{\gamma}(\dot{\gamma}_2(t)) = Par_{t,0}^{\gamma}(\dot{\gamma}''(t))$. Now clearly γ_1 lies in M' and γ_2 lies in M''. Thus define $f(q) := (\gamma_1(1), \gamma_2(1))$. One can show that this map is an isometry, for more details see the reference. $\square$

Corollary 2.2.16 (The de Rham Decomposition)**.** *Every connected simply connected complete Riemannian manifold is isometric to a direct product $M_0 \times M_1 \times ... \times M_k$ of connected, simply connected, complete and irreducibe Riemannian manifolds $\{M_i\}_{i=0}^k$, where M_0 is a Euclidean space (possibly of dimension 0). Such a decomposition is unique up to reordering. Furthermore, if M is a symmetric space then so are M_i's.*

Proof. This is an immediate consequence of the Lemma 2.2.15. $\square$

Lemma 2.2.17 ([19, Theorem 3.5, Ch 7])**.** *Let $M = M_1 \times M_2$ be a direct product of two Riemannian manifolds. It is clear that M is a symmetric space if and only if M_1, M_2 are symmetric. Assume that M is a symmetric space. Let G be the subgroup of isometries of M consisting of geodesic symmetries around points of M, i.e.*

$$G := \{s_x \in Isom(M) \big| \ s_x(x) = x, \ d_x s_x = -id_{T_xM}, \ x \in M\}$$

Then we have $G \cong G_1 \times G_2$ where G_1, G_2 are corresponding subgroups of the isometry groups $I(M_1), I(M_2)$ respectively. Furthermore, if M is an irreducible manifold then G is a simple group.

The upshot of this section is the following corollary which is an easy consequence of the discussion above,

Corollary 2.2.18. *Let M be a complete locally symmetric space and $\widetilde{M}$ be its universal cover. Then $\widetilde{M}$ is isometric to a Riemannian product $M_1 \times M_2 \times ... \times M_k$, where each M_i is an irreducible symmetric space which is either isometric with a Euclidean space or has non-positive curvature or has non-negative curvature.*

After all let us present some examples of symmetric and locally symmetric spaces. The only examples for symmetric spaces with flat type, is the Euclidean spaces and their quotients. A typical example for symmetric spaces of non-compact type is the hyperbolic spaces, i.e. simply connected complete Riemannian manifolds with constant sectional curvature. Symmetric spaces of compact type are compact manifolds. A typical example for the symmetric spaces of compact-type is the spheres. In fact a general symmetric space of compact type is not that far from spheres,

Theorem 2.2.19 ([11])**.** *If M is a simply connected compact Riemannina manifold with non-negative curvature then by de Rham decomposition theorem, it is isometric to a product of irreducible manifolds $M_1 \times ... \times M_k$. All the M_i's are homeomorphic to spheres.*

By Theorem 2.2.7 one can easily check that $\mathbb{H}^2$ is isometric with $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. As a generalization of hyperbolic spaces, $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ is a symmetric space of non-compact type. Any quotient of a hyperbolic space by an arithmetic subgroup is a locally symmetric space of non-compact type. For instance, all Riemannian surfaces with genus at least 2 are quotients of hyperbolic plane, therefore are locally symmetric spcace of non-compact type. As another non-trivial and important example of locally symmetric spaces of non-compact type we should mention the modular curves $\Gamma\backslash\mathbb{H} = \Gamma\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$ where $\Gamma \subset SO_2(\mathbb{Z})$. When $\Gamma = SL_2(\mathbb{Z})$ the space is called the moduli space of elliptic curves.

2.3 Geometric boundaries

In this section we introduce the geodesic(visual) boundary and the Furstenberg boundary of a Riemannian manifold and provide some examples as we shall be familiar with these notions in the construction of Patterson-Sullivan measures.

2.3.1 Geodesic boundary

The geodesic boundary is defined for any simply connected non-positively curved Riemannian manifold X, and will be denoted by $X(\infty)$. But in the following we will restrict ourselves to the symmetric simply connected non-positively curved manifolds. Therefore let $X = G/K$ be a symmetric space of non-compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition associated with K, then the tangent space $T_{x_0}X$ at $K = x_0 \in X$ is canonically identified with $\mathfrak{p}$. We fix the G-equivariant Riemannian metric g on X that is obtained by restricting the killing form to $\mathfrak{p}$ which would be a positive-definite bi-linear form on $\mathfrak{p}$. This metric turns X into a symmetric simply connected Riemannian manifold with non-positive curvature. Now we can define the geodesic boundary as follows,

Definition 2.3.1. *Let $X = G/K$ be a simply connected symmetric space with non-positively curved metric constructed above. Let γ_1, γ_2 be two geodesics in X. Since X is symmetric, hence complete, all geodesics are defined on $\mathbb{R}$. Now we say that $\gamma_1 \sim \gamma_2$ if*

$$\lim_{t \to +\infty} d(\gamma_1(t), \gamma_2(t)) < \infty$$

Clearly $\sim$ is an equivalence relation on the set of all geodesics, we define the geodesic boundary by $\{$geodesics in $X\}/\sim$ and denote it by $X(\infty)$.

Lemma 2.3.2. *The set $X(\infty)$ can be canonically identified with the unit sphere in the tangent space T_xX for any point $x \in X$, in particular $T_{x_0}X = \mathfrak{p}$.*

Proof. Let $x \in X$ be a point. For any unit tangent vector $v \in T_xX$ there is a unique geodesic γ_v with the initial velocity vector v. For any two distinct tangent vector $v, w \in T_xX$ we have $\gamma_v \nsim \gamma_w$. This is because the exponential map $exp : T_xX \to X$ is a norm-increasing map with T_xX equipped with the Euclidean metric. Now let γ be any geodesic in X. Then for every positive integer n there is a unique geodesic γ_n from x to γ_n. Hence there is a subsequence $\gamma_{n'}$ that converges uniformly for t on compact sets to a geodesic γ_∞. then it is easy to check that γ_∞ is a geodesic passing through x and equivalent to γ. $\square$

Definition 2.3.3. *The geodesic compactification of X is defined to be $X \cup X(\infty)$ and the topology of it is defined as follows, let $x_0 \in X$ be fixed and let $[\gamma] \in X(\infty)$ be a point. Then we say an unbounded sequence $\{y_j\}_{j \geqslant 1}$ in X converges to $[\gamma]$ if the geodesic from x_0 to y_j converge to γ. This defines a topology on $X \cup X(\infty)$ and it does not depend on the choice of base point x_0. A base open subset $C(\gamma, t, \epsilon)$ around a point $[\gamma] \in X(\infty)$ with $t, \epsilon > 0$ in this topology is constructed as follows, for any $\epsilon, t > 0$ we define,*

$$C(\gamma, t, \epsilon) := C(\gamma, \epsilon) \backslash B(\gamma(0), t)$$

where $B(\gamma(0), t)$ is the ball with radius t around $\gamma(0)$ and $C(\gamma, \epsilon)$ consists of points x so that the angle between γ and the geodesic from x_0 to x is less than ϵ. This topology on the compactification $X \cup X(\infty)$ is called conic topology.

Lemma 2.3.4. *The isometric action of G on X extends to a continuous action on $X \cup X(\infty)$.*

Proof. Since $\gamma_1 \sim \gamma_2$ if and only if $g(\gamma_1) \sim g(\gamma_2)$, where $g \in G$ arbitrary, so the action of G on X extends to an action of G on $X \cup X(\infty)$. And since the convergent sequences are preserved under isometries, thus the action of G is continuous on $X \cup X(\infty)$. $\square$

Theorem 2.3.5 ([4, Proposition I.2.6])**.** *A proper subgroup of G is a parabolic subgroup of G if and only if it is the stabilizer of a point in $X(\infty)$.*

2.3.2 Furstenberg boundary

We start with introducing the parabolic subalgebras and subgroups. Let G be a real Lie group with maximal compact subgroup K. There is a unique Cartan involution $\theta : G \to G$ with fixed point set K. We shall denote the differential of θ at e with the same notation θ. If $\mathfrak{p}$ is the (-1)-eigenspace of θ on $\mathfrak{g}$ and $\mathfrak{k}$ is the $(+1)$-eigenspace of θ on $\mathfrak{g}$, which is the Lie algebra of K, then we have, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, so that,

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \; [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}, \; [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$$

If B is the Killing form, then B is negative semi-definite on $\mathfrak{k}$, negative defintie on $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$ and also positive definite on $\mathfrak{p}$. The subspace $\mathfrak{p}$ maybe identified with the tangent space $T_{x_0}X$ where $X = G/K$ and $x_0 = K$. The restriction of the Killing form on $\mathfrak{p}$ defined a G-equivariant Riemannian metric on $X = G/K$ with respect to which X is simply connected complete Riemannian symmetric space with non-compact type. The maximal sub-algebras of $\mathfrak{p}$ are all abelian sub-algebras and conjugate under K, i.e. the lie bracket of each two elements vanish and for any two maximal sub-algebra $\mathfrak{a}_1, \mathfrak{a}_2$ there is an element $k \in K$ so that $ad(k)(\mathfrak{a}_1) = \mathfrak{a}_2$. Let $\mathfrak{a}$ be a maximal sub-algebra of $\mathfrak{p}$. A linear 1-form $\alpha : \mathfrak{a} \to \mathbb{R}$ is called a root on $\mathfrak{a}$ if the corresponding root space is non-zero, i.e.,

$$\mathfrak{g}_\alpha := \{V \in \mathfrak{g} : \forall H \in \mathfrak{a}, \quad [H, V] = \alpha(H)V\} \neq 0$$

The set of roots in $\mathfrak{a}^*$ is a root system, i.e. for a root $\alpha \in \mathfrak{a}^*$ and any other root β there is a root that is the reflection of β with respect to the perpendicular hyperplane H_α to α. We shall denote the set of roots by $\Phi(\mathfrak{g}, \mathfrak{a})$ or simply by Φ. Every root α determines a hyperplane H_α on which it vanishes. A connected component of the complement in $\mathfrak{a}$ of the union H_α's is called Weyl chamber. We fix one of the connected components and call it positive Weyl chamber and denote it by $\mathfrak{a}^+$. The choice of $\mathfrak{a}^+$ defines an ordering $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \Phi : \alpha > 0 \ on \ \mathfrak{a}^+\}$. Let Δ denote the set of simple roots, i.e. the roots that are written as sum of two other roots. Define,

$$\mathfrak{n} := \sum_{\alpha > 0} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \sum_{\alpha < 0} \mathfrak{g}_\alpha$$

then we will have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{n}^-$, where $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Moreover $\mathfrak{z}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{m} = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{k}$. If we let $A = exp(\mathfrak{a})$, then A is a maximal totally geodesic sub-manifold of $X = G/K$ which is isometric with a Euclidean space. And any other flat maximal totally geodesic submanifolds is a translation of A by an element of G. Now we are ready to define the parabolic sub-algebras. let $I \subset \Delta$ be a set of simple roots in $\mathfrak{a}^*$. Let $\mathfrak{a}_I = \cap_{\alpha \in I} H_\alpha$. We denote the set of all roots that are linear combinations of simple roots in I by Φ^I and the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$ by $\mathfrak{a}^I$. So then we should have $\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}^I$. Now we define the standard parabolic sub-algebra $\mathfrak{p}_I$ to be a sub-algebra generated with the centerालizer of $\mathfrak{a}_I$, $\mathfrak{z}(\mathfrak{a}_I)$ and $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha$. Therefore we should have,

$$\mathfrak{p}_I = \mathfrak{n} \oplus \mathfrak{z}(\mathfrak{a}_I) = \sum_{\alpha > 0} \mathfrak{g}_\alpha \oplus \mathfrak{m} \oplus \mathfrak{a}_I \oplus \mathfrak{a}^I$$

$$= (\sum_{\alpha \in \Phi^+ \setminus \Phi^I} \mathfrak{g}_\alpha) \oplus (\mathfrak{a}_I) \oplus (\mathfrak{m} \oplus \mathfrak{a}^I \oplus \sum_{\alpha \in \Phi^I} \mathfrak{g}_\alpha)$$

$$= \mathfrak{n}_I \oplus \mathfrak{a}_I \oplus \mathfrak{m}_I$$

Now in the extreme cases we will have, $\mathfrak{p}_\varnothing = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m}$ and $\mathfrak{p}_\Delta = \mathfrak{g}$. A sub-algebra $\mathfrak{p}$ of $\mathfrak{g}$ is said to be parabolic if it is conjugate to a standard parabolic sub-algebra $\mathfrak{p}_I$ for some subset I of simple roots.

Definition 2.3.6. *A subgroup P of G is called parabolic if it is the normalizer of a parabolic sub-algebra $\mathfrak{p}$ in $\mathfrak{g}$. i.e. there is a parabolic sub-algebra $\mathfrak{p}$ in $\mathfrak{g}$ so that the Lie-algebra of P is the following set,*

$$\mathfrak{N}_\mathfrak{g}(\mathfrak{p}) := \{x \in \mathfrak{g} : [x, p] \in \mathfrak{p} \ \forall p \in \mathfrak{p}\}$$

We shall denote the normalizer of the standard parabolic sub-algebra $\mathfrak{p}_I$ by P_I. For $I = \varnothing$ we will call the corresponding normalizer $P_\varnothing$ the minimal parabolic subgroup.

Now the decomposition $\mathfrak{p}_I = \mathfrak{n}_I \oplus \mathfrak{a}_I \oplus \mathfrak{m}_I$ will give us $P_I = N_I \rtimes Z(A_I)$, where N_I is the corresponding connected subgroup of G with $\mathfrak{n}_I$, called the unipotent radical of P_I, and $Z(A_I)$ is the centeralizer of A_I in G, where A_I is the corresponding subgroup with $\mathfrak{a}_I$. Moreover we have, $Z(A_I) = M_I \times A_I$. Here M_I is a subgroup with Lie-algebra $\mathfrak{m}_I$ but not necessarily connected. Finally we can write $P_I \cong N_I \times A_I \times M_I$, as analytic manifolds, which is called the Langlands decomposition of P_I in which M_I, A_I are θ-stable. Recall that θ is the unique involution of G that fixes K. Since M_I is stable under θ, we have $K_I := K \cap M_I \subset M_I$ is a maximal compact subgroup. The quotient $X_I := M_I/K_I = P_I/K_IA_IN_I$ is called the boundary symmetric space associated with P_I and it is a symmetric space of non-compact type. The boundary symmetric space X_I can be identified with the orbit M_IK of the point $K \in X = G/K$ as a subspace of X.

Theorem 2.3.7 (Iwasawa Decomposition)**.** *Let G be a real Lie group, and K be a maximal compact Lie subgroup of G. Let θ be the unique Cartan involution whose fixed point set is K, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian sub-algebra of $\mathfrak{p}$ with a fixed root ordering $\mathfrak{a}^+$. If $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha \oplus \sum_{\alpha<0} \mathfrak{g}_\alpha$ then we have,*

$$G = NAK$$

where N, A are the corresponding Lie subgroups of G with $\mathfrak{n}$ and $\mathfrak{a}$ respectively. Moreover, we have $NA \subset P_I$ for any I subset of simple roots.

As an immediate consequence of Iwasawa decomposition we have $G = P_IK$. Thus P_I acts transitively on $X = G/K$, so $X = P_I/K \cap P_I$ and therefore Langlands decomposition of P_I induces a decomposition of X as follows,

$$X \cong \frac{P_I}{K \cap P_I} = \frac{P_I}{N_I \times A_I \times (K \cap M_I)}$$

$$= N_I \times A_I \times \frac{M_I}{K \cap M_I} = N_I \times A_I \times X_I$$

This decomposition of $X = G/K$ is called the horospherical decomposition.

Example 2.3.8. *Let $G = SL_n(\mathbb{R})$ for $n \geqslant 2$. Fix the maximal compact subgroup $K = SO_2(\mathbb{R})$. A maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ is given by,*

$$\mathfrak{a} = \{diag(t_1, ..., t_n) : t_1 + ... + t_n = 0\}$$

Recall that the Lie algebra of G is the trace-free matrices, the Lie algebra of K is the skew-symmetric matrices and consequently $\mathfrak{p}$ is the set of upper triangular matrices with trace zero. Define $\mathfrak{a}^+ := \{diag(t_1, ..., t_n) : t_1 > ... > t_n\}$, then the nilpotent sub-algebra $\mathfrak{n}$ consists of upper triangular matrices with zero on the diagonal entries, and its corresponding Lie subgroup is the set upper triangular matrices with 1 on the diagonal entries. Then the standard minimal parabolic subgroup normalizing the sub-algebra $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m}$ is the set of upper triangular matrices. The Langlands decomposition is $P_\varnothing = NAM$, where $M = \{diag(\pm1, ..., \pm1)\}$

We have been defining the parabolic subgroups in the context of real Lie groups, but it is more natural to look at them from the point of view of algebraic groups. A subgroup P of a linear algebraic group G defined over an algebraically closed field is called parabolic subgroup if the homogenuous space G/P is a projective variaty. If G is a linear algebraic group, then we can embed it into a $SL_n(\mathbb{R})$ for some n. Then if $G_\mathbb{C}$ is the complexification of G, i.e. smallest linear algebraic subgroup of $SL_n(\mathbb{C})$ that contains G, or the Zariski-closure of G in $SL_n(\mathbb{C})$, then the parabolic subgroups that we have defined are the intersection of parabolic subgroups of $G_\mathbb{C}$, subgroups $P \subset G_\mathbb{C}$ so that $G_\mathbb{C}/P$ is an algebraic variety, with G.

Now we shall introduce the Furstenberg boundary that we will be dealing with in the subsequent chapters.

Definition 2.3.9 (Haar measure)**.** *Let G be a locally compact Hausdorff topological group. The Borel-algebra Σ of G is the σ-algebra generated with the open subsets of G. An element of the Borel-algebra is called Borel set. Then there is a unique non-trivial measure μ, up to a positive multiplicative constant, with the following properties,*

- *μ is left-invariant, $\forall g \in G, S \in \Sigma,\ \mu(gS) = \mu(S)$*
- *μ is finite on every compact Borel set*
- *μ is outer regular on Borel sets, $\forall S \in \Sigma,\ \mu(S) = \inf_{S\subset U}\{\mu(U)\}$, where infimum is taken over all open subsets U containing S*
- *μ is inner regular on Borel sets, $\forall S \in \Sigma,\ \mu(S) = \sup_{K\subset S}\{\mu(K)\}$, where supremum is taken over all compact subsets contained in S*

such a measure in G is called a left Haar measure. Right Haar measure is defined analogously.

Now we first construct the Furstenberg compactification of the unit Poincare disc D in $\mathbb{C}$ and generalize it to the symmetric spaces. Let Δ be the Laplace operator and let $f \in C^0(S^1)$. Then we can extend f to a harmonic map $u : D \to \mathbb{R}$, i.e. one can solve the following Dirichlet problem,

$$\begin{cases} \Delta u = 0 & \textit{in } D \\ u = f & \textit{on } \partial D \end{cases}$$

The solution to the above problem would be,

$$u(z) = \int_{S^1} \frac{1 - |z|^2}{|z - \xi|^2} f(\xi) d\xi$$

where $d\xi$ is a Haar measure on S^1 normalized so that the total measure is 1. Therefore each point of S^1 determines a measure,

$$\mu_z(\xi) := \frac{1 - |z|^2}{|z - \xi|^2} d\xi$$

on S^1. By taking $f \cong 1$ we conclude that μ_z is in fact a probability measure on S^1. If $\mathcal{M}_1(S^1)$ is the space of probability measures on S^1, then we get the following map,

$$i : D \to \mathcal{M}_1(S^1),\ z \mapsto \mu_z$$

where the space $\mathcal{M}_1(S^1)$ is given the weak-* topology. One can see that the map i is an embedding. Now the closure of $i(D)$ in $\mathcal{M}_1(S^1)$ is called the Furstenberg compactification of D and denoted by $\bar{D}^F$. The Furstenberg compactification of D is homeomorphic to the closed unit ball $D \cup S^1$. There is an obvious map $D \cup S^1 \to \mathcal{M}_1(S^1)$ that is defined by $z\mu_z$ on D and takes $\xi \in S^1$ to the delta measure δ_ξ. It is easy to check that if $z_i \to \xi$ then $\mu_{z_i} \to \delta_\xi$. Now we shall generalize this definition of compactification to the symmetric spaces $X = G/K$. If we think of $D = SU(1,1)/U(1)$, then the action of $SU(1,1)$ extends continuously on S^1 and the stabilizer of any point $\xi \in S^1$ is a parabolic subgroup of $SU(1,1)$. In the upper half plane model, $\mathbb{H} = SL_2(\mathbb{R})/SO(2)$, the boundary corresponds to $\mathbb{R} \cup \{i\infty\}$

and the stabilizer of $i\infty$ is the set of upper triangular matrices which is a parabolic subgroup of $SL_2(\mathbb{R})$. Note that we have the Furstengerb boundary of D is in fact the quotient of $SU(1,1)$ by a parabolic subgroup. Similarly in the upper half plan model, Furstenberg boundary is the quotient of $SL_2(\mathbb{R})$ by the subgroup of upper triangular matrices which is a parabolic subgroup.

Definition 2.3.10. *let G be a group. A topological space X is called a homogeneous space of G if G acts transitively on G. Special cases of this is the group G is a Lie/topological group that is a subgroup of automorphisms of X, where automorphism could mean, isometry, diffeomorphism etc. depending on the context.*

Definition 2.3.11. *Let Y be a compact homogeneous space of a group G. We say Y is a G-boundary if for every probability measure $\mu \in \mathcal{M}_1(Y)$ there exist a sequence $\{g_j\}_{j\geqslant 1}$ in G for which the sequence of measures $g_j.\mu$ converges to a delta measure δ_y for some point $y \in Y$.*

Definition 2.3.12. *A homogeneous space M of a group G is called the maximal/universal G-boundary if it is a G-boundary and for any other G-boundary M' there is a surjective G-equivariant map $f : M \to M'$. The universal G-boundaries are isomorphic and hence unique up to isomorphism. A universal G-boundary of G is denoted by $\mathcal{F}(G)$ and called the maximal Furstenberg boundary.*

Note that clearly there is a correspondence between the homogeneous spaces of G and the quotient groups of G. It turns out that the following theorem holds,

Theorem 2.3.13 ([4, page 109])**.** *If P_0 is the minimal parabolic subgroup of G then G/P_0 is the maximal Furstenberg boundary. Moreover, every other G-boundary if G is of the form $G/P_{0,I}$ where $P_{0,I}$ is a standard parabolic subgroup of G containing P_0. In the subsequent chapters we will be denoting the Furstenberg boundary of M by $\partial_F M$.*

Example 2.3.14. *Let $G = SL_2(\mathbb{R})$ and P_0 be the upper triangular matrices in G, the minimal parabolic subgroup of G. Here we confirm that G/P_0 is a G-boundary. Identify G/P_0 with the set $\mathbb{R} \cup \{\infty\} = \mathbb{H}(\infty)$ under the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{c}$. Let $\mu \in \mathcal{M}_1(\mathbb{H}(\infty))$ be a probability measure. Now there exist an element $k \in SO_2(\mathbb{R})$ so that,*

$$\lim_{t\to+\infty} k.\mu\bigg(\{x \in \mathbb{H}(\infty) : |x| > t\}\bigg) = 0$$

In particular $\mu(\{\infty\}) = 0$. Now let $g_t := diag(t^{-1}, t) \in SL_2(\mathbb{R})$. Thus we have, for every subset $E \subset \mathbb{H}(\infty)$

$$g_t k.\mu(E) = k.\mu(t^2 E)$$

therefore $g_t k.\mu \to \delta_0$. So this confirms the Theorem 2.3.13.

2.4 Busemann functions

Definition 2.4.1 (Busemann function)**.** *Let M be a Riemannian manifold with non-positive curvature, $\widetilde{M}$ its universal cover and $\partial\widetilde{M}$ be its geodesic boundary, see Definition 2.3.1. The following map is called the Busemann function on $\widetilde{M}$,*

$$B : \widetilde{M} \times \widetilde{M} \times \partial\widetilde{M} \longrightarrow \mathbb{R}$$

$$(x, y, \theta) \mapsto \lim_{t\to\infty} \left(d_{\widetilde{M}}(\gamma_{x,\theta}(t), y) - t\right)$$

where $\gamma_{x,\theta}$ is the unique geodesic ray from x to θ. As long as the base point x is understood we will be denoting the map $B(x, y, \theta)$ by $B(y, \theta)$.

Theorem 2.4.2. *If M is a complete non-positively curved Riemannian manifold, and γ_1, γ_2 are two geodesics in M, then $d(\gamma_1(t), \gamma_2(t))$ is a convex function. The Busemann function $B = B(x, ., \xi) : \widetilde{M} \to \mathbb{R}$ is also a convex function, where $x \in \widetilde{M}$ and $\xi \in \partial\widetilde{M}$ are fixed.*

Proof. Let γ_1, γ_2 be two geodesic in M. The function $d(\gamma_1(t), \gamma_2(t))$ is a convex function. Let t_1, t_2 be two distinct real number and $t = \frac{1}{2}(t_1 + t_2)$. Let $\sigma : [t_1, t_2] \to M$ be the unique geodesic from $\gamma_1(t_1)$ to $\gamma_2(t_2)$. Now we have,

$$\begin{aligned} d(\gamma_1(t), \gamma_2(t)) &\leqslant d(\gamma_1(t), \sigma(t)) + d(\gamma_2(t), \sigma(t)) \\ &\leqslant \frac{1}{2} d(\gamma_1(t_2), \gamma_2(t_2)) + \frac{1}{2} d(\gamma_1(t_1), \gamma_2(t_1)) \end{aligned}$$

where the second inequality comes from the comparison with the Euclidean case. Now let γ be the geodesic ray starting from x and going towards $\xi \in \partial M$ and let σ be any geodesic. Then for any $t, s \in \mathbb{R}_{>0}$, $m = \frac{1}{2}(s+t)$ and large enough $r \in \mathbb{R}_{>0}$, then to prove the convexity of the Busemann function we need to prove the following

$$(d(\sigma(s), \gamma(r)) - r) + (d(\sigma(t), \gamma(r)) - r) \geqslant 2(d(\sigma(m), \gamma(r)) - r)$$

which is true by comparison with the Euclidean case. $\square$

Chapter 3

Negative curvature

Thurston showed that the simplicial volume of the hyperbolic manifolds, i.e. Riemannian manifolds with negative constant sectional curvature, are proportional to their volumes and thus positive, see Theorem 1.0.13. In the current chapter we will prove the following theorem,

Theorem 3.0.1. *There exist a constant C_n for $n \geqslant 2$, such that the following holds. Let δ be a positive number, and let M be an n-dimensional closed orientable Riemannian manifold with sectional curvature bounded above by $-\delta$. Then the simplicial volume of M is estimated as follows,*

$$||M|| \geqslant C_n \delta^{n/2} vol(M)$$

and in particular it is positive.

Before we start the proof of the above theorem, we need to fix some notations. In the followings in this chapter, δ shall denote a positive number, M a closed n-dimensional Riemannian manifold with sectional curvature bounded from above by $-\delta$ and $p : \widetilde{M} \to M$ shall denote the universal covering of M. For two points p_0, p_1 of $\widetilde{M}$ and $0 \leqslant t \leqslant 1$, by $tp_0 + (1-t)p_1$ we shall mean the point $\gamma(t)$ where γ is the unique geodesic with $\gamma(0) = p_0$ and $\gamma(1) = p_1$, see Theorem 2.1.13.

3.1 The volume of geodesic simplices

In this sectoin we will find an universal upper bound for the volume of the geodesic simplices which totally benefits the presence of the negative curvature. Losing the negative upper bound for the curvature may allow the volume of geodesic simplices become arbitrary large, and this will prevent us extending the idea of the proof to the non-positive curvature case. But we might still be able to get rid of the simplices with large volumes and try to represent the fundamental class of the manifold with universally bounded volume simplices. This will be done through a refined straightening procedure that we shall discuss in the Chapter 4.

Definition 3.1.1 (Geodesic Simplex)**.** *A geodesic k-simplex with vertices $p_0, p_1, ..., p_k$, denoted by $\sigma_{p_1 \dots p_k}$ is defined inductively as follows, define $\Delta^k := \{(x_1, ..., x_{k+1}) \in \mathbb{R}^{k+1} : \sum_i x_i = 1,\ x_i \geqslant 0\}$, and identify Δ^{k-1} with $\{(x_1, ..., x_{k+1}) \in \Delta^k : x_{k+1} = 0\}$. For*

$k = 0$ we let σ_{p_0} be the map $\{1\} \mapsto p_0 \in \widetilde{M}$, and when $\sigma_{p_0 \dots p_{k-1}}$ is obtained, we define $\sigma_{p_0 \dots p_k} : \Delta^k \to \widetilde{M}$ by,

$$\sigma_{p_0 \dots p_k}\big((1-t)s + t(0, ..., 0, 1)\big) := (1-t)\sigma_{p_0 \dots p_{k-1}}(s) + tp_k$$

where $s \in \Delta^{k-1} \subset \Delta^k$.

Definition 3.1.2 (Straightening)**.** *Let σ be a singular k-simplex in M. we put $\lambda(\sigma) = p \circ \bar{\sigma}$ where $\bar{\sigma}$ is the geodesic simplex with the same vertices of some lift of σ. λ extends to a linear map,*

$$\lambda : S_*(M) \longrightarrow S_*(M)$$

For every simplex the term $\lambda(\sigma)$ is called the straightening of σ and The map λ is called straightening.

Remark 3.1.3. *Note that The term $\lambda(\sigma)$ does not depend on the choice of the lift of σ. For simplicity let $k = 1$. Then for any two lifts σ_1, σ_2 of σ, if γ_1, γ_2 are the unique geodesics connecting $\sigma_1(0)$ and $\sigma_2(0)$ to $\sigma_1(1)$ and $\sigma_2(1)$ respectively, then both $p \circ \sigma_1$ and $p \circ \sigma_2$ are geodesics connecting $\sigma(0)$ and $\sigma(1)$, thus by uniqueness of geodesics we must have $p \circ \sigma_1 = p \circ \sigma_2$. The straightening map λ satisfies the following,*

- *λ is a chain map and chain homotopic to $id_{S_*(M)}$*
- *$||\lambda(c)||_1 \leqslant ||c||_1$ for every $c \in S_*(M)$*

To see the proof of the first property of λ see Theorem 5.3.1(condition (2)), the second one is obvious. Considering the second property of the straightening map we have,

$$||M|| = \inf\{||c||_1 : c \in \lambda(S_*(M))\}$$

Now we find a universal bound on the volume of the geodesic simplices, namely we prove the following theorem,

Theorem 3.1.4. *There exist a constant C_k for $k \geqslant 2$, such that for every geodesic k-simplex σ in $\widetilde{M}$, the k-dimensional volume of σ, is estimated as follows,*

$$vol(\sigma(\Delta^k)) \leqslant C_k \delta^{-k/2}$$

Proof. To prove this theorem we proceed by induction. Let $k = 2$. Let σ be a k-simplex. Without loss of generality we may assume $\delta = 1$ and $im(\sigma)$ is a k-dimensional submanifold of M. Recall that $-\delta$ is the negative upper bound for the sectional curvature. Now we have the follwoing inequalities,

$$vol(\sigma(\Delta^k)) = \int_{\sigma(\Delta^k)} dV \leqslant -\int_{\sigma(\Delta^k)} K dV \leqslant \pi$$

where in the first inequality we are using the fact that the Gaussian curvature K of $im(\sigma)$ coincide with the sectional curvature of $\widetilde{M}$ restericted to $im(\sigma)$ and so is bounded above by -1, and in the second inequality we use a very important result so-called Gauss-Bonnet Formula, which only holds in dimension 2, see Theorem 2.1.14. With the notation of Theorem 2.1.14, we consider $\Omega = int(im(\sigma))$, $\gamma = im(\partial\Delta^k)$, so then we get,

$$\int_{int(im(\sigma))} K dV = 2\pi - \int_{\gamma} \kappa_N(s) ds - \sum_{i=0}^{2} \epsilon_i$$

$$= 2\pi - \sum_{i=0}^{2} \epsilon_i \geqslant -\pi =: C_2$$

where the second equality is because the image of the boundary of Δ^k is a union of geodesics so then $\kappa_N N = D_t \gamma = 0$, and the inequality comes from the fact that $\sum_{0 \leqslant i \leqslant 2} \epsilon_i \leqslant 3\pi$. This completes the proof for $k = 2$. Suppose the theorem is verified for dimensions less than k. Let $\tau : \Delta^{k-1} \times [0, 1] \to \Delta^k \subset \widetilde{M}$ defined by,

$$(x, t) \mapsto (1 - t)x + tp_k$$

Here we are identifying Δ^k with its image in $\widetilde{M}$ as a geodesic k-simplex. Now let $\phi : \Delta^{k-1} \times [0, 1] \to \mathbb{R}$ defined by,

$$\tau^* \omega(x, t) = \phi(x, t) dt \wedge \pi^* \bar{\omega}$$

where $\pi : \Delta^{k-1} \times [0, 1] \to \Delta^{k-1}$ is the projection to the first factor, ω is the volume form on $\Delta^k \subset \widetilde{M}$ and $\bar{\omega}$ is the induced volume form on $\Delta^{k-1} \subset \Delta^k \subset \widetilde{M}$.

Lemma 3.1.5. *There is a constant D depending only on k such that for every $p \in \Delta^{k-1}$,*

$$\int_0^1 \phi(p, t) dt \leqslant D$$

Let us first finish the proof of the theorem, then we will come back to the proof of the lemma. Using Lemma 3.1.5, if ω is the volume form on $\widetilde{M}$ then we have,

$$vol(\Delta^k) = \int_{\Delta^k} \omega = \int_{\Delta^{k-1} \times [0,1]} \tau^* \omega$$

$$= \int_{\Delta^{k-1}} \left(\int_0^1 \phi(x, t) dt \right) \bar{\omega}(x) \leqslant D \int_{\Delta^{k-1}} \bar{\omega}(x) \leqslant DC_{k-1} =: C_k$$

□

Proof of the Lemma 3.1.5. Let $\gamma : [0, 1] \to \Delta^k$ be the geodesic defined by $\gamma(t) = \tau(p, t)$ for some $p \in \Delta \in \Delta^{k-1}$. Let $X_0(0), \ldots, X_{n-1}(0)$ be an orthonormal basis for $T_p\widetilde{M}$, such that $X_0(0) = \frac{1}{L}\dot{\gamma}(0)$ where L is the length of γ, and $X_0, \ldots, X_{k-1}$ span $T_p\Delta^k$. We extend $X_i(0)$ to a parallel vector field $X_i(t)$ along γ. Now choose a local coordinate system $(y_1, \ldots, y_{k-1})$ for Δ^{k-1} around p satisfying,

$$\frac{\partial}{\partial y_i}(p) = X_i(0) + b_i X_0(0)$$

and regard $(y_1, ..., y_{k-1}, t)$ as local coordinates around $\gamma([0, 1))$ using τ. Note that the map τ is a diffeomorphism from $\Delta^{k-1} \times [0, 1)$ to its image and takes $\Delta^{k-1} \times \{0\}$ to $\Delta^{k-1} \subset \Delta^k \subset \widetilde{M}$, so then we are able to choose such a local coordinate. Then $\bar{\omega}$ at p is expressed as follows,

$$\bar{\omega}(p) = \sqrt{det\Big(g\big(\frac{\partial}{\partial y_i}(0), \frac{\partial}{\partial y_j}(0)\big)\Big)}\ dy_1 \wedge ... \wedge dy_{k-1}$$

$$= \sqrt{det\Big(I_{k-1} + \big(b_i b_j\big)_{0 \leqslant i,j \leqslant k-1}\Big)}\ dy_1 \wedge ... \wedge dy_{k-1} = \sqrt{1 + \sum_i b_i^2}\ dy_1 \wedge ... \wedge dy_{k-1}$$

where I_{k-1} is the identity $(k-1) \times (k-1)$-matrix. Now for every $1 \leqslant i \leqslant k-1$ and $0 \leqslant j \leqslant n-1$, define $a_{ij} : [0,1] \to \mathbb{R}$ by the formula

$$\frac{\partial}{\partial y_i}(\gamma(t)) = \sum_{j=0}^{n-1} a_{ij}(t) X_j(t)$$

Now let for all $1 \leqslant i \leqslant k-1$,

$$Y_i(t) := \sum_{j=1}^{n-1} a_{ij}(t) X_j(t)$$

It is clear that $Y_i(0) = X_i(0)$. Now note that by construction of the coordinate system $\frac{\partial}{\partial y_i}$ around $\gamma([0,1))$ it is clear that removing the $\dot{\gamma}$ direction from $\frac{\partial}{\partial y_i}$ will give us a variation field of a geodesic variation, thus Y_i's are Jacobi fields.

Let $A(t)$ for $0 \leqslant t < 1$ denote the $(k-1, k-1)$-matrix

$$\big(\langle Y_i(t), Y_j(t)\rangle\big)_{1 \leqslant i,j \leqslant k-1} = \left(\sum_{l=1}^{n-1} a_{il}(t) a_{jl}(t)\right)_{1 \leqslant i,j \leqslant k-1}$$

the volume form ω on Δ^k is written as

$$\omega(\gamma(t)) = \sqrt{det\Big(g\big(\bar{Y}_i(t), \bar{Y}_j(t)\big)\Big)}\ dt \wedge dy_1 \wedge ... \wedge dy_{k-1}$$

where $\bar{Y}_0 = \frac{\partial}{\partial t}$ and $\bar{Y}_i = Y_i$ for $1 \leqslant i \leqslant k-1$. Thus we have,

$$\omega(\gamma(t)) = L\sqrt{det\big(A(t)\big)}\ dt \wedge dy_1 \wedge ... \wedge dy_{k-1}$$

Therefore by definition we have,

$$\phi(p,t) = \frac{L\sqrt{det\big(A(t)\big)}}{\sqrt{1 + \sum_i b_i^2}} \leqslant L\sqrt{det\big(A(t)\big)}$$

$$\leqslant L\sqrt{\left(\sup_{u \neq 0} \frac{u^t A(t) u}{||u||^2}\right)^{k-1}} = L\sqrt{\left(\sup_{u \neq 0} \frac{||U(t)||^2}{||u||^2}\right)^{k-1}}$$

where $u = (u_1, \ldots, u_{k-1})$ and $U(t) = \sum_i u_i Y_i(t)$ which is a Jacobi field along γ as a linear combination of some Jacobi fields. The last equality holds as follows,

$$||U(t)||^2 = g\big(\sum_{i=1}^{k-1} u_i Y_i(t), \sum_{i=1}^{k-1} u_i Y_i(t)\big)$$

$$= \sum_{i,j=1}^{k-1} u_i u_j g(Y_i(t), Y_j(t)) = u^t A(t) u$$

Now we have,

$$\big(||U(t)||^2\big)'' = \langle U(t), U(t)\rangle'' = 2||U'(t)||^2 - 2\Big\langle R\big(U(t), \frac{\partial}{\partial t}\big)\frac{\partial}{\partial t}, U(t)\Big\rangle$$

$$\geqslant 2||U'(t)||^2 + 2||U(t)||^2.||\frac{\partial}{\partial t}||^2 = 2||U'(t)||^2 + 2L^2||U(t)||^2 \geqslant 2L^2||U(t)||^2$$

For all $t \in [0, 1)$. Since $||U(0)||^2 = ||u||^2$ and $||U(1)|| = 0$, we can apply the maximum value principle to the above inequality and obtain the following inequality,

$$||U(t)||^2 \leqslant ||u||^2\big(\sinh(\sqrt{2}L)\big)^{-1}\sinh(\sqrt{2}L(1-t))$$

Namely let $f(t) := ||U(t)||^2/||u||^2$ and $g(t) := \sinh(\sqrt{2}L(1-t))/\sinh(2L)$, then we have $f'' - 2L^2 f \geqslant 0 = g'' - 2L^2 g$. Now if we calculate the first and the second derivative of the function $k(t) := \frac{f(t)/g(t)}{f(0)/g(0)}$, we see that $k''(t) \geqslant 2\sqrt{2}Lk'(t)$, and $k'(0) < 0$. Now if $k'(t_0) = 0$ for some t_0 then the function $k(t)$ keeps increasing after t_0 and since $\lim_{t\to 1} k(t) = 1$ so it has to stay below the line $y = 1$, therefore we have $k(t) \leqslant 1$. Now we get,

$$det(A(t)) \leqslant \big(\sinh(\sqrt{2}L)\big)^{-k+1}\big(\sinh(\sqrt{2}L(1-t))\big)^{k-1}$$

And eventually we can estimate,

$$\int_0^1 \phi(p,t)dt \leqslant L(\sinh(\sqrt{2}L))^{-(k-1)/2}\int_0^1 \big(\sinh(\sqrt{2}L(1-t))\big)^{(k-1)/2}dt$$

$$= \frac{1}{\sqrt{2}}(\sinh(\sqrt{2}L))^{-(k-1)/2}\int_0^{\sqrt{2}L}(\sinh(t))^{(k-1)/2}dt =: X$$

Now as $L \to 0$ we have $X \to 0$ and as $L \to \infty$ we have $X \to \frac{2}{k-1}$. So the right hand side integral is finite and only depends on k. □

3.2 Proof of Theorem 1.0.2

Now we get back to the proof of Theorem 3.0.1

Proof. Let ω be the volume form on M. Suppose that $z = \sum_i a_i\sigma_i \in \lambda(S_n(M))$ represents the real fundamental class $i_*([M])$. Then we have

$$vol(M) = \int_M w \overset{(*)}{=} \int_{\sum_i a_i\sigma_i} w = \sum_i a_i \int_{\sigma_i} w$$

$$\sum_i a_i \int_{\widetilde{\sigma}_i} p^* w = \sum_i a_i vol(\widetilde{\sigma}_i(\Delta^n)) \overset{(*)}{\leqslant} \sum_i |a_i| C_n \delta^{-n/2}$$

where $p : \widetilde{M} \to M$ is the covering map and $\widetilde{\sigma}_i$ is a lift of σ_i to a geodesic simplex in $\widetilde{M}$. The equality $(*)$ holds because there is an isomorphism between the de Rham cohomology and the ordinary singular cohomology defined by

$$H^k_{dr}(M, \mathbb{R}) \to H^k_{sing}(M, \mathbb{R}) = Hom(H_k(M), \mathbb{R})$$

$$\tau \mapsto \big([\sigma] \mapsto \int_{\sigma} \tau\big)$$

where k is a positive integer and if $\sigma : \Delta^k \to M$ is a simplex, then $\int_{\sigma} \tau := \int_{\Delta^k} \sigma^* \tau$. By Stokes' theorem, this isomorphism is well defined, i.e. it does not depend on the representatives of the cycle $[\sigma] \in H_k(M)$. Since any smooth manifold is triangulable, let T be a triangulation of M. Then a linear combination of simplices in T with coeficients ± 1 and a careful choice of signs, denoted by $t \in H_n(M, \mathbb{R})$, represents the fundamental class of M and clearly we have,

$$vol(M) = \int_t \omega$$

Now if $\sum_i a_i \sigma_i$ represents the fundamental class then the equality $(*)$ above holds. The inequality $(*)$ is proved in the Theorem 3.1.4. So that for any representative $z \in \lambda(S_n(M))$ of the fundamental class $i_*([M])$ we have

$$||z||_1 \geqslant \frac{1}{C_n} \delta^{n/2} vol(M)$$

therefore we have, $||M|| = \inf_{z \in \lambda(S_n(M))} ||z||_1 \geqslant \frac{1}{C_n} \delta^{n/2} vol(M)$.

$\square$

Chapter 4

Straightening method

To prove that the simplicial volume of a certain manifold is positive, there is a general approach due to Thurston through which one shows that the manifold admits a straightening and positivity of simplicial volume follows immediately.

Definition 4.0.1 (Straightening)**.** *Let $\widetilde{M}$ be the universal cover of an n-dimensional closed oriented Riemannian manifold M. Denote by Γ the fundamental group of M and by $C_*(\widetilde{M})$ the real singular chain complex of $\widetilde{M}$. Equivalently, $C_k(\widetilde{M})$ is a free $\mathbb{R}$-module generated by $C^0(\Delta^k, \widetilde{M})$, the set of singular k-simplices in $\widetilde{M}$, where Δ^k is equipped with some fixed Riemannian metric induced from a metric on $\mathbb{R}^{k+1}$. We say a collection of maps $st_k : C^0(\Delta^k, \widetilde{M}) \to C^0(\Delta^k, \widetilde{M})$ is a straightening if it satisfies the following conditions,*

1. *the maps st_k are Γ-equivariant*
2. *the maps st_* induce a chain map $st_* : C_*(\widetilde{M}, \mathbb{R}) \to C_*(\widetilde{M}, \mathbb{R})$ that is Γ-equivariantly chain homotopic to the identity.*
3. *top dimensional straightened simplices are C^1 i.e. the image of st_n lies in $C^1(\Delta^n, \widetilde{M})$, where $C^1(\Delta^n, \widetilde{M})$ is the set of continuous maps $f : \Delta^k \to \widetilde{M}$ that can be extended to a differentiable map on a neighborhood of Δ^k in $\mathbb{R}^{k+1}$.*
4. *there exists a constant C depending on $\widetilde{M}$ and the Riemannian metric on Δ^n such that for every $f \in C^0(\Delta^n, \widetilde{M})$ and corresponding straightened simplex $st_n(f) : \Delta^n \to \widetilde{M}$ we have*
$$\forall \delta \in \Delta^n, \quad |Jac(st_n(f))(\delta)| \leqslant C$$

We will see that manifolds with a straightening have positive simplicial volume. To prove this theorem one could replace the conditions (3) and (4) in the definition of straightening by a more general condition that the volume of the image of the top dimensional straightened simplices are uniformly bounded above.

Theorem 4.0.2 (Thurston)**.** *Let $\widetilde{M}$ be the universal cover of an n-dimensional closed oriented Riemannian manifold M. If $\widetilde{M}$ admits a straightening then $||M|| > 0$.*

Proof. Because of the first property of the straightening we can descend the procedure of straightening to a straightening on the compact quotient M. For every simplex $\sigma : \Delta^k \to M$ we define $st_k(\sigma)$ to be $p \circ st_k(\widetilde{\sigma})$ where $\widetilde{\sigma}$ is a lift of σ to $\widetilde{M}$ and $p : \widetilde{M} \to M$ is the covering map. By condition (1) this does not depend on the lift and is well-defined. Condition (2) of the straightening ensures that the homology of M obtained via the complex of straightened chains coincides with the ordinary singular homology of M, where by complex of straightened chains we mean $C^{st}_*(M, \mathbb{R})$ where $C^{st}_k(M, \mathbb{R})$ is a free $\mathbb{R}$-module generated by $C_{st}(\Delta^k, M) := \{f : \Delta^k \to M : \ f = st_k(\sigma)\}$. Now if $\Sigma a_i f_i$ is a real chain representing the real fundamental class of M then so does $\Sigma a_i st_n(f)$ and we have $||\Sigma a_i f_i||_1 \geqslant ||\Sigma a_i st_n(f_i)||_1$. So to prove that the simplicial volume of M is positive it suffices to find a positive lower bound for the L^1-norm of the straightened chains representing the real fundamental class of M. Now by condition (3) of straightening the top dimensional straightened simplices are C^1 and hence we have,

$$vol(M) = \int_{\sum a_i st_n(f_i)} dV_M = \sum a_i \int_{st(f_i)} dV_M$$

where dV_M is the volume form of M. Now we have,

$$\sum a_i \int_{st(f_i)} dV_M \leqslant \sum |a_i| \int_{\Delta^n} |Jac(st_n(f_i))| dV_{\Delta^n}$$

where dV_{Δ^n} is the volume form for the fixed Riemanninan metric on Δ^n. Now by condition (4) of the straightening the Jacobian of straightened simplices are uniformly bounded from above by a constant C, so we have,

$$Vol(M) \leqslant C \ vol(\Delta^n) \sum |a_i|$$

Now taking infimum over all straightened chains we get, $||M|| \geqslant \frac{1}{C \ vol(\Delta^n)} vol(M) > 0$. $\square$

Chapter 5

Closed locally symmetric spaces

The aim of this chapter is to understand the positivity of the simplicial volume of closed locally symmetric spaces. Through this chapter by closed we shall mean compact and without boundary. If M is a locally symmetric space then $\widetilde{M}$ the universal cover of M, is a simply connected symmetric space. Hence it is decomposed isometrically to a product $M_0 \times M_- \times M_+$ where M_0, M_-, M_+ are of Euclidean, non-compact and compact type respectively. As the factor M_+ is a compact manifold, by Theorem 1.0.11 if M_+ is a non-trivial factor then $||M|| = 0$. Thus we may assume that $M_+ = \{*\}$. Furthermore if $M_0 = \mathbb{R}^k$ for some $k \geqslant 1$, then if M' is a closed locally symmetric space with universal cover M_-, hence $(S^1)^k \times M'$ has universal cover $\widetilde{M}$. So by Theorem 1.0.12 and the proportionality principle, Theorem 1.0.6, we get $||M|| = 0$. Therefore we assume that M is a closed locally symmetric space of non-compact type, which means the universal cover of M is of non-compact type. In this chapter we shall prove the following theorem,

Theorem 5.0.1 ([21])**.** *If M is a closed locally symmetric space of non-compact type, then we have $||M|| > 0$.*

5.1 Patterson-Sullivan measures

In this section we explain the construction of the Patterson-Sullivan measures on symmetric spaces of non-compact type. The Patterson-Sullivan measures form the core of the barycentric straightening procedure introduced in Section 5.2. To construct these measures we need some preparation. Let us start with some preliminaries on the decomposition of semi-simple Lie groups. Let $X = G/K$ be a symmetric space with G a semi-simple Lie group and K stabilizer of a base point $x_0 \in X$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, so $T_{x_0}X$ is identified by $\mathfrak{p}$, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. All the maximal abelian subalgebras of $\mathfrak{p}$ are conjugate under the action of $ad(K)$ and there are only finitely many of them. If $A = exp(\mathfrak{a}) \subset G$ is the image of $\mathfrak{a}$ under the exponential map then Ax_0 is a totally geodesic submanifold of X isometric with $\mathbb{R}^k$ with the standard Euclidean metric, where $k = dim(\mathfrak{a}) =: rank(X)$. The submanifold Ax_0 is called a flat submanifold of X. There are only finitely many distinct flat submanifolds of X as they correspond to the maximal abelian subalgebras of $\mathfrak{p}$. Let $\mathfrak{a}^+$ be a Weyl chamber of $\mathfrak{a}$ and $\Delta = \{\alpha_1, ..., \alpha_k\}$ be a set of simple roots, i.e. the fundamental system, see Section 2.3.2. We have $\mathfrak{p} = ad(K)\overline{\mathfrak{a}^+}$ which implies that K acts transitively on the set of Weyl chambers of $\mathfrak{a}$. Every geodesic lies in at least one flat submanifold. A geodesic called regular if it lies in exactly one flat submanifold and singular otherwise. As there are only finitely many flat submanifolds, if F is a flat submanifold through a fixed point $x \in X$, then the singular geodesics through x with initial velocity vector tangent to F form a finite union $\bigcup_\alpha H_\alpha$ of hyperplanes in F. Indeed the tangent spaces of H_α's correspond to the perpendicular hyperplanes to the roots in $\mathfrak{a}$. The image of the Weyl chamber in $\mathfrak{a}$ is one of the connected components of $F \backslash \bigcup_\alpha H_\alpha$ which is also called Weyl chamber. Now let $N^\pm$ be the horospherical subgroup of G corresponding to $\pm\mathfrak{a}^+$, i.e. $N^\pm = exp(\sum_{\pm\alpha>0} \mathfrak{g}_\alpha)$, and let $A = exp(\mathfrak{a})$. Then the demposition $G = KAN^+$ is called the Iwasawa decomposition, see Theorem 2.3.7. Let $I \subset \Delta$ be a subset of simple roots, we denote by f_I the face of type I of the Weyl chamber $\mathfrak{a}^+$ defined by

$$\{H \in \mathfrak{a} : \ \alpha(H) > 0, \ \forall \alpha \in \Delta \backslash I, \ \alpha(H) = 0, \ \forall \alpha \in I\}$$

Note that $f_\varnothing = \mathfrak{a}^+$ and $f_\Delta = 0$. The image of these faces under the exponential map acting on x_0, $exp(f_I)x_0$, is the geometrical faces of the infinite polyhedral of the Weyl chamber without their boundaries. If $I \subset \Delta$, then the connected subgroup of K that fixes the geometrical face $exp(f_I)x_0$ of the Weyl chamber in $F \subset X$ is denoted by M_I whose Lie algebra is equal to the centralizer of $f_I \subset \mathfrak{a}$ in $\mathfrak{k}$. It is clear that $M_\varnothing = M = exp(\mathfrak{m})$, where $\mathfrak{m} = \mathfrak{z}(\mathfrak{a}^+) \cap \mathfrak{k}$, and $M_\Delta = K$ since the corresponding face $exp(f_I)x_0$ is just the point x_0. A subgroup of G that is conjugate to $P_I := M_I AN^+$ for some $I \subset \Delta$ is called a parabolic subgroup of $G = KAN^+$, in particular $P := P_\varnothing = MAN^+$ is called the minimal parabolic subgroup of G, see also Definition 2.3.6. Note that the Lie algebra of gLg^{-1} for a subgroup $L \subset G$ is $Ad(g)\mathfrak{l}$ where $\mathfrak{l}$ is the Lie algebra of L. Now it is clear that the parabolic subgroups of G are the ones that fix a face of a Weyl chamber translated by an element of G. One can also show that for any $\xi \in exp(f_I)x_0(\infty)$ the stabilizer of ξ in G is exactly P_I. In particular the stabilizer of $\xi \in exp(\mathfrak{a}^+)x_0(\infty)$ is the minimal parabolic subgroup P of G. Now let $X(\infty)$ be the geodesic boundary of X. There is a bijection between $T^1_{x_0}X$ and $X(\infty)$. Now since $T_{x_0}X = \mathfrak{p} = Ad(K)\overline{\mathfrak{a}^+}$ we get

$$X(\infty) = \bigcup_{\xi \in \overline{A^+ x_0}(\infty)} G\xi$$

Furthermore we have $G\xi = K\xi$ as the subgroup $AN^+ \subset P_I$ for all $I \subset \Delta$ and consequently fixes ξ. Now since P, the minimal parabolic subgroup of G, is the stabilizer of a point $\xi \in A^+x_0(\infty)$, so then we may identify the orbit $G\xi$ with the quotient G/P. The quotient G/P is called Furstenberg boundary and it can also be described as the set of equivalence classes of asymptotic Weyl chambers since K, and in particular G, acts transitively on the set of Weyl chambers, see also Section 2.3.2. Also G/P_I is identified with the orbit of an element $\xi \in exp(f_I)x_0(\infty)$ and can be described as the set of asymptotic equivalence classes of faces of type I. Now if W is the Weyl group, i.e. the quotient of the normalizer of A^+ in K by its centeralizer in K, and $W_I := W \cap M_I$, then we the generalized Bruhat decomposition can be written as follows,

$$G = \bigcup_{\overline{\omega} \in W/W_I} N^+ \overline{\omega} P_I \text{ (disjoint union)}$$

The N^+-orbits of G/P_I are algebraic varieties which are called standard cells. The standard cell $N^- P_I = N^+ \overline{\omega_{op}} P_I$, where $\omega_{op} \in W$ is the one that takes $\mathfrak{a}^+$ to $-\mathfrak{a}^+$, has the maximal dimension among all the cells and it is called the standard big cell. Cells with smaller dimensions are called small cells. The big cell is an open dense subset of G. Now it is clear that $N^-P\xi = N^-\xi$ is an open dense subset of the orbit $G\xi$ (Furstenberg boundary), its complement consists of finite union of subvarieties with smaller dimensions. The subvarieties $gN^-\xi$ are called the big cells of $G\xi$ and the complementary varieties the small cells. Let us shortly introduce the Finsler metrics (on symmetric spaces of non-comapct type).

Now for a symmetric space of non-compact type X we construct the Patterson-Sullivan measures. Let $x_0 \in X$ be a fixed point and $\Gamma \subset G$ be a discrete subgroup of the identity component of the isometry group of X. define $\delta(\Gamma) := \inf\{s \in \mathbb{R} : \psi(s) = \sum_{\gamma\in\Gamma} e^{-sd(x_0,\gamma x_0)} < \infty\}$ and call it critical exponent of Γ. On the geodesic compactification of X we define the following family of measures,

$$\mu_{s,x} := \frac{1}{\psi(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x_0)} \delta_{\gamma x_0}, \quad s > \delta(\Gamma), \quad x \in X$$

where δ_x is the Dirac mass at x. Clearly these measures are all supported on the orbit Γx_0. Now a Patterson-Sullivan density is an accumulation point $\mu = \{\mu_x\}_{x\in X}$ of the family of maps $\{\mu_s : X \to \mathcal{M}(\partial X) \bigm| \mu_s(x) = \mu_{s,x},\ s \in (\delta(\Gamma), \delta(\Gamma)+1)\}$ as $s \to \delta(\Gamma)$. One can see that the support of the Patterson-Sullivan density lies inside the limit set $\overline{\Gamma x_0} \cap X(\infty)$. The Patterson-Sullivan measures give us information about distribution of the orbit Γx_0 in the geodesic boundary $X(\infty)$. Now let $X_{reg}(\infty)$ be the set of regular elements of $X(\infty)$. A discrete subgroup $\Gamma \subset G$ is called generic if it is Zariski-dense and if the support of any Patterson-Sullivan density lies in $X_{reg}(\infty)$.

Definition 5.1.1. *A Γ-invariant conformal density μ of dimension β is a continuous Γ-equivariant map $\mu : X \to M^+(X(\infty))$ so that,*

$$\frac{d\mu(x)}{d\mu(y)}(\xi) = e^{-\beta h_\xi(x)}$$

for all $x \in X$ and $\xi \in X(\infty)$ where by $M^+(X(\infty))$ we mean the cone of positive finite Borel measures on $X(\infty)$.

The Patterson-Sullivan density that we introduced above is a conformal density of dimension $\delta(\Gamma)$ that is supported in $\overline{\Gamma x_0} \cap X(\infty)$.

Theorem 5.1.2 ([1, Theorem A])**.** *Let $\Gamma \subset G$ be a generic subgroup. There is a single regular G-orbit $\mathfrak{D}_\Gamma$ in $X(\infty)$ such that the support of any Γ-invariant conformal density of dimension $\delta(\Gamma)$ on $X(\infty)$ lie is $\mathfrak{D}_\Gamma$. Moreover the support of the Patterson-Sullivan measures lie in $\mathfrak{D}_\Gamma \cap \overline{\Gamma x_0}$.*

5.2 Barycentric straightening

As we have seen in Chapter 4, a manifold that admits a straightening has positive simplicial volume. In this section our goal is to introduce a straightening procedure for closed locally symmetric manifolds with non-compact type. In the subsequent sections we will prove that it is actually a straightening, i.e. it satisfies the conditions (1) to (4) of Definition 4.0.1. Then the positivity of simplicial volume follows immediately from Theorem 4.0.2. Let M be a closed locally symmetric space of non-compact type, $\widetilde{M}$ be the universal cover of M

and fix a point $p \in \widetilde{M}$. In the following we shall denote the identity component of the isometry group of $\widetilde{M}$ by G and the isotropy group of the point p by K. Thus by Theorem 2.2.7 we have $\widetilde{M} \cong G/K$. The geodesic boundary of $\widetilde{M}$ will be denoted by $\partial\widetilde{M}$ and the Furstenberg boundary will be denoted by $\partial_F\widetilde{M}$, see Definition 2.3.1, Definition 2.3.12 and Theorem 2.3.13. Indeed the Furstenberg boundary of $\widetilde{M}$ is G/P where P is a minimal parabolic subgroup of G, see also Section 5.1.

Definition 5.2.1. *Under the assumptions for M and $\widetilde{M}$ in the above paragraph, we define the Busemann function $B_p : \widetilde{M} \times \partial\widetilde{M} \to \mathbb{R}$ by,*

$$(x,\xi) \mapsto \lim_{t\to+\infty} \big(d(\gamma_{p,\xi}(t),x) - t\big)$$

The Busemann function satisfies $B_p(.,.) = B_{gp}(g.,g.)$ where $g \in \Gamma = \pi_1(M)$. Because,

$$\begin{aligned} B_{gp}(gx,g\xi) &= \lim_{t\to\infty} d(gx,\gamma_{gp,g\xi}(t)) - t \\ &= \lim_{t\to\infty} d(gx,g\gamma_{p,\xi}(t)) - t \\ &= \lim_{t\to\infty} d(x,\gamma_{p,\xi}(t)) - t = B_p(x,\xi) \end{aligned}$$

Before we continue let us point out that since $\widetilde{M}$ is a symmetric space of non-compact type, if it has rank $= 1$ then it must be a negatively curved manifold and consequently M is a negatively curved manifold. The negative curvature case was explained in Chapter 3.

Lemma 5.2.2. *If the Theorem 5.0.1 holds for irreducible closed locally symmetric spaces of non-compact type then it holds for every closed locally symmetric spaces of non-compact.*

Proof. To prove the positivity of the simplicial volume of a manifold M, it is enough to prove that there exist a manifold M' with universal cover isometric with the universal cover of M and has positive simplicial volume, see Theorem 1.0.6. Let $G = G_1 \times \cdots \times G_k$ be the product decomposition of G corresponding to the de Rham deomposition of M, see Theorem 2.2.16 and Lemma 2.2.17. There exist cocompact lattices $\Gamma_i \subset G_i$ for all i, [3]. Let M' be the locally symmetric space $M_1 \times \cdots \times M_k$ obtained by the quotient group $G/(\Gamma_1 \times \cdots \times \Gamma_k)$. By the Theorem 1.0.7, If we prove the main theorem for irreducible cases, then using the inequality $||M|| = ||M'|| \geqslant ||M_1|| \times \cdots \times ||M_k||$ the main theorem follows. □

So far we have reduced the problem to irreducible higher rank locally symmetric spaces of non-compact type. Now by Theorem 1.0.9 we may also assume that $\widetilde{M} \ncong SL_3(\mathbb{R})/SO_3(\mathbb{R})$. Therefore the main theorem will follow from the following claim,

Claim 5.2.3. *If M is a compact quotient of an irreducible symmetric space of non-compact type $\widetilde{M} \ncong SL_3(\mathbb{R})/SO_3(\mathbb{R})$ with rank at least 2 then the simplicial volume of M is positive.*

To prove the claim we will use Thurston's approach, see Chapter 4, thus we need to define a straightening procedure on these manifolds.

Definition 5.2.4. *Let μ be a measure on the geodesic boundary of $\widetilde{M}$. Define $g_\mu : \widetilde{M} \to \mathbb{R}$ by*

$$g_\mu(.) := \int_{\partial\widetilde{M}} B_p(.,\theta)d\mu(\theta)$$

where B_p is the Busemann function. If g_μ has a unique minimizing point, then we call the minimizing point the barycenter of μ and denote it by $bar(\mu) \in \widetilde{M}$.

Remark 5.2.5. *Note that if we denote the first and the second derivative of the Busemann function B_p at (x,θ) by,*

$$dB_{(x,\theta)} : T_x\widetilde{M} \to \mathbb{R}$$

$$DdB_{(x,\theta)} : T_x\widetilde{M} \times T_x\widetilde{M} \to \mathbb{R}$$

then the map g_μ has a unique minimizing point if $DdB_{(x,\theta)}$ is a positive definite bi-linear form, which is equivalent to say that B_p is a strictly convex function. Here convexity means that for any geodesic $\gamma : \mathbb{R} \to \widetilde{M}$ the map $B_0(.,\theta) \circ \gamma$ is a convex map. Since the space $\widetilde{M}$ is a complete non-positively curved manifold so the Busemann function $B_p(.,\theta)$ is a convex function on $\widetilde{M}$ where $\theta \in \partial\widetilde{M}$ is fixed, see Theorem 2.4.2. But it is not clear why the Busemann function should be strictly convex.

Denote the space of atomless probability measures on the geodesic boundary $\partial\widetilde{M}$ by $\mathcal{M}(\partial\widetilde{M})$. And let $\nu : \widetilde{M} \to \mathcal{M}(\partial\widetilde{M})$ denote the $h(g_0)$-conformal density given by the family of Patterson-Sullivan measures, see Section 5.1 for the construction of the Patterson-Sullivan measures. Now by Proposition 3.1 in [7] we know that for a Patterson-Sullivan measure μ the map g_μ in the Definition 5.2.4 is a strictly convex map and consequently it has a well-defined barycenter. Now we are ready to define the barycentric straightening procedure.

Definition 5.2.6. *Spherical k-simplex Δ_s^k is the following subset of $\mathbb{R}^{k+1}$,*

$$\Delta_s^k := \left\{(a_1, ..., a_{k+1}) \in \mathbb{R}^{k+1} : \sum_{i=1}^{k+1} a_i^2 = 1,\ a_i \geqslant 0\right\}$$

equipped with the standard Riemannian metric induced from $\mathbb{R}^{k+1}$ with Euclidean metric.

Definition 5.2.7 (Barycenteric straightening)**.** *Let M be a Riemannian manifold with the properties in the Claim 5.2.3. Given a singular k-simplex $f \in C^0(\Delta_s^k, \widetilde{M})$ with vertices $x_i := f(e_i)$, where e_i's are the standard basis for $\mathbb{R}^{k+1}$, define $st_k(f) : \Delta_s^k \to \widetilde{M}$ by $st_k(f)(\sum_i a_i e_i) := bar(\sum_i a_i^2 \nu(x_i))$. Here ν is the $h(g_0)$-conformal density given by the family of Patterson-Sullivan measures constructed in Section 5.1. So the collection of straightening maps $\{st_k\}_{k=1}^n$ is called barycentric straightening. As the definition of straightened simplex $st_k(f)$ only depends on the vertices of f, for a collection V of k vertices in $\widetilde{M}$ we define $st_V : \Delta \to \widetilde{M}$ by $st_V(\sigma) := st_k(f)(\sigma)$ for some simplex f with the same vertices as V.*

5.3 Condition (1) and (2)

Lemma 5.3.1. *The barycentric straightening satisfies the conditions* (1) *and* (2) *of the Definition 4.0.1.*

Proof. **Condition(1):** The maps st_k are Γ-equivariant: Fix $\sigma = \Sigma_i a_i e_i \in \Delta_s^k$. Then for every $\gamma \in \Gamma$, $st_{\gamma V}(\sigma)$ is the unique minimizing point of the function B_ν defined below, where $\nu = \nu_{\gamma f(\Sigma_i a_i e_i)} := \Sigma_i a_i^2 \nu(\gamma x_i)$ and $x_i = f(e_i)$. Now we have,

$$\begin{aligned}
B_\nu := \int_{\partial\widetilde{M}} B_p(.,\theta)d\nu(\theta) &= \int_{\partial\widetilde{M}} B_p(.,\theta)d\Big(\sum_i a_i^2\nu(\gamma x_i)\Big)(\theta) \\
&= \int_{\partial\widetilde{M}} B_p(.,\theta)d(\sum_i a_i^2\gamma_*\nu(x_i))(\theta) \\
&= \int_{\partial\widetilde{M}} B_p(.,\gamma^{-1}\theta)d(\sum_i a_i^2\nu(x_i))(\theta) \\
&= \int_{\partial\widetilde{M}} B_{\gamma^{-1}\gamma p}(\gamma^{-1}\gamma.,\gamma^{-1}\theta)d(\sum_i a_i^2\nu(x_i))(\theta) \\
&\overset{(*)}{=} \int_{\partial\widetilde{M}} B_{\gamma p}(\gamma.,\theta)d(\sum_i a_i^2\nu(x_i))(\theta)
\end{aligned}$$

where the equality $(*)$ holds since the Busemann function satisfies $B_{\gamma p}(\gamma.,\gamma.) = B_p(.,.)$ for every $\gamma \in \Gamma$. The map $B_{\gamma p}(.,.) - B_p(.,.)$ does not depend on the first factor. To see this let $x \in \widetilde{M}$, $\theta \in \partial\widetilde{M}$, and let $\xi_{p,\theta}$ denote the unique geodesic starting from p and going towards θ. Then we have

$$\begin{aligned}
B_{\gamma p}(x,\theta) = B_p(\gamma^{-1}x,\gamma^{-1}\theta) &= \lim_{t\to\infty}\Big(d\big(\gamma^{-1}x,\xi_{p,\gamma^{-1}\theta}(t)\big) - t\Big) \\
&= \lim_{t\to\infty}\Big(d\big(\gamma^{-1}x,k.\xi_{p,\theta}(t)\big) - t\Big)
\end{aligned}$$

where $k \in \Gamma$ is a stabilizer of p that takes $\xi_{p,\theta}$ to $\xi_{p,\gamma^{-1}\theta}$, it exist because the orbit of G on $\partial\widetilde{M}$ coincides with the orbit of the stabilizer of p. To prove that the last term is equal to $B_p(x,\theta)$, we show that for any $t \in \mathbb{R}_{\geqslant 0}$ there exist a $t' \in \mathbb{R}_{\geqslant 0}$ so that

$$d\big(x,\xi_{p,\theta}\big) - t \geqslant d\big(\gamma^{-1}x,k.\xi_{p,\theta}(t')\big) - t'$$

and vice-versa. So let t be a positive real number, then for large enough t' we have,

$$\begin{aligned}
d\big(x,\xi_{p,\theta}\big) + (t'-t) &\geqslant d\big(x,\xi_{p,\theta}\big) + d\big(k^{-1}\gamma^{-1}x,x\big) \\
&\geqslant d\big(k^{-1}\gamma^{-1}x,\xi_{p,\theta}\big) = d\big(\gamma^{-1}x,k.\xi_{p,\theta}\big)
\end{aligned}$$

and therefore we get,

$$d\big(x,\xi_{p,\theta}\big) - t \geqslant d\big(\gamma^{-1}x,k.\xi_{p,\theta}\big) - t'$$

The converse is the same so they have the same limit. Now we have,

$$B_\nu - B'_\nu \overset{(*)}{=} \int_{\partial\widetilde{M}} k(\theta)d(\sum_i a_i^2\nu(x_i))(\theta)$$

where

$$B'_\nu := \int_{\partial\widetilde{M}} B_p(\gamma.,\theta)d(\sum_i a_i^2\nu(x_i))(\theta)$$

since the right hand side of the equation $(*)$ is constant on $\widetilde{M}$ the unique minimizer of B_ν is also the unique minimizer of B'_ν. But if $x \in \widetilde{M}$ is the unique minimizer of B'_ν then $\gamma^{-1}x$

must be the unique minimizer of B_ν which means $\gamma st_V = st_{\gamma V}$.

Condition (2): The maps st_* induce a chain map $st_* : C_*(\widetilde{M}, \mathbb{R}) \to C_*(\widetilde{M}, \mathbb{R})$ which is Γ-equivariantly chain homotopic to identity: To prove that st_* is a chain maps, it is enough to note that if $f : \Delta^k \to \widetilde{M}$ is a simplex then $st_k(f)_{|_{\Delta^{k-1}}} = st_{k-1}(f_{|_{\Delta^{k-1}}})$ where $\Delta^{k-1} \subset \Delta^k$ is a face of Δ^k, together with the fact that the straightening only depends on the vertices. So st_* induces a chain map. We need to prove that st_* is Γ-equivariantly chain homotopic to identity. note that every simplex is canonically homotopic with its straightening. Let $\sigma : \Delta^K \to \widetilde{M}$ be a simplex and $st_k(\sigma)$ be its straightening. For every point $p \in \Delta^k$ there is a unique geodesic γ_p from with $\gamma_p(0) = \sigma(p)$ and $\gamma_p(1) = st_k(\sigma)(p)$. Define the homotopy as follows,

$$H_\sigma : \Delta^k \times [0,1] \to \widetilde{M}$$

$$(p, t) \mapsto \gamma_p(t)$$

Now clearly we have $H_\sigma\big|_{\Delta^{k-1}\times[0,1]} = H_{\sigma_{|_{\Delta^{k-1}}}}$ where Δ^{k-1} is a face of Δ^k. Now the collection of maps $\{H_\sigma\}_\sigma$ gives us a homotopy between simplices and their straightening.

□

5.4 Top dimensional simplices

Lemma 5.4.1. *The Barycenteric straightening defined in Definition 5.2.7 satisfies the condition* (3) *of the Definition 4.0.1.*

Proof. We shall prove that the image of st_n lies in $C^1(\Delta^n_s, \widetilde{M})$, i.e. the straightened top dimensional simplices are C^1. Note that for any simplex $f \in C^0(\Delta^n_s, \widetilde{M})$ and any point $\sigma = \Sigma_i a_i e_i \in \Delta^n_s$ we have an implicit characterization of the point $st_n(f)(\sigma) = st_V(\sigma)$ via the following equation that comes from deriving the map $B_{\nu(\sigma)}$,

$$0 \equiv dB_{(st_V(\sigma),\nu(\sigma))}(.) = \int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)} d\nu(\sigma)(\theta)$$

where $\nu(\sigma) = \Sigma_i a_i^2 \nu(x_i)$ and by $dB_{(x,\mu)}$ where μ is a measure on $\partial\widetilde{M}$ and $x \in \widetilde{M}$ we mean $d_x B_\mu$. Suppose that $dB_{(x_0,\nu(\sigma_0))} \equiv 0$ for a point $x_0 \in \widetilde{M}$ and $\sigma_0 \in \Delta^n_s$. Now for a chart $(\widetilde{U}, \phi)$ around x_0 define the following function,

$$dB : \Delta^n_s \times \widetilde{U} \longrightarrow \mathbb{R}^n$$

$$(\sigma, x) \mapsto (dB_{(x,\nu(\sigma))}(\frac{\partial}{\partial\phi^1}\Big|_x), ..., dB_{(x,\nu(\sigma))}(\frac{\partial}{\partial\phi^n}\Big|_x))$$

Now if this function satisfies the conditions of the implicit function theorem, Theorem 6.5.2, then we conclude that there exist an open subset $V \subset \Delta^n_s$ around (σ_0, x_0) and an open subset $U' \subset \widetilde{U}$ such that there exist a unique continuously differentiable function $g : V \to U'$ for which we have, $dB(\sigma, g(\sigma)) = 0$ for all $\sigma \in V$. But this means that $dB_{(g(\sigma),\nu(\sigma))} = 0$ so that by uniqueness of the minimal point of $B_{\nu(\sigma)}$ we have $g(\sigma) = st_V(\sigma)$ and this means st_V is a C^1 map since so is the map g. To prove that the map dB satisfies the condition of the Implicit function Theorem it suffices to show that for the endomorphism K defined by,

$$\langle K(u), u\rangle := \int_{\partial\widetilde{M}} DdB_{(st_V(\sigma),\theta)}(u,u)d(\nu(\sigma))$$

defined on the tangent space $T_{st_V(\sigma)}M$, the determinant is non-zero. Since $\nu(\sigma)$ is fully supported in $\partial\widetilde{M}$ so by Theorem 5.5.1, we have $det(K) = det(K_{st_V(\sigma)}(\nu(\sigma))) > 0$. □

5.5 Jacobi estimate

Theorem 5.5.1 ([9, Section 4]). *Let M be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$, and let $\widetilde{M}$ be its universal cover. Let $\mu \in M(\partial\widetilde{M})$ be a probability measure fully supported on $\partial_F\widetilde{M}$ and let $x \in \widetilde{M}$. Consider the endomorphism $K_x(\mu), H_x(\mu)$, defined on $T_x\widetilde{M}$ by,*

$$\langle K_x(\mu)(u), u\rangle = \int_{\partial\widetilde{M}} DdB_{(x,\theta)}(u,u)d(\mu)(\theta)$$

and

$$\langle H_x(\mu)(u), u\rangle = \int_{\partial\widetilde{M}} dB^2_{(x,\theta)}(u)d(\mu)(\theta)$$

Then $det(K_x(\mu)) > 0$ and there is a positive constant $C := C(\widetilde{M}) > 0$ depending only on $\widetilde{M}$ such that:

$$J_x(\mu) := \frac{det(H_x(\mu))^{1/2}}{det(K_x(\mu))} \leqslant C$$

Furthermore the constant C is explicitly computable.

Lemma 5.5.2. *The Barycenteric straightening defined in Definition 5.2.7 satisfies the condition (4) of the Definition 4.0.1.*

Proof. We shall prove that there exist a constant $C > 0$, depending only on $\widetilde{M}$, such that for every $f \in C^0(\Delta^n_s, \widetilde{M})$ and the corresponding straightened simplex $st_n(f) : \Delta^n_s \to \widetilde{M}$, we have

$$|Jac(st_n(f))(\sigma)| < C$$

for all $\sigma \in \Delta^n_s$, where the Jacobian is computed relative to Riemannian metric on Δ^n_s induced by Euclidean metric on $\mathbb{R}^{n+1}$. Let us differentiate the following map which is identically zero with respect to directions in $T_\sigma\Delta^n_s$,

$$dB_{(st_V(.),\nu(.))} : \Delta^n_s \longrightarrow T^*M$$

$$\sigma \mapsto dB_{(st_V(\sigma),\nu(\sigma))}(.) = \int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)}(.)d\nu(\sigma)(\theta)$$

So then the differential would look like the following,

$$D_\sigma dB_{(st_V(.),\nu(.))} : T_\sigma(\Delta^n_s) \otimes T_{st_V(\sigma)}M \longrightarrow \mathbb{R}$$

$$D_\sigma dB_{(st_V(\sigma),\nu(\sigma))}(.,.) = D_\sigma\Bigg(\int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)} d\nu(\sigma)(\theta)\Bigg)(.,.)$$
$$= D_\sigma\Bigg(\int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)} d(\sum_i a_i^2 \nu(x_i))(\theta)\Bigg)(.,.)$$
$$= D_\sigma\Bigg(\sum_i a_i^2 \int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)} d(\nu(x_i))(\theta)\Bigg)(.,.)$$
$$= \sum_i 2a_i\langle ., e_i\rangle \int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)}(.) d(\nu(x_i))(\theta)$$
$$+ \int_{\partial\widetilde{M}} DdB_{(st_V(\sigma),\theta)}(D_\sigma(st_V)(.),.) d(\sum_i a_i^2 \nu(x_i))(\theta)$$

Now we define the endomorphisms H_σ and K_σ of $T_{st_V(\sigma)}\widetilde{M}$ by,

$$\langle H_\sigma(u), u\rangle_{st_V(\sigma)} = \int_{\partial\widetilde{M}} dB^2_{(st_V(\sigma),\theta)}(u) d(\sum_i a_i^2 \nu(x_i))(\theta)$$

$$\langle K_\sigma(u), u\rangle_{st_V(\sigma)} = \int_{\partial\widetilde{M}} DdB_{(st_V(\sigma),\theta)}(u,u) d(\sum_i a_i^2 \nu(x_i))(\theta)$$

Now by Theorem 5.5.1, K_σ is a positive definite endomorphism. Let $\{v_j\}_{j=1}^n$ be an orthonormal eigenbasis of $T_{st_V(\sigma)}\widetilde{M}$ for H_σ. At point $\sigma \in \Delta_s^n$ where the Jacobian of st_V is non-zero, let $\{\widetilde{u}_i\}_{i=1}^n$ be the pull back of the basis $\{v_j\}_{j=1}^n$ via $K_{\sigma\circ D(st_V)_\sigma}$ and $\{u_i\}_{i=1}^n$ be the orthonormal basis of $T_\sigma\Delta_s^n$ obtained from $\{\widetilde{u}_i\}_{i=1}^n$ by Gram-Schmidt algorithm. Now we have,

$$det(K_\sigma).|Jac(st_V)(\sigma)| = |det(K_\sigma \circ D(st_V)_\sigma)|$$
$$\stackrel{(1)}{=} \prod_{j=1}^n |\langle K_\sigma \circ D(st_V)_\sigma(u_j), v_j\rangle_{st_V(\sigma)}|$$
$$\stackrel{(2)}{=} \prod_{j=1}^n |\sum_{i=1}^{n+1}\langle u_j, e_i\rangle_\sigma.2a_i \int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)}(v_j) d(\nu(x_i))(\theta)|$$
$$\stackrel{(3)}{\leqslant} \prod_{j=1}^n \Bigg[\sum_{i=1}^{n+1}\langle u_j, e_i\rangle^2\Bigg]^{1/2}\Bigg[\sum_{i=1}^{n+1} 4a_i^2\Bigg(\int_{\partial\widetilde{M}} dB_{(st_V(\sigma),\theta)}(v_j) d(\nu(x_i))(\theta)\Bigg)\Bigg]^{1/2}$$
$$\stackrel{(4)}{\leqslant} 2^n \prod_{j=1}^n \Bigg[\sum_{i=1}^{n+1} a_i^2 \int_{\partial\widetilde{M}} dB^2_{(st_V(\sigma),\theta)}(v_j) d(\nu(x_i))(\theta)\Bigg]^{1/2}$$
$$\stackrel{(5)}{=} 2^n \prod_{j=1}^n \langle H_\sigma(v_j), v_j\rangle^{1/2}_{st_V(\sigma)} = 2^n det(H_\sigma)^{1/2}$$

Where the equation (1) holds since the basis u_i has been obtained by Gram-Schmidt algorithm starting from the pull back basis so the matrix representation of $K_\sigma \circ D(st_V)_\sigma$ with respect to $\{u_j\}$'s and $\{v_j\}$'s is upper triangular so the determinant is the product of diagonal entries. The equality (2) follows from the following equation,

$$0 \equiv D_\sigma dB_{(st_V(\sigma),\nu(\sigma))}(u_j, v_j)$$

$$= \sum_i 2a_i \langle u_j, e_i \rangle \int_{\partial \widetilde{M}} dB_{(st_V(\sigma),\theta)}(.) d(\nu(x_i))(\theta)$$

$$+ \int_{\partial \widetilde{M}} DdB_{(st_V(\sigma),\theta)}(D_\sigma(st_V)(u_j), v_j) d(\sum_i a_i^2 \nu(x_i))(\theta)$$

together with the fact that, by definition of K_σ we have,

$$\langle K_\sigma(D(st_V)_\sigma(u_j)), v_{v_j} \rangle_{st_V(\sigma)}$$

$$= \int_{\partial \widetilde{M}} DdB_{(st_V(\sigma),\theta)}(D(st_V)_\sigma(u_j), v_j) d(\sum_i a_i^2 \nu(x_i))(\theta)$$

The inequality (3) and (4) are just the Cauchy-Schwartz inequality applied in $\mathbb{R}^{n+1}$ and the space $L^2(\partial \widetilde{M}, \nu(x_i))$. And finally the equality (5) is just by definition of H_σ. So eventually we get the following inequality,

$$|Jac(st_V)(\sigma)| \leqslant 2^n \frac{det(H_\sigma)^{1/2}}{det(K_\sigma)}$$

Now again by Theorem 5.5.1, since the right-hand side is exactly $Jac_{st_V(\sigma)}(\Sigma a_i^2 \nu(x_i))$ and the measure $\Sigma a_i^2 \nu(x_i)$ is fully supported in $\partial_F \widetilde{M}$, there is a constant $C > 0$ depending only on $\widetilde{M}$ such that,

$$|Jac(st_V)(\sigma)| \leqslant 2^n \frac{det(H_\sigma)^{1/2}}{det(K_\sigma)} \leqslant C$$

□

5.6 Proof of Theorem 1.0.3

Proof. Now let $\{\nu_x\}_{x \in \widetilde{M}}$ be the Patterson-Sullivan measures supported on the geodesic boundary of $\widetilde{M}$. Consider the barycentric straightening defined in Definition 5.2.7. In the preceding section we saw that the barycentric straightening is a straightening, satisfying the conditions of Definition 4.0.1. Therefore M has a positive simplicial volume by Theorem 4.0.2. □

Chapter 6

Non-positive curvature and negative Ricci-curvature

I this chapter we will be studying The positivity of the simplicial volume of manifolds with geometric rank one in some special cases. Namely non-positively curved manifolds with a negative Ricci-type curvature. Here we define another notion of rank using Jacobi fields for non-positively curved Riemannian manifolds which will coincides with the previous notion of geometric rank defined in the Definition 2.1.15 for symmetric spaces of non-compact type.

Definition 6.0.1 (Geometric Rank)**.** *Let M be a non-positively curved Riemannian manifold and $v \in T^1M$ be a unit tangent vector. Let γ_v be the unique geodesic with initial velocity vector v. The rank of v is denoted by $rank(v)$ and is defined as the dimension of the space of all parallel Jacobi fields Y along γ_v. Now the rank of the manifold is defined by $rank(M) := \inf_{v \in T^1M} rank(v)$.*

Remark 6.0.2. *The notion of the geometric rank defined in both Definition 6.0.1 and Definition 2.1.15 coincide for symmetric spaces of non-compact type. Furthermore one can easily see that for $v \in T^1M$, $X(t)$ a parellel vector field along γ_v and perpendicular to $\dot{\gamma}(t)$, we have $X(t)$ is a Jacobi field if and only if the sectional curvature of the plane spanned by $X(t), \dot{\gamma}(t)$ vanishes for all t.*

Definition 6.0.3. *Let M be a non-positively curved manifold. We denote the upper rank of M by $Rank(M)$ and define it as follows,*

$$Rank(M) := \sup_{v \in T^1M} rank(v)$$

Remark 6.0.4. *Note that for a non-positively curved n-manifold we have $1 \leqslant rank(M) \leqslant Rank(M) \leqslant n$. In Chapter 5 we studied the simplicial volume of manifolds with $1 < rank(M)$. According to Lemma 6.4.5 we have, $Ric_{k+1} < 0$ (see Definition 6.4.4) if and only if $\sup_{v \in T^1M} dim(null(R_v)) \leqslant k$. Now by Remark 6.0.2 it is clear that $rank(v) - 1 \geqslant dim(null(R_v))$. Therefore we have $Rank(M) \leqslant k$ implies $Ric_k < 0$. In the current chapter we will be dealing with manifolds with negative $(\left\lfloor \frac{n}{4} \right\rfloor + 1)$-Ricci curvature and rank $= 1$ which means in addition to the rank one condition we also require them to not have a large dimensional flat submanifolds.*

In this chapter we will prove the following theorem,

Theorem 6.0.5 (Main Theorem)**.** *If M is a non-positively curved manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature, then $||M|| > 0$.*

Note that there are many examples of manifolds with rank one but a great deal of zero sectional curvature. In the followings we construct some of these manifolds,

Example 6.0.6. *Every compact surface of non-positive curvature and negative Euler characteristic has rank one. Such surfaces may contain large flat regions. Consider two unit squares lying parallel, one above the other in Euclidean space $\mathbb{R}^3$. Identify opposite edges of squares to obtain two flat tori, make two identical round holes at the center of each of the squares and connect the tori by a "neck of negative curvature" as shown in the Figure 6.1.*

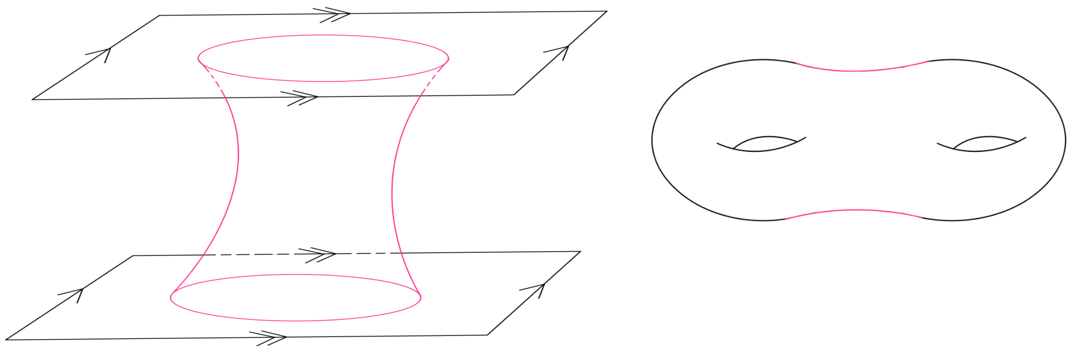

Figure 6.1:

Topologically we get a surface of genus two. Now the curvature is zero on both squares and negative on the neck. Consider a geodesic γ_v which is parallel to one of the edges and does not touch the hole. Clearly rank$(v) = 2$. Moreover, two such parallel geodesics bound a flat strip in the universal cover of M. And obviously the rank of any tangent vector v whose corresponding geodesic goes through the hole is certainly one. Therefore this manifold has rank one and negative 2-Ricci curvature.

Example 6.0.7. *Consider a non-compact n-manifold N of constant negative curvature and finite volume. Such a manifold has only finitely many cusps, [28]. See Definition 6.5.3 for the definition of cusped manifolds and Remark 6.5.4. For simplicity we assume that it only has one cusp. The cross section of the cusp is a compact flat $(n-1)$-dimensional manifold T. Cut off the cusp and flatten the manifold near the cut to make it locally isometric to the direct product of T and the unit interval, see Figure 6.2, Now consider another copy of the same manifold and identify T, T'. The manifold M we obtain has non-positive sectional curvature and an isometrically flat $(n-1)$-torus inside it. The rank of any tangent vector to to a geodesic in torus is n. On the other hand any tangent vector to a geodesic that is transverse along the torus has rank one, so that rank$(M) = 1$. The manifold M that we just constructed does not admit metric with negative curvature but we can certainly make the sectional curvature negative every where but on the torus. Therefore the manifold constructed above has rank one and negative $(n-1)$-Ricci curvature.*

Example 6.0.8. *Let N_0 be a closed k-dimensional manifold with non-positive curvature. By a result in [32], we may construct a $(k+1)$-dimensional manifold N_1 that contains*

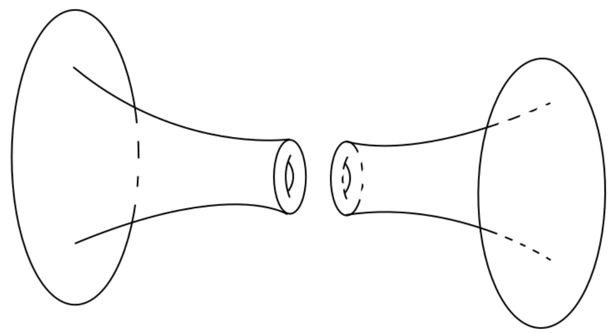

Figure 6.2:

N_0 as a totally geodesic submanifold so that for every tangent vectors $v \in T_pN_1 \backslash T_pN_0$ and $w \in T_pN_0$, where $p \in N_0$ is a point, we have the sectional curvature of the plan spanned by v, w is negative. We iterate this process and obtain a $k + j = n$-dimensional manifold N_j. Then it is clear that N_j has negative k-Ricci curvature. Now if $j \geqslant 3k$ then N_j is an n-dimensional manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature.

We will be following the same method as we used in Chapter 5. First we prove that there is a unique family of finite Borel measures fully supported in the visual boundary, then we use them to define the Barycentric straightening. Then the proof of the satisfaction of the condition (1) and (2) in the Definition 4.0.1 works exactly the same way as in Chapter 5. but we are going to have to use some other tools to prove that the Barycentric straightening satisfies the condition (3) and (4) of the Definition 4.0.1.

6.1 Patterson-Sullivan measures

Theorem/Definition 6.1.1 ([13]). *Let M be a compact non-positively curved geometric rank one manifold, $\widetilde{M}$ the universal cover of M and Γ the fundamental group of M. There exist a unique family of finite Borel measures $\{\mu_x\}_{x \in \widetilde{M}}$ fully supported on $\partial\widetilde{M}$, called Patterson-Sullivan measure, which satisfies the following conditions,*

1. *μ_x is Γ-equivariant for all $x \in \widetilde{M}$*
2. *$\frac{d\mu_x}{d\mu_y}(\theta) = e^{hB(x,y,\theta)}$ for all $x, y \in \widetilde{M}$ and $\theta \in \partial\widetilde{M}$*

where h is the volume entropy of M, see Definition 2.1.2 and $B(x, y, \theta)$ is the Busemann function of $\widetilde{M}$.

Example 6.1.2. *(Harmonic measures in the disk model)*
For the disk $\mathbb{D}$ with hyperbolic metric we have $\partial\mathbb{D} = \mathbb{S}^1$. Consider the following map,

$$P : \mathbb{D} \times \partial\mathbb{S}^1 \longrightarrow \mathbb{R}$$

$$(x, \xi) \mapsto \frac{1 - ||x||^2}{||\xi - x||^2}$$

Now let $\nu_x = P(x, .)\sigma$ be a probability measure on $\mathbb{S}^1$ and call it harmonic measure associated to x, where σ is the uniform probability measure on $\mathbb{S}^1$. Now a direct computation shows *that,*

$$\forall \xi \in \mathbb{S}^1, \quad \forall x, y \in \mathbb{D}, \quad B(x,y,\xi) = log\left(\frac{P(y,\xi)}{P(x,\xi)}\right)$$

In other words, the harmonic measures associated to points of $\mathbb{D}$ satisfy,

$$\forall \xi \in \mathbb{S}^1, \quad \forall x, y \in \mathbb{D}, \quad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-B(x,y,\xi)}$$

which means these harmonic measures are the Patterson-Sullivan measures of $\mathbb{D}$. Note that the volume entropy of the disk model of 2-dimensional hyperbolic space with radius 1 is constant 1.

6.2 Barycentric straightening

We start by a very important theorem which we will be using a lot later o, it estimates the Hessian of the Busemann function in terms of curvature, and it holds for all closed non-positively curved manifolds,

Theorem 6.2.1. *Let $\widetilde{M}$ be the universal cover of some closed non-positively curved manifold M, $x \in \widetilde{M}$ and $\theta \in \partial\widetilde{M}$. If $Y_0 \in T_x^1\widetilde{M}$ is any unit vector in the horocycle direction, that is, $Y_0 \perp \gamma'_{x\theta}(0)$, where $\gamma_{x\theta}$ is the geodesic ray connecting x and θ, then there exist a constant C that depends on quantities $||R||, ||\nabla R||, ||\nabla^2 R||$, such that,*

$$DdB_{(x,\theta)}(Y_0, Y_0) \geqslant C\bigg(-\langle R(\gamma'_{x\theta}(0), Y_0)\gamma'_{x\theta}(0), Y_0\rangle\bigg)^{3/2}$$

Proof. We extend Y_0 along the ray $\gamma_{x\theta}$ to $Y(t)$, the unique stable Jacobi field with $Y(0) = Y_0$. Then the Hassian $DdB_{(x,\theta)}(Y_0, Y_0)$ is the second fundamental form in the direction Y_0 of the horosphere determined by x and θ, which is further equal to $-\langle Y(0), Y'(0)\rangle$. We now take the second covariant derivative along the geodesic ray of $\langle Y(t), Y(t)\rangle$,

$$\langle Y(t), Y(t)\rangle'' = 2\bigg(\langle Y'(t), Y'(t)\rangle + \langle Y(t), Y''(t)\rangle\bigg)$$

$$= 2\bigg(||Y'(t)||^2 + R_{\gamma'_{x\theta}(t)}\big(Y(t)\big)\bigg)$$

Note that since curvature is non-positive so the second covariant derivative above is positive. Therefore since $||Y(t)||^2$ is bounded and the second derivative is positive so we must have $\langle Y(t), Y(t)\rangle' < 0$. So in particular $||Y(t)||^2$ converges to a constant which means $\langle Y(t), Y'(t)\rangle \to 0$. Now integrating along the geodesic ray we obtain,

$$2\int_0^\infty \bigg(||Y'(t)||^2 + R_{\gamma'_{x\theta}(t)}\big(Y(t)\big)\bigg)dt$$

$$= 2\bigg(\lim_{t\to\infty}\langle Y(t), Y'(t)\rangle - \langle Y(0), Y'(0)\rangle\bigg)$$

$$= -2\langle Y(0), Y'(0)\rangle = 2DdB_{(x,\theta)}(Y_0, Y_0)$$

Therefore we get the following inequality,

$$DdB_{(x,\theta)}(Y_0, Y_0) \geqslant \int_0^\infty R_{\gamma'_{x\theta}(t)}\big(Y(t)\big)dt$$

To finish the proof of the theorem we need the following lemma from calculus,

Lemma 6.2.2. *Let F be a C^2 function on $[0,\infty)$. If $F \geqslant 0$ and F'' is bounded above by a constant $L \geqslant 0$, then there is a constant $C > 0$ that depends on L such that,*

$$\int_0^\infty F(t)dt \geqslant C.F(0)^{3/2}$$

First we continue with the proof of the theorem and we will prove the Lemma 7.19 right after. If we can apply Lemma 7.19 to the function $R_{\gamma'_{x\theta}(t)}(Y(t))$ then we get the inequality of the theorem and we are done. So it suffices to show that the second derivative of $R_{\gamma'_{x\theta}(t)}(Y(t))$ is bounded above. In the following we shall write γ', Y, Y' for $\gamma'_{x\theta}(t), Y(t), Y'(t)$ respectively for brevity.

$$\begin{aligned}
\left|\Big(R_{\gamma'_{x\theta}(t)}(Y(t))\Big)''\right| &= \left|\Big(\big\langle(\nabla_{\gamma'}R)(\gamma', Y)\gamma', Y\big\rangle + \big\langle R(\gamma', Y)Y, Y'\big\rangle\Big)'\right| \\
&= \Big|\big\langle(\nabla^2_{(\gamma',\gamma')}R)(\gamma', Y)\gamma', Y\big\rangle + 4\big\langle(\nabla_{\gamma'}R)(\gamma', Y)\gamma', Y'\big\rangle \\
&\quad +2\big\langle R(\gamma', Y)\gamma', Y'\big\rangle + 2\big\langle R(\gamma', Y)\gamma', Y''\big\rangle\Big| \\
&\leqslant C(||Y||^2 + ||Y'||^2)
\end{aligned}$$

where the last inequality is obtained by Cauchy-Schwartz inequality and the Jacobi equation $Y'' + R(\gamma', Y)\gamma' = 0$. And the constant C is only dependant on $||R||, ||\nabla R||, ||\nabla^2 R||$. We also note that $||Y||^2$ is bounded so we only need to prove that $||Y'||$ is bounded above. However we have,

$$\begin{aligned}
\big|\langle Y'(t), Y'(t)\rangle'\big| &= 2\big|\langle Y'(t), Y''(t)\rangle\big| = 2\big|\langle R(\gamma', Y)\gamma', Y'\rangle\big| \\
&\leqslant C_1\sqrt{\langle R(\gamma', Y)\gamma', Y\rangle\langle R(\gamma', Y')\gamma', Y'\rangle} \\
&\leqslant C_2\sqrt{\langle -R(\gamma', Y)\gamma', Y\rangle}||Y'|| \\
&\leqslant C_2\Big(\langle -R(\gamma', Y)\gamma', Y\rangle + ||Y'||^2\Big) \\
&= C_2\langle Y, Y'\rangle'
\end{aligned}$$

where the first inequality is the Cauchy-Schwartz inequality for the positive semi-definite bilinear form $-R$, the second inequality uses the bound $||R||$, the third inequality is again Cauchy-Schwartz inequality in $\mathbb{R}$ and the last equality uses the Jacobi equation. Here the constant C_2 only depends on $||R||$. Integrating the above inequality, we obtain, for any $0 < t < s < \infty$,

$$\big|\langle Y'(t), Y'(t)\rangle - \langle Y'(s), Y'(s)\rangle\big| \overset{(*)}{\leqslant} C_2\big|\langle Y(t), Y'(t)\rangle - \langle Y(s), Y'(s)\rangle\big|$$

As we saw, $\langle Y(s), Y'(s)\rangle$ increases to 0 as $s \to \infty$. Note that,

$$\int_0^\infty ||Y'(t)||^2 dt \leqslant \int_0^\infty \Big(||Y'(t)||^2 + R_{\gamma'_{x\theta}(t)}(Y(t))\Big)dt$$
$$= \int_0^\infty \Big(\langle Y'(t), Y'(t)\rangle + \langle Y''(t), Y(t)\rangle\Big)dt$$
$$= \int_0^\infty \langle Y(t), Y'(t)\rangle' dt < \infty$$

So $\langle Y'(s), Y'(s)\rangle$ goes to 0 as $s \to \infty$. Now if we let s goes to infinity in the inequality $(*)$ we get,

$$\big|\langle Y'(t), Y'(t)\rangle\big| \leqslant C_2\big|\langle Y(t), Y'(t)\rangle\big| \leqslant -C_2\langle Y'(0), Y'(0)\rangle = C_2 DdB_{(x,\theta)}(Y(0), Y(0))$$

But by the comparison theorem the Hassian is bounded above by a constant depending on $||R||$. This shows that $||Y'||$ is bounded by a constant depending on $||R||$, hence the second derivative of $R_{\gamma'_{x\theta}(t)}(Y(t))$ is bounded by a constant on $||R||, ||\nabla R||, ||\nabla^2 R||$ and in view of the Lemma 7.19 we eventually obtain the main iequality stated in the theorem. ☐

Proof. (Proof of Lemma 7.19) Considering the derivative of the function $\sqrt{F(t)}$ we obtain $F(t) \geqslant (\sqrt{F(0)} - L't)^2$ on the interval $[0, \sqrt{F(0)}/L']$ for some constant L' depending on L. If we set $G(t) = F(t) - (\sqrt{F(0)} - L't)^2$, then $G(0) = 0$, and $G(\sqrt{F(0)}/L') = F(\sqrt{F(0)}/L') \geqslant 0$. Moreover $G''(t) = F''(t) - 2L'^2$. So if we choose $L' > \sqrt{L/2}$, then $G'' < 0$. Therefore G would be concave and hence $G \geqslant 0$ on $[0, \sqrt{F(0)}/L']$. Using this and noting that $F \geqslant 0$ we estimate the integral

$$\int_0^\infty F(t)dt \geqslant \int_0^{\sqrt{F(0)}/L'} \Big(\sqrt{F(0)} - L't\Big)^2 dt = \frac{F(0)^{3/2}}{3L'} = C.F(0)^{3/2}$$

where C is a constant only depending on L. ☐

Theorem 6.2.3 (Convexity of Busemann Function)**.** *let M be compact non-positively curved geometric rank one manifold. Fix a base point O in M and denote the Busemann function $B(O, ., .)$ by $B(., .)$, see Definition 2.4.1. The Busemann function is convex and the null space of its Hassian $DdB_{(x,\theta)}$ in direction $v_{x\theta}$-connecting x to θ have zero sectional curvature with $v_{x\theta}$, i.e. $sec(u, v_{x\theta}) = 0$ for $u \in null(DdB_{(x,\theta)})$. Where $v_{x\theta}$ is the velocity vector of the unique geodesic connecting x to θ.*

Proof. If one think of the Busemann function $B(x, \theta)$ as a function that measures the angle $\angle_O(x, \theta)$ then it is clear that it is a convex map. But we will see the detailed calculation soon. The second part of the theorem is an immediate consequence of the Theorem 6.2.1. ☐

Theorem 6.2.4 (Strictly convexity of Busemann Function)**.** *Let M be a compact non-positively curved geometric rank one. If Ricci-curvature is negative and ν is a finite Borel measure fully supported in $\partial\widetilde{M}$ then the function below is strictly convex,*

$$B_\nu : \widetilde{M} \longrightarrow \mathbb{R}$$

$$x \mapsto \int_{\partial \widetilde{M}} B(x, \theta) d\nu(\theta)$$

So consequently there is a unique point in $\widetilde{M}$ such that B_ν attains its minimum on, we denote this point by $bar(\nu)$.

Proof. Considering the definition of convexity of a map on a manifold it is clear that a map is strictly convex if and only if its hessian is positive definite. So we shall prove that the hessian,

$$\int_{\partial \widetilde{M}} DdB_{(x,\theta)}(.,.) d\nu(\theta)$$

is positive definite. To see this let $u \in T_x\widetilde{M}$ be a unit tangent vector. We claim that there is a $\theta_0 \in \partial\widetilde{M}$ such that $DdB_{x,\theta_0}(u,u) > 0$ and $v_{x\theta_0}$ is orthogonal to u. If not then we have $DdB_{(x,\theta)}(u,u) = 0$ for every $\theta \in \partial\widetilde{M}$ that $v_{x\theta}$ is orthogonal to u. By previous theorem since u is in the null space of $DdB_{x,\theta}$ in the direction $v_{x\theta}$ so the sectional curvature of the plane spanned by u and $v_{x\theta}$ is zero for any θ that $v_{x\theta}$ is orthogonal to u(see Theorem 7.18). So the Ricci-curvature at x vanishes that is a contradiction. Therefore there exist a $\theta_0 \in \partial\widetilde{M}$ such that $DdB_{x,\theta_0}(u,u) = \delta_0 > 0$. By continuity there is a neighborhood U of θ_0 such that $DdB_{x,\theta}(u,u) > \delta_0/2$ for all $\theta \in U$. hence we have,

$$\int_{\partial\widetilde{M}} DdB_{x,\theta}(u,u)d\nu(\theta) \geqslant \int_U DdB_{x,\theta}(u,u)d\nu(\theta) > \frac{\delta_0}{2}\nu(U) > 0.$$

Note that here we have used that ν is fully supported in $\partial\widetilde{M}$, and also the Busemann function is (not strictly)convex even without the assumption $Ricci < 0$. So the Hessian is positive definite and hence B_ν is strictly convex. $\square$

Definition 6.2.5. *Standard spherical k-simplex Δ_s^k is defined as follows,*

$$\Delta_s^k := \{(a_1, a_2, ..., a_{k+1}) : a_i \geqslant 0, \quad \sum_{i=1}^{k+1} a_i^2 = 1\} \subset \mathbb{R}^{k+1}$$

with the induced Euclidean Riemannian metric from $\mathbb{R}^{k+1}$ and with ordered vertices $\{e_1, ..., e_{K+1}\}$.

Definition 6.2.6 (Barycentric Straightening)**.** *Let M be a Riemannian manifold. Suppose M is a closed compact manifold with non-positive sectional curvature, negative Ricci-curvature and geometric rank one (Connell, Wang) and let $\widetilde{M}$ be its universal cover. Given any singular k-simplex $f : \Delta_s^k \to \widetilde{M}$, with ordered vertices $(x_1, ..., x_{k+1}) = (f(e_1), ..., f(e_{k+1}))$, we defined the k-straightened simplex,*

$$st_k(f) : \Delta_s^k \longrightarrow \widetilde{M}$$

$$(a_1, ..., a_{k+1}) \mapsto bar(\sum_{i=1}^{k+1} a_i^2 \nu_{x_i})$$

where $\nu_{x_i} = \mu_{x_i}/||\mu_{x_i}||$ is the normalized Patterson-Sullivan measure at x_i and $bar(\nu)$ is defined in the statement of Theorem **??***. Note that $st_k(f)$ is determined only by the ordered vertex set V, so then we denote $st_k(f)(\delta)$ by $st_V(\delta)$ for any $\delta \in \Delta_s^k$.*

6.3 Top dimensional simplices

Lemma 6.3.1 (condition (3))**.** *Let M be a compact manifold with non-positive sectional curvature, negative Ricci curvature and geometric rank one. Then the Barycentric straightening, see Definition 6.2.6, satisfies the condition (3) of the Definition 4.0.1, i.e. the top-dimensional straightened simplices under Barycentric straightening procedure are C^1.*

Proof. Proof is quite similar to the proof of Lemma 5.4.1 once we prove that the following map is strictly convex,

$$x \mapsto \int_{\partial \widetilde{M}} B(x, \theta) d\nu(\theta)$$

where ν is a finite Borel measure fully supported in $\partial \widetilde{M}$. Because then we would have $det(K_\sigma) \neq 0$, hence we would be able to proceed in the proof of Lemma 5.4.1. Strictly convexity of the above map was proved in the Theorem 6.2.4. ☐

6.4 Jacobi estimate

Lemma 6.4.1 (condition (4))**.** *Let M be a compact manifold with non-positive sectional curvature, negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature (which implies negative Ricci curvature) and geometric rank one. Then the Barycentric straightening, see Definition 6.2.6, satisfies the condition (4) of the Definition 4.0.1.*

Proof. Proof is quite similar to the proof of Lemma 5.5.2 up to where we get the following inequality,

$$|Jac(st_V)(\sigma)| \leqslant 2^n \frac{det(H_\sigma)^{1/2}}{det(K_\sigma)}$$

Now to get a uniform bound on the fraction on the right hand side we would need a new method. In the rest of this section we will develop a method to bound the fraction. Recall that the bound in the proof of Lemma 5.5.2 was obtained thanks to the Theorem 5.5.1 for locally symmetric space of non-compact type with no local factors locally isometric to $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$. ☐

Definition 6.4.2. *For any positive semi-definite linear endomorphism $A : V^m \to V^m$ and for any $0 \leqslant k \leqslant m$ we define the $k-th$ trace of A as follows,*

$$Tr_k(A) := \inf_{V_k \subset V^m} Tr(A_{|_{V_k}})$$

where V_k is a k-dimensional subspace of V (not necessarily invariant under A), and A is viewed as a bilinear form when taking restriction. Equivalently it is sum of k least eigenvalues of A.

Definition 6.4.3. *Given an n-dimensional Riemannian manifold M with curvature tensor R, for any $u \in T_xM$ we define a symmetric bilinear form, $R_u(v_1, v_2) := -R(u, v_1, u, v_2) := -\langle R(u, v_1, u), v_2 \rangle$ where $v_1, v_2 \in T_xM$. In particular if the manifold is non-positively curved then R_u defines a positive semi-definite symmetric form on T_xM. Furthermore we define the k-Ricci curvature in direction u as,*

$$Ric_k(u) := -Tr_k(R_u)$$

Definition 6.4.4. *Given an n-dimensional Riemannian manifold with curvature tensor R, $u, v \in T_xM$ and $0 \leqslant k \leqslant n$ define,*

$$Ric_k(u,v) := \sup_{\substack{V \subset T_xM \\ dimV = k}} Tr\Big(R(u,.,v,.)_{|V}\Big)$$

whrere $R(u,.,v,.)_{|V}$ is the restriction of R to $V \times V$. And we also set $Ric_k := \sup_{v \in T_x^1M} Ric_k(v,v)$ where T_x^1M is the set of unit vectors in T_xM.

Lemma 6.4.5. *Let M be a closed manifold with non-positive curvature. Then the followings are equivalent,*

1. *$dim\big(null(R_v)\big) \leqslant n/4$ for all $v \in T_x^1M$.*
2. *$\forall v \in T_x^1M$, there exist a subspace $F_v \subset T_xM$ of dimension at least $3n/4$ such that $\langle v, F_v \rangle = 0$ and $R_v(u,u) \geqslant C_0$ for all $u \in F_v$, where C_0 is some universal constant that only depends on (M,g).*
3. *$\forall v \in T_x^1M$ the k-eigenvalue (in increasing order) of R_v is at least C_0 when $k > n/4$, where C_0 is some universal constant that only deppends on (M,g).*
4. *There is a $\delta > 0$ that only depends on (M,g) such that,*
$$\inf_{v \in T_x^1M} Tr_k(R_v) \geqslant \delta$$
when $k > n/4$.
5. *The manifold has strictly k-th Ricci when $k > n/4$. That is $Ric_k(v) < 0$ for all $v \in T_x^1M$ and $k > n/4$, or equivalently $Ric_{\lfloor n/4 \rfloor + 1} < 0$.*

Proof. We will be proceeding as follows, first we prove (4) $\Leftrightarrow$ (5) and then (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4).

- (4) $\Rightarrow$ (5) : $\forall v \in T_x^1M$ and $k > n/4$ we have,
$$Ric_k(v) = -Tr_k(R_v) \leqslant -\inf_{V_k \subset T_xM} Tr_k(R_v) \leqslant -\delta < 0$$
Equivalently, for $k = \lfloor n/4 \rfloor + 1$,
$$Ric_k = \sup_{v \in T_x^1M} Ric_k(v,v) = \sup_{v \in T_x^1M} \sup_{V_k \subset T_xM} Tr\Big(R(v,.,v,.)_{|V_k}\Big)$$
$$= \sup_{v \in T_x^1M} - \inf_{V_k \subset T_xM} Tr\Big(-R(v,.,v,.)_{|V_k}\Big)$$
$$= \sup_{v \in T_x^1M} -Tr_k(R_v) = -\inf_{v \in T_x^1M} Tr_k(R_v) \leqslant -\delta < 0$$
Note that we are taking the last Infimum over T_x^1M which is a compact set.
- (5) $\Rightarrow$ (4) : Similar to (4) $\Rightarrow$ (5).

- (4) $\Rightarrow$ (3) : Let V_k be the span of the first k eigenvectors of R_v, with associated eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_k$. By definition we have, $\lambda_1 + ... + \lambda_k = Tr(R_{v|_{V_k}}) \geqslant Tr_k(R_v) \geqslant \delta$. So that we have $\lambda_k \geqslant \delta/k$, with constant δ/k only depending on (M, g).
- (3) $\Rightarrow$ (2) : Take F_v to be the span of the last $n-k+1$ eigenvectors of R_v, where $k = \lfloor n/4 \rfloor + 1$. Now since $R_v(v,v) = 0$ so $v \in null(R_v)$ and corresponds to the first eigenvalue of R_v, $\lambda_1 = 0$. Assuming (3) we know that $\lambda_i > 0$ for $i \geqslant k$ so we have $\langle v, F_v \rangle = 0$. Note that $R_v(u,u) \geqslant \lambda_k \geqslant C_0$ for all $u \in F_v$. Also note that $dim(F_v) = n - k + 1 \geqslant n - \lfloor n/4 \rfloor \geqslant 3n/4$.
- (2) $\Rightarrow$ (1) : For any $v \in T_x^1 M$, by property of F_v, $F_v \cap null(R_v) = 0$, Therefore $dim(F_v) + dim(null(R_v)) \leqslant n$, hence $dim(null(R_v)) \leqslant n/4$.
- (1) $\Rightarrow$ (4) : Let $l = \lfloor n/4 \rfloor + 1$ and denote by $\lambda_k(v)$ the k-th eigenvalue of R_v. By (1), $\lambda_l(v) > 0$ for all $v \in T_x^1 M$. Since $\lambda_l(v)$ is continuous on v and $T_x^1 M$ is compact, there exist a universal constant $\delta > 0$ such that $\lambda_l(v) \geqslant \delta$, hence for any $k > n/4$ we have,

$$\inf_{v \in T_x^1 M} Tr_k(R_v) \geqslant \inf_{v \in T_x^1 M} \lambda_k(v) \geqslant \inf_{v \in T_x^1 M} \lambda_l(v) \geqslant \delta$$

$\square$

Definition 6.4.6. *We say that a non-positivelt curved manifold has negative* $(\lfloor \frac{n}{4} \rfloor + 1)$*-negative Ricci-curvature if it satisfies any of the five conditions in Lemma 9.3.*

Theorem 6.4.7. *Under the assumption of Theorem 7.18, If M has negative* $(\lfloor \frac{n}{4} \rfloor + 1)$*-Ricci curvature, then*

$$Tr_{k+1}(DdB_{x,\theta}(.,.)) \geqslant C_0$$

where $k = \lfloor \frac{n}{4} \rfloor$ *and C_0 depends on the negative* $(\lfloor \frac{n}{4} \rfloor + 1)$*-Ricci constant in Lemma 9.4, in particular it depends on (M, g).*

Proof. We choose an orthonormal frame $e_1, ..., e_{k+1}$ of the $k+1$ least eigenvectors of the $DdB_{(x,\theta)}(.,.)$, so that,

$$Tr_{k+1}(DdB_{(x,\theta)}(.,.)) = \sum_{i=1}^{k+1} DdB_{(x,\theta)}(e_i, e_i)$$

Now according to Theorem 7.18 and Holder's inequality we have,

$$\sum_{i=1}^{k+1} DdB_{(x,\theta)}(e_i, e_i) \geqslant C \sum_{i=1}^{k+1} R_{v_{x\theta}}(e_i, e_i)^{3/2} \geqslant C' \sum_{i=1}^{k+1} \bigg(R_{v_{x\theta}}(e_i, e_i) \bigg)^{3/2}$$

Now the negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature condition implies,

$$\sum_{i=1}^{k+1} R_{v_{x\theta}}(e_i, e_i) \geqslant Tr_{k+1}(R_{v_{x\theta}}) \geqslant C''$$

where C'' only depends on (M, g). So we have the inequality of the lemma. $\square$

As a first step to find a bound on the Jacobian of the straightened simplices in barycentric straightening method, we use the Theorem 7.18 to conclude the following lemma which compares pointwise the integrands of H_σ and K_σ. We should remark that the power 2/3 in the following lemma which traces back to Lemma 7.19 directly leade to the imposed $n/4$ condition. If this power can be improved to be closer to 1, then the resulting k-Ricci condition could be slightly weakened, but is still limited to $n/3$ condition.

Lemma 6.4.8. *Suppose M is a closed non-positively curved n-manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature and $\widetilde{M}$ is the Riemannian universal cover of M. Let $x \in \widetilde{M}$, $\theta \in \partial\widetilde{M}$. Then there is a constant C that depends on (M, g) such that for all $v \in T^1_x M$ and all $u \in F_v$ (where F_v satisfies (2) of Lemma 9.4), we have,*

$$dB^2_{(x,\theta)}(u,u) \leqslant C\bigg(DdB_{(x,\theta)}(v,v)\bigg)^{2/3}$$

Proof. We decompose $v = v_1 + v_2$ where v_1 is parallel to $v_{x\theta}$ and v_2 is orthogonal to it, and we denote by α the angle between $v_{x\theta}$ and v. Note that if $sin(\alpha) = 0$, that is v is parallel to $v_{x\theta}$ then $B_O(\gamma_u(t), \theta)$ is a constant function so $dB^2_{(x,\theta)}(u,u) = 0$ and the inequality holds automatically. Now if $sin(\alpha) \neq 0$ then we can estimate,

$$DdB_{(x,\theta)}(v,v) = DdB_{(x,\theta)}(v_2,v_2) \geqslant sin^2(\alpha)\bigg(CR_{v_{x\theta}}(\frac{v_2}{||v_2||}, \frac{v_2}{||v_2||})\bigg)^{3/2}$$

$$= \frac{C^{3/2}}{|sin(\alpha)|}R_{v_{x\theta}}(v_2,v_2)^{3/2} \geqslant C^{3/2}R_{v_{x\theta}}(v_2,v_2)^{3/2}$$

$$= C^{3/2}R_v(v_{x\theta}, v_{x\theta})^{3/2}$$

Note that when restricted to F_v, R_v have eigenvalues at least C_0 according to Lemma 9.4, hence,

$$R_v(v_{x\theta}, v_{x\theta}) \geqslant C_0 cos^2(\angle(v_{x\theta}, F_v)) \geqslant C_0 cos^2(\angle(v_{x\theta}, u)) = C_0 dB^2_{(x,\theta)}(u,u)$$

So then we finally have,

$$dB^2_{(x),\theta}(u,u) \leqslant C\bigg(DdB_{(x,\theta)}(v,v)\bigg)^{2/3}$$

□

Theorem 6.4.9. *Suppose M is a closed non-positively curved n-manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature, and $\widetilde{M}$ is its universal cover. Let $x\widetilde{M}, \theta \in \partial\widetilde{M}$, and ν be any probability measure that has full support in $\partial\widetilde{M}$. Then there exist a universal constant C that only depends on (M, g), so that,*

$$\frac{det\bigg(\int_{\partial\widetilde{M}} dB^2_{x,\theta}(.,.)d\nu(\theta)\bigg)^{1/2}}{det\bigg(\int_{\partial\widetilde{M}} DdB_{(x,\theta)}(.,.)d\nu(\theta)\bigg)} \leqslant C$$

Proof. Set $K_{x,\theta} := DdB_{(x,\theta)}(.,.)$, $H_{x,\theta} := dB^2_{(x,\theta)}(.,.)$ and $K := \int_{\partial\widetilde{M}} K_{x,\theta}(.,.)d\nu(\theta)$, $H := \int_{\partial\widetilde{M}} H_{x,\theta}(.,.)d\nu(\theta)$. Let $0 \leqslant \lambda_1 \leqslant ... \leqslant \lambda_n$ be the eigenvalues of K, and let v be the eigenvector corresponding to λ_1. Then there exist a constant C' depending on (M, g), such that for any $u \in F_v$, we have,

$$H(u,u) = \int_{\partial\widetilde{M}} H_{x,\theta}(u,u)d\nu(\theta) \leqslant C' \int_{\partial\widetilde{M}} K_{x,\theta}(v,v)^{2/3}d\nu(\theta)$$
$$\leqslant C'\left(\int_{\partial\widetilde{M}} K_{x\theta}(v,v)d\nu(\theta)\right)^{2/3} = C'\lambda_1^{2/3}$$

Therefore we can find an orthonormal frame $e_1, ..., e_{n-k}$ at x such that $H(e_i, e_i) \leqslant C'\lambda_1^{2/3}$ for $1 \leqslant i \leqslant n-k$ where $k = \lfloor \frac{n}{4} \rfloor$. This implies that,

$$Tr_{n-k}(H) \leqslant \sum_{i=1}^{n-k} H(e_i, e_i) \leqslant (n-k)C'\lambda_1^{2/3}$$

If we further denote by $\mu_1 \leqslant \mu_2 \leqslant ... \leqslant \mu_n$ the eigenvalues of H then we have,

$$\mu_i \leqslant Tr_{n-k}(H) \leqslant (n-k)C'\lambda_1^{2/3}, \quad 1 \leqslant i \leqslant n-k$$

Since $Tr_{n-k}(H) = \mu_1 + ... + \mu_{n-k}$. Note that the eigenvalues of H are at most 1 and $k = \lfloor \frac{n}{4} \rfloor \leqslant n/4$, so we can estimate the following,

$$\frac{det(H)^{1/2}}{det(K)} = \frac{\prod_{i=1}^n \mu_i^{1/2}}{\prod_{i=1}^n \lambda_i} \leqslant \frac{\left((n-k)C'\lambda_1^{2/3}\right)^{\frac{n-k}{2}}}{\lambda_1^k \lambda_{k+1}^{n-k}} \leqslant \frac{C''}{\lambda_{k+1}^{n-k}}$$

for some constant C'' depending on (M, g). And finally we can bound λ_{k+1} as follows,

$$\lambda_{k+1} \geqslant \frac{1}{k+1} Tr_{k+1}(K) \geqslant \frac{1}{k+1} \inf_{\theta \in \partial\widetilde{M}} Tr_{k+1}(K_{x,\theta}) \geqslant \frac{C_0}{k+1}$$

Therefore by combining the above inequalities we conclude,

$$\frac{(detH)^{1/2}}{detK} \leqslant C''\left(\frac{k+1}{C_0}\right)^{n-k} \leqslant C$$

where C depends on M. $\square$

6.5 Proof of Theorem 1.0.5

Now we restate the main theorem here and prove it,

Theorem 6.5.1 (Main Theorem)**.** *If M is a non-positively curved manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$-Ricci curvature, then $||M|| > 0$.*

Proof. By Theorem 6.1.1 we know that non-positively curved manifolds with geometric rank one admit a unique family of Borel finite measures fully supported in the visual boundary (Patterson-Sullivan measures). Now by Theorem 6.2.4 we know that the following map is strictly convex,

$$B_\nu(.) = \int_{\partial\widetilde{M}} B(., \theta) d\nu(\theta)$$

where ν is a weighted sum of Patterson-Sullivan measures and $B(.,.)$ is the Busemann function, see Definition 2.4.1. Now using this map we define the Barycentric straightening, see Definition 6.2.6. By the Theorem 5.3.1, Theorem 6.3.1 and Theorem 6.4.1 we see that the Barycentric straightening satisfies all the conditions of the straightening defined in the Definition 4.0.1. Now the main theorem follows from the Theorem 4.0.2 by Thurston. □

Appendix

Theorem 6.5.2 (Implicit Function Theorem)**.** *Let M be an n-dimensional smooth manifold and $f : M \times V \to \mathbb{R}^n$ be a continuously differentiable function where $V \subset \mathbb{R}^n$ is an open subset and let $(x_1, ..., x_n, y_1, ..., y_n)$ be a local coordinate on some open subset $U \times V \subset M \times V$. Fix a point $(a, b) \in M \times V$ with $f(a, b) = 0$. If the Jacobian matrix $J_{f,y}(a, b) = \left[\frac{\partial f_i}{\partial y_j}(a, b)\right]$ is invertible then there exist an open subset $\widetilde{U} \subset U$ containing a such that there exist a unique continuously differentiable function $g : \widetilde{U} \to V$ such that $g(a) = b$, and $f(x, g(x)) = 0$ for all $x \in \widetilde{U}$.*

Definition 6.5.3 (Cusped manifold)**.** *A non-compact complete hyperbolic manifold with finite volume is called cusped manifold.*

Remark 6.5.4. *The name "cusp" comes from the fact that these manifolds have the following structure, let M be such a manifold, then M retracts to a compact submanifold M' which has a boundary consisting of flat manifolds $T_1, ..., T_k$, and the rest of the manifold consist of so-called cusps, which are warped products $T_i \times [1, \infty)$ with the metric $\frac{1}{t^2}(dx^2 + dt^2)$ where dx^2 is the flat metric on $T_i's$.*

Definition 6.5.5 (convex function)**.** *Let M be a Riemannian manifold and $f \in C^\infty(M, \mathbb{R})$ be a smooth map. We say f is convex if its restriction to any geodesic is a convex function.*

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