The Painlevé II hierarchy : geometry and applications

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Abstract

The Painlevé II hierarchy : geometry and applications

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The Painlevé II hierarchy is a sequence of nonlinear ODEs, with the Painlevé II equation as first member. Each member of the hierarchy admits a Lax pair in terms of isomonodromic deformations of a rank 2 system of linear ODEs, with polynomial coefficient for the homogeneous case. It was recently proved that the Tracy-Widom formula for the Hastings-McLeod solution of the homogeneous PII equation can be extended to analogue solutions of the homogeneous PII hierarchy using Fredholm determinants of operators acting through higher order Airy kernels. These integral operators are used in the theory of determinantal point processes with applications in statistical mechanics and random matrix theory. From this starting point, this PhD thesis explored the following directions. We found a formula of Tracy-Widom type connecting the Fredholm determinants of operators acting through matrix-valued analogues of the higher order Airy kernels with particular solution of a matrix-valued PII hierarchy. The result is achieved by using a matrix-valued Riemann-Hilbert problem to study these Fredholm determinants and by deriving a block-matrix Lax pair for the relevant hierarchy. We also found another generalization of the Tracy-Widom formula, this time relating the Fredholm determinants of finite-temperature versions of higher order Airy kernels operators to particular solutions of an integro-differential Painlevé II hierarchy. In this setting, a suitable operator-valued Riemann-Hilbert problem is used to study the relevant Fredholm determinant. The study of its solution produces in the end an operator-valued Lax pair that naturally encodes an integro-differential Painlevé II hierarchy. From a more geometrical point of view, we analyzed the Poisson-symplectic structure of the monodromy manifolds associated to a system of linear ODEs with polynomial coefficient, also known as Stokes manifolds. For the rank 2 case, we found explicit log-canonical coordinates for the symplectic 2-form, forming a cluster algebra of specific type. Moreover, the log-canonical coordinates constructed in this way provide a linearization of the Poisson structure on the Stokes manifolds, first introduced by Flaschka and Newell in their pioneering work of 1981.

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To my mother.

Sólo vuela el que se atreve a hacerlo.
(Luis Sepúlveda, *Historia de una gaviota y el gato que le enseñó a volar.*)

Abbasso il nove

Uno scolaro faceva le divisioni:
- Il tre nel tredici sta quattro volte con l'avanzo di uno. Scrivo quattro al quoto. Tre per quattro dodici, al tredici uno. Abbasso il nove...
- Ah no, - gridò a questo punto il nove.
- Come? - domandò lo scolaro.
- Tu ce l'hai con me: perché hai gridato «abbasso il nove»? Che cosa ti ho fatto di male? Sono forse un nemico pubblico?
- Ma io...
- Ah, lo immagino bene, avrai la scusa pronta. Ma a me non mi va giù lo stesso. Grida «abbasso il brodo di dadi», «abbasso lo sceriffo», e magari anche «abbasso l'aria fritta», ma perché proprio «abbasso il nove»?
- Scusi, ma veramente...
- Non interrompere, è cattiva educazione. Sono una semplice cifra, e qualsiasi numero di due cifre mi può mangiare il risotto in testa, ma anch'io ho la mia dignità e voglio essere rispettato. Prima di tutto dai bambini che hanno ancora il moccio al naso. Insomma, abbassa il tuo naso, abbassa gli avvolgibili, ma lasciami stare.

Confuso e intimidito, lo scolaro non abbassò il nove, sbagliò la divisione e si prese un brutto voto. Eh, qualche volta non è proprio il caso di essere troppo delicati.

(Gianni Rodari, *Favole al telefono.*)

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Chapter 1

Introduction

Painlevé equations arose more than one century ago as the solution of a classification problem in ODE theory first posed by Picard ([95]). His aim was to describe all second order ordinary differential equations of a certain prescribed form, for which the solutions have no movable critical points. This property, also known as the Painlevé property, allows indeed to define new functions as the general solutions of these differential equations. The subsequent studies of Painlevé, Fuchs and Gambier ([94, 42, 44]) finally produced a compact list with only six equations satisfying the required properties and for which the general solutions cannot be written in terms of known special functions. All the other equations fulfilling Picard's requirements were shown to be either solvable in terms of known special functions or reduced to one among the six in the list. Nowadays, we call this list of second order nonlinear ordinary differential equations the Painlevé equations, see equations (2.1.1)-(2.1.6). Their solutions, called Painlevé transcendents, are classified as *new* nonlinear transcendental functions and added to the list of the classical special functions (together with the Bessel, Airy, hypergeometric, elliptic functions etc.). The study of their properties increased together with their appearance in different domains involving nonlinear phenomena. During the last fifty years Painlevé equations have been found in connection with many different areas of mathematics and physics thus stimulating their study from many different points of view. Among the physical literature, Painlevé equations appeared in different models of statistical mechanics and quantum field theory ([9, 65, 25] some classical examples and [81, 75] more recent ones particularly related to this thesis). In mathematics, new connections with orthogonal polynomials ([108] a classical reference), random matrices ([102, 104] and subsequent literature) and random growth models (e.g. [41, 5]) are discovered still these days.

Going back in time, one of the first aspects of Painlevé equations to be studied was the dependence of their solutions on the parameters appearing in the coefficients of the equations. It is worth to notice that each of the six equations, apart from the first one, actually depend on some complex parameters (and up to 4 independent ones). For particular choices of the values of these parameters, it is actually possible to construct explicit solutions of the

Painlevé equations in terms of known special or elementary functions. Take the simplest scenario, when there is only one extra parameter. This is indeed realized by our case of interest, the Painlevé II equation

$$\frac{d^2w}{dz^2} = 2w^3 + wz + \alpha, \quad \text{for } \ w = w(z), \ \ \alpha \in \mathbb{C}. \tag{1.0.1}$$

It was first shown by Airault in [4] that for nonzero integer values and semi-integer values of α, the Painlevé II equation admits respectively rational solutions and solutions in terms of the classical Airy function (her results are written in Theorem 2.1.1, 2.1.2). Intriguingly enough, the case $\alpha = 0$ does not fit in either of these classes of solutions. This special case was first handled by Hastings and McLeod in [56] together with specific boundary conditions. The solution of their boundary value problem, known as the Hastings-McLeod solution (details are written in Theorem 2.1.5), appeared some years later in relation with random matrix theory (in the same paper [102] cited above). This result (stated in Theorem 2.1.7), that goes under the name of the Tracy-Widom formula, is just one example among many others describing connections between Painlevé transcendents and the theory of determinantal point processes (that in this specific case applies to random matrix theory). The proof of their formula followed from a study of the properties of the well known Airy kernel. In particular, they proved that the Fredholm determinant of the integral operator acting through the Airy kernel is expressed in terms of the Hastings-McLeod solution of the Painlevé II equation. This Fredholm determinant was already known to express the edge scaling limit of the probability distribution of the largest eigenvalue for the Gaussian Unitary Ensamble (e.g. [40]), thus providing the bridge between random matrix theory and Painlevé transcendents.

Among the many interesting aspects of the Painlevé II equation, in this work we will be particularly interested in these two: its relation with the modified Korteg-De Vries equation and its isomonodromic representation. In a certain way, the first one defines the object at the basis of our study, namely the Painlevé II hierarchy, and the second one gives us the main tool to handle it. The link between the Painlevé II hierarchy and isomonodromic deformations theory was deeply studied in the two subsequent papers of Flaschka and Newell [36, 37] in the eighties, and their work provides in some sense the basis of our work, from both an analytical and a geometrical point of view.

Painlevé equations in general are known to be reduction of integrable (and non) PDEs [1] such as the Korteg De Vries equation, the nonlinear Schroedinger equation and the sine-Gordon equation just to cite some of them. As for the Painlevé II equation, it is obtained as self-similarity reduction of the modified Korteg De Vries equation. This means that while seeking for solutions of the modified KdV equation

$$v_t + v_{xxx} - 6v^2 v_x = 0, \tag{1.0.2}$$

of the type

$$v(t,x) := \frac{w(z)}{(3t)^{\frac{1}{3}}} \quad \text{with} \quad z := \frac{x}{(3t)^{\frac{1}{3}}}, \tag{1.0.3}$$

one obtains exactly that $w(z)$ solves the Painlevé II equation (1.0.1) with α determined as constant of integration (for more details see [1]). This connection is particularly relevant since it allows to define the so called *higher order* analogues of the Painlevé II equation. Indeed, in the study of integrable PDEs, for which the most prominent example is indeed the Korteg De Vries equation, one can often construct in natural way higher order equations that commute among themselves (see [89] for the KdV case). The sequence of equations obtained in this way is the hierarchy associated to the relevant PDE. In our case of interest, starting from the modified KdV hierarchy (2.2.12), which construction is induced by the one of the KdV hierarchy (2.2.8) via a Miura transformation, one can apply a self-similarity reduction (similar to the one defining the reduction of the modified KdV equation to the Painlevé II equation) to all the other members of the modified KdV hierarchy. This procedure results in a sequence of nonlinear ordinary differential equations of increasing order, the first being the Painlevé II equation (1.0.1). Their collection is called the Painlevé II hierarchy (and it is compactly written in equation (2.2.22)).
The relation between Painlevé equations and isomonodromic deformations was first investigated in great generality by the Japanese school in a series of papers [66, 63, 64] and, almost simultaneously, but with specific focus on the Painlevé II case by Flaschka and Newell in [36, 37]. Essentially, isomonodromic deformations describes (for generic rank N) all possible linear system of ODEs

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi \tag{1.0.4}$$

with $A(\lambda)$ a rational matrix with fixed number of poles each one with fixed multiplicity, sharing the same set of *essential monodromy data*. This set of data is composed by some matrices that partially describes the local behaviors of the solution Ψ near the singularities of the matrix coefficient $A(\lambda)$. It turned out that this description can be made by looking at the coefficient matrix $A(\lambda)$ as depending on certain extra parameters $A(\lambda, s)$*, and studying the variations w.r.t. these parameters that preserve the required set of data. One of the main results proved in [66] was that these monodromy preserving deformations are equivalent to some nonlinear equations that the entries of the matrix coefficient $A(\lambda)$ should solve, w.r.t. the deformation parameters. For certain specific cases (choosing number and type of poles), these nonlinear equations coincide with the Painlevé equations. In the modern language, this result is usually stated as the fact that Painlevé equations admit Lax pair representations in terms of isomonodromic deformations. This means that for each of them there exist a pair of matrices $A(\lambda, s), L(\lambda, s)$ such that the compatibility condition of the system

$$\frac{d\Psi}{d\lambda} = A(\lambda, s)\Psi, \quad \frac{d\Psi}{ds} = L(\lambda, s)\Psi, \tag{1.0.5}$$

*In general s could be either a scalar or a finite dimensional vector of independent parameters.

i.e. the equation obtained by cross-differentiation

$$\frac{dA}{ds} - \frac{dL}{d\lambda} + [A, L] = 0, \tag{1.0.6}$$

is equivalent to the relevant Painlevé equation. The existence of Lax pairs for the Painlevé equations allows from a certain perspective to put them into the wide framework of Integrable Systems.

For what concerns the Painlevé II equation (1.0.1), there are actually two *independent*† rank two Lax pairs: one with only one irregular singularity at ∞ and a regular one at 0 (the Flaschka-Newell Lax pair [36]) and one with only one irregular singularity at ∞ (the Jimbo-Miwa-Ueno Lax pair [66]). Some years ago, the work [30] proved that every higher order analogue of the Painlevé II equation admits an isomonodromic Lax pair, that generalizes the Flaschka-Newell one. This is indeed very useful in our studies.

With this panorama in mind, the thesis explored the following directions. On the one hand, we found generalizations of the Tracy-Widom formula for some solutions of *new* Painlevé II equations, in particular matrix-valued and integro-differential higher order analogues, in correspondence with the Fredholm determinants of higher order, matrix-valued and finite-temperature, generalizations of the Airy kernel. Motivations include, but they are not limited to, the fact that these generalizations of the Airy kernel can be used in the theory of determinantal point processes (e.g. [16]), and also in statistical mechanics and random matrix theory (e.g. [81, 5, 68]). The detailed results are stated in Corollary 6.0.2 in Chapter 6 for the matrix-valued case and in Theorem 7.0.7 in Chapter 7 for the finite-temperature case. In order to obtain both of these results, the existence of a Lax pair for the matrix-valued and the integro-differential Painlevé II hierarchies, studied in Chapter 6 and 7 respectively, is fundamental. Their Lax representations are indeed the keys to pass from the study of the relevant generalizations of the Airy kernel, via a Riemann-Hilbert approach, to the definition of some particular solutions of the Painlevé II hierarchy involved. The methodology used in both cases is very similar, even though the one in Chapter 7 is more technical than the one in Chapter 6, and it relies on the well known theory of IIKS integrable operators [60]. This theory can be indeed used or generalized for the study of the Fredholm determinants of the higher order, matrix-valued and finite temperature, analogues of the Airy kernel we are interested in. The fundamental idea is to associate a parametric Riemann-Hilbert problem to the prescribed operator and to study its Fredholm determinant through it. At the same time, the solution of the relevant Riemann-Hilbert problem can also be used to provide the Lax pairs, in our specific case isomonodromic ones, that will be indeed behind the Painelvé II hierarchies of interest. The most prominent difference between Chapter 6 and Chapter 7 is then on the type of Riemann-Hilbert problem that will be associated to the relevant operator: in the first case a standard matrix-valued Riemann-Hilbert problem while in the

†Here independent means that the sets of essential monodromy data of the two systems are not isomorphic, thus there exist no gauge transformation that send one system into the other.

second case an operator-valued one.
However, we notice that the original Tracy-Widom formula was obtained by the authors [103] through a totally different procedure. Other authors later re-derived their formula by using the Riemann-Hilbert approach (e.g. [71]) and this approach has been used in order to derive analogue Tracy-Widom formula for some (scalar) higher order Painlevé II transcendents, in the recent work [26]. For this reason we adopted the same method for our purposes in Chapter 6 and 7.
On the other hand, we studied the Poisson-symplectic structure of the monodromy manifold associated to a system of linear ODEs with polynomial matrix coefficient (thus having only an irregular singularity at ∞), originally introduced by Flaschka and Newell in [37]. This case is indeed underlying the Painlevé II hierarchy (at least the homogeneous one). This particular case of monodromy manifold, called Stokes manifold, is the simplest example of what is known now as a *wild character variety.* For the case of regular singularities, the geometry of monodromy manifolds is encoded by character varieties of (appropriately) punctured Riemann spheres. The character varieties of Riemann surfaces in general are known to be Poisson manifolds, thanks to Goldmann work [49]. Instead, the monodromy manifolds associated to systems carrying on irregular singularities are more complicated, because of the presence of the Stokes phenomenon around each irregular singularity. During the last decades, they were studied in great generality and with particular focus on their Poisson structure by Boalch [17, 18, 19]. In Chapter 8 we prove that this particular case of monodromy manifold, the Stokes manifold, is indeed a symplectic manifold, see Theorem 8.1.5. Moreover, in Lemma 8.2.7 we provide explicit log-canonical coordinates for the symplectic-Poisson structure, that are shown to linearize the original Flaschka-Newell Poisson structure, as follows from Theorem 8.4.3. The log-canonical variables used in this context are related to a cluster algebra of a certain type. Relations between cluster algebras and character varieties, are known and have been largely studied by Fock and Goncharov [38] but without specific reference to monodromy manifolds. Recently, their formalism was also used to find log-canonical coordinates for the Goldmann Poisson structure of character varieties of arbitrary punctured Riemann surfaces [14]. Also, cluster algebras were already known to be connected with the Stokes phenomenon, but the one arising in WKB analysis [72] (not the classical one that we are going to treat here). For all these reasons, cluster algebras were in some way expected to appear also in the context of wild character varieties, such as our Stokes manifolds.

Outline

The thesis is essentially divided in two parts. The first part is composed by the first four chapters which are devoted to introduce the basic objects of the study and to motivate it, to review the fundamental results that relate these objects and to recall the classical methods used to achieve these classical results. The second part contains instead the original contributions obtained in the works [101, 24, 15], that are distributed in the last three chapters. In particular the thesis is organised as follows:

(1) In Chapter 1 we explain how the scalar Painlevé II hierarchy is constructed and we review the Tracy-Widom formula and its generalization for the higher order members of the hierarchy, concerning some Hastings-McLeod type solutions of the hierarchy. In Chapter 2 we summarise some basic facts about the theory of determinantal point processes, with particular focus on its application in random matrix theory. This will be done in order to finally explain how the Tracy-Widom formula relates some Painlevé II transcendent to random matrix theory. These two chapters together essentially introduce the objects we want to study and the main motivations.

(2) Chapters $3, 4$ are focused on the classical techniques that we are going to use or generalize in the chapters thereafter in order to achieve our results. Specifically, Chapter 3 introduces Riemann-Hilbert problems with particular focus on the ones appearing in relation with integrable operators of IIKS type. Chapter 4 is instead a compact review of some results in the theory of isomonodromic deformations. The aim of the chapter is twofold: one is to explain how the Painlevé II hierarchy can be deduced as isomonodromic deformation of certain types of systems, the other is to review the concepts of monodromy data that will be used in Chapter 8 to construct the Stokes manifolds.

(3) In Chapter 6 we give the proof of the result contained in [101]: a generalization of the Tracy-Widom formula relating the Fredholm determinants of matrix-valued higher order Airy kernels analogues to some particular solutions of a matrix-valued Painlevé II hierarchy.

(4) In Chapter 7 we go through the proof of the main result of [24]: this time we obtain a generalization of the Tracy-Widom formula for a finite temperature version of the higher order Airy kernels together with a particular solutions of an integro-differential Painlevé II hierarchy. Even though the results of this chapter and the previous one are comparable, the proof of the second one requires more complicated techniques. Indeed in this case, matrix-valued Riemann-Hilbert problems are replaced by operator-valued ones. Part of the work is then devoted to establish the existence, uniqueness and other properties of their solutions (well-known for the matrix-valued case).

(5) Finally in Chapter 8 we explain most of the content of [15]. We prove that the Stokes manifold associated to a polynomial system of ODEs of generic degree K and rank 2 is indeed a symplectic manifold. In particular we find log-canonical coordinates for the induced Poisson structure, that provide a linearization of the Flaschka-Newell Poisson structure originally discovered on this manifold. The relation with a cluster algebra of A_{2K} type is also discussed.

Chapter 2

The Painlevé II hierarchy

The starting point of our study is the scalar Painlevé II hierarchy, that in this chapter we are going to introduce. Indeed, the construction of the scalar Painlevé II hierarchy will inspire in Chapters 6 and 7 the one of some new Painlevé II hierarchies, analogue of the classical one described in this chapter, but in a matrix context and in an integro-differential context respectively. To start with, we first briefly recall who are the so called Painlevé equations. Then we are going to focus on the second Painlevé equation and after a brief study of its properties, we are going to see how, thanks to its relation with the modified KdV equation, the Painlevé II hierarchy is defined.

2.1 The Painlevé II equation

2.1.1 Introduction to the Painlevé equations

With Painlevé equations we refer to the following list of six nonlinear ordinary differential equations (following [39]) for a certain function $w = w(z)$

$$\text{PI} \quad w'' = 6w^2 + z, \tag{2.1.1}$$

$$\text{PII} \quad w'' = 2w^3 + zw + \alpha, \tag{2.1.2}$$

$$\text{PIII} \quad w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{2.1.3}$$

$$\text{PIV} \quad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{2.1.4}$$

$$\text{PV} \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{2.1.5}$$

$$\begin{aligned} \text{PVI} \quad w'' = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right), \end{aligned} \tag{2.1.6}$$

where we used the notation $' = \dfrac{d}{dz}$ and $\alpha, \beta, \gamma, \delta$ are constant parameters. These equations result as the solution of a classification problem in ODE theory, that was first posed more than one century ago in [95]. The problem was to find all the second order ordinary differential equations of the form

$$w'' = F(w, w', z) \tag{2.1.7}$$

with the function $F(w, w', z)$ being rational in w, w' and analytic in z and with solutions satisfying the so called Painlevé property. A function w, solution of a certain ordinary differential equation, is said to have the Painlevé property, if it does not have movable critical points. This means essentially that the critical points of the solution, if any, only depend on the equation itself and not on the initial or boundary conditions. From another point of view, the Painlevé property allows to construct new special functions as solution of specific second order nonlinear ODEs, as it is done for many special functions coming from linear ODEs, as the Bessel and the hypergeometric functions.

Painlevé first and then Fuchs and Gambier ([94, 42, 44]) studied this problem and concluded that, up to Möbius transformations, there were just fifty equations corresponding to such requests. Furthermore, they proved that these fifty equations can be either integrated in terms of known special functions or reduced to one of the six equations in the list above. The six new nonlinear ordinary differential equations arising out in this way are then called the Painlevé equations. Their solutions, the Painlevé transcendents, are considered as new transcendental functions (for more details we refer to the monograph [57]).

Even though the Painlevé equations arose in the context of a very analytical problem, they appeared then in many other fields of applied mathematics such as statistical mechanics, quantum field theory and nonlinear waves. In particular some of the Painlevé equations were found in connection with partial derivative equations solvable through inverse scattering method ([1] for a classical reference). In the specific case of the Painlevé II equation (2.1.2), that will be our case of study, the relevant PDE is the modified KdV equation. As we will see in the next section, the relation between the modified KdV equation and the Painlevé II equation is indeed the key to construct the Painlevé II hierarchy. But this is far from being an isolated case, and the study of links between Painlevé equations and PDEs is still very popular. Indeed, there is still an open conjecture among Painlevé type equations and integrable PDEs saying that (quoting from [1] pg. 362)

"*Every ODE which arises as similarity reduction of a completely integrable PDE is of Painlevé type, up to transformation of variables*".

One of the first properties of Painlevé transcendentes to be discovered, was that even though the general solutions of the Painlevé equations are transcendental, some particular solutions can be written explicitly. Notice that in every equation (2.1.2)-(2.1.6) there is at least one parameter. Choosing particular values for these parameters it is possible to find rational solutions, or other solutions in terms of known special functions for all the Painlevé equations from II to VI. The presence of these parameters in the Painlevé equations is actually even more relevant. Given a solution of a Painlevé equation for fixed values of the parameters, one can generate other solutions of the same equation with different values of the parameters, or even solutions of a different Painlevé equation, starting from the given one. This phenomenon is usually referred as the Bäcklund transformations of the Painlevé equations, and it is a very useful tool to generate sequences of solutions. These transformations were already discovered by Painlevé and Gambier in the first works on Painlevé equations ([57, 44]) and then studied in the following years. We refer to [39] (Part I, Chapter 6) for a compact review on the subject using the formalism of Lax pairs of Painlevé equations and Schlesinger transformations.

2.1.2 Known solutions of the Painlevé II equation

We are now going to focus on the Painlevé II equation and we start by listing its known solutions. The Painlevé II equation admits two types of Bäcklund tranformations for integer or semi-integers values of the parameter α. These transformations generate respectively sequences of rational solutions and Airy type solutions (that are obtained as ratio of the Airy function and its derivatives). The main results, that were first proven by Airault in [4], are resumed in the following theorems.
For the rational solutions corresponding to integer values of the parameter α the statement is as follows.

Theorem 2.1.1 (Theorem 2 [4]). *The Painlevé II equation* (2.1.2) *has rational solution if and only if α is an integer, in particular for $\alpha = 0$ this solution is trivial. Then for $n \geq 1$, equation* (2.1.2) *admits a solution w_n with $\alpha = n$ that is written as*

$$w_n = -\frac{u_n'}{u_n} + \frac{u_{n-1}'}{u_{n-1}}, \tag{2.1.8}$$

where the functions u_n are obtained through the following recursion

$$u_{n+1}u_{n-1} = C(-2\frac{d^2}{dz^2}\log u_n + z)u_n^2, \tag{2.1.9}$$

with initial conditions $u_0 = 1$ and $u_1(z) = z$. Finally, when $\alpha = -n$ then $w_{-n} = -w_n$.

For the Airy type solutions corresponding instead to semi-integers values of α, we have the following statement. Notice that here the Airy function is defined as a particular solution $\gamma(z)$ of the equation $\gamma'' = -\frac{z}{2}\gamma$.

Theorem 2.1.2 (Theorem 3 [4]). *In the case where α is a semi integer, there is a solution of equation* (2.1.2) *that is a rational function of the Airy function γ and its derivatives. In particular*

- *for $\alpha = -\frac{1}{2}$ then $w_0 = \frac{d}{dz}\log\gamma$;*
- *for $\alpha = -\frac{1}{2} + n$ then $w_n = \frac{u_{n-1}'}{u_{n-1}} - \frac{u_n'}{u_n}$;*

- *for* $\alpha = \frac{1}{2} - n$ *then* $w_{-(n-1)} = -w_n$.

Here the functions u_n *are obtained from the same recursive equation* (2.1.9) *but with initial conditions* $u_0 = \exp\left(\frac{z^3}{24}\right)$ *and* $u_1 = \gamma u_0$.

These results were proved again some years later through a totally different method by Flaschka and Newell in [36]. Their new procedure is called isomonodromy method and it is perhaps the most powerful tool that has been developed in order to study Painlevé transcendents, as the monograph [39] largely shows. This method is based on the fact that the Painlevé II equation has Lax pairs in terms of isomonodromic deformations of certain rank 2 systems of linear ODEs in the complex plane. The precise meaning of that will be discussed in Chapter 5. Using this method, Flaschka and Newell were able to recover the rational and the Airy type solutions found by Airault and they expressed them as finite-size determinants. Their result, first proved in Sec. 3F (*iii*) of [36] for the rational solutions of the Painlevé II equation (2.1.2), can be rewritten as follows.

Theorem 2.1.3 (Theorem 2.4 [29])**.** *Let* $p_k(z)$ *be the polynomial defined by*

$$\sum_{k=0}^{\infty} p_k(z)\lambda^k = \exp\left(z\lambda - \frac{4}{3}\lambda^3\right), \quad \text{with } \ p_k(z) = 0 \ \text{ for } \ k < 0$$

and let τ_n *be the* $n \times n$ *determinant*

$$\tau_n(z) := \begin{vmatrix} p_n(z) & p_{n+1}(z) & \dots & p_{2n-1}(z) \\ p_{n-2}(z) & p_{n-1}(z) & \dots & p_{2n-3}(z) \\ \vdots & & & \vdots \\ p_{-n+2}(z) & p_{-n+3}(z) & \dots & p_1(z) \end{vmatrix}.$$

Then for $\alpha = n$, $n \geq 1$ *the rational solutions of* (2.1.2) *are written as in the form*

$$w_n(z) = \frac{d}{dz} \log \frac{\tau_{n-1}(z)}{\tau_n(z)}. \tag{2.1.10}$$

This result was also proved later in [70], exploiting the relation between the Painlevé II and the KdV equation.
Instead, for the Airy type solutions of equation (2.1.2) the result proved by [36] (Sec. 3F (*iv*)) can be formulated as follows.

Theorem 2.1.4 (Theorem 2.5 [29])**.** *Let* τ_n *be the* $n \times n$ *determinant*

$$\tau_n(z) := \det\left[\frac{d^{j+k}}{dz^{j+k}}\gamma(z)\right]_{j,k=0}^{n-1}, \quad n \geq 1 \ \text{ and } \ \tau_0(z) = 1.$$

Then for $\alpha = n - \frac{1}{2}$ *and* $n \geq 1$ *the Airy type solutions of the Painlevé II equation* (2.1.2) *are* *written in the form*

$$w_n(z) = \frac{d}{dz} \log \frac{\tau_{n-1}(z)}{\tau_n(z)}. \tag{2.1.11}$$

The original works done in this thesis and contained in Chapter 6 and 7, are based indeed on the isomonodromy method. However, these original results generalize (to a matrix-valued and integro-differential case) the existence of a third type of solution of the Painlevé II equation, different from the two family of solutions introduced until now. Consider the homogeneous Painlevé II equation (2.1.2)

$$w'' = 2w^3 + zw \tag{2.1.12}$$

i.e. the special case $\alpha = 0$ in equation (2.1.2). It was first discovered in [56] that this equation together with a boundary condition admits a particular solution, nowadays known as the Hastings-McLeod solution. Their main result is resumed in the following theorem.

Theorem 2.1.5 (Theorem 1 [56])**.** *Consider the homogeneous Painlevé II equation* (2.1.12) *together with the boundary condition*

$$w(z) \to 0 \ \ \textit{for} \ \ z \to +\infty. \tag{2.1.13}$$

Then

1. *any solution of the boundary problem* (2.1.12), (2.1.13) *is asymptotic to* $k\mathrm{Ai}(z)$ *at* $z \to +\infty$*, for some* $k \in \mathbb{R}$*.*
2. *Conversely, for any* k *there is a unique solution of* (2.1.12) *which is asymptotic to* $k\mathrm{Ai}(z)$*.*

Furthermore, for $|k| < 1$ *the solution which has asymptotic* $k\mathrm{Ai}(z)$ *exists for every* z *and as* $z \to -\infty$ *it behaves as*

$$w(z) \sim d|z|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{3}{4}d^2 \log|z| - c\right) \tag{2.1.14}$$

for some constants c, d *depending on* k*.*

Remark 2.1.6. *In the statement of Theorem 2.1.5 and below we denote by* $\mathrm{Ai}(z)$ *the Airy function but with a slightly different convention w.r.t. the function* $\gamma(z)$ *defined previously. Here* $\mathrm{Ai}(z)$ *is intended as a particular solution of* $\phi'' = z\phi$*.*

These solutions were not known to admit a (Fredholm) determinantal representation until the work of Tracy and Widom [103]. They proved indeed that also these solutions can be written in terms of certain determinants, but in a very different sense than the determinants for the rational and the Airy type solutions. They proved in [103] that the Hastings-McLeod solution of the Painlevé II equation (2.1.12) with asymptotic $w(z) \sim \mathrm{Ai}(z)$ is related to the Fredholm determinant of the Airy kernel. Their result is resumed in the following theorem.

Theorem 2.1.7 ([103])**.** *The Hastings-McLeod solution of the homogeneous Painlevé II equation* (2.1.12)*, i.e. its distinguished solution with asymptotic behavior* $w(z) \sim k\mathrm{Ai}(z)$*, is written through the formula*

$$\frac{d^2}{dz^2} \log \det(1 - k^2 \mathcal{K}_{Airy}|_{(z,\infty)}) = -w^2(z). \tag{2.1.15}$$

where $\mathcal{K}_{Airy}|_{(z,\infty)}$ *is considered as the integral operator acting on* $L^2((z, +\infty))$ *with kernel*

$$K_{Airy}(x, y) := \int_0^{+\infty} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt. \tag{2.1.16}$$

We will discuss again about this result at the end of Chapter 3 and there we will explain the reason why this result is so interesting from the point of view of applications. Nevertheless, formula (2.1.15) is exactly the one we have generalized in Chapters 6 and 7 for certain solutions of a matrix and an integro-differential Painlevé II hierarchy, with prescribed asymptotic behavior in terms of generalized Airy functions.

Remark 2.1.8. *We stress, again, that the procedure used by Tracy and Widom in [103] does not make use of the Lax pair representation of the Painlevé II equation at all. Nevertheless, the extension of their result to certain solutions of the Painlevé II hierarchy, that was studied in [26] and that we will discuss thereafter, deeply rely on the isomonodromic representation of the hierarchy and the theory of integrable operators of IIKS type. And so also our generalizations discussed in Chapters 6 and 7 do. However, a proof of the Tracy-Widom result based on this technique was already given in some previous papers [71, 55].*

2.2 Construction of the Painlevé II hierarchy

In this section we are going to define the classical scalar Painlevé II hierarchy. In order to do that, we first need to establish the relation between the Painlevé II equation and the modified KdV equation. We will see that the definition of the Painlevé II hierarchy then follows in a very natural way, once that the definition of the modified KdV hierarchy is established.

2.2.1 Self-similarity reduction of mKdV equation

We start by introducing the KdV equation. Given a function of two variables $u = u(t, x)$, the KdV equation is the following nonlinear partial differential equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.2.1}$$

where subscripts denote partial differentiation. This equation was derived from the physical description of the evolution of long, one dimensional, surface waves propagating in shallow waters with small amplitude by Korteweg and De Vries in [74]. One of the scope of their

work was to find wave equations admitting solitary wave solutions, i.e. waves preserving their own form and propagating with uniform velocity, first observed and then studied by Russel [98]. The KdV equation has been largely studied in the years after its discovery and a lot of interesting mathematical properties were proved, here we cite only few of them. The KdV equation is the prototype of PDE solvable through the Inverse Scattering Method, it admits solitonic solutions (solitary waves solutions that do not change their shape and velocity after interaction with other solitary waves), it has infinitely many commuting symmetries and it is perhaps the main example of infinite dimensional integrable Hamiltonian system.
While studying some remarkable transformation of the KdV equation in the paper [89], Miura discovered the modified KdV equation. This equation is defined for a function of two variables $v = v(t, x)$ as the following partial derivative equation

$$v_t + v_{xxx} - 6v^2 v_x = 0. \tag{2.2.2}$$

We say that the modified KdV equation and the KdV equation are related through a Miura transformation. More precisely, this relation means that for any solution v of the modified KdV equation one can define the Miura transform $u \coloneqq v_x - v^2$, and verify that the function u now solves the KdV equation (2.2.1).

Remark 2.2.1. *By direct computation, replacing $u = v_x - v^2$ in the KdV equation* (2.2.1) *then we get the following equation*

$$\left(\frac{\partial}{\partial x} - 2v\right)\left(v_t - 6v^2 v_x + v_{xxx}\right) = 0. \tag{2.2.3}$$

And this is of course an identity since v solves the modified KdV equation (2.2.2)*. This means that from a solution of the modified KdV equation we can always construct a solution of the KdV equation, but the converse is not true (see also [1] pg.* 23*). In particular, not all the solutions of the KdV equation are obtained from solutions of the modified KdV equation (for more details, have a look at [2]).*

Now we are going to show that the Painlevé II equation can be obtained by self-similarity reduction of the modified KdV equation. Indeed, consider a solution v of the modified KdV equation, having the following form

$$v(t, x) \coloneqq \frac{w(z)}{(3t)^{\frac{1}{3}}} \quad \text{with} \quad z \coloneqq \frac{x}{(3t)^{\frac{1}{3}}}. \tag{2.2.4}$$

Then the modified KdV equation solved by this $v(t, x)$ is reduced to an ordinary differential equation for w w.r.t. the new variable z. In particular it coincides with

$$w_{zzz} - z w_z - w - 6w^2 w_z = 0 \tag{2.2.5}$$

that is exactly the derivative of the Painlevé II equation. Thus we conclude that the function w defined in (2.2.4) solves the Painlevé II equation (2.1.2) with α being an arbitrary integration constant.

Remark 2.2.2. *There is another similarity reduction that relates directly the KdV equation to an equation solvable in terms of solutions of the Painlevé II equation (see [1], pg.* 99*, for more details), but for the definition of the Painlevé II hierarchy is easier to proceed with the relation between the Painlevé II equation and the modified KdV equation.*

2.2.2 The KdV and modified KdV hierarchy

The KdV hierarchy is an infinite set of PDEs for a function depending on infinitely-many parameters $u = u(x = -t_1, t = t_2, t_3, t_4, \dots)$. With this notation, the first member of the hierarchy is an identity and the second one coincides with the KdV equation itself. These PDEs have the fundamental property to commute one with another, giving a system of compatible equations. We remark that this is also equivalent to say that the KdV equation admits infinitely many commuting symmetries. Even though the classical definition of the KdV hierarchy requires the introduction of the algebra of pseudo-differential operators (following the classical reference here [90]), we are going to take a shortcut and give an equivalent definition that involves the Lenard recursion, as introduced in [80], [89].

Definition 2.2.3. *The sequence of Lenard recursion operators acting on a function u is obtained through the following recursion*

$$\begin{cases} \dfrac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = \left(\dfrac{\partial^3}{\partial x^3} + 4u\dfrac{\partial}{\partial x} + 2u_x\right)\mathcal{L}_n[u], \quad n \geq 0 \\ \mathcal{L}_0[u] = \dfrac{1}{2}. \end{cases} \tag{2.2.6}$$

The quantities $\mathcal{L}_n[u]$ generated from this recursion relation are all differential polynomials in u and its x-derivatives until order $2n-2$. The proof of this fact is based on the use of the conserved quantities for the KdV equation (see Theorem 3.1 in [80] for more details).

Example 2.2.4. *Here is a list of the differential polynomials $\mathcal{L}_n[u]$ for the first few values of n, setting all the constants of integration to zero.*

$$\begin{aligned} n = 1: &\qquad \mathcal{L}_1[u] = u, \\ n = 2: &\qquad \mathcal{L}_2[u] = u_{xx} + 3u^2, \\ n = 3: &\qquad \mathcal{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3, \end{aligned} \tag{2.2.7}$$

Using Definition 2.2.3 we can finally construct the KdV hierarchy as follows

$$u_{t_{n+1}} + \frac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = 0, \quad n \geq 0, \tag{2.2.8}$$

where the subscript t_{n+1} indicates the partial derivation w.r.t. t_{n+1}.

Example 2.2.5. *Here is a list of the first members of the KdV hierarchy. For $n = 0$ we have a trivial identity, with $t_1 = -x$, and for $n = 1$ we recover the KdV equation, with $t_2 = t$.*

$$n = 0: \qquad u_{t_1} + u_x = 0, \tag{2.2.9}$$

$$n = 1: \qquad u_{t_2} + 6uu_x + u_{xxx} = 0, \tag{2.2.10}$$

$$n = 2: \qquad u_{t_3} + u_{xxxxx} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x = 0. \tag{2.2.11}$$

The modified KdV hierarchy is then constructed from the KdV hierarchy through the same Miura transformation introduced before, i.e. by taking $u = v_x - v^2$, and looking at the equation for v. Indeed, the modified KdV hierarchy is defined as follows

$$v_{t_{n+1}} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_n\left[v_x - v^2\right] = 0, \quad n \geq 1, \tag{2.2.12}$$

where for $n = 1$ the modified KdV equation (2.2.2) is recovered.

Remark 2.2.6. *Consider the Miura transformation $u = v_x - v^2$ and replace it into the definition of the differential operator of order 3 appearing in the Lenard recursion (2.2.6), namely*

$$H := \frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x. \tag{2.2.13}$$

By direct computation, one can check that under the Miura transformation, H is factorized in the following way

$$H = \left(\frac{\partial}{\partial x} - 2v\right)\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right). \tag{2.2.14}$$

Thus, when we replace the Miura transformation in the definition of the KdV hierarchy (2.2.8) we obtain, generalizing what was observed in Remark 2.2.1, that the n-th member of the KdV hierarchy is transformed into

$$\left(\frac{\partial}{\partial x} - 2v\right)\left(v_{t_{n+1}} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_n\left[v_x - v^2\right]\right) = 0, \tag{2.2.15}$$

where we only used the property (2.2.14). As a byproduct we conclude that, given a solution v of the n-th member of the modified KdV hierarchy then $u = v_x - v^2$ solves the n-th member of the KdV hierarchy, but the converse, again, is in general not true.

Example 2.2.7. *The first members of the modified KdV hierarchy are as follows*

$$n = 1: \qquad v_{t_2} + v_{xxx} - 6v^2 v_x = 0, \tag{2.2.16}$$

$$n = 2: \qquad v_{t_3} + v_{xxxxx} - 10v^2 v_{xxx} - 40v_x v_{xx} - 10v_x^3 + 30v^4 v_x = 0, \tag{2.2.17}$$

$$\begin{aligned} n = 3: \qquad & v_{t_4} + v_{xxxxxxx} - 14v^2 v_{xxxxx} - 84vv_x v_{xxxx} - 140vv_{xx}v_{xxx} \\ & - 126v_x^2 v_{xxx} - 182v_x v_{xx}^2 + 70v^4 v_{xxx} + 560v^3 v_x v_{xx} + 420v^2 v_x^3 - 140v^6 v_x = 0 \end{aligned} \tag{2.2.18}$$

2.2.3 The Painlevé II hierarchy

We are now ready to define the Painlevé II hierarchy. We will first follow the construction done in [78] and in the end we will briefly see another construction already given by [4]. In order to do that we will consider an appropriate self-similarity reduction for each member of the modified KdV hierarchy (2.2.12), analogue to the one we considered for the case $n = 1$ in (2.2.4).
For every $n \geq 1$, we define v a solution of the n-th member of the modified KdV hierarchy, of the following form

$$v(x, t_{n+1}) := \frac{w(z)}{((2n+1)t_{n+1})^{\frac{1}{2}}} \quad \text{with} \quad z := \frac{x}{((2n+1)t_{n+1})^{\frac{1}{2}}}. \tag{2.2.19}$$

We also define for every $n \geq 0$ the quantities $\hat{\mathcal{L}}_n[w]$, as the differential polynomials in w obtained by the same recursion relation (2.2.6) but replacing the variable x with the variable z.
One can prove by induction over n (see Proposition 2.2 in [78]) the following equality

$$\mathcal{L}_n\left[v_x - v^2\right] = \frac{1}{((2n+1)t_{n+1})^{\frac{2n}{2n+1}}}\hat{\mathcal{L}}_n\left[w_z - w^2\right] \quad \text{for every} \quad n \geq 1. \tag{2.2.20}$$

By replacing in the n-th member of the modified KdV equation (2.2.12) the form (2.2.19) of v and by using the relation (2.2.20), this equation is transformed into an ordinary differential equation for the function $w(z)$ that is

$$-zw_z - w + \frac{d}{dz}\left(\frac{d}{dz} + 2w\right)\hat{\mathcal{L}}_n\left[w_z - w^2\right] = 0, \tag{2.2.21}$$

that corresponds indeed to the derivative of

$$\left(\frac{d}{dz} + 2w\right)\hat{\mathcal{L}}_n\left[w_z - w^2\right] = zw + \alpha_n, \tag{2.2.22}$$

where α_n is an arbitrary constant of integration.

Definition 2.2.8. *The Painlevé II hierarchy is the infinite set of ODEs given by equation* (2.2.22) *for any* $n \geq 1$, *obtained after integration of the self-similarity reduction of the modified KdV hierarchy.*

We underline that the n-th member of the Painlevé II hierarchy (2.2.22) is a $2n$ order nonlinear ODE for the function $w(z)$. The first member of the hierarchy, as it is shown in the example that follows, coincides with the Painlevé II equation (2.1.2).

Example 2.2.9. *The first members of the Painlevé II hierarchy* (2.2.22) *are given by*

$$n = 1: \qquad w_{zz} - 2w^3 = zw + \alpha_1, \tag{2.2.23}$$

$$n = 2: \qquad w_{zzzz} - 10ww_z^2 - 10w^2 w_{zz} + 6w^5 = zw + \alpha_2, \tag{2.2.24}$$

$$n = 3: \qquad w_{zzzzzz} - 14w^2 w_{zzzz} - 56ww_z w_{zzz} - 70w_z^2 w_{zz} - 42ww_{zz}^2 + 70w^4 w_{zz} + 140w^3 w_z^2 - 20w^7 = wz + \alpha_3. \tag{2.2.25}$$

The definition of the Painlevé II hierarchy through equation (2.2.22) completely relies on the definitions of the KdV and modified KdV hierarchies as given in (2.2.12), (2.2.12). But the formalism given by the Lenard recursion operators is not the only one that is used to describe the KdV and consequently the modified KdV hierarchies. In the following paragraph, we are going to introduce an alternative formalism.

An alternative definition of the PII hierarchy Here we are going to define the Painlevé II hierarchy through the formalism used by Airault in [4]. In this other formalism, introduced by [91], [3], one defines the following pseudo-differential operator of order 2

$$\mathcal{S}_w \coloneqq 4w^2 + 4w_z \left(\frac{d}{dz}\right)^{-1} w - \frac{d^2}{dz^2} \tag{2.2.26}$$

where $\left(\frac{d}{dz}\right)^{-1}$ stands for the formal z-antiderivative, such that $\left(\frac{d}{dz}\right)^{-1} \frac{d}{dz}(f) = f$ for every function f.
The Painlevé II hierarchy is then defined in [4] by the following sequence of equations

$$\left(\frac{d}{dz}\right)^{-1} \mathcal{S}_w^{n-1}\left[w_z\right] + zw + \delta_{n-1} = 0, \quad n \geq 2 \tag{2.2.27}$$

where δ_k are arbitrary constants of integration.

Remark 2.2.10. *One can check that the first members of the Painlevé II hierarchy obtained through the definition* (2.2.22) *and computed in the Example 2.2.9 coincide with the ones obtained through the definition of the hierarchy* (2.2.27)*.*

The procedure followed to obtain this alternative definition of the Painlevé II hierarchy is similar to the previous one, but it starts from a different definition of the KdV hierarchy. Given a function u, define the following pseudo-differential operator

$$\mathcal{R}_u \coloneqq \left(2\left(u + \frac{\partial}{\partial x} u \left(\frac{\partial}{\partial x}\right)^{-1}\right) - \frac{\partial^2}{\partial x^2}\right) \tag{2.2.28}$$

One can then define the KdV hierarchy, as Airault did in [4], through the following equations

$$(2m-1)u_{t_m} = \mathcal{R}_u^{m-1}\left[u_x\right], \quad m \geq 2. \tag{2.2.29}$$

By looking for self-similarity solutions of the form $u(t_m, x) := t_m^{-\frac{2}{2m-1}} q(z)$ where $z := x t_m^{-\frac{1}{2m-1}}$ similarly to (2.2.19), one can reduce the above equation to an ODE for q. In particular, it follows that q solves

$$\mathcal{R}_q^{m-1}\left[q_z\right] + 2q + q'z = 0,$$

where now $\mathcal{R}_q$ is intended as the same operator given in (2.2.28) but replacing x by z and u by q. Finally, using the Miura transformation at this level and writing $q := w_z + w^2$, the function w is then shown to satisfy equation (2.2.27).

Remark 2.2.11. *Using the operator $\mathcal{R}_u$ one can define the following sequence*

$$X_m\left[u\right] = \mathcal{R}_u X_{m-1}\left[u\right], \quad m \geq 2 \quad \textit{with} \quad X_1\left[u\right] = u_x. \tag{2.2.30}$$

Up to changing the sign of the term of order 3 in the differential operator H used in the Lenard recursion (2.2.6), we actually have that the recursion for the operators X_m in (2.2.30) is a sort of integrated version of (2.2.6). This follows from the bi-Hamiltonian structure of the KdV hierarchy for which the equality above can be continued into

$$X_m\left[u\right] = \mathcal{R}_u X_{m-1}\left[u\right] = \frac{\partial}{\partial x}\left(\delta H_m\right) \tag{2.2.31}$$

and where δH_m are the Hamiltonian functionals of the KdV hierarchy and they are (up to the sign) the Lenard differential polynomials. The equivalence between the two definitions of the Painlevé II hierarchy (2.2.22) and (2.2.27) is then explained.

Even though the two different definitions of the Painlevé II hierarchy give rise to the same infinite set of ODEs, they are quite different in their usage. We wanted to introduce both of these formalism, since they both will inspire our constructions in the next chapters. In Chapter 6 we consider a matrix Painlevé II hierarchy, that is obtained as a matrix generalization of equation (2.2.22). We introduce a noncommutative version of the Lenard recursion (2.2.6) and we use it to define the new hierarchy. In Chapter 7 instead we define an integro-differential Painlevé II hierarchy that is a generalization of equation (2.2.27). In particular, in this last definition the recursion operator is written as the composition of two pseudo-differential operators of order 1 that reduces to the operator $\mathcal{S}_w$ in (2.2.26) in the case where all the variables commute.

Solutions of the Painlevé II hierarchy The study of solutions of higher order Painlevé II transcendents is in general much more complicated since it requires to solve $2n$-order ODEs. One can ask for instance, whether the known solutions of the Painlevé II equation, the rational, the Airy type and the Hastings-McLeod ones extend in some way to solutions of the entire Painlevé II hierarchy (2.2.22). One answer was recently given in the papers [81, 26] concerning the Hastings-McLeod type solutions. In particular, in the last paper the authors explicitely construct solutions for each member of the homogeneous Painlevé II hierarchy (2.2.22) in relation to the Fredholm determinants of the generalized Airy kernels. The explicit formula describing these solutions recovers the Tracy-Widom formula (2.1.15) for the first member of the hierarchy. Furthermore, the authors of [26] were able to compute the asymptotic behavior of these solutions at $\pm\infty$, in terms of the generalized Airy function $\mathrm{Ai}_{2n+1}(z)$, defined as the real solution with rapid decaying at $+\infty$ of the $2n$-order ODE

$$\frac{d^{2n}}{dz^{2n}}\phi = (-1)^{n+1} z\phi. \tag{2.2.32}$$

Their result can be thus interpreted as an extension of the Tracy-Widom result cited before in Theorem 2.1.7 to all members of the Painlevé hierarchy and it is resumed in the following statement.

Theorem 2.2.12 (Theorem 1.1 [26])**.** *For every $n \geq 1$ and $0 < \rho \leq 1$, there is a real solution w of the n-th member of the homogeneous Painlevé II hierarchy* (2.2.22) *which satisfies*

$$-w^2(z;\rho) = \frac{d^2}{dz^2} \log\det\left(1 - \rho \mathcal{K}_{\mathrm{Ai}_{2n+1}}|_{(z,\infty)}\right), \tag{2.2.33}$$

where $\mathcal{K}_{\mathrm{Ai}_{2n+1}}|_{(z,\infty)}$ is considered as the integral operator acting on $L^2((z,+\infty))$ with kernel

$$K_{\mathrm{Ai}_{2n+1}}(x,y) := \int_0^{+\infty} \mathrm{Ai}_{2n+1}(x+t)\mathrm{Ai}_{2n+1}(y+t)dt. \tag{2.2.34}$$

Furthermore, its asymptotic behavior for $z \to +\infty$ is given by $w(z;\rho) \sim \sqrt{\rho}\mathrm{Ai}_{2n+1}(z)$.

As already underlined for $\rho = 1, n = 1$ this result recovers the one of Tracy and Widom resumed in Theorem 2.1.7. Nevertheless, the authors of [26] used a completely different procedure, that essentially relies on the isomonodromic reprensentation of the Painlevé II hierarchy (2.2.22), that was first described in [30]. This procedure, also known as the Riemann-Hilbert approach, is in principle the same procedure we will use in Chapter 6 and 7. For this reason, we resume the fundamental concepts of their proof in the following paragraph. The starting point is that the Fredholm determinants of the Airy kernels $\mathcal{K}_{\mathrm{Ai}_{2n+1}}$ are equal to the ones of some integral operators in Fourier spaces that are *integrable*, in the sense of the IIKS operators [34], [60]. Essentially, this implies that the existence of their resolvent operators is equivalent to the solvability of a certain Riemann-Hilbert problem. As a byproduct, their Fredholm determinants can be expressed in terms of a quantity related to the solution of the relevant Riemann-Hilbert problem. These classical facts will be reviewed with more details in Chapter 4.

Finally, the last element of the proof is provided by the fact that the solution of the Riemann-Hilbert problem, after some manipulation and rescaling operations, solves two differential equations w.r.t. the parameters involved in the Riemann-Hilbert problem itself. This system actually coincides with the isomonodromic Lax pair for the PII hierarchy (2.2.22) (the one found in [30]).

Remark 2.2.13. *Notice that, prior to [26], the work [81] gave a similar formula for the Fredholm determinants of the higher order Airy kernels. In that case, the functions w in the left hand side of equation* (2.2.33) *are shown to solve a system of hamiltonian equations that coincide for the first values of n with the first members of the Painlevé II hierarchy. However, the precise equivalence between their system and the Painlevé II hierarchy* (2.2.22) *still has to be proved.*

Remark 2.2.14. *Theorem 2.2.12 is just a part of the results contained in [26]. There the authors studied in detail also the asymptotic behavior of these solutions at $-\infty$. As a byproduct they were able to describe the asymptotic behavior at $-\infty$ of the corresponding Fredholm determinants of the higher order Airy kernels. This estimate is also known as large gap asymptotics, and it is in general much more complicated to obtain than the one at $+\infty$, since it involves a strong use of nonlinear steepest descent method.*

In order to obtain the generalizations of Theorem 2.2.12 for the case of a matrix and then an integro-differential Painlevé II hierarchy, in Chapters 6 and 7, we will implement the analogue procedure of [26], resumed in the paragraph above. Respectively, we will deal with a block-matrix and an operator-valued Riemann-Hilbert problem instead of a classical 2×2 matrix-valued Riemann-Hilbert problem. Finally, in these noncommutative contexts we did not try to study the asymptotic behavior at $-\infty$ of the relevant solutions of these hierarchies, so this computation is left as an open problem.

Chapter 3

Determinantal point processes

In this chapter we recall the notion of determinantal point processes (that we denote with the abbreviation DPP from now on). DPP appear in many different fields of mathematics and mathematical physics, such as orthogonal polynomials, number theory, random permutations, random growth models, random matrix theory and statistical mechanics. The main motivation to study DPP is given indeed by their appearance in all these fields of study. In a nutshell, DPP can be intended as spatial random processes (there is no notion of time) which can be entirely described through their correlation functions, which have the peculiarity to be written as finite dimensional determinants involving the kernel of some integral operators. The integral operators are not generic and have to satisfy certain requirements. The integral operator defined through the Airy kernel, that we already introduced in Theorem 2.2.12 at the end of the previous chapter, is an example of such operators. As a byproduct the Fredholm determinants of these operators have an interpretation in terms of relevant probabilistic quantities describing the DPP. This is perhaps the main reason why results such as Theorem 2.1.7 and Theorem 2.2.12 are highly considered: in both cases the integral operators involved actually define DPP. Furthermore, the relevant DPP appear in random matrix theory and statistical mechanics respectively. This kind of results allows to build a bridge between the probabilistic world of DPP and the integrable systems world of Painlevé equations, and this is a powerful motivation to go deeper in this study. While the Fredholm determinant in Theorem 2.1.7 was known to be connected to random matrix theory [103] since the early '90, the one in Theorem 2.2.12 has appeared recently in a model for non-interacting fermions in anharmonic potential first studied in [81]. Also, all the other integral operators studied in chapters 6, 7 of the thesis define DPP. Moreover, the finite temperature higher order Airy kernel studied in Chapter 7 has been found in relation to the finite temperature version of the same fermionic model described above and also appeared in [81].

The Chapter is organized in two section: in the first one we start with an intuitive example of DPP and then we go through the basic definitions and the main results of DPP theory. In the second section we introduce random matrices, focusing in particular on the Gaussian

Unitary Ensemble. We will show how to compute the main relevant quantities such as correlation functions, distributions functions and gap probabilities for the eigenvalues of this ensemble emphasising the determinantal character of some of these quantities. In the end, we will finally introduce the Tracy-Widom distribution and we briefly re-discuss Theorem 2.1.7, that is our "model" of result, under this new point of view. The main references for the DPP theory are the classical review articles [100, 20, 67], and this very nice introductory paper [52]. For the random matrix theory we refer essentially to the monograph [88], and to the books [53, 8]. Finally for the Tracy-Widom result we recall that even though the original proof was first given from the authors in [103], we found other useful explanations in [106].

3.1 Basic knowledge on DPP

3.1.1 An introductory example

Inspired by [52], we start our discussion on DPP by treating a very nice example of determinantal point process that is called the *descent point process* (for more details, we refer to Sec.1 of [52]). Even though it is mathematically simple, it is very useful to explain the basic ideas and concepts behind DPP.
The descent point process is defined as follows: consider a column of digits $S_0, \dots, S_n$ independent and identically distributed on $[[0,9]]$. For each line $i = 1, \dots, n$ then consider the random variables

$$X_i \coloneqq \chi_{\{S_i < S_{i-1}\}} = \begin{cases} 1 & \text{if} \quad S_i < S_{i-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

The descent point process is given by all the possible random sequences of natural numbers i for which $X_i = 1$ in the integer segment $i \in [[1,n]]$, namely

$$D_n \coloneqq \{i \in [[1,n]] \;\;|\;\; X_i = 1\} \tag{3.1.2}$$

To visualize that, we can put on the right of the column of values S_i a black dot for each line $i \in [[1,n]]$ for which the condition $X_i = 1$ is satisfied. In this way the descent point process is described by all the possible configurations of the black dots in the segment $[[1,n]]$. See Figure 3.1 for an example.

In order to know the process in exhaustive way, one should be able to compute the probability of each possible configuration of black points or sequences of numbers in $[[1,n]]$, i.e.

$$\mathbb{P}(\ell \subset D_n), \quad \text{for any} \quad \ell \subset [[1,n]]\,. \tag{3.1.3}$$

In general, higher is the cardinality of the subset ℓ and more complicated is to compute the correspondent probability. If k is the cardinality of ℓ then $\ell = \{s_1, \dots, s_k\}$ and we denote the probability of ℓ being in D_n as $\rho_k(\ell)$; in this case, it will be also the k-correlation function

i	S_i		X_i
0	2		
1	4		0
2	9		0
3	7	•	1
4	6	•	1
5	9		0
6	1	•	1
7	5		0

Figure 3.1: An example of configuration of the descent point process for $n = 7$ given by $\{3, 4, 6\}$.

of the process. The distinguished character of determinantal point process is that all the correlation functions of each order are actually written in terms of a single function of two variables, that is called the kernel of the process. If we start computing the correlation function for $k = 1$ in the descent process, we have

$$\rho_1(\{s\}) = \mathbb{P}(\{s\} \subset D_n) = \mathbb{P}(X_s = 1) = \mathbb{P}(S_s < S_{s-1}) = \frac{1}{10^2}\left(\sum_{k=1}^{9} k\right) = \frac{1}{10^2}\binom{10}{2} = \frac{9}{20}.$$

Then for $k = 2$, the computation is a little more delicate. Indeed, if we consider $\ell = \{s, s+1\}$, then

$$\mathbb{P}(\{s, s+1\} \subset D_n) = \mathbb{P}(S_{s+1} < S_s < S_{s-1}) = \frac{1}{10^3}\left(\sum_{k=1}^{8} k\right) = \frac{1}{10^3}\binom{10}{3} = \frac{3}{25} < \left(\frac{9}{20}\right)^2.$$

Instead, if we take the generic subset $\ell = \{s, t\}$ with $t \neq s + 1$, then

$$\mathbb{P}(\{s, t\} \subset D_n) = \mathbb{P}(S_{s+1} < S_s)\mathbb{P}(S_{t+1} < S_t) = \left(\frac{9}{20}\right)^2.$$

Thus in the case of cardinality $k = 2$ the correlation function is defined by cases

$$\rho_2(\{s, t\}) = \begin{cases} \frac{3}{25} & \text{if } \ |t - s| = 1, \\ \left(\frac{9}{20}\right)^2 & \text{otherwise.} \end{cases}$$

In general, we can prove that for any subset ℓ given by $k \geq 3$ consecutive numbers in $[[1, n]]$, then

$$\mathbb{P}(\ell \subset D_n) = \frac{1}{10^{k+1}}\binom{10}{k+1}.$$

Otherwise, the computation is done by following this idea: first one can split the subset $\ell = \ell_1 \cup \ell_2$ in such a way that ℓ_1 and ℓ_2 have distance more than 1. Then one uses that

$\rho_k(\ell) = \rho_{k_1}(\ell_1)\rho_{k_2}(\ell_2)$ where k_i are the cardinalities of ℓ_i respectively for $i = 1, 2$.
A compact way to write down $\rho_k(\ell)$ for any number $k \in [[1, n]]$ and any sequence $\ell = \{s_1, \ldots, s_k\} \subset [[1, n]]$ (with $s_i \neq s_j$ for any $i \neq j$) was found in [21] and it is realized as follows. Consider the two variables function $K(i, j) : [[1, n]]^2 \to \mathbb{R}$ such that

$$K(i, j) := \kappa(j - i), \quad \text{with} \quad \sum_{m \in \mathbb{Z}} \kappa(m) z^m = \frac{1}{1 - (1 - z)^{10}}.$$

Then the k-correlation function of the descent process is then given by

$$\rho_k(\{s_1, \ldots, s_k\}) = \det(K(s_i, s_j))_{i,j=1}^k.$$

For this reason the descent process is a determinantal point process on $\mathbb{Z}$ (actually on the segment $[[1, n]]$ of $\mathbb{Z}$).

3.1.2 Generalities of DPP theory

With this example in mind, we can now give the general definition of point processes and then we restrict to the study of determinantal ones (for this section we mainly follow the classical references [100, 67]). We consider $E = \mathbb{R}$ (or a finite product of disjoint copies of $\mathbb{R}$) and $\mathcal{X} = \text{Conf}(E)$ the space of all possible finite configurations of particles on E. Notice that one can replace $\mathbb{R}$ by $\mathbb{Z}$ (as we actually did in the previous section) or by another discrete space and the theory of DPP on that space similarly follows, see also [20]. We restrict our discussion on the case $E = \mathbb{R}$ just because the applications we are interested in, actually fit in this case.
A formal definition of point process is given as follows. On $\mathcal{X}$ one can construct a σ-algebra of measurable sets, in the following way. First construct the cylinder sets, for any Borel subset $B \subset E$ and any $n \in \mathbb{N}$

$$C_n^B := \{X \in \mathcal{X} \ \text{ s.t. } \ \#_B(X) := |X \cap B| = n\}, \tag{3.1.4}$$

and then consider $\mathcal{B}$ the σ-algebra generated by these C_n^B on $\mathcal{X}$.

Definition 3.1.1. *A point process on E is given by a probability measure P on $(\mathcal{X}, \mathcal{B})$.*

The way to construct a probability measure on the space of configurations $(\mathcal{X}, \mathcal{B})$ was studied in particular by Lenard in a series of papers [82, 83, 84]. The main idea is that the construction can be reduced to the determination of the joint probability distributions of the random variables $\#_D$ for D some simple subsets of E. This procedure allowed the author to go further and prove the relation between the existence of a probability measure on $(\mathcal{X}, \mathcal{B})$ and the existence of the k-point correlation functions for the random variables $\#_B$, for any B Borel subset of E. It turned out that a point process is uniquely identified by its correlation functions if and only if the probability distribution of the random variables $\#_A$

is determined by its moments. For more details about the construction of a point process from its correlation functions, see Theorem 1 in [100].
In this general (continuous) context, the k-correlation functions are defined as follows.

Definition 3.1.2. *For any $k \in \mathbb{N}$, we define the k-point correlation function of the point process $(\mathcal{X}, \mathcal{B}, P)$ as the locally integrable function $\rho_k : E^k \to \mathbb{R}_+$, such that for any collection of different and disjoint Borel subset $A_i \subset E, i = 1, \dots, k$ then*

$$\mathbb{E}\left(\prod_{j=1}^{m} \#_{A_j}\right) = \int_{A_1 \times \dots \times A_k} \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k, \tag{3.1.5}$$

where $\mathbb{E}$ denotes the mathematical expectation.

As we saw in the introductory example, the k-point correlation function has a meaningful probabilistic interpretation : for the descent process $\rho_k(x_1, \dots, x_k)$ was exactly the probability of having particles at the points x_i in $\mathbb{N}$. But this was because the process was defined on (a subset of) $E = \mathbb{Z}$. In the continuous case (e.g. $E = \mathbb{R}$) we can think to $\rho_k(x_1, \dots, x_k)dx_1 \dots dx_k$ as the probability to find a particle in each infinitesimal box $[x_i, x_i + dx_i]$ for $i = 1, \dots, k$. In this way, formula (3.1.5) actually gives the expectation value of finding a configuration $X = \{x_1, \dots, x_k\} \in \mathcal{X}$ with $x_i \in A_i$ for every $i = 1, \dots, k$.

Remark 3.1.3. *For $k = 1$ we have that $\rho_1(x)$ is the density of particles, indeed*

$$\mathbb{E}(\#_A) = \int_A \rho_1(x) dx,$$

for any bounded borel subset $A \subset E$.

Definition 3.1.4. *A point process (on $E = \mathbb{R}$ or $\mathbb{R}^d$) is called determinantal if its k-point correlation functions, for every $k \geq 1$, is written as*

$$\rho_k(x_1, \dots, x_k) = \det \left(K(x_i, x_j)\right)_{i,j=1}^{k}, \tag{3.1.6}$$

with K a trace-class integral operator on $L^2(\mathbb{R})$ with kernel $K(x, y)$ in case $E = \mathbb{R}$ and matrix-valued kernel $(K_{rs}(x, y))_{r,s=1}^{d}$ if $E \cong \mathbb{R}^d$.

By using the Lenard result about the existence of point process through their correlation functions, one can find necessary and sufficient conditions for a kernel $K(x, y), (x, y) \in \mathbb{E}^2$, to uniquely define a DPP. The result is as follows.

Theorem 3.1.5 (Theorem 3 [100])**.** *Every hermitian, locally trace-class operator K on $L^2(E)$ uniquely defines a determinantal point process if and only if $0 \leq K \leq 1$.*

We will apply this result in Chapter 6 and 7 in order to prove that the matrix-valued Airy kernels and the finite temperature Airy kernels respectively define DPP on $\mathbb{R}^r$ and $\mathbb{R}$. There exists also a weak convergence criteria for DPP.

Theorem 3.1.6 (Theorem 5 in [100]). *Consider P, P_n probability measures on $(\mathcal{X}, \mathcal{B})$ for some determinantal point processes with kernels respectively K, K_n. Suppose that*

- $K_n \rightharpoonup K$ *in the weak operator topology for $n \to \infty$;*
- $\mathrm{Tr}(\chi_B K_n \chi_B) \to \mathrm{Tr}(\chi_B K \chi_B)$ *for $n \to \infty$ and for any Borel subset $B \subset E$;*

then the probability measure P_n converges to P weakly on the cylinder sets.

This result will be useful in the next section, where we are going to compute some scaling limits of certain relevant quantities in DPP arising in some random matrix model.
Knowing the k-point correlation functions of a DPP is fundamental in order to compute other relevant quantities for the process. We are in particular interested in the computation of the so called *gap probabilities*, i.e. the probabilities that no particles lie in a certain subset of E. The computation required is based on the following result.

Proposition 3.1.7 (Proposition 2.2 of [67]). *Consider a point process with existing k-point correlation functions and let ϕ be a mesurable, bounded, complex-valued function with bounded support on E. Also, supposing that $\mathrm{supp}(\phi) \subset B$ for B a Borel subset of E, assume that*

$$\sum_{k=0}^{\infty} \frac{||\phi||_{\infty}^{k}}{k!} \int_{B^k} \rho_k(x_1, \dots, x_k) dx_1 dx_k < \infty. \tag{3.1.7}$$

Then we have

$$\mathbb{E}\left(\prod_{j=1}^{\#_B} (1 + \phi(x_j))\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^k} \prod_{k=1}^{n} \phi(x_k) \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k. \tag{3.1.8}$$

Consider now B a bounded Borel subset of E and χ_B its characteristic function. Replacing $\phi = -\chi_B$ in the above formula we get

$$\mathbb{P}\left(\text{no particles in } B\right) = \mathbb{E}\left(\prod_{j} (1 - \chi_B(x_j))\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \rho_k(x_1, \dots, x_n) dx_1 \dots dx_n, \tag{3.1.9}$$

the gap probability distribution. In particular, looking at a point process on $\mathbb{R}$ we can consider $B = (t, \infty)$. Supposing that there exist a $\tilde{t}$ for which $\#_{(\tilde{t},\infty)} < \infty$, then we can say that for every t the property holds (since for every finite subset it is always true). We order then the particles in the interval (t, ∞) as $x_1 < \dots < x_{\#_{(\tilde{t},\infty)}} = x_{\max}$ and we want to study the probability distribution of the largest particle, namely $\mathbb{P}(x_{\max} \leq t)$.

Proposition 3.1.8 (Proposition 2.4 of [67]). *Consider a point process on $\mathbb{R}$ for which all k-point correlation functions exist and respect the condition*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{(t,\infty)^n} \rho_n(x_1, \dots, x_n) dx_1 \dots dx_n < \infty \tag{3.1.10}$$

for any $t \in \mathbb{R}$. Then the process has a last particle and

$$\mathbb{P}(x_{\max} \leq t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t,\infty)^n} \rho_n(x_1, \dots, x_n) dx_1 \dots dx_n. \tag{3.1.11}$$

When the point process is determinantal with kernel $K(x, y)$ defining a trace-class integral operator on $L^2(\mathbb{R})$, the proposition above becomes even more explicit. In fact the right hand side of equation (3.1.11) is written as Fredholm determinant of the operator K.

Corollary 3.1.9 (Proposition 2.9 of [67])**.** *Consider a determinantal point process on $\mathbb{R}$ with hermitian kernel $K(x, y)$ such that: it defines a trace-class integral operator K on $L^2((t,\infty))$ for any $t \in \mathbb{R}$ and so that*

$$\int_t^{\infty} K(x, x) dx < \infty. \tag{3.1.12}$$

Then the process almost surely has a largest particle and

$$\mathbb{P}(x_{\max} \leq t) = \det\left(1 - K|_{(t,\infty)}\right). \tag{3.1.13}$$

This last corollary gives a first connection between the first and the second chapter of the thesis. Indeed certain Painlevé trascendents such as the ones found in Theorem 2.1.7 and Theorem 2.2.12 for the Painlevé II equation and hierarchy, are expressed as Fredholm determinants of the Airy kernels given in equation (4.2.4). For each n, these operators actually satisfy the hypothesis of Theorem 3.1.5 and thus uniquely define some DPP. As a byproduct the relevant Painlevé trascendents can be related to the largest particle distribution of the correspondent DPP. Moreover, in the case $n = 1$ the DPP associated to the Airy kernel corresponds to a certain limit of the DPP describing the eigenvalue distribution of a distinguished random matrix model : the Gaussian Unitary Ensemble, that we are going to treat in the next section.

Remark 3.1.10. *In analogue way, the new Painlevé trascendents that we are going to study in Chapters 6, 7 will also be related to the largest particle probability distribution of some DPP defined through a matrix-valued analogue of the Airy kernels and to a finite temperature versions of the Airy kenrels respectively.*

3.2 Random matrices and DPP

This section aims to introduce some random matrix models and to see how DPP arise out in this context. In particular, we are going to focus on the Gaussian Unitary Ensemble with the ultimate goal to study the probability distribution of the eigenvalues of matrices in this ensemble in some specific large N limit, N being the size of the matrices in the model. Indeed it is in this case that the relation to the Painlevé trascendents introduced in Theorem 2.1.7 emerged first.

We start by defining the Gaussian Unitary Ensemble, GUE from now on. Recall that the vector space (over $\mathbb{R}$) of hermitian matrices, namely

$$\mathcal{H}_N := \{H \in \mathrm{Mat}(N \times N, \mathbb{C}) \ | \ H = H^\dagger\}$$

has real dimension N^2. In particular we can take as coordinates the N diagonal entries H_{ii} (that are real) and the real and imaginary part respectively of the upper triangular entries $\Re H_{jk}, \Im H_{jk}$ (that are exactly $N^2 - N$). Now, an element of GUE is essentially an hermitian matrix H whose entries H_{ii} for $i = 1, \ldots, N$ and $\Re H_{jk}, \Im H_{jk}$ for $j, k = 2, \ldots, N$ are random variables, specifically independent identically distributed (i.i.d.) normal random variables. More precisely GUE is built as follows.

Definition 3.2.1 (Definition 2.5.1 [88]). *The Gaussian Unitary ensemble is defined taking the space of hermitian matrices equipped with a probability measure $P(H)dH$ such that*

1. *the probability $P(H)dH$ of being in the volume element*

$$dH := \prod_{i=1}^{N} dH_{ii} \prod_{j<k} d\Re H_{jk} d\Im H_{jk} \tag{3.2.1}$$

is invariant under conjugation by unitary elements, i.e.

$$P(H) = P(U^{-1}HU) \tag{3.2.2}$$

for every unitary matrix U;

2. *all the linearly independent entries of an element H are also statistically independent, i.e. the function $P(H)$ is a product of independent functions, each of them depending on one of the linearly independent coordinates*

$$P(H) = \prod_{i=1}^{N} f_i(H_{ii}) \prod_{j<k} f_{jk}(\Re H_{jk}) \tilde{f}_{jk}(\Im H_{jk}). \tag{3.2.3}$$

These two requirements together fix in some sense the function $P(H)$. In particular, we have the following result.

Theorem 3.2.2 (Theorem 2.6.3 [88]). *The only possibility for the form of the function $P(H)$ is restricted to*

$$P(H) = \exp\left(-a \operatorname{Tr} H^2 + b \operatorname{Tr} H + c\right) \tag{3.2.4}$$

where $a \in \mathbb{R}_+$ and $b, c \in \mathbb{R}$.

In particular the standard choice for GUE is to consider $P(H) = \exp\left(-\operatorname{Tr} H^2\right)$, since up to rescaling operations and origin translation every choice of a, b, c can be reduced to this one. Now, for any given random matrix ensemble, one fundamental point to develop is to study the probabilistic behavior of the spectra of the elements of the given ensemble. For GUE the classical result is as follows.

Theorem 3.2.3 (Theorem 3.3.1 [88]). *The joint probability density function of the eigenvalues for GUE is given by*

$$P(x_1,\dots,x_N) = C_{N,2}\exp\left(-\sum_{i=1}^{N} x_i^2\right)\prod_{j<k}(x_j-x_k)^2 \tag{3.2.5}$$

where the constant $C_{N,2}$ is taken in such a way that

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} P(x_1,\dots,x_N)dx_1\dots dx_N = 1.$$

Remark 3.2.4. *For the other classical ensembles: the Gaussian Orthogonal one and the Gaussian Symplectic one Theorem 3.2.2 also holds exactly with the same statement, while Theorem 3.2.3 holds with some little changements. The form of the joint probability distribution function in those cases has the same form of* (3.2.5) *but the constant in front of the argument of the exponential function and the power of the second factor change as well as the constant $C_{N,2}$.*

We are now going to see that the probabilistic behavior of the eigenvalues of GUE is indeed a DPP on $\mathbb{R}$. To do this, we need the definition of the n-point correlation functions for the eigenvalues of GUE.

Definition 3.2.5 ([35, 88]). *The n-point correlation function for the eigenvalues of GUE is defined as*

$$\rho_n(x_1,\dots,x_n) = \frac{N!}{(N-n)!}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} P(x_1,\dots,x_N)dx_{n+1}\dots dx_N, \tag{3.2.6}$$

where $P(x_1,\dots,x_N)$ is given in (3.2.5)*.*
The function $\rho_n(x_1,\dots,x_n)$ indicates the probability denisity of finding n eigenvalues around $x_1,\dots,x_n$ with the position of the remaining $N-n$ left unknown.

The probability density function $P(x_1,\dots,x_N)$ is a symmetric function, thus it can be associated to a point process over $\mathbb{R}$. The n-point correlation functions of the relevant process can be taken exactly as (3.2.6) with $P(x_1,\dots,x_N)$ given in (3.2.5) (see also Example 2.6 of [67]). The process is then shown to be determinantal.

Theorem 3.2.6 ([88]). *For every $n = 1,\dots,N-1$ the correlation functions* (3.2.6) *are given by*

$$\rho_n(x_1,\dots,x_n) = \det\left(K_N(x_i,x_j)\right)_{i,j=1}^{n} \tag{3.2.7}$$

where

$$K_N(x_i,x_j) = \sum_{k=1}^{N-1}\phi_k(x_i)\phi_k(x_j), \quad \textit{with} \quad \phi_k(x) = \frac{1}{\sqrt{2^k k!\sqrt{\pi}}}\exp\left(-\frac{x^2}{2}\right)H_k(x) \tag{3.2.8}$$

and $H_k(x)$ being the k-th Hermite polynomial.

We recall that Hermite polynomials $\{H_k(x)\}_{k\in\mathbb{N}}$ are a family of orthogonal polynomials over $\mathbb{R}$ with respect to the weight function $\exp(-x^2)$. They can be written as

$$H_k(x) = \exp\left(x^2\right)\left(-\frac{d^k}{dx^k}\right)\exp\left(-x^2\right) = k!\sum_{j=0}^{[k/2]}(-1)^j\frac{(2x)^{k-2j}}{j!(k-2j)!}, \quad k \in \mathbb{N}. \tag{3.2.9}$$

A proof of Theorem 3.2.6 can be found in Section 6.2 of [88]. To summarise, it essentially follows by observing that the joint probability distribution function $P(x_1,\dots,x_N)$ given in (3.2.5) contains a Vandermonde determinant squared and then by performing row or column operations one gets

$$P(x_1,\dots,x_N) = \frac{1}{N!}\det(\phi_{j-1}(x_i))^2 = \frac{1}{N!}\det(K_N(x_i,x_j))_{i,j=1}^N \tag{3.2.10}$$

with $K_N(x_i,x_j)$ given as in (3.2.8). Then one can integrate over the $N-n$ required variables and apply Theorem 5.1.4 of [88] to conclude. For an alternative proof see e.g. Section 3.2 of [53].

Remark 3.2.7. *We underline that in the definition of the kernel K_N in equation* (3.2.8) *there is an explicit dependence on N the size of the random matrices we are analyzing.*

Gap probabilities Doing similar computations, one can compute other interesting quantities of the process like the gap probabilities. For a given interval $J \subset \mathbb{R}$ we denote by $E(n,J)$ the probability that J contains exactly n eigenvalues, so that $E(0,J)$ is the probability that there are no eigenvalues in J. As we saw in the previous section for general DPP and for $J=(s,\infty)$, the quantity $E(0,J)$ is expressed in terms of a certain Fredholm determinant

$$E(0,J) = \det(1-K_N\chi_J) \tag{3.2.11}$$

where χ_J denotes the characteristic function of the interval J and K_N is the kernel written above in (3.2.8). Otherwise, one can directly compute this quantity as done in e.g. Section 3.2 of [53] for a generic interval J. In the following, we summarise the principal ideas contained there. Indeed, one can see

$$E(0,J) = \mathbb{E}\left(\prod_{i=1}^N(1+f(\lambda_i))\right) = c_{N,2}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\prod_{j<k}(x_j-x_k)^2\prod_i\exp\left(-x_i^2\right)(1+f(x_i))dx_1\dots dx_N$$

with $f(\lambda)=-\chi_J(\lambda)$. But the last integral can be explicitly computed by using the Andreief identity, namely

$$\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\det(f_i(x_j))\det(g_i(x_j))d\nu(x_1)\dots d\nu(x_N) = N!\det\left(\int_{\mathbb{R}}f_i(x)g_j(x)d\nu(x)\right). \tag{3.2.12}$$

Again, in our case, by recognizing a squared Vandermonde determinant in the last integral above and by defining $f_i(x) = x^i = g_i(x)$ and $d\nu(x_i) = \exp(-x_i^2)dx_i$, we have that

$$\mathbb{E}\left(\prod_{i=1}^{N}(1+f(\lambda_i))\right) = \tilde{C}_{N,2}\det\left(\int_{\mathbb{R}} x^{i+j}(1+f(x))\exp(-x^2)dx\right) = \tilde{C}_{N,2}\det\left(\delta_{ij} + \int_{\mathbb{R}}\phi_i(x)\phi_j(x)f(x)dx\right) \tag{3.2.13}$$

where the last identity is obtained by performing row and columns operations, in order to replace the monomials x^k with the orthogonal family $\phi_k(x)$ w.r.t. $\exp(-x^2)$. Finally, one can manipulate the last determinant in (3.2.13) in the following way. Construct the two integral operators

$$A : L^2(\mathbb{R}) \to \mathbb{R}^N, \ \text{ s.t. } \ (Af)_i := \int_{\mathbb{R}} A(i,x)f(x)dx = \int_{\mathbb{R}} \phi_i(x)f(x)dx, \ \text{ for } \ f \in L^2(\mathbb{R}) \tag{3.2.14}$$

and

$$B : \mathbb{R}^N \to L^2(\mathbb{R}) \ \text{ s.t. } \ (Bv)(x) := \sum_{j=1}^{N} B(x,j)v_j = \sum_{j=1}^{N}\phi_j(x)v_j \ \text{ for } \ v \in \mathbb{R}^N. \tag{3.2.15}$$

In this way the last determinant in equation (3.2.13) is $\det(1+AB)$. By applying the Sylvester identity (see for instance equation (5.9) of Chapter VI in [48]) i.e. $\det(1+AB) = \det(1+BA)$, we conclude

$$\mathbb{E}\left(\prod_{i=1}^{N}(1+f(\lambda_i))\right) = \det(1+K_N f),$$

and so for $f = -\chi_J$ the wanted result follows.
Of course, there are many other interesting quantities to study but since our focus will be on the gap probabilities and their relation with the Painlevé II trascendents, we do not go any further in this discussion.

Limiting behaviors As underlined before, the determinantal form of the n-points correlation functions ρ_n as well as the one of the gap probabilities $E(0,J)$ is written in terms of a kernel operator depending on the parameter N, which is the size of the matrices in the ensemble. A natural question is then to study the limiting behavior of these quantities for $N \to \infty$. Thanks to their determinantal form, this essentially reduces to the study of the limiting behavior of the kernel $K_N(x,y)$ themselves, in some appropriate scaling. In particular, the so called edge scaling limit (the limit at the edge of the spectra) for the kernel $K_N(x,y)$ is computed as (see e.g. [40])

$$\lim_{N\to\infty}\frac{1}{2^{1/2}N^{1/6}}K_N\left(\sqrt{2N}+\frac{x}{2^{1/2}N^{1/6}}, \sqrt{2N}+\frac{y}{2^{1/2}N^{1/6}}\right) \to K_{Airy}(x,y) \tag{3.2.16}$$

the convergence being in trace norm on every bounded (from below) subsets of $\mathbb{R}$. We highlight that the proof of this result relies on the use of the Christoffel-Darboux formula, that allows to rewrite the kernel $K_N(x,y)$ as

$$K_N(x,y) = \left(\frac{N}{2}\right)^{1/2} \frac{\phi_N(x)\phi_{N-1}(y) - \phi_N(y)\phi_{N-1}(x)}{x-y} \tag{3.2.17}$$

and then the large-N asymptotics for the Hermite polynomials which enters in the wave functions ϕ_N as defined in (3.2.8). Notice that in this context the Airy kernel K_{Airy} is considered as

$$K_{Airy}(x,y) \coloneqq \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{3.2.18}$$

which is, by the way, equivalent to the definition given in (2.1.16). As a byproduct one can write the edge scaling limit of the probability distribution of the largest eigenvalue in GUE as

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{max} \leq \sqrt{2N} + \frac{s}{2^{1/2}N^{1/6}}\right) = \det(1 - K_{Airy}\chi_{(s,\infty)}) \coloneqq F_{TW}(s) \tag{3.2.19}$$

that is also known as the Tracy-widom distribution.

Remark 3.2.8. *Notice that Theorem 3.1.6 applied to this case says that the probability measures P_N of the DPP describing the positions of the eigenvalues of GUE with size N through correlation functions* (3.2.6)*, converges for $N \to \infty$ to the probability measure of the DPP on $\mathbb{R}$ with kernel the Airy kernel.*

The Tracy-Widom distribution and the Painlevé II transcendent Theorem 2.1.7 assumes now new significance, since the Fredholm determinant of the Airy kernel is interpreted as the edge scaling limit of the probability distribution of the largest eigenvalue in GUE, as shown above in equation (3.2.19). In particular, one can express the Tracy-Widom distribution in terms of the Hastings-McLeod Painlevé II transcendents $u(t)$ as

$$F_{TW}(s) = \exp\left(\int_s^\infty (t-s)u^2(t)dt\right), \tag{3.2.20}$$

which is just the integrated version of the formula given in Theorem 2.1.7.
As previously announced in the Introduction and also in Chapter 2, it is on this type of result that we will be interested in: results that relate the integrable systems world, in this specific case Painlevé equations, with the determinantal point processes, that in this case appear in random matrix theory.
The next two chapters aim thus to introduce the two main tools that can be used to achieve this kind of results, and that will be used in Chapters 6, 7. First the Riemann-Hilbert problems for the class of integrable operators (in which the Airy kernel fit in) and second the isomonodromic representation of the Painlevé equations.

Chapter 4

Integrable operators and Riemann-Hilbert problems

Riemann-Hilbert problems are the protagonists of this chapter, in particular the ones connected with a class of integral operators. This class of operators is known in literature as *integrable* operators of IIKS type, since they were first studied using a Riemann-Hilbert approach in [60]. These operators have kernels of a particular form and their resolvents, whether they exist, have kernels of the same form. In particular, the expression for their resolvent is directly related to the solution of a certain Riemann-Hilbert problem. As a byproduct the Fredholm determinants of these integrable operators can be expressed in terms of quantities related to the solution of the Riemann-Hilbert problem. Many integral operators appearing in random matrix theory or statistical mechanics fit in this class of operators, or are in some way related to them, and can be thus treated with this approach. This allows to find more information about their Fredholm determinants that have in these contexts interesting probabilistic interpretation, as underlined in the previous chapter. For us, the interesting case of study will always be given by the Airy kernel and its higher order generalizations, in scalar, matrix-valued and finite temperature versions. As we already underlined in Chapter 3 and we will underline thereafter, the Fredholm determinants of these Airy kernels describe interesting quantities in random matrix models ([103, 51, 68]) in the study of the KPZ universality class ([5, 31]) and in models for non interacting fermions ([81, 85, 32]). Nevertheless there are other popular integrable operators involved in these applications, like the sine kernel and the Bessel kernels, studied for example in [40, 104, 47]. In conclusion, Riemann-Hilbert problems give a powerful tool to study certain integral operators defining determinantal point processes with applications in many different fields. Moreover, the Riemann-Hilbert problems build the bridge between integral operators and integrable systems. Starting from the solution of a given Riemann-Hilbert problem, one can construct Lax pairs for ordinary or partial differential equations, difference equations and hierarchies in a standard way. In our case of study, we will always be interested in recovering the isomonodromic Lax pair for the Painlevé II hierarchy, described at the end of Chapter

5, and its generalizations.
The Chapter is organized as follows: after a brief introduction on generic Riemann-Hilbert problems, we are going to review the standard results of the IIKS theory for integrable operators. Then, we are going to review as this theory can be extended to the case of matrix-valued integral operators, resuming the work [13]. The results contained in this section will be largely used in Chapter 6 in order to achieve the original results about the matrix Painlevé II hierarchy. In Chapter 7 instead, in order to study the finite-temperature version of the Airy kernels, we will need to introduce the theory of operator-valued Riemann-Hilbert problems, as we did in the paper [24]. In the last decades, examples of this kind of problems can be found in only a few papers, e.g. [61, 62]. Very recently the paper [22] re-introduced operator-valued Riemann-Hilbert problem, with the aim to develop a rigorous and quite general theory to treat them. Following this method, we will see in section 7.2 how the operator-valued Riemann-Hilbert problem can be formulated and solved in our specific case. From that, we will recover an operator-valued isomonodromic Lax pair for the integro-differential Painlevé II hierarchy. Of course, results and methods in Chapter 7, are strongly inspired by the classical theory that we are going to review in this chapter.

4.1 Introduction to Riemann-Hilbert problems

In this section we are going to introduce the Riemann-Hilbert formalism and the main results about the existence of a solution for a given Riemann-Hilbert problem. This first section is mainly inspired from Chapter 5 of the monograph [53] and we refer to that for further details and proofs.
A very nice introduction to this topic and its relation to integrable systems is also given in [58]. The main idea of a Riemann-Hilbert problem is to reconstruct a matrix-valued function defined on the complex plane and having prescribed discontinuities. These discontinuities are given in form of jump equations along certain curves, that the boundary values of the function have to satisfy. Thus, from a practical point of view, a Riemann-Hilbert problem is essentially defined through a pair of data: a contour and a matrix-valued function defined on it. Here are the requirements that this pair has to satisfy.

- Let Σ be any oriented contour in the complex λ-plane. One can allow Σ to have a finite number of self-intersection points, even though in our cases of study in Chapter 6 and 7 there are no such points. Also, Σ can count a finite number of connected components and this is indeed the case in both our works in Chapter 6 and 7.

- Let $G : \Sigma \to \mathrm{GL}(p, \mathbb{C})$ be a map defined all along the contour Σ and taking values in the set of $p \times p$ invertible matrices, $p \geq 1$. We call G the *jump matrix.*

Within the orientation of the contour Σ, we denote by $+$ and $-$ respectively the part of the plane that stands on the left and respectively on the right hand side of the contour. Finally, given a pair (Σ, G), the correspondent Riemann-Hilbert problem is settled as follows.

Riemann-Hilbert Problem 4.1.1. *Find a $p \times p$ matrix-valued function Y with the following properties.*

(1) Y is analytic on $\mathbb{C} \setminus \Sigma$;

(2) For any $\lambda \in \Sigma$, the function Y has continuous boundary values $Y_\pm$, denoting respectively the boundary value of Y for $\lambda \in \Sigma$ while approaching Σ respectively from the left $(+)$ or from the right $(-)$ nontangentially. Moreover $Y_\pm$ satisfy the following jump condition

$$Y_+(\lambda) = Y_-(\lambda)G(\lambda) \quad \lambda \in \Sigma; \tag{4.1.1}$$

(3) The funtion Y satisfies the asymptotic condition

$$Y(\lambda) \sim I_p \quad \textit{for} \quad |\lambda| \to \infty, \tag{4.1.2}$$

where I_p denotes the identity matrix of dimension p.

Remark 4.1.2. *One can add more requirements to the pair (Σ, G), for example asking for G to have constant determinant equal 1 and to decay along all the infinite branches of Σ exponentially fast. Further requirements can be added on the jump matrices along the connected components of Σ when there are self intersection points, around each one of them.*

The solvability of Riemann-Hilbert problem 4.1.1 essentially relies on the Plemelj-Sokhotskii formula. This formula actually gives the solution for a scalar Riemann-Hilbert problem with jump function being Hölder continuous, in terms of a contour integral of Cauchy type. Then, for matrix Riemann-Hilbert problems there are some particular case in which the Plemelj-Sokhotskii formula still describes at least some of the entries of the matrix solution. This happens in the so called abelian cases, when the jump matrix $G(\lambda)$ commutes with itself when computed at different values of λ. In the general case, the solution of a matrix Riemann-Hilbert problem can be still written as a contour integral but in terms of the boundary values of the function itself. We resume all these results in the following pages, for the proofs and more details we refer to [96, 43].

Theorem 4.1.3 (Theorem 5.1.3 [53])**.** *Let Σ be an oriented smooth and closed contour and let $g(\lambda)$ be a Hölder continuous function defined on Σ. Define the function $y(\lambda)$ defined as the contour integral of Cauchy type*

$$y(\lambda) := \frac{1}{2\pi i} \int_\Sigma \frac{g(\zeta)}{\zeta - \lambda} d\zeta = (\mathcal{C}g)(\lambda), \tag{4.1.3}$$

where we denoted by $\mathcal{C}$ the Cauchy transform.
The function $y(\lambda)$ has the following properties.

1. *It is analytic in $\mathbb{C} \setminus \Sigma$ and its boundary values $y_\pm(\lambda)$ are continuous up to the boundary Σ.*

2. $\lim_{\lambda \to +\infty} y(\lambda) = 0$.

3. The boundary values $y_{\pm}(\lambda)$ *satisfy the following formulae (Plemelj-Sokhotskii)*

$$y_{\pm}(\lambda) = \pm\frac{1}{2}g(\lambda) + \frac{1}{2\pi i}\mathcal{P}\int_{\Sigma}\frac{g(\zeta)}{\zeta-\lambda}d\zeta, \quad \textit{for} \quad \lambda \in \Sigma \tag{4.1.4}$$

where $\mathcal{P}$ *stand for the principal value of the integral that follows, i.e.*

$$\mathcal{P}\int_{\Sigma}\frac{g(\zeta)}{\zeta-\lambda}d\zeta := \lim_{\epsilon\to 0}\int_{\Sigma_\epsilon}\frac{g(\zeta)}{\zeta-\lambda}d\zeta \tag{4.1.5}$$

where the contour Σ_ϵ *is taken as* $\Sigma_\epsilon = \Sigma \setminus \{\Sigma \cap |\zeta - \lambda| < \epsilon\}$, *for any* $\lambda \in \Sigma$.

From equation (4.1.4) directly follows that the boundary values of y satisfy for every $\lambda \in \Sigma$ the following relation

$$y_+(\lambda) = y_-(\lambda) + g(\lambda) \tag{4.1.6}$$

that can be thought as an additive jump relation. Thus one concludes that the Cauchy transform of $g(z)$ actually gives a solution solution for an additive Riemann-Hilbert problem, as follows.

Corollary 4.1.4 (Corollary 5.1.5 [53])**.** *Let* Σ *and* g *being as in Theorem 4.1.3. Then the Cauchy transform of* g*, namely* $y(\lambda) = (\mathcal{C}g)(\lambda)$ *defined in* (4.1.3)*, solves the additive Riemann-Hilbert problem for a function defined through the three conditions*

1. $y(\lambda)$ *is analytic for* $\lambda \in \mathbb{C} \setminus \Sigma$*;*

2. the boundary values of y *satisfies* $y_+(\lambda) = y_-(\lambda) + g(\lambda)$ *for any* $\lambda \in \Sigma$*;*

3. $y(\lambda) \to 0$ *for* $\lambda \to \infty$*.*

It directly follows that the classical Riemann-Hilbert problem defined in 4.1.1 in the scalar case (for $p = 1$) and for (Σ, G) satisfying the hypothesis of Theorem 4.1.3, admits the explicit solution

$$Y(\lambda) = \exp(\mathcal{C}\ln G)(\lambda), \tag{4.1.7}$$

just by applying the logarithm to the jump condition (4.1.1) and then applying the corollary above. The existence of this solution is guaranteed provided that $G(\lambda) \neq 0$.

Remark 4.1.5. *Theorem 4.1.3 can be extended to cases where the contour* Σ *and the function* g *are more general than in the hypothesis above. In particular one can consider* Σ *as a piece-wise smooth contours having endpoints and* g *as a generic function in some* L^p*-space.*

For $p > 1$, the same formula (4.1.7) holds for the solution of a matrix Riemann-Hilbert problem (4.1.1) with a jump matrix G such that

$$[G(\lambda_1), G(\lambda_2)] = 0, \quad \text{for any} \quad \lambda_1, \lambda_2 \in \Sigma,$$

while seeking for a solution $Y(\lambda)$ in the same multiplicative subgroup. This particular case is also known as the Abelian case. Here is an explicit example.

Example 4.1.6. *Consider the case of the Riemann-Hilbert problem 4.1.1 with* $p = 2$ *and the jump matrix* G *takes the form*

$$G(\lambda) = \begin{pmatrix} 1 & g(\lambda) \\ 0 & 1 \end{pmatrix}.$$

Its solution can be still written as the contour integral (4.1.7) *and can be further simplified to the following form*

$$Y(\lambda) = \begin{pmatrix} 1 & \mathcal{C}g(\lambda) \\ 0 & 1 \end{pmatrix}.$$

Notice that the Riemann-Hilbert problems that we are going to study in Chapters 6 and 7 will have jump matrices that have this form on every connected component of the contour Σ*.*

For the general case, where the jump matrix G does not satisfy the Abelian condition (4.1), the integral representation of the solution of the Riemann-Hilbert problem (4.1.1) is more complicated. The result is resumed in the following theorem.

Theorem 4.1.7 (Corollary 5.1.2 [53])**.** *The Riemann-Hilbert problem 4.1.1 admits solution represented through the following contour integral*

$$Y(\lambda) = I_p + \frac{1}{2\pi i}\int_{\Sigma} \frac{\rho(\zeta)(G(\zeta) - I_p)}{\zeta - \lambda} d\zeta, \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \Sigma, \tag{4.1.8}$$

where $\rho(\lambda) := Y_-(\lambda)$ *satisfies the integral equation*

$$\rho(\lambda) = I_p + \left(\mathcal{C}_- \left(\rho(G - I_p)\right)\right)(\lambda), \quad \lambda \in \Sigma \tag{4.1.9}$$

and $\mathcal{C}_\pm$ *denote the boundary values of the Cauchy transform while approaching* $\lambda \in \Sigma$ *from its left and right hand side, namely*

$$\left(\mathcal{C}_\pm f\right)(\lambda) = \lim_{\eta \to \lambda_\pm} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - \eta} d\zeta \tag{4.1.10}$$

where the limit is taken nontangential.

The idea of the proof is to rewrite equation (4.1.8) modifying its right hand side in this equivalent way

$$Y(\lambda) = I_p + \frac{1}{2\pi i}\int_{\Sigma} \frac{Y_+(\zeta) - Y_-(\zeta)}{\zeta - \lambda} d\zeta, \quad \lambda \in \mathbb{C} \setminus \Sigma \tag{4.1.11}$$

and then use the Cauchy theorem and the asymptotic condition (4.1.2) to show that this last identity actually holds for every $\lambda \in \mathbb{C} \setminus \Sigma$. Furthermore, the proof can be done for any contour Σ, once that Σ has been transformed (through orientation changes, addition of extra contours carrying the identity as jump matrix) into a contour such that $\Sigma = \partial\Omega_+ = -\partial\Omega_-$ with $\Omega_\pm$ disjoint open subsets covering $\mathbb{C} \setminus \Sigma$. The proof of the theorem is explained in Chapter 5 of [53] (pages 364 – 368) and recovered by steps. First the proof is given for the simple case where Σ is a closed simple contour, then for the case where Σ is an unbounded piece-wise smooth contour and finally for the general case described above.

Remark 4.1.8. *Formula* (4.1.8) *will be used in Chapter 6 to study the asymptotic behavior of the solutions of the homogeneous matrix Painlevé II hierarchy, in a similar way of what was done in the work [26] and in many others for the same kind of question.*

We are going to conclude this section by stating the so called *small norm theorem* for Riemann-Hilbert problems. This result is fundamental in the study of asymptotic properties, and we will use it indeed in the end of Chapter 6 to find the asymptotic behavior of the solutions of the matrix Painlevé II hierarchy studied there. The idea of this result can be resumed as follows: first, assume that the jump matrix G for the Riemann-Hilbert problem 4.1.1 depends on some extra parameter $G = G(\lambda, s)$. This is indeed the case in every problem that we will treat in Chapter 6, 7 and 8 and generally speaking in most of the applications. The point is that, if the jump matrix G approximate the identity matrix in a certain matrix norm and for $s \to \infty$, then also the norm of the quantity $Y - I_p$ can be estimated in the same regime for s.

Theorem 4.1.9 (Theorem 5.1.5 [53])**.** *Suppose that we have the following estimate on the jump matrix G*

$$||G - I_p||_{L^2(\Sigma)\cap L^1(\Sigma)} < \frac{C}{s^\epsilon}, \quad for \;\; s \geq s_0, \;\; \epsilon > 0. \tag{4.1.12}$$

for C some positive constant. Then, for s sufficiently large there is a unique solution $Y = Y(\lambda, s)$ of the Riemann-Hilbert problem 4.1.1 with the above jump matrix G, and it is such that

$$||Y(\lambda, s) - I_p||_{L^2(\Sigma)\cap L^1(\Sigma)} \leq \frac{C}{(1 + |\lambda|^{\frac{1}{2}})s^\epsilon}, \quad for \;\; \lambda \in K, s \geq s_0 \tag{4.1.13}$$

where K is a closed subset of $\mathbb{C} \setminus \Sigma$ satisying $\frac{\mathrm{dist}(\lambda,\Sigma)}{1+|\lambda|} \geq c(K)$ for every $\lambda \in K$.

Notice that the last estimate can be improved if the estimate on the jump matrix G is improved (for example if G decays exponentially in s we expect Y to decay at the same way). The proof of this result strongly rely on the formula given in Theorem 6.1.11 for the solution of the Riemann-Hilbert problem 4.1.1 and on the fact that the Cauchy transform, appearing in that formula, is L^2-bounded. This and some other useful properties of the Cauchy transform are stated in Thoerem 5.1.4 of [53] (proofs can be found in [28, 86]).

Remark 4.1.10. *We largely discussed the question of finding a solution for the Riemann-Hilbert problem 4.1.1 and which form and properties this solution has. Although, the question of uniqueness of the solution was left open. One can prove that by fixing the determinant of the jump matrix* $\det G = 1$ *one fixes also the solution of the Riemann-Hilbert problem.*

Essentially, one first proves that the function $d(\lambda) := \det(Y(\lambda))$ is actually constant and equal to 1 and then show by contradiction that there is only one solution to the Riemann-Hilbert problem with such a jump matrix. If the determinant of the jump matrix is not constant, then the uniqueness of the solution should be discussed case by case.

4.2 Riemann-Hilbert problems and IIKS integrable operators

In the previous section we introduced the Riemann-Hilbert problems in the most general setting, and we studied the basic properties of their solutions. In this section we are going to study a specific Riemann-Hilbert problem that is related to the integrable operators, first introduced in [60]. Here the solution of this Riemann-Hilbert problem plays a central role in the construction of the resolvents of these operators. This is particularly useful when the kernel of the relevant operator and thus the associated Riemann-Hilbert problem depend on some further parameters. Then one can express the logarithmic derivative (w.r.t. these parameters) of the Fredholm determinants of such integrable operators, in terms of the asymptotic coefficients of the solution of the Riemann-Hilbert problem. In the following we review the main results of [60, 34, 54] and we refer to these paper for their proofs.

4.2.1 Integrable operators: definitions and examples

To start with, we introduce the two $r \times p$ matrices $\mathbf{f}$ and $\mathbf{g}$ with entries that are smooth functions defined on the connected components of the contour Σ (considered as in the previous section). We also assume that these matrices $\mathbf{f}, \mathbf{g}$ satisfy the diagonal condition

$$\mathbf{f}^T(\lambda)\mathbf{g}(\lambda) = 0.$$

Definition 4.2.1. *An integral operator $\mathcal{K}$ acting on $\mathbb{C}^p$-valued functions $\mathbf{h}(\lambda)$ as*

$$\mathcal{K}f(\lambda) = \int_\Sigma K(\lambda,\mu)\mathbf{h}(\mu)d\mu,$$

is called integrable if its kernel has the form

$$K(\lambda,\mu) = \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda-\mu}. \tag{4.2.1}$$

Remark 4.2.2. *Thanks to the diagonal condition, the kernel $K(\lambda,\mu)$ is nonsingular along the diagonal and there it should be considered as $K(\lambda,\lambda) = (\mathbf{f}')^T(\lambda)\mathbf{g}(\lambda) = -\mathbf{f}^T(\lambda)\mathbf{g}'(\lambda)$.*

Example 4.2.3. *There are many scalar integral kernels (the case $p = 1, r = 2$) that appear in random matrix theory and statistical mechanics taking this integrable form. Here is a list* *of the most popular ones.*

- *The sinus kernel acts on* $L^2(\Sigma)$ *with* Σ *a disjoint union of a finite number of intervals on* $\mathbb{R}$, *through the kernel*

$$K_{sinus}(\lambda,\mu) := \frac{1}{\pi}\frac{\sin(\lambda-\mu)}{\lambda-\mu} \tag{4.2.2}$$

  is indeed an integrable operator, with $\mathbf{f}(\lambda) = \left(e^{i\lambda}, e^{-i\lambda}\right)$ *and* $\mathbf{g}(\lambda) = \frac{1}{2\pi i}\left(e^{-i\lambda}, -e^{i\lambda}\right)$. *This kernel appears in the bulk scaling limit for GUE [88], and was studied by many different authors, e.g. [110, 34].*

- *The Bessel kernels act on* $L^2(\mathbb{R})$ *through the kernel*

$$K_{Bessel}(\lambda,\mu) := \frac{J_\alpha(\sqrt{\lambda})\sqrt{\mu}J'_\alpha(\sqrt{\mu}) - \sqrt{\lambda}J'_\alpha(\sqrt{\lambda})J_\alpha(\sqrt{\mu})}{2(\lambda-\mu)} \tag{4.2.3}$$

  where J_α *is the Bessel function of order* α. *Taking for example* $\mathbf{f}(\lambda) = \frac{1}{2}\left(J_\alpha(\sqrt{\lambda}), -\sqrt{\lambda}J'_\alpha(\sqrt{\lambda})\right)$ *and* $\mathbf{g}(\lambda) = \frac{1}{2}\left(\sqrt{\lambda}J'_\alpha(\sqrt{\lambda}), J_\alpha(\sqrt{\lambda})\right)$ *one recognizes the integrable structure of this kernel, but this is not the only way to see that. This appears in some scaling limit for the LUE or JUE and was first studied in e.g. [104, 40].*

- *The Airy kernel acts on* $L^2(\mathbb{R}_+)$ *through the kernel*

$$K_{Airy}(\lambda,\mu) := \frac{\mathrm{Ai}(\lambda)\mathrm{Ai}'(\mu) - \mathrm{Ai}'(\lambda)\mathrm{Ai}(\mu)}{\lambda-\mu} \tag{4.2.4}$$

  where Ai *is the Airy function, that we already met in Chapter 2 and 3. Writing the kernel in this way, one can take* $\mathbf{f}(\lambda) := (\mathrm{Ai}(\lambda), -\mathrm{Ai}'(\lambda))$ *and* $\mathbf{g}(\lambda) := (\mathrm{Ai}'(\lambda), \mathrm{Ai}(\lambda))$ *to see the integrability structure. Althought, this is not the only way to see that. Indeed, using the alternative description given in* (2.1.16) *for the kernel, one can found another integrable structure for the Airy kernel by passing in Fourier coordinates (as done in [26]). In Chapter 6 and 7 we will follow this second procedure for the study of both the matrix and the finite temperature generalizations of the higher order Airy kernels. Anyway, as already said in the previous chapters, the Airy kernel appears in the edge scaling limit for GUE [103, 88].*

All these kernels can be also found in relation to many different models in statistical mechanics that were studied for instance in [81, 33, 79].

The first interesting property of the integrable operators is that their resolvents, whenever they exist, they are integrable too. This was first observed and proved in [60] and the result is resumed in the following lemma.

Definition 4.2.4. *For an integral operator* $\mathcal{K}$ *as in Definition 4.2.1, the correspondent resolvent operator is defined as* $\mathcal{R} := (1-\mathcal{K})^{-1}\mathcal{K}$, *when* $1-\mathcal{K}$ *is invertible.*

Lemma 4.2.5 ([54])**.** *Consider $\mathcal{K}$ an integrable operator with kernel* (4.2.1) *and suppose that* $(1-\mathcal{K})^{-1}$ *exist. Then the resolvent $\mathcal{R}$ is an integrable operator with kernel given by*

$$R(\lambda,\mu) = \frac{\mathbf{F}^T(\lambda)\mathbf{G}(\mu)}{\lambda-\mu} \tag{4.2.5}$$

where the matrix-valued functions $\mathbf{F},\mathbf{G}$ are recovered through

$$\mathbf{F}^T(\lambda) = (1-\mathcal{K})^{-1}\mathbf{f}^T, \quad \mathbf{G}(\lambda) = (1-\mathcal{K}^T)^{-1}\mathbf{g}. \tag{4.2.6}$$

In particular the diagonal condition holds also for the resolvent, i.e. $\mathbf{F}^T(\lambda)\mathbf{G}(\lambda) = 0$.

4.2.2 The Riemann-Hilbert problem associated to integrable operators

Given an integrable operator $\mathcal{K}$ as in Definition 4.2.1, the associated Riemann-Hilbert problem of the form 4.1.1 is defined through the pair (Σ, G) where Σ is the contour where the integral in the definition of $\mathcal{K}f(\lambda)$ is computed and the jump matrix G is defined as the $r \times r$ matrix

$$G(\lambda) := I_r - 2\pi i \mathbf{f}(\lambda)\mathbf{g}^T(\lambda). \tag{4.2.7}$$

The solution Y of the Riemann-Hilbert problem constructed in this way is then used to recover the kernel of the resolvent of $\mathcal{K}$. The result is resumed in the following theorem.

Theorem 4.2.6 ([54])**.** *Given the integrable operator $\mathcal{K}$, the operator* $(1-\mathcal{K})^{-1}$ *exists if and only if the Riemann-Hilbert problem 4.1.1 defined through the pair (Σ, G) related to $\mathcal{K}$ (described just above) is solvable. In particular, the functions $\mathbf{F},\mathbf{G}$ defining the kernel of the resolvent $\mathcal{R}$ are obtained in terms of the solution Y of the Riemann-Hilbert problem as*

$$\mathbf{F}(\lambda) = Y(\lambda)\mathbf{f}(\lambda), \quad \mathbf{G}(\lambda) = (Y^T(\lambda))^{-1}\mathbf{g}(\lambda) \tag{4.2.8}$$

and the solution Y of the Riemann-Hilbert probelm 4.1.1 for the pair (Σ, G) has integral reprensentation given by

$$Y(\lambda) = I_r - \int_\Sigma Y_-(\zeta)\frac{\mathbf{f}(\zeta)\mathbf{g}^T(\zeta)}{\zeta-\lambda}d\zeta. \tag{4.2.9}$$

In general, the integrable operators we are interested in will have kernels dependending on some auxiliary parameters. Thus, their Fredholm determinats (whether well defined) are functions of these parameters and their dependence on them should then be studied. Moreover, for the operators satisfying Theorem 3.1.5, the Fredholm determinants are interpreted as relevant probabilistic quantities in relation to the DPP defined through the operator, as stated in Proposition 3.1.9. Finding the explicit dependence on the parameters for these Fredholm determinants becomes even more crucial. The Riemann-Hilbert approach is indeed useful in this sense : it allows to derive a formula for the logarithmic derivative of these Fredholm determinants in terms of certain quantities related to the solution of the relevant Riemann-Hilbert problem. This essentially follows from the application of the Jacobi formula, namely

$$\delta \log \det(1 - \mathcal{K}) = - \operatorname{Tr}((1 - \mathcal{K})^{-1} \, \delta\mathcal{K}), \tag{4.2.10}$$

where δ denotes the variation with respect to the parameters on which $\mathcal{K}$ depends on, together with Theorem 4.2.6. Having explicit expression for the Fredholm determinant can be then used for example to study the asymptotic behavior of them.

4.3 Riemann-Hilbert problems and Hankel integral operators

There are cases in which we are interested in Fredholm determinants of operators that are not of integrable form but that can be proved to be equal, after some manipulations, to Fredholm determinants of operators of integrable type. For example, consider the Hankel matrix-valued operators $\mathcal{C}$ acting on $L^2(\mathbb{R}_+, \mathbb{C}^r)$ as

$$\left(\mathcal{C}\phi\right)(x) = \int_{\mathbb{R}_+} \mathbf{C}(x+y)\phi(y)dy, \quad \phi \in L^2(\mathbb{R}_+, \mathbb{C}^r) \tag{4.3.1}$$

with $\mathbf{C}$ a matrix-valued function having form

$$\mathbf{C}(z) := -i \int_{\gamma_+} e^{iz\mu}\mathbf{r}(\mu)d\mu \tag{4.3.2}$$

where $\mathbf{r}(\mu)$ is an integrable function and γ_+ is some curve in the upper complex plane. In [13] the authors proved that this kind of operators can be treated through a Riemann-Hilbert approach too. In this section we will go through the fundamental results obtained in that paper, and we will use them in Chapter 6 in order to relate the Fredholm determinants of a matrix-valued analogue of higher order Airy kernels to certain solutions of a matrix Painlevé II hierarchy.

The first step, is to prove that the Fredholm determinant of these Hankel operators coincides indeed with the Fredholm determinant of some operator on the space $L^2(\gamma_+, \mathbb{C}^r)$, as explained in the following statement.

Theorem 4.3.1 (Corollary 2.1 [13]). *The Hankel operators $\mathcal{C}$ of type* (4.3.1), (4.3.2), *with the function $\mathbf{r}(\mu) := E_1(\mu)E_2^T(\mu)$ and $E_j \in L^2 \cap L^\infty(\gamma_+, Mat(r \times r))$ are trace class on $L^2(\mathbb{R}_+, \mathbb{C}^r)$ and their Fredholm determinants are such that*

$$\det(1 + \mathcal{C}|_{L^2(\mathbb{R}_+,\mathbb{C}^r)}) = \det(1 + \mathcal{K}|_{L^2(\gamma_+,\mathbb{C}^r)}), \tag{4.3.3}$$

where $\mathcal{K} : L^2(\gamma_+, \mathbb{C}^r) \to L^2(\gamma_+, \mathbb{C}^r)$ *are integral operators with kernel*

$$K(\lambda, \mu) = \frac{E_1^T(\lambda) E_2(\mu)}{\lambda + \mu}. \tag{4.3.4}$$

Remark 4.3.2. *Integral operators with kernels of type* (4.3.4) *for some specific choice of the functions* $E_i(\lambda)$ *acting on* $L^2((0, \infty))$ *were previously studied by Tracy and Widom in [105], in relation with some integrable hierarchies.*

The proof is based on the use of the Fourier-Plancherel transform. The conjugation of $\mathcal{C}$ by this transform gives indeed an integral operator that shares its Fredholm determianant with $\mathcal{C}$ and that can be proven to be trace class on the correspondent Hardy space. This last result comes from the fact that the relevant operator can be seen as composition of Hilbert-Schmidt operators defined on appropriate functional spaces. By exchanging the order of the composition, one obtain exactly the operator $\mathcal{K}$ on $L^2(\gamma_+, \mathbb{C}^r)$ in the statement above, that still shares its Fredholm determinant with the operator $\mathcal{C}$ thanks to the Sylvester identity (cfr. [99, 48]).

Remark 4.3.3. *The operator* $\mathcal{K}$ *with kernel given in* (4.3.4) *is not exactly of the integrable form* (4.2.1)*, because of its denominator. Nevertheless, it was proven in [13] that also these operators* $\mathcal{K}$ *and* $\mathcal{K}^2$ *can be studied through a Riemann-Hilbert problem, extending in some way the theory of standard IIKS operators of [60]. Since in Chapter 6 we will be interested just into the square of some particular operator* $\mathcal{K}$*, in the following we will focus on the results that only concerns the squared operator.*

The relevant Riemann-Hilbert problem (Σ, G) of the form 4.1.1 for a function $Y(\lambda)$ with values in $\mathrm{GL}(2r)$ is built by taking as contour Σ the union of the two disjoint contours

$$\Sigma := \gamma_+ \cup \gamma_- \tag{4.3.5}$$

where $\gamma_- := -\gamma_+$, and as jump matrix

$$G(\lambda) := \begin{bmatrix} I_r & -2\pi i \mathbf{r}(\lambda) \chi_{\gamma_+}(\lambda) \\ -2\pi i \mathbf{r}(-\lambda) \chi_{\gamma_-}(\lambda) & I_r \end{bmatrix}. \tag{4.3.6}$$

Based on the IIKS theorem, the authors of [13] proved the following result about relating the solution of the Riemann-Hilbert problem for $Y(\lambda)$ and the operator $1 - \mathcal{K}^2$.

Theorem 4.3.4 (Theorem 3.1 [13])**.** *The resolvent operator* $\mathcal{R} := \mathcal{K}^2(1 - \mathcal{K}^2)^{-1}$ *on* $L^2(\gamma_+, \mathbb{C}^r)$ *has kernel* $R(\lambda, \mu)$ *expressed in terms of the solution* Y *of the Riemann-Hilbert problem* (Σ, G) *defined in* (4.3.5), (4.3.6)*, as follows*

$$R(\lambda, \mu) = \begin{bmatrix} E_1^T(\lambda) & 0_r \end{bmatrix} \frac{Y^T(\lambda) Y^{-T}(\mu)}{\lambda - \mu} \begin{bmatrix} 0_r \\ E_2(\mu) \end{bmatrix}. \tag{4.3.7}$$

The solution Y of the Riemann-Hilbert problem (Σ, G) exists if and only if $1-\mathcal{K}^2$ is invertible.

Now, suppose that the operators $\mathcal{C}$ and thus $\mathcal{K}$ depend on some auxiliary parameters. As a byproduct the jump matrix G and the solution Y of the Riemann-Hilbert problem (Σ, G) associated to these operators also depends on these auxiliary parameters. Denoting by δ the variation with respect to these parameters, the authors of [13] expressed the variation of the Fredholm determinant of $\mathcal{K}^2$ in the following way.

Theorem 4.3.5 (Theorem 4.1 [13])**.** *We have that*

$$\delta \log \det(1-\mathcal{K}^2) = \frac{1}{2\pi i}\int_\Sigma \mathrm{Tr}\left(Y_-^{-1}Y_-'\delta G G^{-1}\right) d\lambda, \tag{4.3.8}$$

where the $'$ denotes the derivative w.r.t. the parameter λ.

The main ingredient for the proof of this result is the application of the formula (4.2.10) to the relevant IIKS operator acting on $L^2(\Sigma) \simeq L^2(\gamma_+)\bigoplus L^2(\gamma_-)$ that is related to the Riemann-Hilbert problem (Σ, G) defined through (4.3.5), (4.3.6). This operator has Fredholm determinant that coincides with the one of $\mathcal{K}^2$ by the very construction, and thus the proof follows.

This is exactly the result that we need in Chapter 6 in order to find the formula that express the Fredholm determinant of the matrix analogue of the higher order Airy kernels, in terms of some distinguished solutions of the matrix Painlevé II hierarchy.

Remark 4.3.6. *Relation* (4.3.8) *allows to explicitely compute the logarithmic derivative of the relevant Fredholm determinants. Indeed, the dependence of the jump matrix G on the auxiliary parameters is explicit thus the quantity inside the integral on the right hand side $\delta G G^{-1}$ is explicit too and so does the entire integral. A very important example of dependence (this is indeed the case we have to deal with in Chapter 6) is when the jump matrix can be factorized as*

$$G(\lambda, \vec{T}) = e^{T(\lambda,\vec{T})}G_0(\lambda)e^{-T(\lambda,\vec{T})},$$

where $T(\lambda, \vec{T}) = \sum_{j=0}^m T_j\lambda^j$ is a matrix depending on the diagonal matrices T_j that are considered here as the deformation parameters.

Remark 4.3.7. *For every parametric family of Riemann-Hilbert problems 4.1.1 depending in a sufficiently smooth way on the auxiliary parameters and having Σ with no self-intersections, one can always define the integral in the right hand side of equation* (4.3.8)*. Over the space of deformations of these Riemann-Hilbert problems this quantity is interpreted as a* 2*-form*

$$\Theta_M^Y(\delta) \coloneqq \frac{1}{2\pi i}\int_\Sigma \mathrm{Tr}\left(Y_-^{-1}Y_-'\delta G G^{-1}\right) d\lambda,$$

and whether it is closed, one can defined up to a constant, its correspondent tau function in such a way that $\delta\tau_Y = \Theta_M^Y(\delta)$. For more details on this topic we refer to [13, 11] and to the previous series of works of the Japanese school [66, 63, 64]. We will use this 2*-form (for another specific Riemann-Hilbert problem) in Chapter 8.*

Chapter 5

Isomonodromic deformations as Lax pairs

The aim of this chapter is to introduce the theory of isomonodromic deformations focusing in particular on its relation with the Painlevé II equation (2.1.2). All the six equations (2.1.1) – (2.1.6), indeed, admit (at least) a Lax pair representation given by the isomonodromic deformations of a specific 2×2 linear ODEs system with rational coefficients. This general result was first proven in the works [66, 63, 64]. For the Painlevé II equation specifically the works of Flaschka and Newell [36, 37] investigated alternative connections between the Painlevé II equation and the theory of isomonodromic deformations. The Flaschka-Newell Lax pair for the Painlevé II equation, given by equations (3.2a,b), (3.3a,b) in [36], was then generalized to a Lax pair for the all members of the Painlevé II hierarchy (2.2.22) in [30]. The construction of analogue Lax pairs for the matrix and then integro-differential Painlevé II hierarchy, that we are going to study respectively in Chapters 6, 7, will be a fundamental element in the proof of our results generalizing the Tracy-Widom formula.
Generally speaking, the existence of isomonodromic Lax pairs for the Painlevé equations has been very useful to study remarkable properties, asymptotics in particular, of certain Painlevé trascendents. Many results have been collected and proved in details in the monograph [39], that will be indeed the main reference for this chapter.
From another point of view, the isomonodromic representation of Painlevé equations opened the way to a new, more geometrical, field of study: the Painlevé *monodromy manifolds* (see e.g. [27] and references therein). Given a linear system of ODEs with rational coefficients, its monodromy manifold is the space of its monodromy data considered together with eventual algebraic relations between them. For the case of regular singularities (as for the isomonodromic Lax pair for the PVI equation (2.1.6)) the monodromy manifold is related to some *character variety* of the Riemann sphere with prescribed punctures (for PVI, the $SL_2(\mathbb{C})$ character variety of the Riemann sphere with 4 punctures). For systems carrying irregular singularities (as all the isomonodromic Lax pairs for the remaining Painlevé equations, including PII), as we will see, the set of monodromy data is more complicated

mainly because of the presence of Stokes phenomena. Thus the geometrical description of the corresponding monodromy manifolds cannot simply be done in terms of character varieties. Their corresponding generalizations are now known under the name of *wild character varieties*, terminology born in [87] and consolidated by Boalch. One of the major aspects in the study of monodromy manifolds is their Poisson (symplectic) structure, in relation with the Poisson-Lie structure on the rational matrices (coefficients of the relevant ODEs) through the monodromy map. The first papers that studied this problem are [37, 107] where the authors focused on monodromy manifolds for some specific systems of ODEs. Some years later, the series of papers [17, 18, 19] by Boalch investigated the problem in greater generality. In Chapter 8 we are going to study the symplectic structure of the monodromy manifolds of a rank 2 polynomial equation, i.e. with only one irregular singularity of arbitrary Poincaré rank at ∞ (which underlies for odd Poincaré ranks the case of the Lax pair for the homogeneous Painlevé II hierarchy [30]), the case studied by Flaschka and Newell [37].
The Chapter is organized as follows: in the first section we are going to review the fundamental results on the theory of linear system of ODEs in the complex plane and we are going to define the main concepts of monodromy data and monodromy map. In the second section we are going to give the definition of isomonodromic deformation and finally in the third section we will see the isomonodromic Lax pairs for the Painlevé II equation and hierarchy.

5.1 System of ODEs with rational coefficients

In the following two sections we are going to resume the main concepts and results of the theory of linear ODEs in the complex plane, contained in Chapters 1 – 4 of [39]. For more details and for the proofs of the statements, we refer thus to them (and references therein). Let us consider $M(\lambda)$ a $N \times N$ matrix-valued rational function, with $N > 1$ and $\lambda \in \mathbb{C}$. We are interested in finding a $N \times N$ matrix-valued solution $\Psi(\lambda)$ of the linear ODE

$$\frac{d\Psi}{d\lambda} = M(\lambda)\Psi. \tag{5.1.1}$$

5.1.1 Description of local solutions

For a given $\lambda_0 \in \mathbb{CP}^1$, the behavior of a local solution Ψ in a neighborhood of λ_0 is essentially determined by the behavior of the coefficient matrix $M(\lambda)$ at the given point λ_0. Given that $M(\lambda)$ is rational in λ we only have three possibilities : λ_0 is a regular point for the differential $M(\lambda)d\lambda$, or it is a simple pole or it is a pole of greater order (we say that it has Poincaré rank $r > 0$ at λ_0, meaning that the Laurent series of $M(\lambda)d\lambda$ at λ_0 has nonzero coefficient up to the power $-r-1$ in the local coordinate near λ_0). In each of these possible configurations we have different local behaviors of Ψ, as described by the following results.

Theorem 5.1.1 ([39])**.** *Consider $\lambda_0 \in \mathbb{CP}^1$ and a given $N \times N$ invertible matrix Ψ_0. If the* *matrix coefficient $M(\lambda)$* is holomorphic in a disk B_{λ_0} centered in λ_0, then there is a unique solution of the ODE* (5.1.1) *holomorphic in the same disk and satisfying the initial condition $\Psi(\lambda_0) = \Psi_0$.*

Thus, as far as we look for solutions of the equation (5.1.1) near points that are regular for the matrix coefficient $M(\lambda)$, we get local solutions that are smooth too.
Consider now the case where $M(\lambda)d\lambda$ has an isolated simple pole at the given point $\lambda_0 \in \mathbb{CP}^1$. For ζ being the local parameter near λ_0 ($\zeta = \lambda - \lambda_0$ in case λ_0 is finite, $\zeta = \frac{1}{\lambda}$ in case λ_0 is ∞), we can then write in a punctured disk centered at λ_0, $B_{\lambda_0} \setminus \{\lambda_0\}$, the following representation

$$M(\lambda)d\lambda = \sum_{k=-1}^{\infty} M_{k+1}\zeta^k d\zeta, \quad M_0 \neq 0. \tag{5.1.2}$$

The behavior of Ψ near λ_0 is then uniquely determined, up to the spectral properties of the matrix M_0, as follows.

Theorem 5.1.2 ([39]). *Given the previous hypothesis on $M(\lambda)d\lambda$, suppose that the coefficient M_0 is diagonalizable, namely $M_0 = PT_0P^{-1}$ with T_0 a diagonal matrix (called the formal monodromy exponent). Also, suppose that M_0 has nonresonant eigenvalues*† *(i.e. the difference of each couple of distinct eigenvalues is not an integer). Then the ODE* (5.1.1) *has a fundamental solution Ψ near λ_0 of the form*

$$\Psi(\lambda) = \hat{\Psi}(\lambda)\zeta^{T_0}, \tag{5.1.3}$$

with $\hat{\Psi}(\lambda)$ holomorphic and invertible in B_{λ_0} and uniquely determined by the value of $\hat{\Psi}(\lambda_0) = P$.

Notice that it is equivalent to say that in the disk B_{λ_0} the solution Ψ is in the form

$$\Psi(\lambda) = P\left(\sum_{k=0}^{\infty} \Psi_k \zeta^k\right)\zeta^{T_0}, \quad \Psi_0 = I_N \tag{5.1.4}$$

where the power series is convergent. This is indeed the main difference between the behavior of a local solution near a simple pole and near a higher order pole of $M(\lambda)d\lambda$, as we are going to explain. Consider now the case where $\lambda_0 \in \mathbb{CP}^1$ is a pole of Poincaré rank $r > 0$ for the differential $M(\lambda)d\lambda$, namely we can write in the punctured disk $B_{\lambda_0} \setminus \{\lambda_0\}$, using the local coordinate ζ near λ_0, the following representation

$$M(\lambda)d\lambda = \sum_{k=-r-1}^{\infty} M_{k+1}\zeta^k d\zeta, \quad M_{-r} \neq 0. \tag{5.1.5}$$

*or equivalently the differential $M(\lambda)d\lambda$ is holomorphic in the same disk.

†In the cases where M_0 is not diagonalizible or it is so but it does have resonant eigenvalues the statement is adapted with a slightly different behavior of Ψ.

Assume again that the leading coefficient M_{-r} is diagonalizable, namely

$$M_{-r} = PT_{-r}P^{-1} \tag{5.1.6}$$

with T_{-r} a diagonal matrix, that has all distinct nonzero eigenvalues $\alpha_i, i = 1, \dots, N$.

Theorem 5.1.3 ([39])**.** *In the above hypothesis for the differential $M(\lambda)d\lambda$, there is a unique formal fundamental solution of the ODE* (5.1.1) *in the punctured disk $B_{\lambda_0} \setminus \{\lambda_0\}$ and it is written in the form*

$$\Psi_f(\lambda) = P\left(\sum_{k=0}^{\infty}\Psi_k\zeta^k\right)\exp\left(\frac{T_{-r}}{-r}\zeta^{-r} + \cdots + \frac{T_{-1}}{-1}\zeta^{-1} + T_0\ln\zeta\right), \quad \Psi_0 = I_N \tag{5.1.7}$$

with T_k all diagonal matrices for $k = -r, \dots, 0$. Both the coefficients Ψ_j, $j \geq 0$, and the exponents T_k, $k = -r, \dots, 0$ are determined recursively as polynomials of the coefficients M_k in (5.1.5)*.*

The solution Ψ is called formal since typically the series in (5.1.7) does not converge. It turns out that Ψ as in (5.1.7) is actually only the asymptotics (for λ approaching the irregular singularity λ_0) of a genuine fundamental solution of (5.1.1) uniquely defined in a certain sector of the punctured disk $B_{\lambda_0} \setminus \{\lambda_0\}$. These sectors are also known as Stokes sectors, and they are defined as the sectors of the disk B_{λ_0} containing exactly one of the lines defined as $\ell_m^{(i,j)} \coloneqq \{\zeta | \ |\zeta| < \rho, \quad \arg\zeta = \frac{1}{r}\arg(\alpha_i - \alpha_j) + \frac{\pi}{r}\left(m + \frac{1}{2}\right)\}$, $m = 0, \dots, 2r-1$ and $i, j = 1, \dots, N$ with $i < j$.[‡] More precisely the result reads as follows.

Theorem 5.1.4 ([39])**.** *In the hypothesis above, inside any Stokes sector contained in the disk B_{λ_0} there exists a unique fundamental solution $\Psi(\lambda)$ of the ODE* (5.1.1) *such that*

$$\Psi(\lambda) \sim \Psi_f(\lambda) \ \ for \quad \lambda \to \lambda_0, \tag{5.1.8}$$

where $\Psi_f(\lambda)$ is given as in (5.1.7) *and the branch of the logarithm in that formula is chosen.*

Notice that the Stokes sectors can be defined in a canonical way, so that $B_{\lambda_0} \setminus \{\lambda_0\}$ is always covered by $2r$ of them. For $\delta > 0$ sufficiently small, consider the sector

$$\mathcal{S} \coloneqq \{\zeta \in \mathbb{C} | \ 0 < |\zeta| < \rho, \quad \theta_1 < \arg\zeta < \theta_1 + \frac{\pi}{r} + \delta\}. \tag{5.1.9}$$

Then $\mathcal{S}$ is a Stokes sector. With that in mind, one constructs

$$\mathcal{S}_n \coloneqq e^{i\frac{\pi}{r}(n-1)}\mathcal{S}, \ \ n = 1, \dots, 2r. \tag{5.1.10}$$

All $\mathcal{S}_n$ defined in this way are Stokes sectors; moreover, they cover the punctured disk and they are such that $\mathcal{S}_1 = \mathcal{S} = \mathcal{S}_{2r+1}$. It follows from Theorem 5.1.4 that we can define $2r$

[‡]This condition follows while looking for the uniqueness of a fundamental solution of (5.1.1) near a higher order pole, with asymptotics given by (5.1.7).

canonical solutions $\Psi_n(\lambda)$ near λ_0 a higher order pole of $M(\lambda)d\lambda$, each one of them uniquely defined by the asymptotic condition (5.1.7) in the correspondent Stokes sector $\mathcal{S}_n$.
From this construction follows the definition of the Stokes matrices

$$S_n \coloneqq \Psi_n^{-1}(\lambda)\Psi_{n+1}(\lambda), \quad \lambda \in \mathcal{S}_n \cap \mathcal{S}_{n+1}, \quad n = 1, \dots, 2r. \tag{5.1.11}$$

These matrices can be shown to be constant upper or lower triangular matrices, with unit diagonals. Together with the exponents $T_k,\ k = -r, \dots, 0$ the Stokes matrices uniquely determine, up to gauge transformations, the system (5.1.1) having at λ_0 an irregular singular point (for more details on this topic, also known as the Stokes phenomena, see Theorem 5.1 of [39]).

From local to global With these four results in mind, one can construct a local solution of the ODE (5.1.1) starting at any point of the punctured Riemann sphere. But what about global solutions? The answer to this question is given by the following Monodromy Theorem.

Theorem 5.1.5 ([39])**.** *Let $m_i \in \mathbb{CP}^1, i = 1, \dots, n$ be the isolated poles of the coefficient matrix $M(\lambda)$ of the ODE* (5.1.1) *and let $\gamma : [0,1] \to \mathbb{CP}^1 \setminus \{m_i\}_{i=1}^n$ a curve. Consider the germ of a solution of* (5.1.1) *at the (regular) point $\gamma(0)$, namely*

$$\Psi(\lambda) = \sum_{k=0}^{\infty} \Psi_k \zeta^k, \quad \zeta \ \textit{the local coordinate near } \ \gamma(0). \tag{5.1.12}$$

Then $\Psi(\lambda)$ admits analytic continuation all along the path γ to the point $\gamma(1)$. Furthermore, its analytic continuation only depends on the homotopy class of γ.

This result gives the recipe to construct global solutions of the ODE (5.1.1) starting from any point of the punctured Riemann sphere : just consider any local solution and then perform analytic continuation. In a certain way, the construction of global solutions essentially relies on the representation of local solutions. The behavior of local solutions was given by formulae (5.1.4), (5.1.7), in which the main ingredients are the formal monodromy exponent T_0 and the exponents $T_k, k = -r, \dots, 0$, together with the Stokes matrices (5.1.11) respectively. This set of data, should be then completed with the description of the passage from one local representation to the other: all together they form a set of global monodromy data that allows us to completely determine the ODE (5.1.1).

5.1.2 Monodromy data of ODEs

We are now going to describe two sets of data: the global monodromy data and the essential monodromy data. Suppose that among the poles m_ν of $M(\lambda)d\lambda$ we have simple poles for $\nu = 1, \dots, p \leq m$ and then for $\nu = p+1, \dots, m$ we have higher order poles, of Poincaré rank $r_\nu, \nu = p+1, \dots, m$. From the previous discussion, we first collect the following data

- $T_0^{(\nu)}$ for $\nu = 1, \dots, p$;
- $T_k^{(\nu)}$ with $k = -r_\nu, \dots, 0$ for $\nu = p+1, \dots, m$, together with $S_l^{(\nu)}$ for $l = 1, \dots, 2r_\nu$ and ν in the same range.

These data describe the local behavior of solutions $\Psi^{(\nu)}$ near all the simple poles $\{m_\nu\}_{\nu=1}^p$ of $M(\lambda)d\lambda$, and solutions $\{\Psi_l^{(\nu)}\}_{l=1}^{2r_\nu}$ in the canonical Stokes sectors near its higher order poles $\{m_\nu\}_{\nu=p+1}^n$. Consider now a generic fundamental solution of (5.1.1) at a point $m_0 \in \mathbb{CP}^1 \setminus \{m_i\}_{i=1}^n$, determined by initial condition $\Psi(m_0) = \Psi_0$, for Ψ_0 any invertible matrix. Each local solution $\Psi, \Psi^{(\nu)}, \Psi_l^{(\nu)}$, thanks to Theorem 5.1.5, can be analytically continued along every path contained in the punctured Riemann sphere, giving back global solution of the same ODE (5.1.1). Thus, every two of these solutions can only differ by right multiplication by a constant matrix, called the connection matrix. In particular, one defines

$$\Psi(\lambda) = \Psi^{(\nu)}(\lambda)E_\nu, \ \text{ and } \ \Psi(\lambda) = \Psi_1^{(\nu)}(\lambda)E_\nu, \tag{5.1.13}$$

for $\nu = 1, \dots, p$ and $\nu = p+1, \dots, m$ respectively. The matrices E_ν exactly describe the passage from a local solution to the other, and thus conclude the global picture we needed for the complete description of the solutions of the ODEs (5.1.1).
The global monodromy data set is then defined as the following collection

$$\mathbf{M} \coloneqq \{m_1, \dots, m_n,\ T_0^{(1)}, \dots, T_0^{(p)},\ (T_k^{(\nu)}, S_l^{(\nu)})_{k=-r_\nu,\dots,0,\ l=1,\dots,2r_\nu}^{\nu=p+1,\dots,m},\ E_1, \dots, E_m\}. \tag{5.1.14}$$

As shown in Proposition 2.2 of [39], this collection of data uniquely defines the ODE (5.1.1) with $M(\lambda)d\lambda$ having exactly m poles with fixed Poincaré rank $r_\nu, \nu = 1, \dots, m$ (meaning $r_\nu = 0$ for $\nu = 1 \dots, p$). This is no longer true when we restrict the set of global monodromy data to the essential monodromy data, i.e. we eliminate from $\mathbf{M}$ the positions of the poles and the coefficients $T_k^{(\nu)}$ for $k = -r_\nu, \dots, 1$ and $\nu = p+1, \dots, m$. This restricted subset, defined as the set of essential monodromy data, it is explicitly given by the collection

$$\mathbf{m} \coloneqq \{T_0^{(\nu)},\ S_l^{(\mu)},\ E_\nu\}_{l=1,\dots,2r_\nu,\mu=p+1,\dots,m}^{\nu=1,\dots,m}. \tag{5.1.15}$$

In particular, we have that the monodromy map

$$\{M(\lambda)d\lambda \text{ with } m \text{ poles of fixed multiplicities } r_\nu\} \ \to \ \{\mathbf{m}, \text{ sets of essential monodromy data}\}$$

is no longer one-to-one. The problem of describing the subset of rational matrices $M(\lambda)$, coefficient of (5.1.1), sharing the same set of essential monodromy data $\mathbf{m}$ is exactly what isomonodromy deformations are about.

Remark 5.1.6. *The monodromy manifold for a given linear system of ODEs can be defined as the space of its Stokes / connection matrices together with eventual constraints among them. The constraints will change from case to case. In Chapter 8 we will study a specific example of monodromy manifold, called Stokes manifold, that is associated to a polynomial* *linear ODE. As we will see, in this case the monodromy manifold is simply given by the collection of some Stokes matrices. Although they are not independent, they should satisfy an algebraic equation (corresponding to the canonical relation in the fundamental group of* $\mathbb{CP}^1 \setminus \{poles\}$*).*

5.2 Isomonodromic deformations

The study of isomonodromic deformations can be formalised as follows: suppose that the coefficient matrix of the ODE (5.1.1) now depends holomorphically on some extra complex parameters $t_1, \dots, t_K$, namely

$$M(\lambda) = M(\lambda, t_1, \dots, t_K) = M(\lambda, t). \tag{5.2.1}$$

Definition 5.2.1. *An isomonodromy deformation is given by a holomorphic family of rational matrices as in* (5.2.1) *which is an admissible deformation and preserves the set of essential monodromy data of* $M(\lambda, t = 0)$*. More specifically the family* (5.2.1) *has to satisfy the following requirements:*

1. *the number* n *of poles does not depend on* $t_i, i = 1, \dots, K$*. Moreover there exist some disks* B_ν*,* $\nu = 1, \dots, n$ *such that each pole* $m_\nu \in B_\nu$ *for all values of the parameters* t_i *and* $B_\nu \cap B_\mu$ *is the empty set for all* $\nu \neq \mu$*;*

2. *the spectral properties of the leading coefficients of the Laurent series of* $M(\lambda)d\lambda$ *at each singular point do not depend on* t_i*;*

3. *for all the poles* $m_\nu(t)$ *with Poincaré rank* $r_\nu > 0$ *the Stokes sectors in the punctured disk centered at the corresponding pole* $m_\nu(t)$ *are* t*-independent under translation* $\lambda \to \lambda - m_\nu(t)$*;*

4. *canonical solutions of the ODE* (5.1.1) *are holomorphic w.r.t.* t *and for local solutions near irregular points, their asymptotic behavior* (5.1.7) *holds uniformly in* t*;*

5. *(isomonodromic condition) all the formal monodromy exponents* $T_0^{(\nu)}$*, the Stokes matrices* $S_l^{(\mu)}$ *and the connection matrices* E_ν*,* $\nu = 1, \dots, n,\ \mu = p+1, \dots, m,\ l = 1, \dots, r_\nu$ *are* t*-independent.*

The list of requirements in the definition above can actually be translated into the fact that the entries of the matrix coefficient $M(\lambda)$ should solve some further system of nonlinear differential equations (w.r.t. the deformation parameters t_i). This result is obtained by looking at the following differential

$$\Xi(\lambda, t) := d\Psi\Psi^{-1} = \sum_{j=1}^{K} \frac{\partial \Psi}{\partial t_j} dt_j \Psi^{-1}, \tag{5.2.2}$$

that, thanks to the last requirement in the above definition, is actually a single-valued analytic function in $\mathbb{CP}^1 \setminus \{m_\nu\}_{\nu=1}^n$. Studying its behavior near the poles m_ν, using the formulae (5.1.4), (5.1.7) for the local solutions of the ODE (5.1.1) near these points and the fact that the essential monodromy data are t-independent, more can be said about $\Xi(\lambda, t)$. This study was first done in [66] and in the following we cite one of their main results.

Main assumption Assume that the pole $m_n(t) = \infty$ for all t and that the leading coefficient of the Laurent series of $M(\lambda)d\lambda$ at ∞ is already diagonal. Also, assume that the essential monodromy data of (5.1.1) are defined by taking as basic fundamental solution Ψ the local solution near $m_n = \infty$ (the canonical solution $\Psi^{(n)}$ if $m_n = \infty$ is a simple pole, the canonical solution in the first Stokes sector $\Psi_1^{(n)}$ if $m_n = \infty$ is of higher order).

Theorem 5.2.2 ([66])**.** *The differential $\Xi(\lambda, t)$ is a rational matrix-valued function in λ with poles coinciding with $m_1, \dots, m_{n-1}, m_\nu = \infty$ and with the same Poincaré rank $r_\nu, \nu = 1, \dots, n$ of $M(\lambda)$. In particular, $\Xi(\lambda)$ can be explicitly and uniquely determined in terms of the coefficients of the Laurent series of $M(\lambda)$ near each one of its singular points. Namely*

$$\Xi(\lambda) = \Xi(\lambda, \{M_k^{(\nu)}\}, \{m_\nu\}), \tag{5.2.3}$$

with $M_k^{(\nu)}$ defined from the principal part decomposition of $M(\lambda)$

$$\begin{aligned} &M(\lambda) = M^{(\infty)}(\lambda) + \sum_{\nu=1}^{n-1} M^{(\nu)}(\lambda), \\ &M^{(\nu)}(\lambda) = \sum_{j=1}^{r_\nu+1} (\lambda - m_\nu)^{-j} M_{-j+1}^{(\nu)}, \quad \nu = 1, \dots, n-1, \\ &M^{(\infty)}(\lambda) = -\sum_{j=0}^{r_\infty-1} \lambda^j M_{-j-1}^{(\infty)}, \quad \textit{if } r_\infty > 0, \quad M^{(\infty)}(\lambda) = 0, \quad \textit{otherwise.} \end{aligned} \tag{5.2.4}$$

This result allows us to rewrite equation (5.2.2) as a differential equation that the function Ψ should satisfy w.r.t. the parameters $t_i, i = 1, \dots, K$. Namely it reads as

$$d\Psi = \Xi(\lambda)\Psi \quad \text{i.e.} \quad \frac{\partial \Psi}{\partial t_j} = \Xi_j(\lambda)\Psi, \quad \text{with} \quad \Xi(\lambda) = \sum_{j=1}^{K} \Xi_j(\lambda) dt_j, \quad \text{and} \quad j = 1, \dots, K. \tag{5.2.5}$$

In conclusion, assuming that $M(\lambda) = M(\lambda, t)$ is an isomonodromic deformation (respecting the main assumption written above) one obtains the coupled overdetermined system

$$\begin{cases} \frac{\partial \Psi}{\partial \lambda} = M(\lambda)\Psi, \\ d\Psi = d\Xi(\lambda)\Psi. \end{cases} \tag{5.2.6}$$

Its cross-differentiation gives rise to the following differential equation for the matrix coefficient $M(\lambda)$, namely the compatibility condition,

$$dM = \frac{\partial \Xi}{\partial \lambda} + [M, \Xi], \tag{5.2.7}$$

that holds identically in λ. This equation becomes then a system of nonlinear differential equations for the coefficients $M_k^{(\nu)}$ illustrated above. We say that the system (5.2.6) is a Lax pair for the equation (5.2.7). Choosing appropriately the type of isomonodromic deformations for $N = 2$ to look at, namely the number of poles of $M(\lambda)$ and their Poincaré ranks, one obtains that equation (5.2.7) gives respectively one of the six Painlevé equations (2.1.1)–(2.1.6) and thus the corresponding system (5.2.6) is the Lax representation of the relevant Painlevé equation.

Remark 5.2.3. *Notice that there is also a converse of the previous result, meaning that equation* (5.2.7) *is also a sufficient condition to describe an isomonodromic deformation of a rational matrix-valued function* $M(\lambda, t)$ *with fixed number of poles and Poincaré ranks, described as in* (5.2.4)*. For more details we refer to Theorem* 4.1 *in [39] (see also [66]).*

5.3 Isomonodromic representations of the Painlevé II equation and hierarchy

In this last section we are only going to collect well known results about the isomonodromic Lax pair representation of the Painlevé II equation and hierarchy. These representations were indeed fundamental in the papers [71] and [26] in order to re-prove and extend to the all Painlevé II hierarchy the Tracy-Widom result (given in Theorems 2.1.7, 2.2.12 respectively) about the Hastings-McLeod solutions of the Painlevé II equation. For the same reason, it will be fundamental in Chapter 6, 7 to construct an analogue Lax pair for the matrix and integro-differential Painlevé II hierarchies. In the following $\sigma_i, i = 1, 2, 3$ denotes the standard Pauli's matrices, while $\sigma_\pm$ are 2×2 matrices having as unique nonzero entry 1 at $(1, 2)$ and $(2, 1)$ respectively.

Theorem 5.3.1 (Appendix I, [36])**.** *The Painlevé II equation* (2.1.2) *for the function* $u(t)$ *follows from the compatibility condition of the* 2×2 *system*

$$\begin{aligned} \frac{\partial \Psi}{\partial \lambda} &= M\Psi, \quad \textit{with} \quad M(\lambda, t) = -i(4\lambda^2 + t + 2u^2)\sigma_3 + \left(4\lambda u + \frac{\alpha}{\lambda}\right)\sigma_1 - 2u_t\sigma_2 \\ \frac{\partial \Psi}{\partial t} &= L\Psi, \quad \textit{with} \quad L(\lambda, t) = -i\lambda\sigma_3 + u\sigma_1, \end{aligned} \tag{5.3.1}$$

describing isomonodromic deformations of a rank 2 ODE with one irregular singularity of Poincaré rank 3 *at* ∞ *and a simple pole at* 0*.*

The above system is known as the Flaschka-Newell Lax pair for the Painlevé II equation. This Lax pair is the one used in [71], in order to recover the Tracy-Widom result (Theorem

2.1.7) for the Hastings-McLeod solution of the Painlevé II equation through the Riemann-Hilbert approach. Another Lax pair was discovered by Jimbo and Miwa and it is reported in the following.

Theorem 5.3.2 (Appendix C, [63])**.** *The Painlevé II equation* (2.1.2) *for the function* $y(t)$ *follows from the compatibility condition of the* 2×2 *system*[§]

$$\begin{aligned}
\frac{\partial\Psi}{\partial\lambda} &= U\Psi, \quad \textit{with} \quad U(\lambda,t) = \left(\lambda^2 + \frac{t}{2} + z\right)\sigma_3 + \left(u(\lambda - y)\right)\sigma_+ - \frac{2}{u}(\lambda z + yz - \alpha + \frac{1}{2})\sigma_- \\
\frac{\partial\Psi}{\partial t} &= V\Psi, \quad \textit{with} \quad V(\lambda,t) = \frac{\lambda}{2}\sigma_3 + \frac{u}{2}\sigma_+ - \frac{z}{u}\sigma_-,
\end{aligned} \tag{5.3.2}$$

describing isomonodromic deformations of a rank 2 ODE with a degree 2 *polynomial matrix coefficient.*

Notice that the Jimbo-Miwa Lax pair was already known by Garnier [45] in an equivalent form. Also, the Lax pairs written above in equations (5.3.1) and (5.3.2) are really independent since their respective set of essential monodromy data are not isomorphic. Therefore, there is no gauge transformation that allows to pass from one to the other. Notice that there exists a third rank 2 Lax pair for the Painlevé II equation, known as the Harnad-Tracy-Widom Lax pair, but it is shown to be gauge equivalent to the Flaschka-Newell Lax pair (for more details see Proposition 5.2 of [39]). Actually, there exists also another Lax pair for the Painlevé II equation, of rank 3, and we refer to the article [69] for more details about that. In the same work the authors also describe the relation between the Jimbo-Miwa Lax pair and the Harnad-Tracy-Widom one in terms of the generalized Laplace transform.
In the paper [30] the authors extended the Flaschka-Newell isomonodromic Lax pair for the entire Painlevé II hierarchy as defined in equation (2.2.22). In particular, the n-th member of the hierarchy has a Lax pair representation given by the isomonodromic deformations of a rank 2 linear ODE having a pole at ∞ of Poincaré rank $2n+1$ and a simple pole at 0.

Theorem 5.3.3. *[Section* 3*, [30]] The n-th member of the Painlevé II hierarchy* (2.1.2) *for the function* $u(t)$ *follows from the compatibility condition of the* 2×2 *system*

$$\begin{aligned}
\frac{\partial\Psi}{\partial\lambda} &= M^{(n)}\Psi, \quad \textit{with} \quad M^{(n)} = \left(\sum_{j=0}^{2n} A_j(i\lambda)^j - it\right)\sigma_3 + \left(\sum_{j=0}^{2n-1} B_j(i\lambda)^j\right)\sigma_+ + \left(\sum_{j=0}^{2n-1} C_j(i\lambda)^j\right)\sigma_- + \frac{\alpha_n}{\lambda}\sigma_1 \\
\frac{\partial\Psi}{\partial t} &= L\Psi, \qquad \textit{with} \quad L = -i\lambda\sigma_3 + u\sigma_1
\end{aligned} \tag{5.3.3}$$

where the coefficients A_j, B_j, C_j *for every* j *are differential polynomials in* u*, described by closed formulae involving the Lenard recursion operators* (2.2.6)*. For their precise form see*

[§]The compatibility condition actually gives a system of three differential equations of first order, for u, z, y that are all functions of t. Differentiating again the equation for u and eliminating the variables y, z and their derivatives one obtains equation (2.1.2), with actually a minus sign in front of the constant term α.

equations (17*a*)*–*(17*g*) *in [30].*

This Lax pair is the one used in the paper [26] in order to achieve the proof of Theorem 2.2.12. In Chapter 6 we are going to construct a Lax pair for a $r \times r$ matrix Painlevé II hierarchy, that can be thought as a block-matrix generalization of the above Lax pair.

Chapter 6

The matrix Painlevé II hierarchy

The results contained in the article [101] will be discussed in this chapter. The aim of this paper is to relate a family of solutions of a noncommutative version of the Painlevé II hierarchy to Fredholm determinants of a matrix version of the n-th higher order Airy kernels. The scalar versions of these operators have been recently studied in [81], in relation with non-interacting fermionic models (as already discussed in the previous chapters).

In order to construct our matrix analogue, we first define a matrix-valued version of the n-th Airy function, in the following way

$$\mathbf{Ai}_{2n+1}(x,\vec{s}) := \Big(c_{j,k}\mathrm{Ai}_{2n+1}(x+s_j+s_k)\Big)_{j,k=1}^{r}, \qquad c_{j,k}\in\mathbb{C}, \qquad x\in\mathbb{R}, \tag{6.0.1}$$

where $\mathrm{Ai}_{2n+1}(x+s_j+s_k)$ is a shift of the n-th scalar Airy function, for some real parameters s_l, $l=1,\dots,r$. We recall that the n-th scalar Airy function, $\mathrm{Ai}_{2n+1}(x)$, is defined as a particular solution of the n-th generalized Airy equation, written in (2.2.32) in Chapter 2, for every $n\geq 1$. In this paper we will consider these functions $\mathrm{Ai}_{2n+1}(x)$ as contour integrals

$$\mathrm{Ai}_{2n+1}(x) := \int_{\gamma_+^n}\frac{1}{2\pi}\exp\left(\frac{\mathrm{i}\mu^{2n+1}}{2n+1}+\mathrm{i}x\mu\right)\mathrm{d}\mu, \qquad x\in\mathbb{R},$$

for γ_+^n an appropriate curve, which we will specify later on.

With the matrix-valued Airy functions $\mathbf{Ai}_{2n+1}(x,\vec{s})$ defined in (6.0.1), the matrix Airy Hankel operators $\mathcal{A}\mathrm{i}_{2n+1}$ are defined in the standard way

$$\left(\mathcal{A}\mathrm{i}_{2n+1}\mathbf{f}\right)(x) := \int_{\mathbb{R}_+}\mathbf{Ai}_{2n+1}(x+y,\vec{s})\mathbf{f}(y)\,\mathrm{d}y, \tag{6.0.2}$$

for any $\mathbf{f}=(f_1,\dots,f_r)^{\mathrm{T}}\in L^2\Big(\mathbb{R}_+,\mathbb{C}^r\Big)$. It is actually on the square of this sequence of operators that we focused our study, and in particular on the Fredholm determinants defined as

$$F^{(n)}(s_1,\dots,s_r) := \det\Big(\mathrm{Id}_{\mathbb{R}_+}-\mathcal{A}\mathrm{i}_{2n+1}^2\Big), \tag{6.0.3}$$

that are well defined since the operators $\mathcal{A}\mathrm{i}_{2n+1}$ are trace-class on $L^2\Big(\mathbb{R}_+, \mathbb{C}^r\Big)$ (as follows from Proposition 4.3.1, i.e. Corollary 2.1 in [13]).

The core of this work is devoted to establish a relation between the Fredholm determinants (6.0.3) and some solution of a noncommutative Painlevé II hierarchy. In particular, the results resumed in Section 4.3 and originally obtained in [13], where the authors extend the theory of integrable operators of Its–Izergin–Korepin–Slavnov [60], can be directly applied to the matrix operators $\mathcal{A}\mathrm{i}_{2n+1}$ defined in (6.0.2). As byproduct, an equality between the Fredholm determinants $F^{(n)}(s_1, \dots, s_r)$ and those of certain integrable operators can be established. Following the Riemann-Hilbert approach introduced in Section 4.3 we will study these integrable operators through Riemann-Hilbert Problem 6.1.5, from which we will deduce the isomonodromic Lax pair of the noncommutative Painlevé II hierarchy, that we are going to define as follows.

We start defining a matrix-valued analogue of the standard Lenard recursion, through the relations written below. In the following, U, W are functions depending on all the parameters s_l, $l = 1, \dots, r$ with values in $\mathrm{Mat}(r \times r, \mathbb{R})$. The symbols $[\,,\,]$ and $[\,,\,]_+$ indicate respectively the standard commutator and anti-commutator between two matrices, since differential polynomials in U are noncommutative quantities.

Then each differential polynomial $\mathcal{L}_n[U]$ is defined by the following recursive relation

$$\mathcal{L}_0[U] = \frac{1}{2} I_r,$$

$$\frac{\mathrm{d}}{\mathrm{d}S}\mathcal{L}_n[U] = \left(\frac{\mathrm{d}^3}{\mathrm{d}S^3} + [U, \cdot]_+ \frac{\mathrm{d}}{\mathrm{d}S} + \frac{\mathrm{d}}{\mathrm{d}S}[U, \cdot]_+ + [U, \cdot] \frac{\mathrm{d}}{\mathrm{d}S}^{-1} [U, \cdot] \right) \mathcal{L}_{n-1}[U], \qquad n \geq 1, \quad (6.0.4)$$

where the differential operator $\frac{\mathrm{d}}{\mathrm{d}S}$ is defined as

$$\frac{\mathrm{d}}{\mathrm{d}S} := \sum_{k=1}^{r} \frac{\partial}{\partial s_k}, \qquad (6.0.5)$$

and $\frac{\mathrm{d}}{\mathrm{d}S}^{-1}$ in intended as the corresponding formal antiderivative. The recursive relation for the noncommutative version of the Lenard operators $\mathcal{L}_n$, $n \geq 1$, is related to the recursion operator for the noncommutative KdV equation, introduced in [92]. There the authors already conjectured about the locality of these operators computed in U, but the formal proof of that was done some years later in [93] (Theorem 6.2 in this last paper).

Finally we define our noncommutative Painlevé II hierarchy as follows

$$\mathrm{PII}_{\mathrm{NC}}^{(\mathrm{n})}\colon \quad \left(\frac{\mathrm{d}}{\mathrm{d}S} + [W, \cdot]_+ \right) \mathcal{L}_n[U] = (-1)^{n+1} 4^n [S, W]_+, \qquad (6.0.6)$$

where $U := \frac{\mathrm{d}}{\mathrm{d}S} W - W^2$ is the Miura transform of W and the variable S is the diagonal matrix $S := \mathrm{diag}(s_1, \dots, s_r)$ so that the anti-commutator in the right hand side is needed (also notice that $\frac{\mathrm{d}}{\mathrm{d}S} S = I_r$). For this reason we refer to our hierarchy as a fully noncommutative one, since in its definition (6.0.6) also the independent variable S is noncommutative. A matrix Painlevé II hierarchy, constructed by using a noncommutative version of Lenard operators as in (6.0.4), was recently studied in [50] but in this paper the independent variable is a scalar.

In this work, first of all, we found out that the hierarchy (6.0.6) admits an isomonodromic Lax pair with Lax matrices that are block-matrices of dimension $2r$. Furthermore, they are explicitly written in terms of the matrix-valued Lenard operators defined in (6.0.4). The result proved in Section 6.3 is summarized in the following proposition.

Proposition 6.0.1. *For each fixed n there exist two polynomial matrices in λ, namely $L^{(n)}$, $M^{(n)}$, respectively of degree 1 and $2n$, such that the following system*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}(\lambda,\vec{s}) &= L^{(n)}(\lambda,\vec{s})\Psi^{(n)}(\lambda,\vec{s}), \\ \frac{\partial}{\partial\lambda}\Psi^{(n)}(\lambda,\vec{s}) &= M^{(n)}(\lambda,\vec{s})\Psi^{(n)}(\lambda,\vec{s}) \end{aligned} \tag{6.0.7}$$

is an isomonodromic Lax pair for the n-th equation of the matrix Painlevé II hierarchy (6.0.6)*.*

In particular the matrices $L^{(n)}$, $M^{(n)}$ have the following forms

$$L^{(n)}(\lambda,\vec{s}) = \begin{pmatrix} \mathrm{i}\lambda I_r & W(\vec{s}) \\ W(\vec{s}) & -\mathrm{i}\lambda I_r \end{pmatrix},$$

and

$$M^{(n)}(\lambda,\vec{s}) = \begin{pmatrix} A(\lambda,\vec{s}) + \mathrm{i}S & \mathrm{i}G(\lambda,\vec{s}) \\ -\mathrm{i}G(\lambda,\vec{s}) & -A(\lambda,\vec{s}) - \mathrm{i}S \end{pmatrix} + \begin{pmatrix} E(\lambda,\vec{s}) & F(\lambda,\vec{s}) \\ F(\lambda,\vec{s}) & E(\lambda,\vec{s}) \end{pmatrix},$$

where

$$\begin{aligned} A(\lambda,\vec{s}) &= \sum_{k=0}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k} A_{2n-2k}(\vec{s}), \qquad \textit{with} \quad A_{2n} = I_r, \\ G(\lambda,\vec{s}) &= \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k} G_{2n-2k}(\vec{s}), \\ E(\lambda,\vec{s}) &= \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k+1} E_{2n-2k+1}(\vec{s}), \\ F(\lambda,\vec{s}) &= \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k+1} F_{2n-2k+1}(\vec{s}). \end{aligned}$$

All the coefficients A_{2n-2k}, G_{2n-2k}, $E_{2n-2k+1}$, $F_{2n-2k+1}$ are expressed in terms of the Lenard operators through the formulae (6.3.4)*.*

This result can be thought as the noncommutative analogue of the well known isomonodromic Lax pair for the scalar Painlevé II hierarchy studied in [30], and resulting from a self-similarity reduction of the Lax pair for the modified KdV hierarchy.

A solution $\Psi^{(n)}$ for the Lax pair (6.0.7) is constructed, by using the solution of the Riemann-Hilbert Problem 6.1.5 involved in the study of the integrable operators associated to the matrix operators squared $\mathcal{A}\mathrm{i}^2_{2n+1}$.

As a byproduct, we obtain the following relation between some solutions of the hierarchy (6.0.6) and the Fredholm determinants (6.0.3). This is indeed the final result of this work and it is proved at the end of Section 6.3.

Corollary 6.0.2. *There exists a solution W of the n-th member of the matrix PII hierarchy* (6.0.6)*, that is connected to the Fredholm determinant of the n-th Airy matrix Hankel operator through the following formula*

$$-\operatorname{Tr}\left(W^2(\vec{s})\right) = \frac{\mathrm{d}^2}{\mathrm{d}S^2}\ln\left(F^{(n)}(s_1,\dots,s_r)\right).$$

Defining $\mathrm{s} := \frac{1}{r}\sum_{j=1}^r s_j$*, and* $\delta_j := s_j - \mathrm{s}$*, this solution W in the regime* $\mathrm{s} \to +\infty$ *with* $|\delta_j| \leq m$ *for every j, has asymptotic behavior* $(W)_{k,l=1}^r \sim -2(c_{kl}\mathrm{Ai}_{2n+1}(s_k+s_l))_{k,l=1}^r$.

We remark that in [13] the above result was actually proved for the first equation of the hierarchy, i.e., for the case $n = 1$. The result above is a generalization of Theorem 2.1.7 (for $n = 1$) and Theorem 2.2.12 (for the generic n case) to the matrix-valued case. We recall that the scalar Airy kernels involved in the Theorem 2.1.7 and 2.2.12 define DPP on $\mathbb{R}$ with applications in random matrix theory ($n = 1$) and statistichal mechanics (generic n). In this work, we see that the matrix Airy Hankel operators squared $\mathcal{A}\mathrm{i}^2_{2n+1}$ can actually be interpreted as kernels for determinantal point processes on the space of configuration $\{1,\dots,r\}\times\mathbb{R}$ (under certain assumptions on the matrix $C = (c_{j,k})_{j,k=1}^r$), and it would be interesting to study whether they describe phenomena in random matrix theory or statistical mechanics.

Here is a more precise list of what it is done in this work.

- In Section 6.1 the general theory developed in [13], and recalled in Section 4.3, is applied to the operators $\mathcal{A}\mathrm{i}^2_{2n+1}$, in order to associate the Fredholm determinants (6.0.3) to the ones of certain integrable operators. The most important consequence of this study is indeed Theorem 6.1.9, that establishes a relation between Fredholm determinant (6.0.3) and the solution of Riemann-Hilbert Problem 6.1.5. Furthermore, in this section it is provided in which hypothesis the solution exists (Theorem (6.1.11)), and so the relation for the Fredholm determinants found in Theorem 6.1.9 holds.
- In Section 6.2 the fully noncommutative Painlevé II hierarchy is introduced and the first equations are explicitly written.
- In the first part of Section 6.3, the proof of Proposition 6.0.1 is given and the construction of the solution $\Psi^{(n)}$ of the isomonodromic Lax pair (6.0.7) for the hierarchy (6.0.6) is implemented. Finally in the end of Section 6.3, Corollary 6.0.2 is proved, by using Theorem 6.1.9 and the properties of the solution $\Psi^{(n)}$ of the isomonodromic Lax pair (6.0.7).

6.1 Riemann Hilbert problems associated to the matrix Airy operators

To start with, we recall some basic fact about the scalar generalized Airy functions Ai_{2n+1}. As already anticipated in the introduction, for each $n \in \mathbb{N}$, we consider these functions Ai_{2n+1} as the contour integrals

$$\mathrm{Ai}_{2n+1}(x) := \int_{\gamma_+^n} \frac{1}{2\pi} \exp\left(\frac{\mathrm{i}\mu^{2n+1}}{2n+1} + \mathrm{i}x\mu\right) \mathrm{d}\mu, \qquad x \in \mathbb{R}, \tag{6.1.1}$$

where $\gamma_\pm^n$ are curves in the upper (lower) complex plane with asymptotic rays at $\pm\infty$ that are $\phi_\pm^n := \frac{\pi}{2} \pm \frac{\pi n}{2n+1}$, and such that $\gamma_-^n = -\gamma_+^n$. An example of these curves for $n = 1$ is given in Fig. 6.1 (but there are also other possible choices for the curve, as we will see in Chapter 7).

Definition 6.1.1. *Recall that as we saw in the introduction, the n-th matrix-valued Airy function is defined as*

$$\mathbf{Ai}_{2n+1}(x, \vec{s}) := \Big(c_{j,k}\mathrm{Ai}_{2n+1}(x + s_j + s_k)\Big)_{j,k=1}^r, \qquad x \in \mathbb{R}.$$

Here $C = (c_{j,k})_{j,k=1}^r \in \mathrm{Mat}(r \times r, \mathbb{C})$ and the parameters $s_l \in \mathbb{R}$, $l = 1, \dots, r$.

With these functions we construct the matrix-valued operators we are going to study in the following.

Definition 6.1.2. *We consider $\{\mathcal{A}\mathrm{i}_{2n+1}\}_{n\in\mathbb{N}}$ the sequence of matrix Hankel operators acting on any $\mathbf{f} = (f_1, \dots, f_r)^{\mathrm{T}} \in L^2\Big(\mathbb{R}_+, \mathbb{C}^r\Big)$ s.t.*

$$(\mathcal{A}\mathrm{i}_{2n+1}\mathbf{f})(x) := \int_{\mathbb{R}_+} \mathbf{Ai}_{2n+1}(x + y, \vec{s})\mathbf{f}(y)\, \mathrm{d}y. \tag{6.1.2}$$

Component wise the n-th Hankel operator $\mathcal{A}\mathrm{i}_{2n+1}$, reads as

$$\left(\mathcal{A}\mathrm{i}_{2n+1}\mathbf{f}\right)_j(x) = \sum_{k=1}^r c_{j,k} \int_{\mathbb{R}_+} \mathrm{Ai}_{2n+1}(x + y + s_j + s_k) f_k(y)\, \mathrm{d}y, \qquad j = 1, \dots, r. \tag{6.1.3}$$

Remark 6.1.3. *One can equivalently define the matrix-valued generalized Airy functions as contours integrals, in the following way. For each $n \in \mathbb{N}$*

- *we take $s_1, \dots, s_r$ real parameters and $S := \mathrm{diag}(s_1, \dots, s_r)$ and we define the matrix-valued complex function*

$$\theta_{2n+1}(\mu, \vec{s}) := \frac{\mathrm{i}\mu^{2n+1}}{2(2n+1)} I_r + \mathrm{i}\mu S, \tag{6.1.4}$$

where I_r is the identity matrix of dimension r.

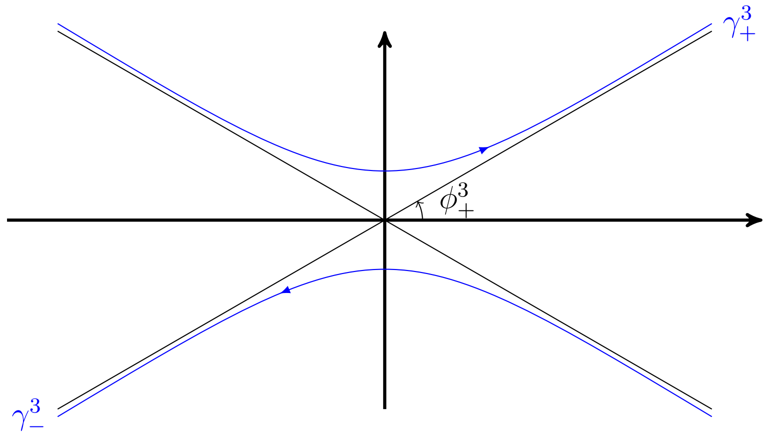

Figure 6.1: These are the contours $\gamma_\pm^3$ for the integral representation (6.1.1) of the Airy function Ai_3 (case $n = 1$). Their asymptotics at $\pm\infty$ are $\phi_\pm^3 := \frac{\pi}{6}, \frac{5\pi}{6}$.

- *Then, we take the matrix* $C = (c_{j,k})_{j,k=1}^r \in \mathrm{Mat}(r \times r, \mathbb{C})$ *we define the matrix-valued function*

$$r^{(n)}(\lambda, \mu, \vec{s}) := \frac{1}{2\pi \mathrm{i}} \exp(\theta_{2n+1}(\lambda, \vec{s})) C \exp(\theta_{2n+1}(\mu, \vec{s})). \tag{6.1.5}$$

- *Finally, we can define the generalized matrix Airy function as*

$$\mathbf{Ai}_{2n+1}(x, \vec{s}) = \int_{\gamma_+^n} \mathrm{i} r^{(n)}(\mu, \mu, \vec{s}) \exp(\mathrm{i} x \mu) \, \mathrm{d}\mu,$$

where the integral is computed entry by entry.

We are now going to define a sequence of Riemann-Hilbert problems related to the matrix-valued analogue of the higher order Airy kernels, obtained as $\mathcal{A}i_{2n+1}^2$. From the solution of these Riemann-Hilbert problems we will deduce the relation between Fredholm determinants of operators $\mathcal{A}i_{2n+1}^2$ and our noncommutative Painlevé II hierarchy.

Remark 6.1.4. *From now on, in order to simplify the notation, the dependence on* $\vec{s}$ *in the quantities* (6.1.4), (6.1.5) *will be omitted and we will use the abbreviation* $r^{(n)}(\lambda, \lambda, \vec{s}) = r^{(n)}(\lambda)$.

Riemann-Hilbert Problem 6.1.5. *Find a* (λ)*-analytic matrix-valued function*

$$\Xi^{(n)}(\lambda) \colon \; \mathbb{C} \setminus \left(\gamma_+^n \cup \gamma_-^n \right) \to \mathrm{GL}(2r, \mathbb{C}),$$

admitting continuous extension to the contour $\gamma_+^n \cup \gamma_-^n$ *from either side and such that it satisfies the following two conditions:*

- *the jump condition for each* $\lambda \in \gamma_+^n \cup \gamma_-^n$

$$\Xi_+^{(n)}(\lambda) = \Xi_-^{(n)}(\lambda) \underbrace{\begin{pmatrix} I_r & -2\pi \mathrm{i} r^{(n)}(\lambda)\chi_{\gamma_+^n}(\lambda) \\ -2\pi \mathrm{i} r^{(n)}(-\lambda)\chi_{\gamma_-^n}(\lambda) & I_r \end{pmatrix}}_{:=J^{(n)}(\lambda,\vec{s})}, \tag{6.1.6}$$

where we denote by $\Xi_\pm^{(n)}$ *the boundary values of* $\Xi^{(n)}$ *for* $\lambda \in \gamma_+^{(n)} \cup \gamma_-^{(n)}$*, approaching the boundary from the left* $(+)$ *and the right* $(-)$ *nontangentially.*

- *the asymptotic condition for* $|\lambda| \to \infty$

$$\Xi^{(n)}(\lambda) \sim I_{2r} + \sum_{j\geq 1} \frac{\Xi_j^{(n)}}{\lambda^j}. \tag{6.1.7}$$

Remark 6.1.6. *In the following we are going to use the Pauli's tensorized matrices, that have the same property as the ones in the usual Clifford algebra. In particular we denote the tensorized matrices by*

$$\hat{\sigma}_1 = \sigma_1 \otimes I_{2r}, \qquad \hat{\sigma}_2 = \sigma_2 \otimes I_{2r}, \qquad \hat{\sigma}_3 = \sigma_3 \otimes I_{2r},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the standard relations hold also in this case:

$$[\hat{\sigma}_1, \hat{\sigma}_2] = -2\mathrm{i}\hat{\sigma}_3, \qquad [\hat{\sigma}_1, \hat{\sigma}_3] = 2\mathrm{i}\hat{\sigma}_2, \qquad [\hat{\sigma}_2, \hat{\sigma}_3] = -2\mathrm{i}\hat{\sigma}_1, \qquad \hat{\sigma}_i^2 = I_{2r}, \quad \forall\, i.$$

The following symmetry property will be useful in the next computations.

Corollary 6.1.7. *The asymptotic coefficients appearing in equation* (6.1.7) *have the following form*

$$\begin{aligned} \Xi_{2j}^{(n)} &= \alpha_{2j}^{(n)} \otimes I_2 + \beta_{2j}^{(n)} \otimes \sigma_1, \\ \Xi_{2j-1}^{(n)} &= \alpha_{2j-1}^{(n)} \otimes \sigma_3 + \beta_{2j-1}^{(n)} \otimes \sigma_2, \qquad j \geq 1. \end{aligned} \tag{6.1.8}$$

Here $\alpha_l^{(n)}$*,* $\beta_l^{(n)}$ *for every* $l \geq 1$ *correspond to the* $r \times r$ *matrices in the entries* $(1,1)$ *and* $(1,2)$ *of the block matrix* $\Xi_l^{(n)}$*.*

An analogue statement is true for the asymptotic coefficients of the inverse of the solution of the Riemann-Hilbert Problem 6.1.5*, namely* $\Theta^{(n)} := \left(\Xi^{(n)}\right)^{-1}$*.*

Proof. We first prove the symmetry condition for the asymptotic coefficients of $\Xi^{(n)}$. We start observing that the jump matrix $J^{(n)}$ for $\lambda \in \gamma_+^n \cup \gamma_-^n$ has the following symmetry

$$\hat{\sigma}_1 J^{(n)}(\lambda, \vec{s})\hat{\sigma}_1 = J^{(n)}(-\lambda, \vec{s}),$$

just using the definition of $\gamma_-^n = -\gamma_+^n$. This directly implies that also the solution of the Riemann-Hilbert Problem 6.1.5 has the same symmetry property. Thus for any λ we have that

$$\Xi^{(n)}(-\lambda) = \hat{\sigma}_1 \Xi^{(n)}(\lambda)\hat{\sigma}_1.$$

Computing the asymptotic expansion at ∞ of both sides of this equation, we have that $(-1)^k \Xi_k^{(n)} = \hat{\sigma}_1 \Xi_k^{(n)} \hat{\sigma}_1$. This directly implies the two equations (6.1.8) for $k = 2j$ or $k = 2j-1$.

Concerning the statement for the asymptotic coefficients of the inverse of $\Xi^{(n)}$, namely $\Theta^{(n)}$, the proof follows by the fact that $\Theta^{(n)}$ solves another Riemann-Hilbert problem, with same symmetry for the jump matrix. Indeed, consider the following problem for a function $\Theta^{(n)}$:

- $\Theta^{(n)}$ is a (λ-)analytic matrix-valued function on $\mathbb{C} \setminus \left(\gamma_+^n \cup \gamma_-^n\right)$ admitting continuous extension from either side to $\gamma_+^n \cup \gamma_-^n$;
- it has a jump condition for each $\lambda \in \gamma_+^n \cup \gamma_-^n$

$$\Theta_+^{(n)}(\lambda) = \underbrace{\begin{pmatrix} I_r & 2\pi \mathrm{i} r^{(n)}(\lambda)\chi_{\gamma_+^n}(\lambda) \\ 2\pi \mathrm{i} r^{(n)}(-\lambda)\chi_{\gamma_-^n}(\lambda) & I_r \end{pmatrix}}_{:=H^{(n)}(\lambda,\vec{s})} \Theta_-^{(n)}(\lambda);$$

- it has the asymptotic condition for $|\lambda| \to \infty$

$$\Theta^{(n)}(\lambda) \sim I_{2r} + \sum_{j \geq 1} \frac{\Theta_j^{(n)}}{\lambda^j}.$$

The function $\Theta^{(n)}$ with these properties is the inverse of the solution of Problem 6.1.5. Indeed: the functions $\Theta^{(n)}\Xi^{(n)}(\lambda)$, and $\Xi^{(n)}\Theta^{(n)}$ have no jumps along $\gamma_+^n \cup \gamma_-^n$ and they both behave like the identity matrix at ∞. Thus by the generalized Liouville theorem, they both have to coincide with the identity matrix.

We then observe that the jump matrix $H^{(n)}$ here has the same symmetry property of $J^{(n)}$, i.e., $\hat{\sigma}_1 H^{(n)}(\lambda, \vec{s})\hat{\sigma}_1 = H^{(n)}(-\lambda, \vec{s})$, for each $\lambda \in \gamma_+^n \cup \gamma_-^n$. Thus, exactly as before, even the function $\Theta^{(n)}$ has the same property:

$$\hat{\sigma}_1 \Theta^{(n)}(\lambda, \vec{s})\hat{\sigma}_1 = \Theta^{(n)}(-\lambda, \vec{s}).$$

We conclude then that the asymptotic coefficients of $\Theta^{(n)}$ have the same form of the Ξ_k, i.e.,

$$\begin{aligned}\Theta^{(n)}_{2j} &= \tilde{\alpha}^{(n)}_{2j} \otimes I_{2r} + \tilde{\beta}^{(n)}_{2j} \otimes \sigma_1,\\ \Theta^{(n)}_{2j-1} &= \tilde{\alpha}^{(n)}_{2j-1} \otimes \sigma_3 + \tilde{\beta}^{(n)}_{2j-1} \otimes \sigma_2, \qquad j \geq 1,\end{aligned} \tag{6.1.9}$$

where, as before, $\tilde{\alpha}^{(n)}_l$ and $\tilde{\beta}^{(n)}_l$ for every $l \geq 1$ correspond to the $r \times r$ matrices in the entries $(1,1)$ and $(1,2)$ of the block matrix $\Theta^{(n)}_l$. ∎

We are now ready to state the fundamental result that connects the matrix Airy Hankel operators to these Riemann-Hilbert problems.

Supposing that the solutions of the Riemann-Hilbert Problem 6.1.5 and its inverse exist, we have the following result.

Remark 6.1.8. *Existence conditions for $\Xi^{(n)}$ (and thus $\Theta^{(n)}$) are given at the end of the section (see Theorem 6.1.11).*

Theorem 6.1.9. *For each $n \in \mathbb{N}$, consider $\Xi^{(n)}$ the solution of the Riemann-Hilbert Problem* 6.1.5 *and its inverse $\Theta^{(n)} := \left(\Xi^{(n)}\right)^{-1}$. Then the following identities hold*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}S} \ln\left(F^{(n)}(s_1, \ldots, s_r)\right) &= \int_{\gamma_+^n \cup \gamma_-^n} \mathrm{Tr}\left(\Theta^{(n)}_-\left(\Xi^{(n)}_-\right)' \frac{\mathrm{d}}{\mathrm{d}S} J^{(n)} \left(J^{(n)}\right)^{-1}\right) \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\\ &= -2\mathrm{i}\,\mathrm{Tr}\left(\alpha^{(n)}_1\right),\end{aligned} \tag{6.1.10}$$

where in the integral in the middle we indicate with $'$ the derivation w.r.t. the complex parameter λ and the differential operator $\frac{\mathrm{d}}{\mathrm{d}S}$ is defined as in (6.0.5).

Proof. The proof follows as an application to this very specific case of some general results obtained in [13] (and written in Section 4.3). We split the proof in two parts, one for each equality in (6.1.10).

In order to obtain the first equality we need essentially two results. The first one establishes the relation between Fredholm determinant of the Airy matrix operator and Fredholm determinant of certain integral kernel operator, thanks to Theorem 4.3.1. In particular, we first get that the Fredholm determinants of $\{\mathcal{A}\mathrm{i}_{2n+1}\}_{n\in\mathbb{N}}$ are equal to the ones of the integral operators acting on $L^2\left(\gamma_+^{(n)}, \mathbb{C}^r\right)$ with kernels

$$\mathcal{K}^{(n)}(\lambda, \mu) = \frac{r^{(n)}(\lambda, \mu)}{\lambda + \mu}, \tag{6.1.11}$$

with $r^{(n)}(\lambda, \mu)$ defined as in (6.1.5).

As by product we then have that

$$F^{(n)}(s_1, \ldots, s_r) = \det\left(\mathrm{Id}_{\gamma_+^{(n)}} - \left(\mathcal{K}^{(n)}\right)^2\right).$$

The second result needed comes from the study of matrix integral kernels of type (6.1.11), through Riemann-Hilbert problems. Indeed, it allows to compute the Fredholm determinants of these integrable operators in terms of the solution of Riemann-Hilbert Problem 6.1.5. In particular, by applying Thorem 4.3.5, we have that

$$\int_{\gamma_+^n \cup \gamma_-^n} \mathrm{Tr}\left(\Theta_-^{(n)}\left(\Xi_-^{(n)}\right)' \frac{\mathrm{d}}{\mathrm{d}S} J^{(n)}\left(J^{(n)}\right)^{-1}\right) \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}S} \ln\det\left(I_{\gamma_+^{(n)}} - \left(\mathcal{K}^{(n)}\right)^2\right).$$

Thus the first identity in the statement holds.

For what concerns the second identity of the statement, we proceed by direct computation of the integral

$$\int_{\gamma_+^n \cup \gamma_-^n} \mathrm{Tr}\left(\Theta_-^{(n)}\left(\Xi_-^{(n)}\right)' \frac{\mathrm{d}}{\mathrm{d}S} J^{(n)}\left(J^{(n)}\right)^{-1}\right) \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}}. \tag{6.1.12}$$

First of all, we observe that the jump matrix $J^{(n)}(\lambda, \vec{s})$ that appears in the jump condition (6.1.6), admits the factorization

$$J^{(n)}(\lambda, \vec{s}) = \exp\left(\theta^{(n)}(\lambda, \vec{s}) \otimes \sigma_3\right) J_0^{(n)} \exp\left(-\theta^{(n)}(\lambda, \vec{s}) \otimes \sigma_3\right),$$

with $J_0^{(n)}$ the constant matrix given by

$$J_0^{(n)} = \begin{pmatrix} I_r & C \\ C & I_r \end{pmatrix}.$$

Thus we can easily compute the second factor appearing under the trace in the integral (6.1.12):

$$\left(\frac{\mathrm{d}}{\mathrm{d}S} J^{(n)}\right)\left(J^{(n)}\right)^{-1} = \mathrm{i}\lambda\hat{\sigma}_3 - J^{(n)}\left(\mathrm{i}\lambda\hat{\sigma}_3\right)\left(J^{(n)}\right)^{-1}. \tag{6.1.13}$$

We are now going to show that the integral in (6.1.12) is actually just the formal residue at ∞ of a certain function. Furthermore in this particular case, due to the form of the matrix $J^{(n)}$, the residue can be explicitly computed using equation (6.1.13).

To start with, we consider the following function

$$\mathrm{Tr}\left(\Theta^{(n)}\left(\Xi^{(n)}\right)' \frac{\mathrm{d}}{\mathrm{d}S}\left(\theta^{(n)} \otimes \sigma_3\right)\right) = \mathrm{Tr}\left(\Theta^{(n)}\left(\Xi^{(n)}\right)' \mathrm{i}\lambda\hat{\sigma}_3\right). \tag{6.1.14}$$

Its formal residue at ∞ can be computed as

$$-\mathrm{Res}_{\lambda=\infty} \mathrm{Tr}\left(\Theta^{(n)}\left(\Xi^{(n)}\right)' \mathrm{i}\lambda\hat{\sigma}_3\right) = \lim_{R\to\infty} \int_{|\lambda|=R} \mathrm{Tr}\left(\Theta^{(n)}\left(\Xi^{(n)}\right)' \mathrm{i}\lambda\hat{\sigma}_3\right) \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}}.$$

Now, this counterclockwise circle for $R \to \infty$, can be deformed like $\gamma_+^{(n)} \cup \gamma_-^{(n)}$. As a byproduct, the formal residue of (6.1.14) can be rewritten, taking into account the boundary values of $\Theta^{(n)}$ and $\left(\Xi^{(n)}\right)'$ along the curves $\gamma_\pm^{(n)}$, as follows

$$\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\left(-\Theta_+^{(n)}\left(\Xi_+^{(n)}\right)' + \Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\right)\mathrm{i}\lambda\hat{\sigma}_3\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}.$$

Now, from the jump condition (6.1.6), by deriving w.r.t. λ, we deduce that all along the curves $\gamma_\pm^{(n)}$ we have the relation

$$\left(\Xi_+^{(n)}\right)' = \left(\Xi_-^{(n)}\right)' J^{(n)} + \left(\Xi_-^{(n)}\right)\left(J^{(n)}\right)'.$$

Thus replacing it in the first integral above we get

$$\begin{aligned}
&\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\left(-\Theta_+^{(n)}\left(\Xi_+^{(n)}\right)' + \Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\right)\mathrm{i}\lambda\hat{\sigma}_3\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\\
&= -\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\left(\left(J^{(n)}\right)^{-1}\Theta_-^{(n)}\left(\left(\Xi_-^{(n)}\right)' J^{(n)} + \Xi_-^{(n)}\left(J^{(n)}\right)'\right) - \Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\mathrm{i}\lambda\hat{\sigma}_3\right)\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\\
&= -\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\left(\left(J^{(n)}\right)^{-1}\Theta_-^{(n)}\left(\Xi_-^{(n)}\right)' J^{(n)} + \left(J^{(n)}\right)^{-1}J' - \Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\right)\mathrm{i}\lambda\hat{\sigma}_3\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\\
&= -\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\left(J^{(n)}\mathrm{i}\lambda\hat{\sigma}_3\left(J^{(n)}\right)^{-1} - \mathrm{i}\lambda\hat{\sigma}_3\right)\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\\
&= \int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\Theta_-^{(n)}\left(\Xi_-^{(n)}\right)'\frac{\mathrm{d}}{\mathrm{d}S}J^{(n)}\left(J^{(n)}\right)^{-1}\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}},
\end{aligned}$$

where in the last passages we used the invariance of the trace by conjugation and the fact that the quantity $\left(J^{(n)}\right)^{-1}\left(J^{(n)}\right)'\mathrm{i}\lambda\hat{\sigma}_3$ is trace free.

Finally, using the asymptotic expansion at ∞ given in (6.1.7), we get that

$$\operatorname{Res}_{\lambda=\infty}\operatorname{Tr}\left(\Theta^{(n)}\left(\Xi^{(n)}\right)'\mathrm{i}\lambda\hat{\sigma}_3\right) = -2\mathrm{i}\operatorname{Tr}\left(\alpha_1^{(n)}\right),$$

and this concludes the proof. ■

Remark 6.1.10. *In the study of isomonodromy deformations, the quantity*

$$\int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\Theta_-\Xi_-'\frac{\mathrm{d}}{\mathrm{d}S}J^{(n)}\left(J^{(n)}\right)^{-1}\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}$$

is associated to the isomonodromic tau function $\tau_{\Xi^{(n)}}$ related to the Riemann-Hilbert Problem 6.1.5 depending on the parameters $\{s_k\}_{k=1}^r$, through the formula

$$\frac{\mathrm{d}}{\mathrm{d}S}\ln\tau_{\Xi^{(n)}} = \int_{\gamma_+^n\cup\gamma_-^n} \operatorname{Tr}\left(\Theta_-\Xi_-'\frac{\mathrm{d}}{\mathrm{d}S}J^{(n)}\left(J^{(n)}\right)^{-1}\right)\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}.$$

This notion was first introduced in [66], and then generalized for example in [11]. With Theorem 6.1.9 we recover for any Airy matrix Hankel operator (6.1.2) *the relation between* *the Fredholm determinant $F^{(n)}(s_1,\dots,s_r)$ and the isomonodromic tau function associated to the Riemann-Hilbert Problem 6.1.5, that was proved in Theorem 4.1 of [13] for Fredholm determinants of generic matrix Hankel operators.*

Finally, in order to use the formula (6.1.10) for the logarithmic derivative of $F^{(n)}(s_1,\dots,s_r)$, we need to find out whether the solution $\Xi^{(n)}$ of the Riemann-Hilbert Problem 6.1.5 exists or not. In particular, we are going to see that under certain assumptions on the constant matrix C, the existence of $\Xi^{(n)}$ is assured. The following result is indeed a generalization of Theorem 5.1 in [13], for the generalized Airy matrix operators defined in (6.1.2), i.e. the case $n > 1$.

Theorem 6.1.11. *Let the matrix C be Hermitian, then the solution $\Xi^{(n)}$ of the Riemann-Hilbert Problem* 6.1.5 *exists if and only if the eigenvalues of C lay in the interval* $[-1, 1]$.

Before starting the proof of Theorem 6.1.11, we state the following lemma. For $n = 1$ the result is known from [10, 56]. In the following we adapted the proof to the case of generic n. For finite $z \in \mathbb{R}$, we introduce the operator

$$\left(\Phi^z_{\mathrm{Ai}_{2n+1}} f\right)(x) = \int_z^{+\infty} \mathrm{Ai}_{2n+1}(x+y) f(y)\,\mathrm{d}y, \qquad f \in L^2(\mathbb{R}).$$

Lemma 6.1.12. *For any $n \in \mathbb{N}$ we consider the Airy transform $\Phi_{\mathrm{Ai}_{2n+1}}$ acting on $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ as*

$$(\Phi_{\mathrm{Ai}_{2n+1}} f)(x) = \lim_{z\to-\infty} \left(\Phi^z_{\mathrm{Ai}_{2n+1}} f\right)(x) = \lim_{z\to-\infty} \left(\int_z^{+\infty} \mathrm{Ai}_{2n+1}(x+y) f(y)\,\mathrm{d}y\right). \qquad (6.1.15)$$

Then $\lim_{z\to-\infty} \left\|\Phi^z_{\mathrm{Ai}_{2n+1}} f\right\| = \|f\|$ for the $L^2(\mathbb{R})$-norm, and thus for any finite z the inequality $\left|\left|\left|\Phi^z_{\mathrm{Ai}_{2n+1}}\right|\right|\right| \le 1$ holds for the $L^2((z,+\infty))$ operator norm.

Proof. We consider $\Phi_{\mathrm{Ai}_{2n+1}}$ the Airy transform acting as defined in (6.1.15), where inside the integral we have the scalar Airy function Ai_{2n+1} defined in (6.1.1), without any shift and for real values of x. We introduce the Fourier transform $\mathfrak{F}$ and its inverse $\mathfrak{F}^{-1}$ defined on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (and extended to $L^2(\mathbb{R})$ by continuity and density argument), in the standard way as

$$(\mathfrak{F}h)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(\lambda) \exp(-\mathrm{i}x\lambda)\,\mathrm{d}\lambda, \qquad \mathfrak{F}^{-1} := \mathfrak{F}\mathfrak{I} = \mathfrak{I}\mathfrak{F},$$

where $(\mathfrak{I}h)(x) = h(-x)$, and the multiplication operator by $\exp\left(\frac{\mathrm{i}x^{2n+1}}{2n+1}\right)$, denoted by $\mathcal{M}_n$. Then we observe that the Airy transform $\Phi_{\mathrm{Ai}_{2n+1}}$ can be rewritten as the composition of these operators, in such a way that

$$\Phi_{\mathrm{Ai}_{2n+1}} = \mathfrak{F}^{-1}\mathcal{M}_n\mathfrak{F}^{-1} = \mathfrak{F}\mathfrak{I}\mathcal{M}_n\mathfrak{I}\mathfrak{F} = \mathfrak{F}\mathcal{M}_n^{-1}\mathfrak{F} = \Phi^{-1}_{\mathrm{Ai}_{2n+1}}.$$

This implies that

$$\lim_{z\to-\infty}\left|\left|\Phi^{z}_{\mathrm{Ai}_{2n+1}}f\right|\right| = \lim_{z\to-\infty}\left(\int_{\mathbb{R}}\left|\Phi^{z}_{\mathrm{Ai}_{2n+1}}f(y)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}}$$
$$= \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\mathrm{Ai}_{2n+1}(y+u)f(u)\mathrm{d}u\right|^2 \mathrm{d}y\right)^{\frac{1}{2}} = ||f||, \tag{6.1.16}$$

the norms being in $L^2(\mathbb{R})$.

Now we prove by contradiction the last inequality $\left|\left|\left|\Phi^{z}_{\mathrm{Ai}_{2n+1}}\right|\right|\right| \leq 1$ for the $L^2((z,+\infty))$ operator norm. Suppose that there exist a scalar μ and an eigenfunction $g^z \in L^2((z,+\infty))$ such that $\Phi^{z}_{\mathrm{Ai}_{2n+1}}g^z = \mu g^z$ and $|\mu| > 1$. Then we can define $g \in L^2(\mathbb{R})$ as

$$g(y) = \begin{cases} g^z(y), & \text{for } y \geq z, \\ 0, & \text{for } y < z, \end{cases}$$

and we obtain for $\tilde{z} \leq z$ that $\Phi^{\tilde{z}}_{\mathrm{Ai}_{2n+1}}g(y) = \Phi^{z}_{\mathrm{Ai}_{2n+1}}g^z(y) = \mu g^z(y) = \mu g(y)$ for $y \geq z$. Finally, since $|\mu| > 1$, we have

$$\left|\left|\Phi^{\tilde{z}}_{\mathrm{Ai}_{2n+1}}g\right|\right|_{L^2(\mathbb{R})} \geq \left|\left|\Phi^{\tilde{z}}_{\mathrm{Ai}_{2n+1}}g\right|\right|_{L^2((z,+\infty))} = |\mu|\,||g||_{L^2((z,+\infty))} > ||g||_{L^2(\mathbb{R})}$$

and this is in contradiction with equation (6.1.16). ■

We can finally provide a complete proof of Theorem 6.1.11.

Proof. By applying Theorem 4.3.4 (i.e. Theorem 3.1 of [13], which generalizes the fundamental result obtained first in [60]) to the sequence of operators $\left(\mathcal{K}^{(n)}\right)^2, n \geq 1$, we have that the solution $\Xi^{(n)}$ of the Riemann-Hilbert Problem 6.1.5 exists if and only if the operator $\mathrm{Id} - \left(\mathcal{K}^{(n)}\right)^2$ is invertible. This is guaranteed by the non vanishing condition of the quantity $\det\left(\mathrm{Id} - \left(\mathcal{K}^{(n)}\right)^2\right) = \det\left(\mathrm{Id} - \mathcal{A}\mathrm{i}^2_{2n+1}\right)$ (the equality follows as before from Theorem 4.3.1 i.e. Corollary 2.1 of [13]) that is verified if the operators $\mathcal{A}\mathrm{i}_{2n+1}$ are such that $|||\mathcal{A}\mathrm{i}_{2n+1}||| < 1$. Here and in the following, $|||\cdot|||$ stands for the operator norm induced from the L^2-norms on the domain and codomain of the relevant operator.

Supposing that the eigenvalues of C are in the interval $[-1,1]$, we are going to show that the inequality for the operator norm of $\mathcal{A}\mathrm{i}_{2n+1}$ holds. Since the operators $\mathcal{A}\mathrm{i}_{2n+1}$ defined in (6.1.2), are constructed by shifting by some component of $\vec{s}$ the Airy function, we first observe that:

$$|||\mathcal{A}\mathrm{i}_{2n+1}||| = \left|\left|\left|\mathrm{P}_s\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}\mathrm{P}_s\right|\right|\right|,$$

where $\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}$ is the operator without any shift, namely

$$\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}\mathbf{f}(x) := \int_{\mathbb{R}_+}\mathbf{Ai}_{2n+1}(x+y,\vec{0})\mathbf{f}(y)\,\mathrm{d}y.$$

considered from and to the space $\bigoplus_{k=1}^r L^2([s_k,+\infty),\mathbb{C})$ and P_s is the orthogonal projection

$$\mathrm{P}_s\colon\ L^2\Big(\mathbb{R},\mathbb{C}^r\Big)\longrightarrow\bigoplus_{k=1}^r L^2([s_k,+\infty),\mathbb{C})$$

acting diagonally as $\mathrm{P}_s:=\mathrm{diag}(\chi_{[s_k,+\infty)})_{k=1}^r$. From equation (6.1.3), we can see the matrix operators $\mathcal{A}\mathrm{i}_{2n+1}$ written in terms of the scalar operators $\Phi^z_{\mathrm{Ai}_{2n+1}}$ through tensor product. In particular, when there is no shift we simply have

$$\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}=C\otimes\Phi^0_{\mathrm{Ai}_{2n+1}}.$$

Finally, using the property of the scalar operator $\Phi^z_{\mathrm{Ai}_{2n+1}}$ proved in Lemma 6.1.12, we conclude that

$$\left|\left|\left|\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}\right|\right|\right|=|||C|||\left|\left|\left|\Phi^0_{\mathrm{Ai}_{2n+1}}\right|\right|\right|\le|||C|||,$$

where the matrix norm of C above is induced by the 2-norm on $\mathbb{C}^r$, i.e., it corresponds to the spectral radius of C. Then we have

$$|||\mathcal{A}\mathrm{i}_{2n+1}|||\le|||\mathrm{P}_s|||\left|\left|\left|\mathcal{A}\mathrm{i}^{\vec{0}}_{2n+1}\right|\right|\right||||\mathrm{P}_s|||<|||C|||\le 1,$$

and this concludes the proof of one of the implications in the statement.

In order to prove the other implication, we suppose that there exists λ_0 eigenvalue of C such that $|\lambda_0|>1$, with corresponding eigenvector $\mathbf{v}_0\in\mathbb{C}^r$. In this case, we will be able to construct a nonzero function $\mathbf{f}_s(x)$ such that there exist a value s_0 for which

$$\mathcal{A}\mathrm{i}^2_{2n+1}\mathbf{f}_{s_0}(x)=\mathbf{f}_{s_0}(x),$$

so we have that the operator $\mathrm{Id}-\mathcal{A}\mathrm{i}^2_{2n+1}$ is not invertible and thus the solution of the Riemann-Hilbert Problem 6.1.5 does not exist.

Indeed, consider $\mathbf{f}(x):=\mathbf{v}_0 f(x)$, for any scalar function $f\in L^2(\mathbb{R})$. Then applying the operator $\mathcal{A}\mathrm{i}^2_{2n+1}$ with a shift $\vec{s}=(s,\dots,s)$ for a certain $s\in\mathbb{R}$ we have

$$\mathcal{A}\mathrm{i}^2_{2n+1}\mathbf{f}(x)=\lambda_0^2\mathbf{v}_0\int_{\mathbb{R}_+}K_{\mathrm{Ai}_{2n+1}}(x+s,y+s)f(y)\,\mathrm{d}y,$$

where $K_{\mathrm{Ai}_{2n+1}}$ is the n-th generalized scalar Airy kernel (cfr. equation (2.2.34)). The corresponding kernel operator is self-adjoint, trace-class and in particular compact, acting on $L^2([s,\infty))$ (see e.g. [26]). We consider its maximum eigenvalue $\mu(s)$ and the corresponding eigenfunction $f_s(x)$. Finally by taking $\mathbf{f}_s(x)=\mathbf{v}_0 f_s(x)$ we get

$$\mathcal{A}\mathrm{i}^2_{2n+1}\mathbf{f}_s(x)=\lambda_0^2\mu(s)\mathbf{f}_s(x).$$

Since $\lambda_0^2>1$ and $\mu(s)$ is a continuous function such that $\mu(s)\to 1$ for $s\to-\infty$ and $\mu(s)\to 0$

for $s \to +\infty$, there exist a value $s_0 \in \mathbb{R}$ for which the above equation reads as

$$\mathcal{A}\mathrm{i}^2_{2n+1}\mathbf{f}_{s_0}(x, \vec{s_0}) = \mathbf{f}_{s_0}(x).$$

And this completes the proof. ■

Remark 6.1.13. *As a byproduct of the theorem above, we have that the operator $\mathcal{A}\mathrm{i}^2_{2n+1}$ is bounded from above by the identity. We can actually show that $\mathcal{A}\mathrm{i}^2_{2n+1}$ is also limited from below: indeed they are all totally positive on $\mathcal{C} := \{1, \dots, r\} \times \mathbb{R}$ (for $n = 1$ [13] already proved it, and here we extend the proof for all n). The main idea to show this is to interpret $\mathcal{A}\mathrm{i}^2_{2n+1}$ as a scalar function on $\mathcal{C} \times \mathcal{C}$, in this way: for any couple $(\xi_1, \xi_2) = ((j_1, x_1), (j_2, x_2)) \in \mathcal{C} \times \mathcal{C}$ we have*

$$\mathcal{A}\mathrm{i}^2_{2n+1}(\xi_1, \xi_2) = \sum_{k=1}^{r} c_{j_1,k} c_{k,j_2} \int_{\mathbb{R}_+} \mathrm{Ai}_{2n+1}(x_1 + z + s_{j_1} + s_k) \mathrm{Ai}_{2n+1}(x_2 + z + s_{j_2} + s_k) \, \mathrm{d}z.$$

In this way the claim is proved if we prove that for any natural L, the quantity

$$\det \left(\mathcal{A}\mathrm{i}^2_{2n+1}(\xi_a, \xi_b) \right)_{a,b \le L}$$

is positive.

In order to do this, we first rewrite $\mathcal{A}\mathrm{i}^2_{2n+1}(\xi_1, \xi_2)$ using the product measure $d\mu(\xi)$ on $\mathcal{C}$ given by the product of the counting measure on $\{1, \dots, r\}$ and the Lebesgue measure on $\mathbb{R}$. Thus

$$\mathcal{A}\mathrm{i}^2_{2n+1}(\xi_a, \xi_b) = \int_{\mathcal{C}_+} F_{2n+1}(\xi_a, \zeta) F_{2n+1}(\zeta, \xi_b) \, \mathrm{d}\mu(\xi), \tag{6.1.17}$$

where we defined the function $F_{2n+1}(\xi_a, \zeta) = c_{j_a,k} \mathrm{Ai}_{2n+1}(x_1 + z + s_{j_a} + s_k)$. In this way we can determine the sign of the determinant, indeed

$$\begin{aligned}
\det \left(\mathcal{A}\mathrm{i}^2_{2n+1}(\xi_a, \xi_b) \right)_{a,b \le L} &= \det \left(\int_{\mathcal{C}_+} F_{2n+1}(\xi_a, \zeta) F_{2n+1}(\zeta, \xi_b) \, \mathrm{d}\mu(\xi) \right)_{a,b \le L} \\
&= \frac{1}{L!} \int_{\mathcal{C}_+^L} \det(F_{2n+1}(\xi_a, \xi_c)) \det(F_{2n+1}(\xi_c, \xi_a)) \prod_{c=1}^{L} \mathrm{d}\mu(\xi_c) \\
&= \frac{1}{L!} \int_{\mathcal{C}_+^L} |\det(F_{2n+1}(\xi_a, \xi_c))|^2 \prod_{c=1}^{L} \mathrm{d}\mu(\xi_c) > 0,
\end{aligned}$$

where in the first passage we used a general property in measure theory, the Andreief identity (see here [8] for details), and in the last one we used the fact that C is hermitian.

In conclusion, by taking C an hermitian matrix with eigenvalues laying in the interval $[-1, 1]$, any $\mathcal{A}\mathrm{i}^2_{2n+1}$ is hermitian and thanks to Theorem 6.1.11 and the previous remark, we can say that any $\mathcal{A}\mathrm{i}^2_{2n+1}$ defines a determinantal point processes on that space of configuration $\mathcal{C}$ (directly by applying Theorem 3.1.5). In particular this implies that the

Fredholm determinants $F^{(n)}(s_1,\dots,s_r)$ are the joint probability of the last points for some multi-process on $\mathbb{R}$ (by Corollary 3.1.9), namely

$$F^{(n)}(s_1,\dots,s_r) = \mathbb{P}\Big(x_i^{\max} < s_i,\, i=1,\dots,r\Big).$$

6.2 Matrix Painlevé II hierarchy

In this section, we are finally going to define our noncommutative Painlevé II hierarchy. In the following, we will consider $U(\vec{s})$, $W(\vec{s})$ as functions depending on the parameters $s_1,\dots,s_r$ with values in $\mathrm{Mat}(r\times r,\mathbb{C})$.

In this context we will use the standard notation for the commutator and anticommutator between two matrices:

$$[A,\cdot] = A\cdot - \cdot A \qquad \text{and} \qquad [A,\cdot]_+ = A\cdot + \cdot A.$$

In order to define a fully noncommutative version of the PII hierarchy, as already anticipated in the introduction, we first define a sequence of differential polynomials $\mathcal{L}_n[U]$ through a matrix version of the Lenard operators. Following [50]:

$$\mathcal{L}_0[U] = \frac{1}{2}I_r,$$
$$\frac{\mathrm{d}}{\mathrm{d}S}\mathcal{L}_n[U] = \left(\frac{\mathrm{d}^3}{\mathrm{d}S^3} + [U,\cdot]_+\frac{\mathrm{d}}{\mathrm{d}S} + \frac{\mathrm{d}}{\mathrm{d}S}[U,\cdot]_+ + [U,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[U,\cdot]\right)\mathcal{L}_{n-1}[U], \qquad n\geq 1. \quad (6.2.1)$$

Here I_r denotes the identity matrix, $\frac{\mathrm{d}}{\mathrm{d}S}$ denotes the differential operator defined in (6.0.5) and $\frac{\mathrm{d}}{\mathrm{d}S}^{-1}$ denotes the corresponding formal antiderivative. The locality of these operators computed in U follows from Theorem 6.2 in [93].

Example 6.2.1. *The first of the differential polynomials in U given by the recursive formula* (6.2.1) *read as follows:*

$$\mathcal{L}_1[U] = U,$$
$$\mathcal{L}_2[U] = U_{2S} + 3U^2,$$
$$\mathcal{L}_3[U] = U_{4S} + 5[U,U_{2S}]_+ + 5U_S^2 + 10U^3.$$

From $n\geq 3$ the "noncommutative" character of these operators appears in form of anticommutators.

Remark 6.2.2. *In the example above and in the following we use the shorter notation $\left(\frac{\mathrm{d}}{\mathrm{d}S}\right)^n U = U_{nS}$ for any $n\in\mathbb{N}$.*

Definition 6.2.3. *We define a matrix PII hierarchy as follows*

$$\mathrm{PII}_{\mathrm{NC}}^{(n)}[\alpha_n]: \quad \left(\frac{\mathrm{d}}{\mathrm{d}S} + [W, \cdot]_+\right) \mathcal{L}_n[U] = (-1)^{n+1} 4^n [S, W]_+ + a_n I_r, \tag{6.2.2}$$

where U is as in the scalar case, the Miura transform of the function W, i.e., $U := \frac{\mathrm{d}}{\mathrm{d}S} W - W^2$, and a_n are scalar constants.

In particular we will study the homogeneous hierarchy, setting $a_n = 0$ for each n.

Remark 6.2.4. *It is also possible to define a more general hierarchy, in the following way*

$$\begin{aligned} \mathrm{PII}_{\mathrm{NC}}^{(n)}[\alpha_n]\colon \quad \left(\frac{\mathrm{d}}{\mathrm{d}S} + [W, \cdot]_+\right) \mathcal{L}_n[U] + \sum_{l=1}^{n-1} t_l \left(\frac{\mathrm{d}}{\mathrm{d}S} + [W, \cdot]_+\right) \mathcal{L}_l[U] \\ = (-1)^{n+1} 4^n [S, W]_+ + a_n I_r, \end{aligned}$$

for some scalars $t_1, \ldots, t_{n-1}$. We recover the hierarchy (6.2.2) setting up these scalars to 0. Another matrix hierarchy was introduced in [50], but there the time variable is a scalar.

Example 6.2.5. *Here are the first three equations of the homogeneous hierarchy (6.2.2).*

- *For $n = 1$ we obtain the noncommutative analogue of the homogeneous PII equation:*

$$\mathrm{PII}_{\mathrm{NC}}\colon \quad W_{2S} = 2W^3 + 4[S, W]_+. \tag{6.2.3}$$

This coincides with the homogeneus version of the fully noncommutative PII equation studied in [97], in a more general context of any noncommutative algebra with derivation.

- *For $n = 2$ we have the 4-th order equation:*

$$\begin{aligned} \mathrm{PII}_{\mathrm{NC}}^{(2)}\colon \quad W_{4S} = 6W^5 + 4\left[W^2, W_{2S}\right]_+ + 2W W_{2S} W + 2\left[W_S^2, W\right]_+ \\ + 6 W_S W W_S - 4^2 [S, W]_+. \end{aligned}$$

- *For $n = 3$ we have the 6-th order equation:*

$$
\begin{aligned}
\mathrm{PII}_{\mathrm{NC}}^{(3)}:\ W_{6S} = {} & 20W^7 - 15\Big[W_{2S}, W^4\Big] - 20W^2W_{2S}W^2 - 10\Big[WW_{2S}W, W^2\Big]_+ \\
& - 10\Big[W_S^2, W^3\Big]_+ - 15\Big[WW_S^2W, W\Big]_+ - 20W_SW^3W_S \\
& - 25\Big[W_SWW_S, W^2\Big]_+ - 5\Big[W_SW^2W_S, W\Big]_+ - 10WW_SWW_SW \\
& + 6\Big[W_{4S}, W^2\Big] + 2WW_{4S}W + 4(W_SW_{3S}W + +WW_{3S}W_S) \\
& + 9(WW_SW_{3S} + W_{3S}W_SW) + 15(W_SWW_{3S} + W_{3S}WW_S) \\
& + 25\Big[W_{2S}, W_S^2\Big]_+ + 20W_SW_{2S}W_S \\
& + 11\Big[W_{2S}^2, W\Big]_+ + 20W_{2S}WW_{2S} + 4^3[S, W]_+.
\end{aligned}
$$

A fundamental property of matrix Lenard operators (that we are going to use in the next section in order to find the Lax pair for the hierarchy (6.2.2)) is given by the following formula (see [50]).

Proposition 6.2.6. *For each $n \in \mathbb{N}$ the matrix-valued Lenard operator acting on the Miura transform factorizes like*

$$\frac{\mathrm{d}}{\mathrm{d}S}\mathcal{L}_{n+1}[U] = \left(\frac{\mathrm{d}}{\mathrm{d}S} - [W,\cdot]_+\right)\left(\frac{\mathrm{d}}{\mathrm{d}S} - [W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S} + [W,\cdot]_+\right)\mathcal{L}_n[U]. \tag{6.2.4}$$

This formula is achieved by the direct computation of the recursive formula for the noncommutative Lenard operators computed in the Miura transform $U = W_S - W^2$. It is exactly the analogue of the factorization formula (2.2.14) that we described in the scalar case treated in Chapter 2.

6.3 The isomonodromic Lax pair

In this section we are finally going to find out a Lax pair for the noncommutative hierarchy (6.2.2), making use of the Riemann-Hilbert Problem 6.1.5 introduced in Section 6.1.

To start with, we consider a new sequence of functions, defined using the solution of the Riemann-Hilbert Problem 6.1.5.

Definition 6.3.1. *For each $n \in \mathbb{N}$, we construct*

$$\Psi^{(n)}(\lambda, \vec{s}) := \Xi^{(n)}(\lambda)\exp\left(\theta^{(n)}(\lambda) \otimes \sigma_3\right).$$

It is easy to check that these functions $\left\{\Psi^{(n)}\right\}_{n\in\mathbb{N}}$ actually solve a new sequence of Riemann-Hilbert problems, with constant jump conditions. Namely, the following problems.

Riemann-Hilbert Problem 6.3.2. *Find a (λ-)analytic matrix-valued function*

$$\Psi^{(n)}(\lambda):\ \mathbb{C} \setminus \left(\gamma_+^n \cup \gamma_-^n\right) \to \mathrm{GL}(2r, \mathbb{C})$$

admitting continuous extension to the contour $\gamma_+^n \cup \gamma_-^n$ from either side and such that it satisfies the following two conditions:

- *the jump condition for each $\lambda \in \gamma_+^n \cup \gamma_-^n$*

$$\Psi_+^{(n)}(\lambda) = \Psi_-^{(n)}(\lambda) \underbrace{\begin{pmatrix} I_r & C\chi_{\gamma_+^n}(\lambda) \\ C\chi_{\gamma_-^n}(\lambda) & I_r \end{pmatrix}}_{:=K^{(n)}};$$

- *the asymptotic condition for $|\lambda| \to \infty$*

$$\Psi^{(n)}(\lambda) \sim \left(I_{2r} + \sum_{j\geq 1} \frac{\Xi_j^{(n)}}{\lambda^j}\right) \exp\left(\theta^{(n)}(\lambda) \otimes \sigma_3\right).$$

As it is standard in the theory of isomonodromic deformations, we deduce the Lax pair for the noncommutative PII hierarchy (6.2.2) from the Riemann-Hilbert problems with piecewise constant jumps solved by $\Psi^{(n)}$. The main idea is the following: using the fact that each $\Psi^{(n)}$ has constant jump condition (i.e., the jump matrix $K^{(n)}$ does not explicitly depend on the spectral parameter λ or the deformations parameters s_i, $i = 1, \dots, r$), we can thus conclude that the quantities

$$\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1} =: L^{(n)} \qquad \text{and} \qquad \frac{\partial}{\partial\lambda}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1} =: M^{(n)} \tag{6.3.1}$$

are matrix-valued polynomials in λ.

Remark 6.3.3. *Here the inverse of $\Psi^{(n)}$ is simply given by*

$$\left(\Psi^{(n)}\right)^{-1}(\lambda) = \exp\left(-\theta^{(n)}(\lambda) \otimes \sigma_3\right)\Theta^{(n)}(\lambda).$$

Furthermore, by using the symmetries of the Riemann-Hilbert Problem 6.1.5, we can compute the exact form of the coefficients of these polynomials $L^{(n)}$, $M^{(n)}$.

The final result is summarized in the proposition below.

Proposition 6.3.4. *There exist two polynomial matrices in λ, which we denote with $L^{(n)}$ and $M^{(n)}$, respectively of degree 1 and $2n$, such that the following system of differential equations is satisfied:*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}(\lambda, \vec{s}) = L^{(n)}(\lambda, \vec{s})\Psi^{(n)}(\lambda, \vec{s}), \\ &\partial_\lambda \Psi^{(n)}(\lambda, \vec{s}) = M^{(n)}(\lambda, \vec{s})\Psi^{(n)}(\lambda, \vec{s}). \end{aligned} \tag{6.3.2}$$

Moreover, $L^{(n)}$ and $M^{(n)}$ have the following forms

$$L^{(n)}(\lambda,\vec{s}) = \begin{pmatrix} \mathrm{i}\lambda I_r & W(\vec{s}) \\ W(\vec{s}) & -\mathrm{i}\lambda I_r \end{pmatrix}, \qquad \textit{with} \quad W(\vec{s}) = 2\beta_1^{(n)}(\vec{s}),$$

and

$$M^{(n)}(\lambda,\vec{s}) = \begin{pmatrix} A(\lambda,\vec{s}) + \mathrm{i}S & \mathrm{i}G(\lambda,\vec{s}) \\ -\mathrm{i}G(\lambda,\vec{s}) & -A(\lambda,\vec{s}) - \mathrm{i}S \end{pmatrix} + \begin{pmatrix} E(\lambda,\vec{s}) & F(\lambda,\vec{s}) \\ F(\lambda,\vec{s}) & E(\lambda,\vec{s}) \end{pmatrix},$$

where

$$A(\lambda,\vec{s}) = \sum_{k=0}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k} A_{2n-2k}(\vec{s}), \qquad \textit{with} \quad A_{2n} = I_r,$$
$$G(\lambda,\vec{s}) = \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k} G_{2n-2k}(\vec{s}),$$
$$E(\lambda,\vec{s}) = \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k+1} E_{2n-2k+1}(\vec{s}),$$
$$F(\lambda,\vec{s}) = \sum_{k=1}^{n} \frac{\mathrm{i}}{2}\lambda^{2n-2k+1} F_{2n-2k+1}(\vec{s}).$$

Proof. We start computing the logarithmic derivative of $\Psi^{(n)}$ w.r.t. S, namely the quantity that we defined in (6.3.1) as

$$\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1} := L^{(n)}.$$

The matrix-valued function $L^{(n)}$ is entire in λ, since it has no jumps along $\gamma_+^n \cup \gamma_-^n$. Furthermore, its asymptotic behavior at infinity is given by a matrix polynomial of degree 1 in λ. Thus, by the generalized Liouville theorem, we conclude that $L^{(n)}$ is exactly a matrix polynomial of degree 1 in λ.

In particular from the asymptotic expansion at ∞, we find an explicit form of its matrix coefficients. Here and in the following series expansions in powers of λ we will use the notation $[\ ]_{\geq 0}$ to indicate that we are taking only the powers λ^r with $r \geq 0$.

$$\begin{aligned} L^{(n)}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1} &= \left[\left(I_{2r} + \sum_{j\geq 1}\frac{\Xi_j^{(n)}}{\lambda^j}\right)\mathrm{i}\lambda\hat{\sigma}_3\left(I_{2r} + \sum_{j\geq 1}\frac{\Theta_j^{(n)}}{\lambda^j}\right)\right]_{\geq 0} \\ &= \mathrm{i}\lambda\hat{\sigma}_3 + \mathrm{i}\left(\Xi_1^{(n)}\hat{\sigma}_3 + \hat{\sigma}_3\Theta_1^{(n)}\right) = \mathrm{i}\lambda\hat{\sigma}_3 + \mathrm{i}\left[\Xi_1^{(n)}, \hat{\sigma}_3\right] = \mathrm{i}\lambda\hat{\sigma}_3 + 2\beta_1^{(n)} \otimes \sigma_1, \end{aligned}$$

where in the last two passages we used the fact that $\Theta_1^{(n)} = -\Xi_1^{(n)}$ and then the symmetry (6.1.8).

We can then consider the second quantity defined in (6.3.1), namely

$$\frac{\partial}{\partial\lambda}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1} =: M^{(n)}.$$

We use the same argument as for $L^{(n)}$. Indeed, also $M^{(n)}$ is entire in λ, since it has no jumps along $\gamma_+^n \cup \gamma_-^n$. Its asymptotic behavior at infinity is given by a matrix polynomial of degree $2n$ in λ. We thus conclude, by the generalized Liouville theorem, that $M^{(n)}$ is exactly a matrix polynomial in λ of degree $2n$. In particular from the asymptotic expansion at ∞ we can find an explicit form of this matrix:

$$
\begin{aligned}
M^{(n)}(\lambda) &= \partial_\lambda \Psi^{(n)} \left(\Psi^{(n)}\right)^{-1} \\
&= \left[\left(I_{2r} + \sum_{j\geq 1} \frac{\Xi_j^{(n)}}{\lambda^j} \right) \left(\left(\frac{\mathrm{i}\lambda^{2n} I_r}{2} + \mathrm{i}S \right) \otimes \sigma_3 \right) \left(I_{2r} + \sum_{j\geq 1} \frac{\Theta_j^{(n)}}{\lambda^j} \right) \right]_{\geq 0} \\
&= \frac{\mathrm{i}\lambda^{2n}}{2} \hat{\sigma}_3 + \mathrm{i}S \otimes \sigma_3 + \sum_{l=1}^{2n} \frac{\mathrm{i}\lambda^{2n-l}}{2} \underbrace{\left(\Xi_l^{(n)} \hat{\sigma}_3 + \hat{\sigma}_3 \Theta_l^{(n)} + \sum_{j:\, j+k=l} \Xi_j^{(n)} \hat{\sigma}_3 \left(\sum_{k=1}^{l-1} \Theta_k^{(n)} \right) \right)}_{=M_{2n-l}^{(n)}} .
\end{aligned}
$$

In order to obtain the remaining part of the statement, we use the following lemma.

Lemma 6.3.5. *The coefficient of the term λ^{2n-l} in the matrix $M^{(n)}$ is such that:*

- *if $l = 2m$, then*
$$M_{2n-2m}^{(n)} = A_{2n-2m}(\vec{s})\hat{\sigma}_3 + G_{2n-2m}(\vec{s})\hat{\sigma}_2;$$
- *if instead $l = 2m-1$, then*
$$M_{2n-2m+1}^{(n)} = E_{2n-2m+1}(\vec{s}) \otimes I_{2r} + F_{2n-2m+1}(\vec{s})\hat{\sigma}_1.$$

Proof. The proof is a direct consequence of the symmetry property that the asymptotics coefficients of $\Xi^{(n)}$, $\Theta^{(n)}$ have. We start with the even case $l = 2m$. The coefficient of the term λ^{2n-2m} in the matrix $M^{(n)}$ is given by the following sum:

$$M_{2n-2m}^{(n)} = \left(\Xi_{2m}^{(n)} \hat{\sigma}_3 + \hat{\sigma}_3 \Theta_{2m}^{(n)} + \sum_{j:\, j+k=2m} \Xi_j^{(n)} \hat{\sigma}_3 \left(\sum_{k=1}^{2m-1} \Theta_k^{(n)} \right) \right),$$

where in the last sum all the terms are of type

$$\Xi_{2s}^{(n)} \hat{\sigma}_3 \Theta_{2(m-s)}^{(n)} \qquad \text{or} \qquad \Xi_{2s-1}^{(n)} \hat{\sigma}_3 \Theta_{2(m-s)+1}^{(n)}.$$

Using the symmetries (6.1.8) and (6.1.9), a direct computation shows that these terms are always linear combinations of the Pauli's matrices $\hat{\sigma}_2$, $\hat{\sigma}_3$.

So we can conclude that

$$M_{2n-2m}^{(n)} = A_{2n-2m}(\vec{s})\hat{\sigma}_3 + G_{2n-2m}(\vec{s})\hat{\sigma}_2.$$

where the functions $A_{2n-2m}(\vec{s})$, $G_{2n-2m}(\vec{s})$ depend on the asymptotic coefficients of $\Xi^{(n)}$,

$\Theta^{(n)}$.

We work in the same way for the odd case, $l = 2m-1$. The coefficient of $\lambda^{2n-2m+1}$ is given by the same formula

$$M^{(n)}_{2n-2m+1} = \left(\Xi^{(n)}_{2m-1}\hat{\sigma}_3 + \hat{\sigma}_3\Theta^{(n)}_{2m-1} + \sum_{j\colon j+k=2m-1} \Xi^{(n)}_j \hat{\sigma}_3 \left(\sum_{k=1}^{2m-2} \Theta^{(n)}_k \right) \right),$$

where in the last sum there are just terms of the two following types

$$\Xi^{(n)}_{2s}\hat{\sigma}_3\Theta^{(n)}_{2(m-s)-1} \qquad \text{or} \qquad \Xi^{(n)}_{2s-1}\hat{\sigma}_3\Theta^{(n)}_{2(m-s)}.$$

In both of the cases, always replacing the symmetries (6.1.8) and (6.1.9), they result to be linear combinations of I_{2r}, $\hat{\sigma}_1$. Thus we can finally conclude that

$$M^{(n)}_{2n-2m+1} = E_{2n-2m+1}(\vec{s}) \otimes I_{2r} + F_{2n-2m+1}(\vec{s})\hat{\sigma}_1. \qquad \blacksquare$$

Thanks to this lemma, the form of the matrix $M^{(n)}$ is exactly the one of the statement and the proposition is completely proved. ■

Remark 6.3.6. *The system* (6.3.2) *for* $\Psi^{(n)}$ *describes the isomonodromic deformations w.r.t. the deformation parameters* s_i*,* $i = 1, \dots, r$*, of the linear differential equation*

$$\frac{\partial}{\partial\lambda}\Psi^{(n)}(\lambda, \vec{s}) = M^{(n)}(\lambda, \vec{s})\Psi^{(n)}(\lambda, \vec{s}),$$

that has only one irregular singular point at ∞ *of Poincaré rank* $r = 2n+1$*, and in the special case of symmetry*

$$-\hat{\sigma}_1 M^{(n)}(\lambda, \vec{s})\hat{\sigma}_1 = M^{(n)}(-\lambda, \vec{s}).$$

We can finally state that the system (6.3.2) is an isomonodromic Lax pair for the matrix PII hierarchy (6.2.2).

Proposition 6.3.7. *For each fixed* n*, the compatibility condition of the system* (6.3.2)*, i.e., the equation*

$$\frac{\partial}{\partial\lambda}L^{(n)}(\lambda, \vec{s}) - \frac{\mathrm{d}}{\mathrm{d}S}M^{(n)}(\lambda, \vec{s}) + \left[L^{(n)}(\lambda, \vec{s}), M^{(n)}(\lambda, \vec{s})\right] = 0 \qquad (6.3.3)$$

is equivalent to the following equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}S} + [W, \cdot]_+\right)\mathcal{L}_n[U] = (-1)^{n+1}4^n[S, W]_+,$$

Furthermore, the coefficients of the matrix $M^{(n)}$ *are written in terms of the matrix Lenard* *operators in the following way*

$$
\begin{aligned}
A_{2n-2k}(\vec{s}) &= -\frac{1}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\mathcal{L}_k[U] - \left(\frac{\mathrm{d}}{\mathrm{d}S} - [W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S} + [W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right),\\
G_{2n-2k}(\vec{s}) &= \frac{\mathrm{i}}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\left(\frac{\mathrm{d}}{\mathrm{d}S} - [W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S} + [W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right),\\
E_{2n-2k+1}(\vec{s}) &= -\mathrm{i}\left(-\frac{1}{4}\right)^{k-1}\frac{\mathrm{d}}{\mathrm{d}S}^{-1}\left([W,\cdot]\left([W,\cdot]_+ + \frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_{k-1}[U]\right),\\
F_{2n-2k+1}(\vec{s}) &= -\mathrm{i}\left(-\frac{1}{4}\right)^{k-1}\left(\left([W,\cdot]_+ + \frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_{k-1}[U]\right), \qquad \textit{for} \quad k = 1,\dots,n. \qquad (6.3.4)
\end{aligned}
$$

In other words the system (6.3.2) *is a Lax pair for the matrix Painlevé II hierarchy* (6.2.2).

Proof. We first rewrite the compatibility condition (6.3.3) as the following system of differential equations for the coefficients A, F, G, E:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}S}E(\lambda,\vec{s}) &= [W, F(\lambda,\vec{s})],\\
\frac{\mathrm{d}}{\mathrm{d}S}A(\lambda,\vec{s}) &= -\mathrm{i}[W, G(\lambda,\vec{s})]_+,\\
\frac{\mathrm{d}}{\mathrm{d}S}F(\lambda,\vec{s}) &= -2\lambda G(\lambda,\vec{s}) + [W, E(\lambda,\vec{s})],\\
\frac{\mathrm{d}}{\mathrm{d}S}G(\lambda,\vec{s}) &= 2\lambda F(\lambda,\vec{s}) + \mathrm{i}[W, A(\lambda,\vec{s})]_+ - [S,W]_+.
\end{aligned}
$$

These equations must be satisfied identically in λ. Thus, by the polinomiality of the coefficients A, F, G, E, this system is equivalent to the following one

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}S}E_{2n-2k+1}(\vec{s}) &= [W, F_{2n-2k+1}(\vec{s})],\\
\frac{\mathrm{d}}{\mathrm{d}S}A_{2n} &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}S}A_{2n-2k}(\vec{s}) &= -\mathrm{i}[W, G_{2n-2k}(\vec{s})]_+,\\
G_{2n-2k}(\vec{s}) &= \frac{1}{2}\left(-\frac{\mathrm{d}}{\mathrm{d}S}F_{2n-2k+1}(\vec{s}) + [W, E_{2n-2k+1}(\vec{s})]\right),\\
F_{2n-1}(\vec{s}) &= -\frac{\mathrm{i}}{2}\left[W, A_{2n}\right]_+,\\
F_{2n-2k-1}(\vec{s}) &= \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}S}G_{2n-2k}(\vec{s}) - \mathrm{i}[W, A_{2n-2k}(\vec{s})]_+\right),\\
\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}S}G_0(\vec{s}) &= -[S,W]_+ - \frac{1}{2}[W, A_0(\vec{s})]_+ \qquad \text{for} \quad k = 1,\dots,n. \qquad (6.3.5)
\end{aligned}
$$

In order to prove the statement, we are going to prove by induction over $l = 2n - j$ that each coefficient A_{2n-2k}, $E_{2n-2k+1}$, G_{2n-2k}, $F_{2n-2k+1}$ is given by the formulae (6.3.4) and that this implies that the last equation in the system (6.3.5) is exactly the n-th member of the PII hierarchy (6.2.2).

We first check that for $l = 2n-1, 2n-2$ the formulae (6.3.4) are solutions of the equations (6.3.5), i.e., the coefficients F_{2n-1}, E_{2n-1}, G_{2n-2}, A_{2n-2}, are given by these formulae.

Since $A_{2n} = I_r$, the equation
$$\frac{\mathrm{d}}{\mathrm{d}S} A_{2n} = 0$$
is satisfied. Then, the equation for F_{2n-1} will be satisfied for

$$F_{2n-1} = -\mathrm{i}W,$$

that is exactly the result of the formula in (6.3.4) for $k = 1$, since

$$-\mathrm{i}\left(-\frac{1}{4}\right)^{0}\left(\left([W,\cdot]_{+} + \frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_0[U]\right) = -\mathrm{i}W.$$

As a consequence, the equation for the coefficient E_{2n-1} in the system (6.3.5) becomes

$$\frac{\mathrm{d}}{\mathrm{d}S} E_{2n-1}(\vec{s}) = 0,$$

thus E_{2n-1} is constant w.r.t. the variable S and it is in particular $E_{2n-1} = 0$, because of the asymptotics of $\Psi^{(n)}$. This is also what is given by the formula for $k = 1$:

$$-\mathrm{i}\left(-\frac{1}{4}\right)^{0} \frac{\mathrm{d}}{\mathrm{d}S}^{-1}\left([W,\cdot]\left([W,\cdot]_{+} + \frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_0[U]\right) = 0.$$

We can then compute the term G_{2n-2} for which the equation in (6.3.5) is now

$$G_{2n-2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}S}(-\mathrm{i}W) = \frac{\mathrm{i}}{2}W_S,$$

that coincides with the formula

$$\frac{\mathrm{i}}{2}\left(-\frac{1}{4}\right)^{0}\left(\left(\frac{\mathrm{d}}{\mathrm{d}S} - [W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S} + [W,\cdot]_{+}\right)\mathcal{L}_0[U]\right) = \frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}S}W.$$

Finally, we can compute the term A_{2n-2}. It is supposed to satisfy, from the system (6.3.5), the equation
$$\frac{\mathrm{d}}{\mathrm{d}S}A_{2n-2} = -\mathrm{i}[W, G_{2n-2}]_{+} = \frac{1}{2}[W, W_S]_{+}.$$
Integrating and taking the constant of integration another time equal 0 (for the same reason used above) we get
$$A_{2n-2} = \frac{1}{2}W^2.$$

The same that is given by the formula

$$-\frac{1}{2}\left(-\frac{1}{4}\right)^{0}\left(\mathcal{L}_1[U]-\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_0[U]\right)$$
$$=-\frac{1}{2}\Big(W_S-W^2-W_S\Big).$$

Thus for $k=1$ the formulas in (6.3.4) gives solutions of the system (6.3.5).

Now we proceed by induction: supposing that for $l=2n-2k+1$ the coefficients $E_{2n-2k+1}$, $F_{2n-2k+1}$ are given by the formulas (6.3.4), we will find that then also the coefficients for $l=2n-2k$ and $l=2n-2k-1$ have the form given by the formulas (6.3.4). Indeed, from the equations in (6.3.5), we have

$$G_{2n-2k}(\vec{s})=\frac{1}{2}\left(-\frac{\mathrm{d}}{\mathrm{d}S}F_{2n-2k+1}(\vec{s})+[W,E_{2n-2k+1}(\vec{s})]\right)=-\frac{1}{2}\left(-\mathrm{i}\left(-\frac{1}{4}\right)^{k-1}\frac{\mathrm{d}}{\mathrm{d}S}\left(\left([W,\cdot]_++\frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_{k-1}[U]\right)\right.$$
$$\left.+\frac{1}{2}\left([W,\cdot]\left(-\mathrm{i}\left(-\frac{1}{4}\right)^{k-1}\frac{\mathrm{d}}{\mathrm{d}S}^{-1}\left([W,\cdot]\left([W,\cdot]_++\frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_{k-1}[U]\right)\right)\right)\right)$$
$$=\frac{\mathrm{i}}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right)$$

that is exactly the formula in (6.3.4) for this coefficient. Then we can compute

$$A_{2n-2k}(\vec{s})=-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,G_{2n-2k}(\vec{s})]_+=\frac{1}{2}\left(-\frac{1}{4}\right)^{k-1}\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]_+$$
$$\times\left(\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right)$$
$$=-\frac{1}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\mathcal{L}_k[U]-\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right),$$

where in the last passage we have integrated (taking the integration's constant 0) after having applied formula (6.2.4). Then the equation for $F_{2n-2k-1}(\vec{s})$ reads as

$$F_{2n-2k-1}=\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}S}G_{2n-2k}(\vec{s})-\mathrm{i}\,[W,A_{2n-2k}(\vec{s})]_+\right)$$
$$=\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}S}\frac{\mathrm{i}}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right)\right)$$
$$-\frac{\mathrm{i}}{2}\left([W,\cdot]_+\frac{1}{2}\left(-\frac{1}{4}\right)^{k-1}\left(\mathcal{L}_k[U]-\left(\frac{\mathrm{d}}{\mathrm{d}S}-[W,\cdot]\frac{\mathrm{d}}{\mathrm{d}S}^{-1}[W,\cdot]\right)\right.\right.$$
$$\left.\left.\times\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_{k-1}[U]\right)\right)=-\mathrm{i}\left(-\frac{1}{4}\right)^{k}\left(\frac{\mathrm{d}}{\mathrm{d}S}+[W,\cdot]_+\right)\mathcal{L}_k[U],$$

where in the last line we used another time property (6.2.4) of the matrix Lenard operators. Finally, the formula for $E_{2n-2k-1}$ directly follows from the equation above and taking the integration contant equal 0, while integrating the equation (6.3.5).

In the end, when we replace the formulas for G_0, A_0 in the last equation of the system (6.3.5), namely

$$\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}S}G_0(\vec{s}) = -[S,W]_+ - \frac{1}{2}[W, A_0(\vec{s})]_+,$$

using another time the property (6.2.4) we get the n-th member of the Painlevé II hierarchy:

$$\left([W,\cdot]_+ + \frac{\mathrm{d}}{\mathrm{d}S}\right)\mathcal{L}_n[U] = (-1)^{n+1}4^n[S,W]_+. \qquad \blacksquare$$

Remark 6.3.8. *The matrices $L^{(n)}$, $M^{(n)}$ obtained here, are the analogue of the Lax pair for the scalar homogeneous Painlevé II hierarchy obtained in [30], written in Theorem 5.3.3, with $W(\vec{s})$ given by*

$$2\beta_1^{(n)}(\vec{s}) = -2\mathrm{i}\lim_{|\lambda|\to\infty}\left(\lambda\Xi^{(n)}(\vec{s})\right)_{1,2} := W(\vec{s}).$$

Also, the proof by induction previously done, it is inspired by the technique used in [30].

We can then state and prove the final result of this study, that links solutions of the homogeneus matrix Painlevé II hierarchy (6.2.2) to Fredholm determinants of the matrix Airy operators.

Corollary 6.3.9. *There exists a solution W of the n-th member of the PII hierarchy* (6.2.2) *connected to Fredholm determinant of the n-th Airy matrix operator* (6.0.3) *through the following formula*

$$-\operatorname{Tr}\left(W^2(\vec{s})\right) = \frac{\mathrm{d}^2}{\mathrm{d}S^2}\ln\left(F^{(n)}(s_1,\ldots,s_r)\right). \tag{6.3.6}$$

This solution W has boundary behavior $(W)_{k,l=1}^r \sim -2(c_{kl}\mathrm{Ai}_{2n+1}(s_k+s_l))_{k,l=1}^r$ in the regime $\mathrm{s}\to+\infty$ with $|\delta_j|\le m$ for every j, where $\mathrm{s} := \frac{1}{r}\sum_{j=1}^r s_j$ is the baricenter of the variables s_j, and $\delta_j := s_j - \mathrm{s}$.

Proof. We first prove the formula (6.3.6). The statement is achieved by Theorem 6.1.9 and the relation between $\alpha_1^{(n)}$, $\beta_1^{(n)}$ given by

$$\frac{\mathrm{d}}{\mathrm{d}S}\alpha_1^{(n)} = -2\mathrm{i}\left(\beta_1^{(n)}\right)^2. \tag{6.3.7}$$

This relation holds for each n and it is obtained by looking at the coefficient of the term λ^{-1} in the asymptotic expansion at ∞ of

$$\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1},$$

and recalling that it must be 0. Indeed, from the asymptotic expansion of $\Psi^{(n)}$ we have that the power λ^{-1} coming from the formal asymptotic expansion of $\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1}$ is*

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}S}\Psi^{(n)}\left(\Psi^{(n)}\right)^{-1}\right]_{-1} &= \left[\left(I_{2r}+\sum_{j\geq 1}\frac{\Xi_j^{(n)}}{\lambda^j}\right)\mathrm{i}\lambda\hat{\sigma}_3\left(I_{2r}+\sum_{j\geq 1}\frac{\Theta_j^{(n)}}{\lambda^j}\right)\right]_{-1}\\ &= \frac{\mathrm{i}}{\lambda}\left(\Xi_2^{(n)}\hat{\sigma}_3+\hat{\sigma}_3\Theta_2^{(n)}+\Xi_1^{(n)}\sigma_3\Theta_1^{(n)}+\frac{\mathrm{d}}{\mathrm{d}S}\Xi_1\right).\end{aligned}$$

And replacing in the coefficient of λ^{-1} the relations between the asymptotic coefficients of $\Theta^{(n)}$ and the ones of $\Xi^{(n)}$, namely

$$\Theta_1^{(n)} = -\Xi_1^{(n)}, \qquad \Theta_2^{(n)} = \left(\Xi_1^{(n)}\right)^2 - \Xi_2^{(n)}$$

the result is exactly the relation (6.3.7).

Now we are going to prove the second part of the statement. We define the scalar variables $\mathrm{s} := \frac{1}{r}\sum_{j=1}^{r} s_j$ and $\delta_j := s_j - \mathrm{s}$ for any $j = 1, \ldots, r$.

We are now going to study the behavior of the solution W for

$$\mathrm{s} \to +\infty \qquad \text{and} \qquad |\delta_j| \leq m \quad \forall\, j. \tag{6.3.8}$$

First, we rewrite the jump matrix $J^{(n)}(\lambda, \vec{s})$ of Riemann-Hilbert Problem 6.1.5 in terms of the rescaled complex parameter $z\mathrm{s}^{\frac{1}{2n}} = \lambda$.

In particular we obtain that the jump matrices along γ_+^n and along γ_-^n, are factorized in a product of commuting matrices, written in terms of the rescaled parameter z and the variables s, δ_j. Namely,

$$\begin{aligned}&I_{2r} - 2\pi\mathrm{i}r^{(n)}\left(\pm z\mathrm{s}^{\frac{1}{2n}}\right)\otimes\sigma_{\pm}\chi_{\gamma_\pm^n}\left(z\mathrm{s}^{\frac{1}{2n}}\right)\\ &= \prod_{k,l=1}^{r}\left(I_{2r} + c_{kl}\mathrm{e}^{\pm\mathrm{i}\mathrm{s}^{\frac{2n+1}{2n}}\left(\frac{z^{2n+1}}{2n+1}+z\left(2+\frac{\delta_k+\delta_l}{\mathrm{s}}\right)\right)}E_{k,l}\otimes\sigma_{\pm}\chi_{\tilde{\gamma}_\pm^n}(z)\right),\end{aligned} \tag{6.3.9}$$

where $E_{k,l}$ are the elementary matrices and $\sigma_+ = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $\sigma_- = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ and $\tilde{\gamma}_\pm^n$ are the transformed contours under the scaling $\lambda = z\mathrm{s}^{\frac{1}{2n}}$.

Now, we are going to show that each matrix in the factorization (6.3.9), that we denote by $F_{kl}^{\pm}$, is close to the identity matrix I_{2r} in the regime fixed in (6.3.8). Remark that every $F_{kl}^{\pm}$ has $2n$ critical points, corresponding to

$$z_0^h = d_{kl}^{\frac{1}{2n}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2n}(2h+1)}, \qquad h = 0, \ldots, 2n-1,$$

where $d_{kl} = 2 + \frac{\delta_k+\delta_l}{\mathrm{s}}$ is real, positive and bounded, while looking at the regime (6.3.8).

We can then split the curves $\tilde{\gamma}_\pm^n$ respectively in the curves $\tilde{\gamma}_{\pm,kl}^n$ one for each factor $F_{kl}^{\pm}$ appearing in the factorization (6.3.9). The curves $\tilde{\gamma}_{\pm,kl}^n$ pass respectively through the points

*Here the notation $[\ \ ]_{-1}$ indicates that we only take the term λ^{-1} in the relevant formal series.

z_0^h with $h = 0, \dots, n-1$ (in the upper plane) and $h = n, \dots, 2n-1$ (in the lower plane).

In this way, we can then evaluate the ∞-norm of each term $F_{kl}^{\pm} - I_{2r}$ and we have

$$\left|\left|F_{kl}^{\pm} - I_{2r}\right|\right|_{\infty} = |c_{kl}| \sup_{z \in \tilde{\gamma}^n_{\pm,kl}} \mathrm{e}^{\mp \mathrm{s}^{\frac{2n+1}{2n}} \Im\left(\frac{z^{2n+1}}{2n+1} + z d_{kl}\right)} = |c_{kl}| \mathrm{e}^{\mp \frac{2n}{2n+1} (\mathrm{s} d_{kl})^{\frac{2n+1}{2n}} \sin\left(\pm \frac{\pi}{2n}\right)} \to 0$$

for $\mathrm{s} \to +\infty$ and $|\delta_j| \leq m \; \forall \, j$.

We can conclude that the rescaled jump matrix itself $J^{(n)}\left(\mathrm{s}^{\frac{1}{2n}} z\right)$ is close to the identity matrix in the regime (6.3.8), since each factor $F_{kl}^{\pm}$ in its factorization shares this property.

Consider now the rescaled function $X^{(n)}(z) := \Xi^{(n)}\left(z \mathrm{s}^{\frac{1}{2n}}\right)$. By using Riemann-Hilbert Problem 6.1.5 solved by $\Xi^{(n)}$, we have that

- $X^{(n)}$ is analytic on $\mathbb{C} \setminus \tilde{\gamma}^n_+ \cup \tilde{\gamma}^n_-$ and it admits continuous extension to these curves from either side;
- its boundary values $X_{\pm}(z)$ while approaching $\tilde{\gamma}^n_+ \cup \tilde{\gamma}^n_-$ from the left and respectively from the right, are related through the jump condition (6.1.6) but with the rescaled jump matrix computed in (6.3.9);
- for $|z| \to +\infty$ we have $X^{(n)} \sim I_{2r} + \sum_{j \geq 1} \frac{X_j^{(n)}}{z^j}$.

Remark that we have $X_1^{(n)} = \mathrm{s}^{-\frac{1}{2n}} \Xi_1^{(n)}$.

By applying the small norm theorem (one version was stated in Theorem 4.1.9, i.e. Theorem 1.5.1 in [59]), we conclude that the function $X^{(n)}(z)$ behaves as

$$X^{(n)}(z) = I_{2r} + \mathcal{O}\left(z^{-1} \mathrm{e}^{-C \mathrm{s}^{\frac{2n+1}{2n}}}\right), \qquad \mathrm{s} \to +\infty, \quad |\delta_j| \leq m \quad \forall \, j, \tag{6.3.10}$$

for a certain value $C > 0$.

Now, using the integral formula [60] for the rescaled solution of the Riemann-Hilbert Problem 6.1.5, namely $X^{(n)}(z)$, we have that

$$X^{(n)}(z) = I_{2r} - \int_{\tilde{\gamma}^n_+} \frac{X_-^{(n)}(w) r^{(n)}\left(w \mathrm{s}^{\frac{1}{2n}}\right) \otimes \sigma_+}{w - z} \mathrm{d}w - \int_{\tilde{\gamma}^n_-} \frac{X_-^{(n)}(w) r^{(n)}\left(-w \mathrm{s}^{\frac{1}{2n}}\right) \otimes \sigma_-}{w - z} \mathrm{d}w,$$

and thus we recover the following expression for the first asymptotic coefficient

$$\left(X_1^{(n)}\right)_{1,2} = \int_{\tilde{\gamma}^n_+} X_-^{(n)}(w) r^{(n)}\left(w \mathrm{s}^{\frac{1}{2n}}\right) \mathrm{d}w.$$

Finally, by recalling the definition of W and using (6.3.10) we conclude that

$$W = -2\mathrm{i}\left(\Xi_1^{(n)}\right)_{1,2} = -2\mathrm{i}\mathrm{s}^{\frac{1}{2n}} \int_{\tilde{\gamma}^n_+} X_-^{(n)}(w) r^{(n)}\left(w \mathrm{s}^{\frac{1}{2n}}\right) \mathrm{d}w \sim -2\left(c_{kl} \mathrm{Ai}_{2n+1}(s_k + s_l)\right)_{k,l=1}^r,$$

in the regime (6.3.8). ■

Remark 6.3.10. *Relation* (6.3.6) *can be thought as the noncommutative analogue of the results proved in [103] for the Painlevé II equation and in [26, 81] for the scalar Painlevé II hierarchy, connecting the theory of Painlevé trascendents to the determinantal point processes theory. For the noncommutative Painlevé II equation* (6.2.3)*, i.e* $n = 1$*, this link was already established in [13] and here we actually extended that result to the noncommutative hierarchy* (6.2.2)*.*

Chapter 7

The integro-differential Painlevé II hierarchy

The aim of this chapter is to prove the main result contained in the joint work with Thomas Bothner and Mattia Cafasso [24]. This paper is devoted to study the Fredholm determinants of a finite temperature version of the Airy kernels previously introduced in Chapter 2, through equation (2.2.34). Specifically, their finite temperature version is defined for any $n \in \mathbb{N}$ and for a given weight function w satisfying the requirements written below.

Definition 7.0.1. *We consider a weight function $w : \mathbb{R} \to \mathbb{R}_+$ as any positive, strictly increasing and differentiable function, such that for some $\omega, x_0 > 0$,*

$$\lim_{x \to +\infty} w(x) = 1, \quad \lim_{x \to -\infty} w(x) = 0 \qquad \text{and} \qquad 0 < w'(x) \leq \mathrm{e}^{-\omega|x|} \qquad \forall\, |x| \geq x_0. \quad (7.0.1)$$

For any fixed weight function with the above properties, we construct the following operators.

Definition 7.0.2. *The finite temperature higher order Airy kernels are integral operators $\mathcal{K}_{t,n} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ acting through the kernel*

$$K_{t,n}(x,y) \coloneqq \int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+z+t)\mathrm{Ai}_{2n+1}(z+y+t)w(z)\,\mathrm{d}z, \quad t \in \mathbb{R}. \quad (7.0.2)$$

These operators $\mathcal{K}_{t,n}$ are proved to be trace class on $L^2(\mathbb{R}_+)$ so that their Fredholm determinants

$$D_n(t,\lambda) \coloneqq \det(1 - \lambda\mathcal{K}_{t,n}) \quad (7.0.3)$$

are well defined for any $(t,\lambda,n) \in \mathbb{R} \times \mathbb{C} \times \mathbb{N}$. As it happens in the scalar case for the Airy kernels (2.2.34), and in the matrix-valued generalization for the square of the Hankel Airy operators defined in (6.1.2), also in this finite temperature case the operators $\lambda\mathcal{K}_{t,n}$ define uniquely a determinantal point process for every $(t,\lambda,n) \in \mathbb{R} \times [0,1] \times \mathbb{N}$, so that the Fredholm determinants $D_n(t,\lambda)$ are the distribution functions of the last particle in this process. In this specific case, the interest in the study of the Fredholm determinants $D_n(t,\lambda)$ is moreover given by the applications that they have in statistical mechanics. Indeed, they were used in the paper [81] to describe some statistical quantities related to a model of free fermions in anharmonic traps at finite temperature. More specifically, in this paper the authors explained how $D_n(t,1)$, when the weight function w is chosen to be the Fermi factor, is equal to the edge scaling limit of the probability distribution of the largest momenta in this specific fermionic model. This was indeed the main motivation for us to study the Fredholm determinants $D_n(t,\lambda)$. For other occurences of these Fredholm determinants see for instance [5, 68]. In particular, our first aim was to find a Tracy-Widom type formula relating the Fredholm determinants $D_n(t,\lambda)$ to some distinguished Painlevé II trascendents of some kind, generalizing the classical result of Tracy and Widom [102]. The process that allowed us to achieve this result has two new remarkable features: the usage of operator-valued Riemann-Hilbert problems to study the Fredholm determinants $D_n(t,\lambda)$ and the definition of an integro-differential Painlevé II hierarchy. The definition of this new hierarchy though, does not use any more the Lenard recursion as in the scalar classical case and in the matrix-valued generalization treated in the previous chapter. It uses instead some recursion operators $\mathcal{L}_\pm^u$ that remind of the Airault's construction [4] of the Painlevé II hierarchy that we saw in equation (2.2.27).

Definition 7.0.3. *Given a function $\mathbb{R}^2 \ni (t,x) \mapsto f(t|x)$, we denote by D_t the ordinary t-derivative and by D_t^{-1} the t-antiderivative, so that $(D_t^{-1}D_t f)(t|x) = f(t|x)$. Now define, for given $u = u(t|x)$,*

$$\begin{aligned}(\mathcal{L}_+^u f)(t|x) &:= \mathrm{i}(D_t f)(t|x) - \mathrm{i}\Big\langle (D_t^{-1}\{u,f\})(t|x,\cdot), u\Big\rangle - 2\mathrm{i}\Big(D_t^{-1}\langle u,f\rangle\Big)u(t|x),\\ (\mathcal{L}_-^u f)(t|x) &:= \mathrm{i}(D_t f)(t|x) + \mathrm{i}\Big\langle (D_t^{-1}[u,f])(t|x,\cdot), u\Big\rangle,\end{aligned}$$

where the rank two integral operators $[\alpha,\beta] := \alpha\otimes\beta - \beta\otimes\alpha$ and $\{\alpha,\beta\} := \alpha\otimes\beta + \beta\otimes\alpha$ have kernels

$$[\alpha,\beta](t|x,y) = \alpha(t|x)\beta(t|y) - \beta(t|x)\alpha(t|y), \quad \{\alpha,\beta\}(t|x,y) = \alpha(t|x)\beta(t|y) + \beta(t|x)\alpha(t|y),$$

and $\langle\cdot,\cdot\rangle$ denotes the weighted bilinear form

$$\langle f,g\rangle := \int_{\mathbb{R}} f(t|x)g(t|x)w'(x)\,\mathrm{d}x, \qquad w'(x) = \frac{\mathrm{d}w}{\mathrm{d}x}(x).$$

The relevant integro-differential Painlevé II hierarchy is then defined as a sequence of integro-differential equations through the recursion operators $\mathcal{L}_\pm^u$ in the following way.

Definition 7.0.4. *For each $n \in \mathbb{N}$, the n-th member of the integro-differential Painlevé II hierarchy is defined, for a function $u = u(t|x)$, as*

$$-(t+x)u(t|x) = \Big((\mathcal{L}_+^u\mathcal{L}_-^u)^n u\Big)(t|x) \tag{7.0.4}$$

In particular, using the shorthand

$$u = u(t|x), \quad u' = (D_t u)(t|x), \quad u'' = (D_t^2 u)(t|x), \quad u''' = (D_t^3 u)(t|x), \quad \dots$$

the first three members read as

$$n = 1: \qquad (t+x)u = u'' - 2u\langle u, u\rangle, \tag{7.0.5}$$

$$n = 2: \quad -(t+x)u = u'''' - 4u''\langle u, u\rangle - 8u'\langle u', u\rangle - 6u\langle u, u''\rangle - 2u\langle u', u'\rangle + 6u\langle u, u\rangle^2, \tag{7.0.6}$$

$$\begin{aligned} n = 3: \qquad (t+x)u = {} & u'''''' - 6u''''\langle u, u\rangle - 8u\langle u'''', u\rangle - 24u'''\langle u', u\rangle - 19u'\langle u, u'''\rangle - 13u\langle u''', u'\rangle \\ & - 31u''\langle u'', u\rangle - 11u\langle u'', u''\rangle - 25u''\langle u', u'\rangle - 45u'\langle u'', u'\rangle + 15u''\langle u, u\rangle^2 \\ & + 55u\langle u, u\rangle\langle u'', u\rangle + 60u'\langle u', u\rangle\langle u, u\rangle + 25u\langle u', u'\rangle\langle u, u\rangle + 55u\langle u', u\rangle^2 - 20u\langle u, u\rangle^3. \end{aligned} \tag{7.0.7}$$

We observe that for the choice of the weight function $w'(x) = \delta_0(x)$ (the delta function at $x = 0$) the classical equations (2.2.23), (2.2.24) and (2.2.25) are recovered from the above ones, at least formally.

Remark 7.0.5. *Even though the operators $\mathcal{L}_\pm^u$ involves t-antiderivatives, the members of the hierarchy* (7.0.4) *are always local. Indeed all the terms involving D_t^{-1}, are shown to be local.*

Remark 7.0.6. *The choice of the weight function w enters in the definition of the recursion operators $\mathcal{L}_\pm^u$ and thus of the hierarchy* (7.0.4) *and its solution $u(t|x)$. But the dependence on w of $u(t|x)$ is not underlined in our notation.*

Even though the definition of this integro-differential Painlevé II hierarchy is new, equations (7.0.5) and (7.0.6) already appeared in different papers. With w being the Fermi factor, equation (7.0.5) appeared in [5], while both equations (7.0.5), (7.0.6) appeared in this recent work [75] where the author was studying the Fredholm determinants $D_n(t, 1)$ in relation to some Painlevé II trascendents but without the underlying Lax pairs. The main statement of this chapter, Theorem 1.2 of [24], is as follows.

Theorem 7.0.7. *For every $(t, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$, with the closed unit disk $\overline{\mathbb{D}_1(0)} := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$,*

$$D_n(t, \lambda) = \exp\left[-\int_t^\infty (s-t)\left(\int_{\mathbb{R}} u^2(s|x) w'(x)\, \mathrm{d}x\right) \mathrm{d}s\right], \tag{7.0.8}$$

where $u(t|x) \equiv u(t|x; n, \lambda)$ is the unique solution of the boundary value problem

$$-(t+x)u(t|x) = \Big((\mathcal{L}_+^u \mathcal{L}_-^u)^n u\Big)(t|x), \qquad u(t|x) \sim \lambda^{\frac{1}{2}} \mathrm{Ai}_{2n+1}(t+x), \quad t \to +\infty. \tag{7.0.9}$$

The mapping $t \mapsto u(t|x; n, \lambda)$ is smooth for any $(x, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$, the asymptotic expansion in (7.0.9) *holds pointwise in $x \in \mathbb{R}$ and we choose an arbitrary fixed branch for $\lambda^{\frac{1}{2}}$.*

Remark 7.0.8. *Our Theorem 7.0.7 recovers for* $n = 1, \lambda = 1$ *Proposition* 1.2 *of [5]. Although the method used in that paper is completely different from the method we are going to use here. The same result for that particular choice of the parameters was proved again in [22] using operator-valued Riemann-Hilbert technique, and this is indeed the paper that mostly inspired our methodology here. However, we notice that the Riemann-Hilbert problem used in [22], for the case* $n = 1, \lambda = 1$*, is different from the one used here.*

The rest of the Chapter is devoted to the proof of Theorem 7.0.7. This requires essentially four steps, each one treated in the following sections.

- In Section 7.1 we prove the main properties of the finite temperature higher order Airy kernels on $L^2(\mathbb{R}_+)$. After that, by using a Fourier technique we prove that the Fredholm determinants $D_n(t, \lambda)$ are equals to the ones of some new integral operators acting on a bigger space $L^2(\Sigma)$, with Σ the contour introduced in (7.1.27). In particular these new operators can be considered as an infinite dimensional versions of the standard *integrable* operators. This kind of operators can be studied through operator-valued Riemann-Hilbert problems, and this is done in Section 7.2.

- In Section 7.3 we deduce an operator-valued system of differential equations, w.r.t. the complex parameter ζ and the deformation parameter t, starting from the solution $\mathbf{X}(\zeta)$ of the Riemann-Hilbert problem 7.2.6. The main ingredient in this computation is the relation proved in Corollary 7.2.14. Moreover, we prove that this system is an operator-valued Lax pair for a coupled system of Painlevé II type equations, involving the operators U, V defined in (7.2.40) in Section 7.2.

- Finally in Section 7.4 we prove that the Lax pair deduced in the previous section, yields the integro-differential Painlevé II hierarchy (7.0.4). This is obtained from the reduction of the coupled systems of differential equations for the operators U, V, by looking at their kernels.

Remark 7.0.9. *In the work [24] we also derived an expression for the Fredholm determinants* $D_n(t, \lambda)$ *similar to equation* (7.0.8) *but involving instead of* $u(t|x)$ *another function* $v(t_1, t_{2n+1}|x)$ *that turns out to be a distinguished solution of an integro-differential modified KdV hierarchy. The result is obtained exactly in the same way as Theorem 7.0.7, but in the case where the weight function* w *actually depends on a positive real parameter* α*. Defining the new variables depending on* α, n, t *as*

$$t_1 := \alpha t \in \mathbb{R}, \quad t_{2n+1} := \frac{\alpha^{2n+1}}{2n+1} \in \mathbb{R}_+, \tag{7.0.10}$$

this new integro-differential modified KdV hierarchy is then defined as

$$\frac{\partial v}{\partial t_{2n+1}}(t_1, t_{2n+1}|x) = \left((\mathcal{L}_-^v \mathcal{L}_+^v)^n \frac{\partial v}{\partial t_1} \right)(t_1, t_{2n+1}|x), \qquad (t_1, t_{2n+1}, x) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}. \tag{7.0.11}$$

The first equation of the hierarchy is written as

$$\frac{\partial v}{\partial t_3} = -\frac{\partial^3 v}{\partial t_1^3} + 3\frac{\partial v}{\partial t_1}\langle v, v\rangle + 3v\left\langle \frac{\partial v}{\partial t_1}, v\right\rangle, \tag{7.0.12}$$

where $\langle \cdot, \cdot \rangle$ *denotes the weighted bilinear form as defined previously in Definition 7.0.3. For the exact statement and its proof we refer to Section* 7 *of [24].*

7.1 Manipulating the finite temperature Airy kernels

First of all, we verify that the Fredholm determinant of the higher order finite temperature Airy kernels defined in (7.0.8) are well defined on $L^2(\mathbb{R}_+)$. This is obtained through a classical argument : we prove that the operator $\mathcal{K}_{t,}$ is obtained as a composition of Hilbert-Schmidt operators on $L^2(\mathbb{R}_+)$ for every $(t, n) \in \mathbb{R} \times \mathbb{N}$.

Lemma 7.1.1. *The operator* $\mathcal{K}_{t,n}$ *is trace-class on* $L^2(\mathbb{R}_+)$ *for every* $(t, n) \in \mathbb{R} \times \mathbb{N}$.

Proof. Recall the definition of the kernel of the operator $\mathcal{K}_{t,n}$ given in (7.0.3). We can directly see that the composition of the two operators $\mathcal{M}_{t,n} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ and $\mathcal{N}_{t,n} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ acting as

$$\left(\mathcal{M}_{t,n} f\right)(x) = \int_{\mathbb{R}_+} \sqrt{w(x)}\mathrm{Ai}_{2n+1}(x+y+t) f(y) dy \quad \text{and} \quad \left(\mathcal{N}_{t,n} g\right)(x) = \int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+y+t) g(y) \sqrt{w(y)} dy \tag{7.1.1}$$

gives exactly the operator $\mathcal{K}_{t,n} = \mathcal{N}_{t,n}\mathcal{M}_{t,n}$. It remains then to prove that the operators $\mathcal{N}_{t,n}, \mathcal{M}_{t,n}$ are both Hilbert-Schmidt. In both cases, we need the following condition to hold

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} |\mathrm{Ai}_{2n+1}(x+y+t)|^2 w(x) \mathrm{d}y \mathrm{d}x < \infty. \tag{7.1.2}$$

The estimate above is essentially obtained by splitting the external integral along $\mathbb{R}$ and by using the asymptotic properties of the n-th Airy function (see for instance equation (30) in [24]). Also, recall the properties of the weight function $w(x)$ given in Definition 7.0.1. In particular, we use here the fact that $w(x) \leq 1$ for every $x \in \mathbb{R}$ and the exponential decay

$w(x) \leq \hat{c}e^{\omega x}$ for all $(-x) \geq x_0 > 0$, with $\hat{c} > 0$. We have

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \left|\mathrm{Ai}_{2n+1}(x+y+t)\right|^2 \mathrm{d}y \right) w(x)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \left|\mathrm{Ai}_{2n+1}(x+y+t)\right|^2 \mathrm{d}y \right) w(x)\,\mathrm{d}x + \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \left|\mathrm{Ai}_{2n+1}(-x+y+t)\right|^2 \mathrm{d}y \right) w(-x)\,\mathrm{d}x \\
&\quad \leq c \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathrm{e}^{-\frac{4n}{2n+1}(x+y+t)}\,\mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{R}_+} \left(c + \int_0^x \left|\mathrm{Ai}_{2n+1}(-x+y+t)\right|^2 \mathrm{d}y \right) w(-x)\,\mathrm{d}x \\
&\quad \leq c \left[1 + \int_{\mathbb{R}_+} \left(1 + x^{\frac{1}{2n}} \right) w(-x)\,\mathrm{d}x \right].
\end{aligned}
\tag{7.1.3}
$$

Remark that the constant c appearing in the passages above changes from line to line and also it depends on the parameters t, n. ■

7.1.1 Some properties of $\mathcal{K}_{t,n}$

We are now going to prove a couple of properties of the operator $\mathcal{K}_{t,n}$ that will be useful in the following. Notice that these properties were proved also in the previous chapter for the matrix-valued analogue of higher order Airy kernels. In particular we have that

1. the operator $\mathcal{K}_{t,n}$ is self-adjoint and such that $0 < \mathcal{K}_{t,n} \leq 1$;
2. $1 - \lambda\mathcal{K}_{t,n}$ is invertible on $L^2(\mathbb{R}_+)$ for every $\lambda \in \overline{\mathbb{D}_1(0)}$.

The first property yields a probabilistic interpretation for the Fredholm determinants $D_n(t, \lambda)$. Indeed, it directly implies (always by applying Theorem 3.1.5 and then Corollary 3.1.9) that for every $\lambda \in [0, 1]$ the operators $\lambda\mathcal{K}_{t,n}$ uniquely defines a determinantal point process and the Fredholm determinant $D_n(t, \lambda)$ describes the probability distribution of the last particle in this determinantal point process. The second property is instead fundamental from a technical point view. Indeed, it assures the solvability of the Riemann-Hilbert problem 7.2.6, as we discuss in Section 7.2.

Lemma 7.1.2. *For every $(t, n) \in \mathbb{R} \times \mathbb{N}$ the operator $\mathcal{K}_{t,n}$ is self-adjoint and it satisfies $0 < \mathcal{K}_{t,n} \leq 1$. Moreover, $1 - \lambda\mathcal{K}_{t,n}$ is invertible on $L^2(\mathbb{R}_+)$ for all $\lambda \in \overline{\mathbb{D}_1(0)}$.*

Proof. The self-adjointness directly follows from the definition of the kernel of $\mathcal{K}_{t,n}$ in (7.0.3). We have then to prove the chain of inequality satisfied by $\mathcal{K}_{t,n}$. To do that, we start by rewriting the kernel of $\mathcal{K}_{t,n}$, using the following trick.
From the properties of the weight function $w(z)$ in Definition 7.0.1 and the asymptotic properties of the n-th Airy function again, we get

$$\frac{dK_{t,n}}{dt}(x, y) = -\int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+z+t)\mathrm{Ai}_{2n+1}(y+z+t)\mathrm{d}\sigma(z), \tag{7.1.4}$$

where we just integrated by parts and used the properties recalled above. Here $\mathrm{d}\sigma(z) = w'(z)\mathrm{d}z$ and it is a probability measure. With this in mind, by applying first dominated convergence theorem and then Fubini's theorem we can finally express $K_{t,n}(x,y)$ in this new fashion

$$K_{t,n}(x,y) = -\int_t^\infty \frac{dK_{s,n}}{ds}(x,y)\mathrm{d}s = \int_{\mathbb{R}}\int_{\mathbb{R}_+} \mathrm{Ai}_{2n+1}(y+z+t+s)\mathrm{Ai}_{2n+1}(x+z+t+s)\mathrm{d}s\mathrm{d}\sigma(z). \tag{7.1.5}$$

Using this formulation we see that for every $f \in L^2(\mathbb{R}_+)$, by denoting $f_+(x) = f(x)\chi_{\mathbb{R}_+}(x)$ then

$$\langle f, \mathcal{K}_{t,n} f\rangle_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}} \left[\int_{z+t}^\infty \left|\int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+s) f_+(x)\,\mathrm{d}x\right|^2 \mathrm{d}s\right] \mathrm{d}\sigma(z) \geq 0, \tag{7.1.6}$$

thus the first inequality for $\mathcal{K}_{t,n}$ holds. For the other one, we start by replacing in the computation above the integral representation of the n-th Airy function with $\mathbb{R}$ as domain of integration. Then by denoting with $\check{f}_+(y) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy} f_+(x)\mathrm{d}x$ and by $g(y) = e^{i\frac{y^{2n+1}}{2n+1}}\check{f}_+(-y)$ we get

$$\int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+s) f_+(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(i\left(\frac{y^{2n+1}}{2n+1} + sy\right)\right)\check{f}(-y)\mathrm{d}y = \check{g}(-s). \tag{7.1.7}$$

Thus we can replace this computation inside the integral in (7.1.6) and then

$$\begin{aligned}\langle f, \mathcal{K}_{t,n} f\rangle_{L^2(\mathbb{R}_+)} &= \int_{\mathbb{R}}\left[\int_{z+t}^\infty |\check{g}(-s)|^2\,\mathrm{d}s\right]\mathrm{d}\sigma(z) \leq \int_{\mathbb{R}}\left[\int_{\mathbb{R}} |\check{g}(-s)|^2\,\mathrm{d}s\right]\mathrm{d}\sigma(z) = ||\check{g}||^2_{L^2(\mathbb{R})} = ||g||^2_{L^2(\mathbb{R})} \\ &= ||\check{f}_+||^2_{L^2(\mathbb{R})} = ||f_+||^2_{L^2(\mathbb{R})} = \langle f, f\rangle_{L^2(\mathbb{R})},\end{aligned} \tag{7.1.8}$$

where we used multiple times the Plancharel's theorem and that $\mathrm{d}\sigma$ is a probability measure. Therefore, also the second inequality for $\mathcal{K}_{t,n}$ holds. Furthermore, this implies that in the $L^2(\mathbb{R}_+)$ operator norm we also have that $||\mathcal{K}_{t,n}|| \leq 1$. This last condition also assures the invertibility on $L^2(\mathbb{R}_+)$ of $1 - \lambda\mathcal{K}_{t,n}$ for any λ having $|\lambda| < 1$.
For the case in which $|\lambda| = 1$, we proceed by contradiction. Suppose that there exists a nonzero function $f \in L^2(\mathbb{R}_+)$ such that $\lambda\mathcal{K}_{t,n} f = f$ for $\lambda = e^{i\theta}$ and some $\theta \in [0, 2\pi)$. In turn we have

$$e^{-i\theta}\langle f, \mathcal{K}_{t,n} f\rangle_{L^2(\mathbb{R}_+)} = \langle f, e^{i\theta}\mathcal{K}_{t,n} f\rangle_{L^2(\mathbb{R}_+)} = ||f||_{L^2(\mathbb{R}_+)} > 0 \tag{7.1.9}$$

thus θ is forced to be zero. Furthermore, the equations above imply that all the sequence in (7.1.8) is actually composed by identities. In particular

$$\int_{\mathbb{R}}\left[\int_{z+t}^\infty |\check{g}(-s)|^2\,\mathrm{d}s\right]\mathrm{d}\sigma(z) = ||\check{g}||_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\check{g}(-s)|^2\mathrm{d}s, \tag{7.1.10}$$

that yields

$$\forall t \in \mathbb{R}: \qquad \int_{\mathbb{R}} \left[\int_{-\infty}^{z+t} |\check{g}_n(-s)|^2 \, \mathrm{d}s \right] \mathrm{d}\sigma(z) = 0. \tag{7.1.11}$$

Since $\mathrm{d}\sigma$ is an absolutely continuous positive Borel measure, this implies automatically that

$$\int_{-\infty}^{y} |\check{g}_n(-s)|^2 \, \mathrm{d}s = 0 \ \text{ a.e..} \tag{7.1.12}$$

Thus $\check{g}(-y) = 0$ a.e. and by recalling the definition of the function $\check{g}(y)$ we obtain

$$\check{g}(-y) = \int_{\mathbb{R}} \mathrm{Ai}_{2n+1}(x+y) f(x) \mathrm{d}x = 0 \text{ a.e..} \tag{7.1.13}$$

Since the integral in the left hand side of the above equation is a continuous function in y and as a byprouduct an entire function. Hence we conclude that $\check{g}(z) = 0$ for every $z \in \mathbb{C}$ and this implies that $f \equiv 0$ in $L^2(\mathbb{R})$. This contradicts the initial assumption, and thus we have that $1 - \lambda\mathcal{K}_{t,n}$ is injective for λ with unitary norm. By Fredholm alternative then $1 - \lambda\mathcal{K}_{t,n}$ is invertible in the same range of the parameter λ. ∎

Corollary 7.1.3. *For every $(t, \lambda, n) \in \mathbb{R} \times [0, 1] \times \mathbb{N}$ there exists a unique determinantal point process with correlation kernel $\lambda K_{t,n}$ and the distribution function of the last particle in this process equals $D_n(t, \lambda)$.*

As underlined before, this follows directly from Lemma 7.1.2, together with the classical results recalled in Chapter 3, namely Theorem 3.1.5 and Corollary 3.1.9.

7.1.2 From $L^2(\mathbb{R}_+)$ to $L^2(\Sigma)$

This last subsection is perhaps the core of the entire section, since we are going to explain how to associate an operator-valued Riemann-Hilbert problem to the higher order finite temperature Airy kernels $\mathcal{K}_{t,n}$. The main idea is to manipulate the kernel of $\mathcal{K}_{t,n}$ through the conjugation of bounded invertible operators, in order to obtain a new integral operator on an enlarged space $L^2(\Sigma)$ that has the same Fredholm determinant $D_n(t, \lambda)$ of $\mathcal{K}_{t,n}$.
Here are resumed the fundamental steps of this procedure

1. First, we consider the operator $\lambda\mathcal{K}_{t,n}\chi_{\mathbb{R}_+}$ on $L^2(\mathbb{R})$, which has Fredholm determinant $D_n(t, \lambda)$. Moreover, this operator is shown to be equal, up to conjugation by the Fourier transform and a multiplication operator, to another trace-class operator called $\lambda\mathcal{J}_{t,n}$ on $L^2(\mathbb{R})$. Thus we also have that the Fredholm determinant of this new operator $\mathcal{J}_{t,n}$ is expressed by $D_n(t, \lambda)$.

2. The operator $\mathcal{J}_{t,n}$ is explicitly factorized in two Hilbert-Schmidt operators $\mathcal{A}_{t,n}, \mathcal{B}_n$ on $L^2(\mathbb{R})$.

3. We can then consider $\mathcal{J}_{t,n}$ as an operator $\mathcal{J}^{\circ}_{t,n}$ on $L^2(\Gamma_\alpha)$ for Γ_α some line in the complex plane parallel to the real line and sufficiently closed to it. The factorization of $\mathcal{J}_{t,n}$ is in

some way preserved for $\mathcal{J}^\circ_{t,n}$ on $L^2(\Gamma_\alpha)$, through operators $\mathcal{A}^\circ_{t,n}, \mathcal{B}^\circ_n$ properly redefining domain and codomain of the operators $\mathcal{A}_{t,n}, \mathcal{B}_n$. Again, $\lambda\mathcal{J}^\circ_{t,n}$ is trace-class and its Fredholm determinant coincides with $D_n(t,\lambda)$.

4. Finally all these operators $\mathcal{J}^\circ_{t,n}, \mathcal{A}^\circ_{t,n}, \mathcal{B}^\circ_n$ can be extended on a bigger space $L^2(\Sigma)$ for Σ a prescribed contour on the complex plane containing the line Γ_α, as $\mathcal{J}^{\text{ext}}_{t,n}, \mathcal{A}^{\text{ext}}_{t,n}, \mathcal{B}^{\text{ext}}_n$. On $L^2(\Sigma)$ the operator $\mathcal{J}^{\text{ext}}_{t,n}$ is still trace-class and factorized through the Hilber-Schmidt operators $\mathcal{A}^{\text{ext}}_{t,n}, \mathcal{B}^{\text{ext}}_n$. But now these last two operator are trace-class too on $L^2(\Sigma)$ with zero operator trace and they are also nilpotent.

5. With these properties of $\mathcal{A}^{\text{ext}}_{t,n}, \mathcal{B}^{\text{ext}}_n$, we can directly conclude the aimed relation

$$D_n(t,\lambda) = \det(1 - \lambda^{\frac{1}{2}} C_{t,n}), \tag{7.1.14}$$

for $C_{t,n} = \mathcal{A}^{\text{ext}}_{t,n} + \mathcal{B}^{\text{ext}}_n$. The operator $C_{t,n}$ obtained in this way has kernel explicitly written in equation (7.1.32), in an *infinite dimensional* integrable form.

The starting point of all these manipulations is, again, the integral representation of the n-th Airy function, that we already used during some proofs in the previous chapter. In this case we are going to use both its integral representations

$$\mathrm{Ai}_{2n+1}(x) = \frac{1}{2\pi}\int_{\Gamma_\alpha} e^{i\psi_n(x)} = \frac{1}{2\pi}\int_{\Gamma_\beta} e^{-i\psi_n(x)}, \quad \text{with} \quad \psi_n(x) = \frac{\lambda^{2n+1}}{2n+1} + \lambda x \tag{7.1.15}$$

where Γ_α, resp. Γ_β, denotes any smooth contour oriented from ∞e^{ia} to ∞e^{ib}, resp. ∞e^{ic} to ∞e^{id}, with

$$a \in \left(\frac{2n\pi}{2n+1}, \pi\right] \text{ and } b \in \left[0, \frac{\pi}{2n+1}\right), \quad \text{resp.} \quad c \in \left(\pi, \frac{(2n+2)\pi}{2n+1}\right) \text{ and } d \in \left(\frac{(4n+1)\pi}{2n+1}, 2\pi\right),$$

such that $0 < \Im(\alpha - \beta) < \frac{\omega}{2}$ and $\Im\beta < 0$ is satisfied for $\alpha \in \Gamma_\alpha$ and $\beta \in \Gamma_\beta$ with $\omega > 0$ as in (7.0.1), see Figure 7.1 below for a possible choice. These constraints for the contours implies in turn from (7.0.1) that

$$\forall\, (\alpha,\beta) \in \Gamma_\alpha \times \Gamma_\beta : \qquad \lim_{\substack{z\to+\infty \\ z\in\mathbb{R}}} e^{iz(\alpha-\beta)} w(z) = 0, \qquad \lim_{\substack{z\to-\infty \\ z\in\mathbb{R}}} e^{iz(\alpha-\beta)} w(z) = 0.$$

We now replace the integral representation of the n-th Airy function inside the definition (7.0.3) of the kernel of $\mathcal{K}_{t,n}$.

$$\begin{aligned} K_{t,n}(x,y) = &\ \frac{1}{(2\pi)^2}\int_{\Gamma_\alpha}\int_{\Gamma_\beta} e^{i(\psi_n(\alpha,x+t)-\psi_n(\beta,y+t))}\left[\int_{\mathbb{R}} e^{iz(\alpha-\beta)} w(z)\,\mathrm{d}z\right]\mathrm{d}\beta\mathrm{d}\alpha \\ = &\ \frac{i}{(2\pi)^2}\int_{\mathbb{R}}\left[\int_{\Gamma_\alpha}\int_{\Gamma_\beta} e^{i\psi_n(\alpha,x+z+t)} e^{-i\psi_n(\beta,z+y+t)} \frac{\mathrm{d}\beta\,\mathrm{d}\alpha}{\alpha-\beta}\right]\mathrm{d}\sigma(z) \end{aligned}$$

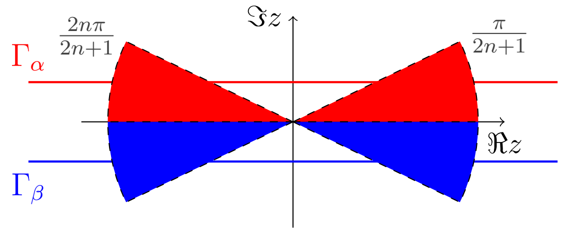

Figure 7.1: An admissible (and very simple) choice for the integration paths Γ_α and Γ_β in (7.1.15), ensuring throughout $0 < \Im(\alpha - \beta) < \frac{\omega}{2}$ and $\Im\beta < 0$ for $(\alpha, \beta) \in \Gamma_\alpha \times \Gamma_\beta$.

where in the last passage we integrated by parts and we used the asymptotic behaviors of the n-th Airy function. Now, as previously explained we are going to consider the operator $\mathcal{K}_{t,n}\chi_{\mathbb{R}_+}$ on $L^2(\mathbb{R})$, keeping in mind that

$$D_n(t,\lambda) = \det(1 - \lambda\mathcal{K}_{t,n}\chi_{\mathbb{R}_+}|_{L^2(\mathbb{R})}). \tag{7.1.16}$$

Using the following integral identity (see also Lemma 2.2 in [7] for a similar one)

$$-\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}y(\gamma-\beta)}\frac{\mathrm{d}\gamma}{\gamma-\beta} = \chi_{\mathbb{R}_+}(y), \quad \text{for} \quad \beta \in \Gamma_\beta, \quad y \in \mathbb{R}\setminus\{0\} \tag{7.1.17}$$

inside the computation for $K_{t,n}(x,y)$, choosing $\Gamma_\alpha = \mathbb{R}$, we obtain

$$K_{t,n}(x,y)\chi_{\mathbb{R}_+}(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}x\alpha}}{\sqrt{2\pi}} \underbrace{\left[\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\Gamma_\beta} \frac{\mathrm{e}^{\mathrm{i}\psi_n(\alpha,z+t)}\mathrm{e}^{-\mathrm{i}\psi_n(\beta,z+t)}}{(\alpha-\beta)(\beta-\gamma)}\,\mathrm{d}\beta\,\mathrm{d}\sigma(z)\right]}_{=:L_{t,n}(\alpha,\gamma)} \frac{\mathrm{e}^{-\mathrm{i}y\gamma}}{\sqrt{2\pi}}\,\mathrm{d}\alpha\,\mathrm{d}\gamma, \tag{7.1.18}$$

just by using Fubini's theorem. Thus we conclude that $\mathcal{F}\mathcal{K}_{t,n}\chi_{\mathbb{R}_+}\mathcal{F}^{-1} = \mathcal{L}_{t,n}$ where $\mathcal{L}_{t,n}$ is the integral operator on $L^2(\mathbb{R})$ with kernel $L_{t,n}$ denoted above and $\mathcal{F}$ is the standard Fourier transform extended unitarily to $L^2(\mathbb{R})$.

Remark 7.1.4. *The operator $\mathcal{L}_{t,n}$ is trace-class on $L^2(\mathbb{R})$ through general trace ideal properties.*

Definition 7.1.5. *We consider the multiplication operator $\mathcal{P}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that acts by multiplying by the function $e^{-\frac{i}{2}\psi_n(\alpha,0)}$.*

This multiplication operator is used in order to construct another integral operator from $\mathcal{L}_{t,n}$ as follows.

Definition 7.1.6. *We consider the integral operator $\mathcal{J}_{t,n} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by conjugation for $\mathcal{P}_n$ of $\mathcal{L}_{t,n}$, namely*

$$\mathcal{J}_{t,n} := \mathcal{P}_n\mathcal{L}_{t,n}\mathcal{P}_n^{-1} \tag{7.1.19}$$

By abstract trace ideal reasoning, since the operator $\mathcal{P}_n$ is bounded in $L^2(\mathbb{R})$ we can conclude that the operator $\mathcal{J}_{t,n}$ is trace-class on $L^2(\mathbb{R})$. We can then state and prove the following proposition about the Fredholm determinant $D_n(t,\lambda)$.

Proposition 7.1.7. *For every* $(t,\lambda,n) \in \mathbb{R}\times\mathbb{C}\times\mathbb{N}$, *on* $L^2(\mathbb{R})$,

$$1 - \lambda\mathcal{K}_{t,n}\chi_{\mathbb{R}_+} = \mathcal{F}^{-1}\mathcal{P}_n^{-1}(1-\lambda\mathcal{J}_{t,n})\mathcal{P}_n\mathcal{F},$$

with the bounded linear operators $\mathcal{F}, \mathcal{P}_n$ *and* $\mathcal{J}_{t,n}$ *on* $L^2(\mathbb{R})$ *defined as above. In particular we record the determinant identity*

$$D_n(t,\lambda) = \det(1-\lambda\mathcal{J}_{t,n}|_{L^2(\mathbb{R})}). \tag{7.1.20}$$

Proof. The operator identity as been proved with the reasoning above. The determinant identity (7.1.20) is obtained by using the operator identity and by applying the Sylvester's identity (see for instance equation (5.9) of Chapter IV in [48]). ■

Before to go ahead, we need some other property of the integral operator $\mathcal{J}_{t,n}$. In particular, we see that this operator is explicitly factored in two Hilbert-Schmidt operators.

Proposition 7.1.8. *The integral operator* $\mathcal{J}_{t,n}$ *is factored as* $\mathcal{J}_{t,n} = \mathcal{A}_{t,n}\mathcal{B}_n$ *where* $\mathcal{A}_{t,n} : L^2(\Gamma_\beta) \to L^2(\mathbb{R})$, *and* $\mathcal{B}_n : L^2(\mathbb{R}) \to L^2(\Gamma_\beta)$ *have kernels*

$$A_{t,n}(\alpha,\beta) := \frac{1}{2\pi}\frac{\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\alpha,2t)-\frac{\mathrm{i}}{2}\psi_n(\beta,2t)}}{\alpha-\beta}\left[\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}z(\alpha-\beta)}\,\mathrm{d}\sigma(z)\right], \qquad B_n(\beta,\gamma) := \frac{1}{2\pi}\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\beta,0)+\frac{\mathrm{i}}{2}\psi_n(\gamma,0)}}{\beta-\gamma}. \tag{7.1.21}$$

Proof. Recall that $\mathcal{J}_{t,n} = \mathcal{P}_n\mathcal{L}_{t,n}\mathcal{P}_n^{-1}$ and that $\mathcal{L}_{t,n}$ has kernel

$$L_{t,n}(\alpha,\gamma) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\Gamma_\beta}\frac{\mathrm{e}^{\mathrm{i}\psi_n(\alpha,z+t)}\mathrm{e}^{-\mathrm{i}\psi_n(\beta,z+t)}}{(\alpha-\beta)(\beta-\gamma)}\,\mathrm{d}\beta\,\mathrm{d}\sigma(z). \tag{7.1.22}$$

Thus we can write down the kernel of $\mathcal{J}_{t,n}$ as

$$J_{t,n}(\alpha,\gamma) = \frac{1}{(2\pi)^2}\int_{\Gamma_\beta}\frac{\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\alpha,2t)-\frac{\mathrm{i}}{2}\psi_n(\beta,2t)}}{\alpha-\beta}\left[\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}z(\alpha-\beta)}\,\mathrm{d}\sigma(z)\right]\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\beta,0)+\frac{\mathrm{i}}{2}\psi_n(\gamma,0)}}{\beta-\gamma}\,\mathrm{d}\beta, \tag{7.1.23}$$

so that $\mathcal{J}_{t,n} = \mathcal{A}_{t,n}\mathcal{B}_n$ with $\mathcal{A}_{t,n}, \mathcal{B}_n$ having kernels as in (7.1.21). ■

We are now ready to construct the extension of the operator $\mathcal{J}_{t,n}$ on some bigger space $L^2(\Sigma)$. To start with, we first look at the operator $\mathcal{J}_{t,n}^{\circ}$ as an operator on $L^2(\Gamma_\alpha)$, for Γ_α some line in the upper complex plane parallel to the real line. This leaves untouched the Fredholm determinant.

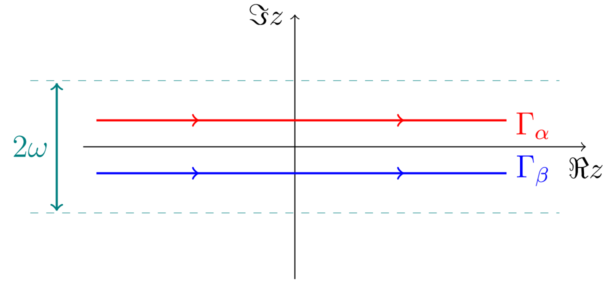

Figure 7.2: Our choice for $\Gamma_{\alpha,\beta}$ with $0 < \Delta = \operatorname{dist}(\Gamma_\alpha, \mathbb{R}) = \operatorname{dist}(\Gamma_\beta, \mathbb{R}) < \frac{\omega}{2}$.

Proposition 7.1.9. *Let Γ_α denote the reflection of Γ_β across the real axis fixing $\Gamma_\beta := \mathbb{R} - \mathrm{i}\Delta$ with $0 < \Delta < \frac{\omega}{2}$. Now define $\mathcal{J}^\circ_{t,n} : L^2(\Gamma_\alpha) \to L^2(\Gamma_\alpha)$ as*

$$(\mathcal{J}^\circ_{t,n} f)(\xi) := \int_{\Gamma_\alpha} \mathcal{J}_{t,n}(\xi, \eta) f(\eta)\, \mathrm{d}\eta, \qquad f \in L^2(\Gamma_\alpha),$$

with kernel $\mathcal{J}_{t,n}(\xi, \eta)$ given in (7.1.23)*. Then $\mathcal{J}^\circ_{t,n}$ is trace class on $L^2(\Gamma_\alpha)$ and we have the equality*

$$D_n(t, \lambda) = \det(I - \lambda \mathcal{J}^\circ_{t,n}|_{L^2(\Gamma_\alpha)}), \qquad (t, \lambda, n) \in \mathbb{R} \times \mathbb{C} \times \mathbb{N}. \tag{7.1.24}$$

Proof. First notice that the operator $\mathcal{J}^\circ_{t,n}$ is well defined on $L^2(\Gamma_\alpha)$ since $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$. Moreover, we can re-define operators $\mathcal{A}^\circ_{t,n} : L^2(\Gamma_\beta) \to L^2(\Gamma_\alpha)$ and $\mathcal{B}^\circ_n : L^2(\Gamma_\alpha) \to L^2(\Gamma_\beta)$ having the same kernels (7.1.21) and they are still Hilbert-Schmidt operators. We also have $\mathcal{J}^\circ_{t,n} = \mathcal{A}^\circ_{t,n} \mathcal{B}^\circ_n$ so that $\mathcal{J}^\circ_{t,n}$ is still trace-class on $L^2(\Gamma_\alpha)$.
Finally in order to obtain the identity for the Fredholm determinant $D_n(t, \lambda)$, we observe that for every $m \in \mathbb{N}$

$$\operatorname*{Tr}_{L^2(\mathbb{R})} \mathcal{J}^m_{t,n} = \operatorname*{Tr}_{L^2(\Gamma_\alpha)} \mathcal{J}^m_{t,n}. \tag{7.1.25}$$

Since $J_{t,n}(\alpha, \gamma)$ is analytic in a neighborhood of $(\alpha, \gamma) \in \Gamma_\alpha \times \Gamma_\alpha$, and

$$\operatorname*{Tr}_{L^2(\mathbb{R})} \mathcal{J}^m_{t,n} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} J_{t,n}(\zeta_1, \zeta_2) \cdot \ldots \cdot J_{t,n}(\zeta_{m-1}, \zeta_m) J_t(\zeta_m, \zeta_1)\, \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_m \tag{7.1.26}$$

we can recursively replace Γ_α instead of $\mathbb{R}$ in each one of the integrals above and conclude (7.1.25). By using the Plemelj-Smithies formula (see for instance Theorem 3.1 in Chapter II of [48]) the identity (7.1.24) then holds. ∎

We now fix the contour Σ in the complex plane as the disjoint union of the horizontal lines, namely

$$\Sigma := \mathbb{R} \sqcup \Gamma_\beta \sqcup \Gamma_\alpha \tag{7.1.27}$$

where $\Gamma_\beta := \mathbb{R} - \mathrm{i}\Delta$, with $0 < \Delta < \frac{\omega}{2}$, and Γ_α is the reflection of Γ_β upon the real line, as in Figure 7.2. Since Σ contains in particular the line Γ_α, we can extend the operator $\mathcal{J}^\circ_{t,n}$ to the bigger space $L^2(\Sigma)$. We define

$$\mathcal{J}_{t,n}^{\text{ext}} : L^2(\Sigma) \to L^2(\Sigma), \quad (\mathcal{J}_{t,n}^{\text{ext}} f)(\xi) = \int_\Sigma J_{t,n}^{\text{ext}}(\xi,\eta) f(\eta)\,\mathrm{d}\eta, \quad J_{t,n}^{\text{ext}}(\xi,\eta) := J_{t,n}(\xi,\eta)\chi_{\Gamma_\alpha}(\xi)\chi_{\Gamma_\alpha}(\eta), \tag{7.1.28}$$

Remark that, again, the extension leaves $D_n(t,\lambda)$ invariant, so that we have

$$D_n(t,\lambda) = \det(1 - \lambda \mathcal{J}_{t,n}^{\text{ext}}|_{L^2(\Sigma)}). \tag{7.1.29}$$

Also remark that $\mathcal{J}_{t,n}^{\text{ext}}$ can be factored in a similar way as before $\mathcal{J}_{t,n}^{\text{ext}} = \mathcal{A}_{t,n}^{\text{ext}}\mathcal{B}_n^{\text{ext}}$, where now

$$\mathcal{A}_{t,n}^{\text{ext}} : L^2(\Sigma) \to L^2(\Sigma), \quad (\mathcal{A}_{t,n}^{\text{ext}} f)(\xi) = \int_\Sigma A_{t,n}^{\text{ext}}(\xi,\eta) f(\eta)\,\mathrm{d}\eta, \quad A_{t,n}^{\text{ext}}(\xi,\eta) := A_{t,n}(\xi,\eta)\chi_{\Gamma_\alpha}(\xi)\chi_{\Gamma_\beta}(\eta),$$

$$\mathcal{B}_n^{\text{ext}} : L^2(\Sigma) \to L^2(\Sigma), \quad (\mathcal{B}_n^{\text{ext}} g)(\eta) = \int_\Sigma B_n^{\text{ext}}(\eta,\zeta) g(\zeta)\,\mathrm{d}\zeta, \quad B_n^{\text{ext}}(\eta,\zeta) := B_n(\eta,\zeta)\chi_{\Gamma_\beta}(\eta)\chi_{\Gamma_\alpha}(\zeta).$$

Moreover, thanks to their construction the operators $\mathcal{A}_{t,n}^{\text{ext}}, \mathcal{B}_n^{\text{ext}}$ on $L^2(\Sigma)$ gain many properties, listed below, with respect to their previous versions.

Lemma 7.1.10. *The operators $\mathcal{A}_{t,n}^{\text{ext}}, B_n^{\text{ext}} : L^2(\Sigma) \to L^2(\Sigma)$ have the following properties*

1. *they are trace class on $L^2(\Sigma)$ for every $(t,n) \in \mathbb{R} \times \mathbb{N}$;*
2. *they have zero operator trace;*
3. *they are nilpotent.*

Proof. We prove the first property first. To see that $\mathcal{A}_{t,n}^{\text{ext}}, \mathcal{B}_n^{\text{ext}}$ are both trace-class on $L^2(\Sigma)$ we find for both a factorization in terms of Hilbert-Schmidt operators. For what concerns $\mathcal{B}_n$, we use the following trick. By residue theorem, for every $(\gamma,\beta) \in \Gamma_\alpha \times \Gamma_\beta$, we have

$$-\frac{1}{2\pi\mathrm{i}} \int_\mathbb{R} \frac{\mathrm{d}\delta}{(\gamma-\delta)(\delta-\beta)} = \frac{1}{\gamma-\beta}.$$

Replacing it in the kernel of $\mathcal{B}_n$ using (7.1.21) we get

$$B_n^{\text{ext}}(\beta,\gamma) = -\frac{\mathrm{i}}{(2\pi)^2} \int_\mathbb{R} \frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\beta,0)+\frac{\mathrm{i}}{2}\psi_n(\gamma,0)}}{(\gamma-\delta)(\delta-\beta)}\,\mathrm{d}\delta\, \chi_{\Gamma_\beta}(\beta)\chi_{\Gamma_\alpha}(\gamma). \tag{7.1.30}$$

and this can seen as the composition $\mathcal{B}_n^{\text{ext}} = \mathcal{B}_{n,1}^{\text{ext}}\mathcal{B}_{n,2}^{\text{ext}}$ where $\mathcal{B}_{n,j}^{\text{ext}} : L^2(\Sigma) \to L^2(\Sigma)$ have Hilbert-Schmidt kernels

$$B_{n,1}^{\text{ext}}(\beta,\delta) := -\frac{\mathrm{i}}{2\pi}\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\beta,0)}}{\beta-\delta}\chi_{\Gamma_\beta}(\beta)\chi_\mathbb{R}(\delta), \qquad B_{n,2}^{\text{ext}}(\delta,\gamma) := \frac{1}{2\pi}\frac{\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\gamma,0)}}{\delta-\gamma}\chi_\mathbb{R}(\delta)\chi_{\Gamma_\alpha}(\gamma).$$

For what concerns $\mathcal{A}_{t,n}^{\text{ext}}$ instead, just integrating by parts (7.1.21) we get

$$A_{t,n}^{\text{ext}}(\alpha,\beta) = -\frac{\mathrm{i}}{2\pi} \int_\mathbb{R} \mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\alpha,2t)-\frac{\mathrm{i}}{2}\psi_n(\beta,2t)}\mathrm{e}^{\mathrm{i}z(\alpha-\beta)} w(z)\,\mathrm{d}z\, \chi_{\Gamma_\alpha}(\alpha)\chi_{\Gamma_\beta}(\beta)$$

and thus $\mathcal{A}^{\text{ext}}_{t,n} = \mathcal{A}^{\text{ext}}_{t,n,1}\mathcal{A}^{\text{ext}}_{t,n,2}$ where $\mathcal{A}^{\text{ext}}_{t,n,j} : L^2(\Sigma) \to L^2(\Sigma)$ have Hilbert-Schmidt kernels

$$A^{\text{ext}}_{t,n,1}(\alpha, z) := -\frac{\mathrm{i}}{2\pi}\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\alpha,2t)+\mathrm{i}z\alpha}\sqrt{w(z)}\,\chi_{\Gamma_\alpha}(\alpha)\chi_{\mathbb{R}}(z), \quad A^{\text{ext}}_{t,n,2}(z,\beta) := \mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\beta,2t)-\mathrm{i}z\beta}\sqrt{w(z)}\,\chi_{\mathbb{R}}(z)\chi_{\Gamma_\beta}(\beta).$$

Finally the last two properties directly comes from the fact that $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$. Indeed, the operator traces are computed as

$$\operatorname*{Tr}_{L^2(\Sigma)} \mathcal{A}^{\text{ext}}_{t,n} = \int_\Sigma A^{\text{ext}}_{t,n}(z,z)\,\mathrm{d}z = 0, \qquad \operatorname*{Tr}_{L^2(\Sigma)} \mathcal{B}^{\text{ext}}_{n} = \int_\Sigma B^{\text{ext}}_{n}(z,z)\,\mathrm{d}z = 0.$$

And we have that for every $(\xi,\zeta) \in \Sigma \times \Sigma$

$$\left(A^{\text{ext}}_{t,n}\right)^2(\xi,\zeta) = A_{t,n}(\xi,\eta)\chi_{\Gamma_\alpha}(\xi)\chi_{\Gamma_\beta}(\eta)A_{t,n}(\eta,\zeta)\chi_{\Gamma_\alpha}(\eta)\chi_{\Gamma_\beta}(\zeta) = 0 \quad \text{thus} \quad \left(\mathcal{A}^{\text{ext}}_{t,n}\right)^2 = 0 \ \text{ on } \ L^2(\Sigma),$$

and the same is true for $\mathcal{B}^{\text{ext}}_n$. ■

Lemma 7.1.11. *For every $(t,\lambda,n) \in \mathbb{R}\times\mathbb{C}\times\mathbb{N}$, we have on $L^2(\Sigma)$,*

$$\left(I + \lambda^{\frac{1}{2}}\mathcal{A}^{\text{ext}}_{t,n}\right)\left(I - \lambda^{\frac{1}{2}}(\mathcal{A}^{\text{ext}}_{t,n} + \mathcal{B}^{\text{ext}}_n)\right)\left(I + \lambda^{\frac{1}{2}}\mathcal{B}^{\text{ext}}_n\right) = I - \lambda\mathcal{A}^{\text{ext}}_{t,n}\mathcal{B}^{\text{ext}}_n = I - \lambda\mathcal{J}^{\text{ext}}_{t,n},$$

with an arbitrary, but throughout fixed, branch for $\lambda^{\frac{1}{2}}$.

Proof. By direct computation, using the nilpotency of the operators $\mathcal{A}^{\text{ext}}_{t,n}, \mathcal{B}^{\text{ext}}_n$. ■

With this operator identity in mind, we can finally prove the final result of this section. We are going to express $D_n(t,\lambda)$ as the Fredholm determinant of a suitable operator on $L^2(\Sigma)$, that is the operator $C_{t,n}$ with kernel written in equations (7.1.32), (7.1.33).

Proposition 7.1.12. *For every $(t,\lambda,n) \in \mathbb{R}\times\mathbb{C}\times\mathbb{N}$,*

$$D_n(t,\lambda) = \det(I - \lambda^{\frac{1}{2}}C_{t,n}|_{L^2(\Sigma)}), \tag{7.1.31}$$

where $C_{t,n} := \mathcal{A}^{\text{ext}}_{t,n} + \mathcal{B}^{\text{ext}}_n : L^2(\Sigma) \to L^2(\Sigma)$ is trace class and has kernel of the form

$$(\xi-\eta)C_{t,n}(\xi,\eta) = \int_{\mathbb{R}}\Big(k_1(\xi|z)m_1(\eta|z) + k_2(\xi|z)m_2(\eta|z)\Big)\mathrm{d}\sigma(z), \tag{7.1.32}$$

where k_i, m_i for $i = 1,2$ are the functions parametrically depending on $\zeta \in \Sigma$, defined as

$$k_1(\zeta|y) := \frac{1}{2\pi}\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\zeta,2t+2y)}\chi_{\Gamma_\alpha}(\zeta), \quad k_2(\zeta|y) := \frac{1}{2\pi}\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\zeta,0)}\chi_{\Gamma_\beta}(\zeta), \quad m_1(\zeta|x) := \mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\zeta,2t+2x)}\chi_{\Gamma_\beta}(\zeta),$$
$$m_2(\zeta|x) := \mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\zeta,0)}\chi_{\Gamma_\alpha}(\zeta), \tag{7.1.33}$$

and with $\psi_n(\zeta,z) := \frac{\zeta^{2n+1}}{2n+1} + z\zeta$, as before.

Proof. First of all, notice that $C_{t,n}$ is trace-class on $L^2(\Sigma)$ since it is the sum of two trace-class operators on the same space. Then, by using properties $1, 2$ of Lemma 7.1.10 and the Plemelj-Smithies formula we compute the following determinant

$$\det(1 + \lambda \mathcal{A}^{\mathrm{ext}}_{t,n}|_{L^2(\Sigma)}) = \exp\left(- \sum_{k=1}^{\infty} (-1)^k \frac{\lambda^k}{k} \operatorname*{Tr}_{L^2(\Sigma)} (\mathcal{A}^{\mathrm{ext}}_{t,n})^k \right) = 1. \tag{7.1.34}$$

With the same reasoning, we obtain also $\det(1 + \lambda \mathcal{B}_n|_{L^2(\Sigma)}) = 1$. Finally, by using the factorization identity ((3.10) in [99]) and recalling (7.1.29), identity (7.1.31) is obtained. ■

We finally proved the relation between the Fredholm determinant of the finite temperature n-th order Airy kernel $D_n(t, \lambda)$ and the Fredholm determinant of the operator $C_{t,n}$, that can be thought as an infinite dimensional version of an integrable operator. Indeed, compare the equation describing the kernel of $C_{t,n}$ (7.1.32) with the classical one for IIKS integrable operators (4.2.1): the structure is the same but an integral now replaces the symbol of summation in the right hand side.

7.2 Finite temperature operators and operator-valued Riemann-Hilbert problems

In this section we introduce the main tool to handle the Fredholm determinant of operators as $C_{t,n}$: the operator-valued Riemann-Hilbert problems. In the present literature there are just a few examples of studies involving operator-valued Riemann-Hilbert problems. They can be found in the two papers [61, 62] and then they were used very recently in the review [22], where the author recovered, through a Riemann-Hilbert approach, the Tracy-Widom formula for the finite temperature Airy kernel with $n = 1$, previously discovered in [5]. Following this approach, we define and study some operator-valued Riemann-Hilbert problems that are now related to the operators $C_{t,n}$.

7.2.1 First definitions and statement of the relevant operator-valued Riemann-Hilbert problem

Essentially, an operator-valued Riemann-Hilbert problem is determined as before by a pair (Σ, G) where now the jump matrix $G(\zeta)$ is a matrix whose entries take values in a particular operator space for any value $\zeta \in \Sigma$. To start with, we are going to define the operator space that is relevant in this case, thus we first have to introduce the following functional space. Recall that we fixed a weight function w as in Definition 7.0.1, so that $\mathrm{d}\sigma(x) = w'(x)\mathrm{d}x$, is a probability measure on the real line.
In the following definition we adopt the same notation of [22].

Definition 7.2.1. *Let $p \geq 1$. We use the below abbreviations for the relevant functional and operator spaces.*

1. *The Hilbert space*

$$\mathcal{H}_p := \bigoplus_{j=1}^{p} L^2(\mathbb{R}, \mathrm{d}\sigma) = \left\{ \mathbf{f} = (f_1, \dots, f_p)^\top \in \mathbb{C}^{p\times 1} : \ f_j \in L^2(\mathbb{R}, \mathrm{d}\sigma) \right\}$$

   equipped with its standard inner product and associated norm.

2. *The space $L^2(\mathbb{R}, \mathrm{d}\sigma; \mathbb{C}^{p\times p})$ of $p \times p$ matrix-valued functions with entries in $L^2(\mathbb{R}, \mathrm{d}\sigma)$, equipped with the induced Frobenius integral norm.*

3. *The space $\mathcal{I}(\mathcal{H}_p)$ of Hilbert-Schmidt integral operators on $\mathcal{H}_p$ of the form*

$$(\mathbf{K}\mathbf{f})(x) = \int_{\mathbb{R}} \mathbf{K}(x, y)\mathbf{f}(y)\, \mathrm{d}\sigma(y),$$

   with kernel $\mathbf{K}(x, y) \in L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma; \mathbb{C}^{p\times p})$.

The operator space of interest for our Riemann-Hilbert problem is the space of integral operators $\mathcal{I}(\mathcal{H}_2)$. This means that we can also see both the jump matrix and the solution of this Riemann-Hilbert problem as 2×2 matrices with entries that are integral operators acting on $\mathcal{H}_1$ with kernels in the functional space $L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma)$.
We are now going to state the operator-valued Riemann-Hilbert problem that is related to our infinite dimensional integrable operator, i.e. the operator $C_{t,n}$ acting on $L^2(\Sigma)$ with kernel of the form (7.1.32), (7.1.33).

Remark 7.2.2. *The structure of the Riemann-Hilbert problem stated below, i.e. its jump condition and its asymptotic condition could be used also in order to study other integral operators having kernel of the same form of $C_{t,n}$ but with different functions k_i, m_i and different contour Σ. Moreover, in the forthcoming work [23], the author intends to show that there is an entire class of suitable weighted Hankel composition operators (in which $\mathcal{K}_{t,n}$ fits) that can be studied through Riemann-Hilbert problems of the same type of the following. This "canonical" association to Riemann-Hilbert problems will no longer depend on the "integrable" shape that the kernel of the operator should have (even after proper manipulation as conjugation by bounded invertible operators, as we did for $\mathcal{K}_{t,n}$ in the previous section).*

We first construct the operator-valued jump matrix that will be used in the Riemann-Hilbert problem, building it up entry by entry.

Definition 7.2.3. *For $i, j = 1, 2$ let $M_i(\zeta) \otimes K_j(\zeta) \in \mathcal{I}(\mathcal{H}_1)$, denote the $\Sigma \ni \zeta$-parametric family of rank one integral operators with kernels*

$$\Big(M_i(\zeta) \otimes K_j(\zeta)\Big)(x, y) := m_i(\zeta|x)k_j(\zeta|y), \qquad x, y \in \mathbb{R},$$

defined in terms of the $\Sigma \ni \zeta$-parametric family of functions k_i, m_i defined in (7.1.33).

Remark 7.2.4. *We underline the following three facts, that will be used later on.*

- *All the operators $M_i(\zeta) \otimes K_j(\zeta)$ also depend on the parameters $(t, n) \in \mathbb{R} \times \mathbb{N}$, but we do not highlight this in our notation.*

- *These integral operators acts on some function $f \in \mathcal{H}_1$ as follows: by multiplying by the correspondent functions $m_i(\eta|x)$ and by integrating $f(y)$ against the kernel $k_j(\zeta|y)$.*

- *Since the contours Γ_α and Γ_β are disjoint, it follows by (7.1.33) that the kernels*

$$M_1(\zeta) \otimes K_1(\zeta)(x, y) = 0 = M_2(\zeta) \otimes K_2(\zeta)(x, y), \tag{7.2.1}$$

  thus the correspondent operators are zero too.

The analogue of the jump matrix G involved in the standard Riemann-Hilbert problem 4.1.1 is replaced here by the following operator $\mathbf{G}$.

Definition 7.2.5. *The integral operator $\mathbf{G}(\zeta)$ acting on $\mathcal{H}_2$ is defined for every $\zeta \in \Sigma$ as*

$$\mathbf{G}(\zeta) = \mathbb{I}_2 + 2\pi \mathrm{i} \lambda^{\frac{1}{2}} \begin{bmatrix} M_1(\zeta) \otimes K_1(\zeta) & M_1(\zeta) \otimes K_2(\zeta) \\ M_2(\zeta) \otimes K_1(\zeta) & M_2(\zeta) \otimes K_2(\zeta) \end{bmatrix} = \mathbb{I}_2 + \mathbf{G}_0(\zeta), \tag{7.2.2}$$

where $\mathbb{I}_2$ denotes the identity operator on $\mathcal{H}_2$, and the branch of $\lambda^{\frac{1}{2}}$ is fixed.

Finally we consider the below $\mathcal{I}(\mathcal{H}_2)$-valued Riemann-Hilbert problem, the central operator-valued Riemann-Hilbert problem of this work.

Riemann-Hilbert Problem 7.2.6. *Given $(t, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$, determine an integral operator $\mathbf{X}(\zeta) = \mathbf{X}(\zeta; t, \lambda, n)$ such that*

(1) $\mathbf{X}(\zeta) = \mathbb{I}_2 + \mathbf{X}_0(\zeta)$ and $\mathbf{X}_0(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ with kernel $\mathbf{X}_0(\zeta|x, y)$ analytic in $\mathbb{C} \setminus \Sigma$.

(2) $\mathbf{X}_0(\zeta)$ admits continuous boundary values $\mathbf{X}_{0\pm}(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ on Σ, oriented as shown in Figure 7.2, such that $\mathbf{X}_\pm(\zeta) = \mathbb{I}_2 + \mathbf{X}_{0\pm}(\zeta)$ satisfy

$$\mathbf{X}_+(\zeta) = \mathbf{X}_-(\zeta)\mathbf{G}(\zeta). \tag{7.2.3}$$

(3) There exists $c = c(n, t) > 0$ such that for $\zeta \in \mathbb{C} \setminus \Sigma$,

$$\|\mathbf{X}_0(\zeta|x, y)\| \leq \frac{c\sqrt{|\lambda|}}{1 + |\zeta|} \Delta^{-\frac{1}{4n}} \mathrm{e}^{-\frac{(-1)^n \Delta}{2(2n+1)} \Delta^{2n}} \mathrm{e}^{\Delta(|x|+|y|+|t|)}, \quad \Delta := \mathrm{dist}(\Gamma_\alpha, \mathbb{R}) = \mathrm{dist}(\Gamma_\beta, \mathbb{R}) > 0, \tag{7.2.4}$$

uniformly in $(x, y) \in \mathbb{R}^2$ and $\lambda \in \overline{\mathbb{D}_1(0)}$.

We notice that the structure of the Riemann-Hilbert problems 4.1.1 and 7.2.6 are exactly the same, with the only difference that in the last one we specified the asymptotic condition $\mathbf{X}(\zeta) \sim \mathbb{I}_2$ for $|\zeta| \to \infty$ by requiring a particular condition on the operator norm of the

operator $\mathbf{X}_0$.
While for the standard matrix-valued Riemann-Hilbert problems 4.1.1 the request of finding a matrix-valued function analytic outside the prescribed contour and with continuous boundary values along the contour itself does not need further explanation, for the operator-valued Riemann-Hilbert problem 7.2.6 the same requests are demanded now for an operator-valued function $\mathbf{X}(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ and we need to revise their precise meaning. In the following definitions we adopt the same notation of [22].

Definition 7.2.7. *We say that an operator $\mathbf{K}(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ with kernel $\mathbf{K}(\zeta|x,y)$ is analytic in $\zeta \in \Omega$ a subset of $\mathbb{C}$, if*

1. *for any $(x,y) \in \mathbb{R}^2$, the map $\zeta \mapsto \mathbf{K}(\zeta|x,y)$ is analytic in Ω.*
2. *for any $\zeta \in \Omega$, the map $(x,y) \mapsto \mathbf{K}(\zeta|x,y)$ is in $L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma; \mathbb{C}^{2\times 2})$.*

Furthermore, if $\Sigma \subset \Omega \subset \mathbb{C}$ is an oriented contour consisting of a finite union of smooth oriented curves in $\mathbb{CP}^1$ with finitely many self-intersections (as it is indeed the case for us), then the continuity of the boundary values of $\mathbf{K}(\zeta)$ along Σ is defined as follows.

Definition 7.2.8. *We say that an analytic in $\zeta \in \Omega \setminus \Sigma$ operator $\mathbf{K}(\zeta) \in \mathcal{I}(\mathcal{H}_p)$ admits continuous boundary values $\mathbf{K}_\pm(\zeta) \in \mathcal{I}(\mathcal{H}_p)$ on Σ with kernels $\mathbf{K}_\pm(\zeta|x,y)$ if*

(1) for any $(x,y) \in \mathbb{R}^2$, the map $\zeta \mapsto \mathbf{K}_\pm(\zeta|x,y)$ is continuous on Σ.

(2) for any $(x,y) \in \mathbb{R}^2$, the non-tangential limits

$$\lim_{\lambda\to\zeta} \mathbf{K}(\lambda|x,y) = \mathbf{K}_\pm(\zeta|x,y), \qquad \lambda \in \pm \text{side of } \Sigma \text{ at } \zeta$$

exist.

With these two last definitions in mind, the statement of the Riemann-Hilbert problem 7.2.6 is now clarified and the next step is to find out whether a solutions exists and whether it is unique.

7.2.2 Existence and uniqueness of the solution of the Riemann-Hilbert problem

In the following we are going to prove that the solution of the Riemann-Hilbert problem 7.2.6 exists and is unique. Furthermore, we are going to prove that it has an integral representation very similar to the one that is known for the generic matrix-valued Riemann-Hilbert problem 4.1.1 from Theorem 4.2.6. We start with the proof of uniqueness of the solution of the Riemann-Hilbert problem 7.2.6. We anticipate that the technique used reminds of the one used in the standard matrix case. Also, notice that the third point in Remark 7.2.4 will be fundamental in the proof.

Theorem 7.2.9. *Whether the solution of the Riemann-Hilbert problem 7.2.6 exists, it is unique.*

Proof. Suppose that a solution $\mathbf{X}(\zeta) = \mathbb{I}_2 + \mathbf{X}_0(\zeta) \in \mathbb{I}_2 + \mathcal{I}(\mathcal{H}_2)$ of the Riemann-Hilbert problem 7.2.6 exists. We start by proving that the solution is invertible. To do that, consider on the space $\mathcal{H}_2$ the following Fredholm determinant

$$d(\zeta) \coloneqq \det(\mathbb{I}_2 + \mathbf{X}_0(\zeta)), \quad \zeta \in \mathbb{C} \setminus \Sigma. \tag{7.2.5}$$

For ζ in this domain the Fredholm determinant is well-defined (thanks to the asymptotic condition (7.2.4)) and also analytic in ζ, since we required $\mathbf{X}_0(\zeta)$ to be analytic away from Σ. For $\zeta \in \Sigma$ we can use the continuous boundary values of the solution $\mathbf{X}(\zeta)$ in order to define the non-tangential boundary values of the function $d(\zeta)$

$$d_\pm(\zeta) = \det(\mathbf{X}_\pm(\zeta)), \quad \zeta \in \Sigma. \tag{7.2.6}$$

We can do the same construction for the operator-valued jump matrix $\mathbf{G}(\zeta) = \mathbb{I}_2 + \mathbf{G}_0(\zeta)$, defined for $\zeta \in \Sigma$. Indeed, $\mathbf{G}_0(\zeta) \in \mathcal{I}(\mathcal{H}_2)$ is trace class and its operator norm can be estimated as follows

$$\|\mathbf{G}_0(\zeta|x,y)\| \leq c\sqrt{|\lambda|}\, \mathrm{e}^{-\frac{(-1)^n \Delta}{2n+1}\Delta^{2n}} \mathrm{e}^{\Delta(|x|+|y|+|t|)}, \quad c = c(n) > 0. \tag{7.2.7}$$

Thus by the Hadamard's inequality, the Fredholm determinant $g(\zeta) \coloneqq \det(\mathbb{I}_2 + \mathbf{G}_0(\zeta))$ exists for $\zeta \in \Sigma$. Moreover, by using Remark 7.2.4 we conclude that

$$\operatorname*{Tr}_{\mathcal{H}_2} \mathbf{G}_0(\zeta) = 0 \quad \text{and} \quad (\mathbf{G}_0(\zeta))^2 = \mathbf{0}. \tag{7.2.8}$$

Thus, expressing $g(\zeta)$ through the Plemelj-Smithies formula (see for instance Theorem 3.1 in Chapter II of [48]) we conclude that $g(\zeta) = 1$ for all $\zeta \in \Sigma$. Finally, the multiplicativity of Fredholm determinants applied on the jump condition (7.2.3) yields

$$d_+(\zeta) = d_-(\zeta), \quad \zeta \in \Sigma. \tag{7.2.9}$$

that assures that the function $d(\zeta)$ is actually entire. Moreover, since $d(\zeta) \to 1$ for $\zeta \to \infty$ from the asymptotic condition (7.2.4), we conclude by the generalized Liouville theorem that $d(\zeta) \equiv 1$. In particular $\mathbf{X}(\zeta)$ is invertible for $\zeta \in \mathbb{C} \setminus \Sigma$ and so are their boundary values $\mathbf{X}_\pm(\zeta)$ for $\zeta \in \Sigma$.

Suppose now that there are two solutions $\mathbf{X}_1(\zeta), \mathbf{X}_2(\zeta)$ of the Riemann-Hilbert problem 7.2.6. We can then consider the following integral operator on $\mathcal{H}_2$

$$\mathbf{Y}(\zeta) \coloneqq \mathbf{X}_1(\zeta)(\mathbf{X}_2(\zeta))^{-1}, \quad \zeta \in \mathbb{C} \setminus \Sigma. \tag{7.2.10}$$

For ζ in this domain the operator is analytic and for $\zeta \in \Sigma$ it admits continuous boundary values $\mathbf{Y}_{\pm}(\zeta)$. Moreover for $\zeta \in \Sigma$ we actually have that $\mathbf{Y}_{+}(\zeta) = \mathbf{Y}_{-}(\zeta)$, meaning that the kernel of this operator $\mathbf{Y}(\zeta)$ is actually an entire function in ζ. Finally, by using that $\mathbf{Y}(\zeta) \to \mathbb{I}_2$ for $\zeta \to \infty$, again thanks to the Liouville theorem we conclude that $\mathbf{Y}(\zeta) \equiv \mathbb{I}_2$, i.e. $\mathbf{X}_1(\zeta) = \mathbf{X}_2(\zeta)$ identically in ζ. ■

We are now going to prove that a solution for the Riemann-Hilbert problem 7.2.6 exists and it admits a convenient contour integral representation. As it arose out in Theorem 4.2.6 for the matrix-valued case, also in this operator-valued case the existence of the solutions $\mathbf{X}(\zeta)$ completely relies on the invertibility of the operator $1 - \lambda^{\frac{1}{2}} C_{t,n}$ on $L^2(\Sigma)$. This last condition indeed holds for any $(t, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$ and the proof follows from Lemma 7.1.2 together with Proposition 7.1.12 both proved in the previous section.

Theorem 7.2.10. *For every $(t, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$ consider the integral operator on $\mathcal{H}_2$*

$$\mathbf{X}(\zeta) = \mathbb{I}_2 + \lambda^{\frac{1}{2}} \int_\Sigma \begin{bmatrix} N_1(\eta) \otimes K_1(\eta) & N_1(\eta) \otimes K_2(\eta) \\ N_2(\eta) \otimes K_1(\eta) & N_2(\eta) \otimes K_2(\eta) \end{bmatrix} \frac{\mathrm{d}\eta}{\eta - \zeta}, \qquad \zeta \in \mathbb{C} \setminus \Sigma, \tag{7.2.11}$$

where $N_i(\eta)$ are the operators on $\mathcal{H}_1$ which multiply by the functions $n_i(\eta|x)$ determined via the integral equation on $L^2(\Sigma)$

$$\Big(I - \lambda^{\frac{1}{2}} C^*_{t,n}\Big) n_i(\cdot|x) = m_i(\cdot|x), \quad i = 1, 2, \tag{7.2.12}$$

with $x \in \mathbb{R}$ and the real adjoint $C^*_{t,n}$ of $C_{t,n}$.
Then (7.2.11) solves the Riemann-Hilbert problem 7.2.6.

Proof. As noticed before, the right hand side of (7.2.11) exists if and only if the solution of the integral equation (7.2.12) exists. This is indeed the case as it follows from Lemma 7.1.2 together with Proposition 7.1.12. With this in mind, we prove that the right hand side of (7.2.11) actually satisfies the three requests in the Riemann-Hilbert problem 7.2.6.
For the first request: we start by observing that each entry of the operator $\mathbf{X}_0(\zeta)$ in the right hand side of (7.2.11) is an integral operator with nontrivial kernel

$$X_0^{ij}(\zeta|x, y) = \lambda^{\frac{1}{2}} \int_\Sigma n_i(\eta|x) k_j(\eta|y) \frac{d\eta}{\eta - \zeta}, \quad (x, y) \in \mathbb{R}^2, \;\; \zeta \notin \Sigma. \tag{7.2.13}$$

In order to prove the first condition of the Riemann-Hilbert problem 7.2.6, we have to prove that these kernels are analytic for $\zeta \notin \Sigma$ and that $(x, y) \to X_0^{ij}(\zeta|x, y)$ is in $L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma)$ (following Definition 7.2.7).
For the second part, remark that

$$\|n_i(\cdot|x)\|_{L^2(\Sigma)} \leq c \|m_i(\cdot|x)\|_{L^2(\Sigma)}, \quad c = c(n, t) > 0, \quad i = 1, 2, \tag{7.2.14}$$

thanks to the fact that the resolvent operator is bounded. Thus, by using the definition of the functions m_i, k_j in (7.1.33), the definition of the contour Σ and the Cauchy-Schwartz inequality we estimate

$$\left|X_0^{ij}(\zeta|x,y)\right| \le \frac{c\sqrt{|\lambda|}}{\mathrm{dist}(\zeta,\Sigma)}\Delta^{-\frac{1}{4n}}\mathrm{e}^{-\frac{(-1)^n\Delta}{2(2n+1)}\Delta^{2n}}\mathrm{e}^{\Delta(|x|+|y|+|t|)}, \quad c=c(n,t)>0, \quad i,j=1,2. \tag{7.2.15}$$

Therefore $(x,y)\to X_0^{ij}(\zeta|x,y)$ is indeed in the space $L^2(\mathbb{R}^2,\mathrm{d}\sigma\otimes\mathrm{d}\sigma)$ for every $\zeta\notin\Sigma$. For what concerns the analyticity property: we first observe that for every (x,y) the function $\eta\to n_i(\eta|x)k_j(\eta|y)$ is Holder continuous and thus its Cauchy transform, by the Plemelji-Sokhotoski theorem, is analytic for $\zeta\notin\Sigma$ and so it is $X_0^{ij}(\zeta)$ for each $i,j=1,2$. Thus the first condition of Riemann-Hilbert problem 7.2.6 is satisfied by the right hand side of formula (7.2.11). Thanks to estimate (7.2.15) also the asymptotic condition (7.2.4) is satisfied by the right hand side of formula (7.2.11).

We only have to prove that the jump condition (7.2.3) is satisfied by the right hand side of formula (7.2.11). First of all, remark that the boundary values $\mathbf{X}_\pm(\zeta)$ exist and are Holder-continuous for $\zeta\in\Sigma$, thanks to the properties of the Cauchy transforms, and they are both in the space $\mathbb{I}_2+\mathcal{I}(\mathcal{H}_2)$. In order to check that $\mathbf{X}_\pm(\zeta)$ satisfies the jump condition (7.2.3), we start by applying the property of the Cauchy transform ($\mathcal{C}_+-\mathcal{C}_-=Id$) to (7.2.11) and we deduce

$$\mathbf{X}_+(\zeta)-\mathbf{X}_-(\zeta)=2\pi\mathrm{i}\lambda^{\frac{1}{2}}\begin{bmatrix}N_1(\zeta)\otimes K_1(\zeta) & N_1(\zeta)\otimes K_2(\zeta)\\ N_2(\zeta)\otimes K_1(\zeta) & N_2(\zeta)\otimes K_2(\zeta)\end{bmatrix}, \qquad \zeta\in\Sigma. \tag{7.2.16}$$

We then compute the composition of operators $\mathbf{X}_-(\zeta)\mathbf{G}(\zeta)$ by using their definitions (7.2.11), (7.2.2)

$$\begin{aligned}\mathbf{X}_-(\zeta)\mathbf{G}(\zeta) &= \mathbf{X}_-(\zeta)\left(\mathbb{I}_2+2\pi i\lambda^{\frac{1}{2}}\begin{bmatrix}M_1(\zeta)\otimes K_1(\zeta) & M_1(\zeta)\otimes K_2(\zeta)\\ M_2(\zeta)\otimes K_1(\zeta) & M_2(\zeta)\otimes K_2(\zeta)\end{bmatrix}\right)\\ &= \mathbf{X}_-(\zeta)+2\pi i\lambda^{\frac{1}{2}}\left(\mathbb{I}_2+\lambda^{\frac{1}{2}}\int_\Sigma\begin{bmatrix}N_1(\eta)\otimes K_1(\eta) & N_1(\eta)\otimes K_2(\eta)\\ N_2(\eta)\otimes K_1(\eta) & N_2(\eta)\otimes K_2(\eta)\end{bmatrix}\frac{\mathrm{d}\eta}{\eta-\zeta_-}\right)\\ &\circ\begin{bmatrix}M_1(\zeta)\otimes K_1(\zeta) & M_1(\zeta)\otimes K_2(\zeta)\\ M_2(\zeta)\otimes K_1(\zeta) & M_2(\zeta)\otimes K_2(\zeta)\end{bmatrix}.\end{aligned} \tag{7.2.17}$$

Now, looking at the definition of the kernel of the operator $C_{t,n}$ in equation (7.1.32) and using general theory of rank 1 integral operators we have that

$$\begin{aligned}&\begin{bmatrix}N_1(\eta)\otimes K_1(\eta) & N_1(\eta)\otimes K_2(\eta)\\ N_2(\eta)\otimes K_1(\eta) & N_2(\eta)\otimes K_2(\eta)\end{bmatrix}\begin{bmatrix}M_1(\zeta)\otimes K_1(\zeta) & M_1(\zeta)\otimes K_2(\zeta)\\ M_2(\zeta)\otimes K_1(\zeta) & M_2(\zeta)\otimes K_2(\zeta)\end{bmatrix}\\ &\quad=(\eta-\zeta)C_{t,n}(\eta,\zeta)\begin{bmatrix}N_1(\eta)\otimes K_1(\zeta) & N_1(\eta)\otimes K_2(\zeta)\\ N_2(\eta)\otimes K_1(\zeta) & N_2(\eta)\otimes K_2(\zeta)\end{bmatrix}, \qquad (\eta,\zeta)\in\Sigma\times\Sigma,\end{aligned} \tag{7.2.18}$$

thus we can rewrite the quantity above describing $\mathbf{X}_-(\zeta)\mathbf{G}(\zeta)$ as follows

$$\begin{aligned}\mathbf{X}_-(\zeta)\mathbf{G}_-(\zeta) = \mathbf{X}_-(\zeta) + 2\pi i\lambda^{\frac{1}{2}} \begin{bmatrix} M_1(\zeta)\otimes K_1(\zeta) & M_1(\zeta)\otimes K_2(\zeta) \\ M_2(\zeta)\otimes K_1(\zeta) & M_2(\zeta)\otimes K_2(\zeta) \end{bmatrix} \\ + 2\pi i\lambda \int_\Sigma C_{t,n}(\zeta,\eta) \begin{bmatrix} N_1(\eta)\otimes K_1(\eta) & N_1(\eta)\otimes K_2(\eta) \\ N_2(\eta)\otimes K_1(\eta) & N_2(\eta)\otimes K_2(\eta) \end{bmatrix} \mathrm{d}\eta.\end{aligned} \tag{7.2.19}$$

Now notice that the integral equation (7.2.12) for the operators $N_i(\zeta), M_i(\zeta)$ reads as

$$N_i(\zeta) - \lambda^{\frac{1}{2}} \int_\Sigma C_{t,n}(\eta,\zeta) N_i(\eta)\,\mathrm{d}\eta = M_i(\zeta), \qquad \zeta\in\Sigma, \tag{7.2.20}$$

and thus by replacing it above and by using equation (7.2.16) we finally obtain that

$$\mathbf{X}_-(\zeta)\mathbf{G}(\zeta) = \mathbf{X}_-(\zeta) + 2\pi \mathrm{i}\lambda^{\frac{1}{2}} \begin{bmatrix} N_1(\zeta)\otimes K_1(\zeta) & N_1(\zeta)\otimes K_2(\zeta) \\ N_2(\zeta)\otimes K_1(\zeta) & N_2(\zeta)\otimes K_2(\zeta) \end{bmatrix} = \mathbf{X}_+(\zeta). \tag{7.2.21}$$

This means that also the jump condition (7.2.3) is satisfied by the right hand side of formula (7.2.11) and thus the proof is completed. ∎

So far, we proved that the solution of the Riemann-Hilbert problem 7.2.6 exists and it is unique. Moreover, we explicitly constructed an integral contour representation for the solution $\mathbf{X}(\zeta)$ for any $\zeta \notin \Sigma$ and we know that the solution $\mathbf{X}(\zeta)$ is invertible on $\mathcal{H}_2$ from Theorem 7.2.9. As a byproduct, it follows that the operator $\mathbf{X}(\zeta)^{-1}$ has an analogue integral representation.

Corollary 7.2.11. *For every* $(t,\lambda,n)\in\mathbb{R}\times\overline{\mathbb{D}_1(0)}\times\mathbb{N}$ *the inverse on* $\mathcal{H}_2$ *of the solution* $\mathbf{X}(\zeta)$ *of the Riemann-Hilbert problem 7.2.6 has the following integral representation*

$$(\mathbf{X}(\zeta))^{-1} = \mathbb{I}_2 - \lambda^{\frac{1}{2}} \int_\Sigma \begin{bmatrix} M_1(\zeta)\otimes L_1(\zeta) & M_1(\zeta)\otimes L_2(\zeta) \\ M_2(\zeta)\otimes L_1(\zeta) & M_2(\zeta)\otimes L_2(\zeta) \end{bmatrix} \frac{d\eta}{\eta-\zeta}, \qquad \zeta\in\mathbb{C}\setminus\Sigma, \tag{7.2.22}$$

where $L_i(\eta)$ *are integral operators on* $\mathcal{H}_1$ *with kernel* $\ell_i(t|y)$ *determined from the* $L^2(\Sigma)$ *integral equation*

$$(I-\lambda^{\frac{1}{2}}C_{t,n})\ell_i(\cdot|y) = k_i(\cdot|y), \quad i=1,2. \tag{7.2.23}$$

Remark 7.2.12. *Notice again that the right hand side of equation* (7.2.22) *exists because the integral equation* (7.2.23) *admits solution, for the same reason explained before.*

Proof. It is enough to prove that the right hand side of equation (7.2.22), that we will denote by $\mathbf{Y}(\zeta)$ in the following, it is the actual right inverse of $\mathbf{X}(\zeta)$. We start by computing

$$\mathbf{X}(\zeta)\mathbf{Y}(\zeta) = \mathbb{I}_2 + \lambda^{\frac{1}{2}} \int_\Sigma \left(\overline{\mathbf{X}}(\eta) - \overline{\mathbf{Y}}(\eta)\right)\frac{\mathrm{d}\eta}{\eta-\zeta} - \lambda \int_\Sigma\int_\Sigma \overline{\mathbf{X}}(\eta_1)\overline{\mathbf{Y}}(\eta_2)\frac{\mathrm{d}\eta_1}{\eta_1-\zeta}\frac{\mathrm{d}\eta_2}{\eta_2-\zeta}, \quad \zeta\notin\Sigma, \tag{7.2.24}$$

where we denoted by $\overline{\mathbf{X}}(\zeta), \overline{\mathbf{Y}}(\zeta)$ the finite rank integrand appearing in the right hand side of (7.2.11) and (7.2.22) respectively. The aim is now to prove that the sum of the last two terms in the equation above is zero (the zero operator on $\mathcal{H}_2$).
To start with, notice that by the definition of the kernel of the operator $C_{t,n}$ we have that

$$\overline{\mathbf{X}}(\eta_1)\overline{\mathbf{Y}}(\eta_2) = (\eta_1 - \eta_2)C_{t,n}(\eta_1, \eta_2)\begin{bmatrix} N_1(\eta_1) \otimes L_1(\eta_2) & N_1(\eta_1) \otimes L_2(\eta_2) \\ N_2(\eta_1) \otimes L_1(\eta_2) & N_2(\eta_1) \otimes L_2(\eta_2) \end{bmatrix}. \tag{7.2.25}$$

Replacing this equation in the double integral term appearing above, we can compute it as

$$\begin{aligned}\int_\Sigma\int_\Sigma \overline{\mathbf{X}}(\eta_1)\overline{\mathbf{Y}}(\eta_2)\frac{\mathrm{d}\eta_1}{\eta_1-\zeta}\frac{\mathrm{d}\eta_2}{\eta_2-\zeta} &= \int_\Sigma\int_\Sigma C_{t,n}(\eta_1,\eta_2)\begin{bmatrix} N_1(\eta_1) \otimes L_1(\eta_2) & N_1(\eta_1) \otimes L_2(\eta_2) \\ N_2(\eta_1) \otimes L_1(\eta_2) & N_2(\eta_1) \otimes L_2(\eta_2) \end{bmatrix}\mathrm{d}\eta_1\frac{d\eta_2}{\eta_2-\zeta} \\ &- \int_\Sigma\int_\Sigma C_{t,n}(\eta_1,\eta_2)\begin{bmatrix} N_1(\eta_1) \otimes L_1(\eta_2) & N_1(\eta_1) \otimes L_2(\eta_2) \\ N_2(\eta_1) \otimes L_1(\eta_2) & N_2(\eta_1) \otimes L_2(\eta_2) \end{bmatrix}d\eta_2\frac{d\eta_1}{\eta_1-\zeta} \\ &= -\int_\Sigma \overline{\mathbf{Y}}(\eta_2)\frac{d\eta_2}{\eta_2-\zeta} + \int_\Sigma \overline{\mathbf{X}}(\eta_1)\frac{\mathrm{d}\eta_1}{\eta_1-\zeta}\end{aligned} \tag{7.2.26}$$

where in the last passage we used both the integral equations (7.2.12), (7.2.23). This concludes the proof. ■

Remark 7.2.13. *The main ideas in the construction of the Riemann-Hilbert problem 7.2.6 and the proofs of the theorems for its solution have been already developed in [22], and they are indeed due to the work of Thomas Bothner.*

From the construction of the integral representation of $\mathbf{X}(\zeta)$ and its inverse given in Theorem 7.2.10 and Corollary 7.2.11 one can deduce a relation among the multiplication operators on $\mathcal{H}_1$ called $N_i(\zeta), M_i(\zeta)$ for $i = 1, 2$, the integral operators $L_i(\zeta), K_i(\zeta)$ for $i = 1, 2$ and the solution $\mathbf{X}(\zeta)$ of the Riemann-Hilbert problem 7.2.6. The derivation of a Lax pair from the solution $\mathbf{X}(\zeta)$, that will be treated in the following section, completely relies on this relation. In order to express it in a compact form, we define the following vector-valued operators on $\mathcal{H}_2$

$$\begin{gathered}\mathbf{N}(\zeta) := \Big[N_1(\zeta), N_2(\zeta)\Big]^\top, \quad \mathbf{M}(\zeta) := \Big[M_1(\zeta), M_2(\zeta)\Big]^\top, \quad \mathbf{L}(\zeta) := \Big[L_1(\zeta), L_2(\zeta)\Big], \\ \mathbf{K}(\zeta) := \Big[K_1(\zeta), K_2(\zeta)\Big].\end{gathered} \tag{7.2.27}$$

Corollary 7.2.14. *For every $\zeta \in \Sigma$, independently on the choice of the boundary values of $\mathbf{X}(\zeta)$ we have that*

$$\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta), \quad \mathbf{L}(\zeta) = \mathbf{K}(\zeta)(\mathbf{X}(\zeta))^{-1}. \tag{7.2.28}$$

Proof. Here we are going to prove only the first equation, since it is the only one that is actually needed in the derivation of the Lax pair. The second one is obtained in a similar way and we refer to the proof of Corollary 4.11 in [24] for the details.

Recall that we proved in the last passages of proof of Theorem 7.2.10 the following identity

$$\mathbf{X}_{-}(\zeta)\mathbf{G}(\zeta) = \mathbf{X}_{-}(\zeta) + 2\pi \mathrm{i}\lambda^{\frac{1}{2}} \begin{bmatrix} N_1(\zeta) \otimes K_1(\zeta) & N_1(\zeta) \otimes K_2(\zeta) \\ N_2(\zeta) \otimes K_1(\zeta) & N_2(\zeta) \otimes K_2(\zeta) \end{bmatrix}. \tag{7.2.29}$$

On the other hand, since $\mathbf{G}(\zeta)$ is invertible on $\mathcal{H}_2$ with inverse

$$\Big(\mathbf{G}(\zeta)\Big)^{-1} = \mathbb{I}_2 - 2\pi \mathrm{i}\lambda^{\frac{1}{2}} \begin{bmatrix} M_1(\zeta) \otimes K_1(\zeta) & M_1(\zeta) \otimes K_2(\zeta) \\ M_2(\zeta) \otimes K_1(\zeta) & M_2(\zeta) \otimes K_2(\zeta) \end{bmatrix}, \quad \zeta \in \Sigma, \tag{7.2.30}$$

we can then compute in a similar way the quantity $\mathbf{X}_{+}(\zeta)\Big(\mathbf{G}(\zeta)\Big)^{-1}$ and it follows that

$$\mathbf{X}_{+}(\zeta)\Big(\mathbf{G}(\zeta)\Big)^{-1} = \mathbf{X}_{+}(\zeta) - 2\pi \mathrm{i}\lambda^{\frac{1}{2}} \begin{bmatrix} N_1(\zeta) \otimes K_1(\zeta) & N_1(\zeta) \otimes K_2(\zeta) \\ N_2(\zeta) \otimes K_1(\zeta) & N_2(\zeta) \otimes K_2(\zeta) \end{bmatrix}. \tag{7.2.31}$$

Finally, combining (7.2.29), (7.2.31), the definitions of $\mathbf{G}(\zeta)$ and its inverse yields

$$\mathbf{X}_{\pm}(\zeta) \begin{bmatrix} M_1(\zeta) \otimes K_1(\zeta) & M_1(\zeta) \otimes K_2(\zeta) \\ M_2(\zeta) \otimes K_1(\zeta) & M_2(\zeta) \otimes K_2(\zeta) \end{bmatrix} = \begin{bmatrix} N_1(\zeta) \otimes K_1(\zeta) & N_1(\zeta) \otimes K_2(\zeta) \\ N_2(\zeta) \otimes K_1(\zeta) & N_2(\zeta) \otimes K_2(\zeta) \end{bmatrix}, \quad \zeta \in \Sigma, \tag{7.2.32}$$

from which $\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta)$ directly follows. ■

On the asymptotic expansion of the solution $\mathbf{X}(\zeta)$ In this paragraph we are going to discuss some technicalities about the asymptotic representation of $\mathbf{X}(\zeta)$ and $(\mathbf{X}(\zeta))^{-1}$. In particular we prove two symmetry properties and an estimate on the operator norm of the asymptotic coefficients. These results are technical, but they are crucial in order to explicitly recover the Lax pair from relation (7.2.28). Nevertheless, while the statement of the Riemann-Hilbert problem 7.2.6 and the results about its solution $\mathbf{X}(\zeta)$ contained in the previous section can be extended to an arbitrary operator of the same kind of $C_{t,n}$, the statements in this paragraph strictly depend on the exact definition of $C_{t,n}$.
To start with, we recall for every $k \geq 1$ the following formula

$$\frac{1}{\eta - \zeta} = -\frac{1}{\zeta}\sum_{j=0}^{k-1}\left(\frac{\eta}{\zeta}\right)^{j} + \frac{\eta^k}{\zeta^k(\eta-\zeta)} \quad \text{for} \quad \zeta \neq \eta. \tag{7.2.33}$$

By replacing this formula in the integral representation of the solution $\mathbf{X}(\zeta)$ and its inverse, for $|\zeta| \to \infty$, we obtain their asymptotic representation. In particular we have respectively

$$\mathbf{X}(\zeta) = \mathbb{I}_2 - \sum_{j=1}^{k-1}\frac{\mathbf{X}_j}{\zeta^j} + \mathcal{O}(\zeta^k), \quad \text{and} \quad (\mathbf{X}(\zeta))^{-1} = \mathbb{I}_2 + \sum_{j=1}^{k-1}\frac{\mathbf{Y}_j}{\zeta^j} + \mathcal{O}(\zeta^k), \ \text{for} \ \zeta \in \mathbb{C}\backslash\Sigma \tag{7.2.34}$$

with $\mathbf{X}_j = \{X_j^{ml}\}_{m,l=1,2}, \mathbf{Y}_j = \{Y_j^{ml}\}_{m,l=1,2}$ that are integral operators on $\mathcal{H}_2$ not depending any more on the complex parameter ζ, given by

$$X_j^{ml} = \int_\Sigma N_m(\eta) \otimes K_l(\eta)\eta^{j-1} d\eta, \quad Y_j^{ml} = \int_\Sigma M_m(\eta) \otimes L_l(\eta)\eta^{j-1} d\eta \quad j \geq 1, \ \ m,l \in \{1,2\}. \tag{7.2.35}$$

Corollary 7.2.15. *For every $i,j \in \{1,2\}$ we have on $\mathcal{H}_1$ the following identities*

$$\int_\Sigma M_i(\eta) \otimes L_j(\eta) \mathrm{d}\eta = \int_\Sigma N_i(\eta) \otimes K_j(\eta) \mathrm{d}\eta, \tag{7.2.36}$$

and also

$$\int_\Sigma M_i(\eta) \otimes L_j(\eta)\eta \mathrm{d}\eta = \int_\Sigma N_i(\eta) \otimes K_j(\eta)\eta \mathrm{d}\eta + \lambda^{\frac{1}{2}} \left(\int_\Sigma N_i(\eta) \otimes K_j(\eta) \mathrm{d}\eta \right)^2. \tag{7.2.37}$$

Proof. During the proof of Corollary 7.2.11 we proved the following identity

$$\lambda^{\frac{1}{2}} \int_\Sigma \left(\overline{\mathbf{X}}(\eta) - \overline{\mathbf{Y}}(\eta) \right) \frac{\mathrm{d}\eta}{\eta - \zeta} - \lambda \int_\Sigma \int_\Sigma \overline{\mathbf{X}}(\eta_1) \overline{\mathbf{Y}}(\eta_2) \frac{\mathrm{d}\eta_1}{\eta_1 - \zeta} \frac{\mathrm{d}\eta_2}{\eta_2 - \zeta} = \mathbf{0}, \qquad \zeta \notin \Sigma. \tag{7.2.38}$$

Replacing in both terms formula (7.2.33) for $k = 2$, and collecting the powers $1, 2$ of ζ^{-1} as $|\zeta| \to \infty$ gives exactly the two identity stated. ■

Equivalently the asymptotic coefficients of $\mathbf{X}(\zeta)$ and $(\mathbf{X}(\zeta))^{-1}$ are related in the following way

$$Y_1^{ml} = X_1^{ml}, \quad Y_2^{ml} = X_2^{ml} + \left(X_1^{ml} \right)^2, \quad m,l \in \{1,2\}. \tag{7.2.39}$$

Remark 7.2.16. *One could in principle replace formula* (7.2.33) *for $k > 2$ and obtain more complicated relations for the higher order asymptotic coefficients X_i^{ml}, Y_j^{ml}. But for our future scopes the two relations above are sufficient.*

We are now going to prove another important symmetry relation, this time at the level of the kernels of some operators on $\mathcal{H}_1$. In particular, we are going to consider the operators filling the off-diagonal entries of the first asymptotic coefficient of $\mathbf{X}(\zeta)$, and we denote them as follows

$$U := \lambda^{\frac{1}{2}} \int_\Sigma N_1(\eta) \otimes K_2(\eta) \mathrm{d}\eta = X_1^{12}, \quad V := \lambda^{\frac{1}{2}} \int_\Sigma N_2(\eta) \otimes K_1(\eta) \mathrm{d}\eta = X_1^{21}. \tag{7.2.40}$$

Proposition 7.2.17. *For every $(t, \lambda, n) \in \mathbb{R} \times \overline{\mathbb{D}_1(0)} \times \mathbb{N}$ and for every $(x,y) \in \mathbb{R}^2$, we have that*

$$U(x,y) = V(y,x). \tag{7.2.41}$$

In order to prove this statement, we need to review some of the properties of the operator $C_{t,n}$, defined in (7.1.32), (7.1.33). Recall that the operator $C_{t,n}$ on $L^2(\Sigma)$ is defined as the sum of two operators on the same space acting with the following kernels

$$(\zeta-\eta)A_{t,n}^{\mathrm{ext}}(\zeta,\eta)=\int_{\mathbb{R}}k_1(\zeta|z)m_1(\eta|z)\mathrm{d}\sigma(z),\quad(\xi-\eta)B_n^{\mathrm{ext}}(\xi,\eta)=\int_{\mathbb{R}}k_2(\xi|z)m_2(\eta|z)\mathrm{d}\sigma(z).\tag{7.2.42}$$

These operators are both nilpotent on $L^2(\Sigma)$, thanks to equation (7.1.33), as proved in Lemma 7.1.10. Also, thanks to the symmetry $\overline{\Gamma}_\beta=\Gamma_\alpha$ and to the fact that $\lambda\to\psi_n(\lambda,\cdot)$ is odd, we have that $B_n^{\mathrm{ext}}(-\xi,-\eta)=B_n^{\mathrm{ext}}(\eta,\xi)$ for every $(\xi,\eta)\in\Gamma_\alpha\times\Gamma_\beta$ and $A_{t,n}^{\mathrm{ext}}(-\eta,-\zeta)=A_{t,n}^{\mathrm{ext}}(\zeta,\eta)$ for every $(\eta,\zeta)\in\Gamma_\beta\times\Gamma_\alpha$. Using the nilpotency of the operators $\mathcal{A}_{t,n}^{\mathrm{ext}},\mathcal{B}_n^{\mathrm{ext}}$ the powers of the operator $C_{t,n}$ are computed as

$$C_{t,n}^k=\begin{cases}(\mathcal{A}_{t,n}^{\mathrm{ext}}\mathcal{B}_n^{\mathrm{ext}})^m\mathcal{A}_{t,n}^{\mathrm{ext}}+(\mathcal{B}^{\mathrm{ext}}\mathcal{A}_{t,n}^{\mathrm{ext}})^m\mathcal{B}^{\mathrm{ext}}, & k=2m+1\\ (\mathcal{A}_{t,n}^{\mathrm{ext}}\mathcal{B}^{\mathrm{ext}})^m+(\mathcal{B}^{\mathrm{ext}}\mathcal{A}_{t,n}^{\mathrm{ext}})^m, & k=2m\end{cases}.\tag{7.2.43}$$

Finally, using the properties of the kernels of $\mathcal{A}_{t,n},\mathcal{B}_n$ we conclude that

$$C_{t,n}^{2m+1}(-\xi,-\eta)=0,\quad C_{t,n}^{2m}(-\xi,-\eta)=C_{t,n}^{2m}(\eta,\xi),\quad\text{for any}\quad(\xi,\eta)\in\Gamma_\alpha\times\Gamma_\alpha,\tag{7.2.44}$$

thus for any $k\in\mathbb{N}$ we have that $C_{t,n}^k(-\xi,-\eta)=C_{t,n}^k(\eta,\xi)$, for every $(\xi,\eta)\in\Gamma_\alpha\times\Gamma_\alpha$. Having this property of the operator $C_{t,n}$ in mind, we can finally give the proof of the above proposition.

Proof. In order to prove the proposition, we start by computing the left hand side of equation (7.2.41).

$$\begin{aligned}U(x,y)&=\lambda^{\frac{1}{2}}\int_\Sigma\Big(N_1(\eta)\otimes K_2(\eta)\Big)(x,y)\,\mathrm{d}\eta=\int_\Sigma n_1(\eta|x)k_2(\eta|y)\,\mathrm{d}\eta\\&=\frac{1}{2\pi}\int_{\Gamma_\beta}\left[\int_{\Gamma_\beta}(I-\lambda^{\frac{1}{2}}C_{t,n})^{-1}(\xi,\eta)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\xi,2t+2x)}\,\mathrm{d}\xi\right]\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi_n(\eta,0)}\,\mathrm{d}\eta\\&=\frac{1}{2\pi}\int_{\Gamma_\alpha}\left[\int_{\Gamma_\alpha}(I-\lambda^{\frac{1}{2}}C_{t,n})^{-1}(-\xi,-\eta)\,\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\xi,2t+2x)}\,\mathrm{d}\xi\right]\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\eta,0)}\,\mathrm{d}\eta,\end{aligned}$$

where we used equations (7.1.33), (7.2.12) and the conjugation symmetry $\overline{\Gamma}_\beta=\Gamma_\alpha$. Now, by rewriting the operator $(I-\lambda^{\frac{1}{2}}C_{t,n})^{-1}$ with its Neumann series expansion, and by using that for all $k\in\mathbb{N}$ $C_{t,n}^k(-\xi,-\eta)=C_{t,n}^k(\eta,\xi)$ for every $(\xi,\eta)\in\Gamma_\alpha\times\Gamma_\alpha$ we can conclude that

$$\begin{aligned}U(x,y)&=\frac{1}{2\pi}\int_{\Gamma_\alpha}\left[\int_{\Gamma_\alpha}(I-\lambda^{\frac{1}{2}}C_{t,n}|_{L^2(\Sigma)})^{-1}(\eta,\xi)\,\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\xi,2t+2x)}\,\mathrm{d}\xi\right]\mathrm{e}^{\frac{\mathrm{i}}{2}\psi_n(\eta,0)}\,\mathrm{d}\eta\\&=\int_\Sigma\ell_1(\eta|x)m_2(\eta|y)\,\mathrm{d}\eta=\int_\Sigma\Big(M_2(\eta)\otimes L_1(\eta)\Big)(y,x)\;=\int_\Sigma\Big(N_2(\eta)\otimes K_1(\eta)\Big)(y,x)\,\mathrm{d}\eta,\end{aligned}\tag{7.2.45}$$

where in the last passages we used the integral equation (7.2.23) and the symmetry condition (7.2.36). ∎

The last technical property of the asymptotic coefficients of the solution $\mathbf{X}(\zeta)$ is given in the following statement.

Corollary 7.2.18. *Let $i, j \in \{1,2\}$ and $m \in \mathbb{Z}_{\geq 0}$. Then*

$$\int_\Sigma N_i(\eta) \otimes K_j(\eta)\, \eta^m \,\mathrm{d}\eta \to 0 \quad \text{and} \quad \int_\Sigma M_i(\eta) \otimes L_j(\eta)\, \eta^m \,\mathrm{d}\eta \to 0 \tag{7.2.46}$$

exponentially fast as $t \to +\infty$ in operator norm on $\mathcal{H}_1$.

For the proof, we refer to the proof of Corollary 4.14 in [24].

The machinery of operator-valued Riemann-Hilbert problem 7.2.6 associated to the operator $C_{t,n}$ can then be used to study properties of $D_n(t, \lambda)$, and this is what we are going to do. Let summarise what we have proved until now: the unique solution of the Riemann-Hilbert problem 7.2.6 is denoted by $\mathbf{X}(\zeta), \zeta \in \mathbb{C}$ and it is an integral operator acting on the space $\mathcal{H}_2$ with kernel being in the functional space $L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma; \mathbb{C}^{2\times 2})$. Moreover, $\mathbf{X}(\zeta)$ admits the integral representation (7.2.11) for every $\zeta \notin \Sigma$, with continuos boundary values from both sides of Σ. Finally, from Corollary 7.2.14, this solution $\mathbf{X}(\zeta)$ satisfies for every $\zeta \in \Sigma$ and independently on the choice of its boundary value, the following identity

$$\mathbf{N}(\zeta) = \mathbf{X}(\zeta)\mathbf{M}(\zeta), \tag{7.2.47}$$

with $\mathbf{N}(\zeta), \mathbf{M}(\zeta)$ some vector-valued multiplication operators on $\mathcal{H}_2$, defined in (7.2.27). This equation above, together with all the other properties of the solution $\mathbf{X}(\zeta)$ of Riemann-Hilbert problem 7.2.6 will be largely used in the following section, in order to deduce a Lax pair.

7.3 The Lax pair for an operator-valued Painlevé II hierarchy

The main ingredient, in order to deduce the Lax pair, is the relation between $\mathbf{N}(\zeta), \mathbf{M}(\zeta), \mathbf{X}(\zeta)$ in (7.2.47). We recall the definition of the vector-valued operators $\mathbf{M}(\zeta), \mathbf{N}(\zeta)$

$$\mathbf{N}(\zeta) := \Big[N_1(\zeta), N_2(\zeta)\Big]^\top, \;\; \mathbf{M}(\zeta) := \Big[M_1(\zeta), M_2(\zeta)\Big]^\top,$$

where $M_i(\zeta), N_i(\zeta)$ are multiplication operators on $\mathcal{H}_1$, that multiply respectively by the functions $m_i(\zeta|x)$ defined in (7.1.33) and $n_i(\zeta|x)$ defined through the integral equation (7.2.12). Given that, we can also interpret these operators M_i, N_i as integral operators on $\mathcal{H}_1$ with distributional kernels given by

$$\begin{aligned} m_i(\zeta|x) &\mapsto m_i(\zeta|x,y) := m_i(\zeta|x)\delta(x-y)(w'(y))^{-1}, \\ n_i(\zeta|x) &\mapsto n_i(\zeta|x,y) := n_i(\zeta|x)\delta(x-y)(w'(y))^{-1}, \end{aligned} \tag{7.3.1}$$

for any $(x,y) \in \mathbb{R}^2$.

Remark 7.3.1. *We recall that by definition,*

$$\int_{-\infty}^{\infty} \delta(x-y)(w'(y))^{-1} f(y)\, \mathrm{d}\sigma(y) := f(x), \qquad f \in \mathcal{H}_1,$$

so that $(M_i f)(x) = m_i(\zeta|x) f(x)$ *and* $(N_i f)(x) = n_i(\zeta|x) f(x)$ *for any* $f \in \mathcal{H}_1$.

The aim is to prove that the vector-valued operator $\mathbf{N}(\zeta)$ satisfies a couple of operator-valued differential equations w.r.t. the complex parameter ζ and the real parameter t, by using relation (7.2.47). Thus we are going to need the computation of the derivative w.r.t ζ and t of $\mathbf{M}(\zeta)$, written below. Recalling the definition of the functions $m_i(\zeta|x)$ given in (7.1.33), we find the kernel identity

$$\frac{\partial}{\partial \zeta} \mathbf{M}(\zeta|x,y) = \begin{bmatrix} -\mathrm{i}(\frac{1}{2}\zeta^{2n} + t + x) & 0 \\ 0 & \frac{\mathrm{i}}{2}\zeta^{2n} \end{bmatrix} \mathbf{M}(\zeta|x,y), \qquad (\zeta, x, y) \in \Sigma \times \mathbb{R}^2,$$

or equivalently the operator identity

$$\frac{\partial}{\partial \zeta} \mathbf{M}(\zeta) = \left(\zeta^{2n} \mathbf{A}_0 + \widehat{\mathbf{A}}_{2n}\right) \mathbf{M}(\zeta), \quad \zeta \in \Sigma, \tag{7.3.2}$$

where the operators $\mathbf{A}_0, \widehat{\mathbf{A}}_{2n} : \mathcal{H}_2 \to \mathcal{H}_2$ are ζ-independent and have kernels

$$\mathbf{A}_0(x,y) := \delta(x-y)\frac{1}{2}\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix} \left(w'(y)\right)^{-1}, \quad \widehat{\mathbf{A}}_{2n}(x,y) := \delta(x-y)\begin{bmatrix} -\mathrm{i}(t+x) & 0 \\ 0 & 0 \end{bmatrix} \left(w'(y)\right)^{-1}. \tag{7.3.3}$$

Similarly,

$$\frac{\partial}{\partial t} \mathbf{M}(\zeta) = \left(\zeta \mathbf{B}_0\right) \mathbf{M}(\zeta), \qquad \zeta \in \Sigma, \tag{7.3.4}$$

where $\mathbf{B}_0 : \mathcal{H}_2 \to \mathcal{H}_2$ has kernel

$$\mathbf{B}_0(x,y) := \delta(x-y)\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & 0 \end{bmatrix} \left(w'(y)\right)^{-1}. \tag{7.3.5}$$

With this in mind, we now proceed through the following steps

1. we first prove that $\mathbf{N}(\zeta)$ solves linear differential equations w.r.t. both ζ and t with operator-valued coefficients $\mathbf{A}(\zeta)$, $\mathbf{B}(\zeta)$ that are analytic operator-valued functions in ζ;

2. we prove then that $\mathbf{A}(\zeta)$, $\mathbf{B}(\zeta)$ are actually polynomials in ζ of degree $2n$ and 1 with operator-valued coefficients;

3. by exploiting the compatibility condition of the system for $\mathbf{N}(\zeta)$, we prove that all the coefficients of the operator-valued polynomials $\mathbf{A}(\zeta)$, $\mathbf{B}(\zeta)$ are determined in terms of U, V the integral operators on $\mathcal{H}_1$ defined in (7.2.40) and their t-derivatives;

4. we finally conclude that the system of differential equations for $\mathbf{N}(\zeta)$ is a Lax pair for a coupled operator-valued PII hierarchy involving the operators U and V.

Proposition 7.3.2. *There exist (t,λ,n)-dependent, analytic in $\zeta\in\mathbb{C}$ integral operators $\mathbf{A}(\zeta),\mathbf{B}(\zeta)$ on $\mathcal{H}_2$ such that for every $\zeta\in\Sigma$ and $(t,\lambda,n)\in\mathbb{R}\times\overline{\mathbb{D}_1(0)}\times\mathbb{N}$,*

$$\frac{\partial\mathbf{N}}{\partial\zeta}(\zeta)=\mathbf{A}(\zeta)\mathbf{N}(\zeta),\qquad \frac{\partial\mathbf{N}}{\partial t}(\zeta)=\mathbf{B}(\zeta)\mathbf{N}(\zeta). \tag{7.3.6}$$

Proof. We ζ-differentiate the first identity in (7.2.47)

$$\frac{\partial\mathbf{N}}{\partial\zeta}(\zeta)=\underbrace{\left(\frac{\partial\mathbf{X}}{\partial\zeta}(\zeta)\Big(\mathbf{X}(\zeta)\Big)^{-1}+\mathbf{X}(\zeta)\Big(\zeta^{2n}\mathbf{A}_0+\widehat{\mathbf{A}}_{2n}\Big)\Big(\mathbf{X}(\zeta)\Big)^{-1}\right)}_{=:\mathbf{A}(\zeta)}\mathbf{N}(\zeta). \tag{7.3.7}$$

Here, $\mathbf{A}(\zeta)$ is an integral operator acting on $\mathcal{H}_2$ by Theorem 7.2.10, Corollary 7.2.11, and $\mathbf{A}(\zeta)$ is analytic for $\zeta\in\mathbb{C}\setminus\Sigma$ with continuous boundary values $\mathbf{A}_\pm(\zeta)$ on Σ by the same reasoning. Recalling (7.2.3) we then compute on Σ,

$$\begin{aligned}\mathbf{A}_+(\zeta)=\left[\frac{\partial\mathbf{X}_-}{\partial\zeta}(\zeta)\mathbf{G}(\zeta)+\mathbf{X}_-(\zeta)\frac{\partial\mathbf{G}}{\partial\zeta}(\zeta)\right]\Big(\mathbf{G}(\zeta)\Big)^{-1}\Big(\mathbf{X}_-(\zeta)\Big)^{-1}&\\ +\mathbf{X}_-(\zeta)\mathbf{G}(\zeta)\Big(\zeta^{2n}\mathbf{A}_0+\widehat{\mathbf{A}}_{2n}\Big)\Big(\mathbf{G}(\zeta)\Big)^{-1}\Big(\mathbf{X}_-(\zeta)\Big)^{-1},&\end{aligned} \tag{7.3.8}$$

and with (7.2.2),(7.3.3) we derive for $\zeta\in\Sigma$,

$$\frac{\partial\mathbf{G}}{\partial\zeta}(\zeta|x,y)=\int_{-\infty}^{\infty}\left\{\Big(\zeta^{2n}\mathbf{A}_0(x,z)+\widehat{\mathbf{A}}_{2n}(x,z)\Big)\mathbf{G}_0(\zeta|z,y)-\mathbf{G}_0(\zeta|x,z)\Big(\zeta^{2n}\mathbf{A}_0(z,y)+\widehat{\mathbf{A}}_{2n}(z,y)\Big)\right\}\mathrm{d}\sigma(z).$$

Here we abbreviate, as in the definition of $\mathbf{G}(\zeta)$ given in (7.2.2), $\mathbf{G}(\zeta)=\mathbb{I}_2+\mathbf{G}_0(\zeta)$. Notice that the last kernel identity is equivalent to the operator commutator identity

$$\frac{\partial\mathbf{G}}{\partial\zeta}(\zeta)=\Big[\zeta^{2n}\mathbf{A}_0+\widehat{\mathbf{A}}_{2n},\mathbf{G}(\zeta)\Big]\in\mathcal{I}(\mathcal{H}_2),\qquad\zeta\in\Sigma. \tag{7.3.9}$$

Inserting (7.3.9) into (7.3.8) we find at once

$$\mathbf{A}_+(\zeta)=\frac{\partial\mathbf{X}_-}{\partial\zeta}(\zeta)\Big(\mathbf{X}_-(\zeta)\Big)^{-1}+\mathbf{X}_-(\zeta)\Big(\zeta^{2n}\mathbf{A}_0+\widehat{\mathbf{A}}_{2n}\Big)\Big(\mathbf{X}_-(\zeta)\Big)^{-1}=\mathbf{A}_-(\zeta),\quad\zeta\in\Sigma,$$

i.e. $\mathbf{A}(\zeta)$ extends analytically across Σ. In turn, $\mathbf{A}(\zeta)$ is analytic for every $\zeta\in\mathbb{C}$ given that $(x,y)\mapsto\mathbf{A}(\zeta|x,y)$ is in $L^2(\mathbb{R}^2,\mathrm{d}\sigma\otimes\mathrm{d}\sigma;\mathbb{C}^{2\times2})$ for every $\zeta\in\mathbb{C}$ by construction. This proves our first identity and the reasoning for the second one is analogous: first differentiate

(7.2.28) using (7.3.4),

$$\frac{\partial \mathbf{N}}{\partial t}(\zeta) = \underbrace{\left(\frac{\partial \mathbf{X}}{\partial t}(\zeta)\Big(\mathbf{X}(\zeta)\Big)^{-1} + \mathbf{X}(\zeta)\Big(\zeta\mathbf{B}_0\Big)\Big(\mathbf{X}(\zeta)\Big)^{-1}\right)}_{=:\mathbf{B}(\zeta)}\mathbf{N}(\zeta). \tag{7.3.10}$$

Since $\mathbf{B}(\zeta)$ is an integral operator on $\mathcal{H}_2$ and $\mathbf{B}(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \Sigma$ with continuous boundary values $\mathbf{B}_\pm(\zeta)$ on Σ, again from Theorem 7.2.10 and Corollary 7.2.11, we simply compute for $\zeta \in \Sigma$

$$\mathbf{B}_+(\zeta) = \left[\frac{\partial \mathbf{X}_-}{\partial t}\mathbf{G}(\zeta) + \mathbf{X}_-(\zeta)\frac{\partial \mathbf{G}}{\partial t}(\zeta)\right]\Big(\mathbf{G}(\zeta)\Big)^{-1}\Big(\mathbf{X}_-(\zeta)\Big)^{-1} + \mathbf{X}_-(\zeta)\mathbf{G}(\zeta)\Big(\zeta\mathbf{B}_0\Big)\Big(\mathbf{G}(\zeta)\Big)^{-1}\Big(\mathbf{X}_-(\zeta)\Big)^{-1}. \tag{7.3.11}$$

But from (7.2.3), (7.3.5),

$$\frac{\partial \mathbf{G}}{\partial t}(\zeta|x,y) = \int_{-\infty}^{\infty}\left\{\Big(\zeta\mathbf{B}_0(x,z)\Big)\mathbf{G}_0(\zeta|z,y) - \mathbf{G}_0(\zeta|x,z)\Big(\zeta\mathbf{B}_0(z,y)\Big)\right\}\mathrm{d}\sigma(z),$$

leading to the following operator commutator identity

$$\frac{\partial \mathbf{G}}{\partial t}(\zeta) = \Big[\zeta\mathbf{B}_0, \mathbf{G}(\zeta)\Big] \in \mathcal{I}(\mathcal{H}_2), \quad \zeta \in \Sigma.$$

Once substituted back into (7.3.11) we find at once $\mathbf{B}_+(\zeta) = \mathbf{B}_-(\zeta)$ for $\zeta \in \Sigma$, i.e. $\mathbf{B}(\zeta)$ is analytic for $\zeta \in \mathbb{C}$. This concludes our proof. ■

The next step is to prove that the coefficient operators $\mathbf{A}(\zeta), \mathbf{B}(\zeta)$ introduced in Proposition 7.3.2 are actually polynomials in ζ and to express their coefficients in terms of quantities related to the solution of Riemann-Hilbert problem 7.2.6.

Proposition 7.3.3. *We have*

$$\mathbf{B}(\zeta) = \zeta\mathbf{B}_0 + \mathbf{B}_1, \qquad \mathbf{A}(\zeta) = \zeta^{2n}\mathbf{A}_0 + \sum_{k=1}^{2n}\mathbf{A}_k\zeta^{2n-k} + \widehat{\mathbf{A}}_{2n}, \tag{7.3.12}$$

where $\mathbf{B}_j : \mathcal{H}_2 \to \mathcal{H}_2$ are the ζ-independent integral operators with kernels written in (7.3.5) *and* (7.3.13) *below. Likewise, $\mathbf{A}_j : \mathcal{H}_2 \to \mathcal{H}_2$ are ζ-independent, the kernels of $\mathbf{A}_0$ and $\widehat{\mathbf{A}}_{2n}$ are written in* (7.3.3) *and the entries of $\mathbf{A}_k$ are polynomials in the asymptotic coefficients of $\mathbf{X}(\zeta)$ introduced in* (7.2.35), *namely $\int_\Sigma N_i(\eta) \otimes K_j(\eta)\eta^m \mathrm{d}\eta$ and $\int_\Sigma M_i(\eta) \otimes L_j(\eta)\eta^m \mathrm{d}\eta$ with $m \in \mathbb{Z}_{\geq 0}, i,j \in \{1,2\}$.*

Proof. Recall the definition of the operator-valued function $\mathbf{A}(\zeta), \mathbf{B}(\zeta)$ given during the previous proof. The main idea is to replace in them the asymptotic representations of $\mathbf{X}(\zeta)$

and $\mathbf{X}(\zeta)^{-1}$ that we gave in equations (7.2.34), (7.2.35) for $k = 2n$. In particular we have

$$\mathbf{X}(\zeta) = \mathbb{I}_2 - \lambda^{\frac{1}{2}} \sum_{k=1}^{2n} \frac{1}{\zeta^k} \int_\Sigma \begin{bmatrix} N_1(\eta) \otimes K_1(\eta) & N_1(\eta) \otimes K_2(\eta) \\ N_2(\eta) \otimes K_1(\eta) & N_2(\eta) \otimes K_2(\eta) \end{bmatrix} \eta^{k-1} \, \mathrm{d}\eta + \mathcal{O}(\zeta^{-2n-1}) \quad \zeta \notin \Sigma$$

and

$$\Big(\mathbf{X}(\zeta)\Big)^{-1} = \mathbb{I}_2 + \lambda^{\frac{1}{2}} \sum_{k=1}^{2n} \frac{1}{\zeta^k} \int_\Sigma \begin{bmatrix} M_1(\eta) \otimes L_1(\eta) & M_1(\eta) \otimes L_2(\eta) \\ M_2(\eta) \otimes L_1(\eta) & M_2(\eta) \otimes L_2(\eta) \end{bmatrix} \eta^{k-1} \, \mathrm{d}\eta + \mathcal{O}(\zeta^{-2n-1}), \quad \zeta \notin \Sigma.$$

Replacing these formulae in the definition of $\mathbf{B}(\zeta)$ given in (7.3.10) and applying the generalized Liouville theorem, we conclude that

$$\mathbf{B}(\zeta) = \zeta \mathbf{B}_0 + \mathbf{B}_1, \quad \zeta \in \mathbb{C},$$

with $\mathbf{B}_0$ the integral operator on $\mathcal{H}_2$ with distributional kernel (7.3.5) and $\mathbf{B}_1$ the integral operator on $\mathcal{H}_2$ with kernel $\mathbf{B}_1(x, y) = \Big[B_1^{ij}(x, y)\Big]_{i,j=1}^2$ and

$$B_1^{11}(x, y) = B_1^{22}(x, y) = 0, \qquad B_1^{12}(x, y) = -\mathrm{i}U(x, y), \quad B_1^{21}(x, y) = \mathrm{i}V(x, y), \quad (7.3.13)$$

where we $U(x, y)$ and $V(x, y)$ are the kernels of $U = \lambda^{\frac{1}{2}} \int_\Sigma N_1(\eta) \otimes K_2(\eta) \mathrm{d}\eta$ and $V = \lambda^{\frac{1}{2}} \int_\Sigma N_2(\eta) \otimes K_1(\eta) \mathrm{d}\eta$, as defined in (7.2.40). In the same way, we replace the asymptotic representations of $\mathbf{X}(\zeta), (\mathbf{X}(\zeta))^{-1}$ in the definition of $\mathbf{A}(\zeta)$ given in (7.3.7) and we apply the generalized Liouville theorem, concluding that

$$\mathbf{A}(\zeta) = \zeta^{2n} \mathbf{A}_0 + \sum_{k=1}^{2n} \mathbf{A}_k \zeta^{2n-k} + \widehat{\mathbf{A}}_{2n}, \quad \zeta \in \mathbb{C}$$

with $\mathbf{A}_0, \hat{\mathbf{A}}_{2n}$ operators on $\mathcal{H}_2$ with kernels as in (7.3.3). ∎

This last result does not determine explicitly the coefficients $\mathbf{A}_k$ for $k = 1, \ldots, 2n$. Nevertheless, by looking at the compatibility condition of the system (7.3.6), namely the operator identity

$$\mathbf{A}(\zeta)\mathbf{B}(\zeta) - \mathbf{B}(\zeta)\mathbf{A}(\zeta) = \frac{\partial \mathbf{B}}{\partial \zeta}(\zeta) - \frac{\partial \mathbf{A}}{\partial t}(\zeta), \qquad \zeta \in \mathbb{C}, \qquad (7.3.14)$$

we can see that the entries of $\mathbf{A}_k$ for any k are recursively determined in terms of the operators U, V and their t-derivatives. This first result is resumed in the following lemma.

Lemma 7.3.4. *Recall $U, V : \mathcal{H}_1 \to \mathcal{H}_1$ in* (7.2.40) *and introduce the integral operator $M_t : \mathcal{H}_1 \to \mathcal{H}_1$ with distributional kernel $M_t(x, y) := (t + x)\delta(x - y)(w'(y))^{-1}$. Then* (7.3.14) *is equivalent to the operator-valued system* (7.3.15), (7.3.16) *and* (7.3.17) *written out below where A_k^{ij} are the entries of $\mathbf{A}_k$ in* (7.3.12).

Proof. The polynomial equation (7.3.14) yields at once (given that $\mathbf{B}_0$ and $\mathbf{A}_0$ as well as $\mathbf{B}_0$ and $\widehat{\mathbf{A}}_{2n}$ commute)

$$\sum_{k=1}^{2n} \frac{\partial \mathbf{A}_k}{\partial t}\zeta^{2n-k} = \left[\mathbf{B}_1, \mathbf{A}_{2n} + \widehat{\mathbf{A}}_{2n}\right] + \sum_{k=0}^{2n-1}\left(\left[\mathbf{B}_0, \mathbf{A}_{k+1}\right] + \left[\mathbf{B}_1, \mathbf{A}_k\right]\right)\zeta^{2n-k}, \quad \zeta \in \mathbb{C},$$

and therefore, after matching powers in ζ, first to order $\mathcal{O}(\zeta^{2n})$,

$$A_1^{12} = -\mathrm{i}U, \quad A_1^{21} = \mathrm{i}V, \tag{7.3.15}$$

followed by all orders $\mathcal{O}(\zeta^{2n-k})$ for $k = 1, \ldots, 2n-1$,

$$\begin{cases} \dfrac{\partial A_k^{11}}{\partial t} = -\mathrm{i}(UA_k^{21} + A_k^{12}V), & \dfrac{\partial A_k^{12}}{\partial t} = -\mathrm{i}(A_{k+1}^{12} + UA_k^{22} - A_k^{11}U) \\ \dfrac{\partial A_k^{22}}{\partial t} = \mathrm{i}(VA_k^{12} + A_k^{21}U), & \dfrac{\partial A_k^{21}}{\partial t} = \mathrm{i}(A_{k+1}^{21} + VA_k^{11} - A_k^{22}V) \end{cases}, \tag{7.3.16}$$

and finally the order $\mathcal{O}(\zeta^0)$,

$$\begin{cases} \dfrac{\partial A_{2n}^{11}}{\partial t} = -\mathrm{i}(UA_{2n}^{21} + A_{2n}^{12}V), & \dfrac{\partial A_{2n}^{12}}{\partial t} = -\mathrm{i}(UA_{2n}^{22} - A_{2n}^{11}U + \mathrm{i}M_tU) \\ \dfrac{\partial A_{2n}^{22}}{\partial t} = \mathrm{i}(VA_{2n}^{12} + A_{2n}^{21}U), & \dfrac{\partial A_{2n}^{21}}{\partial t} = \mathrm{i}(VA_{2n}^{11} - A_{2n}^{22}V - \mathrm{i}VM_t) \end{cases}. \tag{7.3.17}$$

This completes our proof of the Lemma. ■

Notice that equations (7.3.16) together with the initial condition (7.3.15), allows to compute recursively the entries A_k^{ij} for $k = 1, \ldots, 2n$ (or $2n-1$ for the diagonal entries). For each k, first t-integrating the equations for the diagonal entries A_k^{ii} from the equations on the left and then using them to compute the off-diagonal entries A_{k+1}^{ij} with $i \neq j$ from the equations on the right of (7.3.16). The first system in (7.3.17) is used to determine the last diagonal entries A_{2n}^{ii}. Instead, the second system in (7.3.17) gives a further condition that A_{2n}^{ij}, U, V should satisfy.

Remark 7.3.5. *As explained above, the diagonal entries A_k^{ii} are obtained by t-integrating some equations. The constant of integration in this procedure is fixed to zero thanks to Lemma 7.3.4 and Corollary 7.2.18. The fact that the integration gives always local terms is shown through the following lemma, for which the proof relies on a technique used in [109].*

Lemma 7.3.6. *We have on $\mathcal{H}_1$ for $k = 1, 2, \ldots, 2n$,*

$$A_k^{11} = -\mathrm{i}\sum_{j=1}^{k-1}\left(A_j^{11}A_{k-j}^{11} + A_j^{12}A_{k-j}^{21}\right) \qquad \text{and} \qquad A_k^{22} = \mathrm{i}\sum_{j=1}^{k-1}\left(A_j^{22}A_{k-j}^{22} + A_j^{21}A_{k-j}^{12}\right),$$

and thus in particular $A_1^{11} = A_1^{22} = 0$.

Proof. We start by computing the composition operator $\mathbf{C}(\zeta) = \mathbf{A}(\zeta)\mathbf{A}(\zeta)$ on $\mathcal{H}_2$ from (7.3.12),

$$\mathbf{C}(\zeta) = \sum_{k=0}^{4n}\left(\sum_{j=0}^{k}\mathbf{A}_j\mathbf{A}_{k-j}\right)\zeta^{4n-k} + \sum_{k=0}^{2n}\left(\mathbf{A}_k\widehat{\mathbf{A}}_{2n} + \widehat{\mathbf{A}}_{2n}\mathbf{A}_k\right)\zeta^{2n-k} + \widehat{\mathbf{A}}_{2n}\widehat{\mathbf{A}}_{2n} \equiv \sum_{k=0}^{4n}\mathbf{C}_k\zeta^{4n-k}, \tag{7.3.18}$$

and then use the compatibility constraint (7.3.14),

$$\frac{\partial \mathbf{C}}{\partial t}(\zeta) = \left\{\mathbf{A}(\zeta), \mathbf{B}_0\right\} + \left[\mathbf{B}(\zeta), \mathbf{C}(\zeta)\right], \tag{7.3.19}$$

where the curly brackets indicate the anticommutator. Matching powers $\mathcal{O}(\zeta^{4n-k})$ for $k = 0, \ldots, 2n-1$ in (7.3.19) while using (7.3.18) and (7.3.12) yields at once

$$\begin{cases} \dfrac{\partial C_k^{11}}{\partial t} = -\mathrm{i}(UC_k^{21} + C_k^{12}V), \quad \dfrac{\partial C_k^{12}}{\partial t} = -\mathrm{i}(C_{k+1}^{12} + UC_k^{22} - C_k^{11}U) \\ \dfrac{\partial C_k^{22}}{\partial t} = \mathrm{i}(VC_k^{12} + C_k^{21}U), \quad \dfrac{\partial C_k^{21}}{\partial t} = \mathrm{i}(C_{k+1}^{21} + VC_k^{11} - C_k^{22}V) \end{cases}, \quad k = 0, \ldots, 2n-1, \tag{7.3.20}$$

and

$$\frac{\partial C_{2n}^{11}}{\partial t} = -\mathrm{i}(B_0^{11} + UC_{2n}^{21} + C_{2n}^{12}V), \qquad \frac{\partial C_{2n}^{22}}{\partial t} = \mathrm{i}(VC_{2n}^{12} + C_{2n}^{21}U), \tag{7.3.21}$$

for some of the coefficient operator entries of $\mathbf{C}_k$ with $k = 0, 1, \ldots, 2n$. In turn, system (7.3.20) shows that the operators $\mathbf{C}_k$ are trivial for $k = 1, \ldots, 2n-1$ and

$$C_{2n}^{12} = C_{2n}^{21} = C_{2n}^{22} = 0.$$

Indeed, using (7.3.15),(7.3.16) and Corollary 7.2.18 we find that $A_1^{11} = A_1^{22} = 0$ on $\mathcal{H}_1$ and so by direct computation from (7.3.20),

$$\mathbf{C}_1 = \sum_{j=0}^{1}\mathbf{A}_j\mathbf{A}_{1-j} = \mathbf{0} \quad \text{on } \mathcal{H}_2,$$

where we just replaced equations (7.3.3) and (7.3.17). Hence, proceeding inductively and assuming $\mathbf{C}_j = \mathbf{0}$ for all $j = 1, \ldots, k$ with arbitrary $k \in \{1, \ldots, 2n-2\}$ we first use the off-diagonal equations in (7.3.20) to conclude that

$$C_{j+1}^{12} = \mathrm{i}\frac{\partial C_j^{12}}{\partial t} - UC_j^{22} + C_j^{11}U = 0, \quad C_{j+1}^{21} = -\mathrm{i}\frac{\partial C_j^{21}}{\partial t} - VC_j^{11} + C_j^{22}V = 0,$$

by induction hypothesis. Hence, again by (7.3.20), this time through the diagonal equations,

$$\frac{\partial C_{j+1}^{11}}{\partial t} = -\mathrm{i}(UC_j^{21} + C_j^{12}V) = 0, \quad \frac{\partial C_{j+1}^{22}}{\partial t} = \mathrm{i}(VC_j^{12} + C_j^{21}U) = 0,$$

yielding $C_{j+1}^{11} = C_{j+1}^{22} = 0$ on $\mathcal{H}_1$ by Corollary 7.2.18 and Proposition 7.3.3 since $\mathbf{C}_k = \sum_{j=0}^{k} \mathbf{A}_j \mathbf{A}_{k-j}$ for $k = 1, \ldots, 2n-1$ by (7.3.18) vanishes uniformly as $t \to +\infty$. Moving ahead the proclaimed vanishing of C_{2n}^{12}, C_{2n}^{21} and C_{2n}^{22} follows now from the off-diagonal equations in (7.3.20) as well as the second equation in (7.3.21). We are now prepared to prove the stated formulæ for A_k^{11} and A_k^{22}. First, from (7.3.18),

$$\mathbf{C}_{2n} = \sum_{j=1}^{2n-1} \mathbf{A}_j \mathbf{A}_{2n-j} + \mathbf{A}_0(\mathbf{A}_{2n} + \widehat{\mathbf{A}}_{2n}) + (\mathbf{A}_{2n} + \widehat{\mathbf{A}}_{2n})\mathbf{A}_0,$$

$$\mathbf{C}_k = \sum_{j=1}^{k-1} \mathbf{A}_j \mathbf{A}_{k-j} + \mathbf{A}_0 \mathbf{A}_k + \mathbf{A}_k \mathbf{A}_0, \qquad k = 2, \ldots, 2n-1,$$

so reading off (22)-entries, with the aforementioned fact that $C_k^{22} = 0$ for $k = 1, \ldots, 2n$ and with (7.3.3),

$$0 = C_k^{22} = \sum_{j=1}^{k-1} \left(A_j^{22} A_{k-j}^{22} + A_j^{21} A_{k-j}^{12} \right) + \mathrm{i} A_k^{22}, \quad k = 2, \ldots, 2n. \tag{7.3.22}$$

Combined with the (22)-equation in (7.3.16), identity (7.3.15) and again Corollary 7.2.18, (7.3.22) yields the desired equation for $A_k^{22}, k = 1, \ldots, 2n$. By similar logic

$$0 = C_k^{11} = \sum_{j=1}^{k-1} \left(A_j^{11} A_{k-j}^{11} + A_j^{12} A_{k-j}^{21} \right) - \mathrm{i} A_k^{11}, \qquad k = 2, \ldots, 2n-1; \quad \frac{\partial A_1^{11}}{\partial t} = 0$$

which confirms the stated equation for A_k^{11} provided $k = 1, \ldots, 2n-1$ after another application of Corollary 7.2.18. The A_{2n}^{11} formula has to be treated slightly different since by (7.3.21), after our above workings,

$$\frac{\partial C_{2n}^{11}}{\partial t} = -\mathrm{i} B_0^{11},$$

and in addition

$$C_{2n}^{11} = \sum_{j=1}^{2n-1} \left(A_j^{11} A_{2n-j}^{11} + A_j^{12} A_{2n-j}^{21} \right) - \mathrm{i} A_{2n}^{11} - \mathrm{i} \widehat{A}_{2n}^{11}.$$

However $\frac{\partial}{\partial t} \widehat{A}_{2n}^{11} = B_0^{11}$, so the last two identities yield

$$0 = \frac{\partial}{\partial t} \left[\sum_{j=1}^{2n-1} \left(A_j^{11} A_{2n-j}^{11} + A_j^{12} A_{2n-j}^{21} \right) - \mathrm{i} A_{2n}^{11} \right]$$

and hence after t-integration and an application of Corollary 7.2.18 indeed the stated identity for A_{2n}^{11}. This concludes our proof of the Lemma. ■

We can finally resume all the results found until now in the following corollary, that gives a recursive recipe to find all the coefficients $\mathbf{A}_k$ for $k = 1, \dots, 2n$ in terms of U, V and their t-derivatives and to write in a compact fashion the last two differential equations of the compatibility condition at the level ζ^0.

Corollary 7.3.7. *On $\mathcal{H}_2$,*

$$\mathbf{A}_1 = \begin{bmatrix} 0 & -\mathrm{i}U \\ \mathrm{i}V & 0 \end{bmatrix}, \qquad \mathbf{A}_{k+1} = \begin{bmatrix} A^{11}_{k+1} & A^{12}_{k+1} \\ A^{21}_{k+1} & A^{22}_{k+1} \end{bmatrix}, \quad k = 1, \dots, 2n-1, \tag{7.3.23}$$

where

$$\begin{cases} A^{12}_{k+1} = \mathrm{i}\dfrac{\partial A^{12}_k}{\partial t} - UA^{22}_k + A^{11}_k V \\ A^{21}_{k+1} = -\mathrm{i}\dfrac{\partial A^{21}_k}{\partial t} - VA^{11}_k + A^{22}_k V \end{cases}, \qquad \begin{cases} A^{11}_{k+1} = -\mathrm{i}\displaystyle\sum_{j=1}^{k}(A^{11}_j A^{11}_{k+1-j} + A^{12}_j A^{21}_{k+1-j}) \\ A^{22}_{k+1} = \mathrm{i}\displaystyle\sum_{j=1}^{k}(A^{22}_j A^{22}_{k+1-j} + A^{21}_j A^{12}_{k+1-j}) \end{cases}. \tag{7.3.24}$$

Moreover,

$$\frac{\partial A^{12}_{2n}}{\partial t} = -\mathrm{i}(UA^{22}_{2n} - A^{11}_{2n}U + \mathrm{i}M_t U), \qquad \frac{\partial A^{21}_{2n}}{\partial t} = \mathrm{i}(VA^{11}_{2n} - A^{22}_{2n}V - \mathrm{i}VM_t), \tag{7.3.25}$$

and (7.3.23),(7.3.24),(7.3.25) *combined together yield the following* $(2n)$*-th order coupled, operator-valued system for U and V,*

$$\mathcal{D}^{2n}\begin{bmatrix} -\mathrm{i}U \\ \mathrm{i}V \end{bmatrix} = \begin{bmatrix} \mathrm{i}M_t U \\ -\mathrm{i}VM_t \end{bmatrix}, \qquad \mathcal{D}\begin{bmatrix} A \\ B \end{bmatrix} := \begin{bmatrix} \mathrm{i}\frac{\partial A}{\partial t} - \mathrm{i}UD_t^{-1}(VA + BU) - \mathrm{i}D_t^{-1}(UB + AV)U \\ -\mathrm{i}\frac{\partial B}{\partial t} + \mathrm{i}VD_t^{-1}(UB + AV) + \mathrm{i}D_t^{-1}(VA + BU)V \end{bmatrix} \tag{7.3.26}$$

where $\mathcal{D}$ acts entrywise on operators A and B on $\mathcal{H}_1$ and D_t^{-1} denotes the formal t-antiderivative.

Proof. The only thing that is actually left to prove is that equations (7.3.25) can be rewritten by using equations (7.3.24) and the operator $\mathcal{D}$ as (7.3.26). To see this, we first rewrite the recursion for the off-diagonal entries of $\mathbf{A}_k$ by using the operator $\mathcal{D}$. By (7.3.16) and (7.3.24),

$$\mathcal{D}\begin{bmatrix} A^{12}_k \\ A^{21}_k \end{bmatrix} = \begin{bmatrix} A^{12}_{k+1} \\ A^{21}_{k+1} \end{bmatrix}, \quad k = 1, \dots, 2n-1 \tag{7.3.27}$$

since $D_t^{-1}(VA^{12}_k + A^{21}_k U) = -\mathrm{i}A^{22}_k$ and $D_t^{-1}(UA^{21}_k + A^{12}_k V) = \mathrm{i}A^{11}_k$. Likewise, by (7.3.25) and (7.3.15),

$$\mathcal{D}\begin{bmatrix} A^{12}_{2n} \\ A^{21}_{2n} \end{bmatrix} = \begin{bmatrix} \mathrm{i}M_t U \\ -\mathrm{i}VM_t \end{bmatrix}, \tag{7.3.28}$$

where we use $D_t^{-1}(VA^{12}_{2n} + A^{21}_{2n}U) = -\mathrm{i}A^{22}_{2n}$ and $D_t^{-1}(UA^{21}_{2n} + A^{12}_{2n}V) = \mathrm{i}A^{11}_{2n}$. Hence, iterating (7.3.27),(7.3.28) with the initial data (7.3.23) we arrive at the desired system (7.3.26) which does not contain any antiderivative terms because of the iterative formulæ for A_k^{11} and A_k^{22} written in (7.3.24). ∎

In this section we proved that the system solved by $\mathbf{N}(\zeta)$ given in (7.3.6) can be seen as the Lax pair for a coupled system of differential equations of order $2n$ for the operators U, V. These equations can be seen as a noncommutative (operator-valued) coupled analogue of the Painlevé II hierarchy. We write the equations for the first values of n in the example below.

Example 7.3.8. *For $n = 1$ the coupled system of differential equations for the operators U, V on $\mathcal{H}_1$ given in* (7.3.26) *reads as*

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = (2UV + M_t)U, \\ \\ \frac{\partial^2 V}{\partial t^2} = V(2UV + M_t). \end{cases}$$

While for $n = 2$ it reads as

$$\begin{cases} \frac{\partial^4 U}{\partial t^4} = -6UVUVU + 4\frac{\partial^2 U}{\partial t^2}VU + 4UV\frac{\partial^2 U}{\partial t^2} + 2U\frac{\partial^2 V}{\partial t^2}U + 2\frac{\partial U}{\partial t}\frac{\partial V}{\partial t}U + 2U\frac{\partial V}{\partial t}\frac{\partial U}{\partial t} + 6\frac{\partial U}{\partial t}V\frac{\partial U}{\partial t} + M_tU, \\ \\ \frac{\partial^4 V}{\partial t^4} = -6VUVUV + 4\frac{\partial^2 V}{\partial t^2}UV + 4VU\frac{\partial^2 V}{\partial t^2} + 2V\frac{\partial^2 U}{\partial t^2}V + 2\frac{\partial V}{\partial t}\frac{\partial U}{\partial t}V + 2V\frac{\partial U}{\partial t}\frac{\partial V}{\partial t} + 6\frac{\partial V}{\partial t}U\frac{\partial V}{\partial t} + VM_t. \end{cases}$$

In order to see that the system (7.3.6) is actually the Lax pair for the integro-differential Painlevé II hierarchy, we still have some work to do.

7.4 The derivation of the integro-differential Painlevé II hierarchy

In this last section we are first going to show that the Lax pair (7.3.6) naturally encodes the integro-differential Painlevé II hierarchy introduced at the beginning of the chapter in (7.0.4). After that, we finally complete the proof of Theorem 7.0.7.
In order to recognize the integro-differential Painlevé II hierarchy behind the compatibility condition (7.3.14), the idea is simply to look at the compatibility condition (7.3.14) at the level of the kernels of the operators involved U, V, A_k^{ij}, instead of the operators themselves. In doing so, we can prove a fundamental symmetry property of the kernels of the off-diagonal operators A_k^{ij}.

Lemma 7.4.1. *Let $k \in \{1, \dots, 2n\}$, then $A_k^{12}(x, y)$ and $A_k^{21}(y, x)$ are y-independent and we have*

$$A_k^{12}(x, y) = (-1)^k A_k^{21}(y, x), \qquad (x, y) \in \mathbb{R}^2. \tag{7.4.1}$$

Proof. We have $A_1^{12}(x,y) = -\mathrm{i}U(x,y)$ and $A_1^{21}(y,x) = \mathrm{i}V(y,x)$ by (7.3.23). Using Proposition 7.2.17, we thus obtain (7.4.1) for $k=1$ and since

$$U(x,y) = \lambda^{\frac{1}{2}} \int_{\Sigma} n_1(\eta|x) k_2(\eta|y)\,\mathrm{d}\eta,$$

the y-independence of $A_1^{12}(x,y)$ directly follows from the definitions of the functions n_1, k_2, see (7.1.33). But $U(x,y) = V(y,x)$, so the y-independence of $A_1^{21}(y,x)$ follows similarly. Proceeding inductively, we assume that the claims have been proven for $k \in \{1,\dots,m\}$ and some $1 \le m \le 2n-1$. Since by (7.3.27),

$$\begin{aligned} A_{k+1}^{12}(x,y) =& \mathrm{i}\frac{\partial A_k^{12}}{\partial t}(x,y) - \mathrm{i}\int_{\mathbb{R}} U(x,z) \int^t \int_{\mathbb{R}} \Big(V(z,w)A_k^{12}(w,y) + A_k^{21}(z,w)U(w,y)\Big)\mathrm{d}\sigma(w)\mathrm{d}t\,\mathrm{d}\sigma(z) \\ &- \mathrm{i}\int^t \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(U(x,z)A_k^{21}(z,w) + A_k^{12}(x,z)V(z,w)\Big)U(w,y)\,\mathrm{d}\sigma(z)\mathrm{d}\sigma(w)\mathrm{d}t \end{aligned}$$

we see that $A_{k+1}^{12}(x,y)$ is y-independent by the induction hypothesis and base case. Moreover, using explicitly the induction hypothesis in the form $A_k^{12}(x,y) = (-1)^k A_k^{21}(y,x)$, we obtain

$$\begin{aligned} A_{k+1}^{12}(x,y) =& (-1)^{k+1}\Bigg[- \mathrm{i}\frac{\partial A_k^{21}}{\partial t}(y,x) + \mathrm{i}\int_{\mathbb{R}} U(x,z) \int^t \int_{\mathbb{R}} \Big(V(z,w)A_k^{21}(y,w) + A_k^{12}(w,z)U(w,y)\Big) \times \\ &\times \mathrm{d}\sigma(w)\mathrm{d}t\,\mathrm{d}\sigma(z) + \mathrm{i}\int^t \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(U(x,z)A_k^{12}(w,z) + A_k^{21}(z,x)V(z,w)\Big)U(w,y)\,\mathrm{d}\sigma(z)\mathrm{d}\sigma(w)\mathrm{d}t\Bigg]. \end{aligned} \tag{7.4.2}$$

On the other hand, (7.3.27) also says

$$\begin{aligned} A_{k+1}^{21}(x,y) =& -\mathrm{i}\frac{\partial A_k^{21}}{\partial t}(x,y) + \mathrm{i}\int_{\mathbb{R}} V(x,z) \int^t \int_{\mathbb{R}} \Big(U(z,w)A_k^{21}(w,y) + A_k^{12}(z,w)V(w,y)\Big)\mathrm{d}\sigma(w)\mathrm{d}t\,\mathrm{d}\sigma(z) \\ &+ \mathrm{i}\int^t \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(V(x,z)A_k^{12}(z,w) + A_k^{21}(x,z)U(z,w)\Big)V(w,y)\,\mathrm{d}\sigma(w)\mathrm{d}\sigma(z)\mathrm{d}t, \end{aligned}$$

and thus $A_{k+1}^{21}(x,y)$ is x-independent by the induction hypothesis and base case. Finally, relabelling the integration variables $z \leftrightarrow w$ in the last equality and using the induction base case six times in the form $U(x,y) = V(y,x)$ we see at once that with (7.4.2),

$$A_{k+1}^{21}(x,y) = (-1)^{k+1} A_{k+1}^{12}(y,x), \quad (x,y) \in \mathbb{R}^2.$$

■

This Lemma is the key to simplify the equations given from the compatibility condition and resumed in Corollary 7.3.7. Indeed, by defining the following functions

$$\begin{aligned} u(t|x) &:= U(x,x) = U(x,y) = V(y,x) = V(x,x), \\ a_k(t|x) &:= A_k^{12}(x,x) = (-1)^k A_k^{21}(x,x), \end{aligned} \tag{7.4.3}$$

for all $(t,x)\in\mathbb{R}^2$, the recursion for the operators A_k^{12} given in (7.3.16) becomes

$$a_{k+1}(t|x)=\begin{cases}(\mathcal{L}_+^u a_k)(t|x), & k\equiv 0 \mod 2\\ (\mathcal{L}_-^u a_k)(t|x), & k\equiv 1 \mod 2\end{cases}, \qquad k=1,2\dots,2n-1; \qquad a_1(t|x):=-\mathrm{i}u(t|x) \tag{7.4.4}$$

where the recursion operators $\mathcal{L}_u^{\pm}$ are given in Definition 7.0.3. Furthermore, the coupled system of differential equations for U,V, that was given in (7.3.26), actually coincides with a unique equation that is now rewritten as

$$-(t+x)a_1(t|x)=(\mathcal{L}_+^u a_{2n})(t|x). \tag{7.4.5}$$

Thus iterating backward the right hand side through (7.4.4) we get

$$-(t+x)a_1(t|x)=\Big((\mathcal{L}_+^u\mathcal{L}_-^u)^n a_1\Big)(t|x)$$

and replacing the initial condition for $a_1(t|x)$, the last equation of the compatibility condition is exactly

$$(t+x)u(t|x)=-\Big((\mathcal{L}_+^u\mathcal{L}_-^u)^n u\Big)(t|x)$$

that is the n-th member of the integro-differential Painlevé II hierarchy.
We are now ready to prove the formula that expresses the Fredholm determinant $D_n(t,\lambda)$ in terms of distinguished solution of the integro-differential Painlevé II hierarchy (7.0.4). We are going to prove it in two steps: first we have this lemma.

Lemma 7.4.2. *For every* $(t,\lambda,n)\in\mathbb{R}\times\overline{\mathbb{D}_1(0)}\times\mathbb{N}$,

$$\frac{\partial}{\partial t}\ln D_n(t,\lambda)=-\mathrm{i}\lambda^{\frac{1}{2}}\operatorname*{Tr}_{\mathcal{H}_1}\int_\Sigma N_1(\xi)\otimes K_1(\xi)\,\mathrm{d}\xi,$$

followed by

$$\frac{\partial^2}{\partial t^2}\ln D_n(t,\lambda)=-\operatorname*{Tr}_{\mathcal{H}_1}(UV)$$

Proof. We start by computing the first t-derivative of the logarithm of $D_n(t,\lambda)$. To do that, we recall equation (7.1.31) and we apply the Jacobi formula

$$\frac{\partial}{\partial t}\ln D_n(t,\lambda)=\frac{\partial}{\partial t}\ln\det(I-\lambda^{\frac{1}{2}}C_{t,n}|_{L^2(\Sigma)})=-\lambda^{\frac{1}{2}}\operatorname*{Tr}_{L^2(\Sigma)}\left[(I-\lambda^{\frac{1}{2}}C_{t,n}|_{L^2(\Sigma)})^{-1}\frac{\partial}{\partial t}C_{t,n}\right]. \tag{7.4.6}$$

Then by using the definition of the operator $C_{t,n}$ given in (7.1.32) we get the kernel derivative

$$\frac{\partial}{\partial t}C_{t,n}(\xi,\eta)=\frac{\mathrm{i}}{2\pi}\int_\mathbb{R}\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi_n(\xi,2t+2z)-\psi_n(\eta,2t+2z))}\chi_{\Gamma_\alpha}(\xi)\chi_{\Gamma_\beta}(\eta)\,\mathrm{d}\sigma(z)=\mathrm{i}\int_\mathbb{R}k_1(\xi|z)m_1(\eta|z)\,\mathrm{d}\sigma(z),$$

where in the last passage we just replaced (7.1.33). Hence back in (7.4.6),

$$\begin{aligned}\frac{\partial}{\partial t}\ln D_n(t,\lambda) &= -\lambda^{\frac{1}{2}}\int_\Sigma\int_\Sigma (I-\lambda^{\frac{1}{2}}C_{t,n})^{-1}(\eta,\xi)\frac{\partial}{\partial t}C_{t,n}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= -\mathrm{i}\lambda^{\frac{1}{2}}\int_{\mathbb{R}}\left[\int_\Sigma\Big(N_1(\xi)\otimes K_1(\xi)\Big)(z,z)\,\mathrm{d}\xi\right]\mathrm{d}\sigma(z) = -\mathrm{i}\lambda^{\frac{1}{2}}\operatorname*{Tr}_{\mathcal{H}_1}\int_\Sigma N_1(\xi)\otimes K_1(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where in the second passage we just used the integral equation (7.2.23). Thus the first identity in the statement holds. We notice that in its right hand side we actually have a multiple of the $\mathcal{H}_1$-trace of the $(1,1)$-entry of the first asymptotic coefficient of the solution of the Riemann-Hilbert problem 7.2.6 $\mathbf{X}(\zeta)$. For the second identity in the statement we need then the t-derivative of this quantity. This is obtained in a classical way, revisiting our proof of Proposition 7.3.3 and explicitly computing the $\mathcal{O}(\zeta^{-1})$ correction when inserting the asymptotic representations of $\mathbf{X}(\zeta)$ and $(\mathbf{X}(\zeta))^{-1}$ into the defining equation of $\mathbf{B}(\zeta)$ in (7.3.10). The same $\mathcal{O}(\zeta^{-1})$ correction has to vanish identically by generalized Liouville's theorem and this yields the operator commutator identity

$$(\mathbf{X}_1)_t = [\mathbf{B}_0,\mathbf{X}_2] - \mathbf{X}_1[\mathbf{B}_0,\mathbf{X}_1],$$

where $\mathbf{B}_0$ is written in (7.3.5). Taking the entry $(1,1)$ of the above identity and using the symmetries proved in Corollary 7.2.15 yields in particular

$$\frac{\partial}{\partial t}\left(\lambda^{\frac{1}{2}}\int_\Sigma N_1(\xi)\otimes K_1(\xi)\,\mathrm{d}\xi\right) = -\mathrm{i}\lambda\int_\Sigma\int_\Sigma\Big(N_1(\eta)\otimes K_2(\eta)\Big)\Big(N_2(\xi)\otimes K_1(\xi)\Big)\,\mathrm{d}\eta\,\mathrm{d}\xi = -\mathrm{i}UV$$

where in the last passage we just split the double integral and recognize the definition of U,V as in (7.2.40). Therefore the second identity holds once derived the first one and replaced the above relation. ■

The last step essentially just require to actually compute the operator trace appearing in the second equation of the above lemma and to compute the asymptotic behavior of the solution $u(t|x)$ of the n-th member of the integro-differential Painlevé II hierarchy (7.0.4).

Lemma 7.4.3. *For every* $(t,\lambda,n)\in\mathbb{R}\times\overline{\mathbb{D}_1(0)}\times\mathbb{N}$,

$$D_n(t,\lambda) = \exp\left[-\int_t^\infty (s-t)\left(\int_{\mathbb{R}} u^2(s|x)\mathrm{d}\sigma(x)\right)\mathrm{d}s\right]. \tag{7.4.7}$$

where $u(t|x)\equiv u(t|x;n,\lambda)$ *solves the dynamical system* (7.0.4) *and it is such that* $u(t|x)\sim\lambda^{\frac{1}{2}}\mathrm{Ai}_{2n+1}(t+x)$ *as* $t\to+\infty$*, pointwise in* $x\in\mathbb{R}$.

Proof. By Lemma 7.4.2,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\ln D_n(t,\lambda) &= -\operatorname*{Tr}_{\mathcal{H}_1}(UV) = -\int_{\mathbb{R}}\int_{\mathbb{R}} U(x,y)V(y,x)\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(x) = -\int_{\mathbb{R}}\int_{\mathbb{R}} U^2(x,y)\,\mathrm{d}\sigma(x)\mathrm{d}\sigma(y) \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}} U^2(x,x)\,\mathrm{d}\sigma(x)\mathrm{d}\sigma(y) = -\int_{\mathbb{R}} u^2(t|x)\,\mathrm{d}\sigma(x),\end{aligned} \tag{7.4.8}$$

where we used the symmetry condition given in Proposition 7.2.17, the definition of $u(t|x)$ and its y-independence and the fact that $\mathrm{d}\sigma$ is a probability measure. However,

$$u(t|x) = \lambda^{\frac{1}{2}} \int_{\Sigma} n_1(\eta|x) k_2(\eta|x)\,\mathrm{d}\eta = \frac{\lambda^{\frac{1}{2}}}{2\pi} \int_{\Gamma_\beta} \mathrm{e}^{-\mathrm{i}\psi_n(\eta,t+x)}\,\mathrm{d}\eta + \lambda \int_{\Sigma} \Big(C_{t,n}^{*} m_1(\cdot|x)\Big)(\eta) k_2(\eta|x)\,\mathrm{d}\eta$$
$$+ \lambda^{\frac{1}{2}} \int_{\Sigma} \Big[n_1(\eta|x) - m_1(\eta|x) - \lambda^{\frac{1}{2}} \Big(C_{t,n}^{*} m_1(\cdot|x)\Big)(\eta) \Big] k_2(\eta|x)\,\mathrm{d}\eta,$$

so by using the integral representation of the n-th Airy function, indeed $u(t|x) \sim \lambda^{\frac{1}{2}} \mathrm{Ai}_{2n+1}(t+x)$ as $t \to +\infty$ once we estimate the two remaining integrals involving $m_1(\cdot|x)$ as in our proof of Corollary 7.2.18 (cfr. [24]). All together, (7.4.7) follows from (7.4.8) after integration since $u(t|x) \sim \lambda^{\frac{1}{2}} \mathrm{Ai}_{2n+1}(t+x)$ yields $\int_{\mathbb{R}} u^2(t|x) \mathrm{d}\sigma(x) \to 0$ exponentially fast as $t \to +\infty$ because of the asymptotic properties of the n-th Airy function and of the weight function w. This completes our proof of the Lemma. ■

Theorem 7.0.7 is finally proved.

Chapter 8

Stokes manifolds and cluster algebras

In this last chapter we discuss some of the original results contained in the joint work with Marco Bertola [15]. The aim of this work is to study the symplectic-Poisson structure of certain Stokes manifolds defined as the monodromy manifolds of a linear system of ODEs with polynomial (sl_N-valued) coefficient of generic degree. In particular, for the case $N = 2$ we found explicit log-canonical coordinates for the symplectic two from, and we studied their relation with the emergent field of cluster algebras. The induced Poisson structure in these coordinates turns out to be the linearization of the Flaschka-Newell Poisson structure, defined almost 40 years ago in their paper [37], where the first concrete example of wild character variety was introduced.
The adjective *wild* here is used to underline the difference with the classical character varieties, involved in the study of the monodromy map for ODEs having only simple poles. Indeed, the monodromy map connects the space of rational matrices, giving the coefficient of a linear system of ODEs, to some representations of the fundamental group of the punctured Riemann sphere. Looking at ODEs with only simple poles, this connection is explained in terms of character varieties of the punctured Riemann sphere. Instead, if the ODEs matrix coefficient has higher order poles, the Stokes phenomena makes the set of monodromy data more complicated, thus complicating the studying of the monodromy map. The new geometrical object arising in this study goes under the name of wild character variety. The interest in its Poisson structure comes naturally from the following fact. On the side of the ODEs, there is a well known Lie-Poisson structure defined on the space of coefficient matrices. It seems natural to ask whether and how the monodromy map "transfers" this structure on the relevant monodromy manifold. The pioneering works addressing this question were first the already cited [37] and then the one of Ugaglia [107], who studied the case of rank N ODEs with a simple pole at 0 and a pole at ∞ with Poincaré rank 2. Concerning the general Fuchsian case, it was shown in [73] that the Lie-Poisson structure on the space of coefficient matrices induces the Goldmann Poisson structure (the classical Poisson structure on character varieties [49]) on the space of monodromy matrices. For what concerns the irregular case instead, it was the series of papers of Boalch [17, 18, 19] that provided in very

a general setting, the description of the Poisson-symplectic structure on the space of extended monodromy data (and re-derived the cases studied by Flaschka-Newell and Ugaglia).
For us, the Stokes manifold of interest $\mathfrak{S}_K$, is the following algebraic variety

$$\mathfrak{S}_K = \left\{ \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_{2K+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{2K+2} & 1 \end{pmatrix} \lambda^{\sigma_3} = \mathbf{1} \quad \text{with } s_i \in \mathbb{C}, \quad \lambda \in \mathbb{C}^\times \right\} \tag{8.0.1}$$

of complex dimension $2K$, for every $K \geq 1$. We proved in two different ways that $\mathfrak{S}_K$ is indeed a symplectic manifold, with symplectic 2-form given by

$$\mathcal{W}_K := \frac{1}{2} \sum_{\ell=1}^{2K+3} \mathrm{Tr} \left(H_\ell^{-1} \mathrm{d} H_\ell \wedge S_\ell^{-1} \mathrm{d} S_\ell \right), \quad H_\ell := S_1 \cdots S_\ell, \quad S_{2K+3} := \mathrm{e}^{2i\pi L}, \tag{8.0.2}$$

where S_ℓ, for $\ell = 1, \ldots, 2K+2$ denote the upper and lower triangular matrices with unit diagonal, appearing in equation (8.0.1), and $\mathrm{e}^{2i\pi L} = \lambda^{\sigma_3}$, for the rank 2 case. In one way, we proved that the 2-form (8.0.2) has pull-back (via the monodromy map) that coincides with the "universal symplectic structure" of Krichever and Phong [77], [76], (induced by the Poisson-Lie structure on the space of coefficient matrices over its symplectic leafs) thus providing that $\mathcal{W}_K$ is symplectic. In the other one way we built, for the case of rank $N = 2$, explicit coordinates $y_i, i = 1, \ldots, 2K$ that parametrize the Stokes manifolds $\mathfrak{S}_K$ as (see Lemma 8.2.5)

$$s_1 = -y_1^{-2}, \quad s_{2k+1} = -(1+y_{2k+1}^2) \prod_{1 \leq j \leq 2k+1} y_j^{(-1)^j 2}, \quad k = 1, \ldots, K-1, \quad s_{2K+1} = -\prod_{1 \leq j \leq 2K} y_j^{(-1)^j 2},$$

$$s_{2k} = (1+y_{2k}^2) \prod_{1 \leq j \leq 2k} y_j^{(-1)^{j+1} 2}, \quad k = 1, \ldots, K, \quad s_{2K+2} = y_1^2 \left(1 + y_2^2 \left(\ldots \left(1 + y_{2K}^2\right) \ldots \right)\right) \prod_{j=1}^{K} y_{2j}^{-4},$$

$$\lambda = (-1)^K \prod_{j=1}^{K} y_{2j}^2.$$

As a byproduct the form $\mathcal{W}_K$ is expressed in log-canonical form within these variables and it is in particular non-degenerate. Moreover, its associated Poisson bracket (Lemma 8.2.7) is described by this constant coefficient matrix (for the logarithms of the coordinates y_i)

$$\mathbf{P}_K = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & & -1 & 0 & 1 \\ 0 & 0 & \ldots & & 0 & -1 & 0 \end{pmatrix}. \tag{8.0.3}$$

The construction of the log-canonical variables y_i is based on the choice of a certain triangulation of a $2(K+1)$ regular polygon, in a way similar to the one used for the Grasmannian of 2-planes (in [46], Chapter II). The explicit computation of the 2-form $\mathcal{W}_K$ follows instead the techniques developed in the recent work [14], relying on the theory of standard 2-forms associated to oriented graph with connection. The connection with cluster algebras comes from the simple observation that the matrix $\mathbf{P}_K$ is (up to a constant factor) the matrix representing the simple quiver of type A_{2K} (with prescribed orientation); this means that the variables y_j^2 form a *seed* for the cluster algebra of type A_{2K}. To complete the picture we need to show that different choices of triangulations of the regular $(2K+2)$–gon yield parametrizations of the Stokes' data that are obtained from the initial seed by applying a suitable sequence of *mutations*, i.e. simple birational maps from one chart to another (see the subsection 8.3.1). The appearance of cluster algebras in this kind of context is not surprising: in the last decades the works of Fock and Goncharov [38] already shown the deep connection between cluster algebras and the geometry of character varieties. Thus similar connections should be expected to appear also in the context of wild character varieties.
Finally, the Flaschka-Newell Poisson bracket defined for the original monodromy parameters describing $\mathfrak{S}_K$, namely

$$\begin{aligned} \Big\{s_j, s_l\Big\}_{FN} &= \delta_{j,l-1} - \frac{\delta_{j,1}\delta_{l,2K+2}}{\lambda^2} + (-1)^{j-l+1} s_j s_l, \qquad j < l. \\ \Big\{s_j, \lambda\Big\}_{FN} &= (-1)^j s_j \lambda. \end{aligned} \tag{8.0.4}$$

is showed to coincide with the Poisson bracket described above in Theorem 8.4.3, under the parametrization given in (8.0.3). All these results can be resumed in the following compact statement

Theorem 8.0.1. *The wild character variety of an sl_2 polynomial connection of degree K on the Riemann sphere is a cluster manifold of type A_{2K} with one frozen variable. The log–canonical Poisson (symplectic) structure on this cluster variety coincides with the push–forward by the monodromy map of the Lie-Poisson structure.*

The Chapter is organized as follows: in the first section we describe the symplectic structure on the space of rational polynomial matrices, and we prove its relation with the symplectic structure on the Stokes manifolds. In the second section we analyze the rank 2 case and we construct the log canonical coordinates for the symplectic 2-form on the Stokes manifolds. In the third section we study the connection between these log-canonical coordinates and cluster algebras. Finally the last section is devoted to recover the original Flaschka-Newell Poisson structure from the linearized one in the coordinates y_i for the Stokes manifold.

8.1 Symplectic structure on the Stokes matrices

Consider a polynomial ODE of the form

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi, \quad A(\lambda) := \sum_{j=1}^{K} A_j \lambda^j. \tag{8.1.1}$$

For the sake of this discussion we can consider the case of $N \times N$ matrices (without real loss of generality, we consider the sl_N case with $\mathrm{Tr}(A(\lambda)) \equiv 0$). Keeping in mind that all the results can be extended to an arbitrary semisimple Lie algebra. We assume that A_K has simple eigenvalues (i.e. it is regular semisimple). Under this hypothesis, using Theorem 5.1.3, one can find a solution in the class of formal series of the form

$$\Psi_{form}(\lambda) = \widehat{Y}(\lambda)\lambda^{-L}\mathrm{e}^{T(\lambda)}, \quad \widehat{Y}(\lambda) := G_0\left(\mathbf{1} + \sum_{j\geq 1}\frac{Y_j}{\lambda^j}\right) \in SL_N[[\lambda^{-1}]], \tag{8.1.2}$$

where G_0 is a chosen diagonalizing matrix for A_K and $L, T(\lambda)$ are diagonal traceless matrices. In this case, the entries of L are the *formal monodromy exponents* and the matrix T is a polynomial of the form

$$T(\lambda) = T_{K+1}\frac{\lambda^{K+1}}{K+1} + \cdots + T_1\lambda, \quad T_j \in \mathfrak{h}, \tag{8.1.3}$$

where $\mathfrak{h}$ denotes the Cartan subalgebra of sl_N, namely diagonal traceless matrices. The coefficients of $T(\lambda)$ are the (higher formal) *Birkhoff invariants*. The matrix T_{K+1} is the diagonal form of the leading coefficient A_K, so that

$$A_K = G_0 T_{K+1} G_0^{-1}. \tag{8.1.4}$$

Poisson structure on the space of matrices $A(\lambda)$. The Lie-Poisson structure on the set of rational matrices can be expressed as (for a review see [6])

$$\{A(\lambda) \overset{\otimes}{,} A(\mu)\} = \left[\frac{\Pi}{\lambda-\mu}, \overset{1}{A}(\lambda) + \overset{2}{A}(\mu)\right] \tag{8.1.5}$$

where $A^1(\lambda) := A(\lambda) \otimes \mathbf{1}$, $A^2(\mu) := \mathbf{1} \otimes A(\mu)$ and $\Pi : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ is the tensor effecting the flip:

$$\Pi(v \otimes f) = f \otimes v, \quad v, f \in \mathbb{C}^n. \tag{8.1.6}$$

It can be explicitly written as $\Pi = \sum_{k,j=1}^n \mathbb{E}_{k,j} \otimes \mathbb{E}_{j,k}$, with $\mathbb{E}_{ij}$ the elementary matrices. In our case $A(\lambda)$ is a polynomial; the matrix A_K is easily seen to consist entirely of Casimir functions for this Poisson structure. The symplectic leaves are thus described; let $G(\lambda)$ be the matrix of eigenvectors for $A(\lambda)$ of the form

$$G(\lambda) = G_0\left(\mathbf{1} + \sum_{j\geq 1} \frac{B_j}{\lambda^j}\right). \tag{8.1.7}$$

(The Laurent series has a finite radius of convergence). Then

$$A(\lambda) = G(\lambda)D(\lambda)G(\lambda)^{-1}, \quad D(\lambda) = T_{K+1}\lambda^K + \dots + T_1 - \frac{L}{\lambda} + \dots \tag{8.1.8}$$

where the matrices T_j are all diagonal traceless matrices; as the choice of letters suggests, they coincide (a simple exercise) with the Birkhoff invariants and the exponents of formal monodromy, while the rest of the Laurent tail plays no role in our present considerations. Then the Casimir functions are $T_1, \dots, T_{K+1}$ and $A_K = G_0 T_{K+1} G_0^{-1}$ (see also [6], Ch. III).

On the symplectic leaves, the Poisson structure (8.1.5) has the form of the "universal symplectic structure" of Krichever and Phong [77], [76]:

$$\begin{aligned}\omega_{KK} &= -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(D(\lambda)G(\lambda)^{-1}\delta G(\lambda)\wedge G(\lambda)^{-1}\delta G(\lambda)\right)\mathrm{d}\lambda \\ &= -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(A(\lambda)\delta G(\lambda)G(\lambda)^{-1}\wedge \delta G(\lambda)G(\lambda)^{-1}\right)\mathrm{d}\lambda \end{aligned} \tag{8.1.9}$$

The two-form is invariant under gauge action of right multiplication of G by diagonal matrices of the form

$$F(\lambda) = \mathbf{1} + \sum_{j\geq 1}\frac{F_j}{\lambda^j} \quad \in \quad \mathfrak{h}[[\lambda^{-1}]]. \tag{8.1.10}$$

To see this we introduce the symplectic potential

$$\theta := \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(D(\lambda)G(\lambda)^{-1}\delta G(\lambda)\right) \tag{8.1.11}$$

which has the property that $\delta\theta = \omega_{KK}$. Now observe that under the gauge transformation $G(\lambda) \mapsto G(\lambda)F(\lambda)$ we have

$$\theta \mapsto \theta + \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(D(\lambda)F^{-1}(\lambda)\delta F(\lambda)\right)\mathrm{d}\lambda. \tag{8.1.12}$$

In the latter term, since $F(\lambda) = \mathbf{1} + \mathcal{O}(\lambda^{-1})$ only the non-negative powers of $D(\lambda)$ contribute (since $F^{-1}(\lambda)\delta F(\lambda) = \mathcal{O}(\lambda^{-1})$). Given that the parameters $T_1, \dots, T_{K+1}$ in (8.1.8) are constants, we can express the last term in (8.1.12) as the total derivative of the function

$$\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(D(\lambda)F^{-1}(\lambda)\delta F(\lambda)\right)\mathrm{d}\lambda = \delta \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(D(\lambda)\ln F(\lambda)\right)\mathrm{d}\lambda, \tag{8.1.13}$$

which implies that $\omega_{_{KK}} = \delta\theta$ is indeed invariant. It is also invariant under left multiplication $G(\lambda) \mapsto HG(\lambda)$ with H a constant (in λ): indeed, the left multiplication by a constant matrix H leaves θ completely invariant:

$$\theta \mapsto \theta + \operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(G(\lambda)D(\lambda)G^{-1}(\lambda)H^{-1}\delta H\right)\mathrm{d}\lambda = \theta \tag{8.1.14}$$

where we have used that $G(\lambda)D(\lambda)G^{-1}(\lambda) = A(\lambda)$ is a polynomial.

The core of the idea of the "extended coadjoint orbit" of [19] is the following: while $A_K = G_0 T_{K+1} G_0^{-1}$ is a Casimir for the KKS symplectic structure, G_0 itself is not because right multiplications by a *constant* diagonal matrix do not leave the symplectic form invariant.

Thus we allow G_0 to be kinematical variables: fix the Birkhoff invariants $T(\lambda) = \sum_{j=1}^{K+1} T_j \lambda^j / j$ (i.e. the diagonal traceless matrices $T_1, \ldots, T_{K+1}$) and consider the set

$$\widehat{\mathcal{O}}_T := \left\{(G_0, A(\lambda)) \in SL_N \times \mathcal{A}_K : \quad G_0^{-1} A_K G_0 = T_{K+1}, \quad (G(\lambda)^{-1}A(\lambda)G(\lambda))_+ = T'(\lambda)\right\}, \tag{8.1.15}$$

where $()_+$ denotes the Taylor part of a Laurent series (here is a polynomial part).

The dimension of $\widehat{\mathcal{O}}_T$ is

$$\dim_{\mathbb{C}}\left(\widehat{\mathcal{O}}_T\right) = (K+1)(N^2-1) + (N-1) - (K+1)(N-1) = KN(N-1) + N^2 - 1 \tag{8.1.16}$$

The extended orbit $\widehat{\mathcal{O}}_T$ carries the following SL_N–action:

$$(G_0, A(\lambda)) \mapsto (HG_0, HA(\lambda)H^{-1}), \quad H \in SL_N. \tag{8.1.17}$$

Then the quotient $\widehat{\mathcal{O}_T}/SL_N$ is a symplectic manifold of dimension $KN(N-1) = \dim_{\mathbb{C}} \mathfrak{S}_K$.

In order to connect the Lie–Poisson structure with the Flaschka–Newell structure on the Stokes' matrices we need first a lemma and to justify the definition of Stokes manifolds given in (8.0.1).

Lemma 8.1.1. *The first $K+1$ coefficient matrices $Y_1, \ldots, Y_{K+1}$ in the expansion of the* *formal solution* Ψ_{form} (8.1.2) *coincide with the expansion of the eigenvector matrix, to wit*

$$\widehat{Y}(\lambda) := G_0\left(\mathbf{1} + \sum_{j\geq 1}\frac{Y_j}{\lambda^j}\right) = G(\lambda) + \mathcal{O}(\lambda^{-K-2}). \tag{8.1.18}$$

Proof. The formal series $\widehat{Y}$ satisfies the ODE

$$\widehat{Y}'(\lambda) + \widehat{Y}(\lambda)\left(T'(\lambda) - \frac{L}{\lambda}\right) = A(\lambda)\widehat{Y}(\lambda), \tag{8.1.19}$$

where we abbreviated with $'$ the derivation w.r.t. λ. Since $\widehat{Y}'(\lambda) = \mathcal{O}(\lambda^{-2})$, the matrices $T(\lambda), L$ are diagonal and since the degree of A is K we deduce that $\widehat{Y}$ matches the Laurent expansion of the eigenvector matrix $G(\lambda)$ up to the indicated order. ∎

Description of the Stokes manifolds. Recall the results stated in Section 5.1 about the behavior of local solutions of linear ODEs near singular points, in particular Theorem 5.1.4. In our case of study, namely equation (8.1.1), there is only one pole at ∞ of Poincaré rank $K+1$ for each $K \geq 1$. Thus the complex plane can be partitioned into $2K+2$ canonical Stokes sectors of equal angular width $\mathcal{S}_\mu$, arranged in counterclockwise order. Within each such sector, Theorem 5.1.4 assures that there exists a unique analytic solution $\Psi_\mu(\lambda)$ to the ODE (8.1.1) such that

$$\Psi_\mu(\lambda) \simeq \Psi_{form}(\lambda), \quad |\lambda| \to \infty, \;\; \arg\lambda \in \mathcal{S}_\mu, \tag{8.1.20}$$

with Ψ_{form} given in (8.1.2). In these asymptotics, the determination of the matrix of formal exponents λ^L is the same, –say– the principal one. In this setting, we have then $2K+2$ Stokes' matrices S_μ as defined in (5.1.11); if the entries $t_1, \ldots, t_n$ of T_{K+1} are arranged in increasing order of $\Re(t_j e^{\theta_0})$ (for a generic θ_0 so that this order is unique), then the Stokes' matrices are all triangular matrices with unit diagonal, namely they belong to $N_\pm \subset SL_n$. Specifically, they alternate the triangularity as we move counterclockwise.

The entries of these matrices are not independent; they must satisfy the monodromy relation

$$S_1 S_2 \cdots S_{2K+2} e^{2i\pi L} = \mathbf{1} \tag{8.1.21}$$

which is a consequence of the fact that the ODE has no singularities in the finite part of the plane and therefore each of the solutions Ψ_μ extends uniquely to an entire matrix–valued function. We thus define the Stokes' manifold as the set of these data:

Definition 8.1.2. *The Stokes' manifold is the following set*

$$\mathfrak{S}_K := \left\{(S_1, \ldots, S_{2K+2}, L) \in (N_+ \times N_-)^{K+1} \times \mathfrak{h} : \quad S_1 \cdots S_{2K+2} e^{2i\pi L} = \mathbf{1}.\right\} \tag{8.1.22}$$

where $N_\pm$ denote the solvable subgroups of upper/lower triangular matrices with ones on the diagonal and $\mathfrak{h}$ denotes the subalgebra of diagonal traceless matrices. The dimension of this manifold is

$$\dim_{\mathbb{C}}(\mathfrak{S}_K) = KN(N-1). \tag{8.1.23}$$

It is apparent that the dimension is even; in fact Boalch [19] shows that these type of manifolds are symplectic. We are going to give a self–contained description, adapted to this case, of this structure. In particular, in the next paragraph we are going to prove that for the general N case, the 2-form $\mathcal{W}_K$ defined in (8.0.2) has (up to a constant factor) pull-back that coincides with the symplectic form ω_{KK} written in (8.1.9). Thus implying that $\mathfrak{S}_K$ equipped with $\mathcal{W}_K$ is a symplectic manifold. Then in the next sections, we will treat the case $N = 2$ finding explicit log-canonical coordinates in which $\mathcal{W}_K$ is in non-degenerate form and proving that the induced Poisson bracket indeed coincide with the Flaschka-Newell one, written in equation (8.0.4).

The Malgrange form associated to an analytic family of Riemann Hilbert problems. We describe here the gist of [11, 12]. Suppose that $\Sigma \subset \mathbb{C}$ is a collection of oriented smooth arcs (intersecting transversally) and $J : \Sigma \to SL_N$ a smooth matrix–valued function (the "jump matrix") depending analytically on parameters that we denote collectively by $\mathbf{s}$. As discussed in Section 4.1, this pair of data defines a family of Riemann-Hilbert problems ($\mathbf{s}$ depending). In case the contours Σ has some self-intersections, the matrix $J(z;\mathbf{s})$ must satisfy suitable assumptions (see [12] for details). The most important one for the description here is the "local monodromy free" condition: let v be a "vertex" of the graph, namely, a point of intersection of the smooth arcs of Σ. Let $e_1, \dots e_n$ be the sub-arcs of Σ entering a small disk $\mathbb{D}_v$ centered at v and enumerated counterclockwise from an arbitrarily chosen one. We denote by

$$J_\ell(v;\mathbf{s}) = \lim_{\substack{\lambda\to v\\ \lambda\in e_\ell}} J^{\pm 1}(\lambda;\mathbf{s}), \tag{8.1.24}$$

where the power is $+1$ if the edge e_ℓ is oriented away from v and -1 viceversa. Then the matrices must satisfy

$$J_1(v;\mathbf{s})\cdots J_n(v;\mathbf{s}) = \mathbf{1} \tag{8.1.25}$$

for all the vertices v of Σ, identically with respect to the deformation parameters $\mathbf{s}$. Suppose now that there exists (generically with respect to $\mathbf{s}$) the solution of the Riemann–Hilbert problem*

$$\Gamma_+(\lambda;\mathbf{s}) = \Gamma_-(\lambda;\mathbf{s})J(z;\mathbf{s}), \quad z\in\Sigma, \quad \Gamma(\infty;\mathbf{s})\equiv C_0. \tag{8.1.26}$$

The normalization condition at $\lambda=\infty$ is usually taken to be the identity, but it will be convenient to consider a more general one. Then we recall the definition anticipated at the end of Chapter 4.

Definition 8.1.3. *The Malgrange form is defined by the formula*

$$\Theta_M := \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda;\mathbf{s})\Gamma_-'(z;\mathbf{s})\Xi(\lambda;\mathbf{s})\right)\frac{\mathrm{d}\lambda}{2i\pi} \tag{8.1.27}$$

where $\Xi(\lambda;\mathbf{s}) := \delta J(\lambda;\mathbf{s})J^{-1}(\lambda;\mathbf{s})$ is the Maurer–Cartan form, the prime denotes the differentiation w.r.t. λ and δ is the total differential in the deformation parameters $\mathbf{s}$.

We observe that the Malgrange form Θ_M is independent of the normalization at $\lambda=\infty$, which corresponds to a left multiplication of Γ by a λ–independent matrix. Then one has

Theorem 8.1.4 (Thm. 2.1 in [12])**.** *The exterior derivative of the Malgrange form Θ_M is*

$$\delta\Theta_M = -\frac{1}{2}\int_\Sigma \frac{\mathrm{d}\lambda}{2i\pi}\,\mathrm{Tr}\left(\Xi'(\lambda)\wedge\Xi(\lambda)\right) - \frac{1}{4i\pi}\sum_{v\in\mathbf{V}(\Sigma)}\sum_{\ell=1}^{n_v}\mathrm{Tr}\left(H_\ell^{-1}(v)\delta H_\ell(v)\wedge J_\ell^{-1}(v)\delta J_\ell(v)\right) \tag{8.1.28}$$

where $H_\ell(v) = J_1(v)\cdots J_\ell(v)$ and the matrices $J_\ell(v)$ are defined prior to (8.1.25).†

We now come to the main statement of the section.

Theorem 8.1.5. *The following two-form is a (complex) symplectic structure on $\mathfrak{S}_K$:*

$$\mathcal{W}_K := \frac{1}{2}\sum_{\ell=1}^{2K+3}\mathrm{Tr}\left(H_\ell^{-1}\mathrm{d}H_\ell\wedge S_\ell^{-1}\mathrm{d}S_\ell\right), \quad H_\ell := S_1\cdots S_\ell, \quad S_{2K+3} := \mathrm{e}^{2i\pi L}. \tag{8.1.29}$$

Its pull-back by the (extended) monodromy map coincides with the Lie–Poisson structure (8.1.9) *times $-2i\pi$.*

Before discussing the proof, we point out that this form is written in a different way from [19] (Thm 5, formula (7)) and rather reflects the general theory of "canonical form associated to a graph" developed in [14]. The two expressions (a posteriori) can be verified to give the same two-form when restricted to the constraint (8.1.21). In principle, in our explicit computation in Section 8.2 for the SL_2 case, this theorem is verified *ex post facto*.

*To simplify the mental picture, the reader may assume here that Σ is compact: if some rays extend to infinity, the assumption is that $J(\lambda)$ tends to the identity matrix faster than any power of λ^{-1} as $\lambda\to\infty$, $\lambda\in\Sigma$, so that the RHP can be posed consistently. Details are in [12].

†In loc. cit. the form is presented in a different, but equivalent, way.

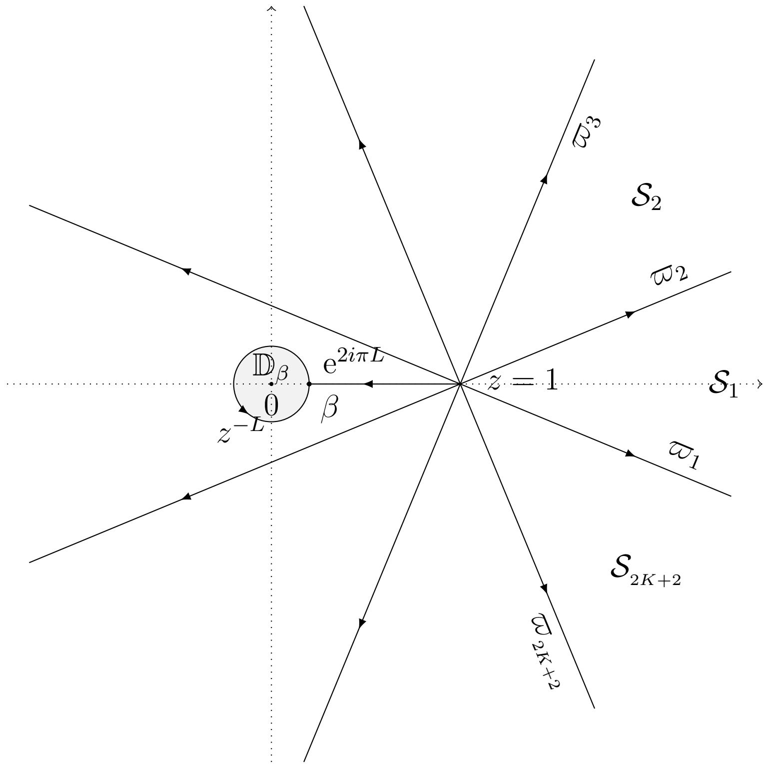

Figure 8.1: An example of Stokes' graph Σ used in Theorem 8.1.5.

Proof. We show that the symplectic form (8.1.9) coincides with the pull-back by the monodromy map of the form $\mathcal{W}_K$ in (8.0.2) and hence showing that the latter is also symplectic (or, to put it more plainly, we write (8.1.9) in the coordinates provided by the Stokes' matrices). The proof here is completely different from [19]; rather than computing the two-form $\mathcal{W}_K$ in the coordinates of the Stokes' matrices, we directly compute the symplectic potential (8.1.11).

Let Σ be graph indicated in Fig. 8.1: the vertex of the star is at $\lambda = 1$ and the small circle is centered at the origin $\lambda = 0$. The Stokes' rays are the lines $\varpi_1, \dots \varpi_{2K+2}$ issuing from $\lambda = 1$ and extending to infinity along the Stokes' directions. In the Fig. 8.1 we have drawn them for the case $K = 3$ under the assumption that the real parts $\Re(it_j)$ are ordered increasingly, so that the Stokes' rays ϖ_ℓ have asymptotic directions $\arg \lambda = \frac{i\pi}{2(K+1)} + \frac{i\pi}{K+1}(\ell - 1)$ and the Stokes' matrix S_1 is then upper triangular.

We now define a piecewise analytic function Γ in each of the connected components of $\mathbb{C} \setminus \Sigma$; in the sector $\mathcal{S}_1$, Γ is given by

$$\Gamma(\lambda) = \Gamma_1(\lambda) := \Psi_1(\lambda) e^{-T(\lambda)+T(1)} \lambda^L, \tag{8.1.30}$$

where the determination of λ^L is the principal one. In the other unbounded components (including the one that contains the disk $\mathbb{D}_r$) the matrix Γ is defined by multiplying $\Gamma_1(\lambda)$

by the jump matrices

$$J_\ell(\lambda) := \mathrm{e}^{T(\lambda)-T(1)}\lambda^{-L} S_\ell \lambda^L \mathrm{e}^{-T(\lambda)+T(1)}, \quad \lambda \in \varpi_\ell. \tag{8.1.31}$$

The triangularity of S_ℓ is such that $J_\ell(\lambda) = \mathbf{1} + \mathcal{O}(\lambda^{-\infty})$ as $|\lambda| \to \infty,\ \lambda \in \varpi_\ell$. Within the disk $\mathbb{D}_r$ we define

$$\Gamma(\lambda) = \Gamma_0(\lambda) := \Gamma_{j_0}(\lambda)\lambda^{-L} = \Psi_{j_0}(\lambda)\mathrm{e}^{T(1)-T(\lambda)}, \tag{8.1.32}$$

where j_0 is the index of the sector containing $\mathbb{D}_\beta$. Note that Γ_0 is locally analytic near $\lambda = 0$.

In the sector containing the disk $\mathbb{D}_\beta$ the matrix Γ does not have a jump on the ray $(-\infty, -\beta]$ because of the monodromy relation (8.1.21) and combined with the monodromy of the factor λ^L. There is, however the jump $\Lambda = \mathrm{e}^{2i\pi L}$ on the segment $[\beta, 1]$. A straightforward exercise shows that the piecewise analytic matrix function Γ satisfies a RHP on the graph Σ shown in Fig. 8.1:

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)J(\lambda), \quad \lambda \in \Sigma, \qquad\qquad \Gamma(\lambda) \simeq \mathrm{e}^{T(1)}\widehat{Y}(\lambda), \quad |\lambda| \to \infty, \tag{8.1.33}$$

where $\simeq$ denotes the asymptotic equivalence in the Poincaré sense, $\widehat{Y}(\lambda)$ is the formal series as in Lemma 8.1.1 and the jump matrix $J(\lambda)$ is given by

$$J(\lambda) = \begin{cases} J_\ell(\lambda) & \lambda \in \varpi_\ell \quad \text{(see (8.1.31))} \\ \lambda^{-L} & \lambda \in \partial\mathbb{D}_\beta. \end{cases} \tag{8.1.34}$$

The jump matrix on $\partial\mathbb{D}_\beta$ is the function λ^{-L} and the determination is (recall that $\beta \in \mathbb{R}_+$) with $\arg\lambda \in [0, 2\pi)$, which is not the same used earlier but we do not want to overload the notation by using a different symbol for the power. Using Lemma 8.1.1 we can write the symplectic potential (8.1.11) as the formal residue

$$\begin{aligned} \theta &= \operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(A(\lambda)\delta G(\lambda)G^{-1}(\lambda)\right)\mathrm{d}\lambda = \text{“}\operatorname*{res}_{\lambda=\infty}\text{''} \operatorname{Tr}\left(A(\lambda)\delta\widehat{Y}(\lambda)\widehat{Y}^{-1}(\lambda)\right)\mathrm{d}\lambda = \\ &= \operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(A(\lambda)\delta\Gamma(\lambda)\Gamma^{-1}(\lambda) - \Gamma^{-1}(\lambda)A(\lambda)\Gamma(\lambda)\delta T(1)\right)\mathrm{d}\lambda. \end{aligned} \tag{8.1.35}$$

Since the expansion at ∞ of Γ coincides with that of the eigenvectors up to order λ^{-K-1} (included), the second term in the residue yields (recall that $\operatorname{res}_{\lambda=\infty}$ extracts the coefficient of λ^{-1} with a *minus* sign)

$$-\operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\left(T' - \frac{L}{\lambda}\right)\delta T(1)\right)\mathrm{d}\lambda = -\operatorname{Tr}(L\delta T(1)). \tag{8.1.36}$$

The first term in (8.1.35) is a formal residue and can be realized as the following limit of an actual integral

$$\lim_{r\to\infty}\oint_{|\lambda|=r}\frac{\mathrm{d}\lambda}{2i\pi}\operatorname{Tr}\left(A(\lambda)\delta\Gamma\Gamma^{-1}\right) \tag{8.1.37}$$

where the contour runs counterclockwise. Note that the integrand is actually an analytic function defined piecewisely for each sector. Applying Cauchy's theorem, we can reduce the integration along the support of the jumps of Γ and we obtain

$$\theta=\int_\Sigma\frac{\mathrm{d}\lambda}{2i\pi}\operatorname{Tr}\left(A(\lambda)\Delta_\Sigma(\delta\Gamma\Gamma^{-1})\right)-\operatorname{Tr}\left(L\delta T(1)\right) \tag{8.1.38}$$

where Δ_Σ is the jump operator $\Delta_\Sigma F(\lambda)=F_+(\lambda)-F_-(\lambda)$, $\lambda\in\Sigma$. Now observe that

$$\Gamma_+=\Gamma_-J \quad\Rightarrow\quad \delta\Gamma_+=\delta\Gamma_-J+\Gamma_-\delta J \quad\Rightarrow\quad \delta\Gamma_+\Gamma_+^{-1}=\delta\Gamma_-\Gamma_-^{-1}+\Gamma_-\delta JJ^{-1}\Gamma_-^{-1}. \tag{8.1.39}$$

and hence we have

$$\Delta_\Sigma(\delta\Gamma\Gamma^{-1})=\Gamma_-\delta JJ^{-1}\Gamma_-^{-1}. \tag{8.1.40}$$

Plugging (8.1.40) into (8.1.38) gives

$$\theta=\int_\Sigma\frac{\mathrm{d}\lambda}{2i\pi}\operatorname{Tr}\left(\Gamma_-^{-1}A\Gamma_-\delta JJ^{-1}\right)-\operatorname{Tr}\left(L\delta T(1)\right). \tag{8.1.41}$$

The above expression suggest a relationship with the Malgrange form Θ_M in Def. 8.1.3 which we now investigate. Using the definition $\Gamma(\lambda)=\Psi(\lambda)\mathrm{e}^{T(1)-T(\lambda)}\lambda^L$ (piecewise sectorially), we find that

$$A(\lambda)\Gamma(\lambda)=\Psi'(\lambda)\mathrm{e}^{T(1)-T(\lambda)}\lambda^L=\Gamma'(\lambda)+\Gamma\left(T'(\lambda)-\frac{L}{\lambda}\right). \tag{8.1.42}$$

Thus the expression (8.1.41) is recast into:

$$\theta=\int_\Sigma\frac{\mathrm{d}\lambda}{2i\pi}\operatorname{Tr}\left(\Gamma_-^{-1}\Gamma_-'\delta JJ^{-1}\right)+\int_\Sigma\frac{\mathrm{d}\lambda}{2i\pi}\operatorname{Tr}\left(\left(T'(\lambda)-\frac{L}{\lambda}\right)\delta JJ^{-1}\right)-\operatorname{Tr}\left(L\delta T(1)\right) \tag{8.1.43}$$

The integrand in the second integral is zero on each of the Stokes' rays ϖ_ℓ because the matrices $\delta J_\ell J_\ell^{-1}$ are strictly triangular (upper or lower), with zeros on the diagonal and

L, T' are diagonal, so that the product is diagonal-free. Thus the second integral reduces to

$$\int_\Sigma \frac{\mathrm{d}\lambda}{2i\pi} \operatorname{Tr}\left(\left(T'(\lambda) - \frac{L}{\lambda}\right)\delta J J^{-1}\right) =$$
$$= \int_\beta^{\beta \mathrm{e}^{2i\pi}} \frac{\mathrm{d}\lambda}{2i\pi} \operatorname{Tr}\left(\left(T'(\lambda) - \frac{L}{\lambda}\right)\left(-\delta T(1) - \delta L \ln\lambda\right)\right) + \int_1^\beta \left(\left(T'(\lambda) - \frac{L}{\lambda}\right)\delta L\right)\mathrm{d}\lambda =$$
$$= -\sum_{j=1}^{K+1} \frac{\operatorname{Tr}(T_j \delta L)}{2i\pi}\left(\frac{\ln\lambda}{j} - \frac{1}{j^2}\right)\lambda^j \Big|_\beta^{\beta\mathrm{e}^{2i\pi}} + \frac{\delta \operatorname{Tr}(L^2)}{4i\pi}\frac{(\ln z)^2}{2}\Big|_\beta^{\beta\mathrm{e}^{2i\pi}} + \operatorname{Tr}\Big(L\delta T(1)\Big)$$
$$+ \operatorname{Tr}\left(\Big(T(\beta) - T(1)\Big)\delta L - \delta\left(\frac{L^2}{2}\right)\ln\beta\right) =$$
$$= -\operatorname{Tr}\Big(T(1)\delta L\Big) - \frac{i\pi}{2}\delta \operatorname{Tr}\Big(L^2\Big). \tag{8.1.44}$$

Thus we have shown that

$$\theta = \Theta_M - \operatorname{Tr}\Big(T(1)\delta L\Big) - 2i\pi\delta \operatorname{Tr}\Big(L^2\Big) - \operatorname{Tr}\Big(L\delta T(1)\Big) = \Theta_M - \delta \operatorname{Tr}\left(T(1)L + \frac{i\pi}{2}L^2\right). \tag{8.1.45}$$

This means that the Kirillov-Kostant form θ coincides with the Malgrange form up to an exact differential. We now compute the exterior derivative of θ using Theorem 8.1.4. It is clear that the last term in (8.1.45) does not contribute to the exterior differentiation because it is an exact form. The integral in (8.1.28) has no contribution because

- on the rays ϖ_ℓ the integrand is traceless (given the triangularity of the jump matrices (8.1.31));
- on the segment issuing from $\lambda = 1$ and directed to the disk, the matrix Ξ is constant in λ;
- on the boundary of the disk $\Xi'(\lambda) \wedge \Xi(\lambda) = \frac{\ln\lambda}{\lambda}\delta L \wedge \delta L = 0$ since L is diagonal.

Thus we are left only with the contributions from the two vertices of the graph in Fig. 8.1, which are $v_0 = \beta$ and $v = 1$. At v_0 we have three incident edges and the matrices J_1, J_2, J_3 are $J_1 = \mathrm{e}^{2\pi L}$, $J_2 = \beta^L$, $J_3 = \beta^{-L}\mathrm{e}^{-2i\pi L}$. Since they commute, it is easy to see that there is no contribution (each term contains $\delta L \wedge \delta L$, which vanishes identically since L is diagonal).

Thus the only contribution comes from $v = 1$; here the jumps are:

$$J_\ell(v) = S_\ell, \quad \ell = 1, \ldots, 2K+2 \tag{8.1.46}$$

and $J_{2K+3} = \mathrm{e}^{-2i\pi L}$. Then the Theorem 8.1.4 gives precisely (8.0.2) divided by $-2i\pi$. Thus we conclude that $\mathcal{W}_K$ in (8.0.2) is a symplectic form. ■

Remark 8.1.6. *To be explicit, the coordinates on the quotient of the extended orbit* (8.1.15) *are as follows; one writes*

$$G = G_0 \exp\left(\frac{H_1}{z} + \frac{H_2}{z^2} + \cdots + \frac{H_K}{z^K} + \mathcal{O}(z^{-K-1})\right) \tag{8.1.47}$$

where $H_1, \ldots, H_K$ can be chosen diagonal free (i.e. with zeros on the diagonal), using the gauge freedom (8.1.12). *Then the $KN(N-1)$ entries of $H_1, \ldots, H_K$ are the coordinates.*

8.2 Stokes manifolds for $n = 2$

Our goal now is twofold:

1. provide explicit parametrization in terms of patches of free coordinates for the complex manifold $\mathfrak{S}_K$ (8.1.22);

2. show that the coordinates introduced above are log–canonical for the two-form (8.0.2).

We recall here the terminology; a coordinate system $(x_1, \ldots, x_{2n})$ on a symplectic manifold $(\mathcal{M}, \omega)$ is called *log-canonical* if the symplectic form is expressed as follows in the coordinate system

$$\omega(\mathbf{x}) = \sum_{i<j} \omega_{ij} \frac{\mathrm{d}x_i}{x_i} \wedge \frac{\mathrm{d}x_j}{x_j} \tag{8.2.1}$$

with ω_{ij} constants. If P_{ij} denotes the inverse transposed of the matrix ω_{ij} then the Poisson brackets read

$$\{x_i, x_j\} = P_{ij} x_i x_j \quad \text{(no summation)}, \tag{8.2.2}$$

namely the logarithms of the coordinates have constant Poisson brackets amongst themselves (whence the terminology). At this point the problem of finding Darboux coordinates reduces to a simple problem of linear transformation in the logarithmic coordinates to find the canonical symplectic matrix for the Poisson brackets.

We are going to carry out the two steps above in the case of SL_2, which corresponds to the historically first case ever studied in [37]. The higher case can be handled in a similar way but we defer the computation to a later work since it would unnecessarily obfuscate the computation behind a plethora of indices.

As anticipated in the introduction, the Stokes' manifold (8.1.22) specializes for any $K \geq 1$ and $N = 2$ to the following

$$\mathfrak{S}_K = \left\{ \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_{2K+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{2K+2} & 1 \end{pmatrix} \lambda^{\sigma_3} = I_2 \text{ with } s_i \in \mathbb{C}, \quad \lambda \in \mathbb{C}^\times \right\}. \tag{8.2.3}$$

We will denote by S_{2l-1} the upper triangular matrices and by S_{2l} the lower triangular matrices appearing in the equation above for $l = 1, \dots, K+1$.

Remark 8.2.1. *The matrix equation in* (8.2.3) *is equivalent to three algebraically independent scalar equations for the Stokes parameters s_j and the formal monodromy exponent α so that* $\dim(\mathfrak{S}_K) = 2(K+1) + 1 - 3 = 2K$*, as it follows from* (8.1.23) *for $N = 2$.*

8.2.1 Construction of the log-canonical coordinates

We consider on $\mathfrak{S}_K$ the 2-form (8.0.2). Following [14] we introduce some basic definitions and properties of the 2-form associated to a graph embedded in a surface, and we will see that the Stokes 2-form can be conveniently interpreted within that formalism. This is indeed the key in order to compute it explicitly and find the log-canonical coordinates.

Graph theory We briefly recall the definition of the standard 2-form associated to an oriented graph on a surface (we refer to Section 2 of [14] for more details). Let Σ be an oriented graph on a surface, we denote with $\mathbb{V}(\Sigma)$ the set of its vertices, $\mathbb{E}(\Sigma)$ the set of its edges and $\mathbb{F}(\Sigma)$ the set of its faces. A "jump matrix" J is a map from $\mathbb{E}(\Sigma)$ to SL_n with the properties that:

1. for any edge $e \in \mathbb{E}(\Sigma)$ we have
$$J(-e) = J(e)^{-1} \tag{8.2.4}$$
with $-e$ denoting the same edge e with opposite orientation;

2. for any vertex $v \in \mathbb{V}(\Sigma)$ of valence n_v we have that the ordered counterclockwise product of the matrices associated to each edge oriented away from v is the identity. Namely:
$$J(e_1) \dots J(e_{n_v}) = I_n, \tag{8.2.5}$$
where we ordered the edges $e_1, \dots, e_{n_v}$ incident at v then counting them counterclockwise.

To the pair (Σ, J), we can then associate the standard 2-form $\Omega(\Sigma)$ defined hereafter.

Definition 8.2.2. *The standard 2-form $\Omega(\Sigma)$ associated to the graph Σ is defined as follows (we omit explicit reference to the dependence on J from the notation)*

$$\Omega(\Sigma) := \sum_{v \in \mathbb{V}(\Sigma)} \sum_{\ell=1}^{n_v - 1} \mathrm{Tr}\left(\left(H^{(v)}_{[1:\ell]}\right)^{-1} \mathrm{d}H^{(v)}_{[1:\ell]} \wedge \left(J^{(v)}_\ell\right)^{-1} \mathrm{d}J^{(v)}_\ell \right). \tag{8.2.6}$$

where in this formula for any vertex $v \in \mathbb{V}(\Sigma)$ we have taken the incident edges $e_1, \dots, e_{n_v}$ oriented away from v and enumerated in counterclockwise order, starting from any of them.

Here $H^{(v)}_{[1:\ell]} = J_1 \dots J_\ell$ *with* $J_i = J(e_i)$ *for* $i = 1, \dots, n_v$. *Thanks to the property* (8.2.5)*, this 2-form is well defined, namely, independent of the choice of first edge in the cyclic order at each vertex.*

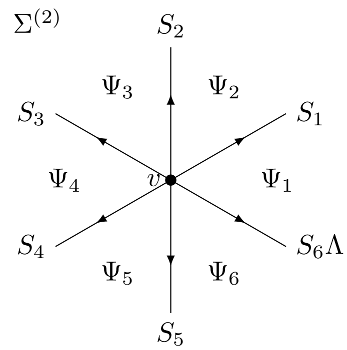

Figure 8.2: The Stokes graph $\Sigma^{(2)}$.

The form $\Omega(\Sigma)$ in Def. 8.2.2 is shown to be invariant under certain transformations $(\Sigma, J) \mapsto (\Sigma', J')$ (called *moves*, see Section 2 of [14]); these moves consist in the self–describing titles of

1. edge contractions;
2. merging edges;
3. attaching edges to vertices (and the converse)

The star-graph for the Stokes' phenomenon. Given the formula (8.0.2) we surmise that the form $\mathcal{W}_K$ can be represented as $2\mathcal{W}_K = \Omega(\Sigma^\star)$ where $\Sigma^\star$ (the "star-graph") is simply the collection of $2K+3$ rays, each carrying the matrices $J_1 := S_1, \dots, J_{2K+2} := S_{2K+2}, J_{2K+3} := \Lambda = \mathrm{e}^{2i\pi L}$ as jumps. We can actually merge the last two rays and corresponding jump matrices to obtain a simpler star-graph $\Sigma^{(K)}$ indicated by the way of example in Fig. 8.2 for $K = 2$. This is not quite one of the generally allowed moves listed in [14] but we now verify directly that it leaves the form invariant. Let thus $\widetilde{J}_\ell = J_\ell,\ \ \ell = 1, \dots, 2K+1$ and $\widetilde{J}_{2K+2} := J_{2K+2} J_{2K+3} = S_{2K+2}\Lambda$. Recall that $S_{2K+2} \in N_-$ and Λ is diagonal. Note that $H_\ell = \widetilde{H}_\ell$ up to $\ell = 2K+1$, while $\widetilde{H}_{2K+2} = H_{2K+2}\Lambda = \mathbf{1}$. Then the difference between the two-forms is

$$\Omega(\Sigma^\star) - \Omega(\Sigma^{(K)}) = \mathrm{Tr}\left(H^{-1}_{2K+2}\mathrm{d}H_{2K+2} \wedge S^{-1}_{2K+2}\mathrm{d}S_{2K+2}\right). \tag{8.2.7}$$

Since $H_{2K+3} = H_{2K+2}\Lambda = \mathbf{1}$ we must have that $H_{2K+2} = \Lambda^{-1}$, namely, it is diagonal. But S_{2K+2} is unipotent triangular and hence $S^{-1}_{2K+2}\mathrm{d}S_{2K+2}$ is strictly lower triangular, so that the matrix in (8.2.7) is diagonal–free and the trace gives zero. Thus, in conclusion, we only

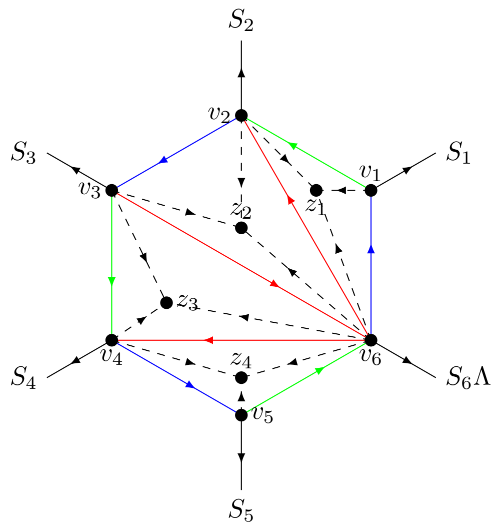

Figure 8.3: The modified graph $\Sigma_0^{(2)}$. Here we take the triangulation T_0 of the hexagon that connects any of its vertices to v_6.

need to analyze the two-form associated to the graphs of the form $\Sigma^{(K)}$ depicted in Fig. 8.2, since we proved that

$$2\mathcal{W}_K = \Omega(\Sigma^{(K)}). \tag{8.2.8}$$

The idea is to realize the simple graph $\Sigma^{(K)}$ as the complete contraction of all the (finite length) edges of another graph with explicit, simple jump matrices that depend on *free* parameters (contrary to the Stokes' parameter that are subject to algebraic relations).

Consider the graph $\Sigma_0^{(K)}$, exemplified in Figure 8.3 for $K = 2$: then it is apparent that $\Sigma^{(K)}$ is the total contraction of $\Sigma_0^{(K)}$. The jump matrices for this graph are described in the following paragraph. The key fact is that the computation of the symplectic form associated to $\Sigma_0^{(K)}$ is then a straightforward exercise.

Since the graphs $\Sigma_0^{(K)}$ and $\Sigma^{(K)}$ are related by the "moves" hinted before and described in [14], the corresponding associated forms coincide: $\Omega\left(\Sigma_0^{(K)}\right) = \Omega\left(\Sigma^{(K)}\right)$. Then, by using the definition of the 2-form associated to a graph, we will compute explicitly the Stokes form, showing directly that it is indeed symplectic.

The graph $\Sigma_0^{(K)}$ and its jump matrices. The graph $\Sigma_0^{(K)}$ (see Fig. 8.3 for the example with $K = 2$) is the graph consisting of $2(K+1)$ infinite rays emanating from the vertices of a regular $2K+2$–gon. The polygon is subdivided into triangles with a common vertex v_{2K+2}. We denote by T_0 this precise triangulation of the polygon. Inside each triangle we have a vertex z_j and three edges from the three vertices bounding the triangle to the vertex z_j. We describe the jump matrices for T_0 with the understanding that, *mutatis mutandis*, the same matrices are defined for an arbitrary triangulation. To each oriented edge of $\Sigma_0^{(K)}$ we associate a matrix that is constant or depends on complex parameters $y_j \in \mathbb{C}^*, j = 1, \ldots, 2K$. The orientation is defined as follows: the perimeter of the polygon is oriented counterclockwise and as for the vertices z_j, each edge is oriented towards the vertex z_j. The internal diagonals of the triangulation are oriented in such a way that for every even perimetric vertex the internal diagonal is exiting from the vertex v_{2K+2} and for every odd vertex the internal diagonal is instead entering in the vertex v_{2K+2}. The Stokes rays are kept with the same orientation as in the Stokes graph. The matrices for each edge are defined as follows:

- on the perimetric edges connecting $v_{2k} \to v_{2k+1}$ for $k = 1, \dots K$ and $v_{2K+2} \to v_1 \sim v_{2K+3}$ (the blue edges in Figure 8.3), we take diagonal matrices of the form

$$D\left(x_{2k}\right) := \begin{pmatrix} x_{2k}^{-1} & 0 \\ 0 & x_{2k} \end{pmatrix}, \tag{8.2.9}$$

  where x_l is the following product of y_j's variables

$$x_l := y_1 \prod_{2 \le k \le l} \prod_{d_j \perp v_k} y_j^{(-1)^{k+1}}, \quad l = 2, \dots, 2K+1, \quad x_{2K+2} := y_1 \prod_{d_j \perp v_1} y_j^{-1} \tag{8.2.10}$$

- on the perimetric edges connecting $v_{2k+1} \to v_{2k+2}$ (the green edges in Figure 8.3), we take off-diagonal matrices of the form

$$V\left(x_{2k+1}^{-1}\right) := \begin{pmatrix} 0 & -x_{2k+1}^{-1} \\ x_{2k+1} & 0 \end{pmatrix}, \tag{8.2.11}$$

  and along the edge $v_1 \to v_2$ we impose the jump matrix $V(y_1^{-1})$;

- on the three edges incident to z_j (each of the dashed lines in Figure 8.3) we associate the constant matrix

$$A := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \tag{8.2.12}$$

  that has the property $A^3 = \mathbf{1}$.

  Remark 8.2.3. *In the SL_n case, the matrix A would be replaced by matrices $A_{1,2,3}$ that depend on $(n-1)(n-2)/2$ additional parameters for each triangle.*

- on each internal diagonal edge d_j for $j = 2, \dots, 2K$ defining the original triangulation T_0, we associate off-diagonal matrices of the form $V(y_j)$ given by

$$V(y_j) := \begin{pmatrix} 0 & -y_j \\ y_j^{-1} & 0 \end{pmatrix} \tag{8.2.13}$$

  for $j = 2, \dots, 2K$ (these are the red edges of Figure 8.3). In this way each internal diagonal d_j is uniquely associated to the free variable y_j, for $j = 2, \dots, 2K$.

Remark 8.2.4. *In this construction one among the boundary edges plays a distinguished role, namely, the one laying to the left of the first Stokes ray. Indeed, the matrix associated* *to this edge is of the same type of the matrices associated to the internal diagonal edges of the triangulation T_0 and it depends only on y_1. It would be possible to choose an arbitrary distinguished boundary edge for our variable y_1, while retaining the same triangulation. Then one may verify (but we do not report the details here) that the new distiguished variable $\widetilde{y}_1$ is a monomial containing y_1, while the other variables are unchanged.*

8.2.2 Computation of $\mathcal{W}_K$

The Stokes' matrices S_j on the unbounded rays are then uniquely determined in terms of the remaining ones by the condition (8.2.5) at the corresponding vertex v_j. In this way each S_j is expressed in terms of the y_j variables. Of course, for each triangulation, we will obtain different parametrization of the Stokes parameters and the transformation of coordinates will be investigated later.

The initial triangulation Consider now the triangulation T_0, underlying the graph $\Sigma_0^{(K)}$, where the last vertex v_{2K+2} is connected to each other vertex starting from v_2, and with alternated orientation of the internal diagonals (as in Figure 8.3 for the case $K = 2$). Then the Stokes matrices are given by

$$\begin{aligned}
S_1 &= \left(V(y_1^{-1})AD(y_1)^{-1}\right)^{-1} \\
S_2 &= \left(D(x_2)AV(y_2)^{-1}AV(y_1^{-1})^{-1}\right)^{-1}, \\
S_{2k} &= \left(D(x_{2k})AV(y_{2k})^{-1}AV(x_{2k-1}^{-1})^{-1}\right)^{-1}, \quad k = 2, \ldots, K \\
S_{2k+1} &= \left(V(x_{2k+1}^{-1})AV(y_{2k+1}))AD(x_{2k})^{-1}\right)^{-1}, \quad k = 1, \ldots, K-1 \\
S_{2K+1} &= \left(V(x_{2K+1}^{-1})AD(x_{2K})^{-1}\right)^{-1} \\
S_{2K+2}\Lambda &= \left(D(y_1)\prod_{j=2}^{2K}\left(AV(y_j)^{(-1)^j}\right)AV(x_{2K+1}^{-1})^{-1}\right)^{-1}.
\end{aligned} \tag{8.2.14}$$

The choice of the triangulation of the polygon also defines the variables x_l. According to the general rule (8.2.10) with the triangulation T_0 fixed here, this definition reduces to

$$x_l \coloneqq \prod_{j=1}^{l} y_j^{(-1)^{j+1}} \quad l = 2, \ldots, 2K, \quad x_{2K+1} = x_{2K}, \quad x_{2K+2} \coloneqq y_1. \tag{8.2.15}$$

These considerations are summarized in the following lemma.

Proposition 8.2.5. *The Stokes parameters are written in terms of the y_j variables, w.r.t.*

the fixed triangulation T_0 described above, as follows

$$
\begin{aligned}
&s_1 = -y_1^{-2}\\
&s_{2k} = (1+y_{2k}^2)\prod_{1\le j\le 2k} y_j^{(-1)^{j+1}2}, \quad k=1,\dots,K\\
&s_{2k+1} = -(1+y_{2k+1}^2)\prod_{1\le j\le 2k+1} y_j^{(-1)^{j}2}, \quad k=1,\dots,K-1\\
&s_{2K+1} = -\prod_{1\le j\le 2K} y_j^{(-1)^{j}2},\\
&s_{2K+2} = y_1^2\left(1+y_2^2\left(\dots\left(1+y_{2K}^2\right)\dots\right)\right)\prod_{j=1}^{K} y_{2j}^{-4},\\
&\lambda = (-1)^K\prod_{j=1}^{K} y_{2j}^2.
\end{aligned}
\tag{8.2.16}
$$

Proof. Just computing explicitly the parametrizations given from equations (8.2.14) and using the definition of the variables x_{2k}, x_{2k+1} given in (8.2.15). ■

With this parametrization of the Stokes matrices we can then proceed to the computation of the Stokes form.

Proposition 8.2.6. *The 2-form associated to the graph $\Sigma_0^{(K)}$ coincide with*

$$
\Omega\left(\Sigma_0^{(K)}\right) = +8\sum_{\substack{j=1\\ l\ge j}}^{K} d\log y_{2j-1}\wedge d\log y_{2l}. \tag{8.2.17}
$$

In particular it is symplectic.

Proof. The fact that the form is symplectic follows from Theorem 8.1.5 and the fact that the contraction of $\Sigma_0^{(K)}$ coincides with the graph Σ_K (see Fig. 8.2); however the explicit expression (8.2.17) is manifestly a nondegenerate form and so it could be used directly as a proof. By using the definition of the 2-form (8.2.6), we have to compute the contributions coming from each vertex $v_j, j=1,\dots 2K+2$ in the graph $\Sigma_0^{(K)}$. The vertices $z_j, j=1,\dots,2K$ do not give any contribution since all their incident edges carry constant matrices.
We start with the vertex v_1. Since the valence of v_1 is 4 and A is a constant matrix, there is only one contribution to take into account from v_1, and it is

$$
\mathrm{Tr}\left(\underbrace{\left(V(y_1^{-1})AD(y_1)^{-1}\right)^{-1} d\left(V(y_1^{-1})AD(y_1)^{-1}\right)}_{=S_1 d(S_1^{-1})}\wedge\underbrace{\left(D(y_1)dD(y_1)^{-1}\right)}_{=-d\log y_1\sigma_3}\right) = 0 \tag{8.2.18}
$$

that turns out to be also zero, thanks to the form of the Stokes matrices given in (8.2.14). Thus the total contribution of the vertex v_1 is actually zero.

Since the vertex v_{2K+1} is in the same configuration of v_1, but replacing $D(y_1)$ by $D(x_{2K})$, by the same reasoning we can conclude that its contribution is also zero.
Now we compute the contributions of the vertices v_{2k} for $k = 1, \dots, K$. For each of them there is only one nonzero contribution and it is coming from the term

$$\mathrm{Tr}\left(\underbrace{\left(\left(D(x_{2k})AV(y_{2k})^{-1}\right)^{-1} d\left(D(x_{2k})AV(y_{2k})^{-1}\right)\right)}_{-d\log(x_{2k-1})+E_{21}f(\vec{y})d\vec{y}} \wedge \underbrace{\left(V(y_{2k})d(V(y_{2k})^{-1})\right)}_{=-d\log y_{2k}\sigma_3}\right) =$$

$$\begin{aligned}
&= 2d\log x_{2k-1} \wedge d\log y_{2k} = \\
&= 2d\log\left(\prod_{j=1}^{2k-1} y_j^{(-1)^{j+1}}\right) \wedge d\log y_{2k} = \\
&= 2d\log y_1 \wedge d\log y_{2k} + 2\sum_{l=2}^{k} d\log y_{2l-1} \wedge d\log y_{2k} - 2\sum_{l=1}^{k-1} d\log y_{2l} \wedge d\log y_{2k}.
\end{aligned} \tag{8.2.19}$$

Notice that for the case $k = 1$ we only have the term $2d\log y_1 \wedge d\log y_2$.
A similar computation shows that the only nonzero contribution for the vertices v_{2k+1} for $k = 1, \dots, K-1$ is given by

$$\mathrm{Tr}\left(\underbrace{\left(\left(V(x_{2k+1}^{-1})AV(y_{2k+1})\right)^{-1} d\left(D(x_{2k+1})JAV(y_{2k+1})\right)\right)}_{=d\log x_{2k}\sigma_3+E_{21}g(\vec{y})d\vec{y}} \wedge \underbrace{\left(V(y_{2k+1})^{-1}d(V(y_{2k+1}))\right)}_{=-d\log y_{2k+1}}\right) =$$

$$\begin{aligned}
&= -2d\log x_{2k} \wedge d\log y_{2k+1} = \\
&= -2d\log\left(\prod_{j=1}^{2k} y_j^{(-1)^{j+1}}\right) \wedge d\log y_{2k+1} = \\
&= -2d\log y_1 \wedge d\log y_{2k+1} - 2\sum_{j=2}^{k} d\log y_{2j-1} \wedge d\log y_{2k+1} + 2\sum_{j=1}^{k} d\log y_{2j} \wedge d\log y_{2k+1}.
\end{aligned} \tag{8.2.20}$$

It only remains to compute the contribution of the vertex v_{2K+2}. The internal diagonals carrying the variables y_{2k} for $k = 1, \ldots, K$ give the contribution

$$\begin{aligned} C_1 &:= \sum_{k=1}^{K} \operatorname{Tr}\left(\left(V(y_{2k})^{-1} d(V(y_{2k}))\right) \wedge \left(D(y_1)\prod_{j=2}^{2k} A\left(V(y_j)\right)^{(-1)^j}\right)^{-1} d\left(D(y_1)\prod_{j=2}^{2k} A\left(V(y_j)\right)^{(-1)^j}\right)\right) = \\ &= \sum_{k=1}^{K} \operatorname{Tr}\left(-d\log y_{2k}\sigma_3 \wedge \left(-d\log y_1 - \sum_{j=2}^{2k} d\log y_j\right)\sigma_3\right) = \\ &= -2\sum_{k=1}^{K} d\log y_1 \wedge d\log y_{2k} + 2\sum_{\substack{k=1\\ j\le k}}^{K}\left(d\log y_{2k}\wedge d\log y_{2j} + d\log y_{2k}\wedge d\log y_{2j-1}\right). \end{aligned} \tag{8.2.21}$$

The internal diagonals carrying on the variables y_{2k+1} give instead the contribution

$$\begin{aligned} C_2 &:= \sum_{k=1}^{K-1} \operatorname{Tr}\left(\left(D(y_1)\prod_{j=2}^{2k+1} A\left(V(y_j)\right)^{(-1)^j}\right)^{-1} \mathrm{d}\left(D(y_1)\prod_{j=2}^{2k+1} A\left(V(y_j)\right)^{(-1)^j}\right) \wedge \left(V(y_{2k+1})\mathrm{d}(V(y_{2k+1})^{-1})\right)\right) = \\ &= \sum_{k=1}^{K-1} \operatorname{Tr}\left(\left(-d\log y_1 - \sum_{j=2}^{2k+1} d\log y_j\right)\sigma_3 \wedge (-d\log y_{2k+1}\sigma_3)\right) = \\ &= 2\sum_{k=1}^{K-1} d\log y_1 \wedge d\log y_{2k+1} - 2\sum_{\substack{k=2\\ j\le k}}^{K-1}\left(d\log y_{2k+1}\wedge d\log y_{2j} + d\log y_{2k+1}\wedge d\log y_{2j-1}\right). \end{aligned} \tag{8.2.22}$$

Finally the last edge on the right of the Stokes ray of v_{2K+2} also gives a nonzero contribution, that is

$$\begin{aligned} C_3 &:= \operatorname{Tr}\left((S_{2K+2}\Lambda)\mathrm{d}(S_{2K+2}\Lambda)^{-1} \wedge \left(V(x_{2K+1}^{-1})\mathrm{d}(V(x_{2K+1}^{-1})^{-1})\right)\right) = \\ &= \operatorname{Tr}\left(\left(2\sum_{l=1}^{K}\mathrm{d}\log y_{2l}\right)\sigma_3 \wedge \left(-\mathrm{d}\log y_1 + \sum_{j=1}^{K}(-\mathrm{d}\log y_{2j+1} + \mathrm{d}\log y_{2j})\right)\sigma_3\right) = \\ &= 4\sum_{l=1}^{K} d\log y_1 \wedge d\log y_{2l} + 4\sum_{j=2}^{K} d\log y_{2j-1} \wedge \sum_{l=1}^{K} d\log y_{2l} - 4\underbrace{\sum_{j=1}^{K} d\log y_{2j}\wedge\sum_{l=1}^{K} d\log y_{2l}}_{=0} = \\ &= 4\sum_{l=1}^{K} d\log y_1 \wedge d\log y_{2l} + 4\sum_{j=2}^{K} d\log y_{2j-1} \wedge \sum_{l=1}^{K} d\log y_{2l} \end{aligned} \tag{8.2.23}$$

where in the last equality we used the skew-symmetry of the wedge product. Now we can sum up all the nonzero contributions coming from $v_l, l = 2, \ldots, 2K+2$ and we obtain

$$\Omega\left(\Sigma_0^{(K)}\right) = 2\sum_{k=1}^{K} \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_{2k} + 2\sum_{\substack{2\leq l\leq k\\ k=2}}^{K} \mathrm{d}\log y_{2l-1} \wedge \mathrm{d}\log y_{2k} - 2\sum_{\substack{1\leq l\leq k-1\\ k=2}}^{K} \mathrm{d}\log y_{2l} \wedge \mathrm{d}\log y_{2k}$$

$$-2\sum_{k=1}^{K-1} d\log y_1 \wedge \mathrm{d}\log y_{2k+1} - 2\sum_{\substack{2\leq j\leq k\\ k=2}}^{K-1} \mathrm{d}\log y_{2j-1} \wedge \mathrm{d}\log y_{2k+1} + 2\sum_{\substack{1\leq j\leq k\\ k=1}}^{K-1} d\log y_{2j} \wedge d\log y_{2k+1}$$

$$+2\sum_{k=1}^{K} \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_{2k} - 2\sum_{\substack{k=1\\ j\leq k}}^{K} (\mathrm{d}\log y_{2k} \wedge \mathrm{d}\log y_{2j} + \mathrm{d}\log y_{2k} \wedge \mathrm{d}\log y_{2j-1}) \tag{8.2.24}$$

$$+2\sum_{k=1}^{K-1} \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_{2k+1} - 2\sum_{\substack{k=2\\ j\leq k}}^{K-1} (\mathrm{d}\log y_{2k+1} \wedge \mathrm{d}\log y_{2j} + \mathrm{d}\log y_{2k+1} \wedge \mathrm{d}\log y_{2j-1}) \tag{8.2.25}$$

$$+4\sum_{l=1}^{K} \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_{2l} + 4\sum_{j=2}^{K} \mathrm{d}\log y_{2j-1} \wedge \sum_{l=1}^{K} \mathrm{d}\log y_{2l} \tag{8.2.26}$$

$$= -8\sum_{\substack{k=1\\ j\geq k}}^{K} \mathrm{d}\log y_{2k-1} \wedge \mathrm{d}\log y_{2j} \tag{8.2.27}$$

■

By using relation (8.2.8), we can finally conclude that the Stokes 2-form $\mathcal{W}_K$ is written in terms of these y_j variables as

$$\mathcal{W}_K = \frac{1}{2}\Omega\left(\Sigma^{(K)}\right) = \frac{1}{2}\Omega\left(\Sigma_0^{(K)}\right) = 4\sum_{\substack{k=1\\ j\geq k}}^{K} d\log y_{2k-1} \wedge d\log y_{2j}, \tag{8.2.28}$$

and since it has maximal rank, it is a symplectic 2-form.
The Poisson structure induced by the the symplectic structure in the same variables will be then written as

$$\{y_i, y_j\} = \mathbf{P}_K^{ij} y_i y_j \tag{8.2.29}$$

where $\mathbf{P}_K = \mathbf{\Omega}_K^{-t}$ and $\mathbf{\Omega}_K$ is the matrix of coefficient of the Stokes 2-form w.r.t. the logarithmic variables $\log y_l$.

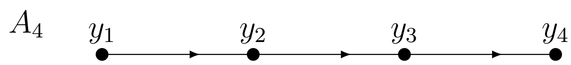

Figure 8.4: The Dynkin diagram associated to the 4×4 matrix $\mathbf{B}_2$. This quiver can also be obtained following the construction described in the paragraph below with the triangulation of the hexagon fixed to be T_0.

Lemma 8.2.7. *The matrix $\mathbf{P}_K$ is the $2K \times 2K$ tridiagonal matrix given by*

$$\mathbf{P}_K = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & & -1 & 0 & 1 \\ 0 & 0 & \dots & & 0 & -1 & 0 \end{pmatrix} \tag{8.2.30}$$

8.3 Comparison between $\mathbf{P}_K$ and Poisson structure on Y-cluster manifold

Let focus our attention on the matrix $\mathbf{B}_K := 4\mathbf{P}_K$.

Definition 8.3.1. *Given a quiver Q with labeled vertices $q_i, i = 1, \dots, \#\mathbb{V}(Q)$, we call B its adjacency matrix the skew-symmetric, integer-valued square matrix, of dimension $\#\mathbb{V}(Q)$, given by*

$$B_{kl} := \#\{\textit{edges oriented from } q_k \textit{ to } q_l\} - \#\{\textit{ edges oriented from } q_l \textit{ to } q_k\} \tag{8.3.1}$$

for $k, l = 1, \dots, \#\mathbb{V}(Q)$.

Then the matrix $\mathbf{B}_K$ can be identified as the directed adjacency matrix of a Dynkin graph of type A_{2K} with specified orientation. An example for $K = 2$ is given in Figure 8.4. There is a classical way to associate a directed graph to a triangulation of a given polygon (see for instance paragraph 2.1 of [46]). We slightly modify this construction, taking into account the fact that there is an edge along the perimeter of the polygon (the edge at the left of the first Stokes ray) that has a distinguished role in our case. We end up with the following graph $Q(T)$ for a given triangulation T of the polygon:

- the vertices of $Q(T)$ are defined one for each of the following edges of T: the edge along the perimeter at the left of the first Stokes ray and every internal diagonal edge of the triangulation T;
- the edges of $Q(T)$ are are build between each pair of vertices that lies on edges of the triangulation T that share one of the endpoints and are immediately adjacent;

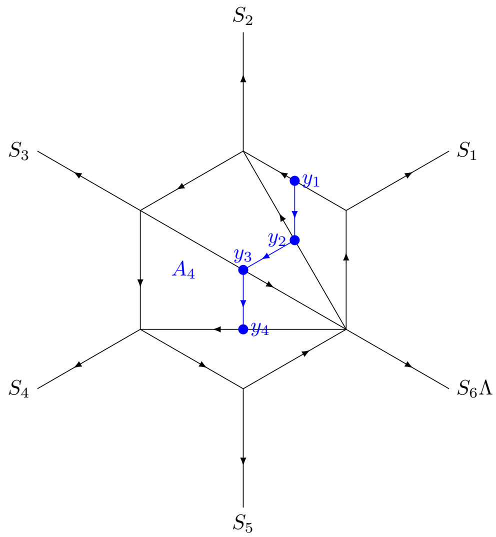

Figure 8.5: Here the triangulation T_0 of the hexagon and the variables y_j assigned to the relevant edges induce the Dynkin diagram with variables y_1, y_2, y_3, y_4 in blue.

- the orientation of the edges of $Q(T)$ is defined as follows: an edge connecting the vertices q_i and q_j on the adjacent edges of T d_i and d_j is oriented $q_i \to q_j$ if the edge d_i immediately precedes d_j counting counterclockwise the edges incident to their common endpoint. Otherwise it is oriented in the opposite way. For the vertex y_1 along the edge on the right of the first Stokes ray (since on this edge we actually used the variable y_1^{-1}) we reverse the orientation of all the edges of $Q(T)$ that have y_1 as endpoint.

With this construction, we obtain that for the initial triangulation T_0 underlying $\Sigma_K^{(0)}$ the quiver $Q(T_0)$ is a Dynkin graph of type A_{2K} with the orientation induced from T_0 (but each orientation of the same type of Dynkin graph is mutation equivalent, see Theorem 3.29 of [46]).
The matrix $\mathbf{B}_K$ gives a compatible Poisson structure on the Y-cluster manifold which is defined by the ring of functions that are polynomials in all of the seeds obtained by subsequent *mutations* (of Y-type), defined below.

Definition 8.3.2. *A mutation $\mu_k(Q)$ w.r.t. a vertex $q_k \in \mathbb{V}(Q)$ of the quiver Q is a new quiver defined by*

- *the same set of vertices, namely $\mathbb{V}(Q) = \mathbb{V}(\mu_k(Q))$;*
- *the set of edges constructed as follows*
    1. *for any sequence $q_i \to q_k \to q_l$ add an edge $q_i \to q_l$,*
    2. *reverse any edge having source or end in the vertex q_k,*
    3. *remove every 2-cycle if any.*

Equivalently we can define the mutation $\mu_k(Q)$ of Q through its adjacency matrix $\mu_k(B)$ that is given by the following equations

$$\mu_k(B)_{st} = \begin{cases} -B_{st}, & \text{for } s = k \text{ or } s = t, \\ B_{st} + sign(B_{sk})\,[B_{sk}, B_{kt}]_+\,, & \text{otherwise.} \end{cases} \tag{8.3.2}$$

In our case of study, a set of variables $y_i \in \mathbb{C}^*$ one of each vertex q_i is associated to the quiver, for $i = 1, \ldots, 2K$. To each mutation $\mu_k(Q)$ of the quiver is then associated a new set of variables $\mu_k(\vec{y})$ following the equations in the definitions that we recall below (see also (1.30) in e.g. [46]).

Definition 8.3.3. *A Y-mutation for the variables y_i of the couple $(Q, \vec{y})$ is a new set of variables $(\mu_k(\vec{y}))_{i=1}^{2K}$, for $i = 1, \ldots 2K$ defined as rational functions of the y_i in the following way*

$$y_i' := (\mu_k(\vec{y}))_i = \begin{cases} y_k^{-1}, & \textit{for } i = k, \\ y_i \frac{y_k^{[B_{ik}]_+}}{(1+y_k)^{B_{ik}}}, & \textit{otherwise.} \end{cases} \tag{8.3.3}$$

Every new pair $\mu_k(\vec{y}, Q) = (\vec{y}', Q')$ obtained by an allowed mutation is called a *seed*. In our case, we have that the initial quiver $Q(T)$ is the Dynkin graph of A_{2K}-type (for every $n \geq 1$) that is related to the triangulation T of the polygon in $\Sigma_0^{(K)}$. The allowed mutations in this case are with respect to all the vertices with variables $y_2, \ldots, y_{2K}$ (the ones associated to the internal diagonals of the triangulation T of the polygon).

Definition 8.3.4. *Given a pair $(\vec{y}, Q)$ where Q is a quiver with labeled vertices $q_i, i = 1, \ldots, \#\mathbb{V}(Q)$ and the variables $y_i \in \mathbb{C}^*$ are associated to each q_i, we call the Y-cluster algebra $\mathcal{A}_Y(Q)$ the sub-ring of all polynomials in y_i and all their possible seeds $\mu_k(\vec{y}, Q)$ where μ_k is a mutation w.r.t. the vertex q_k with assigned variable y_k.*

Definition 8.3.5. *Given a Y-cluster algebra, its correspondent Y-cluster manifold is defined as the smooth part of $Spec(\mathcal{A}_Y(Q))$.*

Denoting by $\mathcal{A}_{Y,i\neq 1}(A_{2K})$ the Y-cluster algebra described above for our case, then on its correspondent Y-cluster manifold $\mathcal{M} := Spec(\mathcal{A}_{Y,i\neq 1}(A_{2K}))$ there is a compatible Poisson structure having the form

$$\{y_i, y_j\} = \mathbf{B}_K y_i y_j. \tag{8.3.4}$$

Therefore we reach the conclusion that the Poisson structure (induced by the symplectic 2-form $\mathcal{W}_K$) on the Stokes manifold $(\mathfrak{S}_K, \mathbf{P}_K)$ coincides with the Poisson structure of $(\mathcal{M}, \mathbf{B}_K)$, up to a constant multiplicative factor.

8.3.1 Flipping the edges

In the previous section we have established how to define the matrices and the variables y_j, x_l associated to each edge of a given triangulation, in order to get a parametrization of

the Stokes matrices. We also computed the Stokes matrices and the Stokes 2-form for a fixed triangulation, seeing that its matrix coefficient is related to the matrix coefficient of the Poisson structure of the Y-cluster manifold of A_{2K}-type.
We are now going to show that the y-variables associated to two triangulations T and $\tilde{T}$ that are related by a single flip of one of their internal diagonal edges d_j, are related by the rules of the mutation of seed variables (Def. 8.3.3). Subsequent flips give different systems of equations for the variables, so we are going to study separately all the possible cases of flip. The equations between the old and the new y variables are obtained by requiring that the Stokes matrices remain the same, independently of the triangulation.

Consider a generic triangulation of the $2(K+1)$-gon, and consider any quadrilateral inside the triangulation consisting of two triangles sharing an edge. For the case $K \geq 2$ we have the following possibilities for the sides of the quadrilateral:

1. three sides lie along the perimeter of the polygon, one side is an internal diagonal;
2. two sides lie along the perimeter of the polygon and two sides are internal diagonals;
3. one side is along the perimeter and the three others are internal diagonals;
4. all the four sides are internal diagonals.

With the two last cases only occurring for $K > 2$. Moreover, the number of y_j variables directly and nontrivially involved in the flip is equal to the number of sides of the quadrilateral that are internal diagonals. We are going to analyze the flip for each case. After the flip, we define some new variables associated to each edge of the new triangulation and we find the corresponding parametrizations of the Stokes matrices in these new variables denoted $\tilde{y}_j$. Finally, by imposing the equality between these Stokes matrices, the ones parametrized w.r.t. the first triangulation and the other ones, we obtain an over-determined but compatible system of equations for the old variables and the new ones, y_j and $\tilde{y}_j$. Indeed, notice that the y_j variables are always $2K$ and we have an equation for each Stokes matrix, thus we have a system of $2K+2$ equations in $2K$ variables. We will see that this system is equivalent to the y-mutation correspondent to the vertex on the flipped edge, in the quiver $Q(T)$ associated with the triangulation T.

Case 1. This is the case where three edges of the quadrilateral are along the perimeter. This means that we have only two variables y that are directly and nontrivially involved in the flip. We can suppose that the first vertex, denoted by v_{2i} (in even position, the odd case is analogous) have valence only 6 and that the last one have valence 9, see Figure 8.6. Every other case can be reduced to this one after an appropriate simplification in the equations we are going to obtain. We denote by S_j the Stokes matrices obtained through the triangulation T and by $\tilde{S}_j$ the ones obtained by the flip of T.
First, we observe that for every $j \leq 2i$ the Stokes matrices are parametrized exactly in the

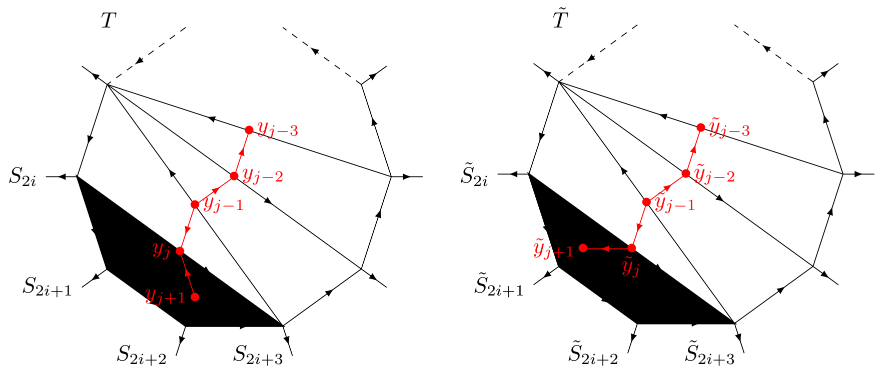

Figure 8.6: A flip of a quadrilateral inside the triangulation T with 3 sides along the perimeter of the polygon and the new triangulation $\tilde{T}$ obtained in this way.

same way w.r.t. the y_j variables and the $\tilde{y}_j$. Thus the equations $S_j(y_k) = \tilde{S}_j(\tilde{y}_k)$ tell us that $y_k = \tilde{y}_k$ for every k that is not incident to $v_{2i}, v_{2i+1}, v_{2i+2}$. As a byproduct also the variables $x_l = \tilde{x}_l$ for every $l \leq 2i$ they remain invariant.
We focus on the equations $S_j(y_k) = \tilde{S}_j(\tilde{y}_k)$ for $k = 2i, 2i+1, 2i+2, 2i+3$. We obtain an over-determined system of four equations from the following four matrix equations

$$D(x_{2i})AV(y_j)^{-1}AV(x_{2i-1}^{-1})^{-1} = D(\tilde{x}_{2i})AV(\tilde{y}_{j+1})^{-1}AV(\tilde{y}_{j+1})^{-1}AV(x_{2i-1}^{-1})^{-1}$$

$$V(x_{2i+1}^{-1})AV(y_{j+1})AD(x_{2i})^{-1} = V(\tilde{x}_{2i+1}^{-1})AD(\tilde{x}_{2i})^{-1}$$

$$D(x_{2i+2})AV(x_{2i+1}^{-1})^{-1} = D(\tilde{x}_{2i+1}^{-1})AV(\tilde{y}_{j+1})^{-1}AV(\tilde{x}_{2i+2})^{-1}$$

$$V(x_{2i+3}^{-1})AV(y_{j-1})^{-1}AV(y_j)AV(y_{j+1})^{-1}AD(x_{2i+2})^{-1} = V(\tilde{x}_{2i+3}^{-1})AV(\tilde{y}_{j-1})AV(\tilde{y}_j)AD(\tilde{x}_{2i+2})^{-1} \tag{8.3.5}$$

It follows then the following relations between the old and the new variables must hold

$$\tilde{y}_j^2 = (1 + y_{j+1}^2)y_j^2, \quad \tilde{y}_{j+1}^2 = \frac{1}{y_{j+1}^2} \tag{8.3.6}$$

where y_j is the variable on the diagonal $v_{2i} - v_{2i+3}$ and y_{j+1} is the one on the diagonal $v_{2i+1} - v_{2i+3}$ as show in Figure 8.6. One obtains these results from the second and third equation directly, then the other equations are automatically satisfied replacing these relations.

Case 2. Now we consider the case where there are two edges of the quadrilateral on the perimeter of the polygon, and the other two edges are internal diagonals. We can suppose as before that the first vertex is even v_{2i}. Also, we can assume that v_{2i}, v_{2i+4} both have valence 8 and v_{2i+3} has valence 4. Then all the other cases (when the valences of these vertices are higher) can be reduced to this one, after appropriate simplification. In this case three

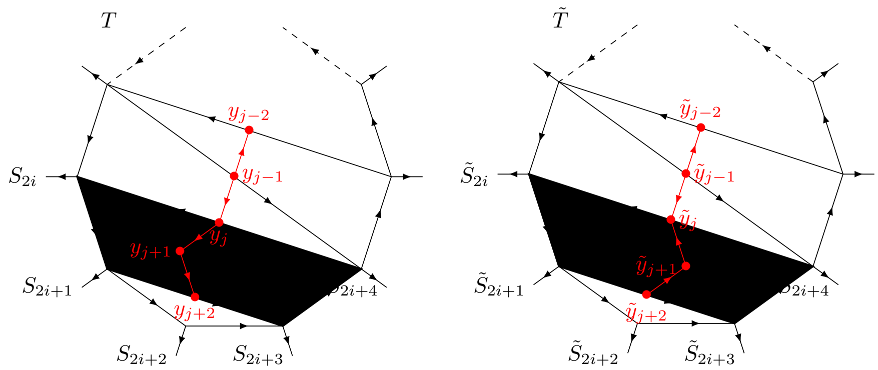

Figure 8.7: A flip of a quadrilateral inside the triangulation T with 2 sides along the perimeter of the polygon and the new triangulation $\tilde{T}$ obtained in this way.

variables y are directly involved in the flip. Indeed, by the fact that $S_j(y_k) = \tilde{S}_j(\tilde{y}_k)$ for every j, we obtain that $y_l = \tilde{y}_l$ for any index l that is not incident to $v_{2i}, v_{2i+1}, v_{2i+2}, v_{2i+3}$ and also for all the variables that stay on the right of the y_j diagonal, see Figure 8.7. Furthermore, by looking at $j = 2i, 2i+1, 2i+2, 2i+3$ we obtain the following over-determined system of four equations, from the four matrix equations

$$\begin{aligned}
&D(x_{2i})AV(y_{j+1})AV(y_j)^{-1}AV(x_{2i-1}^{-1})^{-1} = D(\tilde{x}_{2i})AV(\tilde{y}_j)AV(\tilde{x}_{2i-1}^{-1})^{-1} \\
&V(x_{2i+1}^{-1})AV(y_{j+2})AD(x_{2i})^{-1} = V(\tilde{x}_{2i+1}^{-1})AV(\tilde{y}_{j+2})AV(\tilde{y}_{j+1})AD(\tilde{x}_{2i})^{-1} \\
&D(x_{2i+2})AV(x_{2i+1})^{-1} = D(\tilde{x}_{2i+2})AV(\tilde{x}_{2i+1})^{-1} \\
&V(x_{2i+3}^{-1})AV(y_{j+1})^{-1}AV(y_{j+2})^{-1}AD(x_{2i+2})^{-1} = V(\tilde{x}_{2i+3}^{-1})AV(\tilde{y}_{j+2})AD(\tilde{x}_{2i+2})^{-1}.
\end{aligned} \tag{8.3.7}$$

In particular, from the first three equations we obtain the following relations between the old and the new variables

$$\tilde{y}_j^2 = y_j^2 \frac{y_{j+1}^2}{1 + y_{j+1}^2}, \quad \tilde{y}_{j+1}^2 = \frac{1}{y_{j+1}^2}, \quad \tilde{y}_{j+2}^2 = y_{j+2}^2(1 + y_{j+1}^2), \tag{8.3.8}$$

and all the other equations are then satisfied by replacing these quantities (included the equation for $j = 2i+4$).

Case 3. Here we consider the case where three edges of the quadrilateral are internal diagonals of the polygon and only one edge is on its perimeter. Notice that this means that there are four variables y that are nontrivially involved in the flip. We suppose as before that the first edge considered is even v_{2i} and that all the vertices involved in the quadrilateral and their adjacent vertices have minimal valence, as in Figure 8.8. As in the previous cases,

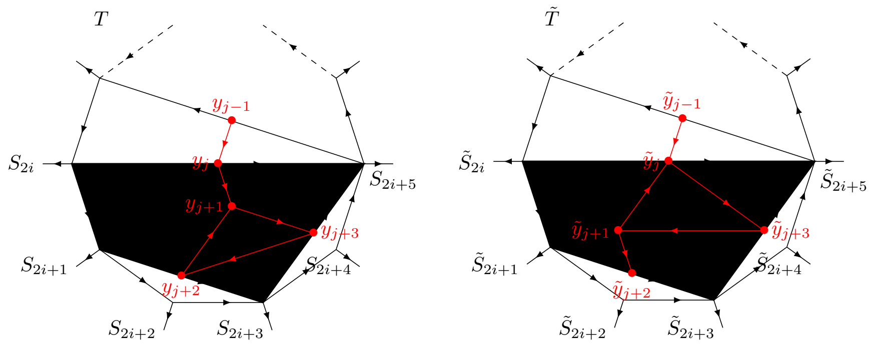

Figure 8.8: A flip of a quadrilateral inside the triangulation T with only 1 side along the perimeter of the polygon and the new triangulation $\tilde{T}$ obtained in this way.

the equations $S_l(y_k) = \tilde{S}_l(\tilde{y}k)$ for the indices $l \neq 2i, \ldots, 2i+4$ give that the variables $y_k = \tilde{y}_k$ for the k that are not incident to the vertices $v_{2i}, \ldots, v_{2i+4}$. Then looking at the matrix equations for $l = 2i, \ldots 2i+3$ we have the four matrix equations

$$D(\tilde{x}_{2i})AV(\tilde{y}_j)AV(\tilde{x}_{2i-1}^{-1}) = D(x_{2i})AV(y_{j+1})AV(y_j)^{-1}AV(x_{2i-1}^{-1})$$

$$V(\tilde{x}_{2i+1})AV(\tilde{y}_{j+2})AV(\tilde{y}_{j+1})AD(\tilde{x}_{2i})^{-1} = V(x_{2i+1}^{-1})AV(y_{j+2})AD(x_{2i})^{-1}$$

$$D(\tilde{x}_{2i+2})AV(\tilde{x}_{2i+1}^{-1})^{-1} = D(x_{2i+2})AV(x_{2i+1}^{-1})^{-1}$$

$$V(\tilde{x}_{2i+3}^{-1})AV(\tilde{y}_{j+3})^{-1}AV(\tilde{y}_{j+2})^{-1}AD(\tilde{x}_{2i+2})^{-1} = V(x_{2i+3}^{-1})AV(y_{j+3})^{-1}AV(y_{j+1})^{-1}AV(y_{j+2})^{-1}AD(x_{2i+2})^{-1}. \tag{8.3.9}$$

From these equations we obtain that the old variables and the new variables are related through the following relations

$$\tilde{y}_j^2 = y_{j+1}^2 \frac{y_j^2}{1+y_{j+1}^2}, \quad \tilde{y}_{j+1}^2 = \frac{1}{y_{j+1}^2}, \quad \tilde{y}_{j+2}^2 = y_{j+2}^2 \frac{y_{j+1}^2}{1+y_{j+1}^2}, \quad \tilde{y}_{j+3}^2 = y_{j+3}^2(1+y_{j+1}^2) \tag{8.3.10}$$

and all the other equations (included for the vertices v_{2i+4}, v_{2i+5}) are identically satisfied once we replace the relations above.

Case 4. Here we consider the case where all the sides of the quadrilateral are internal diagonals. We suppose, as always, to have the first vertex that is even v_{2i} and that each vertex has minimal valence, as in Figure 8.9. Every other case, with higher order valence for the vertices involved, can be reduced to this one after appropriate simplification. In this case, we have five variables y directly involved in the flip, thus we will have one more equation than in the other cases.
By looking at the equations $S_l(y_k) = \tilde{S}_l(\tilde{y}_k)$ for $l \neq 2i, \ldots, 2i+5$, we get that $y_k = \tilde{y}_k$ for

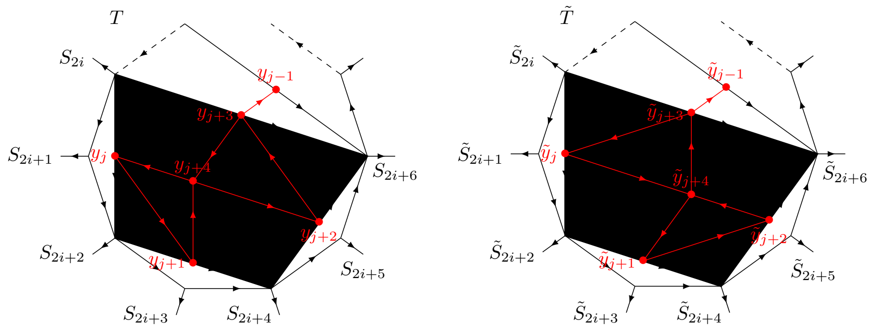

Figure 8.9: A flip of a quadrilateral inside the triangulation T with no sides along the perimeter of the polygon and the new triangulation $\tilde{T}$ obtained in this way.

every index k that is not adjacent to the flipped edge with coordinate y_{j+4}. Then by looking at the equations for $l = 2i, \dots, 2i+4$ we have the following matrix-valued system

$$D(x_{2i})AV(y_j)AV(y_{j+1})^{-1}AV(y_{j+3})AV(x_{2i-1})^{-1} = D(\tilde{x}_{2i})AV(\tilde{y}_j)AV(\tilde{y}_{j+3})AV(\tilde{x}_{2i-1})^{-1}$$

$$V(x_{2i+1})AD(x_{2i})^{-1} = V(\tilde{x}_{2i+1})AD(\tilde{x}_{2i})^{-1}$$

$$D(x_{2i+2})AV(y_{j+1})AV(y_j)^{-1}AV(x_{2i+1})^{-1} = D(\tilde{x}_{2i+2})AV(\tilde{y}_{j+1})AV(\tilde{y}_{j+4})AV(\tilde{y}_j)^{-1}AV(\tilde{x}_{2i-1})^{-1}$$

$$V(x_{2i+3})AD(x_{2i+2})^{-1} = V(\tilde{x}_{2i+3})AD(\tilde{x}_{2i+2})^{-1}$$

$$D(x_{2i+4})AV(y_{j+2})^{-1}AV(y_{j+4})AV(y_{j+1})^{-1}AV(x_{2i+3})^{-1} = D(\tilde{x}_{2i+2})AV(\tilde{y}_{j+2})AV(\tilde{y}_{j+1})^{-1}AV(\tilde{x}_{2i+3})^{-1}. \tag{8.3.11}$$

This system is solved through the following relations between the old and the new variables

$$\tilde{y}_j^2 = y_j^2(1+y_{j+4}^2), \quad \tilde{y}_{j+1}^2 = y_{j+1}^2 \frac{y_{j+4}^2}{1+y_{j+4}^2}, \quad \tilde{y}_{j+2}^2 = y_{j+2}^2(1+y_{j+4}^2), \quad \tilde{y}_{j+3}^2 = y_{j+3}^2 \frac{y_{j+4}^2}{1+y_{j+4}^2}, \quad \tilde{y}_{j+4}^2 = \frac{1}{y_{j+4}^2} \tag{8.3.12}$$

and they also satisfy the equations for $l = 2i+5, 2i+6$.

Notice that in each case we obtained that the system of equations for the old and new y variables obtained from the matrix equations $S_l(y_k) = \tilde{S}_l(\tilde{y}_k)$ is solved by some y-mutation relations of the Dynkin diagram of A_{2K}-type, as in equation (8.3.3). In particular, every set of equations (8.3.6), (8.3.8), (8.3.10), (8.3.12) coincide with the y-mutation w.r.t. the vertex y_l associated to the flipped edge of the triangulation T of the polygon, of the Dynkin diagram of A_{2K}-type associated to the triangulation T for the square of its variables.

Remark 8.3.6. *For what concerns the flip of the internal diagonal of the triangulation T_0 associated to the variable y_2, analogue considerations hold. In particular, by looking at the equations $S_l(y_k) = \tilde{S}_l(\tilde{y}_k)$ for $l = 1, 2, 3$, one obtains that the squares of the variables y_k and*

$\tilde{y}_k$ for $k = 1, 2, 3$[‡] are related by the Y-mutation relations for the mutation of the quiver $Q(T_0)$ of type A_{2K} with respect to the vertex y_2. The other equations $S_l(y_k) = \tilde{S}_l(\tilde{y}_k), l > 3$ directly implies that all the other variables $y_k, k \neq 1, 2, 3$ do not change under this flip.

8.3.2 Example: the case $K = 2$

We work out on the case $K = 2$, i.e. the case of the hexagon. In particular, we are going to take the fixed triangulation T_0 of the hexagon (e.g. the one in Figure 8.5), and we consider the variables and the matrices associated to each edge of the graph in the common way explained before. We compute then the Stokes matrices and the Stokes 2-form $\mathcal{W}_2$ in these variables.

Then, we consider all the possible flip of this triangulation, w.r.t. the edges with variables y_2, y_3, y_4 as in Figure 8.10, and we perform the same computations above with the new variables associated to each new triangulation obtained in that way. We will see that in each case, the inverse of the matrix coefficient of the Stokes 2-form is, up to the same factor $\frac{1}{4}$ the adjacency matrix of a certain mutation of the A_4 Dynkin diagram, the one given in Figure 8.4.

- For the triangulation T_1 the variables x_l are

$$x_2 = y_1 y_2^{-1}, \quad x_3 = y_1 y_2^{-1} y_3, \quad x_4 = y_1 y_2^{-1} y_3 y_4^{-1}, \quad x_5 = x_4, \quad x_6 = y_1. \tag{8.3.13}$$

  The 2-form ${\mathcal{W}_2}^{T_1}$ is log-canonical in the variables y_i and such that its matrix coefficient has inverse

$$\mathbf{P}_2^{T_1} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \frac{1}{4} Adj_{A_4}. \tag{8.3.14}$$

- For the triangulation T_2 the variables x_l are

$$x_2 = u_1, \quad x_3 = u_1 u_2 u_3, \quad x_4 = u_1 u_2 u_3 u_4^{-1}, \quad x_5 = u_1 u_2 u_3 u_4^{-1}, \quad x_6 = u_1 u_2^{-1}. \tag{8.3.15}$$

  The 2-form ${\mathcal{W}_2}^{T_2}$ is log-canonical in the variables y_i and such that the inverse of its coefficient matrix, namely $\mathbf{P}_2^{T_2}$ gives

$$\mathbf{P}_2^{T_2} = \frac{1}{4}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \frac{1}{4} Adj_{\mu_2(A_4)}.$$

[‡]The correct Y-mutation formula is actually obtained for y_1^{-1}, y_2, y_3 and $\tilde{y}_1^{-1}, \tilde{y}_2, \tilde{y}_3$, but this is just a matter of notation, due to the fact that we associated the matrix $V(y_1^{-1})$ to the edge $v_1 \to v_2$ in the graph $\Sigma_0^{(K)}$.

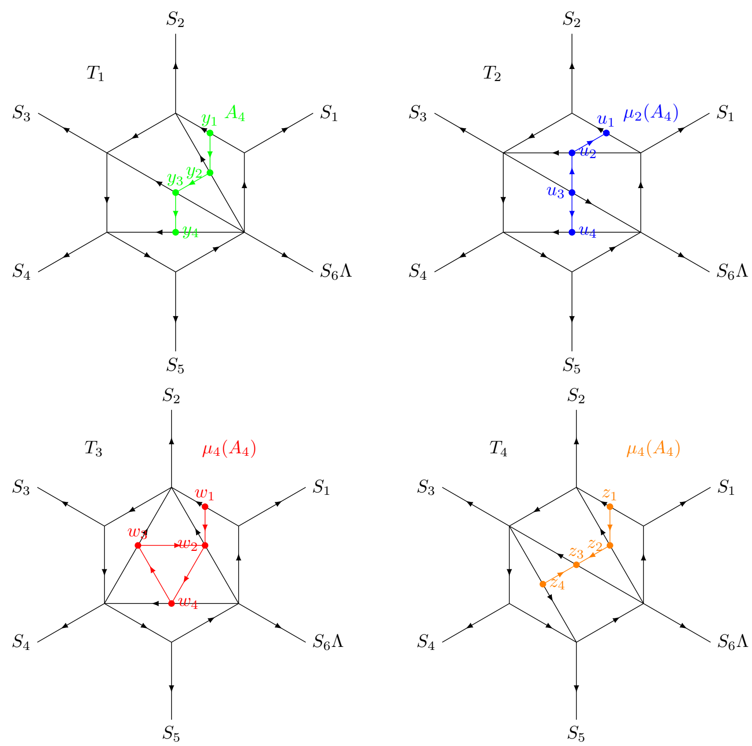

Figure 8.10: The 4 triangulations considered are T_1 and then all the others obtained from T_1 by a flip of one of the diagonals d_j for $j = 2, 3, 4$.

- For the triangulation T_3 the variables x_l are

$$x_2 = w_1 w_2^{-1} w_3^{-1}, \quad x_3 = x_2, \quad x_4 = w_1 w_2^{-1} w_3^{-2} w_4^{-1}, \quad x_5 = x_4, \quad x_6 = w_1. \tag{8.3.16}$$

The 2-form ${\mathcal{W}_2}^{T_3}$ is such that the inverse of its coefficient matrix, namely $\mathbf{P}_2^{T_3}$ gives

$$\mathbf{P}_2^{T_3} = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = \frac{1}{4} Adj_{\mu_3(A_4)}.$$

- For the triangulation T_4 the variables x_l are

$$x_2 = t_1 t_2^{-1}, \quad x_3 = t_1 t_2^{-1} t_3 t_4, \quad x_3 = t_4, \quad x_5 = t_1 t_2^{-1} t_3 t_4^2. \tag{8.3.17}$$

The 2-form ${\mathcal{W}_2}^{T_4}$ is such that the inverse of its coefficient matrix, namely $\mathbf{P}_2^{T_4}$ gives

$$\mathbf{P}_2^{T_4} = \frac{1}{4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{4} Adj_{\mu_4(A_4)}.$$

Furthermore the equations $S_i(\vec{y}) = \tilde{S}_i(\vec{u})$ that impose the Stokes equations parametrized in the 2 triangulations T_1 and T_j to be equal, give exactly that u_i^2, w_i^2 or t_i^2 respectively for $j = 2, 3, 4$ are y-mutation of y_i^2 related to A_4 w.r.t. the vertices y_2, y_3, y_4.

8.4 Computation of the Poisson brackets for the original monodromy parameters

In the previous sections we have parametrized the Stokes manifold $\mathfrak{S}_K$ of dimension $2K$, by using the variables y_j for $j = 1, \ldots, 2K$ of the A_{2K} cluster algebra type. Using this parametrization, explicitly computed in Lemma 8.2.5, we also proved that the two-form $\mathcal{W}_K$ defined on $\mathfrak{S}_K$ is symplectic and that the variables y_j are log-canonical for this two-form. We also computed the Poisson brackets $\mathbf{P}_K$ induced by the symplectic structure $\mathcal{W}_K$ on $\mathfrak{S}_K$. Now, we want to compute these Poisson brackets $\mathbf{P}_K$ on the parametrization of the original monodromy parameters s_j, for $j = 1, \ldots, 2K+2$ and λ describing $\mathfrak{S}_K$. In particular, we are going to show that the Poisson brackets $\mathbf{P}_K$ for the y_j defined in (8.2.29) are a log-canonical formulation of the following bracket, called Flaschka-Newell Poisson bracket in the introduction.

Definition 8.4.1. *Consider the nonlinear Poisson bracket on $\mathbb{C}^{2K+2} \times \mathbb{C}^*$ with coordinates $(s_1, \ldots, s_{2K+2}, \lambda)$ given by*

$$\begin{aligned} \Big\{s_j, s_l\Big\}_{FN} &= \delta_{j,l-1} - \frac{\delta_{j,1}\delta_{l,2K+2}}{\lambda^2} + (-1)^{j-l+1} s_j s_l, \qquad j < l. \\ \Big\{s_j, \lambda\Big\}_{FN} &= (-1)^j s_j \lambda. \end{aligned} \tag{8.4.1}$$

These Poisson structure first appeared in [37] (see section 3, 5).

Proposition 8.4.2. *Let*

$$F = F_K = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & s_{2K+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_{2K+2} & 1 \end{pmatrix} \lambda^{\sigma_3}. \tag{8.4.2}$$

Let σ_3, σ_+, s_- be the matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{8.4.3}$$

(1) *The matrix* F *satisfies*

$$\begin{aligned}
\{s_1, F\}_{FN} &= \frac{s_1}{2}[\sigma_3, F] + [\sigma_-, F] \\
\{s_{2K+2}, F\}_{FN} &= \frac{s_{2K+2}}{2}[F, \sigma_3] + \frac{1}{\lambda^2}[\sigma_+, F] \\
\{s_\ell, F\}_{FN} &= (-1)^\ell [F, \sigma_3], \quad 2 \le \ell \le 2k+1 \\
\{\lambda, F\}_{FN} &= \frac{1}{2}[\sigma_3, F].
\end{aligned} \tag{8.4.4}$$

(2) *The unique Casimir function for the bracket* (8.0.4) *is* $\mathfrak{C} = \mathrm{Tr}(F)$*;*
(3) *The sub-varieties* $\mathfrak{S}_K = \{F_K = \mathbf{1}\}$ *are Poisson sub-varieties.*

We defer the proof to the Appendix of the paper [15].

Theorem 8.4.3. *The parametrization given in Lemma 8.2.5 for the Stokes parameters* $s_j, j = 1, \ldots, 2K+2$ *and the formal monodromy exponent* λ *transforms the Poisson bracket* (8.4.1) *in the bracket* (8.2.29)*.*

Proof. We start by observing that the bracket (8.2.29) is such that all even-indexed variables commute amongst themselves, and so do the odd ones. We now verify that the bracket (8.2.29) yields the bracket (8.4.1) under the map (8.2.16). We will verify some of the brackets explicitly and leave the rest of the verification to the reader. Let us start with the case $\{s_{2k+1}, \lambda\}$ for $k < K$: since λ is a function of only the even variables it commutes with the even ones and we can write

$$\{s_{2k+1}, \lambda\} = -\prod_{j=1}^{k} y_{2j}^2 \left\{ \prod_{j=0}^{k-1} y_{2j+1}^{-2} + \prod_{j=0}^{k} y_{2j+1}^{-2}, (-1)^K \prod_{j=1}^{K} y_{2j}^2 \right\}. \tag{8.4.5}$$

This computation is easily done by passing to the logarithms of the variables y_j's, in which the Poisson bracket (8.2.29) is constant: thus both terms inside the bracket are log–canonical. Then one observes that the bracket above involves a telescopic sum and only the term y_1 yields a contribution and we obtain

$$\{s_{2k+1}, \lambda\} = -s_{2k+1}\lambda. \tag{8.4.6}$$

The case $\{s_{2K+1}, \lambda\}$ is handled similarly. Consider now an even variable s_{2k} for $k < K$; since λ is a function of only the even variables we can write

$$\{s_{2k}, \lambda\} = \frac{(1 + y_{2k}^2)}{\prod_{j=1}^{k} y_{2j}^2} \left\{ \prod_{j=0}^{k-1} y_{2j+1}^2, (-1)^K \prod_{j=1}^{K} y_{2j}^2 \right\} = s_{2k}\lambda, \tag{8.4.7}$$

where we have used the same telescopic-sum argument. Again, the case $\{s_{2K+2}, \lambda\}$ is handled similarly observing that $s_{2K+2} = y_1^2$ times a function of only even variables.

Let us now consider the bracket $\{s_a, s_b\}$; suppose both $a = 2k, b = 2l$ are even.

$$\left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2} \prod_{j=1}^k y_{2j-1}^2, \frac{(1+y_{2l}^2)}{\prod_{j=1}^l y_{2j}^2} \prod_{j=1}^l y_{2j-1}^2 \right\} = \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2} \left\{ \prod_{j=1}^k y_{2j-1}^2, \frac{(1+y_{2l}^2)}{\prod_{j=1}^l y_{2j}^2} \right\} \prod_{j=1}^l y_{2j-1}^2 + \prod_{j=1}^k y_{2j-1}^2 \left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2}, \prod_{j=1}^l y_{2j-1}^2 \right\} \frac{(1+y_{2l}^2)}{\prod_{j=1}^l y_{2j}^2}. \tag{8.4.8}$$

The computation relies on the following simple observation, which can be used for both terms by interchanging the roles of k and l:

$$\left\{ \prod_{j=1}^k y_{2j-1}^2, \frac{1}{\prod_{j=1}^l y_{2j}^2} \right\} = \begin{cases} -\frac{\prod_{j=1}^k y_{2j-1}^2}{\prod_{j=1}^l y_{2j}^2} & k \le l \\ 0 & k > l. \end{cases} \tag{8.4.9}$$

Now let $k \le l-1$: then the second bracket in (8.4.8) is zero and the first yields back $s_{2k}s_{2l}$ which is consistent with (8.4.1). The odd-odd case is similarly handled.
We still have to check the case even-odd. For that, consider the case $\{s_{2k}, s_{2l+1}\}$ for $k \le l$:

$$\left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2} \prod_{j=1}^k y_{2j-1}^2, -\frac{(1+y_{2l+1}^2)}{\prod_{j=0}^l y_{2j+1}^2} \prod_{j=1}^l y_{2j}^2 \right\} = \\ = -\frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2} \left\{ \prod_{j=1}^k y_{2j-1}^2, \prod_{j=1}^l y_{2j}^2 \right\} \frac{(1+y_{2l+1}^2)}{\prod_{j=1}^l y_{2j+1}^2} - \prod_{j=1}^k y_{2j-1}^2 \left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2}, \frac{(1+y_{2l+1}^2)}{\prod_{j=0}^l y_{2j+1}^2} \right\} \prod_{j=1}^l y_{2j}^2. \tag{8.4.10}$$

The first bracket in (8.4.10) gives $\prod_{j=1}^k y_{2j-1}^2 \prod_{j=1}^l y_{2j}^2$ and hence

$$\{s_{2k}, s_{2l+1}\} = s_{2k}s_{2l+1} - \prod_{j=1}^k y_{2j-1}^2 \left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2}, \frac{(1+y_{2l+1}^2)}{\prod_{j=0}^l y_{2j+1}^2} \right\} \prod_{j=1}^l y_{2j}^2. \tag{8.4.11}$$

The several contributions in (8.4.11) can all be accounted for by the formula (8.4.9): if $l \ge k+1$ then one sees immediately that all terms in the bracket in (8.4.11) vanish. The only case when the bracket gives a nonzero contribution is for $k = l$:

$$\left\{ \frac{(1+y_{2k}^2)}{\prod_{j=1}^k y_{2j}^2}, \frac{(1+y_{2k+1}^2)}{\prod_{j=0}^k y_{2j+1}^2} \right\} = \left\{ \frac{1}{\prod_{j=1}^k y_{2j}^2}, \frac{1}{\prod_{j=0}^{k-1} y_{2j+1}^2} \right\} = -\frac{1}{\prod_{j=1}^k y_{2j}^2} \frac{1}{\prod_{j=0}^{k-1} y_{2j+1}^2}. \tag{8.4.12}$$

Combining this with (8.4.11) gives finally

$$\{s_{2k}, s_{2l+1}\} = 1+s_{2k}s_{2l+1} \tag{8.4.13}$$

To complete the verification remains only to check the case

$$\begin{aligned}\{s_1, s_{2K+2}\} &= \left\{-y_1^{-2}, \sum_{l=1}^{K} \frac{y_1^2}{\prod_{j=1}^{K} y_{2j}^2 \prod_{j=l}^{K} y_{2j}^2}\right\} = -\sum_{l=1}^{K} y_1^2 \left\{y_1^{-2}, \frac{1}{\prod_{j=1}^{K} y_{2j}^2 \prod_{j=l}^{K} y_{2j}^2}\right\} = \\ &= -\sum_{l=1}^{K} y_1^2 \left\{y_1^{-2}, \frac{1}{\prod_{j=1}^{K} y_{2j}^2}\right\} \frac{1}{\prod_{j=l}^{K} y_{2j}^2} - \sum_{l=1}^{K} y_1^2 \left\{y_1^{-2}, \frac{1}{\prod_{j=l}^{K} y_{2j}^2}\right\} \frac{1}{\prod_{j=1}^{K} y_{2j}^2}.\end{aligned} \tag{8.4.14}$$

In the second sum only the term $l = 1$ contributes and the result of this is $\frac{1}{\lambda^2}$; the first sum instead contributes $-s_1 s_{2K+2}$ and in total we find

$$\{s_1, s_{2K+2}\} = -\frac{1}{\lambda^2} + s_1 s_{2K+2}. \tag{8.4.15}$$

The verification is thus complete. ∎

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