Sharing Cupcakes on a Number Line: The Effects of Instructional Models and Problem Context in Equal-Sharing Situations

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ABSTRACT

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Rebecca MacCaul

The present study investigates participants' ability to solve discrete and continuous context equal-sharing problems using either area or number line models. The experiment also tested whether area or number line models helped participants to learn the part-whole and quotient interpretations of fraction. Sixty-eight undergraduates were randomly assigned to an area (n = 34) or number line (n = 34) condition, in which they used their assigned model to solve equal-sharing problems. Participants first completed a fractions assessment designed to measure their part-whole and quotient understanding of fractions. Then participants solved equal-sharing questions using their assigned model, after which they completed an isomorphic fractions assessment. Results revealed that the area model resulted in better performance on discrete context items than continuous items. In contrast, participants' performance on problems couched in discrete and continuous contexts did not differ when they used the number line. There was no growth from pretest to posttest on part-whole or quotient understanding of fractions, regardless of condition. The present study provides valuable information for how teachers can support the use of the area model in equal sharing situations.

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Chapter 1: Statement of the Problem

Learning fractions is important for a variety of reasons. First and foremost, fractions are necessary and unavoidable in everyday life (Barnett-Clarke, 2013). One way fractions are used everyday is to communicate about quantities, such as stating that there is only a quarter of a tank of gas left or that a recipe calls for a third of a cup of flour to make dough. Time can also be described using fractions, stating that there is only half an hour left before recess. Overall, fractions allow for the precision that is necessary for measured quantities that is not possible with whole numbers.

Furthermore, fractions are closely related to other important mathematical concepts and representations, such as decimals and percent, which are also commonly used in everyday life. An understanding of fractions also provides critical foundations for mathematical concepts encountered in high school, such as algebra (Barnett-Clarke, 2013). Finally, competence in mathematics, including a strong understanding of fractions, has been shown to be predictive of success in higher education and opens the door to careers in STEM fields. However, this knowledge is best acquired in the elementary years and the problem is that these concepts are not being mastered early or deeply enough to grant these opportunities to students.

Despite the importance of learning fractions, children often find that learning them is challenging (Empson, 1999; Siegler et al., 2011). In part, this is because fraction concepts are more nuanced and complex than whole number concepts (Barnett-Clarke, 2013). Indeed, a fraction can mean many different things (Empson & Levi, 2011). For example, a fraction can represent a quantity (e.g., $\frac{1}{3}$ of a pie), or a relationship, such as the multiplicative relationship in the ratio of two eggs for every cup of flour. To make matters more complex, fractions can be seen as the result of an operation such as division, a point on a number line, a shaded area, a symbol, or just a definition. What one of the most difficult ideas for children to understand is that a fraction is indeed a number, with its value to be understood through the multiplicative relationship between the numerator and denominator.

To support the teaching and learning of fractions, there are a variety of external representations, or models, that can be used during instruction. A very popular model used in the classroom is the area model (Moss & Case, 1999; Zhang et al., 2015). There is a debate among researchers, however, as to what models are the most effective for children's learning (Hamdan & Gunderson, 2017; Keijzer & Terwel, 2003; Pearn & Stephens, 2007; Tunç-Pekkan, 2015). The research tends to focus on the models that are best for the development of two specific fraction interpretations, namely part-whole (i.e., a fraction as a part of a whole) and measure (i.e., a fraction as a distance from 0). In particular, area models are often used to study children's learning of part-whole understanding and the number line is often used to investigate children's understanding of fraction as measure. In contrast, little research has focused on the models that support other fractions meanings, such as the quotient interpretation (i.e., a fraction being the result of the numerator divided by the denominator). More research on the quotient interpretation of fractions is needed because children find it particularly challenging. One reason it is difficult to learn is because it requires understanding, in some cases, that smaller numbers can be divided by larger numbers (Barnett-Clarke, 2013).

Furthermore, instructional models can vary in the degree to which they look like the quantities in the problem being solved. An example of a model being perceptually similar to the quantity described in a problem would be circles representing cookies that are to be shared equally. Another example might be to use a number line to represent a length of ribbon that is to be shared equally. Models that look like the objects described in a problem can be considered "grounded" in the sense that they refer to the real-world objects described in the problem (Belenky & Schalk, 2014). However, it is unclear what the effect would be on children's learning if the model used during instruction and problem solving is perceptually dissimilar to the quantity described in the problem. For example, it is possible that children's problem-solving strategies would be different, or even compromised, if a number line were used to represent cupcakes or if circular area models were used to represent a length of ribbon. In such case, models become more "idealized" when they share fewer perceptual features with the objects described in the problem (Belenky & Schalk, 2014). Thus, the context of the problem, defined here as the real-world item that is to be partitioned in the problem, is an important factor to consider when investigating the instructional effects of different fraction models.

The main focus of this research is on children's learning of the quotient interpretation of fractions in the context of solving problems involving partitioning real-world quantities. The present study constitutes a pilot study examining the same research questions with adults. Equal-sharing problems are word problems that require a quantity to be divided into equal groups where the size of the group is unknown (e.g., Six children are sharing 4 brownies equally. How much will each child get?). The mathematical structure of equal-sharing problems is called partitive division, and the strategies children use to solve equal-sharing problems lead to a quotient understanding of fractions (Empson & Levi, 2011). Previous research on children's thinking about partitive division tends to focus on the types of quantities being shared – that is, whether they are discrete or continuous (Adrien et al., 2020; Hiebert & Tonnessen, 1978; Kornilaki & Nunes, 2005). To my knowledge, however, no previous research investigated which pictorial models are most appropriate for solving partitive division problems, and more so, no research has examined the interactive effects of model type and problem context. Investigating the combined effects of model and context on students' solutions to partitive division problems will provide more information about what conditions encourage appropriate strategies and help cultivate a quotient interpretation of fractions.

Some teachers may not be aware that different models can afford different learning and teaching opportunities (Dreher & Kuntze, 2015; Stylianou, 2010), and by extension, different problem-solving strategies as well. To maximize student learning, teachers need to have more research-based information on specific instructional objectives—in this case the quotient understanding of fractions through partitive division—and which tools best support the students' transfer of these concepts. This research is important because there is barely any on how to support the interpretation of fractions as quotients. The results of the current study could be a starting point from which teachers can make more informed decisions regarding which representations to use for specific outcomes.

To fill the gap in research in this area, I propose a study that will aim to determine which of two models—the area and number line model— will best foster the quotient understanding of fractions through equal-sharing word problems. I chose to contextualize the equal-sharing problems in either discrete (e.g., sharing cookies) or continuous (e.g., sharing string) contexts to determine the circumstances under which area and length models may best support problem solving and quotient understanding. This research promises to inform teachers regarding their fraction instruction, with the ultimate goal of helping students better understand the quotient interpretation of fractions.

Chapter 2: Literature Review

Fractions form a large and important part of the elementary mathematics curriculum. Barnett-Clarke (2013) defined a fraction as a symbolic representation taking the form of $\frac{a}{b}$, expressing the quotient of two quantities, where b does not equal zero. There are many reasons children experience difficulty learning fractions, including a weak understanding of whole numbers (Pearn & Stephens, 2007) and viewing fractions as images and not as numbers or quantities (Empson & Levi, 2011). Moreover, interpretations of numbers that children originally learn (i.e., counting and whole numbers) must shift to accommodate fractions and other rational numbers (Siegler et al., 2011). Thus, children are required to integrate fractions into a known number system, and this can be difficult. For example, some children think that, as is the case with whole numbers, fractions can be placed in a specific sequence (Vamvakoussi & Vosniadou, 2010). Additionally, some children adopt a whole number bias (Siegler et al., 2011), which is often seen in several of their misconceptions, such as viewing fractions as two numbers as opposed to one with a single value. Additionally, children experience several challenges in representing fractions, whether it is symbolically, pictorially, or on number lines (Tunç-Pekkan, 2015; Zhang et al., 2015).

There are several conceptual interpretations of fractions that are useful in a variety of situations where rational numbers are required. Charalambous and Pitta-Pantazi (2007) explained that fractions can be interpreted as parts of wholes, as ratios (which relate to concepts of equivalence), operators (which rely on multiplication concepts), measures (which are related to addition), and quotients (which are used in problem solving, including equal sharing). The complexity of fraction understanding can be attributed in part to these different interpretations and can also explain why teaching and learning fractions is so challenging.

As previously mentioned, fractions can be interpreted as quotients and measures. Barnett-Clarke (2013) explained that interpreting a fraction as a quotient means that the value the fraction represents is equivalent to the number obtained when the numerator is divided by the denominator. As such, a quotient interpretation represents a division. An example of how the quotient interpretation can be used in everyday life is through equal-sharing problems. If four children want to share three cookies equally, and the problem asks how much each child would receive, the answer could be found by dividing 3 by 4, which equals $\frac{3}{4}$. Understanding fractions as quotients is useful because it opens the door to learning about different representations of numbers (e.g., the quotient of $\frac{1}{2}$ is 0.5) as well as how operations, division in this case, can be represented. Furthermore, it promotes the understanding that fractions have an infinite number of equivalent representations (e.g., $\frac{1}{4} = \frac{2}{8} = 0.25$).

Fractions can also be interpreted as measures. The measure interpretation is understanding that a fraction can be used to describe a quantity, usually a continuous quantity, such as area, speed, or distance. The quantity being measured must always be considered relative to the unit, whether it is specified or not. An example of a fraction being interpreted as a measure with a specified unit would be stating that a car travelled 3/4 of a kilometer: Here, the kilometer is the unit to which 3/4 refers. An example in which the unit is not specified is a situation where 5/8 of a movie has been viewed: Here, the duration of the movie is the unit to which the fraction refers, but the specific duration is unknown. Interpreting fractions as measures is important because it helps learners interpret division in a certain way (i.e., the measurement interpretation), and it more generally allows for accurate descriptions of quantities (Barnett-Clark, 2013).

There is little research that aims to determine how the interpretations of fraction as quotient and measure can be best fostered in children. A common way of teaching fractions in elementary school is by providing pictorial models (Cramer et al., 2002; Cramer et al., 2008), which serve to help teachers more clearly represent various aspects of fraction concepts for children to visualize. There is evidence to suggest, however, that certain models are more conducive to specific fraction concepts than others (Gersten et al., 2017; Hamdan & Gunderson, 2017; Hiebert & Tonnessen, 1978). For example, research has shown that number lines are useful for tasks supporting the interpretation of fractions as measures (Fuchs et al., 2013; Gersten et al., 2017; Hamdan & Gunderson, 2017), but to my knowledge, there is no research examining what tools should be selected by teachers to foster students' quotient interpretation of fractions. Therefore, one of objective of this study is to examine the effects of different models on children's quotient interpretation of fractions.

Two Interpretations of Division

Partitive Division

One useful way to approach fractions instruction is through equal-sharing problems. Equal-sharing (or partitive division) word problems are those that require a specified quantity to be equally distributed among a known number of sharers (Empson & Levi, 2011). An example of an equal-sharing problem is: *Nine children want to share 4 cakes equally. How much cake will each child get*? The total quantity and number of groups are the known values, and the size of each group is what is left to determine.

Equal-sharing problems are an excellent platform from which to introduce fractions in the early grades because they promote in-depth understandings of what fractions are and ultimately lead to a quotient interpretation of fractions (Empson & Levi, 2011). Equal-sharing problems are beneficial for students because the contexts in the problems are relevant, thereby encouraging students to use strategies based on their everyday experiences. When students are first introduced to equal sharing, they can use intuitive strategies, such as halving and repeated halving, that are meaningful and connected to fractional quantities commonly introduced in the early years (i.e., halves and quarters). Additionally, because equal-sharing problems are designed to have fractional answers, and the fractions are the result of partitioning countable sets that children are familiar with (e.g., 6 cookies), students learn that the partitions (i.e., fractional parts) are also quantities (Empson & Levi, 2011). Indeed, Empson (1999) found that children were better able to consider fractions in mathematically correct ways when they used their knowledge about informal partitioning prompted by equal-sharing problems.

Measurement Division

Barnett-Clarke et al. (2013) described measurement division, also known as quotative division, which is a second way to interpret division. For example, the measurement interpretation of $6 \div \frac{1}{3}$ would be, *How many times does one third fit into six?* This way of thinking about division is called measurement division because of its relation to measurement itself. In the preceding example, for instance, the quantity 6 is *measured* using a unit of size $\frac{1}{3}$. Thus, the measurement interpretation of division is favorable to measurement contexts, such as determining how many lengths/groups of one-third meters would fit into a field six meters long. The number of groups is the unknown value in measurement division problems (Empson & Levi, 2011).

Instructional Models for Fractions

With regard to teaching fractions, there are several external representations that are described in the literature to support fraction learning and understanding. Common external representations—or models—are fraction strips and charts, pattern blocks, circular and rectangular diagrams. These visual aids are classified as area models because they share area attributes and are easily partitioned into sets or sections that represent fractions as parts of wholes (in which the shaded section represents the part, and all the partitions together represent the whole). Area models are a popular pictorial representation in fractions instruction (Moss & Case, 1999; Zhang et al., 2015) and Cramer et al. (2002) posited that teaching with area model representations reap many benefits for students' learning of various fraction concepts.

It is clear that area models can be beneficial because they have been shown to support a variety of conceptual understandings about fractions. However, evidence suggests that a sole reliance on area models can lead to a whole number bias (i.e., thinking of a fraction as two separate whole numbers rather than a single value expressed by the relationship between numerator and denominator) when learning fractions (Moss & Case, 1999). Additionally, relying too much on area models has been found to have a negative impact on the understanding of unit fractions (i.e., a fraction with a numerator of one; Zhang et al., 2015).

The number line is another external model that can also be used to teach fractions to children. It is characterized by a linear scale segmented into equidistant partitions (Barnett-Clarke, 2013). The space between 0 and 1 (and all equivalent spaces on the number line) represents the unit, which can be further segmented into any desired number of partitions. The number line is a powerful tool for fraction learning because of its ability to accurately represent magnitudes as measured quantities, including those greater than 1. The flexibility of the number line makes such representations possible because it can show precisely how much larger one fraction is than another and is easily adaptable to any number of unit subdivisions.

Number lines have been shown to be beneficial for students' learning about the measure interpretation of fractions and corresponding concepts such as fraction magnitude. Indeed, there has been a recent surge of research about the effects of the number line on children's fractions learning (Gersten et al., 2017; Hamdan & Gunderson, 2017; Sidney et al., 2019; Tunç-Pekkan, 2015; Witherspoon, 2019). Tunç-Pekkan (2015), for example, administered the "reconstructing the unit" task to assess fourth- and fifth-graders' reversible thinking. Reconstructing the unit from a fraction essentially requires the participant to iterate a given fraction until it reaches 1. This task requires measurement concepts, in which one iterates the divisor to determine how many times it "fits" into the dividend. The authors found that when participants were asked to reconstruct the unit pictorially when given a proper fraction in the form of a symbol (e.g., 2/5), the number line was more helpful. In contrast, on questions that aimed to target the part-whole interpretation of fractions, the number line was less helpful than circular and rectangular area representations.

Hamdan and Gunderson (2017) conducted a study examining the effect of the number line on fraction magnitude by comparing it to area models. The researchers gathered 114 children in grades two and three, and randomly assigned them to three training conditions: number line, area model, and non-numerical representations (control group). In the training, students used the model corresponding to the condition, and students in the non-numerical condition worked on crossword puzzles. After the participants completed the training, they were given a fraction estimation task and a magnitude comparison task. They found that participants in both area and number line conditions showed significant improvement from pretest to posttest on the estimation task using the model they were trained on. This suggests that training the use of a specific model will increase performance on that model, but children in the number line condition outperformed those in the other conditions on the magnitude comparison task.

In another study, Sidney and colleagues (2019) examined whether the features of various external representations supported children's reasoning about division with fractions. The sample consisted of 123 participants from the fifth and sixth grade. Participants were randomly assigned to one of four conditions that differed by the external model provided to them for solving fraction problems: circular area, rectangular area, number lines, and no model. A research assistant first demonstrated how to use the assigned model using a whole number example, and did so using the measurement interpretation of division. For example, the research assistant modeled how to show her thinking about 8 ÷ 4 by explaining how to make a group of four, how to think about how big four is, and how to determine how many groups of four can go into eight— which is a measurement interpretation of division. The model drawn for the children during the introductory task corresponded to condition (i.e., circle, rectangle, number line) and all components of the division equation were represented and made apparent. In the no model condition, the research assistant used the same whole number example and verbal instruction as in the model conditions, but only wrote the digits with no accompanying model. Participants were then required to use their assigned model to solve a variety of fraction division problems presented symbolically. An example problem from the study was $4\frac{1}{6} \div \frac{5}{6}$, and participants solved it using either a circular area model, a rectangular area model, a number line, or no model, depending on condition.

All the models used in the Sidney et al. study depicted six whole units (e.g., six circles would be presented in the circular model condition and a number line with six demarcations would be provided for the number line condition) and the units in all models were pre-partitioned (i.e., segmented) into as many segments as the denominator of the fractional divisor, or dividend when the divisor was a whole number. The researchers used the students' written work on the models to determine if their understanding of division was conceptually sound. Conceptually sound models of division required participants to correctly represent the magnitude of the numbers in the division equation, as well as the relationship between the numbers. For example, subtracting $\frac{1}{3}$ from 4 would not demonstrate a conceptually sound understanding of the equation $4 \div \frac{1}{3}$. Problem-solving accuracy was also assessed by the numerical quotient the children provided, regardless of strategy.

The results showed that regardless of model, participants were more accurate on problems with fractional divisors than whole number divisors, regardless of whether the dividend was a whole number, mixed number, proper fraction, or unit fraction. More importantly, however, students in the number line condition generated the most accurate solutions relative to the other conditions. Furthermore, conceptually sound models of division—whether measurement or partitive—were most frequently produced by number line condition participants. It is important to note, however, that very few problems had whole number divisors, thereby potentially reducing the number of partitive division interpretations used by the participants. The authors attributed the number line advantage to its continuous nature. They speculated that the number line offers the opportunity to depict more accurate representations of magnitude on a continuum, as opposed to separate units, as reflected in the circular and rectangular area models.

In summary, evidence suggests that different models afford different kinds of fractions learning. Notably, area models can foster a part-whole interpretation of fractions, whereas number lines can help learners develop the interpretation of fraction as measure. One reason number lines appear to be beneficial for the measurement interpretation is because of the relation between measurement division and the interpretation of fraction as measure. Another reason is that number lines are visually similar to several commonly-used measuring instruments (e.g., rulers). Thus, it is reasonable to expect that an interpretation of fraction as measure should be supported by the number line because of its perceptual similarity to commonly-used measurement instruments (Charalambous & Pitta-Pantazi, 2007).

The challenge here, however, is helping children to use number lines in any measurement situation, particularly because children also struggle to measure quantities with rulers. Using a ruler is challenging for children because it requires the coordination and understanding of unit (Barrett et al., 2006), thereby resulting in inappropriate strategies, such as counting the demarcations in lieu of the spaces between them or interpreting the entire number line as the unit. Extending this to fractions, Patahuddin et al. (2018) documented that it can be difficult for students to reinterpret fractions as measures on a number line after exposure to area models.

Discrete vs Continuous

Quantities are either discrete or continuous, and the models used to represent them in mathematics instruction often mirror these characteristics. For example, a discrete quantity such as 11 apples can be represented as 11 blocks, or chips, or pictures of circles. Each apple, or circular unit, is separated by physical space in the representation, indicating that each apple is a discrete unit. A rope that is 14 meters long, on the other hand, can be represented by models that are continuous in nature, such as a line or a number line. There are no empty spaces between the units that measure the rope because the rope is continuous. In both cases of the apples and the rope, the models are perceptually similar to the real-world object described in the problem.

In equal-sharing problems, the quantity that is shared can also either be discrete or continuous. An example of a discrete equal-sharing problem would be sharing seven pies among 10 children. Although the pies would need to be partitioned according to their area attribute, which is itself continuous, they will be considered discrete in the context of this study because they are often represented, both by students and teachers, as separate circles. Rapp et al. (2015)

defined discrete quantities as sets of objects that cannot be naturally broken down (e.g., marbles), which they called "discretized" quantities. On the other hand, equal-sharing problems could involve objects that are continuous, and in the context of this study, be represented not as several separate objects, but as one quantity with continuous measurement characteristics. An example of a continuous equal-sharing problem would be sharing a rope that is seven lengths long between two people. Representing the continuous nature of the rope would entail drawing a single line partitioned into seven equal pieces (i.e., 7 units), much like a number line. Therefore, one could see how representing the quantities in a discrete equal-sharing problem (e.g., 7 pancakes) on a number line could present a discontinuity for a student because of the lack of similarity between the objects in the problem and how they are represented. The same can be said for representing continuous objects in an equal-sharing problem using an area model.

Indeed, research suggests that framing fraction problems in the context of discrete and continuous models results in different cognitive demands. This is because the processes are different for identifying the unit and partitioning it. For example, in a continuous pictorial model, units are contiguous (i.e., they touch each other), but in discrete models, each unit is separate and stands alone (Behr et al., 1988; Hiebert & Tonnessen, 1978).

Additionally, there is evidence to suggest that different contexts for partitive division can afford different problem-solving strategies. Hiebert and Tonnessen (1978), for instance, investigated continuous and discrete partitive division problems by having participants share a clay pie or piece of liquorice—both continuous objects in nature by area or length attributes— or candies (discrete objects) among stuffed animals. The results of Hiebert and Tonnessen's experiment demonstrated that the strategies children used to solve the problems differed by the type of quantity to be shared. Overall, the researchers found that children had more difficulty partitioning the continuous quantities, particularly the length quantities. Partitioning and distributing the length representations (i.e., liquorice) was more difficult than the other representations as the number of partitions required increased (i.e., it was easier for children to partition the length into fewer shares as opposed to more). The area representation (i.e., clay pie), in contrast, allowed for halving and repeated halving, a strategy observed in very young children (Empson, 1999), and the strategies afforded by the discrete set representation involved counting and distributing units one at a time.

The authors speculated that children's difficulties with length representations was because they need well developed anticipatory skills, which are necessary to partition continuous quantities because there is only one object: The child must think deeply about how to best partition it into equal pieces to distribute, which is more difficult than distributing a set of discrete objects, which does not require thinking about the set as a whole. As such, with discrete models, children are not initially required to anticipate how to partition the object to be shared because they can simply count the number of objects to be distributed and deal out the objects. The authors also speculated that the ability to partition continuous quantities accurately requires an understanding of the concept of measurement. Because measurement is thought to be difficult for children (Barrett et al., 2006; Smith et al., 2011), this is another reason why sharing continuous quantities can be more challenging than sharing discrete ones.

Because of children's inherent difficulty sharing continuous quantities, Kornilaki and Nunes (2005) aimed to explore whether five- to seven-year old children would use the same reasoning to solve continuous and discrete equal-sharing problems. The discrete quantities to be shared were sets of fish and fishcakes, their operationalization of continuous quantities, both of which were to be shared among a group of cats. All representations of fish and fishcakes were images cut out on paper. The number of fishcakes to be shared were always fewer than the number of sharers, but there were always more fish than sharers.

The researchers found that the vast majority of children in their study were able to successfully share both discrete and continuous objects. Despite most research that warns that sharing continuous quantities is difficult for children, Kornilaki and Nunes claimed that the participants were able to use their knowledge about sharing discrete quantities to think about division regardless of the type of quantity being shared. If children understood that more sharers resulted in each sharer receiving a smaller amount (i.e., the inverse principle of division), for example, they were able to use this logic to solve partitive division problems, and the type of quantity—discrete or continuous—was not relevant to their reasoning.

It is important to note that in the Kornilaki and Nunes study, the discrete and continuous quantities were represented with materials that had similar physical features—the fish were represented with individually cut out fish, and each fishcake was represented as a circle, also individually cut out. In other words, the models looked like the quantities to be shared, and as such can be considered grounded (Belenky & Schalk, 2014). It is therefore possible that the perceptual similarity of fish and fishcakes to the models the participants used may have supported their strategies rather than their thinking about division concepts.

When discrete area models—such as sets of circles or rectangles—are used to solve equal-sharing problems involving discrete items, they can be thought of as grounded representations because the model itself visually mimics, to one extent or another, the objects to be shared and partitioned. On the other hand, number lines—a model more continuous in nature—can be interpreted as a more idealized model when solving problems with discrete to-be-partitioned quantities. Some research suggests that grounded representations appear to support learning because the students' knowledge of the real world is activated (Belenky & Schalk, 2014; Koedinger et al., 2008). In the case of equal-sharing problems, grounded models may also support students' learning because they are aligned with the physical attributes of the real-world items described in the problems. This suggests that perhaps area models would be more useful for children solving discrete equal-sharing problems and that number lines might be less suitable as a tool for these types of problems because they are more idealized. Note that the number line can also be grounded if it is used to solve continuous length problems. As such, it may be that number lines will be more useful than area models for children solving continuous equal-sharing problems.

Present Study

Sidney et al. (2019) found that the number line model was more conducive than area models for problem-solving accuracy and conceptually-grounded strategies for symbolically presented division problems where the divisor was a fraction. Because it is difficult to interpret such problems using a partitive interpretation (because the number of groups would not be a whole number), it is likely that participants interpreted the problems using measurement division. I interpret the findings of Sidney et al. to mean that the number line model is most effective for measurement division fraction problems. My research aimed to examine the affordances of area and number line models on partitive division problems. Specifically, I tested the effects of model type (area, number line) in discrete and continuous contexts on the performance of participants when they solved equal-sharing problems. I also tested the effects of model on participants' part-whole and quotient interpretation of fractions.

This was an exploratory study using undergraduate students as participants to uncover the potential factors that may impact strategy use and performance on children's partitioning in equal-sharing contexts. The data from this study will be used as a springboard for future research with children.

Children have been documented to struggle with various concepts related to fractions and similar challenges have been found in the adult population. Alghazo and Alghazo (2017) studied college students and found that they held several misconceptions about fractions. Research has also shown that preservice teachers tend to struggle to solve computations involving fractions (Osana & Royea, 2011), and even more so when the computations involved division (Hedges et al., 2005; Newton, 2008). Because fraction concepts are challenging for both children and adults, the data from this study will be useful for further developing research on children's understanding of fractions.

Using an online experiment software (Gorilla.sc), participants first completed a pretest that assessed their knowledge of two interpretations of fractions: part-whole and quotient. Following the pretest was the experimental manipulation, in which participants completed the Model Task. Students were randomly assigned to two conditions: the area model condition and the number line condition. Using the models corresponding to their condition, they solved partitive division word problems, half of which were couched in a discrete context (e.g., partitioning items such as cupcakes) and the other half in a continuous context (e.g., partitioning items such as ropes). After the Model Task, the participants completed posttest measures, which consisted of an isomorphic version of the pretest to again assess their part-whole and quotient understanding.

As such, the main goal of my research was to determine the affordances of area and number line models for accuracy when solving partitive division problems. More specifically:

1. Is there an effect of model type (i.e., area, number line) on participants' performance on equal-sharing problems? Will problem context (discrete vs. continuous quantities) moderate the effects of the model on performance?
2. Is there an effect of model type (i.e., area, number line) on participants' interpretations of fractions? In particular, will the area model be more conducive to quotient and part-whole interpretations of fractions?

With respect to the first research question, I hypothesized that area models would be more beneficial for partitive division because one can imagine distributing units and their partitions better than moving or distributing parts of a number line. I also predicted that the context of the problem (i.e., the type of item that is partitioned) would moderate this effect. The area model better reflects how the objects to be partitioned look in the discrete context questions, so using the area model to solve discrete context problems would be better than using the area model to solve continuous context problems. On the other hand, number lines are more grounded representations for continuous problems that involve ropes, string, or objects with length attributes because they look similar. Therefore, the pattern would be the reverse of that predicted for area models – that is, participants who used number lines to solve continuous problems would outperform participants who used number lines to solve discrete problems.

With respect to condition effects on participants' learning about quotient and part-whole interpretations of fractions, I predicted a three-way interaction between time, condition, and FIT problem type. On the quotient problem type, I predicted improved performance from pretest to posttest in both conditions, but participants' growth in the area condition would be greater than the growth in the number line condition. I predicted this because it is easier to partition using an area model, particularly because children have experience partitioning items like pizzas in real life. Although partitioning quantities on a number line is more difficult (Heron, 2014), I also predicted improved performance in the number line condition on quotient problems. Because of the exposure to equal-sharing problems participants would receive by completing the Model Task, their understanding of fraction as quotient should improve, even if they use a number line.

Regarding performance on part-whole problems, I predicted improved performance in the area condition, but not in the number line condition. I predicted this for the area condition because equal-sharing problems require partitioning, and this helps children understand foundational fractions concepts that can be applied to a wide range of fractions tasks (e.g., wholes can be divided into parts; partitions must be the same size; Empson & Levi, 2011), including problems that rely on part-whole interpretations. I predicted however, that participants' part-whole understanding would not grow in the number line condition because number lines are more difficult to partition and thus not as conducive to partitioning actions. As such, practicing with a number line would hinder further learning about parts and wholes because of the inherent difficulties partitioning on a number line.

Chapter 3: Method

Participants

Sixty-eight ($N = 68$) undergraduate students from a large, urban university in Canada participated in this study. The participants were recruited from a participant pool and those who participated earned course credit. In the sample, 70.6% were female ($N = 48$), 25% were male ($N = 17$), and 4.4% identified as other ($N = 3$). There were no significant differences between conditions in the proportions of males, females, and other, χ^2 (2, $N = 68$) = 2.55, $p = .28$. The mean age of the participants was 24.76 years ($SD = 5.9$). The youngest participant was 20 years and 8 months old and the oldest was 53 and nine months. Due to a software glitch, 32% of participant birthdate data were not recorded. All participants had to be enrolled in an undergraduate program and participated in this study for course credit. The mean number of credits earned at the time of the study was 57.09 ($SD = 26.49$).

In the area condition, no participants stated that they had a diagnosed mathematics difficulty whereas in the number line condition, one participant claimed to have a diagnosed mathematics difficulty. In the area condition 64.7% of participants said they had no trouble learning mathematics compared to 70.6% in the number line condition. Finally, in the area condition 20.6% reported mathematics anxiety compared to 23.5% in the number line condition. There were no significant differences between conditions in the proportion of participants reporting trouble learning mathematics, χ^2 (1, $N = 67$) = .12, $p = .73$, or in the proportion reporting math anxiety, χ^2 (1, $N = 68$) = .08, $p = .77$.

Design

This study is a 2 (model: area, number line) x 2 (problem type: discrete, continuous) mixed experimental design (see Figure 1 for a graphical representation of the study design). Half of the participants were randomly assigned to an area condition and the other half to a number line condition to solve partitive division (i.e., equal sharing) problems. The repeated measure in this design was the problem context (discrete or continuous) – that is, half of the partitive division problems in each condition had a discrete context, and half had a continuous context.

Figure 1

Study Design

		Condition (between groups)	
		Area	Number Line
Problem Context (within groups)	Discrete		
	Continuous		
	Total	*n* = 34	*n* = 34

Participants were asked to solve a series of partitive division problems using the model corresponding to their condition (area model or number line). This task is called the Model Task. The pretest evaluated the participants' knowledge of the quotient and part-whole interpretation of fractions. At posttest, their quotient and part-whole knowledge was re-evaluated.

Measures

Demographic Data

To ensure that there were no prior demographic differences between the two conditions, demographic data were collected on each participant through Gorilla. Information collected was birthdate, gender, level of study (credits earned), and math experiences and perceptions.

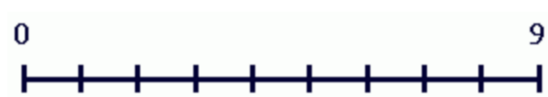
Model Task

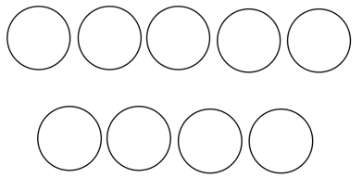
The Model Task (based on Sidney et al., 2019) was used to assess participants' performance on partitive division problems. Within each condition, the participants were asked to solve partitive division problems in two different contexts. Three partitive division questions had a discrete context (e.g., brownies) and three partitive division questions had a continuous context (e.g., rope). Participants were instructed to use their assigned model and show their work on all problems.

All models used in both conditions showed the dividend for each question. For example, the area condition model used circles and the number of circles corresponded to the dividend in the problem. The number line condition presented a number line with the dividend indicated on the right of the line and the units represented by the partitions (Figure 2).

Figure 2

Model Task Sample Items

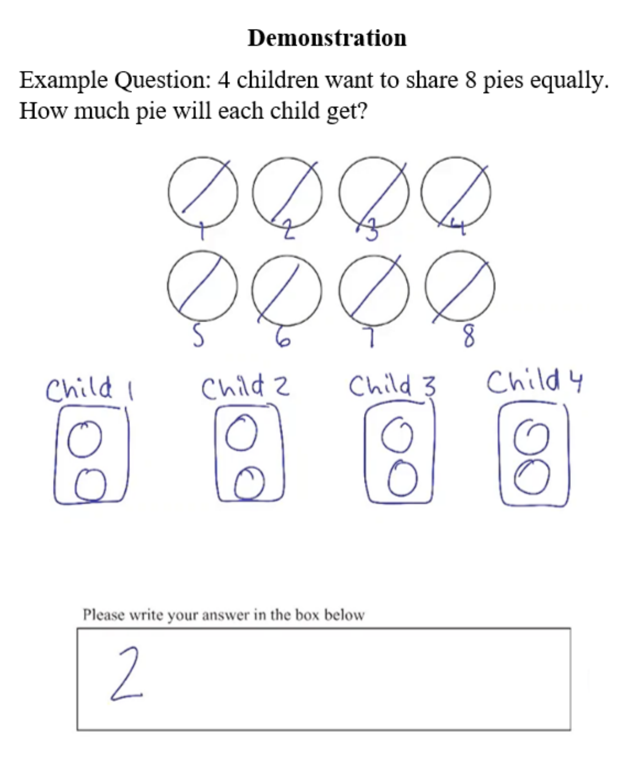
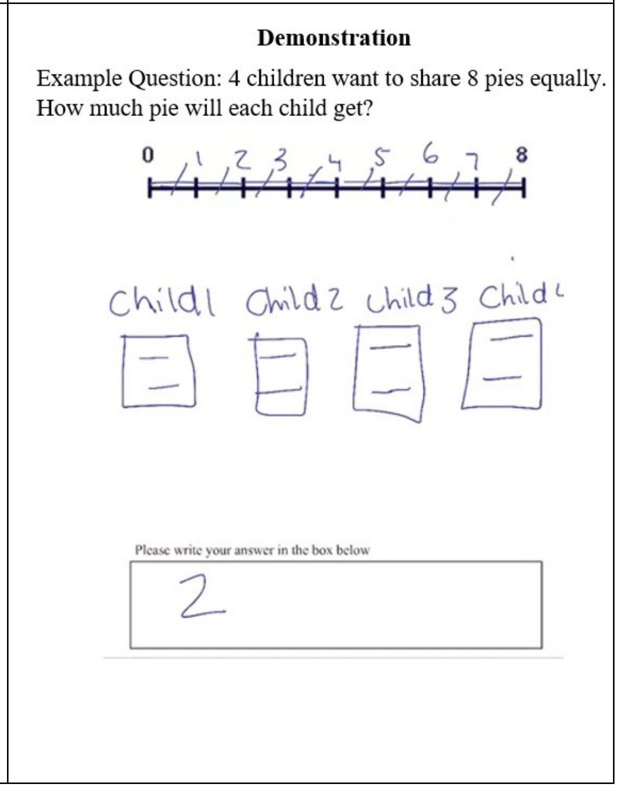
	Condition	
Problem Context	Area	Number Line
Discrete	Harper has 9 brownies that she wants to share equally between her 4 best friends. How much brownie will each friend get?	Harper has 9 brownies that she wants to share equally between her 4 best friends. How much brownie will each friend get? 

		
Continuous	Avery has a rope that is 6 lengths long. He wants to share the rope equally between his 4 friends so they can practice their rope-tying. How much rope will each friend get? 	Avery has a rope that is 6 lengths long. He wants to share the rope equally between his 4 friends so they can practice their rope-tying. How much rope will each friend get? 

The Model Task began immediately after a training video that corresponded to the assigned condition. The video was a recording of a researcher explaining how a partitive division question with no remainder can be solved using the model provided (e.g., 4 children want to share 8 pies. How much pie will each child get?). Drawings that demonstrated how the problem was solved accompanied the explanations. In each condition, the RA demonstrated a partitive strategy (i.e., dealing). In the area condition, the researcher showed how to cross off each circle one at a time as she distributed them to the sharers. In the number line condition, the researcher identified individual units one at a time by crossing them off and distributing them one-by-one to the sharers (see Figure 3).

Figure 3

RA Model Task Demonstration Samples

<table>
<tr><td colspan="2">4 children want to share 8 pies. How much pie will each child get?</td></tr>
<tr><td>Area model sample</td><td>Number line model sample</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

After the training video, the participants were prompted to practice using the pen provided on the screen with the mouse. A practice page with a model corresponding to the assigned condition was provided to practice partitioning and was accompanied by a corresponding division equation (e.g., the equation 6 ÷ 3 = ___ accompanied a number line partitioned into six sections on a number line or six circles). The participant was also prompted to practice writing their answer in the text box in both numerical and written form. The participant had the opportunity to restart practice task several times before proceeding to the

Model Task (the practice task always showed the same question and model) by clicking the "Practice Again" button.

At the start of the Model Task, the participants were asked to use the pen on the screen to show their work and to move the mouse slowly to maximize clarity. Participants were encouraged to try their best to answer each question using the diagrams given. The participants were also reminded to write their answer in numerical and written form (e.g., 2, two) to aid in the interpretation of numerical responses (e.g., 11/4 could be interpreted as eleven fourths or one and one quarter). For each question, the model was placed directly under the question, which appeared at the top of each page. Each problem had an empty text box at the bottom with "*Please enter your answer in the box below"* written above it. The participants each answered six questions, and were given as much time as they needed.

The Model Task was counterbalanced by creating two versions. Half of the participants within each condition were randomly assigned to Version A and the other half to Version B. Both versions alternated between discrete and continuous problems, with Version A starting with a discrete problem and Version B starting with a continuous problem. The order of the discrete problems remained the same across both versions, and the order of the continuous problems also remained the same.

Fraction Interpretation Task

The Fraction Interpretation Task (FIT) was designed to assess two aspects of the participants' fraction understanding, namely the quotient and part-whole interpretations. The items were 24 multiple choice items that required participants to choose among four possible responses to each question. Sample item types on the FIT (quotient and part-whole) are displayed in Figure 4.

Figure 4

FIT Problem Items Categorized by Context (discrete, continuous) and Subscale (quotient, part-whole)

	Subscale		
	Quotient		Part-Whole
Context	Answer < 1	Answer > 1	Answer < 1
Discrete	8 chocolate bars are given to 10 children. The children want to equally share the chocolate bars, how much chocolate bar will each child get? (4 items)	4 children want to share 10 sandwiches equally, how much sandwiches will each child get? (4 items)	There is a garden plot with 9 flowers, 4 of them are red. What fraction of the flowers are red? (4 items)
Continuous	A rope that is 3 lengths long is given to 2 children to share equally. How many lengths of rope will each child get? (4 items)	5 children want to equally share 2 lengths of wire for a craft, how much wire will each child get? (4 items)	6 lengths of a piece of bow-tying ribbon are red, the ribbon is 15 lengths long. What fraction of the ribbon is green? (4 items)
Total number of items	8	8	8

At pretest, the FIT was administered through Gorilla and completed individually by the participants. The instructions for the FIT appeared on the screen stating that to answer the questions, they must select the box with the correct answer, and they were informed that they would have one minute to respond before the next question appears. Before beginning the task, participants had two practice problems: $5 + 2$, and $6 - 3$ so they could practice selecting their response from a list of 4 choices. Each question appeared on its own after the previous question had been answered or one minute had elapsed (whichever came first). The pretest could take a maximum of 24 minutes.

The quotient subscale of the FIT had 16 items designed to assess the quotient interpretation in partitive division contexts. The part-whole subscale had eight items designed to assess part-whole understanding. Moreover, in both subscales, half of the items were phrased so that the divisor appeared first, and the other half had the dividend stated as the first value in the question (e.g., 6 children want to equally share 10 cupcakes [divisor first] vs. 10 cupcakes were given to 6 children to share equally [dividend first]) to reduce practice effects. Additionally, half of all items were set in a discrete context (e.g., 6 chocolate bars) and half used a continuous context (e.g., a rope that is 3 lengths long). All correct answers were fractions in their simplest form. Therefore, all correct answers were unique to each item, meaning that no two problems had the same correct response. To make sure that the quotient and part-whole subscale items were similar in terms of difficulty level, the numerator and denominator in the correct answer were always stated in the problem. The FIT posttest was an isomorphic version of the pretest and was administered immediately after the Model Task using the same procedures.

Quotient Interpretation Subscale. Sixteen of the 24 multiple choice items on the FIT were designed to assess participants' knowledge of the quotient interpretation of fractions. Eight items had quotients greater than one, and eight had quotients less than one. Furthermore, eight items were set in a discrete context, and the remaining eight were set in a continuous context. An example of an item set in a discrete context is: *6 children want to share 4 pieces of fudge, how much fudge will each child get?* An example of an item set in a continuous context is: *2 children want to equally share a string that is 5 lengths long. How much string will each child get?*

Multiple choice alternatives for quotient interpretation problems were constructed using three numbers: the dividend (e.g., 10), the divisor (e.g., 4), and the number 1. Possible answers for a question where 10 children share 4 objects would be (a) the correct answer (e.g., 4/10) (b) the reciprocal of the correct answer (e.g., 10/4), (c) a unit fraction with the dividend quantity in the denominator (e.g., 1/10), and (d) a unit fraction with the divisor in the denominator (e.g., ¼). The answers to each question were ordered randomly for each question. Correct answers and distractors were presented as fractions in all problems.

Part-Whole Subscale. There was a total of eight part-whole items. The setting for the discrete context items (4 items) was a garden plot. Participants had to select one of four possible answers that correctly represented the fraction of the specified color of flowers in relation to all the flowers in the plot (e.g., A plot contains 5 green flowers. There are 14 flowers in total. What fraction of the flowers in the plot are green?). The setting for the continuous context items (4 items) was a piece of bow-tying ribbon, of which a certain amount was a specific color (e.g., A bow-tying ribbon is 14 lengths long. Five of the lengths are red. What fraction of the ribbon is red?). Additionally, half the items were phrased so that the part appeared first in the problem text (e.g., Four flowers in a garden plot are yellow. In total the garden plot has 10 flowers. What fraction of the flowers are yellow?), and the other half had the whole amount appear first (e.g., A piece of bow-tying ribbon is 8 lengths long. Six of the lengths are blue. What fraction of the ribbon is blue?). Solutions to all part-whole items were less than one to reduce the possibility of floor effects. Possible multiple-choice answers to part-whole questions were: (a) the correct answer (e.g., 6/8), (b) the reciprocal of the correct answer (e.g., 8/6), (c) a part-part interpretation (e.g., 6/2), and (d) the reciprocal part-part interpretation (e.g., 2/6). The order of the potential answers was randomized.

Scoring

The Model Task was scored for solution accuracy. Correct answers were assigned 1 point and incorrect answers were assigned 0 points. A Model Task score was computed by converting the total number of points to percent. The FIT was scored to assess the participants' quotient and part-whole knowledge of fractions. Correct answers were assigned 1 point and incorrect answers were assigned 0 points. If the question timed-out and no response was recorded, it was considered missing data. A FIT quotient score and a FIT part-whole score were computed by converting the total number of points on each subscale to percent.

Procedure

The experiment was conducted through Gorilla.sc, which is an online software for psychological experiments. Each participant completed the experiment in one individual session on their own time. First, the consent form informed participants about the nature of the study and briefly explained the aim of the study. After consent was obtained and the demographic questionnaire was completed, participants completed the FIT pretest. Following this was the Model Task, and the FIT posttest concluded the experiment. Participants could take as long as necessary to complete the Model Task. The FIT pre and posttest had a time limit of one minute per question.

Chapter 4: Results

Analytic Plan

To answer the first research question about the effects of the model on participants' accuracy for equal-sharing problems and the potential moderating effects of problem context, I conducted a 2(condition: area, number line) x 2(context: discrete, continuous) analysis of covariance (ANCOVA). The dependent measure was the Model Task score. Condition was the between groups factor and context was the within groups factor. The covariate was the FIT equal-sharing score at pretest.

To answer the second research question about the effect of model on the participants' growth in their quotient and part-whole fractions understanding, I ran a 2(time: pretest, posttest) x 2(condition: area, number line) x 2(problem type: part-whole, equal-sharing) ANOVA, using the quotient and part-whole subscale scores of the FIT as the dependent measures.

Descriptives

The mean number of minutes it took participants to complete the experiment (not including the demographic questionnaire and consent form) was 19.90 minutes (SD = 9.6). The shortest time to complete the experiment was 5.17 minutes and the longest was 50.78 minutes.

The correlations between performance on the FIT at pretest by problem type and performance on the Model Task by problem context are displayed in Table 1. In the area condition, performance on the FIT at pretest for both equal-sharing and part-whole problem types was significantly correlated with performance on both the discrete and continuous Model Task items. In the number line condition, only performance on the FIT equal-sharing items at pretest was correlated with performance on the discrete items in the Model Task.

Table 1

Correlations Between FIT Pretest Scores and Model Task Scores by Problem Context

FIT problem type	Model Task					
	Area			Number line		
	Discrete	Continuous	Total	Discrete	Continuous	Total
Part-whole	.68**	.68**	.71**	.10	-.06	.02
Equal-sharing	.60**	.68**	.66**	.39*	.32	.37*
Total	.71**	.77**	.76**	.35*	.24	.31

*$p < 0.05$. **$p < 0.01$.

Effect of Model and Problem Context on Model Task Performance

The adjusted means (as percents) and standard errors of performance on the Model Task by condition for both discrete and continuous context questions are presented in Table 2.

Table 2

Adjusted Means and Standard Errors for the Model Task by Condition

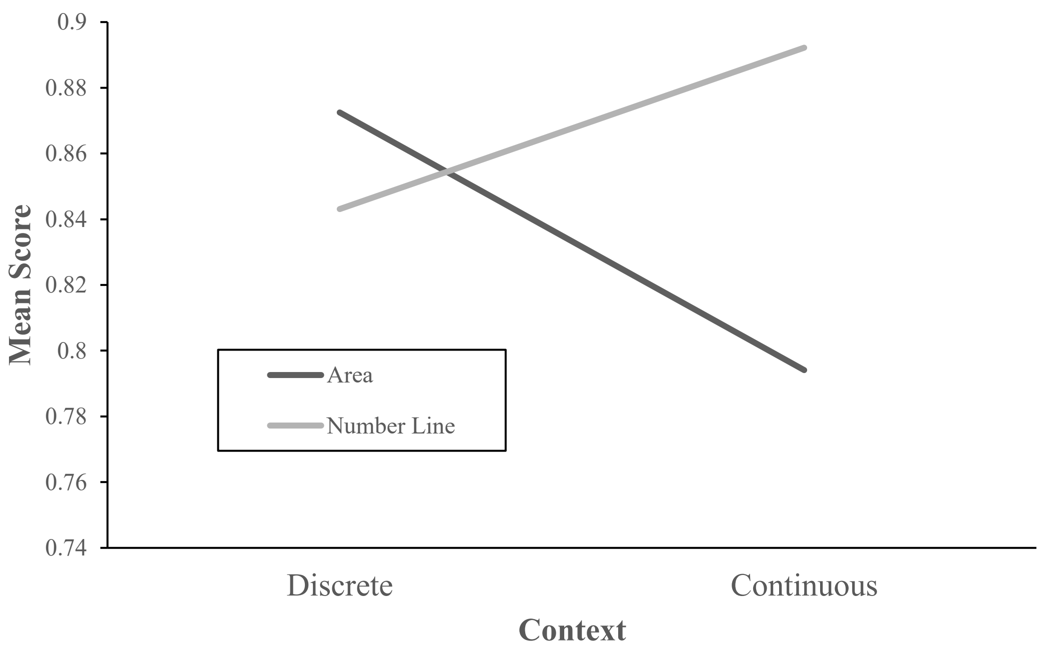
Problem context	Condition			
	Area (n = 34)		Number line (n = 34)	
	M	*SE*	*M*	*SE*
Discrete	.89	.04	.82	.04
Continuous	.82	.05	.87	.05

A mixed design ANCOVA was conducted to test the effects of the model used on participants' problem-solving performance on the Model Task. The model (condition: area, number line) was the between groups variable, and context (discrete, continuous) was the within groups variable. Performance on the equal-sharing subscale of the FIT at pretest was used as the covariate. The mean FIT score on the equal-sharing subscale at pretest was used as the covariate in the analysis because it was correlated with total performance on the Model Task at .66 in the area condition and .37 in the number line condition. For the FIT part-whole subscale, in contrast, the correlations between performance on the part-whole subscale and the Model Task (total scores) were .71 in the area condition and almost zero in the number line condition.

The results of the ANCOVA revealed no main effects, but a significant interaction between context and condition on Model Task performance was found, $F(1, 65) = 8.94$, $p = .004$, which is graphically displayed in Figure 5. Follow-up simple effects analysis with Bonferroni corrections revealed that only participants in the area condition performed differently on the discrete and continuous context questions ($p = .01$). In particular, participants in the area condition performed better on discrete context questions than continuous context questions. There was no significant difference between discrete and continuous context performance in the number line condition ($p = .11$), although the means were in the predicted direction.

Figure 5

Interaction Between Context and Condition on Model Task Performance



I re-ran the same analysis using only the participants who used the model at least once during the Model Task. Because there were six questions to answer, each participant had six opportunities to use the model provided to them. If the participant drew on at least one of the six models provided, I included them in this analysis. If the participant did not draw on any of the six models, they were considered non-drawers and were excluded from the analysis. A participant was considered to have used the model if they drew on it or marked it up in any way that was relevant to the problem. Examples of not using the model would be if the participant wrote only the fractional answer near the model with the pen, if they used the pen for any computations (such as long division), or if there were no pen marks at all.

Three participants were excluded from the area condition ($N = 31$) and five participants were excluded from the number line condition ($N = 29$). The ANCOVA revealed the same pattern of results: No main effects were found, but participants in the area condition performed better on discrete context questions compared to continuous context questions, whereas there was no significant difference between context type in the number line condition.

Effect of Model on FIT Performance from Pretest to Posttest

The means and standard deviations of the FIT scores (as percents) by problem type and condition at pre- and posttest are presented in Table 3.

Table 3

Means and Standard Deviations of FIT Scores by Condition at Pretest and Posttest

Problem type	Condition			
	Area (n = 34)		Number line (n = 34)	
	M	*SD*	*M*	*SD*
	Pretest			
Part-whole	.86	.30	.94	.15
Equal-sharing	.74	.28	.81	.21
	Posttest			
Part-whole	.88	.25	.92	.19
Equal-sharing	.76	.29	.82	.27

I wanted to determine if using either the area or number line model on the Model Task helped participants to perform better on the FIT at posttest than at pretest. In particular, I predicted a three-way interaction between time, condition, and FIT problem type. Therefore, I ran a 2(Time: pretest, posttest) x 2(Condition: area, number line) x 2(FIT problem type: part-whole, equal-sharing) ANOVA with time and FIT problem type as the repeated measures and FIT mean scores as the dependent variable. Condition was the between groups factor. No main effects or interactions were found, except for a main effect of problem type, $F(1, 66) = 18.17$, $p < .001$; part-whole: $M = .90$, $SD = .22$; equal-sharing: $M = .78$, $SD = .26$. This finding means that regardless of condition and time, participants performed better on part-whole questions than equal-sharing questions.

The same analysis was performed with the non-drawers excluded from the sample, and the same pattern was found. Specifically, part-whole performance ($M = .92$, $SD = .21$) was higher than equal-sharing performance ($M = .80$, $SD = .26$) regardless of condition and time of testing.

Chapter 5: Discussion

The main objective of the present study was to investigate the impact of area and number line models on equal-sharing problem solving performance. In addition, I also wanted to explore if problem context (discrete or continuous) would moderate the effect of model. As such, participants were assigned to either an area or number line condition and completed a series of equal-sharing problems with both discrete and continuous problem contexts. The pretest and posttest measured the understanding of the part-whole and quotient interpretation of fractions.

The first research question asked if area or number line models had an effect on equal-sharing problem solving performance and if problem context (discrete or continuous) would moderate this effect. I hypothesized that because area models lend themselves better to imagining distributing objects, and because number lines are difficult to partition, that area models would be more beneficial to partitive division performance compared to number lines on the Model Task. I also predicted a moderating effect of context; specifically, area models would be better for sharing discrete items compared to continuous ones, and number line models would be better for sharing continuous items compared to discrete items. I made this prediction because area models look similar to discrete items (e.g., muffins), and number lines look similar to continuous items (e.g., string).

With regard to my first hypothesis, there was no main effect of condition on average, but a condition by problem context interaction was revealed. In particular, using the area model resulted in better performance on problems that required partitioning discrete items, such as pies, than on those requiring the partitioning of continuous items, such as rope. On the contrary, there was no difference in performance on problems couched in discrete and continuous contexts in the number line condition. Thus, the moderating effect of context on the effects of condition is in line with my hypothesis that the models would have a different effect on problems that required partitioning discrete and continuous items.

The predicted pattern of means in the interaction was found in the area condition, but not in the number line condition. I originally thought that performance in the number line condition would be superior on items that required partitioning continuous items relative to discrete items because of the likeness of the context to the model. Continuous quantities look similar to number lines, but less so to area models. However, the predicted difference in the number line condition did not materialize, and therefore, the likeness of the model to the to-be-partitioned item cannot explain all the results of the present study.

Hiebert and Tonnessen (1978) found that sharing a discrete set of objects was easier than sharing objects with a continuous attribute in equal-sharing situations. Therefore, why did participants not do better when sharing discrete quantities on a number line compared to continuous items? I speculate that the participants' differing prior experience with area and number line models may also account for the findings. The area model is the most popular choice for fraction instruction (Moss & Case, 1999; Zhang et al., 2015), and as a result, students throughout the elementary and secondary grades have less exposure to the number line, especially with regard to learning fractions. Furthermore, Rapp et al.'s (2015) textbook analysis revealed that fractions are more often represented using countable entities as opposed to continuous quantities. I speculate, therefore, that the participants' lack of familiarity and knowledge with number line models may have leveled the performance in the number line condition on both discrete and continuous items on the Model Task.

Another potential reason there were no differences between discrete and continuous contexts in the number line condition is that the number line is more difficult to use than area models (Bright et al., 1988; Wong, 2013), regardless of whether the to-be-partitioned items in the problem “look like” the model or not. Heron (2014) identifies three main struggles students experience with the number line. The first is recognizing the unit. The area model makes the unit easily identifiable because each object is one unit, but units are attached on the number line, so a single unit is difficult to identify. The second struggle is understanding the hash marks on the number line. Children often count the hash marks in lieu of the spaces in between. The third struggle is related to proportional reasoning. It is difficult to partition a number line like one would partition an area model, especially when it is already difficult to identify the unit (Heron, 2014). Moreover, in a study by Yanik et al. (2006), seventh graders applied the part-whole interpretation of fractions inappropriately to the number line because they have difficulty interpreting it as a continuous representation. Thus, the research suggests using the number line is generally difficult because it contains many features that are necessary to comprehend before one can successfully use it. Therefore, the finding that continuous contexts did not help participants to use the number line in the same way that discrete contexts helped with the area model could be explained by the number line being more challenging to use than area models, especially when partitioning is required. In contrast, the area model is more familiar and easier to use when physically constructing fair shares.

My second research question asked if using area or number line models while solving equal sharing problems helped to foster a deeper understanding of quotient and part-whole interpretations of fractions. For this, I hypothesized that using the area model would be more beneficial for learning the quotient interpretation of fractions than the number line model. With respect to part-whole understanding, I predicted that the performance of participants in the area condition would also improve from pretest to posttest relative to those in the number line condition, who would not show improved performance.

Contrary to my second hypothesis, there was no effect of model on participants' quotient interpretation of fractions. This means that there was no evidence that using either model during the Model Task helped participants to better understand fractions as a quotient. This could have been due to participants not having had enough practice solving equal-sharing problems with the models to make a difference in their learning about the quotient interpretation. Empson and Levi (2011) stated that children are able to develop the mental strategy that coordinates division and fractions in a way that supports a quotient interpretation, but this requires practice solving and discussing equal-sharing strategies. The participants in the present study used a model to solve only six equal-sharing problems. Furthermore, they did not have the opportunity to engage in any discussion about the problems they solved. The little amount of practice with the models may be one reason that there was no difference in performance on the quotient problems between models.

My last hypothesis that relative to pretest performance, participants in the area condition would outperform those in the number line condition at posttest on part-whole questions was also not confirmed. This means that the model played no role in participants' understanding of the part-whole interpretation of fractions either. The only finding was that regardless of time and condition, the participants performed significantly better on part-whole type questions compared to equal-sharing type questions. This is not a surprising finding given that the part-whole interpretation of fractions is often the only interpretation learned by students in school. Congruent with this finding, Van Steenbrugge et al.'s (2014) study revealed that the part-whole understanding was strongest when compared to other fraction interpretations among undergraduate preservice teachers. Preservice elementary teachers have also been found to rely on their robust part-whole understanding of fractions, as opposed to other interpretations of fractions (e.g., measure, quotient) when attempting to explain fractions (Reeder & Utley, 2017). Additionally, because performance on part-whole problems was high even at pretest, I do not believe there was much room to grow with regards to the participants' part-whole understanding, thereby resulting in no effect of either model on part-whole problems.

Strengths and Limitations of the Study

In regard to learning fractions and their different interpretations, there is considerable research on the effectiveness of area and number line models (e.g., Gersten et al., 2017; Hamdan & Gunderson, 2017; Patahuddin et al., 2018; Sidney et al., 2019). Furthermore, research has shown that fractions are more often represented as discrete objects in textbooks as opposed to continuous objects (Rapp et al., 2015). Moreover, in equal-sharing situations, children are better at sharing discrete items compared to continuous items (Hiebert & Tonnessen, 1978). However, there is little research that aims to determine how to best develop the quotient interpretation of fractions. To my knowledge, this study is the first to investigate the area and number line models in discrete and continuous contexts, and how they can impact the quotient interpretation of fractions specifically.

One strength of this study is the experimental design used. The random assignment to different conditions allowed me to determine if using the area and number line models impacted participants' performance on equal-sharing problems and their interpretations of fractions. Another strength was large number of participants. The ability to gather a large number of participants in this case was a result of the online format that was easy to access for participants who met the participation criteria.

One limitation of the study was that because of the online platform for collecting data, I was not able to analyze the strategies used by the participants to solve the equal-sharing problems in the Model Task. Hence, I cannot know the nature or quality of the participants' strategies, which could have provided some insight about the impact of different models and contexts, as well as their learning of the quotient interpretation of fractions. The online platform also does not allow the researchers to know if the participants were following the instructions given for each task in the same way as if a research assistant were present. Furthermore, the participants did not have the opportunity to ask questions that a research assistant would have been able to clarify.

Another limitation was the number of items in the Model Task. Increasing the number of items in this task would have provided more exposure to the assigned model and the participants may have had the opportunity to see the pattern in coordinating the to-be-shared quantities and the number of sharers that eventually leads to a quotient understanding. For example, a student who practices solving equal-sharing problems will eventually realize that a fraction can be understood as a quantity (numerator) divided by a number of groups (denominator). In addition, Empson and Levi (2011) posited that discussing equal-sharing situations in the classroom is a steppingstone to learning the quotient interpretation. Participants in this study were not given the opportunity to discuss their work for the Model Task with a research assistant or peer. Creating a setting where participants spent more time solving equal-sharing problems and giving them the opportunity to share their strategies and discuss their solutions with each other may have been a better approach to fostering learning about fraction as quotient.

Future Research

The results of this pilot study with adults are promising enough to warrant repeating the experiment with children. Repeating this study with children would be ideal so that the results could help teachers make more informed decisions about their practice, such as pairing discrete context equal-sharing questions with the area model to maximize student success in these situations.

Something else worth considering would be using familiar fractions in a future study. All correct responses in the Model Task were mixed numbers (e.g., 2 ¼) or improper fractions (e.g., 18/8), whereas the FIT tasks included a mix of improper and proper fractions (e.g., 1/3), but no mixed numbers. The most familiar fractions are $\frac{1}{2}$ and $\frac{1}{4}$ because they are easily found using repeated halving, which is one of the first intuitive partitive strategies to develop in young children (Empson & Levi, 2011). It would be worth investigating whether the quotient interpretation is more easily adopted if the majority of solutions to the equal-sharing problems are familiar, namely fractions less than one. Because the correct answers to the Model Task questions were mixed numbers or improper fractions, various representations of one value was possible (i.e., 2 ¼ can also be represented as 9/4). Therefore, if a participant was inclined to think of the solution to 9 ÷ 4 as 2 ¼, it may have been more difficult to recognize the pattern between shares and sharers than had they only seen the solution as 9/4. The fact that 9 ÷ 4 is equivalent to 9/4 may have been more apparent if 9/4 were the only possible solution, which would be the case with proper fractions.

Educational Implications

It is important to attempt to determine under what conditions the quotient understanding is best fostered. Because fractions have been challenging for children and adults alike (Alghazo & Alghazo, 2017; Barnett-Clarke, 2013), it is important that teachers are provided with information on how to scaffold the quotient interpretation. Given the variety of representative models available, in combination with various problem contexts, there are many ways one could approach instruction.

The results of the present study suggest that when students are given area models to solve equal-sharing problems, they will be more successful when the problems involve discrete quantities rather than continuous quantities. With this information, teachers can be more selective about the problems that they give their students and the models they provide to solve them. The findings also imply that if teachers decide to frame the problems in continuous contexts, students may require more scaffolding from the teacher if they are using area models. For example, a teacher may try to draw links between the units in continuous quantities, such as rope, and the area models that students are using. Similarly, the number line can still be used in equal-sharing contexts, but it would likely require explicit instruction on the use of the model.

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Appendix

Script for Model Task Training Video

Have you ever seen a diagram like this before? Let me show you how you might use it to solve this problem. So, the question is *4 children want to share 8 pies equally. How much pie will each child get?* First, because we have 8 pies, we can **show how big 8 is** [count out 8 circles or partitions on number line while drawing numbers]

Now, we can represent the 4 children because we are sharing the pies between 4 children [write child 1, 2…].

Now, we want to distribute the 8 pies equally among the 4 children so that each child can get the same amount of pie. To do that, I will give out 1 pie at a time to each child until I have none left. *So first I'll take this pie and give it to child 1, I'll take this pie and give it to child 2, and I cross them out as I go to know that I've used them. I'll give this one to child 4 and I'll keep going until I have none left*

Now I can see that each child gets two pies [circle each group of 2].

So, we used the diagram to show when 8 pies are shared equally among 4 children, each child gets 2 pies. So now I write that in the answer box below. Now, you'll see some new diagrams and I would like you to use the diagrams to come up with the answers to the new problems.