Moments of Cubic Hecke L-Functions

Arihant Jain

A Thesis
in
The Department
of
Mathematics and Statistics

Presented in the Partial Fulfillment of the Requirements
for the Degree of Master of Science (Mathematics) at
Concordia University
Montreal, Quebec, Canada

August, 2021

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared
By : Arihant Jain
Entitled : Moments of Cubic Hecke L-Functions
and submitted in partial fulfillment of the requirements for the degree of

Master of Science (Mathematics and Statistics)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee

________________________ Chair
Prof. Giovanni Rosso

________________________ Examiner
Prof. Hershy Kisilevsky

________________________ Supervisor
Prof. Chantal David

Approved by

____
Prof. Galia Dafni, Graduate Program Director

August 17, 2021

____
Prof. Pascale Sicotte, Dean of Faculty

ABSTRACT

Moments of Cubic Hecke L-Functions

Arihant Jain

Moments of families of L-functions provide understanding of their size and also about their distribution. The aim of this thesis is to calculate the asymptotics of the first moment of L-functions associated to primitive cubic Hecke characters over $\mathbb{Q}(\omega)$ and upper bounds for $2k$-th moments for the same family. Both of these results assume Generalized Riemann Hypothesis. We consider the full family of characters which results in a main term of order $x \log x$. We also calculate conditional upper bounds for $2k$-th moments for the same family and conclude that there are $\gg x$ primitive characters of conductor at most x for which the L-function doesn't vanish at the central point.

Acknowledgements

First and foremost I would like to express my deep gratitude to my supervisor Professor Chantal David. None of this would have been possible without her valuable suggestions and helpful discussions along with constant support and encouragement throughout the program. I also thank the faculty at the Department of Mathematics for the knowledge and insights that I have gained during the last two years. I extend my thanks to the Number Theory group in Montreal. Thanks are also due to my friends who have been a source of great moral support and comfort, especially during difficult times. My family has been a pillar of strength for me and I will always be grateful for their love and affection. Finally, I thank Concordia University for the financial support during my stay in Montréal.

Contents

Chapter 1

Introduction

Finding moments of Riemann zeta-function and families of L-function is an old and interesting problem in number theory. There are several applications of moments to the theory of Riemann zeta-functions and L-functions. Using Random Matrix Theory [CFK$^+$05] gave a recipe to conjecture moments for a very large variety of L-functions. Their conjectures are very far from proven completely. For the L-functions associated to quadratic characters first four moments can be computed ([Jut81], [Sou00], [SY10]). In contrast, only the first moment was calculated for the family of cubic characters. In this article we present a result for cubic characters. Let ψ be a Hecke character and the associated Hecke L-function is

$$L(s, \psi) = \sum_{\mathcal{A} \neq 0} \frac{\psi(\mathcal{A})}{N(\mathcal{A})^s}$$

where the sum runs over non-zero integral ideals of $\mathbb{Z}[\omega]$. Our result is the following theorem.

Theorem 1.1. *Let ψ be a primitive cubic Hecke character such that $\psi \in \mathcal{F}$* (see (2.1) for definition) *and $L(s, \psi)$ be a Hecke L-function. Assuming GRH we have*

$$\sum_{\psi \in \mathcal{F}(x)} L\left(\tfrac{1}{2}, \psi\right) = C_1 x \log x + C_2 x + O_\epsilon(x^{\frac{23}{24}+\epsilon})$$

where C_1, C_2 are absolute constants described in (3.16).

This theorem is similar to results of [Luo04] and [BY10]. However [Luo04] considered a thin subfamily of the primitive cubic characters over $\mathbb{Q}(\omega)$ and [BY10] considered L-series over rationals, which also reduces the size of family of characters. Both of these results are unconditional. The difference in our case is that we consider full family of primitive Hecke characters. Due to the increase in size of family, we need to assume GRH to bound error terms. Over function field, the first moment in the Kummer case was computed by [DFL19]. Their result also requires GRH which is not a hypothesis in function field (see Theorem 1.2 of [DFL19]). The exponent in their error term is $\frac{1+\sqrt{7}}{4} = 0.9114378\ldots$ which is smaller than $\frac{23}{24} = 0.958333\ldots$. One reason for their better error term is that they computed the residue in dual term and saved on both sums instead of just saving over one sum.

We also prove the upper bounds for moments of absolute values of L-functions. The methods are based on Harper's proof [Har13] which is a refinement of the work of Soundararajan [Sou09]. In proving the upper bounds, we have also followed the work of [LR19] and [DFL20].

Theorem 1.2. *Let $\psi \in \mathcal{F}$, then assuming GRH*

$$\sum_{\psi \in \mathcal{F}(x)} |L(1/2, \psi)|^{2k} \ll_k (x \log x)(\log x)^{k^2}$$

Upper bounds for Dirichlet L-functions associated with cubic characters are also obtained in Theorem 1.3 of [GZ21] for Dirichlet characters over $\mathbb{Q}$ and then their bound is $x(\log x)^{k^2}$. In [DFL20], the authors have calculated upper bounds for all mollified moments of L-functions associated to cubic characters in the function field setting and further obtained non-vanishing results in the non-Kummer case. Using Theorem 1.1 and Theorem 1.2, along with Cauchy Schwarz inequality, we get the following result.

Corollary 1.3. *Assuming GRH we have*

$$\#\{\psi : \psi \in \mathcal{F}(x) \ \textit{and} \ \ L(1/2, \psi) \neq 0\} \gg x.$$

Since the family of characters is of size $x \log x$ (up to a constant), this is not a positive proportion.

1.1 Outline of the proof

In chapter 2 we discuss the family of primitive cubic Hecke characters. We also state the approximate functional equation for L-functions which is a sum of two quantities : principal term and dual term.

In chapter 3 we estimate the principal term. The main term is $C_1 x \log x$ and for the error term we assume GRH which is where we differ from [Luo04] and [BY10] as stated before. At the end of this chapter we briefly state the difficulty encountered while trying to get rid of GRH assumption. Further in chapter 4 we estimate the dual term. Here we get cancellation and therefore this contributes to the error in Theorem 1.1. The proofs here rely on results of [Pat77], [HBP79], and ideas from [BY10].

Proof of Theorem 1.2 appears in chapter 5. We use an L-function inequality due to [Cha09] and an important lemma (Lemma 5.5). Using the work of [Sou09] and [DFL20] we establish slightly weaker upper bounds which are then used in §5.4 to yield the required sharp upper bounds. In obtaining sharp bounds we follow [Har13], [LR19] and [DFL20].

Chapter 2

Preliminaries

This chapter is divided in three sections. We start by describing Hecke characters and the Hecke L-functions, approximate functional equation and some results on cubic Gauss sums. All of the content presented in this chapter is standard and well-known. We also include Proposition 2.1 which is a result on smooth character sums and will be useful in obtaining upper bounds.

2.1 Primitive Cubic Hecke Characters

All of what we present here is mentioned in [IK04] (section 3.8), [IR90] (Chapter 9) and [BY10] (section 2). We first start by describing the cubic residue symbol. For a prime $\pi \in \mathbb{Z}[\omega]$ (ω is a cube root of unity other than 1) and $\pi \nmid 3$ we define the cubic residue symbol

$$\left(\frac{\alpha}{\pi}\right)_3 \equiv \alpha^{\frac{N(\pi)-1}{3}}(mod\ \pi).$$

This is well defined since $N(\pi) \equiv 1(mod\ 3)$. Also, by Proposition 9.3.2 of [IR90], for any α and prime $\pi \nmid 3\alpha$ we have $\alpha^{\frac{N(\pi)-1}{3}} \equiv \omega^m(\pi)$ for $m = 0, 1$, or 2. Therefore the residue symbol is a third root of unity. We extend the definition to all $\alpha \in \mathbb{Z}[\omega]$ using periodicity and by assigning 0 whenever $\pi|\alpha$. Note that the conjugate character $\overline{\left(\frac{\cdot}{\pi}\right)_3}$ is a different character than $\left(\frac{\cdot}{\pi}\right)_3$ and both are are primitive with norm of the conductor equal to $N(\pi)$. For $c \in \mathbb{Z}[\omega]$, square free, $\left(\frac{\cdot}{c}\right)_3$ and $\overline{\left(\frac{\cdot}{c}\right)_3}$ are primitive characters defined as

$$\left(\frac{\cdot}{c}\right)_3 := \prod_{\pi|c}\left(\frac{\cdot}{\pi}\right)_3 \qquad and \qquad \overline{\left(\frac{\cdot}{c}\right)_3} := \prod_{\pi|c}\overline{\left(\frac{\cdot}{\pi}\right)_3}$$

Now we move to the discussion of Hecke characters. To see a complete detailed discussion, we refer the reader to section 3.8 of [IK04]. Let $\mathfrak{m} = (m)$ be a non-zero integral ideal of $\mathbb{Z}[\omega]$ such that $m \equiv 1(mod\ 3)$. A Hecke character $\psi(mod\ \mathfrak{m})$ is a homomorphism on the group of ideals coprime to $\mathfrak{m}$ for which there exist two characters $\widetilde{\psi} : (\mathbb{Z}[\omega]/(m))^* \to \mathbb{C}^*$ and $\psi_\infty : \mathbb{C}^* \to \mathbb{C}^*$, satisfying

$$\psi((a)) = \widetilde{\psi}(a)\psi_\infty(a) \qquad and \qquad |\widetilde{\psi}(a)| = |\psi_\infty(a)| = 1$$

for every $a \in \mathbb{Z}[\omega]$, $(a, m) = 1$. When we consider the case of cubic Hecke characters of $\mathbb{Z}[\omega]$, the group homomorphism $\widetilde{\psi}$ is a cubic residue symbol. To simplify calculations we choose ψ_∞ to be a trivial character which forces $\widetilde{\psi}$ to be trivial on units since for a unit $u \in \mathbb{Z}[\omega]$, $\psi((u)) = 1 = \widetilde{\psi}(u)\psi_\infty(u)$. To achieve this we choose $c_1, c_2 \equiv 1(mod\ 9)$ and thus we get $N(c_1), N(c_2) \equiv 1(mod\ 9)$ and such cubic residue character is trivial on units (see the remarks below Theorem 1 in Chapter 9 of [IR90]). So we will consider the following family of primitive cubic Hecke characters

$$\mathcal{F} := \left\{ \psi_{c_1c_2}(\cdot) := \left(\frac{\cdot}{c_1}\right)_3 \overline{\left(\frac{\cdot}{c_2}\right)_3} : c_1, c_2 \equiv 1(mod\ 9); c_1, c_2 \text{ square free}; (c_1, c_2) = 1 \right\} \tag{2.1}$$

$$\mathcal{F}(x) := \{\psi_{c_1c_2} \in \mathcal{F} : N(c_1c_2) \leq x\} \tag{2.2}$$

In [Luo04], the author considered a thin subfamily by taking $c_2 = 1$. Further, the definition of cubic residue symbol can be generalized to any modulus by multiplicativity. For $a = \pi_1^{\alpha_1}\pi_2^{\alpha_2} \cdots \pi_k^{\alpha_k}$, we define

$$\left(\frac{\cdot}{a}\right)_3 := \left(\frac{\cdot}{\pi_1}\right)_3^{\alpha_1} \left(\frac{\cdot}{\pi_2}\right)_3^{\alpha_2} \cdots \left(\frac{\cdot}{\pi_k}\right)_3^{\alpha_k}.$$

and for a primitive Hecke cubic character ψ,

$$L(s, \psi) = \sum_{\substack{\mathcal{A}:\text{Integral ideal of } \mathbb{Z}[\omega] \\ \mathcal{A} \neq 0}} \frac{\psi(\mathcal{A})}{N(\mathcal{A})^s}$$

is the Hecke L-function associated to ψ. In [BY10], the authors considered Dirichlet L-function and consequently the size of family is reduced. For $\Re(s) > 1$, the Hecke L-functions have an Euler product. Let us use p to denote rational primes and $\mathfrak{p}$ to denote prime ideals of $\mathbb{Z}[\omega]$ lying over p.

$$L(s, \psi) = \prod_{\mathfrak{p}} \left(1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} = \prod_{p} \left(1 - \frac{\alpha_1(p, \psi)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_2(p, \psi)}{p^s}\right)^{-1} \tag{2.3}$$

where

$$\alpha_1(p, \psi) = \begin{cases} \psi(\pi) & p \equiv 1(mod\ 3) \\ \overline{\psi}(p) = \psi(p^2) & p \equiv 2(mod\ 3) \\ \psi(1-\omega) & p = 3 \end{cases} \tag{2.4}$$

$$\alpha_2(p, \psi) = \begin{cases} \psi(\overline{\pi}) & p \equiv 1(mod\ 3) \\ -\overline{\psi}(p) = -\psi(p^2) & p \equiv 2(mod\ 3) \\ 0 & p = 3 \end{cases} \tag{2.5}$$

and $\pi, \overline{\pi}$ are prime ideals lying over p when $p \equiv 1(mod\ 3)$.

Cubic Reciprocity. For primes $a, b \in \mathbb{Z}[\omega]$ and $a \equiv b \equiv \pm 1 (mod\ 3)$, then we have the following relation

$$\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3.$$

Now we prove a proposition on character sums which is analogous to Remark 1 of [LR20] and is based on Poisson Summation Formula.

Proposition 2.1. *Let χ be a Dirichlet character on $\mathbb{Z}[\omega]$ of modulus $m \in \mathbb{Z}[\omega]$ and F be a Schwartz function whose Fourier transform has compact support contained in $(-A, A)$. Then for $X > A^2 N(m)$ we have*

$$\sum_{c \in \mathbb{Z}[\omega]} \chi(c) F\left(\frac{N(c)}{X}\right) = \begin{cases} \dfrac{X}{N(m)} \widehat{F}(0)\phi(m) & \chi \text{ is trivial} \\ 0 & \chi \text{ is non-trivial} \end{cases}$$

Proof. Using Poisson Summation Formula (see the proof of Lemma 10 in [HB00a]) we have

$$\sum_{c \in \mathbb{Z}[\omega]} \chi(c) F\left(\frac{N(c)}{X}\right) = \frac{X}{N(m)} \sum_{k \in \mathbb{Z}[\omega]} \widehat{F}\left(\sqrt{\frac{N(k)X}{N(m)}}\right) \sum_{r (mod\ m)} \chi(r) e(-kr/m\sqrt{-3})$$

where the terms corresponding to $k \neq 0$ vanish since $N(k) \geq 1 \Rightarrow \frac{N(k)X}{N(m)} \geq A^2$. If χ is non-trivial then $\sum_{r(mod\ m)} \chi(r) = 0$ by orthogonality otherwise $\sum_{r(mod\ m)} \chi(r) = \phi(m)$ and we get $X\widehat{F}(0)\phi(m)/N(m)$. □

2.2 Approximate Functional Equation

We state the approximate functional equation based on [IK04]. For any $\sigma > 1/2$ we have

$$\begin{aligned} L(1/2, \psi_{c_1c_2}) &= \frac{1}{2\pi i} \sum_{\mathcal{A} \neq 0} \frac{\psi_{c_1c_2}(\mathcal{A})}{\sqrt{N(\mathcal{A})}} \int_{(\sigma)} \left(\frac{y}{N(\mathcal{A})}\right)^u \frac{\Gamma(1/2+u)}{\Gamma(1/2)} \frac{du}{u} \\ &+ w(\psi_{c_1c_2}) \frac{1}{2\pi i} \sum_{\mathcal{A} \neq 0} \frac{\overline{\psi_{c_1c_2}}(\mathcal{A})}{\sqrt{N(\mathcal{A})}} \int_{(\sigma)} \left(\frac{3}{4\pi^2} \frac{N(c_1c_2)}{yN(\mathcal{A})}\right)^u \frac{\Gamma(1/2+u)}{\Gamma(1/2)} \frac{du}{u}. \end{aligned} \tag{2.6}$$

where $\mathcal{A}$ denotes the integral ideal of $\mathbb{Z}[\omega]$ and $w(\psi_{c_1c_2})$ is the normalized Gauss sum given by

$$\begin{aligned} w(\psi_{c_1c_2}) &= \frac{1}{\sqrt{N(c_1c_2)}} \sum_{a (mod\ c_1c_2)} \psi_{c_1c_2}(a) e\left(Tr\left(\frac{a}{(1-\omega)\ c_1c_2}\right)\right) \\ &= \frac{1}{\sqrt{N(c_1c_2)}} \psi_{c_1c_2}(1-\omega) \sum_{a (mod\ c_1c_2)} \psi_{c_1c_2}(a) e\left(Tr\left(\frac{a}{c_1c_2}\right)\right) \\ &= \frac{1}{\sqrt{N(c_1c_2)}} \sum_{a (mod\ c_1c_2)} \psi_{c_1c_2}(a) e\left(Tr\left(\frac{a}{c_1c_2}\right)\right) \end{aligned}$$

where we have used that $\psi_{c_1c_2}(1-\omega) = 1$ for $c_1, c_2 \equiv 1(mod\ 9)$ (Remark(c) below Theorem 1 in Chapter 9 of [IR90]). The first sum is generally referred to as the principal term and second as dual term. We follow the same terminology in this article. Let us define the integrals appearing in (2.6) as

$$V(X) := \frac{1}{2\pi i}\int_{(\sigma)} X^{-u}\ \frac{\Gamma(1/2+u)}{\Gamma(1/2)}\frac{du}{u}.$$

Using Proposition 5.4 of [IK04] we have

$$V(N(\mathcal{A})/y) = \begin{cases} 1 + O((N(\mathcal{A})/y)^{\alpha}) & N(\mathcal{A}) \le y \quad and \quad 0 < \alpha < 1/2 \\ O((y/N(\mathcal{A}))^{A}) & N(\mathcal{A}) > y \quad and \quad A > 0 \end{cases}. \tag{2.7}$$

For finding moments we need to calculate sum over all characters. We evaluate the principal term and the dual term in §3 and §4 respectively.

2.3 Results Related to Gauss Sums

Lemma 2.2. *For c_1c_2 square free and $(c_1, c_2) = 1$ we have*

$$\sum_{d(mod\ c_1c_2)} \left(\frac{a^2d}{c_1}\right)_3 \overline{\left(\frac{a^2d}{c_2}\right)_3}\, e\left(Tr\left(\frac{d}{c_1c_2}\right)\right)$$
$$= \left(\sum_{d_1(mod\ c_1)} \left(\frac{a^2d_1}{c_1}\right)_3 e\left(Tr\left(\frac{d_1}{c_1}\right)\right)\right)\left(\sum_{d_2(mod\ c_2)} \overline{\left(\frac{a^2d_2}{c_2}\right)_3}\, e\left(Tr\left(\frac{d_2}{c_2}\right)\right)\right).$$

Proof. Multiplying the right hand side

$$\sum_{\substack{d_1(mod\ c_1)\\ d_2(mod\ c_2)}} \left(\frac{a^2d_1}{c_1}\right)_3 \overline{\left(\frac{a^2d_2}{c_2}\right)_3}\, e\left(Tr\left(\frac{c_2d_1 + c_1d_2}{c_1c_2}\right)\right). \tag{2.8}$$

Note that for the terms inside the summation to be non-zero, both d_1 and d_2 should be coprime to c_1 and c_2 respectively. Let us consider the following map

$$(\mathbb{Z}[\omega]/c_1)^* \times (\mathbb{Z}[\omega]/c_2)^* \to (\mathbb{Z}[\omega]/(c_1c_2))^*$$
$$(\alpha, \beta) \mapsto (c_2\alpha + c_1\beta).$$

It is not difficult to see that this map is one-one and onto. So the summation over d_1 and d_2 can be rewritten in terms of $d(mod\ c_1c_2)$ and the d_1, d_2 appearing in the terms can be replaced using the the inverse of the map described above.

$$a^2d_1 \equiv a^2(dc_2^{-1})(mod\ c_1) \quad and \quad a^2d_2 \equiv a^2(dc_1^{-1})(mod\ c_2)$$

Also for c_1 and c_2 we have

$$\left(\frac{c_2^{-1}}{c_1}\right)_3 = \overline{\left(\frac{c_2}{c_1}\right)_3}, \qquad \left(\frac{c_1^{-1}}{c_2}\right)_3 = \overline{\left(\frac{c_1}{c_2}\right)_3}$$

Therefore

$$\left(\frac{a^2 d_1}{c_1}\right)_3 = \left(\frac{a^2 d c_2^{-1}}{c_1}\right)_3 = \overline{\left(\frac{c_2}{c_1}\right)_3}\left(\frac{a^2 d}{c_1}\right)_3 \qquad \left(\frac{a^2 d_2}{c_2}\right)_3 = \left(\frac{a^2 d c_1^{-1}}{c_2}\right)_3 = \overline{\left(\frac{c_1}{c_2}\right)_3}\left(\frac{a^2 d}{c_2}\right)_3$$

Using cubic reciprocity we rewrite (2.8) as

$$\begin{aligned} &\sum_{d(mod\ c_1c_2)} \overline{\left(\frac{c_2}{c_1}\right)_3}\left(\frac{a^2 d}{c_1}\right)_3 \left(\frac{c_2}{c_1}\right)_3 \overline{\left(\frac{a^2 d}{c_2}\right)_3} e\left(Tr\left(\frac{d}{c_1c_2}\right)\right) \\ = &\sum_{d(mod\ c_1c_2)} \left(\frac{a^2 d}{c_1}\right)_3 \overline{\left(\frac{a^2 d}{c_2}\right)_3} e\left(Tr\left(\frac{d}{c_1c_2}\right)\right). \end{aligned}$$

□

We define the following notation

$$g(c) := \sum_{d(mod\ c)} \left(\frac{d}{c}\right)_3 e\left(Tr\left(\frac{d}{c}\right)\right)$$

and its conjugate is

$$\overline{g(c)} = \sum_{d\ (mod\ c)} \overline{\left(\frac{d}{c}\right)_3} e\left(Tr\left(\frac{-d}{c}\right)\right) = \sum_{d\ (mod\ c)} \overline{\left(\frac{d}{c}\right)_3} e\left(Tr\left(\frac{d}{c}\right)\right)$$

where we replaced d by $-d$ and used $\left(\frac{-1}{c}\right)_3 = 1$. Using these definitions we can write the result of the Lemma 2.2 as

$$\frac{1}{N(c_1c_2)^{-u}} w(\psi_{c_1c_2})\overline{\psi_{c_1c_2}}(a) = \overline{\left(\frac{a}{c_1}\right)_3} \frac{g(c_1)}{N(c_1)^{\frac{1}{2}-u}} \left(\frac{a}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{\frac{1}{2}-u}}. \tag{2.9}$$

We also collect three important properties (stated in Proof of Theorem 2 and in beginning of Section 4 of [HBP79]) related to $g(\cdot)$ and its generalizations, which will be useful later. The proofs are application of the definitions, cubic reciprocity and properties of cubic characters.

1. $\overline{g(c)} = g(\overline{c})$
2. $g(r\pi, \pi^2) = N(\pi)\overline{g(r,\pi)}$ (given $(r,\pi) = 1$) where $g(r,c)$ is defined as
$$g(r,c) = \sum_{d(mod\ c)} \left(\frac{d}{c}\right)_3 e\left(Tr\left(\frac{rd}{c}\right)\right). \tag{2.10}$$
3. For $(r,c) = 1$ we have
$$g(r,c) = \overline{\left(\frac{r}{c}\right)_3} \sum_{d(mod\ c)} \left(\frac{d}{c}\right)_3 e\left(Tr\left(\frac{d}{c}\right)\right) \tag{2.11}$$

Chapter 3

Principal Term

In this chapter we prove the following proposition.

Proposition 3.1. *Let x, y be positive real numbers such that $y < x$. Then for any $\epsilon > 0$ we have the following result*

$$\sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1 (mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2) \leq x}} \sum_{\mathcal{A} \neq 0} \frac{\psi_{c_1c_2}(\mathcal{A})}{N(\mathcal{A})^{1/2}} V\left(\frac{N(\mathcal{A})}{y}\right) = C_1 x \log x + C_2 x + O_\epsilon\left(\frac{x \log x}{y^{\frac{1}{6}-\epsilon}}\right) + O_\epsilon(x^{\frac{1}{2}+\epsilon} y^{\frac{1}{2}+\epsilon})$$

where C_1, C_2 are described in (3.16).

The main term of $x \log x$ comes from a double pole and for the error term $O_\epsilon(x^{\frac{1}{2}+\epsilon} y^{\frac{1}{2}+\epsilon})$ we use GRH. We complete the proof in first four sections. At the end of this chapter we briefly mention the difficulty while trying to remove the GRH assumption.

3.1 Setting up the Double Sum

The principal term in (2.6) is

$$\sum_{\mathcal{A} \neq 0} \frac{\psi_{c_1,c_2}(\mathcal{A})}{\sqrt{N(\mathcal{A})}} V\left(\frac{N(\mathcal{A})}{y}\right)$$

where $\mathcal{A}$ runs over all non-zero ideals in $\mathbb{Z}[\omega]$. The family of characters is parameterized by c_1, c_2 and we work towards evaluating the following sum

$$S_{princ} := \sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1 (mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2) \leq x}} \sum_{\mathcal{A} \neq 0} \frac{\psi_{c_1c_2}(\mathcal{A})}{N(\mathcal{A})^{1/2}} V\left(\frac{N(\mathcal{A})}{y}\right).$$

The remaining of this section consists of proving this proposition. For every non-zero ideal $\mathcal{A}$, we have $\mathcal{A} = (1-\omega)^r(a)$ where $(a, 1-\omega) = 1, r \geq 0$ and we write

$$S_{princ} = \sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1(mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \sum_{\substack{r\geq 0 \\ a\equiv 1(mod \ 3)}} \frac{\psi_{c_1c_2}(a) V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}} N(a)^{\frac{1}{2}}} = \sum_{\substack{r\geq 0 \\ a\equiv 1(mod \ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}} N(a)^{\frac{1}{2}}} \Bigg(\sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1(mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \psi_{c_1c_2}(a) \Bigg). \tag{3.1}$$

We rewrite the sum over c_1, c_2 in (3.1) as

$$\begin{aligned}
&\sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1(mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3} \\
&= \sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1(mod \ 9) \\ N(c_1c_2)\leq x}} \sum_{\substack{h|(c_1,c_2) \\ h\equiv 1(mod \ 3)}} \mu(h) \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3} \\
&= \sum_{\substack{h\equiv 1(mod \ 3) \\ N(h)\leq \sqrt{x}}} \mu(h) \sum_{\substack{c_1,c_2 \ sq \ free \\ (h,c_1c_2)=1 \\ c_1,c_2\equiv h^{-1}(mod \ 9) \\ N(c_1c_2)\leq \frac{x}{N(h)^2}}} \left(\frac{hc_1}{a}\right)_3 \overline{\left(\frac{hc_2}{a}\right)_3} \\
&= \sum_{\substack{h\equiv 1(mod \ 3) \\ N(h)\leq \sqrt{x} \\ (a,h)=1}} \mu(h) \sum_{\substack{c_1,c_2 \ sq \ free \\ (h,c_1c_2)=1 \\ c_1,c_2\equiv h^{-1}(mod \ 9) \\ N(c_1c_2)\leq \frac{x}{N(h)^2}}} \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3} \\
&= \sum_{\chi_1,\chi_2 (mod \ 9)} \sum_{\substack{h\equiv 1(mod \ 3) \\ N(h)\leq \sqrt{x} \\ (a,h)=1}} \mu(h) \frac{\chi_1(h)\chi_2(h)}{\# h^2_{(9)}} \sum_{\substack{c_1,c_2 \ sq \ free \\ (h,c_1c_2)=1 \\ c_1,c_2\equiv 1(mod \ 3) \\ N(c_1c_2)\leq \frac{x}{N(h)^2}}} \chi_1(c_1)\chi_2(c_2) \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3} \qquad (3.2)
\end{aligned}$$

where χ_1, χ_2 are ray class characters mod 9. From here we consider two cases depending whether a is a cube or not.

3.2 CASE I : a is a cube

This case provides the main term. In this case, we encounter poles depending on the following cases

$$\begin{cases} \textit{double pole} & \chi_1, \chi_2 \textit{ are trivial} \\ \textit{single pole} & \textit{exactly one of } \chi_1, \chi_2 \textit{ is trivial} \\ \textit{no pole} & \textit{if both } \chi_1 \textit{ and } \chi_2 \textit{ are non-trivial} \end{cases}$$

Both ray class characters are trivial : Double pole. Using Perron's formula (see chapter 7 of [Kou19]), the sum over c_1, c_2 in (3.2) is equal to

$$\sum_{\substack{c_1,c_2\ sq\ free\\(ah,c_1c_2)=1\\c_1,c_2\equiv 1(mod\ 3)\\N(c_1c_2)\leq\frac{x}{N(h)^2}}} 1 = \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-2} \frac{x^s}{N(h)^{2s}}\frac{ds}{s} + O\left(\frac{x\log x}{N(h)^2T}\right)$$

Using Cauchy's residue theorem the main term will come from the residue at $s = 1$. We postpone the calculation of the residue to §3.4 and bound horizontal and vertical integrals here, using the convexity bounds. For the vertical integral, this gives for any $\epsilon > 0$

$$\int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} \frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-2} \frac{x^s}{N(h)^{2s}}\frac{ds}{s} \ll_\epsilon T^{1+\epsilon}\frac{x^{\frac{1}{2}+\epsilon}}{N(h)^{1+\epsilon}}N(a)^\epsilon$$

and for the horizontal integral

$$\int_{\frac{1}{2}+\epsilon+iT}^{1+\frac{1}{\log x}+iT} \frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-2} \frac{x^s}{N(h)^{2s}}\frac{ds}{s} \ll_\epsilon \frac{x}{N(h)^{1+\epsilon}T}N(a)^\epsilon$$

Taking $T = x^{\frac{1}{4}}$ we get

$$\sum_{\substack{c_1,c_2\ sq\ free\\(ah,c_1c_2)=1\\c_1,c_2\equiv 1(mod\ 3)\\N(c_1c_2)\leq\frac{x}{N(h)^2}}} 1 = \underset{s=1}{res}\left(\frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-2} \frac{x^s}{N(h)^{2s}}\frac{1}{s}\right) + O_\epsilon\left(\frac{x^{\frac{3}{4}+\epsilon}}{N(h)^{1+\epsilon}}N(a)^\epsilon\right).$$

Exactly one of the ray class characters is trivial : Single Pole. Let us assume that χ_2 is trivial so the sum over c_1, c_2 in (3.2) is

$$\sum_{\substack{c_1,c_2\ sq\ free\\(ah,c_1c_2)=1\\c_1,c_2\equiv 1(mod\ 3)\\N(c_1c_2)\leq\frac{x}{N(h)^2}}} \chi_1(c_1)$$

$$= \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{\zeta_{\mathbb{Z}[\omega]}(s)}{\zeta_{\mathbb{Z}[\omega]}(2s)}\frac{L(s,\chi_1)}{L(2s,\chi_1^2)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-1} \prod_{\pi|ah}\left(1+\frac{\chi_1((\pi))}{N(\pi)^s}\right)^{-1} \frac{x^s}{N(h)^{2s}}\frac{ds}{s}$$

$$+ O\left(\frac{x\log x}{N(h)^2T}\right)$$

$$= \underset{s=1}{res}\left(\frac{\zeta_{\mathbb{Z}[\omega]}(s)}{\zeta_{\mathbb{Z}[\omega]}(2s)}\frac{L(s,\chi_1)}{L(2s,\chi_1^2)} \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)^s}\right)^{-1} \left(1+\frac{\chi_1((\pi))}{N(\pi)^s}\right)^{-1} \frac{x^s}{N(h)^{2s}s}\right)$$

$$+ O_\epsilon\left(\frac{x^{\frac{3}{4}+\epsilon}}{N(h)^{1+\epsilon}}N(a)^\epsilon\right)$$

where, as before, we chose $T = x^{\frac{1}{4}}$.

None of the ray class characters are trivial : No Pole. For χ_1, χ_2 non-trivial, the sum in (3.2) is

$$\sum_{\substack{c_1,c_2 \ sq \ free \\ (ah, c_1c_2)=1 \\ c_1,c_2 \equiv 1 (mod \ 3) \\ N(c_1c_2) \le \frac{x}{N(h)^2}}} \chi_1(c_1)\chi_2(c_2)$$

$$= \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{L(s,\chi_1)}{L(2s,\chi_1^2)} \frac{L(s,\chi_2)}{L(2s,\chi_2^2)} \prod_{\pi | ah} \left(1 + \frac{\chi_1((\pi))}{N(\pi)^s}\right)^{-1} \prod_{\pi | ah} \left(1 + \frac{\chi_2((\pi))}{N(\pi)^s}\right)^{-1} \frac{x^s}{N(h)^{2s}} \frac{ds}{s}$$
$$+ O\left(\frac{x \log x}{N(h)^2 T}\right)$$
$$= O_\epsilon\left(\frac{x^{\frac{3}{4}+\epsilon}}{N(h)^{1+\epsilon}} N(a)^\epsilon\right)$$

where $T = x^{\frac{1}{4}}$. We recall that our aim is to evaluate S_{princ} and in this section we have partially dealt with the case when a is a cube. Using (3.1), (3.2) and the discussion in this section, we get

$$\sum_{\substack{r \ge 0 \\ a \equiv 1 (mod \ 3) \\ a = cube}} \frac{V(\frac{3^r N(a)}{y})}{3^{r/2} N(a)^{1/2}} \left(\sum_{\substack{c_1,c_2 \ sq \ free \\ c_1,c_2 \equiv 1 (mod \ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2) \le x}} \psi_{c_1c_2}(a) \right)$$

$$= \sum_{\substack{r \ge 0 \\ a \equiv 1 (mod \ 3) \\ a = cube}} \frac{V(\frac{3^r N(a)}{y})}{3^{r/2} N(a)^{1/2}} \sum_{\chi_1,\chi_2 (mod \ 9)} \sum_{\substack{h \equiv 1 (mod \ 3) \\ N(h) \le \sqrt{x} \\ (a,h)=1}} \mu(h) \frac{\chi_1(h)\chi_2(h)}{\# h_{(9)}^2} \sum_{\substack{c_1,c_2 \ sq \ free \\ (h, c_1c_2)=1 \\ c_1,c_2 \equiv 1 (mod \ 3) \\ N(c_1c_2) \le \frac{x}{N(h)^2}}} \chi_1(c_1)\chi_2(c_2) \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3}$$

$$= \sum_{\substack{r \ge 0 \\ a \equiv 1 (mod \ 3) \\ a = cube}} \frac{V\left(\frac{3^r N(a)}{y}\right)}{3^{\frac{r}{2}} \sqrt{N(a)}} \sum_{\chi_1,\chi_2 (mod \ 9)} \sum_{\substack{h \equiv 1 (mod \ 3) \\ N(h) \le \sqrt{x} \\ (a,h)=1}} \mu(h) \frac{\chi_1(h)\chi_2(h)}{\# h_{(9)}^2} \left(res(\chi_1,\chi_2) + O_\epsilon\left(\frac{x^{\frac{3}{4}+\epsilon} N(a)^\epsilon}{N(h)^{1+\epsilon}}\right)\right)$$

$$= \left[\sum_{\substack{r \ge 0 \\ a \equiv 1 (mod \ 3) \\ a = cube}} \frac{V\left(\frac{3^r N(a)}{y}\right)}{3^{\frac{r}{2}} \sqrt{N(a)}} \sum_{\chi_1,\chi_2 (mod \ 9)} \sum_{\substack{h \equiv 1 (mod \ 3) \\ N(h) \le \sqrt{x} \\ (a,h)=1}} \mu(h) \frac{\chi_1(h)\chi_2(h)}{\# h_{(9)}^2} res(\chi_1,\chi_2) \right] + O_\epsilon\left(x^{\frac{3}{4}}\right)$$

where $res(\chi_1, \chi_2)$ is the residue depending on whether χ_1, χ_2 are trivial as discussed in this section. As stated before we will calculate the residues in §3.4 and thus completely evaluate this case of $a = cube$.

3.3 CASE II : a is not a cube

If a is neither a cube nor a unit, then for $c \equiv 1 (mod\ 3)$, $\chi_a : (c) \to \left(\frac{a}{c}\right)_3$ is a Hecke character of modulus $9a$. The Hecke L-function associated to this character is

$$L(s, \chi_a) = \sum_{\mathcal{A} \neq 0} \frac{\chi_a(\mathcal{A})}{N(\mathcal{A})^s} = \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c \equiv 1 (mod\ 3)}} \left(\frac{a}{c}\right)_3 N(c)^{-s}.$$

Therefore for the case when a is not a cube we have

$$\sum_{\substack{c_1, c_2\ sq\ free \\ (h, c_1 c_2) = 1 \\ c_1, c_2 \equiv 1 (mod\ 3) \\ N(c_1 c_2) \leq \frac{x}{N(h)^2}}} \chi_1(c_1) \chi_2(c_2) \left(\frac{c_1}{a}\right)_3 \overline{\left(\frac{c_2}{a}\right)_3}$$

$$= \frac{1}{2\pi i} \int_{1 + \frac{1}{\log x} - iT}^{1 + \frac{1}{\log x} + iT} \frac{L(s, \chi_1 \chi_a)}{L(2s, \chi_1^2 \chi_a^2)} \frac{L(s, \chi_2 \overline{\chi_a})}{L(2s, \chi_2^2 \overline{\chi_a^2}))}$$
$$\times \prod_{\substack{\pi | h \\ \pi \equiv 1 (mod\ 3)}} \left(1 + \frac{\chi_1 \chi_a((\pi))}{N(\pi)^s}\right)^{-1} \left(1 + \frac{\chi_2 \overline{\chi_a}((\pi))}{N(\pi)^s}\right)^{-1} \frac{x^s}{N(h)^{2s}} \frac{ds}{s} + O\left(\frac{x \log x}{N(h)^2 T}\right) \tag{3.3}$$

The character $\chi_1 \chi_a$ is not necessarily primitive so we first establish that $L(s, \chi_1 \chi_a)$ is entire in $\Re(s) > 0$ using arguments similar to those from Chapter 5 of [Dav80]. Let $\chi_{1,a}$, $\chi_{2,a}$ be the primitive characters that induce $\chi_1 \chi_a$ and $\chi_2 \overline{\chi_a}$, respectively. Then we have

$$L(s, \chi_1 \chi_a) = L(s, \chi_{1,a}) \prod_{\substack{\pi | 9a \\ \pi \equiv 1 (mod\ 3)}} \left(1 - \frac{\chi_{1,a}((\pi))}{N(\pi)^s}\right)$$
$$L(s, \chi_2 \overline{\chi_a}) = L(s, \chi_{2,a}) \prod_{\substack{\pi | 9a \\ \pi \equiv 1 (mod\ 3)}} \left(1 - \frac{\chi_{2,a}((\pi))}{N(\pi)^s}\right).$$

The functions $L(s, \chi_{1,a})$, $L(s, \chi_{2,a})$ and the products over primes dividing $9a$ are entire in the region $\Re(s) > 0$ and thus the same is true for $L(s, \chi_1 \chi_a)$ and $L(s, \chi_2 \overline{\chi_a})$. We define the following notation for the Euler products appearing above

$$F(m, s, \chi, \widetilde{\chi}) := \prod_{\substack{\pi | m \\ \pi \equiv 1 (mod\ 3)}} \left(1 - \frac{\chi((\pi))}{N(\pi)^s}\right) \left(1 - \frac{\widetilde{\chi}((\pi))}{N(\pi)^s}\right). \tag{3.4}$$

Using Cauchy's Theorem, the integral in (3.3) is

$$= \frac{1}{2\pi i}\Bigg(- \int_{1/2+\epsilon+iT}^{1+\frac{1}{\log x}+iT} + \int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} + \int_{1/2+\epsilon-iT}^{1+\frac{1}{\log x}-iT} \Bigg)$$
$$\frac{L(s,\chi_{1,a})L(s,\chi_{2,a})}{L(2s,\chi_1^2\chi_a^2)L(2s,\chi_2^2\overline{\chi_a^2})}\ F(9a,s,\chi_{1,a},\chi_{2,a})\frac{F(h,s,\chi_1\chi_a,\chi_2\overline{\chi_a})}{F(h,2s,\chi_1^2\chi_a^2,\chi_2^2\overline{\chi_a^2})}\frac{x^s}{N(h)^{2s}}\ ds$$

and for the function $F(\cdot,\cdot,\cdot)$ we have

$$F(m,\tfrac{1}{2}+\epsilon,\chi,\widetilde{\chi}) = O_\epsilon(N(m)^\epsilon) \qquad \textit{and} \qquad \frac{1}{F(m,1+\epsilon,\chi,\widetilde{\chi})} = O_\epsilon(1)$$

We are assuming GRH which implies the Lindelöf Hypothesis, so we have the bound

$$L\left(\tfrac{1}{2}+\epsilon+it,\chi_{1,a}\right) \ll_\epsilon (N(a)|t|^2)^\epsilon$$

and similarly for $L(\frac{1}{2}+\epsilon+it,\chi_{2,a})$. Thus we have for the vertical integral

$$\int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} \frac{L(s,\chi_{1,a})L(s,\chi_{2,a})}{L(2s,\chi_1^2\chi_a^2)L(2s,\chi_2^2\overline{\chi_a^2})}\ F(9a,s,\chi_{1,a},\chi_{2,a})\frac{F(h,s,\chi_1\chi_a,\chi_2\overline{\chi_a})}{F(h,2s,\chi_1^2\chi_a^2,\chi_2^2\overline{\chi_a^2})}\frac{x^s}{N(h)^{2s}}\frac{ds}{s}$$

$$\ll_\epsilon \frac{x^{\frac{1}{2}+\epsilon}N(a)^\epsilon}{N(h)^{1+\epsilon}}\int_{-T}^{T}\frac{1}{|t|}(|t|)^\epsilon dt \ll \frac{2x^{\frac{1}{2}+\epsilon}N(a)^\epsilon}{N(h)^{1+\epsilon}}|T|^\epsilon$$

and for the horizontal integral

$$\int_{\frac{1}{2}+\epsilon+iT}^{1+\frac{1}{\log x}+iT} \frac{L(s,\chi_{1,a})L(s,\chi_{2,a})}{L(2s,\chi_1^2\chi_a^2)L(2s,\chi_2^2\overline{\chi_a^2})}F(9a,s,\chi_{1,a},\chi_{2,a})\frac{F(h,s,\chi_1\chi_a,\chi_2\overline{\chi_a})}{F(h,2s,\chi_1^2\chi_a^2,\chi_2^2\overline{\chi_a^2})}\frac{x^s}{N(h)^{2s}}\frac{ds}{s}$$

$$\ll_\epsilon\ N(a)^\epsilon T^\epsilon N(h)^\epsilon \frac{1}{T}\int_{\frac{1}{2}+\epsilon}^{1+\frac{1}{\log x}} \frac{x^\sigma}{N(h)^{2\sigma}}\ d\sigma$$
$$\ll_\epsilon\ N(a)^\epsilon T^\epsilon N(h)^\epsilon \frac{1}{T}\frac{1}{N(h)^{1+2\epsilon}}\int_{\frac{1}{2}+\epsilon}^{1+\frac{1}{\log x}} x^\sigma d\sigma\ \ \ll_\epsilon\ \frac{xN(a)^\epsilon}{N(h)^{1+\epsilon}}T^{-1+\epsilon}.$$

Taking $T=\sqrt{x}$, the contribution from horizontal and vertical integral is $O_\epsilon\left(\frac{N(a)^\epsilon}{N(h)^{1+\epsilon}}x^{\frac{1}{2}+\epsilon}\right)$. We now sum over a and h to get the complete error term

$$\sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}} \sum_{\chi_1,\chi_2(mod\ 9)} \sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ (a,h)=1\\ N(h)\leq\sqrt{x}}} \mu(h)\frac{\chi_1(h)\chi_2(h)}{\#h_{(9)}^2}\left(\frac{N(a)^\epsilon}{N(h)^{1+\epsilon}}x^{\frac{1}{2}+\epsilon}\right) \tag{3.5}$$

The sum over h is bounded by a constant depending on ϵ, thus

$$(3.5) \ll_\epsilon \sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}} \sum_{\chi_1,\chi_2(mod\ 9)} x^{\frac{1}{2}+\epsilon} N(a)^\epsilon \ll \quad x^{\frac{1}{2}+\epsilon} \sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}} N(a)^{\frac{1}{2}-\epsilon}}.$$

Thus we need to estimate

$$\sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}} N(a)^{\frac{1}{2}-\epsilon}} = \Bigg(\sum_{r\leq \log y} \sum_{\substack{a\equiv 1(mod\ 3)\\ N(a)\leq y/3^r}} + \sum_{r\leq \log y} \sum_{\substack{a\equiv 1(mod\ 3)\\ N(a)\geq y/3^r}} + \sum_{r>\log y} \sum_{a\equiv 1(mod\ 3)} \Bigg) \frac{1}{3^{\frac{r}{2}}} \frac{V(\frac{3^r N(a)}{y})}{N(a)^{\frac{1}{2}-\epsilon}}$$

where $\log y$ is the logarithm of y to base 3. Using (2.7) for $3^r N(a) \leq y$ and taking $\alpha = 1/3$ we have

$$\sum_{r\leq \log y} \sum_{\substack{a\equiv 1(mod\ 3)\\ N(a)\leq y/3^r}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}} N(a)^{\frac{1}{2}-\epsilon}} = \sum_{r\leq \log y} O\left(\left(\frac{y}{3^r}\right)^{\frac{1}{2}+\epsilon} + \frac{3^{r\alpha} y^{\frac{1}{2}+\alpha}}{3^{r(\alpha+\frac{1}{2})} y^\alpha} \right) = O_\epsilon(y^{\frac{1}{2}+\epsilon}).$$

For the range in the second and third sum , $3^r N(a) > y$, we again use (2.7)

$$\sum_{r\leq \log y} \sum_{\substack{a\equiv 1(mod\ 3)\\ N(a)\geq y/3^r}} \frac{V(\frac{3^r N(a)}{y})}{N(a)^{\frac{1}{2}-\epsilon}} + \sum_{r>\log y} \sum_{a\equiv 1(mod\ 3)} \frac{V(\frac{3^r N(a)}{y})}{N(a)^{\frac{1}{2}-\epsilon}} = O_\epsilon(y^{\frac{1}{2}+\epsilon})$$

Therefore the error term for the case when a is a cube is $O_\epsilon(x^{\frac{1}{2}+\epsilon} y^{\frac{1}{2}+\epsilon})$.

3.4 Calculation of Residue

As promised before, we compute the residue in this section. Let us first do the case when both χ_1, χ_2 are trivial. In this case we encounter a double pole (at $s = 1$) of the following function

$$\frac{x^s}{N(h)^{2s}} \frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \prod_{\pi|9ah} \left(1 + \frac{1}{N(\pi)^s}\right)^{-2} \frac{1}{s}. \tag{3.6}$$

Let us define

$$R_1 := \underset{s=1}{res} \frac{\zeta_{\mathbb{Z}[\omega]}(s)}{\zeta_{\mathbb{Z}[\omega]}(2s)}\ , \qquad R_2 := \underset{s=1}{res} \frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)} \qquad and \qquad f(m) := \prod_{\pi|m} \left(1 + \frac{1}{N(\pi)^s}\right)^{-1}. \tag{3.7}$$

So the residue of (3.6) at $s=1$ is equal to

$$\frac{x}{N(h)^2}\log\left(\frac{x}{N(h)^2}\right)\ R_1^2\ \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)}\right)^{-2}+\ \frac{x}{N(h)^2}\ \underset{s=1}{res}\frac{\zeta^2_{\mathbb{Z}[\omega]}(s)}{\zeta^2_{\mathbb{Z}[\omega]}(2s)}\ \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)}\right)^{-2}$$
$$-\frac{x}{N(h)^2}\ R_1^2\ \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)}\right)^{-2}\left[\sum_{\pi|9ah}\frac{2\log N(\pi)}{1+N(\pi)}\right]-\ \frac{x}{N(h)^2}\ R_1^2\ \prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)}\right)^{-2}$$
$$=x\log\ x\ \frac{f(9ah)^2}{N(h)^2}R_1^2+x\ \frac{f(9ah)^2}{N(h)^2}\left[-R_1^2\log N(h)^2+R_2-R_1^2\ \sum_{\pi|9ah}\frac{2\log N(\pi)}{1+N(\pi)}-R_1^2\right]. \tag{3.8}$$

Therefore the term we have to evaluate is the following

$$x\log x\ \frac{R_1^2\ f(9)^2}{\#h^2_{(9)}}\ \sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})}{\sqrt{N(a)}}\ f(a)^2\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\leq\sqrt{x}\\ (a,h)=1}}\frac{\mu(h)f(h)^2}{N(h)^2}$$
$$+\ x\frac{f(9)^2}{\#h^2_{(9)}}\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\leq\sqrt{x}\\ (a,h)=1}}\frac{\mu(h)f(h)^2}{N(h)^2}\Bigg[$$
$$-R_1^2\log N(h)^2+R_2+R_1^2\sum_{\pi|9ah}\frac{2\log N(\pi)}{1+N(\pi)}-R_1^2\Bigg]. \tag{3.9}$$

We simplify the above expression in two parts : one for the sum with $x\ \log\ x$ and another with x. Let us define

$$H_1(a):=\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ (a,h)=1}}\frac{\mu(h)f(h)^2}{N(h)^2}. \tag{3.10}$$

then the term with $x\log x$ in (3.9) is

$$x\ \ \log x\ \frac{R_1^2\ f(9)^2}{\#h^2_{(9)}}\ \sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})}{\sqrt{N(a)}}\ f(a)^2\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\leq\sqrt{x}\\ (a,h)=1}}\frac{\mu(h)f(h)^2}{N(h)^2}$$
$$=x\log\ x\left(\frac{R_1^2f(9)^2}{\#h^2_{(9)}}\ \sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}f(a)^2H_1(a)\right)+O\left(x^{\frac{1}{2}+\epsilon}\log x\right). \tag{3.11}$$

We break the main term in (3.11) as

$$\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\ H_1(a) = \Bigg(\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)\\ 3^r N(a)\leq y}} + \sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)\\ 3^r N(a)>y}} \Bigg) \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\ H_1(a). \quad (3.12)$$

Using (2.7) we have

1. For $3^r N(a)\leq y$

$$\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)\\ 3^r N(a)\leq y}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\ H_1(a) = \sum_{r\leq \log_3 y} \sum_{\substack{a=cube\\ a\equiv 1(mod\ 3)\\ 3^r N(a)\leq y}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\ H_1(a)$$

$$= \sum_{r\leq \log_3 y} \frac{1}{3^{\frac{r}{2}}} \sum_{\substack{a=cube\\ a\equiv 1(mod\ 3)}} \frac{f(a)^2\ H_1(a)}{\sqrt{N(a)}} + O(y^{-\frac{1}{6}}).$$

2. For $3^r N(a) > y$

$$\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)\\ 3^r N(a)>y}} \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\ f(a)^2\ H_1(a) = \Bigg(\sum_{\substack{r\leq \log\ y\\ a=cube\\ a\equiv 1(mod\ 3)\\ N(a)>y/3^r}} + \sum_{\substack{r>\log\ y\\ a=cube\\ a\equiv 1(mod\ 3)}} \Bigg) \frac{V(\frac{3^r N(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}} f(a)^2\ H_1(a)$$

$$\ll_\epsilon \sum_{r\leq \log\ y} \frac{1}{3^{\frac{r}{3}}} \frac{1}{y^{\frac{1}{6}-\epsilon}} + \sum_{r\leq \log\ y} \frac{1}{3^{\frac{r}{2}}} \ll y^{-\frac{1}{6}+\epsilon}$$

Therefore (3.11) is equal to

$$x \log\ x \left(\frac{\sqrt{3}}{\sqrt{3}-1}\ \frac{R_1^2\ f(9)^2}{\# h_{(9)}^2} \sum_{\substack{a=cube\\ a\equiv 1(mod\ 3)}} \frac{f(a)^2\ H_1(a)}{\sqrt{N(a)}} \right) + O_\epsilon\left(\frac{x\ \log\ x}{y^{\frac{1}{6}-\epsilon}} \right). \quad (3.13)$$

Now we evaluate the residue term with x in (3.9). Let us define

$$H_2(a) := \sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ (a,h)=1}} \frac{\mu(h) f(h)^2}{N(h)^2} \log N(h)^2 \qquad H_3(a) := \sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ (a,h)=1}} \frac{\mu(h) f(h)^2}{N(h)^2} \sum_{\pi | 9ah} \frac{2\log N(\pi)}{1+N(\pi)}$$

Then the residue term with x is

$$x\left(\frac{f(9)^2}{\#h_{(9)}^2}\right)\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})}{3^{\frac{r}{2}}\sqrt{N(a)}}\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ (a,h)=1\\ N(h)\leq\sqrt{x}}}\frac{\mu(h)f(ah)^2}{N(h)^2}\Bigg[$$

$$-R_1^2\log N(h)^2+R_2+R_1^2\sum_{\pi|9ah}\frac{2\log N(\pi)}{1+N(\pi)}-R_1^2\Bigg]$$

$$=x\left(\frac{f(9)^2}{\#h_{(9)}^2}\right)\sum_{\substack{r\geq 0\\ a=cube\\ a\equiv 1(mod\ 3)}}\frac{V(\frac{3^rN(a)}{y})f(a)^2}{3^{\frac{r}{2}}\sqrt{N(a)}}\big((R_2-R_1^2)H_1(a)-R_1^2H_2(a)-R_1^2H_3(a)\big)+O(x^{\frac{1}{2}+2\epsilon})$$

We already saw the procedure to evaluate the sum over r and a in the case of $x\log x$, so we write the results directly.

$$x\left(\frac{\sqrt{3}}{\sqrt{3}-1}\frac{f(9)^2}{\#h_{(9)}^2}\right)\Bigg[\sum_{\substack{a=cube\\ a\equiv 1(mod\ 3)}}\frac{f(a)^2}{\sqrt{N(a)}}((R_2-R_1^2)H_1(a)-R_1^2H_2(a)-R_1^2H_3(a))\Bigg]+O_\epsilon\left(\frac{x}{y^{\frac{1}{6}-\epsilon}}\right).$$

Now we move to the case when exactly one of the ray class character is trivial. Let us assume that χ_1 is trivial. In this case we encounter a single pole of the following function

$$\frac{x^s}{N(h)^{2s}}\frac{\zeta_{\mathbb{Z}[\omega]}(s)}{\zeta_{\mathbb{Z}[\omega]}(2s)}\prod_{\pi|9ah}\left(1+\frac{1}{N(\pi)}\right)^{-1}\frac{L(s,\chi_2)}{L(2s,\chi_2^2)}\prod_{\pi|ah}\left(1+\frac{\chi_2(\pi)}{N(\pi)}\right)^{-1}\frac{1}{s}$$

where χ_2 is a non-trivial ray class character modulo 9. Let

$$f_\chi(m):=\prod_{\pi|ah}\left(1+\frac{\chi(\pi)}{N(\pi)}\right)^{-1}\tag{3.14}$$

and the residue is equal to

$$\frac{x}{N(h)^2}R_1f(9ah)\frac{L(1,\chi_2)}{L(2,\chi_2^2)}f_{\chi_2}(ah).$$

We need to sum for all h,a and r and also over all non-trivial χ_2 to get a complete expression for residue. Let us define

$$H(a,\chi):=\sum_{\substack{h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\leq\sqrt{x}\\ (a,h)=1}}\frac{\mu(h)f(h)f_\chi(h)}{N(h)^2}$$

and we get that the complete expression for residue is

$$R_1 x \frac{f(9)}{\# h_{(9)}^2} \sum_{\substack{\chi (mod\ 9) \\ \chi \neq trivial}} \frac{L(1,\chi)}{L(2,\chi^2)} \sum_{\substack{r \geq 0 \\ a = cube}} \frac{V\left(\frac{3^r N(a)}{y}\right)}{3^{\frac{r}{2}} \sqrt{N(a)}} f(a) f_\chi(a) \sum_{\substack{h \equiv 1 (mod\ 3) \\ h\ sq\ free \\ N(h) \leq \sqrt{x} \\ (a,h)=1}} \frac{\mu(h) f(h) f_\chi(h)}{N(h)^2}$$

$$= R_1 x \frac{f(9)}{\# h_{(9)}^2} \sum_{\substack{\chi (mod\ 9) \\ \chi \neq trivial}} \frac{L(1,\chi)}{L(2,\chi^2)} \sum_{\substack{r \geq 0 \\ a = cube}} \frac{V\left(\frac{3^r N(a)}{y}\right)}{3^{\frac{r}{2}} \sqrt{N(a)}} f(a) f_\chi(a) H(a,\chi) + O(x^{\frac{1}{2}+\epsilon})$$

$$= x \ \frac{\sqrt{3}}{\sqrt{3}-1} \ \frac{R_1 \ f(9)}{\# h_{(9)}^2} \sum_{\substack{\chi (mod\ 9) \\ \chi \neq trivial}} \frac{L(1,\chi)}{L(2,\chi^2)} \sum_{\substack{a = cube \\ a \equiv 1 (mod\ 3)}} \frac{f(a) f_\chi(a) H(a,\chi)}{\sqrt{N(a)}} + O_\epsilon\left(\frac{x}{y^{\frac{1}{6}-\epsilon}}\right).$$

Finally we conclude the proof by combining all the cases and the error terms obtained earlier.

$$\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2 \equiv 1 (mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2) \leq x}} \sum_{\mathcal{A} \neq 0} \frac{\psi_{c_1c_2}(\mathcal{A})}{N(\mathcal{A})^{1/2}} V\left(\frac{N(\mathcal{A})}{y}\right) = C_1 \ x \log \ x \ + C_2 x + O_\epsilon\left(\frac{x \log \ x}{y^{\frac{1}{6}-\epsilon}}\right) + O_\epsilon(x^{\frac{1}{2}+\epsilon} \ y^{\frac{1}{2}+\epsilon}). \tag{3.15}$$

where C_1, C_2 are as follows

$$C_1 := \frac{\sqrt{3}}{\sqrt{3}-1} \ \frac{R_1^2 \ f(9)^2}{\# h_{(9)}^2} \Bigg(\sum_{\substack{a \equiv 1 (mod\ 3) \\ a = cube}} \frac{f(a)^2 \ H_1(a)}{\sqrt{N(a)}} \Bigg)$$

$$C_2 := \frac{\sqrt{3}}{\sqrt{3}-1} \ \frac{R_1 f(9)^2}{\# h_{(9)}^2} \Bigg(\sum_{\substack{a \equiv 1 (mod\ 3) \\ a = cube}} \frac{f(a)^2}{\sqrt{N(a)}} \Bigg[\frac{2R}{f(9)} \sum_{\substack{\chi (mod\ 9) \\ \chi \neq trivial}} \frac{L(1,\chi)}{L(2,\chi^2)} f(a) f_\chi(a) H(a,\chi) + (R_2 - R_1^2) H_1(a) + R_1^2 H_2(a) - R_1^2 H_3(a) \Bigg] \Bigg). \tag{3.16}$$

where $f(a)$ and $f_\chi(a)$ are defined by (3.7) and (3.14), respectively.

Remark 3.2. *We assumed GRH to bound the product of L-functions $L(s,\chi_1\chi_a)L(s,\chi_2\overline{\chi_a})$ in terms of powers of $N(a)$ at $\Re(s) = 1/2+\epsilon$. In [BY10] and [Luo04] the GRH was replaced by cubic large sieve which is equivalent to GRH on average, and they computed the first moment without any hypothesis. We try to use such a sieve, but the double sum over c_1, c_2 seems to cause trouble, and we did not succeed in getting an unconditional result. The large sieve for cubic character is proved in [HB00b] under the following form. Let c_n be an arbitrary* *sequence of complex numbers, where n runs over $\mathbb{Z}[\omega]$. Then*

$$\sum_{N(m)\leq M}^{*}\left|\sum_{N(n)\leq N}^{*}\left(\frac{n}{m}\right)_3 c_n\right|^2 \ll_\epsilon (N+M+(MN)^{2/3})(QM)^\epsilon \sum_n^{*} |c_n|^2$$

for any $\epsilon > 0$, where $\sum^*$ denotes that the sum is over square free elements of $\mathbb{Z}[\omega]$.

Chapter 4

Dual Term

In this chapter we prove the following proposition which gives an upper bound for the dual term.

Proposition 4.1. *Let σ be a real number such that $\sigma > 1/2$. Then, for any $\epsilon > 0$, we have the following estimate of dual term*

$$\sum_{\mathcal{A}\neq 0} \frac{1}{\sqrt{N(\mathcal{A})}} \int_{(\sigma)} \left(\frac{3}{4\pi^2\ yN(\mathcal{A})}\right)^u \left[\frac{\Gamma(1/2+u)}{\Gamma(1/2)} \left(\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \frac{1}{N(c_1c_2)^{-u}}\ \omega(\psi_{c_1c_2})\overline{\psi_{c_1c_2}}(\mathcal{A})\right)\right] \frac{du}{u} \ll_\epsilon x^{\frac{11}{12}} \left(\frac{x}{y}\right)^{\frac{1}{2}+\epsilon}.$$

In the first section we remove coprimality conditions on c_1, c_2 and then deduce this proposition assuming some bounds which we prove in §4.3. In the last section we will deduce Theorem 1.1 using Proposition 3.1 and Proposition 4.1.

4.1 Removing the Interdependence of c_1 and c_2

We recall from (2.6) that the dual term is the sum

$$\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} w(\psi_{c_1c_2}) \sum_{\mathcal{A}\neq 0} \frac{\overline{\psi_{c_1c_2}}(\mathcal{A})}{\sqrt{N(\mathcal{A})}} \int_{(\sigma)} \left(\frac{3}{4\pi^2}\ \frac{N(c_1c_2)}{yN(\mathcal{A})}\right)^u \frac{\Gamma(1/2+u)}{\Gamma(1/2)} \frac{du}{u}$$

$$= \sum_{\mathcal{A}\neq 0} \frac{1}{\sqrt{N(\mathcal{A})}} \int_{(\sigma)} \left(\frac{3}{4\pi^2\ yN(\mathcal{A})}\right)^u \frac{\Gamma(1/2+u)}{\Gamma(1/2)} \left(\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} N(c_1c_2)^u\ \omega(\psi_{c_1c_2})\overline{\psi_{c_1c_2}}(\mathcal{A})\right) \frac{du}{u} \tag{4.1}$$

where

$$\overline{\psi_{c_1c_2}}(\mathcal{A})\omega(\psi_{c_1c_2}) = \overline{\psi_{c_1c_2}}(a)\omega(\psi_{c_1c_2}) = \overline{\left(\frac{a}{c_1}\right)_3} \frac{g(c_1)}{N(c_1)^{\frac{1}{2}}} \left(\frac{a}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{\frac{1}{2}}}$$

For the first equality we have used that $\mathcal{A}$ is generated by $(1-\omega)^r a$ for some $r \geq 0$ and $a \equiv 1(mod\ 3)$ and $\psi_{c_1c_2}(1-\omega) = 1$. For the second equality we have used (2.9). In this section we will prove the following proposition and thus establish an upper bound for the dual term.
Our main task is to estimate the following

$$\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \overline{\left(\frac{a}{c_1}\right)_3} \frac{g(c_1)}{N(c_1)^{\frac{1}{2}-u}} \left(\frac{a}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{\frac{1}{2}-u}}$$

but we first prove the following lemma which removes the condition $(c_1, c_2) = 1$ from this summation.

Lemma 4.2. *For $a \in \mathbb{Z}[\omega]$, $a \equiv 1(mod\ 3)$ and $u \in \mathbb{C}$, we have*

$$\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ (c_1,c_2)=1 \\ N(c_1c_2)\leq x}} \overline{\left(\frac{a}{c_1}\right)_3} \frac{g(c_1)}{N(c_1)^{\frac{1}{2}-u}} \left(\frac{a}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{\frac{1}{2}-u}}$$

$$= \sum_{\substack{(a,h)=1 \\ h\equiv 1(mod\ 3) \\ N(h)\leq\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(ah,c_1)=1 \\ c_1\equiv h^{-1}(mod\ 9) \\ N(c_1)\leq x/N(h)^2}} \frac{g(ah,c_1)}{N(c_1)^{u'}} \sum_{\substack{(\overline{a}\overline{h},c_2)=1 \\ c_2\equiv \overline{h}^{-1}(mod\ 9) \\ N(c_2)\leq x/N(c_1)N(h)^2}} \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}}$$

where $u' = \frac{1}{2} - u$ and $g(r,c)$ is defined in (2.10).

Proof. We remove the condition $(c_1, c_2) = 1$

$$\sum_{\substack{c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1(mod\ 9) \\ N(c_1c_2)\leq x}} \left(\sum_{\substack{h|(c_1,c_2) \\ h\equiv 1(mod\ 3)}} \mu(h)\right) \overline{\left(\frac{a}{c_1}\right)_3} g(c_1) \left(\frac{a}{c_2}\right)_3 \overline{g(c_2)} \frac{1}{N(c_1c_2)^{u'}}$$

$$= \sum_{\substack{h\equiv 1(mod\ 3) \\ N(h)\leq\sqrt{x}}} \mu(h) \sum_{\substack{(h,c_1c_2)=1 \\ c_1,c_2\ sq\ free \\ c_1,c_2\equiv h^{-1}(mod\ 9) \\ N(c_1c_2)\leq x/N(h)^2}} \overline{\left(\frac{a}{hc_1}\right)_3} g(hc_1) \left(\frac{a}{hc_2}\right)_3 \overline{g(hc_2)} \frac{1}{N(h)^{2u'} N(c_1c_2)^{u'}}.$$

For $(h, c) = 1$ we have (see section 4 of [HBP79]) $g(hc) = \overline{\left(\frac{h}{c}\right)_3}\, g(h)g(c)$. Replacing above, we obtain

$$= \sum_{\substack{h\equiv 1 (mod\ 3)\\ N(h)\le\sqrt{x}}} \mu(h) \sum_{\substack{(h,c_1c_2)=1\\ c_1,c_2\ sq\ free\\ c_1,c_2\equiv h^{-1}(mod\ 9)\\ N(c_1c_2)\le x/N(h)^2}} \frac{1}{N(h)^{2u'}N(c_1c_2)^{u'}}\Big[\overline{\left(\frac{a}{hc_1}\right)_3} \ \overline{\left(\frac{h}{c_1}\right)_3} g(h)g(c_1) \left(\frac{a}{hc_2}\right)_3 \left(\frac{h}{c_2}\right)_3 \overline{g(h)g(c_2)}\Big]$$

Using

$$\left(\frac{a}{hc}\right)_3\left(\frac{h}{c}\right)_3 = \left(\frac{a}{h}\right)_3\left(\frac{a}{c}\right)_3\left(\frac{h}{c}\right)_3 = \left(\frac{a}{h}\right)_3\left(\frac{ah}{c}\right)_3.$$

in the last equation we get

$$\sum_{\substack{h\equiv 1 (mod\ 3)\\ N(h)\le\sqrt{x}}} \mu(h) \sum_{\substack{(h,c_1c_2)=1\\ c_1,c_2\ sq\ free\\ c_1,c_2\equiv h^{-1}(mod\ 9)\\ N(c_1c_2)\le x/N(h)^2}} \frac{1}{N(h)^{2u'}N(c_1c_2)^{u'}}\Big[\overline{\left(\frac{a}{h}\right)_3} \overline{\left(\frac{ah}{c_1}\right)_3} g(h)g(c_1) \left(\frac{a}{h}\right)_3 \left(\frac{ah}{c_2}\right)_3 \overline{g(h)g(c_2)}\Big]$$

Since

$$\overline{\left(\frac{a}{h}\right)_3}\left(\frac{a}{h}\right)_3 = \begin{cases} 1 & (a,h)=1\\ 0 & otherwise\end{cases}$$

replacing we get

$$\sum_{\substack{(a,h)=1\\ h\equiv 1 (mod\ 3)\\ N(h)\le\sqrt{x}}} \frac{\mu(h)}{N(h)^{2u'}} \sum_{\substack{(h,c_1c_2)=1\\ c_1,c_2\ sq\ free\\ c_1,c_2\equiv h^{-1}(mod\ 9)\\ N(c_1c_2)\le x/N(h)^2}} \overline{\left(\frac{ah}{c_1}\right)_3} g(h)g(c_1) \left(\frac{ah}{c_2}\right)_3 \overline{g(h)g(c_2)} \frac{1}{N(c_1c_2)^{u'}}.$$

$$= \sum_{\substack{(a,h)=1\\ h\equiv 1 (mod\ 3)\\ N(h)\le\sqrt{x}}} \frac{\mu(h)}{N(h)^{2u'}} N(h) \sum_{\substack{(h,c_1)=1\\ c_1\ sq\ free\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\le x/N(h)^2}} \overline{\left(\frac{ah}{c_1}\right)_3}\frac{g(c_1)}{N(c_1)^{u'}} \sum_{\substack{(h,c_2)=1\\ c_2\ sq\ free\\ c_2\equiv h^{-1}(mod\ 9)\\ N(c_2)\le x/N(c_1)N(h)^2}} \left(\frac{ah}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{u'}}$$

$$= \sum_{\substack{(a,h)=1\\ h\equiv 1 (mod\ 3)\\ N(h)\le\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(h,c_1)=1\\ c_1\ sq\ free\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\le x/N(h)^2}} \overline{\left(\frac{ah}{c_1}\right)_3}\frac{g(c_1)}{N(c_1)^{u'}} \sum_{\substack{(h,c_2)=1\\ c_2\ sq\ free\\ c_2\equiv h^{-1}(mod\ 9)\\ N(c_2)\le x/N(c_1)N(h)^2}} \left(\frac{ah}{c_2}\right)_3 \frac{\overline{g(c_2)}}{N(c_2)^{u'}}.$$

where we have used $g(h)\overline{g(h)} = |g(h)|^2 = N(h)$. Since $g(c) \neq 0$ if and only if c is not square free we can drop the condition that c_1, c_2 are square free. Using $\overline{g(c)} = g(\overline{c})$ and $\overline{\left(\frac{A}{B}\right)_3} = \left(\frac{\overline{A}}{\overline{B}}\right)_3$ we have

$$= \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ N(h)\leq\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(h,c_1)=1\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\leq x/N(h)^2}} \overline{\left(\frac{ah}{c_1}\right)_3} \frac{g(c_1)}{N(c_1)^{u'}} \sum_{\substack{(h,c_2)=1\\ c_2\equiv h^{-1}(mod\ 9)\\ N(c_2)\leq x/N(c_1)N(h)^2}} \overline{\left(\frac{\overline{a}\overline{h}}{\overline{c_2}}\right)_3} \frac{g(\overline{c_2})}{N(c_2)^{u'}}$$

$$= \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ N(h)\leq\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(a,c_1)=1\\ (h,c_1)=1\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\leq x/N(h)^2}} \frac{g(ah,c_1)}{N(c_1)^{u'}} \sum_{\substack{(a,c_2)=1\\ (h,c_2)=1\\ c_2\equiv h^{-1}(mod\ 9)\\ N(c_2)\leq x/N(c_1)N(h)^2}} \frac{g(\overline{a}\overline{h},\overline{c_2})}{N(\overline{c_2})^{u'}}.$$

$$= \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ N(h)\leq\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(ah,c_1)=1\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\leq x/N(h)^2}} \frac{g(ah,c_1)}{N(c_1)^{u'}} \sum_{\substack{(\overline{a}\overline{h},c_2)=1\\ c_2\equiv \overline{h}^{-1}(mod\ 9)\\ N(c_2)\leq x/N(c_1)N(h)^2}} \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}}$$

where $g(ah, c)$ for $(a, h) = 1$, is described in (2.11) . Thus, the proof is completed. □

4.2 Proof of Proposition

Using the above lemma we removed the coprimality condition of c_1, c_2 and now we use ray class characters to get the condition of $c_1, c_2 \equiv 1(mod\ 3)$. Let χ_1, χ_2 be ray class characters mod 9 then

$$\sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ N(h)\leq\sqrt{x}}} \frac{\mu(h)}{N(h)^{-2u}} \sum_{\substack{(ah,c_1)=1\\ c_1\equiv h^{-1}(mod\ 9)\\ N(c_1)\leq x/N(h)^2}} \frac{g(ah,c_1)}{N(c_1)^{u'}} \sum_{\substack{(\overline{a}\overline{h},c_2)=1\\ c_2\equiv \overline{h}^{-1}(mod\ 9)\\ N(c_2)\leq x/N(c_1)N(h)^2}} \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}}$$

$$= \sum_{\chi_1,\chi_2(mod\ 9)} \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ N(h)\leq\sqrt{x}}} \frac{\mu(h)\chi_1(h)\chi_2(\overline{h})}{N(h)^{-2u}(\#h_{(9)})^2} \left(\sum_{\substack{(ah,c_1)=1\\ (\overline{a}\overline{h},c_2)=1\\ c_1,c_2\equiv 1(mod\ 3)\\ N(c_1c_2)\leq x/N(h)^2}} \chi_1(c_1) \frac{g(ah,c_1)}{N(c_1)^{u'}} \chi_2(c_2) \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}} \right). \tag{4.2}$$

Let us define

$$S_1 := \sum_{\substack{c_2\equiv 1(mod\ 3)\\ N(c_2)\leq\sqrt{x}/N(h)}} N(c_2)^{\Re(u)} \left| \sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\leq x/N(c_2)N(h)^2}} \chi_1(c_1) \frac{g(ah,c_1)}{N(c_1)^{u'}} \right| \tag{4.3}$$

$$S_2 := \sum_{\substack{c_1\equiv 1(mod\ 3)\\ N(c_1)\leq\sqrt{x}/N(h)}} N(c_1)^{\Re(u)} \left| \sum_{\substack{(\overline{a}\overline{h},c_2)=1\\ c_2\equiv 1(mod\ 3)\\ N(c_2)\leq x/N(c_1)N(h)^2}} \chi_2(c_2) \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}} \right| \tag{4.4}$$

$$S_3 := \Bigg| \sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\le\sqrt{x}/N(h)}} \chi_1(c_1)\frac{g(ah,c_1)}{N(c_1)^{u'}}\Bigg| \qquad S_4 := \Bigg| \sum_{\substack{(\overline{a}\overline{h},c_2)=1\\ c_2\equiv 1(mod\ 3)\\ N(c_2)\le\sqrt{x}/N(h)}} \chi_2(c_2)\frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}}\Bigg| \tag{4.5}$$

then using Dirichlet's Hyperbola method, for the sum over c_1, c_2 in (4.2), we have

$$\Bigg| \sum_{\substack{(ah,c_1)=1\\ (\overline{a}\overline{h},c_2)=1\\ c_1,c_2\equiv 1(mod\ 3)\\ N(c_1c_2)\le x/N(h)^2}} \chi_1(c_1)\ \frac{g(ah,c_1)}{N(c_1)^{u'}}\chi_2(c_2)\ \frac{g(\overline{a}\overline{h},c_2)}{N(c_2)^{u'}}\Bigg| \le\ S_1+S_2+S_3S_4. \tag{4.6}$$

We will estimate S_1, S_2 and S_3S_4 in §4.3. However we first deduce Proposition 4.1 by using the bounds for S_1, S_2 from (4.13) and bound for S_3S_4 from (4.15). So (4.2) is less than

$$\sum_{\chi_1,\chi_2(mod\ 9)} \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\le\sqrt{x}}} \frac{1}{N(h)^{-2\Re(u)}\# h^2_{(9)}}\Bigg| \sum_{\substack{(ah,c)=1\\ (\overline{a}\overline{h},c_2)=1\\ c_1,c_2\equiv 1(mod\ 3)\\ N(c_1c_2)\le x/N(h)^2}} \frac{\chi_1(c_1)\chi_2(c_2)}{N(c_1c_2)^{u'}}g(ah,c_1)g(ah,c_2)\Bigg|$$

$$\ll_\epsilon |u|x^{\Re(u)} \sum_{\substack{(a,h)=1\\ h\equiv 1(mod\ 3)\\ h\ sq\ free\\ N(h)\le\sqrt{x}}} \Bigg[\frac{x^{\frac{11}{12}}}{N(a_1)^{1/6}N(h)^2} + \frac{x^{\frac{5}{6}}N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{13}{12}}} +$$

$$|u|\frac{x^{\frac{5}{6}}}{N(a_1)^{1/3}N(h)^2} + |u|\frac{x^{\frac{3}{4}+\epsilon}N(a)^{\frac{1}{4}+\epsilon}}{N(a_1)^{1/6}N(h)^{\frac{17}{12}}} + |u|\frac{x^{\frac{2}{3}+\epsilon}N(a)^{\frac{1}{2}+2\epsilon}}{N(h)^{\frac{5}{6}}}\Bigg].$$

It is easy to see that first four terms are converging when summed over h and the term containing $N(h)^{5/6}$ will contribute a $x^{1/12}$. Thus the above expression is bounded by

$$|u|^2x^{\Re(u)}\left[x^{\frac{11}{12}} + x^{\frac{5}{6}}N(a)^{\frac{1}{4}+\epsilon} + x^{\frac{3}{4}+\epsilon}N(a)^{\frac{1}{2}+2\epsilon}\right].$$

Using these results we have

$$\sum_{\mathcal{A}\ne 0}\frac{1}{\sqrt{N(\mathcal{A})}}\int_{(\sigma)}\left(\frac{3}{4\pi^2\ yN(\mathcal{A})}\right)^u \frac{\Gamma(1/2+u)}{\Gamma(1/2)}\left(\sum_{\substack{c_1,c_2\ sq\ free\\ c_1,c_2\equiv 1(mod\ 9)\\ (c_1,c_2)=1\\ N(c_1c_2)\le x}} N(c_1c_2)^u\ \omega(\psi_{c_1c_2})\overline{\psi_{c_1c_2}}(\mathcal{A})\right)\frac{du}{u}$$

$$\ll_\epsilon \sum_{\substack{r\ge 0\\ a\equiv 1(mod\ 3)}} \frac{1}{\sqrt{3^rN(a)}}\int_{(\sigma)}\left(\frac{3}{4\pi y3^rN(a)}\right)^{\Re(u)}\left|\frac{\Gamma(1/2+u)}{\Gamma(1/2)}\right|$$

$$|u|x^{\Re(u)}[x^{\frac{11}{12}} + x^{\frac{5}{6}}N(a)^{\frac{1}{4}+\epsilon} + x^{\frac{3}{4}+\epsilon}N(a)^{\frac{1}{2}+2\epsilon}]\ du$$

$$\ll_\epsilon \sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{1}{\sqrt{3^r N(a)}}\left(\frac{x}{3^r y N(a)}\right)^{\frac{1}{2}+\epsilon} x^{\frac{11}{12}} \int_{\left(\frac{1}{2}+\epsilon\right)} |u| \left|\frac{\Gamma(1/2+u)}{\Gamma(1/2)}\right| du$$

$$+ \sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{1}{\sqrt{3^r N(a)}}\left(\frac{x}{3^r y N(a)}\right)^{\frac{3}{4}+\epsilon} x^{\frac{5}{6}} N(a)^{\frac{1}{4}+\epsilon} \int_{\left(\frac{3}{4}+\epsilon\right)} |u| \left|\frac{\Gamma(1/2+u)}{\Gamma(1/2)}\right| du$$

$$+ \sum_{\substack{r\geq 0\\ a\equiv 1(mod\ 3)}} \frac{1}{\sqrt{3^r N(a)}}\left(\frac{x}{3^r y N(a)}\right)^{1+\epsilon} x^{\frac{3}{4}} N(a)^{\frac{1}{2}+\epsilon} \int_{(1+\epsilon)} |u| \left|\frac{\Gamma(1/2+u)}{\Gamma(1/2)}\right| du$$

where we have chosen different values of σ in each of the above integrals. In each of these three terms, the sum over r, a contributes $O_\epsilon(1)$ and therefore the dual term is bounded by

$$x^{\frac{11}{12}}\left(\frac{x}{y}\right)^{\frac{1}{2}+\epsilon} + x^{\frac{5}{6}}\left(\frac{x}{y}\right)^{\frac{3}{4}+\epsilon} + x^{\frac{3}{4}}\left(\frac{x}{y}\right)^{1+\epsilon}$$

We need to choose $y > x^{5/6}$ so that all three are less than x. Thus $x^{\frac{11}{12}}(x/y)^{\frac{1}{2}+\epsilon}$ is the dominant term and the dual term is bounded by $\ll_\epsilon x^{\frac{11}{12}}(x/y)^{\frac{1}{2}+\epsilon}$.

4.3 Proofs of Estimates

In this section we establish the bounds on S_1, S_2 and $S_3 S_4$. These sums are very similar except some parameters. We will estimate these using Perron's formula (Lemma 4.5) and using the properties of Gauss sums as established in section 4 of [HBP79]. Let us define the following notation for the generating series of Gauss sums

$$G_b(r,\chi,s) := \sum_{\substack{(b,c)=1\\ c\equiv 1(mod\ 3)}} \chi(c)\frac{g(r,c)}{N(c)^s}$$

where χ is a ray class character modulo 9, $r, b \equiv 1(mod\ 3) \in \mathbb{Z}[\omega]$ and $g(r,c)$ is defined in (2.10). It is clear that we need to know about the analytic behaviour of $G_{ah}(ah,\chi,s)$, so we use the results for $G_1(ah,\chi,s)$ from the work of [Pat77], [HBP79] and [Pat87] and the following lemma which relates $G_{ah}(ah,\chi,s)$ and $G_1(ah,\chi,s)$.

Lemma 4.3. *Let us write $a = a_1 a_2^2 a_3^3$ for a_1, a_2 square free, $(a_1,a_2) = (a_2,a_3) = (a_1,a_3) = 1$ and define $a_3' := \prod_{\pi|a_3}\pi$, the product over primes dividing a_3. Let χ_1 be a ray class character then for $h \in \mathbb{Z}[\omega]$ such that h is square free, $\equiv 1(mod\ 3)$, and coprime to a we have the* *following equality relating $G_{ah}(\cdot,\chi_1,s)$ to $G_1(\cdot,\chi_1,s)$*

$$\begin{aligned}
&G_{a_1a_2^2a_3^3h}(a_1a_2^2a_3^3h,\chi_1,s)\\
&=\prod_{\pi|a_2}(1-\chi_1(\pi)^3N(\pi)^{2-3s})^{-1}\\
&\times\sum_{d|a_3'}\frac{\mu(d)\chi_1(d)g(a_1a_2^2h,d)}{N(d)^s}\Bigg(\prod_{\pi|a_1dh}(1-\chi_1(\pi)^3N(\pi)^{2-3s})^{-1}\\
&\qquad\times\sum_{e|a_1dh}\mu(e)N(e)^{1-2s}\chi_1(\pi)^2\overline{g(a_1a_2^2dh/e,e)}G_1(a_1a_2^2dh/e,\chi_1,s)\Bigg).
\end{aligned}\tag{4.7}$$

This is almost identical to Lemma 3.6 in [BY10]. There are some changes in notation so we briefly state the path to prove it without going into details. Using $g(a_1a_2^2a_3^3h,c)=g(a_1a_2^2h,c)$ we establish that $G_{a_1a_2^2a_3^3h}(a_1a_2^2a_3^3h,\chi_1,s)=G_{a_1a_2^2a_3^3h}(a_1a_2^2h,\chi_1,s)$. Also by definition of $G_{(\cdot)}(\cdot,\cdot,\cdot)$, the powers in the subscript are redundant and we can replace the quantity in subscript by product of all primes dividing $a_1a_2^2a_3^3h$ or equivalently by $a_1a_2a_3'h$. The next task is to remove a_2 and a_3' from the subscript and is described in [BY10] quite clearly (see proof of (25) and (27) in their Lemma 3.6). Finally we get rid of a_1h in the subscript using Lemma 3(i) of [HBP79].

Lemma 4.4. *Let $a=a_1a_2^2a_3^3$ with a_1,a_2,a_3,a_3' and h as defined in Lemma 4.3. The function $G_{ah}(ah,\chi,s)$ is holomorphic in the region $\Re(s)>1$ except possibly a pole at $s=4/3$. Further, for $s=\beta+it$ and $1+\epsilon\le\beta\le 3/2+\epsilon$, $|s-4/3|>1/12$,*

$$|G_{ah}(ah,\chi_1,s)|\ll_\epsilon N(ah)^{\frac{1}{2}(\frac{3}{2}+\epsilon-\beta)}(1+t^2)^{\frac{3}{2}+\epsilon-\beta}$$

and if $a_3=1$ then the residue satisfies

$$\underset{s=4/3}{res}\ G_{ah}(ah,\chi_1,s)\ll N(a_1h)^{-\frac{1}{6}+2\epsilon}N(a_3')^{\epsilon}=:G(a,h).\tag{4.8}$$

Proof. From [BY10], we have for $s=\beta+it$ and $1+\epsilon\le\beta\le\frac{3}{2}+\epsilon$, $|s-4/3|>1/12$,

$$|G_1(ah,\chi_1,s)|\ll N(ah)^{\frac{1}{2}(\beta_1-\beta)}(1+t^2)^{\beta_1-\beta}\tag{4.9}$$

and if a is cube free then the residue satisfies

$$\underset{s=4/3}{res}\,G_1(ah,\chi_1,s)\ll N(a_1h)^{-\frac{1}{6}+\epsilon}.$$

Using Lemma 4.3 for $s=\beta+it$ and $1+\epsilon\le\beta\le\frac{3}{2}+\epsilon$,

$$\begin{aligned}
|G_{ah}(ah,\chi_1,s)|\le&\left|\prod_{\pi|a_2}\left(1-\frac{\chi_1(\pi)^3}{N(\pi)^{3s-2}}\right)^{-1}\right|\\
&\times\sum_{d|a_3'}\frac{1}{N(d)^{\beta-\frac{1}{2}}}\left[\left|\prod_{\pi|a_1dh}\left(1-\frac{\chi_1(\pi)^3}{N(\pi)^{3s-2}}\right)^{-1}\right|\times\sum_{e|a_1dh}\frac{1}{N(e)^{2\beta-\frac{3}{2}}}|G_1(a_1a_2^2dh/e,\chi_1,s)|\right]
\end{aligned}$$

Using (4.9), we bound the innermost sum over divisors of a_1dh. Thus we have the following upper bound for $|G_{ah}(ah, \chi_1, s)|$

$$\begin{aligned}
&\prod_{\pi|a_2}\left(1-\frac{1}{N(\pi)^{3\beta-2}}\right)^{-1}\\
&\times \sum_{d|a_3'}\frac{1}{N(d)^{\beta-\frac{1}{2}}}\left[\prod_{\pi|a_a dh}\left(1-\frac{1}{N(\pi)^{3\beta-2}}\right)^{-1}\times\sum_{e|a_1dh}\frac{N(a_1a_2^2dh)^{\frac{1}{2}(\frac{3}{2}+\epsilon-\beta)}}{N(e)^{2\beta-\frac{3}{2}+\frac{1}{2}(\frac{3}{2}+\epsilon-\beta)}}(1+t^2)^{\frac{3}{2}+\epsilon-\beta)}\right]\\
&\ll_\epsilon (1+t^2)^{\frac{3}{2}+\epsilon-\beta}N(a_1a_2^2h)^{\frac{1}{2}(\frac{3}{2}+\epsilon-\beta)}N(a_3')^\epsilon N(a_1h)^\epsilon \le N(ah)^{\frac{1}{2}(\frac{3}{2}+\epsilon-\beta)}(1+t^2)^{\frac{3}{2}+\epsilon-\beta}.
\end{aligned}$$

Now we calculate the bounds for the residue. Again, we make use of (4.7)

$$\begin{aligned}
\underset{s=4/3}{res}\, G_{ah}(ah,\chi_1,s) \le& \prod_{\pi|a_2}(1-N(\pi)^{-2})^{-1}\\
&\times\sum_{d|a_3'}N(d)^{-5/6}\left(\prod_{\pi|a_1dh}(1-N(\pi)^{-2})^{-1}\times\sum_{e|a_1dh}N(e)^{-7/6}N(a_1dh/e)^{-1/6+\epsilon}\right)\\
&\ll_\epsilon N(a_1h)^{-1/6+2\epsilon}N(a_3')^\epsilon.
\end{aligned}$$

□

We now estimate the sums S_1, S_2 and S_3S_4 using Perron's formula. Since these steps are going to be similar for each of the four sums, we mention the general results in the form of following lemma and then specialize to different cases.

Lemma 4.5. *Let χ_1 be a ray class character modulo 9 then we have the following result*

$$\begin{aligned}
\sum_{\substack{(ah,c)=1\\ c\equiv 1(mod\ 3)\\ N(c)\le X}}\chi_1(c)\frac{g(ah,c)}{N(c)^{\frac{1}{2}}} &= \int_{1+\epsilon-iT}^{1+\epsilon+iT}G_{ah}(ah,\chi_1,\tfrac{1}{2}+s)X^s\frac{ds}{s}+O\left(\frac{X^{1+\epsilon}\log X}{T}\right)\\
&= X^{\frac{5}{6}}\underset{s=5/6}{res}\,G_{ah}(ah,\chi_1,\tfrac{1}{2}+s)+E_{Hor}(a,h,T)+E_{Ver}(a,h,T)+O\left(\frac{X^{1+\epsilon}\log X}{T}\right)
\end{aligned}$$

where

$$E_{Ver}(a,h,T) \ll N(ah)^{\frac{1}{4}+\epsilon}X^{\frac{1}{2}+\epsilon}\sqrt{T} \tag{4.10}$$

$$E_{Hor}(a,h,T) \ll N(ah)^{\frac{1}{4}+\epsilon}(1+T^2)^{1+\epsilon}\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\frac{X^\sigma}{(1+T^2)^\sigma}\frac{d\sigma}{T}. \tag{4.11}$$

The value of X for S_1, S_2 will be different from that for S_3, S_4 and accordingly our choice of T will change as well.

Proof. The statement is basically Perron's formula. The only thing non-trivial is the bound on the vertical and horizontal integrals. Let us first estimate the vertical integral using Lemma 3 (see (20)) of [HB00a]

$$\int_{\frac{1}{2}+\epsilon-iT}^{\frac{1}{2}+\epsilon+iT} G_{ah}(ah,\chi_1,\tfrac{1}{2}+s)X^s\frac{ds}{s} \ll \; N(ah)^{\frac{1}{4}+\epsilon}X^{\frac{1}{2}+\epsilon}\sqrt{T}$$

and for the horizontal integral, using result of Lemma 4.4.

$$\begin{aligned}\int_{\frac{1}{2}+\epsilon+iT}^{1+\epsilon+iT} G_{ah}(ah,\chi_1,\tfrac{1}{2}+s)X^s\frac{ds}{s} &\ll_\epsilon \; N(ah)^{\frac{1}{2}+\epsilon}(1+T^2)^{1+\epsilon}\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\frac{X^\sigma}{(\sqrt{N(ah)}(1+T^2))^\sigma}\frac{d\sigma}{T}\\ &\ll_\epsilon \; N(ah)^{\frac{1}{4}+\epsilon}(1+T^2)^{1+\epsilon}\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\frac{X^\sigma}{(1+T^2)^\sigma}\frac{d\sigma}{T}\end{aligned}$$

□

Recall (4.3),

$$S_1 = \sum_{\substack{(ah,c_2)=1\\ c_2\equiv 1(mod\ 3)\\ N(c_2)\le\sqrt{x}/N(h)}} N(c_2)^{\Re(u)}\left|\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\le x/N(c_2)N(h)^2}}\chi_1(c_1)\frac{g(ah,c_1)}{N(c_1)^{u'}}\right|$$

We apply partial summation to get

$$\begin{aligned}\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\le x/N(c_2)N(h)^2}}\chi_1(c_1)\frac{g(ah,c_1)}{N(c_1)^{u'}} &= \left(\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\le x/N(c_2)N(h)^2}}\chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\right)\left(\frac{x}{N(c_2)N(h)^2}\right)^u\\ &\quad -\int_1^{\frac{x}{N(c_2)N(h)^2}}\left(\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1(mod\ 3)\\ N(c_1)\le t}}\chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\right) ut^{u-1}\;dt\end{aligned}$$

where we have used $u' = 1/2-u$. In Lemma 4.5 we take $X = x/(N(c_2h^2))$ and $T = \left(\frac{x}{N(c_2)}\right)^{1/3}$, so in the horizontal integral we have

$$\begin{aligned}\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\frac{x^\sigma}{(N(c_2)N(h)^2(1+T^2))^\sigma}\frac{d\sigma}{T} &\le \frac{1}{(N(h)^2)^{\frac{1}{2}+\epsilon}}\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\frac{x^\sigma}{N(c_2)^\sigma(1+T^2)^\sigma}\frac{d\sigma}{T}\\ &\ll \frac{x^{1+\epsilon}}{(N(h)N(c_2)(1+T^2))^{1+\epsilon}T}\end{aligned}$$

Therefore

$$E_{Hor}(a,h,T) \ll \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{3}{4}}N(c_2)^{\frac{2}{3}+\epsilon}} x^{\frac{2}{3}+\epsilon}$$
$$E_{Ver}(a,h,T) \ll \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{3}{4}}N(c_2)^{\frac{2}{3}+\epsilon}} x^{\frac{2}{3}+\epsilon}.$$

Lastly, the error from Perron's formula is $O(x^{\frac{2}{3}+\epsilon}\log x/N(c_2)^{\frac{2}{3}+\epsilon}N(h)^{\frac{4}{3}+2\epsilon})$. Using the notation of the residue as defined in Lemma 4.4 we have the following relation

$$\Bigg(\sum_{\substack{(ah,c_1)=1 \\ c_1\equiv 1 \ (mod\ 3) \\ N(c_1)\le x/N(c_2)N(h)^2}} \chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)$$
$$= \underset{s=4/3}{res}\, G_{ah}(ah,\chi_1,s)\left(\frac{x}{N(c_2)N(h)^2}\right)^{\frac{5}{6}} + O\left(\frac{N(a)^{\frac{1}{4}+\epsilon}}{N(c_2)^{\frac{2}{3}}N(h)^{\frac{3}{4}}}x^{\frac{2}{3}+\epsilon}\log x\right).$$

Using the similar procedure we can also bound the term for a general t, for this case we take $T = t^{\frac{1}{3}}$.

$$\Bigg(\sum_{\substack{(ah,c_1)=1 \\ c_1\equiv 1 \ (mod\ 3) \\ N(c_1)\le t}} \chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg) = t^{\frac{5}{6}}\underset{s=4/3}{res}\, G_{ah}(ah,\chi_1,s) + O(N(ah)^{\frac{1}{4}+\epsilon}t^{\frac{2}{3}+\epsilon}\log t). \qquad (4.12)$$

We can always choose our ϵ so we include the terms of $\log x$ and $\log t$ in $x^{\frac{2}{3}+\epsilon}$ and $t^{\frac{2}{3}+\epsilon}$, respectively. Therefore

$$\int_1^{\frac{x}{N(c_2)N(h)^2}} \Bigg(\sum_{\substack{(ah,c_1)=1 \\ c_1\equiv 1 \ (mod\ 3) \\ N(c_1)\le t}} \chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)\frac{ut^u}{t}dt$$
$$\ll_\epsilon \int_1^{\frac{x}{N(c_2)N(h)^2}} \left(t^{\frac{5}{6}}\ G(a,h) + N(ah)^{\frac{1}{4}+\epsilon}t^{\frac{2}{3}+\epsilon}\right)\frac{|u|t^{\Re(u)}}{t}dt$$
$$\ll |u|\ G(a,h)\left(\frac{x}{N(c_2)N(h)^2}\right)^{\frac{5}{6}+\Re(u)} + |u|\ N(ah)^{\frac{1}{4}+\epsilon}\left(\frac{x}{N(c_2)N(h)^2}\right)^{\frac{2}{3}+\Re(u)+\epsilon}$$
$$= |u|\ \frac{x^{\Re(u)}}{N(c_2)^{\Re(u)}N(h)^{2\Re(u)}}\left[\frac{G(a,h)}{N(c_2)^{\frac{5}{6}}N(h)^{\frac{5}{3}}}x^{\frac{5}{6}} + \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(c_2)^{\frac{2}{3}+\epsilon}N(h)^{\frac{13}{12}}}x^{\frac{2}{3}+\epsilon}\right]$$

where $G(a,h)$ is the bound for residue of $G_{ah}(ah,\chi_1,s)$ at $s=4/3$ (see (4.8)). So

$$S_1 \ll \sum_{\substack{(ah,c_2)=1\\ c_2\equiv 1(mod\ 3)\\ N(c_2)\le\sqrt{x}/N(h)}} N(c_2)^{\Re(u)}|u|\frac{x^{\Re(u)}}{N(c_2)^{\Re(u)}N(h)^{2\Re(u)}}\left[\frac{G(a,h)}{N(c_2)^{\frac{5}{6}}N(h)^{\frac{5}{3}}}\ x^{\frac{5}{6}}+\frac{N(a)^{\frac{1}{4}+\epsilon}}{N(c_2)^{\frac{2}{3}}N(h)^{\frac{3}{4}}}\ x^{\frac{2}{3}+\epsilon}\right]$$

$$\ll \ |u|\frac{x^{\Re(u)}}{N(h)^{2\Re(u)}}\left[\left(\frac{\sqrt{x}}{N(h)}\right)^{\frac{1}{6}}\times\frac{G(a,h)}{N(h)^{\frac{5}{3}}}\ x^{\frac{5}{6}}+\left(\frac{\sqrt{x}}{N(h)}\right)^{\frac{1}{3}}\ \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{3}{4}}}\ x^{\frac{2}{3}+\epsilon}\right]$$

$$\ll_\epsilon |u|\ \frac{x^{\Re(u)}}{N(h)^{2\Re(u)}}\ \left[\frac{G(a,h)}{N(h)^{\frac{11}{6}}}\ x^{\frac{11}{12}}+\ \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{13}{12}}}\ x^{\frac{5}{6}}\right]$$

$$\ll_\epsilon |u|\ \frac{x^{\Re(u)}}{N(h)^{2\Re(u)}}\ \left[\frac{1}{N(a_1)^{\frac{1}{6}}N(h)^2}\ x^{\frac{11}{12}}+\ \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{13}{12}}}\ x^{\frac{5}{6}}\right]. \tag{4.13}$$

Analysis of S_2 is same as S_1 except the fact that the generating series we get is $G_{\overline{ah}}(\overline{ah},\chi_2,s)$ instead of $G_{ah}(ah,\chi_1,s)$ but that doesn't affect all these calculations and we get the same bound for S_2 as well.

For S_3S_4 we follow the same procedure as we did for S_1 above. Since S_3 and S_4 are similar we show the process only for S_3.

$$S_3=\Bigg(\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1\ (mod\ 3)\\ N(c_1)\le\sqrt{x}/N(h)}}\chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)\frac{(\sqrt{x})^u}{N(h)^u}-\int_1^{\frac{\sqrt{x}}{N(h)}}\Bigg(\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1\ (mod\ 3)\\ N(c_1)\le t}}\chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)\frac{ut^u}{t}du. \tag{4.14}$$

Again, we use Lemma 4.5 with $X=\sqrt{x}/N(h)$ and $T=X^{1/3}$. Using (4.10) for vertical integral

$$E_{Ver}(a,h,T)\ll N(ah)^{\frac{1}{4}+\epsilon}\frac{1}{N(h)^{\frac{2}{3}+\epsilon}}(\sqrt{x})^{\frac{2}{3}+\epsilon}=\frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{5}{12}}}(\sqrt{x})^{\frac{2}{3}+\epsilon}$$

and using (4.11) for horizontal

$$\int_{\frac{1}{2}+\epsilon}^{1+\epsilon}\left(\frac{\sqrt{x}}{N(h)(1+T^2)}\right)^{\sigma}\frac{d\sigma}{T}\ll\frac{1}{(1+T^2)^{1+\epsilon}}\frac{(\sqrt{x})^{1+\epsilon}}{N(h)^{1+\epsilon}T}$$

Hence

$$E_{Hor}(a,h,T)\ll N(ah)^{\frac{1}{4}+\epsilon}\frac{(\sqrt{x})^{1+\epsilon}}{N(h)^{1+\epsilon}}\frac{1}{T}=N(a)^{\frac{1}{4}+\epsilon}\frac{(\sqrt{x})^{1+\epsilon}}{N(h)^{\frac{3}{4}}}\frac{N(h)^{\frac{1}{3}}}{x^{\frac{1}{6}}}\quad\le\frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{5}{12}}}(\sqrt{x})^{\frac{2}{3}+\epsilon}.$$

Lastly, the error from Perron's is $O((\sqrt{x})^{\frac{2}{3}+\epsilon}/N(h)^{2/3})$. Therefore

$$\Bigg(\sum_{\substack{(ah,c_1)=1\\ c_1\equiv 1\ (mod\ 3)\\ N(c_1)\le\sqrt{x}/N(h)}}\chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)=\underset{s=4/3}{res}\,G_{ah}(ah,\chi_1,s)\left(\frac{(\sqrt{x})}{N(h)}\right)^{\frac{5}{6}}+O\left(\frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{5}{12}}}(\sqrt{x})^{\frac{2}{3}+\epsilon}\right).$$

Using (4.12) in (4.14) we get

$$\int_1^{\frac{\sqrt{x}}{N(h)}} \Bigg(\sum_{\substack{(ah,c_1)=1 \\ c_1\equiv 1 \ (mod \ 3) \\ N(c_1)\le t}} \chi_1(c_1)\frac{g(ah,c_1)}{\sqrt{N(c_1)}}\Bigg)\frac{ut^u}{t}dt$$
$$\ll |u|G(a,h)\left(\frac{\sqrt{x}}{N(h)}\right)^{\frac{5}{6}+\Re(u)} + |u|N(ah)^{\frac{1}{4}+\epsilon}\left(\frac{\sqrt{x}}{N(h)}\right)^{\frac{2}{3}+\Re(u)}.$$

Thus for S_3 we have

$$S_3 = \Bigg| \sum_{\substack{(ah,c_1)=1 \\ c_1\equiv 1 (mod \ 3) \\ N(c_1)\le\sqrt{x}/N(h)}} \chi_1(c_1)\frac{g(ah,c_1)}{N(c_1)^{u'}}\Bigg|$$
$$\ll_\epsilon \ |u|\left[\frac{G(a,h)}{N(h)^{\frac{5}{6}+\Re(u)}}(\sqrt{x})^{\frac{5}{6}+\Re(u)} + \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{5}{12}+\Re(u)}}(\sqrt{x})^{\frac{2}{3}+\Re(u)}\right].$$

Same bounds for S_4 and thus multiplying the two sums we get the following upper bound

$$\begin{aligned} & |u|^2\left[\frac{G(a,h)^2}{N(h)^{\frac{5}{3}+2\Re(u)}}x^{\frac{5}{6}+\Re(u)} + \frac{G(a,h)N(a)^{\frac{1}{4}+\epsilon}}{N(h)^{\frac{5}{4}+2\Re(u)}}x^{\frac{3}{4}+\Re(u)} + \frac{N(a)^{\frac{1}{2}+2\epsilon}}{N(h)^{\frac{5}{6}+2\Re(u)}}x^{\frac{2}{3}+\Re(u)}\right] \\ = \quad & |u|^2\frac{x^{\Re(u)}}{N(h)^{2\Re(u)}}\left[\frac{1}{N(a_1)^{\frac{1}{3}}N(h)^2}x^{\frac{5}{6}} + \frac{N(a)^{\frac{1}{4}+\epsilon}}{N(a_1)^{\frac{1}{6}}N(h)^{\frac{17}{12}}}x^{\frac{3}{4}} + \frac{N(a)^{\frac{1}{2}+2\epsilon}}{N(h)^{\frac{5}{6}}}x^{\frac{2}{3}}\right] \end{aligned} \tag{4.15}$$

4.4 Proof of Theorem 1.1

We deduce Theorem 1.1 using (2.6), Proposition 3.1 and Proposition 4.1.

$$\sum_{\substack{c_1,c_2\equiv 1(mod \ 9) \\ (c_1,c_2)=1 \\ c_1,c_2 \ sq \ free \\ N(c_1c_2)\le x}} L(1/2,\psi_{c_1c_2}) = C_1 x\log x + C_2 x + O\left(x^{\frac{1}{2}+\epsilon}y^{\frac{1}{2}+\epsilon}\right) + O\left(\frac{x}{y^{\frac{1}{6}-\epsilon}}\right)$$
$$+ O\left(x^{\frac{11}{12}}\left(\frac{x}{y}\right)^{\frac{1}{2}+\epsilon}\right)$$

The main term is clear and for the error term we choose $y = x^{\frac{11}{12}}$ which gives $(xy)^{\frac{1}{2}} = x^{\frac{23}{24}} = x^{\frac{11}{12}}(x/y)^{\frac{1}{2}}$. □

Chapter 5

Conditional Upper Bounds for Moments of L-Functions

In this chapter we prove Theorem 1.2. In §5.2 we set up notations and useful lemmas. The last ingredient is slightly weaker upper bounds of moments and we establish these bounds in §5.3. Finally in §5.4 we complete the proof using the strategy of [Har13] and useful ideas from [LR19].

5.1 L-function Inequality

In the following proposition we state Chandee's inequality ([Cha09]) written as a sum over prime ideals.

Proposition 5.1. *Assume GRH. Let ψ be a Hecke character with norm of conductor equal to C and let $L(s,\psi)$ be a Hecke L-function. Then*

$$|L(1/2,\psi)| \ll \exp\left(\sum_{N(\mathfrak{p})\leq X} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2} a'(\mathfrak{p},X) + \sum_{\mathfrak{p}|p\Rightarrow p\leq\sqrt{X}} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2} a''(\mathfrak{p},X) + \frac{\log C}{\log X}\right) \tag{5.1}$$

where

$$a'(\mathfrak{p},X) := \begin{cases} N(\mathfrak{p})^{-\frac{1}{2}-\frac{1}{\log X}}\left(1-\frac{\log N(\mathfrak{p})}{\log X}\right) & \mathfrak{p} \mid p \Rightarrow p\equiv 1 (mod\ 3) \\ 0 & \mathfrak{p} \mid p \Rightarrow p \equiv 2(mod\ 3) \end{cases} \tag{5.2}$$

$$a''(\mathfrak{p},X) := \begin{cases} \frac{1}{2}N(\mathfrak{p})^{-1-\frac{2}{\log X}}\left(1-\frac{2\log N(\mathfrak{p})}{\log X}\right) & \mathfrak{p} \mid p \Rightarrow p\equiv 1 (mod\ 3) \\ N(\mathfrak{p})^{-\frac{1}{2}-\frac{1}{\log X}}\left(1-\frac{\log N(\mathfrak{p})}{\log X}\right) & \mathfrak{p} \mid p \Rightarrow p \equiv 2(mod\ 3) \end{cases} \tag{5.3}$$

and both functions are extended to all ideals of $\mathbb{Z}[\omega]$ by defining $a'(\mathcal{I}\mathcal{J},X) = a'(\mathcal{I},X)a'(\mathcal{J},X)$ for any $\mathcal{I},\mathcal{J}$ and similarly $a''(\mathcal{I}\mathcal{J},X) = a''(\mathcal{I},X)a''(\mathcal{J},X)$.

Proof. Using Theorem 2.1 of [Cha09], taking $\lambda = 1$ and for $X \geq 10$ we have

$$|L(1/2,\psi)| \leq \exp\left(\Re \sum_{p^n \leq X} \frac{a(p^n,\psi)}{p^{n(\frac{1}{2}+\frac{1}{\log x})} n \log p} \frac{\log X/p^n}{\log X} + \frac{\log C}{\log X} + O(1)\right) \tag{5.4}$$

where

$$a(p^n,\psi) := (\alpha_1(p,\psi)^n + \alpha_2(p,\psi)^n) \log p \tag{5.5}$$

and $\alpha_1(p^n,\psi)$, $\alpha_2(p^n,\psi)$ are described in (2.4) and (2.5). Thus

$$\begin{aligned}
&\Re \sum_{p^n \leq X} \frac{a(p^n,\psi)}{p^{n(\frac{1}{2}+\frac{1}{\log X})} n \log p} \frac{\log X/p^n}{\log X} \\
&= \Re \sum_{\substack{p^n \leq X \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} \frac{(\psi(\pi)^n + \psi(\overline{\pi})^n)}{p^{n(\frac{1}{2}+\frac{1}{\log X})} n} \frac{\log X/p^n}{\log X} + \Re \sum_{\substack{p^n \leq X \\ p \equiv 2 (mod\ 3)}} \frac{(\psi(p^2)^n + (-\psi(p)^2)^n)}{p^{n(\frac{1}{2}+\frac{1}{\log X})} n} \frac{\log X/p^n}{\log X}.
\end{aligned} \tag{5.6}$$

Clearly for $n \geq 3$, the contribution is $O(1)$. In the summation over primes congruent to 2 modulo 3, terms corresponding to $n = 1$ vanish and only primes congruent to 1 modulo 3 contribute. Hence for $n = 1$ we get

$$\begin{aligned}
&\sum_{\substack{p \leq X \\ p \equiv 1 (mod\ 3)}} \frac{\Re(\psi(\pi) + \psi(\overline{\pi}))}{p^{\frac{1}{2}+\frac{1}{\log x}}} \times \left(1 - \frac{\log p}{\log X}\right) \\
&= \sum_{\substack{N(\mathfrak{p}) \leq X \\ \mathfrak{p}|p \Rightarrow p \equiv 1 (mod\ 3)}} \frac{\Re\psi(\mathfrak{p})}{N(\mathfrak{p})^{\frac{1}{2}+\frac{1}{\log X}}} \left(1 - \frac{\log N(\mathfrak{p})}{\log X}\right) = \sum_{N(\mathfrak{p}) \leq x} \frac{\psi(\mathfrak{p}) + \overline{\psi}(\mathfrak{p})}{2} a'(\mathfrak{p}, X).
\end{aligned}$$

Now for $n = 2$ we get

$$\sum_{\substack{p \leq \sqrt{X} \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} \frac{\Re\ (\psi(\pi)^2 + \psi(\overline{\pi})^2)}{2p} \frac{1}{p^{\frac{2}{\log X}}} \left(1 - \frac{2\log p}{\log X}\right) + \sum_{\substack{p \leq \sqrt{X} \\ p \equiv 2 (mod\ 3)}} \frac{\Re\ \psi(p)}{p} \frac{1}{p^{\frac{2}{\log X}}} \left(1 - \frac{2\log p}{\log X}\right) \tag{5.7}$$

$$= \sum_{\substack{N(\mathfrak{p}) \leq \sqrt{X} \\ \mathfrak{p}|p \Rightarrow p \equiv 1 (mod\ 3)}} \frac{\Re\psi(\mathfrak{p})}{2N(\mathfrak{p})^{1+\frac{2}{\log X}}} \left(1 - \frac{2\log N(\mathfrak{p})}{\log X}\right) + \sum_{\substack{N(\mathfrak{p}) \leq X \\ \mathfrak{p}|p \Rightarrow p \equiv 2 (mod\ 3)}} \frac{\Re\psi(\mathfrak{p})}{N(\mathfrak{p})^{\frac{1}{2}+\frac{1}{\log X}}} \left(1 - \frac{\log N(\mathfrak{p})}{\log X}\right) \tag{5.8}$$

$$= \sum_{\mathfrak{p}|p \Rightarrow p \leq \sqrt{X}} \frac{\psi(\mathfrak{p}) + \overline{\psi(\mathfrak{p})}}{2} a''(\mathfrak{p}, X). \tag{5.9}$$

□

The following lemma shows that the sum over prime squares (5.7) can be shortened significantly. This lemma is based on Lemma 2.1 of [Sou09] and [Har13].

Lemma 5.2. *Let ψ be a Hecke character of conductor less than or equal to x and $y \in \mathbb{R}$ be such that $\log x < y < x$ then*

$$\left(\sum_{\substack{p \le y \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} - \sum_{\substack{p \le \log x \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} \right) \frac{\psi(\pi)^2 + \psi(\overline{\pi})^2}{2p} \frac{1}{p^{\frac{1}{\log y}}} \left(1 - \frac{\log p}{\log y}\right)$$
$$+ \left(\sum_{\substack{p \le y \\ p \equiv 2 (mod\ 3)}} - \sum_{\substack{p \le \log x \\ p \equiv 2 (mod\ 3)}} \right) \frac{\psi((p))}{p} \frac{1}{p^{\frac{1}{\log y}}} \left(1 - \frac{\log p}{\log y}\right) = O(1).$$

Proof. The argument is based on explicit formula proof. By following the argument in [Dav80] (Chapter 19,20) and assuming GRH to calculate the contribution of zeros we have for $z \le x$

$$\begin{aligned} S(\psi, z) &:= \sum_{\substack{p \le z \\ p \equiv 1 (mod\ 3)}} (\psi(\pi) + \psi(\overline{\pi})) \log p + \sum_{\substack{p \le z \\ p \equiv 2 (mod\ 3)}} 2\psi((p)) \log p \\ &= - \sum_{\substack{|\Im(\rho)| < T \\ \rho : L(\rho, \psi) = 0}} \frac{z^\rho}{\rho} + O\left(\frac{z}{T} \log^2 xz\right) \\ &\ll \sqrt{z} \log^2 xT + \frac{z}{T} \log^2 xz \end{aligned}$$

where $L(s, \psi)$ is the Hecke L-function associated to Hecke character ψ. Taking $T = \sqrt{z} \le \sqrt{x}$ we get

$$S(\psi, z) \ll \sqrt{z} \log^2 xz. \tag{5.10}$$

By partial summation we have $S(\psi, y) - S(\psi, (\log x)^6) = o(1)$ and using $\sum_{\log x \le p \le (\log x)^6} \frac{1}{p} = O(1)$, we obtain the result.

□

Let us use this lemma to shorten the sum over prime squares. In (5.7), the length of the sum is $\sqrt{X}$ so for $\log x < \sqrt{X} < x$ we divide (5.7) in two parts

$$(5.7) = \sum_{\substack{p \le \log x \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} \frac{\Re\ (\psi(\pi)^2 + \psi(\overline{\pi})^2)}{2p} \frac{1}{p^{\frac{2}{\log X}}} \left(1 - \frac{2 \log p}{\log X}\right) + \sum_{\substack{p \le \log x \\ p \equiv 2 (mod\ 3)}} \frac{\Re\ \psi(p)}{p} \frac{1}{p^{\frac{2}{\log X}}} \left(1 - \frac{2 \log p}{\log X}\right)$$
$$+ \sum_{\substack{\log x < p \le \sqrt{X} \\ p \equiv 1 (mod\ 3) \\ p = \pi\overline{\pi}}} \frac{\Re\ (\psi(\pi)^2 + \psi(\overline{\pi})^2)}{2p} \frac{1}{p^{\frac{1}{\log \sqrt{X}}}} \left(1 - \frac{\log p}{\log \sqrt{X}}\right) + \sum_{\substack{\log x < p \le \sqrt{X} \\ p \equiv 2 (mod\ 3)}} \frac{\Re\ \psi(p)}{p} \frac{1}{p^{\frac{1}{\log \sqrt{X}}}} \left(1 - \frac{\log p}{\log \sqrt{X}}\right)$$

The second sum is $O(1)$ by the lemma 5.2 proved above. Hence the sum over prime squares is essentially of length $\log x$. Therefore in (5.1), for $\log x < \sqrt{X} < x$, we have

$$|L(1/2,\psi)| \ll \exp\left(\sum_{N(\mathfrak{p})\leq X} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2} a'(\mathfrak{p},X) + \sum_{\mathfrak{p}|p\Rightarrow p\leq \log x} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2} a''(\mathfrak{p},X) + \frac{\log C}{\log X}\right) \tag{5.11}$$

5.2 Notations and Useful Combinatorics

Following [Har13] and [LR19], our strategy is to partition the set of primes less than or equal to X into multiple intervals $I_{1,n}$ for $1 \leq n \leq J$, each of which is $(x^{\theta_{n-1}}, x^{\theta_n}]$ where

$$\theta_n := \frac{e^n}{(\log\log x)^4} \quad \textit{for } 0 \leq n \leq J$$

where J is chosen such that it satisfies $\eta_1 \leq \theta_J \leq e\eta_1$ for $0 < \eta_1 \leq \frac{(e^{\frac{1}{4}}-1)^4}{2^{32}e}$ (see Remark 5.9). We also define $I_{1,0} := (1, x^{\theta_0}]$. We also define intervals $I_{2,m} := (2^{m-1}, 2^m]$ for $1 \leq m \leq M_2$ where $M_2 := \left\lceil \frac{\log\log x}{\log 2} \right\rceil$. We partition the sum in (5.1) by defining

$$P(I,\psi,B,x^{\theta_u}) := \sum_{\mathfrak{p}\in I} \frac{\psi(\mathfrak{p})+\overline{\psi(\mathfrak{p})}}{2} B(\mathfrak{p},x^{\theta_u}) \tag{5.12}$$

where I is an interval which can be $I_{1,n}$ or $I_{2,m}$ and $\mathfrak{p} \in I$ means $\mathfrak{p} \mid p \Rightarrow p \in I$. $B(\mathfrak{p}, x^{\theta_u})$ is a completely multiplicative function such that $B(\mathcal{I}\mathcal{J}, x^{\theta_u}) = B(\mathcal{I}, x^{\theta_u})B(\mathcal{J}, x^{\theta_u})$ for any ideals $\mathcal{I}$ and $\mathcal{J}$. For instance, B can be a' or a'' as defined in (5.2) and (5.3) respectively. Hence using these notations in (5.11) we have for a fixed $0 \leq j \leq J$ and $X = x^{\theta_j}$ and $\psi \in \mathcal{F}(x)$

$$|L(1/2,\psi)| \ll \exp\left(\sum_{n=0}^{j} P(I_{1,n},\psi,a',x^{\theta_j}) + \sum_{m=1}^{M_2} P(I_{2,m},\psi,a'',x^{\theta_j}) + \frac{1}{\theta_j}\right) \tag{5.13}$$

$$= e^{1/\theta_j} \prod_{n=0}^{j} \exp(P(I_{1,n},\psi,a',x^{\theta_j})) \prod_{m=1}^{M_2} \exp(P(I_{2,m},\psi,a'',x^{\theta_j})). \tag{5.14}$$

Lemma 5.3. *Let ν be a multiplicative function on ideals defined on powers of prime ideals as*

$$\nu(\mathfrak{p}^n) := \frac{1}{n!}$$

where n is non-negative. Then for any even integer $s \geq 0$ we have

$$\sum_{m=0}^{s} \frac{2^m\ (P(I,\psi,B,x^{\theta_u}))^m}{m!} = \sum_{\substack{\mathcal{I},\mathcal{J} \\ \mathfrak{p}|\mathcal{I},\mathcal{J}\Rightarrow \mathfrak{p}\in I \\ \Omega(\mathcal{I}\mathcal{J})\leq s}} B(\mathcal{I}\mathcal{J},x^{\theta_u})\nu(\mathcal{I})\nu(\mathcal{J})\psi(\mathcal{I})\overline{\psi}(\mathcal{J}). \tag{5.15}$$

Proof. We calculate powers of $P(I,\psi,B,x^{\theta_u})$ to get

$$P(I,\psi,B,x^{\theta_u})^m = \left(\sum_{\mathfrak{p}\in I} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2}\ B(\mathfrak{p},x^{\theta_u})\right)^m$$

$$= \sum_{\substack{\mathcal{I},\mathcal{J} \\ \mathfrak{p}|\mathcal{I},\mathcal{J}\Rightarrow \mathfrak{p}\in I \\ \Omega(\mathcal{I}\mathcal{J})=m}} \frac{\psi(\mathcal{I})}{2^{\Omega(\mathcal{I})}}\frac{\overline{\psi}(\mathcal{J})}{2^{\Omega(\mathcal{J})}} B(\mathcal{I}\mathcal{J},x^{\theta_u}) \sum_{\mathfrak{p}_1\mathfrak{p}_2\ldots\mathfrak{p}_m=\mathcal{I}\mathcal{J}} 1$$

$$= \frac{m!}{2^m} \sum_{\substack{\mathcal{I},\mathcal{J} \\ \mathfrak{p}|\mathcal{I},\mathcal{J}\Rightarrow \mathfrak{p}\in I \\ \Omega(\mathcal{I}\mathcal{J})=m}} B(\mathcal{I}\mathcal{J},x^{\theta_u})\nu(\mathcal{I})\nu(\mathcal{J})\psi(\mathcal{I})\overline{\psi}(\mathcal{J}). \tag{5.16}$$

On summing these expressions for $m=0$ to $m=s$, the proof follows. □

Remark 5.4. *For an even integer s and $t\in\mathbb{R}$ we have $\sum_{k=0}^{s}\frac{t^k}{k!}>0$. Thus the expression in (5.15) is non-negative whenever s is even.*

Lemma 5.5. *Let j be a fixed non-negative integer and $0<T_{-1}<T_0<T_1<T_2<\cdots<T_j$ be real numbers. For $0\le n\le j$ we define $I_n:=(T_{n-1},T_n]$ and let $a_n(\cdot)$ be a completely multiplicative function on ideals of $\mathbb{Z}[\omega]$ and s_n be an even integer. If there exist $A>0$ such that*

$$\prod_{n=0}^{j} T_n^{s_n} \le \frac{\sqrt{x}}{3A}$$

then for $\psi_{c_1c_2}\in\mathcal{F}$

$$\sum_{\substack{(c_1,c_2)=1 \\ c_1,c_2\ sq\ free \\ c_1,c_2\equiv 1\ (mod\ 9) \\ N(c_1c_2)\le x}} \prod_{n=0}^{j}\left(\sum_{m=0}^{s_n} \frac{\left(\sum_{\mathfrak{p}\in I_n}\left(\psi_{c_1c_2}(\mathfrak{p})+\overline{\psi}_{c_1c_2}(\mathfrak{p})\right)a_n(\mathfrak{p})\right)^m}{m!}\right) \tag{5.17}$$

$$\ll (x\log x)\prod_{n=0}^{j}\sum_{\substack{\mathcal{I}\mathcal{J}^2=cube \\ \mathfrak{p}|\mathcal{I}\mathcal{J}\Rightarrow\mathfrak{p}\in I_n \\ \Omega(\mathcal{I}\mathcal{J})\le s_n}} a_n(\mathcal{I}\mathcal{J})\nu(\mathcal{I})\nu(\mathcal{J}). \tag{5.18}$$

Proof. Remark 5.4 ensures that the sum from $m=0$ to s_n is non-negative. Let F be a Schwartz function, which is greater than or equal to 1 on $[-1,1]$ and of finite support such that its Fourier transform has compact support contained in $(-A,A)$. Then (5.17) is bounded above by

$$\le \sum_{\substack{c_1\equiv 1(mod\ 9) \\ N(c_1)\le x}}\ \sum_{\substack{c_2\equiv 1(mod\ 9) \\ N(c_2)\le \frac{x}{N(c_1)}}} \prod_{n=0}^{j}\left(\sum_{m=0}^{s_n}\frac{\left(\sum_{\mathfrak{p}\in I_n}\left(\psi_{c_1c_2}(\mathfrak{p})+\overline{\psi}_{c_1c_2}(\mathfrak{p})\right)a_n(\mathfrak{p})\right)^m}{m!}\right) F\left(\frac{N(c_1c_2)}{x}\right)$$

where we have lifted the restrictions of c_1, c_2 square free and $(c_1, c_2) = 1$ using positivity of the sums from $m = 0$ to s_n. For prime ideals $\mathfrak{p}$, let $\pi \equiv 1(mod\ 3)$ be the generator of ideal $\mathfrak{p}$. Since $c_1, c_2 \equiv 1(mod\ 9)$, $\psi_{c_1 c_2}(\mathfrak{p}) = \left(\frac{\pi}{c_1 c_2^2}\right)_3$, and the above term is equal to

$$\sum_{\substack{c_1 \equiv 1(mod\ 9) \\ N(c_1) \le x}} \sum_{\substack{c_2 \equiv 1(mod\ 9) \\ N(c_2) \le \frac{x}{N(c_1)}}} \prod_{n=0}^{j} \left(\sum_{m=0}^{s_n} \frac{\left(\sum_{(\pi) \in I_n} \left(\left(\frac{\pi}{c_1 c_2}\right)_3 + \overline{\left(\frac{\pi}{c_1 c_2}\right)_3} \right) a_n((\pi)) \right)^m}{m!} \right) F\left(\frac{N(c_1 c_2)}{x}\right)$$

Let χ_0 be the principal ray class character modulo 9. Then above quantity is

$$\le \sum_{N(c_1) \le x} \sum_{N(c_2) \le \frac{x}{N(c_1)}} \chi_0(c_1 c_2) \prod_{n=0}^{j} \left(\sum_{m=0}^{s_n} \frac{\left(\sum_{(\pi) \in I_n} \left(\left(\frac{\pi}{c_1 c_2}\right)_3 + \overline{\left(\frac{\pi}{c_1 c_2}\right)_3} \right) a_n((\pi)) \right)^m}{m!} \right) F\left(\frac{N(c_1 c_2)}{x}\right)$$

$$= \sum_{N(c_1) \le x} \sum_{N(c_2) \le \frac{x}{N(c_1)}} \chi_0(c_1 c_2) \prod_{n=0}^{j} \sum_{\substack{r,t \equiv 1(mod\ 3) \\ \mathfrak{p}|(rt) \Rightarrow \mathfrak{p} \in I_n \\ \Omega(rt) \le s_n}} a_n((rt)) \nu((r)) \nu((t)) \left(\frac{r}{c_1 c_2^2}\right)_3 \overline{\left(\frac{t}{c_1 c_2^2}\right)_3} F\left(\frac{N(c_1 c_2)}{x}\right)$$

$$= \sum_{\substack{r_0, \dots, r_j \equiv 1(mod\ 9) \\ t_0, \dots, t_j \equiv 1(mod\ 9) \\ \mathfrak{p}|(r_n t_n) \Rightarrow \mathfrak{p} \in I_n \\ \Omega(r_n t_n) \le s_n}} a_0((r_0 t_0)) \dots a_j((r_j t_j)) \nu((r_0))) \nu((t_0)) \dots \nu((r_j))) \nu((t_j)))$$

$$\sum_{N(c_1) \le x} \chi_0(c_1) \left(\frac{r_0 t_0^2 \cdots r_j t_j^2}{c_1}\right)_3 \sum_{N(c_2) \le \frac{x}{N(c_1)}} \chi_0(c_2) \overline{\left(\frac{r_0 t_0^2 \cdots r_j t_j^2}{c_2}\right)_3} F\left(\frac{N(c_2)}{x/N(c_1)}\right).$$

For any $c \in \mathbb{Z}[\omega]$ and $(c, 3) = 1$, $a \equiv b(mod\ 9c) \Rightarrow \left(\frac{c}{a}\right)_3 = \left(\frac{c}{b}\right)_3$ using cubic reciprocity. Therefore $\chi_0(\cdot) \overline{\left(\frac{r_0 t_0^2 \cdots r_j t_j^2}{\cdot}\right)_3}$ is a Dirichlet character of modulus $9 r_0 t_0^2 \cdots r_j t_j^2$. Since $N(r_0 t_0^2 \cdots r_j t_j^2) \le \prod_{n=0}^{j} T_n^{2s_n} \le x/9A^2$, we apply Proposition 2.1 and get contributions only when $r_0 t_0^2 \cdots r_j t_j^2 = cube$ which is equivalent to $r_n t_n^2 = cube$ for all $0 \le n \le j$. Therefore the term above is

$$\ll \sum_{\substack{r_0, \dots, r_j \equiv 1(mod\ 9) \\ t_0, \dots, t_j \equiv 1(mod\ 9) \\ \mathfrak{p}|(r_n t_n) \Rightarrow \mathfrak{p} \in I_n \\ r_0 t_0^2 \cdots r_j t_j^2 = cube \\ \Omega(r_n t_n) \le s_n}} a_0((r_0 t_0)) \dots a_j((r_j t_j)) \nu((r_0))) \nu((t_0)) \dots \nu((r_j))) \nu((t_j))) \sum_{N(c_1) \le x} \frac{x}{N(c_1)}$$

$$\ll (x \log x) \prod_{n=0}^{j} \sum_{\substack{r,t \equiv 1(mod\ 3) \\ \mathfrak{p}|(rt) \Rightarrow \mathfrak{p} \in I_n \\ rt^2 = cube \\ \Omega(rt) \le s_n}} a_n((rt)) \nu((r))) \nu((t)) = (x \log x) \prod_{n=0}^{j} \sum_{\substack{\mathcal{I}\mathcal{J}^2 = cube \\ \mathfrak{p}|\mathcal{I}\mathcal{J} \Rightarrow \mathfrak{p} \in I_n \\ \Omega(\mathcal{I}\mathcal{J}) \le s_n}} a_n(\mathcal{I}\mathcal{J}) \nu(\mathcal{I}) \nu(\mathcal{J}).$$

□

Remark 5.6. *By following the arguments in the proof, we can also prove the following results.*

1. *For an even integer s, $a(\cdot)$ be a completely multiplicative function on ideals and $I = (T_0, T]$ such that $T^{2s} \leq x/9A^2$. Then we have*

$$\sum_{\psi \in \mathcal{F}(x)} \left(\sum_{\mathfrak{p} \in I} (\psi(\mathfrak{p}) + \overline{\psi}(\mathfrak{p})) a(\mathfrak{p}) \right)^s \ll (x \log x)(s!) \sum_{\substack{\mathcal{I}\mathcal{J}^2 = cube \\ \mathfrak{p} | \mathcal{I}\mathcal{J} \Rightarrow \mathfrak{p} \in I \\ \Omega(\mathcal{I}\mathcal{J}) = s}} a(\mathcal{I}\mathcal{J}) \nu(\mathcal{I}) \nu(\mathcal{J}) \tag{5.19}$$

2. *For a non-negative integer s, let $a(\cdot)$ be a completely multiplicative function on ideals and $I = (T_0, T]$ such that $T^{4s} \leq x/9A^2$. Then we have*

$$\sum_{\psi \in \mathcal{F}(x)} \left| \sum_{\mathfrak{p} \in I} \psi(\mathfrak{p}) a(\mathfrak{p}) \right|^{2s} \ll (x \log x) \, (s!)^2 \sum_{\substack{\mathcal{I}\mathcal{J}^2 = cube \\ \mathfrak{p} | \mathcal{I}\mathcal{J} \Rightarrow \mathfrak{p} \in I \\ \Omega(\mathcal{I}) = \Omega(\mathcal{J}) = s}} a(\mathcal{I}\mathcal{J}) \nu(\mathcal{I}) \nu(\mathcal{J}) \tag{5.20}$$

5.3 Almost Sharp Upper Bound

Before proving the complete result, we first prove a slightly weaker version that will be very useful in our proof.

Proposition 5.7. *For any real positive k and $\epsilon > 0$*

$$\sum_{\substack{c_1, c_2 \equiv 1 (mod\ 9) \\ c_1, c_2\ sq\ free \\ (c_1, c_2) = 1 \\ N(c_1 c_2) \leq x}} |L(1/2, \psi_{c_1 c_2})|^{2k} \ll_k (x \log x)(\log x)^{k^2 + \epsilon}.$$

Our proof of this proposition here is obtained by following the work of [DFL20](Proposition 6.1) and [Sou09]. The difference in our case, as compared to [DFL20], is that we work over $\mathbb{Q}(\omega)$ and there is a double sum which gives $x \log x$ instead of x in the RHS. However this difference is not really significant in terms of ideas involved in the proof. We start with proving an important lemma.

Lemma 5.8. *Let ℓ be a positive integer divisible by 3 and $y', y \in \mathbb{R}$ such that $1 < y' < y \leq (\sqrt{x}/3A)^{1/2\ell}$ where A is constant.*

$$\sum_{\substack{c_1, c_2 \equiv 1 (mod\ 9) \\ c_1, c_2\ sq\ free \\ (c_1, c_2) = 1 \\ N(c_1 c_2) \leq x}} \left| \sum_{N(\mathfrak{p}) \in (y', y]} \psi(\mathfrak{p}) a'(\mathfrak{p}, y) \right|^{2\ell} \ll (x \log x) \frac{(\ell!)^2 (25/9)^{\ell/3}}{(2\ell/3)!} \left(\sum_{N(\mathfrak{p}) \in (y', y]} a'(\mathfrak{p}^2, y) \right)^{\ell} \tag{5.21}$$

where $a'(\mathfrak{p}, y)$ is as defined in (5.2). If $\ell \leq |\sum_{N(\mathfrak{p}) \in (y', y]} a'(\mathfrak{p}^2, y)|^{3-\epsilon}$ then

$$\sum_{\substack{c_1, c_2 \equiv 1 (mod\ 9) \\ c_1, c_2\ sq\ free \\ (c_1, c_2) = 1 \\ N(c_1 c_2) \leq x}} \left| \sum_{N(\mathfrak{p}) \in (y', y]} \psi(\mathfrak{p}) a'(\mathfrak{p}, y) \right|^{2\ell} \ll (x \log x)(\ell!) \left(\sum_{N(\mathfrak{p} \in (y', y]} a'(\mathfrak{p}^2, y) \right)^{\ell} \tag{5.22}$$

Proof. Since $y^{4\ell} \le x/9A^2$, we apply (5.20) to get

$$\Bigg|\sum_{N(\mathfrak{p}\in(y',y]}\psi(\mathfrak{p})a'(\mathfrak{p},y)\Bigg|^{2\ell} \ll (x\log x)(\ell!)^2 \sum_{\substack{\mathcal{I}\mathcal{J}^2=cube\\ \mathfrak{p}|\mathcal{I}\mathcal{J}\Rightarrow N(\mathfrak{p})\in(y',y]\\ \Omega(\mathcal{I})=\Omega(\mathcal{J})=\ell}} a'(\mathcal{I}\mathcal{J},y)\nu(\mathcal{I})\nu(\mathcal{J}) \tag{5.23}$$

Let $\mathcal{I} = \mathcal{G}\mathcal{I}_1$ and $\mathcal{J} = \mathcal{G}\mathcal{J}_1$ such that $(\mathcal{I}_1,\mathcal{J}_1) = 1$. $\mathcal{I}\mathcal{J}^2 = cube$ implies $\mathcal{I}_1 = cube$, $\mathcal{J}_1 = cube$ and the above bound is same as

$$\begin{aligned}
&(x\log x)(\ell!)^2 \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow N(\mathfrak{p})\in(y',y]\\ \Omega(\mathcal{G})\le\ell}} a'(\mathcal{G}^2,y) \sum_{\substack{\mathcal{I}_1=cube;\mathcal{J}_1=cube\\ \mathfrak{p}|\mathcal{I}_1,\mathcal{J}_1\Rightarrow N(\mathfrak{p})\in(y',y]\\ \Omega(\mathcal{I}_1)=\Omega(\mathcal{J}_1)=\ell-\Omega(\mathcal{G})}} a'(\mathcal{I}_1\mathcal{J}_1,y)\nu(\mathcal{G}\mathcal{I}_1)\nu(\mathcal{G}\mathcal{J}_1)\\
\le&(x\log x)(\ell!)^2 \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow N(\mathfrak{p})\in(y',y]\\ \Omega(\mathcal{G})\le\ell\\ 3|\Omega(\mathcal{G})}} a'(\mathcal{G}^2,y)\nu(\mathcal{G})\left(\sum_{N(\mathfrak{p})\in(y',y]}\frac{a'(\mathfrak{p}^3,y)}{3}\right)^{\frac{2(\ell-\Omega(\mathcal{G}))}{3}} \frac{1}{(\frac{\ell-\Omega(\mathcal{G})}{3}!)^2}\\
\ll&(x\log x)(\ell!)^2 \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow N(\mathfrak{p})\in(y',y]\\ \Omega(\mathcal{G})\le\ell\\ 3|\Omega(\mathcal{G})}} \frac{a'(\mathcal{G}^2,y)\nu(\mathcal{G})}{3^{\frac{2}{3}(\ell-\Omega(\mathcal{G}))}(\frac{\ell-\Omega(\mathcal{G})}{3}!)^2}
\end{aligned} \tag{5.24}$$

where we have used $\nu(\mathcal{I}\mathcal{J}) \le \nu(\mathcal{I})\nu(\mathcal{J})$, $\nu(\mathcal{I}) \le 1$, and $\nu(\mathcal{I}^3) \le \nu(\mathcal{I})/3^{\Omega(\mathcal{I})}$. In the last step we used $a'(\mathfrak{p}^3,y) \le N(\mathfrak{p})^{-\frac{3}{2}}$ and bounded the innermost sum over $\mathfrak{p}$ by a constant. So the bound in (5.24) is

$$\begin{aligned}
=&(x\log x)(\ell!)^2 \sum_{\substack{i=0\\ 3|i}}^{\ell} \sum_{\substack{\Omega(\mathcal{G})=i\\ \mathfrak{p}|\mathcal{G}\Rightarrow N(\mathfrak{p})\in(y',y]}} \frac{a'(\mathcal{G}^2,y)\nu(\mathcal{G})}{3^{\frac{2}{3}(\ell-i)}(\frac{\ell-i}{3}!)^2}\\
=&(x\log x)(\ell!)^2 \sum_{\substack{i=0\\ 3|i}}^{\ell} \frac{1}{(\frac{\ell-i}{3}!)^2}\frac{1}{3^{\frac{2}{3}(\ell-i)}(i!)}\left(\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)\right)^i\\
\ll&(x\log x)\frac{(\ell!)^2}{9^{\ell/3}}\left(\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)\right)^\ell \sum_{\substack{i=0\\ 3|i}}^{\ell}\frac{3^{\frac{2i}{3}}}{(i!)(\frac{\ell-i}{3}!)^2}\\
\le&(x\log x)(\ell!)^2\left(\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)\right)^\ell \frac{(25/9)^{\ell/3}}{(\frac{2\ell}{3})!}.
\end{aligned}$$

This establishes (5.21). For the case when $\ell \le |\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)|^{3-\epsilon}$, we need to show for $3 \mid i$ and $i \le \ell$

$$\left(\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)\right)^i \frac{3^{\frac{2i}{3}}}{(\frac{\ell-i}{3})!^2(\frac{2i}{3}!)} \ll \frac{1}{\ell!}\left(\sum_{N(\mathfrak{p})\in(y',y]} a'(\mathfrak{p}^2,y)\right)^\ell$$

or equivalently

$$\frac{x^i 3^{\frac{2i}{3}}}{(\frac{\ell-i}{3}!)^2(\frac{2i}{3}!)} \ll \frac{x^\ell}{\ell!}$$

for $\ell \leq x^{3-\epsilon}$ which is shown in [DFL20] and thus our proof is complete. □

We define

$$n(x,V) := \#\{\psi : \psi \in \mathcal{F}(x), \log|L(1/2,\psi)| \geq V\}$$

and note that

$$\sum_{\psi\in\mathcal{F}(x)} |L(1/2,\psi)|^{2k} = 2k\int_{-\infty}^{\infty} \exp(2kV)n(x,V)\ dV. \tag{5.25}$$

We will use the L-function inequality but we first bound the contribution from prime squares in (5.11).

$$\begin{aligned}
&\sum_{\mathfrak{p}|p\Rightarrow p\leq\log x} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2}a''(\mathfrak{p},X) \\
&= \sum_{\substack{p\leq\log x\\ p\equiv1(mod\ 3)\\ p=\pi\overline{\pi}}} \frac{\frac{\Re\ (\psi(\pi)^2+\psi(\overline{\pi})^2)}{2}}{p}\frac{1}{p^{\frac{2}{\log X}}}\left(1-\frac{2\log p}{\log X}\right) + \sum_{\substack{p\leq\log x\\ p\equiv2(mod\ 3)}} \frac{\Re\ \psi(p)}{p}\times\frac{1}{p^{\frac{2}{\log X}}}\left(1-\frac{2\log p}{\log X}\right)\\
&\leq \sum_{p\leq\log x}\frac{1}{p} \leq C\log\log\log x.
\end{aligned}$$

for an absolute constant C. Therefore using this bound in (5.11) we get

$$|L(1/2,\psi)| \leq \exp\left(\sum_{\substack{N(\mathfrak{p})\leq X\\ p|\mathfrak{p}\Rightarrow p\equiv1(mod\ 3)}} \frac{\psi(\mathfrak{p})+\overline{\psi}(\mathfrak{p})}{2}a'(\mathfrak{p},X) + \frac{\log x}{\log X}\right)(\log\log x)^C. \tag{5.26}$$

We will use this bound in (5.25) and we choose X according to x and V as follows :

$$X := x^{B/V} \qquad \textit{and} \qquad B := \begin{cases} \frac{1}{30}\log\log\log x & \sqrt{\log\log x}\leq V\leq\log\log x\\ \frac{\log\log x}{30V}\log\log\log x & \log\log x < V \leq \frac{\log\log x}{180}(\log\log\log x)\\ 6 & \frac{\log\log x}{180}(\log\log\log x) < V.\end{cases}$$

For the remaining case of $V < \sqrt{\log\log x}$ we use $n(x,V) \ll x(\log x)$ to get the contribution to the integral in (5.25) $\ll_k x(\log x)(\log x)^\epsilon$.

To compute the contribution to (5.25) in the remaining range of values of V, we define $1 \leq z := X^{1/\log\log x} \leq X$ and

$$S_1(\psi) := \left|\sum_{N(\mathfrak{p})\leq z}\psi(\mathfrak{p})a'(\mathfrak{p},X)\right| \qquad S_2(\psi) := \left|\sum_{z<N(\mathfrak{p})\leq X}\psi(\mathfrak{p})a'(\mathfrak{p},X)\right|.$$

If $\log |L(1/2,\psi)| \geq V$ then by (5.26), for large x,

$$S_1(\psi) + S_2(\psi) \geq V\left(1 - \frac{2}{B}\right)$$

and there are two possibilities

1. $S_2(\psi) > V/B$
2. $S_2(\psi) \leq V/B \iff S_1(\psi) \geq V(1-3/B) := V_1$.

We define

$$n_1(x,V) := \#\{\psi : \psi \in \mathcal{F}(x) \textit{ and } S_1(\psi) \geq V_1\}$$
$$n_2(x,V) := \#\{\psi : \psi \in \mathcal{F}(x) \textit{ and } S_2(\psi) \geq V/B\}.$$

For $\ell \leq \frac{V}{10B}$, using Lemma 5.8 and Stirling's formula we have

$$n_2(x,V) \leq \sum_{C(\psi)\leq x} \left(\frac{S_2(\psi)}{V/B}\right)^{2\ell} \ll \left(\frac{B}{V}\right)^{2\ell} \left(\frac{25\ell^4}{4e^4}\right)^{\ell/3} \sqrt{\ell}(x\log x) \left(\sum_{N(\mathfrak{p})\in(z,X]} \frac{a'(\mathfrak{p}^2,X)}{N(\mathfrak{p})}\right)^{\ell}.$$

The sum over $\mathfrak{p}$ can be trivially bounded as $O(\log \frac{\log X}{\log z}) = O(\log\log\log x)$. Taking $\ell = 3\left\lfloor \frac{V}{30B} \right\rfloor$ and using $\ell^{4\ell/3} \leq (V/B)^{4\ell/3}$, we get

$$n_2(x,V) \ll \exp\left(\frac{2\ell}{3}\log\frac{B}{V} + \frac{\ell}{3}\log\frac{25(\log\log\log x)^3}{2e^4} + \frac{1}{2}\log\ell\right) \ll x(\log x)\ \exp\left(-\frac{V}{30B}\log V\right).$$

Now we do similar steps for $n_1(x,V)$. If $V < (\log\log x)^{\frac{7}{4}}$ we pick $\ell = 3\lfloor \frac{V_1^2}{3\log\log x} \rfloor$ which ensures that we can apply Lemma 5.8. Also for this choice of ℓ we have $\ell \leq (\log\log x)^{\frac{5}{2}} \leq (\sum_{N(\mathfrak{p})\leq z} a'(\mathfrak{p}^2,z))^{3-\epsilon}$ since $\sum_{N(\mathfrak{p})\leq z} a'(\mathfrak{p}^2,z) = \log\log x + o(\log\log x)$. So we use (5.22) to get

$$\begin{aligned} n_1(x,V) \ll \sum_{\psi\in\mathcal{F}(x)} \left(\frac{S_1(\psi)}{V_1}\right)^{2\ell} &\ll (x\log x)\sqrt{\ell}\left(\frac{1}{e}\right)^{\ell}\left(\frac{\ell\log\log x}{V_1^2}\right)^{\ell} \\ &\ll (x\log x)\frac{V_1}{\sqrt{\log\log x}}\exp\left(-\frac{V_1^2}{\log\log x}\right) \end{aligned}$$

The other case if $V \geq (\log\log x)^{\frac{7}{4}}$ we pick $\ell = 3\lfloor V \rfloor$ and apply Lemma 5.8 to get

$$n_1(x,V) \ll \sum_{\psi\in\mathcal{F}(x)} \left(\frac{S_1(\psi)}{V_1}\right)^{2\ell} \ll (x\log x)\sqrt{\ell}\left(\frac{25\ell^4(\log\log x)^3}{4e^4V_1^6}\right)^{\ell/3} \ll \exp\left(-\frac{V}{7}\log V\right).$$

where we have used $\ell^4(\log\log x)^3 \leq 81V^{\frac{12}{7}}$ and $V/2 \leq V_1$. We now calculate estimates for $n(x,V)$ in all three ranges of V.

1. If $\sqrt{\log\log x} \leq V \leq \log\log x$

$$n(x,V) \ll (x\log x)\sqrt{\log\log x}\exp\left(-\frac{V^2}{\log\log x}\left(1-\frac{30}{\log\log\log x}\right)^2\right)$$

2. If $\log\log x < V \leq \frac{\log\log x}{180}\log\log\log x$

$$n(x,V) \ll (x\log x)(\log\log x)\exp\left(-\frac{V^2}{\log\log x}\left(1-\frac{90V}{(\log\log x)(\log\log\log x)}\right)^2\right)$$

3. If $V > \frac{\log\log x}{180}\log\log\log x$

$$n(x,V) \ll (x\log x)(\log\log x)^{\frac{5}{4}}\exp\left(-\frac{V\log V}{1260}\right)$$

To calculate $2k$-th moments using (5.25) we use $n(x,V) \ll_\epsilon x(\log x)^{1+\epsilon}\exp\left(-V^2/\log\log x\right)$ if $V \leq 4k\log\log x$ and $n(x,V) \ll_\epsilon x(\log x)^{1+\epsilon}\exp(-4kV)$ if $V > 4k\log\log x$. This completes the proof of Proposition 5.7.

□

5.4 Sharp Upper Bounds

We partition the set of characters $\mathcal{F}(x)$ in different sets depending on the values taken by $P(I_{1,n},\psi,a',x^{\theta_u})$ and $P(I_{2,m},\psi,a'',x^{\theta_u})$. Let $\ell_n := 2\lfloor\theta_n^{-3/4}\rfloor$ for $0 \leq n \leq J$ and we define

$$\mathcal{P}_1(J) := \left\{\psi \in \mathcal{F}(x) : |2kP(I_{1,n},\psi,a',x^{\theta_u})| \leq \frac{\ell_n}{e^2} \ \textit{for all} \ \ 0 \leq n \leq u \leq J\right\} \tag{5.27}$$

$$\mathcal{P}_1(0) := \left\{\psi \in \mathcal{F}(x) : |2kP(I_{1,0},\psi,a',x^{\theta_u})| > \frac{\ell_0}{e^2} \ \textit{for some} \ u \leq J\right\} \tag{5.28}$$

and for $0 < j < J$,

$$\mathcal{P}_1(j) := \Bigg\{\psi \in \mathcal{F}(x) : |2kP(I_{1,n},\psi,a',x^{\theta_u})| \leq \frac{\ell_n}{e^2} \ \textit{for all} \ \ 0 \leq n \leq j \ \textit{and} \ n \leq u \leq J$$
$$\textit{but} \ |2kP(I_{1,j+1},\psi,a',x^{\theta_u}))| > \frac{\ell_{j+1}}{e^2} \ \textit{for some} \ \ j \leq u < J\Bigg\}. \tag{5.29}$$

The sets $\mathcal{P}_1(j)$ are mutually disjoint. Further we define for $1 \le j \le J$ and $1 \le m \le M_2$,

$$\mathcal{P}_2(m,j) := \Big\{ \psi \in \mathcal{P}_1(j) : |P(I_{2,m}, \psi, a'', x^{\theta_j})| > 2^{-c_m} \textit{ but } |P(I_{2,n}, \psi, a'', x^{\theta_j})| \le 2^{-c_n}$$
$$\textit{for all } m+1 \le n \le M_2 = \left\lfloor \frac{\log\log x}{\log 2} \right\rfloor \Big\}$$
$$\mathcal{Q}_2(j) := \{\psi \in \mathcal{P}_1(j) : |P(I_{2,n}, \psi, a'', x^{\theta_j})| \le 2^{-c_n} \textit{for all } 0 \le n \le M_2\}$$

where $c_m := \frac{m}{10}$. Therefore $\mathcal{F}(x) = \bigsqcup_{j=0}^{J} \mathcal{P}_1(j)$ and for $1 \le j \le J$,

$$\mathcal{P}_1(j) = \left(\bigcup_{m=1}^{M_2} \mathcal{P}_2(m,j) \right) \bigsqcup \mathcal{Q}_2(j)$$

and

$$\begin{aligned} &\sum_{\psi \in \mathcal{F}(x)} |L(1/2,\psi)|^{2k} \\ &= \sum_{\psi \in \mathcal{P}_1(0)} |L(1/2,\psi)|^{2k} + \sum_{j=1}^{J} \sum_{\psi \in \mathcal{P}_1(j)} |L(1/2,\psi)|^{2k} \\ &\le \sum_{\psi \in \mathcal{P}_1(0)} |L(1/2,\psi)|^{2k} + \sum_{j=1}^{J} \sum_{\psi \in \mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k} + \sum_{j=1}^{J} \sum_{m=1}^{M_2} \sum_{\psi \in \mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k} \end{aligned} \tag{5.30}$$

We first consider the simplest case, $\psi \in \mathcal{P}_1(0)$, which provides a good glimpse of the methods involved. Let s_0 be an even integer such that $\ell_0 \le s_0 \le \theta_0^{-7/8}$ then

$$\begin{aligned} \sum_{\psi \in \mathcal{P}_1(0)} |L(1/2,\psi)|^{2k} &\le \sum_{\psi \in \mathcal{F}(x)} |L(1/2,\psi)|^{2k} \left(\frac{2ke^2}{\ell_0} P(I_{1,0}, \psi, a', x^{\theta_u}) \right)^{s_0} \\ &\le \sqrt{ \left(\sum_{\psi \in \mathcal{F}(x)} |L(1/2,\psi)|^{4k} \right) \left(\sum_{\psi \in \mathcal{F}(x)} \left(\frac{2ke^2}{\ell_0} P(I_{1,0}, \psi, a', x^{\theta_u}) \right)^{2s_0} \right) } \end{aligned} \tag{5.31}$$

Note that the first sum is bounded above by $(x \log x)(\log x)^{4k^2+\epsilon}$ using Proposition 5.7. Since $\ell_0 \le s_0 \le \theta_0^{-7/8} \Rightarrow (\log\log x)^3 \ll s_0 \ll (\log\log x)^{7/2} \Rightarrow x^{4\theta_0 s_0} \ll x^{\delta}$ for any $\delta > 0$, so we use (5.19) with $a = a'(\mathfrak{p}, x^{\theta_u})/2$, $S = I_{1,0}$ and $s = 2s_0$ to get

$$\sum_{\psi \in \mathcal{F}(x)} P(I_{1,0}, \psi, a', x^{\theta_u})^{2s_0} \ll (x \log x) \frac{(2s_0!)}{2^{2s_0}} \sum_{\substack{\mathcal{I}\mathcal{J}^2 = cube \\ \mathfrak{p} | \mathcal{I}\mathcal{J} \Rightarrow \mathfrak{p} \in I_{1,0} \\ \Omega(\mathcal{I}\mathcal{J}) = 2s_0}} a'(\mathcal{I}\mathcal{J}, x^{\theta_u}) \nu(\mathcal{I}) \nu(\mathcal{J}) \tag{5.32}$$

The following steps are similar to what we did in the proof of Lemma 5.8. Therefore the right hand side of the above inequality is

$$= (x\log x)\frac{2s_0!}{2^{2s_0}} \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow\mathfrak{p}\in I_{1,0}\\ \Omega(\mathcal{G})\le s_0\\ \Omega(\mathcal{G})\equiv s_0 (mod\ 3)}} \sum_{\substack{\mathcal{I},\mathcal{J}=cube\\ \mathfrak{p}|\mathcal{IJ}\Rightarrow\mathfrak{p}\in I_{1,0}\\ \Omega(\mathcal{IJ})=2s_0-2\Omega(\mathcal{G})}} a'(\mathcal{G}^2\mathcal{IJ},x^{\theta_u})\nu(\mathcal{GI})\nu(\mathcal{GJ})$$

$$\le (x\log x)\frac{2s_0!}{2^{2s_0}} \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow\mathfrak{p}\in I_{1,0}\\ \Omega(\mathcal{G})\le s_0\\ \Omega(\mathcal{G})\equiv s_0 (mod\ 3)}} a'(\mathcal{G}^2)\nu(\mathcal{G}) \sum_{\substack{\mathfrak{p}|\mathcal{IJ}\Rightarrow\mathfrak{p}\in I_{1,0}\\ \Omega(\mathcal{IJ})=\frac{2}{3}(s_0-2\Omega(\mathcal{G}))}} a'(\mathcal{I}^3\mathcal{J}^3,x^{\theta_u})\nu(\mathcal{I})\nu(\mathcal{J})\frac{1}{3^{\Omega(\mathcal{IJ})}}$$

$$= (x\log x)\frac{2s_0!}{2^{2s_0}} \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow\mathfrak{p}\in I_{1,0}\\ \Omega(\mathcal{G})\le s_0\\ \Omega(\mathcal{G})\equiv s_0 (mod\ 3)}} a'(\mathcal{G}^2)\nu(\mathcal{G}) \frac{2^{\frac{2}{3}(s_0-\Omega(\mathcal{G}))}}{3^{\frac{2}{3}(s_0-\Omega(\mathcal{G}))}(\frac{2(s_0-\Omega(\mathcal{G}))}{3}!)} \left(\sum_{\mathfrak{p}\in I_{1,0}} a'(\mathfrak{p}^3,x^{\theta_u})\right)^{\frac{2}{3}(s_0-\Omega(\mathcal{G}))}$$

$$\ll (x\log x)\frac{2s_0!}{2^{2s_0}}\left(\sum_{\mathfrak{p}\in I_{1,0}} a'(\mathfrak{p}^2,x^{\theta_u})\right)^{s_0} \sum_{\substack{0\le i\le s_0\\ 3|i}} \frac{(2/3)^{2i/3}}{(s_0-i)!(\frac{2i}{3}!)} \ll (x\log x)\frac{(2s_0)!\lceil s_0/3\rceil!}{s_0!}\left(\frac{2}{3}\right)^{s_0}(\log\log x)^{s_0}$$

Therefore replacing in (5.31)

$$\sum_{\psi\in\mathcal{P}_1(0)} |L(1/2,\psi)|^{2k} \ll_k x(\log x)^{O_k(1)} s_0^{1/4}\left(\frac{32k^2e^4}{(3e)^{4/3}}\frac{s_0^{\frac{4}{3}}}{\ell_0^2}\log\log x\right)^{s_0/2}$$

$$\ll_k x(\log x)^{O_k(1)}\left(\frac{C'}{(\log\log x)^{1/3}}\right)^{s_0/2}$$

where $C' = \frac{32k^2e^4\eta^{1/3}}{(3e)^{4/3}}$ and for the second inequality we have used $\frac{s_0^{4/3}}{\ell_0^2} \le e^{1/3}(\log\log x)^{-4/3}$. Finally using $s_0 \gg (\log\log x)^3$ we get that the bound above is $\ll_k x(\log x)^{-D}$ for any $D>0$.

Now we move to the cases when $\psi\in\mathcal{P}_1(j)$ for $0<j\le J$. For each j we have two cases. The first is when $\psi\in\mathcal{Q}_2(j)$ which implies that the contribution from prime squares is just $O(1)$. Another case is $\psi\in\mathcal{P}_2(m,j)$ for some m which means that there are sums over prime squares which are large. We consider both these cases in separate sections.

The sum over prime squares is small. For $j>0$, take $X=x^{\theta_j}$ in (5.1), so we can use (5.13). Also, $\psi\in\mathcal{Q}_2(j)\Rightarrow\sum_{m=1}^{M_2} P(I_{2,m},\psi,a'',x^{\theta_j}) = O(1)$ and we have

$$\sum_{\psi\in\mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k} \ll_k \sum_{\psi\in\mathcal{Q}_2(j)} \exp\left(\sum_{n=0}^{j} 2kP(I_{1,n},\psi,a',x^{\theta_j}) + \frac{2k}{\theta_j}\right).$$

Let ℓ be an even integer then for $t\le\ell/e^2$ we have

$$e^t \le (1+e^{-\ell/2})\left(\sum_{s=0}^{\ell}\frac{t^s}{s!}\right). \tag{5.33}$$

Since $|2kP(I_{1,n},\psi,a',x^{\theta_j})| \le \ell_n/e^2$ for $0 \le n \le j$ we have

$$\exp\left(\sum_{n=0}^{j} 2kP(I_{1,n},\psi,a',x^{\theta_j})\right) \ll \prod_{n=0}^{j}\left(\sum_{s=0}^{\ell_n}\frac{2kP(I_{1,n},\psi,a',x^{\theta_j})^s}{s!}\right)$$

and

$$\sum_{\psi\in\mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k} \ll_k \sum_{\psi\in\mathcal{F}(x)}\prod_{n=0}^{j}\left(\sum_{s=0}^{\ell_n}\frac{2kP(I_{1,n},\psi,a',x^{\theta_j})^s}{s!}\right)\left(\frac{2ke^2}{\ell_{j+1}}P(I_{1,j+1},\psi,a',x^{\theta_{u_j}})\right)^{s_{j+1}} \tag{5.34}$$

where u_j is any u for which $|2kP(I_{1,j+1},\psi,a',x^{\theta_u})| \ge \ell_{j+1}/e^2$ and $s_{j+1} = 2\lfloor\frac{1}{128\theta_{j+1}}\rfloor$. We remark that when $j = J$, there is no extra term of $P(I_{1,j+1},\psi,a',x^{\theta_{u_j}})$. The choice of η_1 along with the choices of ℓ_n, s_{j+1} ensure that $\sum_{n=0}^{j} 2\theta_n\ell_n + 2\theta_{j+1}s_{j+1} < 1 - \frac{2\log 3A}{\log x}$, as explained in Remark (5.9), and we use Lemma 5.5 together with (5.19) to get

$$\sum_{\psi\in\mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k} \ll_k (x\log x)(e^{2k/\theta_j})\left(\prod_{n=0}^{j}\sum_{\substack{\mathcal{I}\mathcal{J}^2=cube\\ \mathfrak{p}|\mathcal{I}\mathcal{J}\Rightarrow N(\mathfrak{p})\in I_{1,n}\\ \Omega(\mathcal{I}\mathcal{J})\le\ell_n}} k^{\Omega(\mathcal{I}\mathcal{J})}a'(\mathcal{I}\mathcal{J},x^{\theta_n})\nu(\mathcal{I})\nu(\mathcal{J})\right)$$
$$\times\left(\frac{2ke^2}{\ell_{j+1}}\right)^{s_{j+1}}\left(\frac{s_{j+1}!}{2^{s_{j+1}}}\right)\left(\sum_{\substack{\mathcal{I}\mathcal{J}^2=cube\\ \mathfrak{p}|\mathcal{I}\mathcal{J}\Rightarrow N(\mathfrak{p})\in I_{1,j+1}\\ \Omega(\mathcal{I}\mathcal{J})=s_{j+1}}} a'(\mathcal{I}\mathcal{J},x^{\theta_{u_j}})\nu(\mathcal{I})\nu(\mathcal{J})\right). \tag{5.35}$$

Again, for $j = J$, all the terms involving index $j+1$ are not there in the above expression. For $j < J$, the factors involving s_{j+1} can be estimated by following the steps done for the term on right of (5.32).

We now focus on the sum over ideals appearing inside the product from $n = 0$ to j. Let $(\mathcal{I},\mathcal{J}) = \mathcal{G}$, $\mathcal{I} = \mathcal{G}\mathcal{I}_1$ and $\mathcal{J} = \mathcal{G}\mathcal{J}_1$ with $(\mathcal{I}_1,\mathcal{J}_1) = 1$. The condition of $\mathcal{I}\mathcal{J}^2 = cube$ forces $\mathcal{I}_1 = cube$ and $\mathcal{J}_1 = cube$. Therefore for a fixed n,

$$\sum_{\substack{\mathcal{I}\mathcal{J}^2=cube\\ \mathfrak{p}|\mathcal{I}\mathcal{J}\Rightarrow N(\mathfrak{p})\in I_{1,n}\\ \Omega(\mathcal{I}\mathcal{J})\le\ell_n}} k^{\Omega(\mathcal{I}\mathcal{J})}a'(\mathcal{I}\mathcal{J},x^{\theta_n})\nu(\mathcal{I})\nu(\mathcal{J})$$
$$\le \sum_{\substack{\mathfrak{p}|\mathcal{G}\Rightarrow\mathfrak{p}\in I_{1,n}\\ \Omega(\mathcal{G})\le\ell_n/2}} k^{2\Omega(\mathcal{G})}a'(\mathcal{G}^2,x^{\theta_n})\sum_{\substack{\mathcal{I}_1=cube\\ \mathfrak{p}|\mathcal{I}_1\Rightarrow\mathfrak{p}\in I_{1,n}\\ \Omega(\mathcal{G}^2\mathcal{I}_1)\le\ell_n}} k^{\Omega(\mathcal{I}_1)}a'(\mathcal{I}_1,x^{\theta_n})\nu(\mathcal{G}\mathcal{I}_1)\sum_{\substack{\mathcal{J}_1=cube\\ \mathfrak{p}|\mathcal{J}_1\Rightarrow\mathfrak{p}\in I_{1,n}\\ \Omega(\mathcal{G}^2\mathcal{I}_1\mathcal{J}_1)\le\ell_n}} k^{\Omega(\mathcal{J}_1)}a'(\mathcal{J}_1,x^{\theta_n})\nu(\mathcal{G}\mathcal{J}_1) \tag{5.36}$$

For any $\mathcal{I}$, $a'(\mathcal{I}, x^{\theta_n}) \le \frac{1}{\sqrt{N(\mathcal{I})}}$. We also use $\nu(\mathcal{G}\mathcal{I}) \le \nu(\mathcal{G})\nu(\mathcal{I})$ for $\mathcal{I}_1$ and $\mathcal{J}_1$ in the inner two sums. Also we have

$$\sum_{\substack{\mathcal{J}_1 = cube \\ \mathfrak{p} | \mathcal{J}_1 \Rightarrow N(\mathfrak{p}) \in I_{1,n} \\ \Omega(\mathcal{J}_1) \le \ell_n - \Omega(\mathcal{I}_1 \mathcal{G}^2)}} k^{\Omega(\mathcal{J}_1)} a'(\mathcal{J}_1, x^{\theta_n}) \nu(\mathcal{J}_1) \le \sum_{\substack{\mathfrak{p} | \mathcal{J}_1 \Rightarrow N(\mathfrak{p}) \in I_{1,n} \\ \Omega(\mathcal{J}_1) \le \frac{\ell_n}{3}}} k^{3\Omega(\mathcal{J}_1)} a'(\mathcal{J}_1^3, x^{\theta_n}) \nu(\mathcal{J}_1)$$

$$\le \sum_{m=0}^{\ell_n} \frac{\left(\sum_{N(\mathfrak{p}) \in I_{1,n}} k^3 a'(\mathfrak{p}^3, x^{\theta_n})\right)^m}{m!}$$

Since $a'(\mathfrak{p}^3, x^{\theta_n}) \le N(\mathfrak{p})^{-3/2}$, the quantity above is bounded by $\exp(k^3 \sum_{\mathfrak{p} \in I_{1,n}} N(\mathfrak{p})^{-3/2}) = O_k(1)$. Hence (5.36) is bounded above by

$$\sum_{\substack{\mathfrak{p} | \mathcal{G} \Rightarrow \mathfrak{p} \in I_{1,n} \\ \Omega(\mathcal{G}) \le \ell_n / 2}} k^{2\Omega(\mathcal{G})} a'(\mathcal{G}^2, x^{\theta_n}) \nu(\mathcal{G}) \le \exp\left(k^2 \sum_{\mathfrak{p} \in I_{1,n}} \frac{1}{N(\mathfrak{p})}\right) \ll_k \exp\left(k^2 \log \frac{\log x^{\theta_n}}{\log x^{\theta_{n-1}}}\right)$$

$$= \left(\frac{\log x^{\theta_n}}{\log x^{\theta_{n-1}}}\right)^{k^2} = \left(\frac{\theta_n}{\theta_{n-1}}\right)^{k^2}$$

for $n > 0$. As $I_{1,0} = (c, x^{\theta_0})$, for $n = 0$, the bound is $(\log x^{\theta_0} / \log c)^{k^2}$. We now use this result in (5.35) and get

$$\sum_{\psi \in \mathcal{Q}_2(j)} |L(1/2, \psi)|^{2k}$$

$$\ll_k (x \log x) e^{2k/\theta_j} \left(\frac{\log x^{\theta_0}}{\log c}\right)^{k^2} \left(\frac{\theta_j}{\theta_0}\right)^{k^2} s_{j+1}^{1/2} \left(\frac{32k^2 e^4}{(6e)^{4/3}} \frac{s_{j+1}^{4/3}}{\ell_{j+1}^2}\right)^{\frac{s_{j+1}}{2}} \left(\sum_{\mathfrak{p} \in I_{1,j+1}} a'(\mathfrak{p}^2, x^{\theta_{j+1}})\right)^{\frac{s_{j+1}}{2}}$$

$$\ll_k (x \log x)(\log x)^{k^2} \theta_j^{k^2} \exp\left(\frac{2k}{\theta_j}\right) s_{j+1}^{\frac{1}{2}} \left(\frac{160k^2 e^4}{(6e)^{4/3}} \frac{s_{j+1}^{4/3}}{\ell_{j+1}^2}\right)^{\frac{s_{j+1}}{2}} \tag{5.37}$$

where we used the fact that the sum over $\mathfrak{p} \in I_{j+1}$ is simply bounded by a constant, say 5. If $j = J$ then $e^{2k/\theta_J} = O_k(1)$, there are no s_{j+1} terms and $\theta_J^{k^2} = O_k(1)$ and for $j = J$ we have

$$\sum_{\psi \in \mathcal{Q}_2(J)} |L(1/2, \psi)|^{2k} \ll_k (x \log x)(\log x)^{k^2}. \tag{5.38}$$

For the case of $0 < j < J$ we use $\frac{s_{j+1}^{4/3}}{\ell_{j+1}^2} \le \frac{\theta_{j+1}^{1/6}}{256}$ and $\theta_{j+1}^{-1} \le 128 s_{j+1} \le 2\theta_{j+1}^{-1}$ to get

$$\sum_{\psi \in \mathcal{Q}_2(j)} |L(1/2, \psi)|^{2k}$$

$$\ll (x \log x) \exp\left(\frac{2ke}{\theta_{j+1}} + k^2 \log\log x + \frac{1}{2} \log \frac{1}{\theta_{j+1}} + \frac{1}{128\theta_{j+1}} C_1(k) - \frac{1}{256\theta_{j+1}} \log \frac{1}{\theta_{j+1}^{1/6}}\right)$$

$$\ll_k (x \log x) \exp\left(k^2 \log\log x - \frac{\log \frac{1}{\theta_{j+1}}}{2^{10}(3)\theta_{j+1}}\right) \le (x \log x) \exp\left(k^2 \log\log x - \frac{1}{2^{10}(3)\theta_{j+1}}\right)$$

where $C_1(k) = \left|\log \frac{160k^2e^4}{(6e)^{4/3}}\right|$. Combining the above estimate with (5.38) we get

$$\sum_{j=1}^{J} \sum_{\psi \in \mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k} = \sum_{\psi \in \mathcal{Q}_2(J)} |L(1/2,\psi)|^{2k} + \sum_{j=1}^{J-1} \sum_{\psi \in \mathcal{Q}_2(j)} |L(1/2,\psi)|^{2k}$$

$$\ll_k (x\log x)(\log x)^{k^2} + (x\log x)(\log x)^{k^2} \sum_{j=1}^{J-1} \exp\left(-\frac{1}{2^{10}(3\theta_{j+1})}\right) \ll (x\log x)(\log x)^{k^2}.$$

Remark 5.9. *We need to ensure* $2\sum_{n=0}^{j} \theta_n \ell_n + 2\theta_{j+1}s_{j+1} < 1 - \frac{2\log 3A}{\log x}$. *So*

$$2\sum_{n=0}^{j} \theta_n \ell_n + 2\theta_{j+1}s_{j+1} \le 4\sum_{n=0}^{j} \theta_n^{\frac{1}{4}} + \frac{1}{32} \le 4\theta_0^{\frac{1}{4}}\left(\frac{e^{(1/4)J}-1}{e^{1/4}-1}\right) + \frac{1}{32} \le 4\left(\frac{(\theta_0 e^J)^{1/4}}{e^{1/4}-1}\right) + \frac{1}{32}$$

$$\le \frac{4\theta_J^{1/4}}{e^{1/4}-1} + \frac{1}{32}$$

$$\le \frac{4(e\eta_1)^{1/4}}{e^{1/4}-1} + \frac{1}{32}$$

where $\theta_J \le e\eta_1$. *Thus for large* x,

$$\eta_1 \le \frac{(e^{\frac{1}{4}-1})^4}{2^{32}e} \Rightarrow \frac{4(\eta_1 e)^{\frac{1}{4}}}{e^{\frac{1}{4}-1}} + \frac{1}{32} \le \frac{4}{256} + \frac{1}{32} \le \frac{3}{64} < 1 - \frac{2\log 3A}{\log x}$$

The sum over prime squares is big. We move to the final case when for some $1 \le j \le J$ and $0 \le m \le M_2$, $\psi \in \mathcal{P}_2(m,j)$. We consider two cases depending on the size of m.

Case I Consider the case when $2^m > \frac{\ell_0^2}{32k^2e^4} \asymp_k (\log\log x)^6$. Then with $t_m := 2\lceil 2^{9m/16}\rceil$

$$\sum_{\psi \in \mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k} \le \sum_{\psi \in \mathcal{F}(x)} |L(1/2,\psi)|^{2k} \left(2^{m/10} P(I_{2,m},\psi,a'',x^{\theta_j})\right)^{t_m} \tag{5.39}$$

$$\ll \sqrt{x(\log x)^{O_k(1)}\left(2^{\frac{2mt_m}{10}} \sum_{\psi \in \mathcal{F}(x)} P(I_{2,m},\psi,a'',x^{\theta_j})^{2t_m}\right)} \tag{5.40}$$

where we applied Cauchy Schwarz inequality and Proposition 5.7. Since $m \le \log\log x/\log 2$ and $t_m \le 4(\log x)^{9/16}$ we have $(m+1)t_m \le \frac{1}{8}\log x$ for large enough x and thus we can use Lemma 5.5 for the sum in (5.40). Since this is similar to the sum on LHS of (5.32), we directly write the estimates in this case which is

$$\sum_{\psi \in \mathcal{F}(x)} P(I_{2,m},\psi,a'',x^{\theta_j})^{2t_m} \ll (x\log x)\frac{2t_m!}{2^{2t_m}} \sum_{\substack{\mathcal{I}\mathcal{J}^2 = cube \\ \mathfrak{p}|\mathcal{I}\mathcal{J} \Rightarrow \mathfrak{p} \in I_{2,m} \\ \Omega(\mathcal{I}\mathcal{J}) = 2t_m}} a''(\mathcal{I}\mathcal{J},x^{\theta_u})\nu(\mathcal{I})\nu(\mathcal{J})$$

$$\ll (x\log x)\frac{2t_m!\lfloor t_m/3\rfloor!}{t_m!}\left(\frac{2}{3}\right)^{t_m}\left(\sum_{\mathfrak{p} \in I_{2,m}} a''(\mathfrak{p}^2,x^{\theta_u})\right)^{t_m}.$$

Note that $a''(\mathfrak{p}^2, x^{\theta_j}) \leq \frac{1}{p^2}$ for $\mathfrak{p}|p$, and the sum over primes can be bounded trivially as $(\frac{1}{2^m})^{t_m}$. Replacing in (5.40),

$$\sum_{\psi \in \mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k} \ll_k x(\log x)^{O_k(1)} t_m^{\frac{1}{4}} \left(\frac{8}{(3e)^{4/3}} \frac{t_m^{\frac{4}{3}}}{2^{\frac{4m}{5}}} \right)^{t_m/2}. \tag{5.41}$$

Using $2^{\frac{9m}{16}} < t_m \leq 4(2^{\frac{9m}{16}})$ and $2^m > C(k)(\log\log x)^6$ (where $C(k) := (32\eta^{\frac{3}{2}} k^2 e^4)^{-1}$), we have

$$\begin{aligned} t_m^{\frac{1}{4}} \left(\frac{8}{(3e)^{4/3}} \frac{t_m^{\frac{4}{3}}}{2^{\frac{4m}{5}}} \right)^{t_m/2} &\leq \exp\left(\frac{1}{4}\log t_m - \frac{t_m}{2} \log 2^{\frac{m}{20}} \right) \ll \exp\left(-\frac{t_m}{4} \log 2^{\frac{m}{20}} \right) \\ &\ll \exp\left(-\frac{C(k)^{\frac{9}{16}} (\log\log x)^{27/8}}{4} \log C(k)^{\frac{1}{20}} (\log\log x)^{3/10} \right) \end{aligned} \tag{5.42}$$

and this is $\ll (\log x)^{-D}$ for any $D > 0$. Replacing this estimate in (5.41) we have

$$\sum_{j=1}^{J} \sum_{\substack{m \leq M_2 \\ 2^m > \frac{\ell_0^2}{32k^2e^4}}} \sum_{\psi \in \mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k} \ll_k x(\log x)^{-D}(JM_2) \ll x(\log x)^{-D/2}$$

for any $D > 0$.

Case II Finally we consider the case where $2^m \leq \frac{\ell_0^2}{32k^2e^4} \asymp_k (\log\log x)^6 \ll x^{\theta_0}$. For $\mathfrak{p}|p$ we have $|a'(\mathfrak{p}, x^{\theta_j})| \leq p^{-\frac{1}{2}}$ and $|a''(\mathfrak{p}, x^{\theta_j})| \leq p^{-1}$ thus we have the following trivial estimate

$$P((1, 2^{m+1}], \psi, a', x^{\theta_j}) + \sum_{n=0}^{m} P(I_{2,n}, \psi, a'', x^{\theta_j}) \leq (3\sqrt{2})2^{m/2}.$$

Also for $\psi \in \mathcal{P}_2(m,j)$, $|P(I_{2,n}, \psi, a'', x^{\theta_j})| \leq 2^{-n/10}$ for $n \geq m+1$. Hence using these bounds in (5.13) we have for $\psi \in \mathcal{P}_2(m,j)$

$$\begin{aligned} &|L(1/2,\psi)|^{2k} \\ &\ll e^{(2^{\frac{m}{2}} 6\sqrt{2}k)} \exp\left(\sum_{N(\mathfrak{p}) \in (2^{m+1}, x^{\theta_0}]} 2kP((2^{m+1}, x^{\theta_0}], \psi, a', x^{\theta_j}) + \sum_{n=1}^{j} 2kP(I_{1,n}, \psi, a', x^{\theta_j}) + \frac{2k}{\theta_j} \right). \end{aligned}$$

Further by triangle inequality

$$\begin{aligned} |P(2^{m+1}, x^{\theta_0}], \psi, a', x^{\theta_j})| &\leq |P(I_{1,0}, \psi, a', x^{\theta_j})| + |P(1, 2^{m+1}], \psi, a', x^{\theta_j})| \\ &\leq \frac{\ell_0}{2ke^2} + (2\sqrt{2})2^{m/2} \leq \frac{\ell_0}{2ke^2} + \frac{\ell_0}{2ke^2} = \frac{\ell_0}{ke^2} \end{aligned}$$

where we used $|P(I_{1,0},\psi,a',x^{\theta_j})| \le \frac{\ell_0}{2ke^2}$ and trivially bounded $P((1,2^{m+1}],\psi,a',x^{\theta_j})$. Thus for $\psi \in \mathcal{P}_2(m,j)$

$$|L(1/2,\psi)|^{2k} \ll e^{(2^{\frac{m}{2}}6\sqrt{2}k)+\frac{2k}{\theta_j}} \left(\sum_{s=0}^{2\ell_0} \frac{(2kP((2^{m+1},x^{\theta_0}],\psi,a',x^{\theta_j}))^s}{s!}\right)\left(\prod_{n=1}^{j}\sum_{s=0}^{\ell_n} \frac{(2kP(I_{1,n},\psi,a',x^{\theta_j}))^s}{s!}\right)$$

and we have for $t_m = 2\lceil 2^{9m/16}\rceil$ as defined in Case I

$$\begin{aligned}&\sum_{\psi\in\mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k}\\ &\ll \sum_{\psi\in\mathcal{F}(x)} e^{6\sqrt{2}k2^{\frac{m}{2}}+\frac{2k}{\theta_j}} \; (2^{\frac{m}{10}}P(I_{2,m},\psi,a'',x^{\theta_j}))^{t_m} \left(\sum_{s=0}^{2\ell_0} \frac{(2kP((2^{m+1},x^{\theta_0}],\psi,a',x^{\theta_j}))^s}{s!}\right)\\ &\times \left(\prod_{n=1}^{j}\sum_{s=0}^{\ell_n} \frac{(2kP(I_{1,n},\psi,a',x^{\theta_j}))^s}{s!}\right) \left(\frac{2ke^2}{\ell_{j+1}}P(I_{1,j+1},\psi,a',x^{\theta_{u_j}})\right)^{s_{j+1}}\end{aligned}$$

where, as done in the previous section, the factor $P(I_{1,j+1},\psi,a',x^{\theta_{u_j}})^{s_{j+1}}$ is only present when $j < J$. We now apply Lemma 5.5. Since we have carried out calculations for each factor in one of the previous cases, we write the results directly here, so

$$\begin{aligned}&\sum_{\psi\in\mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k}\\ &\ll_k e^{6\sqrt{2}(k2^{\frac{m}{2}})+\frac{2k}{\theta_j}} \; (x\log x)\;\sqrt{t_m}\left(\frac{4}{(3e)^{4/3}2^{1/3}}\frac{t_m^{\frac{4}{3}}}{2^{\frac{4m}{5}}}\right)^{\frac{t_m}{2}} \; (\log x)^{k^2}\theta_j^{k^2}s_{j+1}^{\frac{1}{2}}\left(\frac{160k^2e^4}{(6e)^{4/3}}\;\frac{s_{j+1}^{4/3}}{\ell_{j+1}^2}\right)^{\frac{s_{j+1}}{2}}\end{aligned}$$

There is no difference from (5.37) for the factors involving s_{j+1} and θ_j. For the factors involving m, using $2^{\frac{9m}{16}} \le t_m \le 2(2^{\frac{9m}{16}})$ we have the bound

$$\exp\left(6\sqrt{2}k2^{m/2}+\frac{1}{2}\log t_m+\frac{t_m}{2}\log\frac{t_m^{\frac{4}{3}}}{2^{\frac{4m}{5}}}\right) \ll \exp\left(6\sqrt{2}k2^{m/2}+\frac{1}{2}\log 2^{\frac{9m}{16}}-\frac{2^{\frac{9m}{16}}}{2}\log 2^{\frac{m}{20}}\right).$$

Therefore

$$\begin{aligned}&\sum_{j=1}^{J}\sum_{2^m\le\frac{\ell_0}{32k^2e^4}}\sum_{\psi\in\mathcal{P}_2(m,j)} |L(1/2,\psi)|^{2k}\\ &\ll_k (x\log x)(\log x)^{k^2}\sum_{j=1}^{J}\sum_{2^m\le\frac{\ell_0}{32k^2e^4}} \exp\left(6\sqrt{2}k2^{m/2}+\frac{1}{4}\log 2^{\frac{9m}{16}}-\frac{2^{\frac{9m}{16}}}{2}\log 2^{\frac{m}{20}}\right)\exp\left(-\frac{1}{2^{10}(3\theta_{j+1})}\right)\\ &\ll_k (x\log x)(\log x)^{k^2}\end{aligned}$$

Hence the proof of Theorem 1.2 is complete.

Bibliography

[BY10] Stephan Baier and Matthew P Young. Mean values with cubic characters. *Journal of Number Theory*, 130(4):879–903, 2010.

[CFK+05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Integral moments of L-functions. *Proceedings of the London Mathematical Society*, 91(1):33–104, 2005.

[Cha09] Vorrapan Chandee. Explicit upper bounds for L-functions on the critical line, 2009.

[Dav80] Harold Davenport. *Multiplicative Number Theory.* Graduate Texts in Mathematics. Springer-Verlag New York, 1980.

[DFL19] Chantal David, Alexandra Florea, and Matilde Lalin. The mean values of cubic L-functions over function fields, 2019.

[DFL20] Chantal David, Alexandra Florea, and Matilde Lalin. Non-vanishing for cubic L–functions, 2020.

[GZ21] Peng Gao and Liangyi Zhao. Bounds for moments of cubic and quartic Dirichlet L-functions, 2021.

[Har13] Adam J Harper. Sharp conditional bounds for moments of the Riemann zeta function. *arXiv preprint arXiv:1305.4618*, 2013.

[HB00a] D.R. Heath-Brown. Kummer's conjecture for cubic gauss sums. *Isr. J. Math. 120, 97–124*, 2000.

[HB00b] D.R. Heath-Brown. Kummer's conjecture for cubic gauss sums. *Isr. J. Math.*, 120:97–124, 2000.

[HBP79] D.R. Heath-Brown and S.J. Patterson. The distribution of kummer sums at prime arguments. *Journal für die reine und angewandte Mathematik*, 310:111–130, 1979.

[IK04] H. Iwaniec and E. Kowalski. *Analytic Number Theory.* American Mathematical Society Colloquium Publications. American Mathematical Society, 2004.

[IR90] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory.* Graduate Texts in Mathematics. Springer-Verlag New York, 1990.

[Jut81] M. Jutila. On the mean value of $L(1/2, \chi)$ for real characters. *Analysis, 1(2)*, 1981.

[Kou19] Dimitris Koukoulopolous. *The Distribution of Prime Numbers*, volume 203 of *Graduate Texts in Mathematics*. American Mathematical Society, 2019.

[LR19] Stephen Lester and Maksym Radziwiłł. Signs of Fourier coefficients of half-integral weight modular forms. *arXiv preprint arXiv:1903.05811*, 2019.

[LR20] Stephen Lester and Maksym Radziwiłł. Quantum unique ergodicity for half-integral weight automorphic forms. *Duke Mathematical Journal*, 169(2), Feb 2020.

[Luo04] Wenzhi Luo. On hecke l-series associated with cubic characters. *Compositio Mathematica*, Volume 140 , Issue 5:1191 – 1196, September 2004.

[Pat77] S.J. Patterson. A cubic analogue of the theta series. *Journal für die reine und angewandte Mathematik*, 296:125–161, 1977.

[Pat87] S. J. Patterson. The Distribution of General Gauss Sums and Similar Arithmetic Functions at Prime Arguments. *Proceedings of the London Mathematical Society*, s3-54(2):193–215, 03 1987.

[Sou00] K. Soundararajan. Nonvanishing of quadratic Dirichlet L-functions at s = 1/2. *Annals of Mathematics*, 152(2):447–488, 2000.

[Sou09] Kannan Soundararajan. Moments of the Riemann zeta function. *Annals of Mathematics 130, 981-993*, 2009.

[SY10] K. Soundararajan and Matthew Young. The second moment of quadratic twists of modular L-functions. *Journal of the European Mathematical Society*, page 1097–1116, 2010.