Studies in Mechanism Design

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Abstract

Studies in Mechanism Design

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This thesis consists of three studies on mechanism design with and without monetary transfers. Following a brief introductory chapter, the second chapter contributes to the auction design literature and studies bidders with heterogeneous risk attitudes. The third and fourth chapters focus on matching couples to jobs without monetary transfers. The third chapter proposes a new method of preference aggregation for couples that enter the labour market together, and the fourth chapter proposes a new mechanism that accommodates couples in entry-level labour markets which relies on the preference aggregation studied in the third chapter.

In the second chapter I study the sale of a single indivisible good to two bidders with heterogeneous attitudes towards risk. Optimal auctions for risk neutral or risk averse bidders have been studied in the literature, but bidders are assumed to be either risk neutral or risk averse. My objective is to study the heterogeneity of bidders in terms of their risk attitude. In my model the valuations of the bidders are private information, however, one bidder is risk averse with a publicly known degree of risk aversion, while the other bidder is risk neutral. I derive the revenue maximizing Bayesian incentive compatible auction in this environment.

The third chapter focuses on the aggregation of a couple's preferences over their respective jobs when they enter a centralized labor market jointly, such as the market for assigning hospital residencies to medical students. Usually in such markets couples need to submit joint preferences over pairs of residency positions. Starting from two individual preference orderings over positions, we first study the Lexicographic and the Rank-Based Leximin rules, and then propose a family of aggregation rules, the k-Lexi-Pairing rules, and provide an axiomatic characterization of these rules. The parameter k indicates the degree of selfishness for one partner (and altruism for the other partner), with the least selfish Rank-Based Leximin rule at one extreme and the most selfish Lexicographic rule at the other extreme. Since couples care about geographic proximity, we also identify a simple parametric family of preference aggregation rules, (k, t)-Lexi-Pairing rules, which also reflect the couple's preference for togetherness.

In the fourth chapter we study centralized entry-level labour markets with couples and require couples to submit only (k, t)-Lexi-Pairing joint preferences as their input to the matching procedure. We introduce a new matching mechanism which takes advantage of the known preference structure of the joint preferences submitted by couples. This mechanisms resolves the cycles that typically arise in matching procedures with couples by working with the parameter t which indicates the degree of preference to be employed in the same geographic area for a couple. We also analyze the stability and efficiency properties of this new mechanism for couples' markets. This is the first study that takes into account how couples form their paired preference orderings when participating in a centralized matching procedure, and the first mechanism which makes explicit use of the preference aggregation parameters that indicate the preferences of couples for togetherness.

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Contribution of Authors

Chapter 2 is not co-authored.

Chapter 3 is a joint work with my supervisor, Dr. Szilvia Pápai.

Chapter 4 is a joint work with my supervisor, Dr. Szilvia Pápai.

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Chapter 1

Introduction

1.1 Optimal Seller's Revenue With Asymmetric Bidders

In the first study (Chapter 2), I consider an environment with a single indivisible object and two bidders. The valuations of the bidders are private information, so the bidders know that they can behave strategically. The bidders have asymmetric attitudes towards risk. One bidder is risk averse with a publicly known degree of risk aversion, while the other bidder is risk neutral. My objective is to study the selling mechanism that maximizes the seller's expected revenue. Since the bidders can behave strategically, any successful selling mechanism must have the property that it induces the bidders to be truthful in their report of respective valuations in the equilibrium. The equilibrium notion that is widely used in the literature is the notion of Bayesian Incentive Compatibility. It requires that truthful reporting by any bidder maximize his expected utility under the assumption that all other bidders are reporting truthfully. The expectation is computed with respect to the bidder's prior belief. In short, truthful reporting must constitute Bayes Nash equilibrium for the underlying incomplete information games. A successful selling mechanism also have the property that it is individually rational or in other words it induces the bidders to participate in the mechanism voluntarily. In this environment I derive the revenue maximizing Bayesian incentive compatible auction.

I identify two regions in the valuation space marked by cut-off valuations θ_*, $\bar{\theta}$ with $\theta_* < \bar{\theta} \leqslant 1$. There is no allocation if both bidders report values less than θ_*. Between $\left[\theta_*, \bar{\theta}\right]$ region, the risk neutral bidder wins the object. Thus, the auction would be as if there is a single risk neutral bidder.

Above the cut-off point $\bar{\theta}$, the object is allocated to the bidder with the higher valuation, thus, the auction is an all-pay auction that , it is deterministic. The payments for the bidders are completely determined by their respective allocation probability functions.

1.2 Preference Aggregation for Couples

In the second study (Chapter 3), we study the aggregation of a couple's preferences over their respective jobs when they enter a centralized labor market jointly, such as the market for assigning hospital residencies to medical students. Usually in such markets couples need to submit joint preferences over pairs of jobs, but it has not been studied previously how couples form their joint preference orderings.

The preference aggregation issue for a couple has not been addressed by the social choice theory literature either, which mainly focuses on aggregating preferences over social outcomes. Our problem formally differs from the preference aggregation literature in several aspects. The aggregation for couples has two preference rankings over individual positions as input, and a joint preference ranking over pairs of positions as an output. Our model also differs from usual models in that we only have two agents, while preference aggregation is typically considered for an arbitrary number of agents. This makes our task simpler, but it also renders tie-breaking more important since ties arise frequently with two agents only. Finally, in the labor market context we need to worry about complements in couples' preferences due to geographical considerations, which is absent from the preference aggregation literature and only pertains to aggregating preferences over private assignments with complementarities.

Starting from two individual preference orderings over individual jobs, we first study the Lexicographic and the Rank-Based Leximin rules, and then propose a family of aggregation rules, the k-Lexi-Pairing rules, and provide an axiomatic characterization of these rules. There are two parameters, k and t, for selfishness and togetherness respectively. The parameter k indicates the degree of selfishness for one partner (and altruism for the other partner), with the least selfish Rank-Based Leximin rule at one extreme and the most selfish Lexicographic rule at the other extreme. This parameter can also be seen as a trade-off parameter between the two partners. Since couples care about geographic proximity, we also identify a simple parametric family of preference aggregation rules which build on the k-Lexi-Pairing rules and reflect the couple's preference for togetherness.

1.3 Matching Couples in the Labour Market

In the third study (Chapter 4), we consider how to match couples in a centralized labour market by making use of the information about the couples' preference aggregation as studied in Chapter 3. For example, the National Resident Matching Program (NRMP) in the US assigns thousands of medical school graduates to hospital residency positions through a central clearing house each year. Such markets with couples pose notoriously difficult market design questions, since the existence of a stable matching is no longer guaranteed in the presence of couples, and the matching algorithm itself has to accommodate couples. While the matching algorithm that is currently used in the NRMP works quite well in practice, as demonstrated by simulations and justified by findings for large markets, it does not select a stable matching even when the preferences are responsive (which means, essentially, that hospital jobs are not complements), and thus a stable matching exists. The main reason for this issue is that couples wish to work in the same geographical area, which introduces complementarities in their preferences. Consequently, algorithms for couples markets tend to lead to inefficient outcomes or even wastefulness, and they tend to cycle if the algorithm attempts to circumvent wastefulness.

We propose a new mechanism, the Lexi Couples mechanism, for couples markets which is shown to be non-wasteful. Couples are required to submit only their individual preference orderings and the two parameters k and t, expressing the trade-offs between the partners and their preferences for togetherness, respectively. Thus, the mechanism satisfies informational efficiency and calculates the paired preferences based on these inputs, using the (k, t)-Lexi-Pairing rules from Chapter 3. Moreover, the mechanism makes explicit use of the togetherness parameter in order to resolve the arising cycles effectively. The Lexi Couples mechanism is shown to be responsive-stable, which means that if all couples have responsive preferences the mechanism selects a stable matching, a property that is not satisfied by the NRMP algorithm. Specifically, when all couples have responsive preferences, the Lexi Couples mechanism coincides with the Deferred Acceptance mechanism and picks the doctor-optimal matching.

Chapter 2

Optimal Seller's Revenue with Asymmetric Bidders

2.1 Introduction

The problem of finding the revenue-maximizing selling scheme for allocation of resources is a well studied problem. Consider the selling of a single indivisible object by a seller. In the most standard formulation of the problem, each potential bidder's valuation for the object is private information. Consequently, the bidders know that by behaving strategically, they can extract rent (because of their private information) from the seller. Any selling scheme that sets to maximize the revenue of the seller, must therefore provide adequate incentives to the bidder to report their private information truthfully in equilibrium.

At the same time the selling mechanism or scheme (throughout the paper, I use the two terms interchangeably), must induce the bidders to voluntarily participate in the scheme.

The equilibrium notion for truth-telling, that is widely used in the literature is Bayesian Incentive Compatibility (BIC). BIC requires truth-telling to maximize the expected utility being computed with respect to the bidder's prior belief and under the assumption that all other bidders are reporting their private information truthfully.

When there is a single indivisible object for sale, a selling mechanism that has been widely studied in the literature is an auction mechanism. Beginning with Harris and Raviv (1981 (a))

[12] and Harris and Raviv, (1981 (b)) [13] there have been a plethora of papers studying optimal auctions. In his classic paper on auctions, Myerson (1981) [30] studies optimal auction for a single indivisible object under the following assumptions, (a) the bidders are risk neutral, (b) the bidders have no budget constraints (i.e. no income effect) and (c) the bidders are symmetric in the sense that the only asymmetry in the bidders' preferences arrive through their private valuations. Since the classic result by Myerson, a vast number of papers have addressed the issue of optimal auction after relaxing some of the assumptions listed above. In particular, Matthews(1983) [29] and Maskin and Riley (1984) [28] consider selling an indivisible object when all bidders are risk averse. however, in either setting, the bidders are symmetric in the sense that the bidders have identical attitudes towards risk.

In the present paper, I continue this line of research and study optimal auctions in an environment where bidders are asymmetric in their attitudes towards risk. More specifically, I consider a simple benchmark model where there is one risk neutral bidder (n) and one risk averse bidder (a). The identity of the risk neutral bidder and the risk averse bidder as well as the utility function representing their preferences is commonly known. The valuation of the bidders are private information. The bidders drew their valuations identically and independently from a one dimensional set Θ. Since the bidders are asymmetric in their attitudes towards risk, they have to be treated asymmetrically. To capture this feature, I consider an all pay auction format which has been done in many other papers in optimal auction in different context (see for example Laffont and Robert (1996) [26]).

In this environment, I characterize the optimal auction as an all-pay auction. My main results are the following. I identify two regions in the valuation space marked by cut-off valuations θ_*, $\bar{\theta}$ with $\theta_* < \bar{\theta} \leqslant 1$. If both bidders report values less than θ_* the object is not allocated. If the reported valuation of the risk neutral bidder lies in the region $\left[\theta_*, \bar{\theta}\right]$, the risk neutral bidder is awarded the object. Thus, in this region, the auction proceeds as if there is a single risk neutral bidder. Beyond the cut-off point $\bar{\theta}$, the object is allocated to the bidder with the higher valuation, thus, the auction is an all-pay auction that , unlike in Matthews (1983) [29] is deterministic. The payments for the bidders are completely determined by their respective allocation probability functions.

My result, contrasts with the result by Matthews (1983) [29] , in the sense that unlike Matthews(1983) [29] , my optimal mechanism is deterministic. In this sense, my result is more in accord with the result by Myerson(1981) [30] that characterizes the optimal mechanism for risk neutral bidders. However the cutoff types and the interpretation of the cutoff types are different in my model. Moreover, in my model, there exist significant region in the valuation space where the object is allocated to the risk neutral bidder irrespective of whether he is of the higher valuation or not. Consequently the asymmetry in the bidders attitude towards risk introduces significant inefficiency in my model. Moreover the expected payoff to the seller in my model is lower than the expected payoff when both bidders are risk neutral or when both bidders share the same attitude towards risk. To summarize, the presence of the risk-averse bidder reduces competition in my model. For a significant range of valuations, the optimal mechanism takes the form of a monopoly fixed price. In this "lower" range of valuation, the risk averse bidder is not competitive and the object is allocated to the risk neutral bidder. Beyond a high threshold valuation, the mechanism takes the form of a competitive auction.

The literature on optimal auctions with risk averse bidders originates with the seminal papers by Matthews (1983) [29] and Maskin and Riley (1984) [28]. Both papers consider optimal allocation when the bidders are risk averse, but have identical attitudes towards risk. In my paper, I consider the preference for the risk averse bidder to be identical to the one present in Matthews (1983) [29]. I introduce asymmetry, by incorporating one additional risk-neutral bidder. In the context of aversion, Hu *et.al.* (2010) [15] consider a model where bidders and seller have asymmetric attitude towards risk, but they focus on revenue comparison between first and second price auctions. More recently Hu *et.al.* (2018) [16] study ascending price English auctions where bidders have asymmetric attitude towards risk. To the best of my knowledge there has been no paper that has considered optimal selling mechanisms with bidders with asymmetric attitude towards risk.

Relatedly, there is an extensive literature on optimal auctions with asymmetric bidders. However, the focus in that literature is on asymmetry that arises from budget constraint (see for example, Pai and Vohra (2014) [31] , and the references therein). More recently there has been a small literature on auctions with non-quasilinear preferences (see for example, Kazumura *et.al*(2020) [19] and the references therein). Notice that, in my model there is obvious non-quasilinearity , since one bidder is risk averse. However, the focus of this strand of literature is on more general non quasilinear preferences. On the other hand in my model, my focus is on a specific problem with one risk averse bidder and one risk neutral bidder, and my aim is to study the consequences on the optimal selling mechanism in this simple model.

The paper is organized as follows. In section 2 I introduce the model. In section 3 I derive the seller's problem and present my results in section 4. Section 5 concludes.

2.2 Model

Risk neutral seller sells one unit of an indivisible object. The seller's cost of producing the object is set at 0. There are two potential buyers, $i = \{a, n\}$. where I denote by a the risk-averse bidder and by n the risk-neutral bidder. Each bidder knows his own private valuation but is unaware of the realized valuation of the other bidder. Each bidder $i \in \{a, n\}$ draws his private valuation or "type" θ_i identically and independently from the set $\Theta = [0, 1]$ according to the distribution function $F(.)$. We assume $F(.)$ is differentiable everywhere with $f(.) = F'(.)$. In addition the distribution $F(.)$ satisfies *monotone hazard rate* condition, i.e., $\frac{f(\theta)}{1-F(\theta)}$ is non-decreasing in θ. All the information mentioned above is common knowledge.

For the risk neutral bidder n the utility from wealth level x is given by $u_n(x) = x$. On the other hand, the risk averse bidder a has a preference given by *constant absolute risk aversion (CARA)* utility function. In other words, for bidder a the utility from wealth level x is given by

$$u_a(x) = \frac{1 - e^{-Rx}}{R}$$

where $R \geq 0$ is the parameter for risk aversion. I assume that there exists a $\bar{R} > 0$, such that $R \in [0, \bar{R}]$. Since there is only one risk averse bidder, we drop the subscript a from u_a, to economize on notation, and simply denote the utility function of the risk averse bidder as $u(.)$.

Following Myerson (1981) [30], we restrict our attention to the direct selling mechanisms.

Definition 1: **(Direct Selling Mechanism)** A direct selling mechanism is a collection $\langle p_i(.,.), q_i(.,.)\rangle_{i\in\{a,n\}}$ where for each $q_i : [0,1]^2 \to [0,1]$ and for each $i \in \{a, n\}$, $p_i : [0,1]\times[0,1] \to \mathfrak{R}$ with the restriction that for all $(\theta_a, \theta_n) \in [0,1] \times [0,1]$,

$$0 \leq q_a(\theta_a, \theta_n) + q_n(\theta_a, \theta_n) \leq 1.$$

In a direct selling mechanism each bidder i is asked to report his type θ_i to the seller. Given a vector of reports (θ_a, θ_n), for each bidder i, $q_i(\theta_a, \theta_n)$ is the probability with which bidder i is allocated the object. For each vector of types $(\theta_a, \theta_n) \in [0,1] \times [0,1]$, $p_i(\theta_a, \theta_n)$ is the payment for bidder i. Given that the two bidders are asymmetric in their attitudes towards risk, we restrict our attention to the case where for each bidder i, and for any vector of types $(\theta_i, \theta_j), i \neq j$, $p_i(\theta_i, \theta_j) = p_i(\theta_i)$. In other words, the payment function for each bidder depends *only* on his reported type.

In the determination of an optimal selling mechanism the crucial elements are *expected probabilities of allocation* for the two bidders. Below I specify the expected allocation probability expressions for the two bidders.

Suppose that the risk averse bidder a reports type $\hat{\theta}_a$ to the mechanism. Then *expected allocation probability* of bidder a in the mechanism, denoted as Q_a is,

$$Q_a\left(\hat{\theta}_a\right) = \int_0^1 q_a\left(\hat{\theta}_a, \theta\right) f(\theta)\, d\theta. \tag{1}$$

Likewise the expected allocation probability of bidder n, denoted by Q_n, when he reports $\hat{\theta}_n$ is

$$Q_n\left(\hat{\theta}_n\right) = \int_0^1 q_n\left(\theta, \hat{\theta}_n\right) f(\theta)\, d\theta. \tag{2}$$

Given the expected allocation probability, I now present the expected utility for the two bidders. The expected utility of the risk-neutral bidder n, (denoted by U_n), who is of type θ_n and reports $\hat{\theta}_n$ is,

$$\begin{aligned} U_n\left(\theta_n, \hat{\theta}_n\right) &= Q_n\left(\hat{\theta}_n\right) u_n\left(\theta_n - p_n\left(\hat{\theta}_n\right)\right) + \left(1 - Q_n\left(\hat{\theta}_n\right)\right) u_n\left(\theta_n - p_n\left(\hat{\theta}_n\right)\right) \\ &= Q_n\left(\hat{\theta}_n\right)\left(\hat{\theta}_n - p_n\left(\hat{\theta}_n\right)\right) + \left(1 - Q_n\left(\hat{\theta}_n\right)\right)\left(-p_n\left(\hat{\theta}_n\right)\right) \\ &= Q_n\left(\hat{\theta}_n\right)\theta_n - p_n\left(\hat{\theta}_n\right). \end{aligned} \tag{3}$$

The expected utility of the risk averse bidder a, (denoted by $\hat{U}$), who is of type θ_a and reports

$\hat{\theta}_a$ is,

$$\hat{U}_a\left(\theta_a, \hat{\theta}_a\right) = Q_a\left(\hat{\theta}_a\right) u\left(\theta_a - p_a\left(\hat{\theta}_a\right)\right) \\ + \left(1 - Q_a\left(\hat{\theta}_a\right)\right) u\left(\theta_a - p_a\left(\hat{\theta}_a\right)\right). \tag{4}$$

Following Matthews (1983) [29], I now make a monotonic transformation of $\hat{U}$, that will be easier to work with. For any θ, define,

$$\psi(\theta, Q(\theta)) = -\frac{1}{R} ln\left[1 - Q\left(\hat{\theta}\right) + Q\left(\hat{\theta}\right) u'\left(\hat{\theta}\right)\right]. \tag{5}$$

Now noting that, $0 \leq \hat{U}(\theta, \hat{\theta}) < 1$ for all $\theta, \hat{\theta}$, I define, for every $\theta, \hat{\theta}$, $U(\theta, \hat{\theta})$ as follows:

$$U(\theta, \hat{\theta}) = -\frac{1}{R} \ln\left[-R\hat{U}\left(\theta, \hat{\theta}\right) + 1\right].$$

Then from equation (5) we can write,

$$U_a(\theta_a, \hat{\theta}_a) = \psi(\theta_a, Q_a(\hat{\theta}_a)) - p_a(\hat{\theta}_a). \tag{6}$$

Here, for any $y \geq 0$ and any $Q \in [0, 1]$, $\psi(y, Q)$ is the certainty equivalence of the lottery offering y with probability Q and 0 with probability $1 - Q$.

Equipped with the expected allocation probabilities, I can now proceed to the issue of incentives. All our constraints as well as the analysis of optimality will be conducted in terms of the expected allocation probabilities Q_i's and the payment functions p_i's. To that end, I will call a collection $\langle Q_a, Q_n, p_a, p_n \rangle$ a *scheme*. My first objective will be to find the *optimal scheme*. Finding an optimal scheme is not enough to characterize an optimal selling mechanism; I need then, to find the appropriate q_i's that generate the optimal Q_i's. In other words, I need to find the appropriate q_i's that implement Q_i's. Next I turn to the incentive issues.

2.2.1 Incentive Compatibility Constraints

One primary requirement for any scheme $\langle Q_a, Q_n, p_a, p_n \rangle$ is to provide appropriate incentives to the bidders to report their private valuations truthfully in equilibrium. The equilibrium concept that is standard in the literature is *Bayesian incentive compatibility (BIC).* Truthtelling is required to maximize the expected utility of every bidder under the assumption that all other bidders are reporting truthfully. Expected utility is computed with respect to the expected allocation probabilities specified by the scheme i.e., the Q_i's. In short, truthtelling is required to be a *Bayes Nash equilibrium* of the underlying incomplete information game. Formally,

DEFINITION 2: **(BIC)** A scheme $\langle Q, p\rangle \equiv \langle Q_a, Q_n, p_a, p_n\rangle$ is BIC if for every bidder $i \in \{a, n\}$, and every θ_i, θ_i',

$$U_i(\theta_i, \theta_i) \geq U_i(\theta_i', \theta_i). \tag{7}$$

Denote by $V_i(\theta_i) = U_i(\theta_i, \theta_i)$ the indirect expected utility function of bidder i of valuation θ_i from telling the truth.

The following celebrated lemma from Myerson (1981) [30], provides a characterization of the incentive compatibility of a scheme for the risk-neutral bidder.

Lemma 1. *(Myerson (1981)). Let $\langle Q, p\rangle$ be a scheme. The scheme $\langle Q, p\rangle$ is BIC for the risk neutral bidder if and only if,*

1. *the expected allocation probability Q_n for the risk neutral bidder is nondecreasing in the risk neutral bidder's type; that is, for every $\theta, \theta' \in [0,1]$,*

$$\theta > \theta' \Rightarrow Q_n(\theta) \geq Q_n\left(\theta'\right). \tag{8}$$

2. *For every $\theta \in [0,1]$, the following integral condition is satisfied:*

$$V_n\left(\theta\right) = V_n\left(0\right) + \int_0^{\theta_n} Q_n\left(x\right) dx. \tag{9}$$

The above lemma is a well known result and I skip the proof. Instead, I now turn to the characterization of incentive compatibility for the risk averse bidder. For the risk averse bidder, recall that for any $\theta \in [0,1]$, $V_a(\theta) = U_a(\theta, \theta) = \psi(\theta, Q_a(\theta)) - p_a(\theta)$. Also note that, for any

$\theta \in [0,1]$,

$$\begin{aligned}\psi_1(\theta, Q_a(\theta)) &= \frac{\partial}{\partial\theta}\psi(\theta, Q_a(\theta)) \\ &= \frac{1}{1+\frac{1-Q_a(\theta))}{Q_a(\theta)u'(\theta)}}.\end{aligned}$$

The following lemma from Matthews (1983, *Lemma 4, page 378,*) [29] characterizes the incentive compatibility for the risk averse bidder.

Lemma 2. *(Matthews, (1983)). A scheme $\langle Q, p\rangle$ is BIC for the risk averse bidder, if and only if,*

1. *for all $\theta, \theta' \in [0,1]$,*
$$\theta > \theta' \Rightarrow \frac{Q_a(\theta)}{1-Q_a(\theta)} \geq \frac{Q_a(\theta')}{1-Q_a(\theta')}. \tag{10}$$

2. *for all $\theta \in [0,1]$,*
$$V_a\left(\theta_a\right) = V_a\left(0\right) + \int_0^{\theta_a} \psi_1\left(z, Q_a\left(z\right)\right) dz. \tag{11}$$

Note that,
$$Q_a(\theta) \geq Q_a(\theta') \Rightarrow \frac{Q_a(\theta)}{1-Q_a(\theta)} \geq \frac{Q_a(\theta')}{1-Q_a(\theta')}.$$

The above two lemmas are standard and well established in the literature and I skip the proofs.

From (9) one obtains the payment function for the risk neutral bidder: for every $\theta \in [0,1]$,

$$p_n\left(\theta\right) = p_n\left(0\right) + \theta Q_n\left(\theta\right) - \int_0^{\theta} Q_n\left(z\right) dz. \tag{12}$$

Likewise, from (11), one notices that the expected allocation probability for the risk-averse bidder completely specifies the payment function for him. Formally, for any $\theta \in [0,1]$,

$$\begin{aligned}U_a(\theta,\theta) = V_a(\theta) &= \psi(\theta, Q_a(\theta)) - p_a(\theta) \\ \Rightarrow p_a(\theta) &= \psi(\theta, Q_a(\theta)) - V_a(\theta).\end{aligned}$$

Substituting (11) into the above equation we obtain,

$$p_a\left(\theta\right) = p_a\left(0\right) + \psi\left(\theta, Q_a\left(\theta\right)\right) - \int_0^{\theta} \psi_1\left(z, Q_a\left(z\right)\right) dz. \tag{13}$$

I now turn to the second set of constraints that are important- namely, the requirement that bidders must be induced to participate in the mechanism *voluntarily*.

2.2.2 Individual Rationality Constraints

The *individual rationality (IR)* constraints impose the restriction that each bidder of each type gets at least as much utility from participating in the auction as from not participating. The payoff from the outside option of not participating is normalized to zero. Formally,

DEFINITION 3: **(IR)** A scheme $\langle Q, p \rangle$, staisfies individual rationality if for every bidder $i \in \{a, n\}$, and for every type θ,

$$V_i(\theta) = U_i(\theta, \theta) \geq 0. \tag{14}$$

The above two sets of constraints – the BIC constaints and the IR constrainst are not the only constraints one needs to take into account. It has to be the case that the scheme must be *feasible*. I next turn to the feasibility constraint.

2.2.3 Feasibility Constraint

DEFINITION 4: **Feasibility Constraint (F)** A scheme $\langle Q_a, Q_n, p_a, p_n \rangle$ satisfies feasibility constraint (F), if for every $\theta \in [0, 1]$,

$$\begin{aligned} \int_\theta^1 Q_a(\alpha) dF(\alpha) + \int_\theta^1 Q_n(\alpha) dF(\alpha) &\leq 1 - \left[\int_0^\theta dF(\alpha) \right]^2 \\ &= 1 - \left[F(\theta) \right]^2 \end{aligned} \tag{15}$$

In other words,

$$Y(\theta) = 1 - \left[F(\theta) \right]^2 - \int_\theta^1 Q_a(\alpha) dF(\alpha) - \int_\theta^1 Q_n(\alpha) dF(\alpha) \geq 0 \tag{16}$$

with $Y(1) = 0$.

The left hand side of (15) is the expected probability that the object is allocated to a bidder with type θ which must be less than or equal to the probability that at least one bidder has a type θ (see, Border (1991) [3]). The general condition is for all possible subsets of $[0, 1]$, however as it is shown in Border (1991) [3], this is equivalent to (15).

2.3 Seller's problem

If the seller uses a scheme $\langle Q_a, Q_n, p_a, p_n \rangle$ then the expected payoff obtained from a risk averse bidder of type θ is $p_a(\theta)$. Likewise, the expected payment from a risk neutral bidder of valuation θ is $p_n(\theta)$. Therefore the seller's expected revenue is,

$$R = \int_0^1 p_a(\theta) f(\theta) \, d\theta + \int_0^1 p_n(\theta) f(\theta) \, d\theta. \tag{17}$$

Replacing the values of $p_a(\theta)$ and $p_n(\theta)$ we obtain,

$$\begin{aligned} R = & \int_0^1 \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) Q_n(\theta) f(\theta) \, d\theta \\ & + \int_0^1 \left(\frac{\psi(\theta, Q_a(\theta))}{\psi_1(\theta, Q_a(\theta))} - \frac{1 - F(\theta)}{f(\theta)} \right) \psi_1(\theta, Q_a(\theta)) f(\theta) \, d\theta. \end{aligned} \tag{18}$$

Notice in this case, the seller's problem is an optimal control problem. The problem of the seller is to maximize (18) by choosing control functions Q_n and Q_a. Also, note that the Q_a and Q_n completely determine the payment functions p_a and p_n. Thus the seller's objective is to maximize (18) subject to the incentive compatibility constraints for the risk neutral and risk averse bidders, the individual rationality constraints and the feasibility constraint (8), (9), (10), (11), the set of IR constraints, (14) and (15). Of these the integral condition in the incentive compatibility constraints (9) and (11) have already been incorporated in the objective function. The remaining constraints are the monotonicity as part of the incentive compatibility constraints (8) and (10), the individual rationality constraints and the feasibility constraint. I will solve the relaxed problem involving only the feasibility constraint and ignoring the monotonicity constraints and the individual rationality constraints and then establish that the solution satisfies the monotonicity and individual rationality constraints.

Let us recall the feasibility constraint, (16): for all $\theta \in [0, 1]$,

$$Y(\theta) \geq 0.$$

In the seller's maximization problem, Y is the state variable with equation of motion

$$Y'(\theta) = \Big[Q_n(\theta) + Q_a(\theta) - 2F(\theta)\Big] f(\theta) \text{ with } Y(1) = 0. \tag{19}$$

For (Q_a, Q_n) to be implementable, the constraints $Y(\theta) \geq 0$ and (19) are necessary as shown in Matthews (1983) [29] and Maskin and Riley (1984) [28]. For implementation it is necessarily the case that $\forall i \in \{a, n\}$,

$$0 \leqslant Q_i(\theta) \leqslant 1. \tag{20}$$

The solution, if it exists, will be in terms of (Q_a, Q_n). The Q_is completely determine the payment functions p_is and together $\langle Q_i, p_i \rangle_{i \in \{a,n\}}$ will constitute an optimal scheme. The scheme then has to be shown to be feasible by showing that there exist actual allocation probability functions q_is that *implement* the Q_is.

In order to do so, following Maskin and Riley (1984) [28] and Matthews (1983) [28] , we will need to impose a further regularity condition.

Definition 4: **Regularity Condition (RC)** For any $R \in [0, \bar{R}]$, the term $u(\theta) - \frac{1-F(\theta)}{f(\theta)}$ is increasing in $\theta \in [0,1]$. In other words, for any $\theta, \alpha \in [0,1]$, $\theta > \alpha$,

$$u(\theta) - \frac{1 - F(\theta)}{f(\theta)} \geq u(\alpha) - \frac{1 - F(\alpha)}{f(\alpha)}. \tag{21}$$

We denote by $\tau(\theta)$ the following:

$$\tau(\theta) = u(\theta) - \frac{1 - F(\theta)}{f(\theta)} \tag{22}$$

In addition we need to define two other variables. For any $\theta \in [0,1]$, define $\gamma(\theta)$ as follows:

$$\gamma(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

Thus $\gamma(\theta)$ is the *virtual valuation* of a bidder of type θ. The second variable is defined as follows:

for any $\theta \in [0, 1]$, define,

$$\alpha(\theta) = \frac{u(\theta)}{u'(\theta)} - \frac{1 - F(\theta)}{f(\theta)} \tag{23}$$

It is immediate that for any $\theta \in (0, 1)$,

$$\tau(\theta) < \gamma(\theta) < \alpha(\theta). \tag{24}$$

A first step in our maximization of the objective function (18) subject to the constraint (15) is to define the Hamiltonian,

$$\begin{aligned} H(\theta, Q_a, Q_n, Y, \mu) &= \left(\theta_n - \frac{1 - F(\theta)}{f(\theta)}\right) Q_n(\theta_n) f(\theta) \\ &+ \left(\frac{\psi(\theta_a, Q_a(\theta_a))}{\psi_1(\theta_a, Q_a(\theta_a))} - \frac{1 - F(\theta)}{f(\theta)}\right) \psi_1(\theta_a, Q_a(\theta_a)) f(\theta) \\ &+ \mu(\theta) \Big[Q_n(\theta) + Q_a(\theta) - 2F(\theta)\Big] f(\theta), \end{aligned} \tag{25}$$

and the extended Hamiltonian or the Lagrangian,

$$L = H + \eta Y(\theta).$$

Here μ is the co-state variable corresponding to the equation of motion condition (19), and η is the Lagrange multiplier corresponding to the feasibility condition (16). Note that when Q_i's are chosen to maximize H, for fixed (θ, Y, μ) the maximized Hamiltonian is concave in Y. We now invoke, Theorem 8 in Seierstad and Sydsaeter (1977, page 380) [35] to say the following: The constraint conditions (20), (16) and (19) along with the conditions listed below are necessary and sufficient for an optimal solution to exist. The conditions are as follows:

(i). for any θ,

$$\mu'(\theta) = -\frac{\partial L}{\partial Y} = -\eta, \quad \mu(0) \leq 0, \quad \mu(0) Y(0) = 0. \tag{26}$$

(ii). if μ is discontinuous at θ, then θ is an entry or exit point of an interval upon which $Y = 0$ and

$$\mu(\theta^-) > \mu(\theta^+) \tag{27}$$

(iii). for any θ,

$$\eta(\theta) \geq 0, \eta(\theta) Y(\theta) = 0. \tag{28}$$

(iv). $H(\theta, Y(\theta), (Q_i(\theta))_{i \in \{a,n\}}, \mu(\theta))$ is continuous.

(v). At the solution (Q_n^*, Q_a^*), for all $\theta \in [0, 1]$,

$$H(\theta, Y(\theta), (Q_i^*(\theta))_{i \in \{a,n\}}, \mu(\theta)) \geqslant H(\theta, Y(\theta), (Q_i(\theta))_{i \in \{a,n\}}, \mu(\theta)). \tag{29}$$

We are now in a position to state and prove our results.

2.4 Result

I present two propositions. In Proposition (1), I characterize the optimal scheme $\langle Q_a^*, Q_n^*, p_a^*, p_a^* \rangle$ and in Proposition (2), I show the existence of the associated q_i's.

One more function needs to be defined. Recall that for any θ

$$\gamma(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

Definition 5: For any $\theta, \alpha \in [0, 1]$ such that $0 \leq \gamma(\alpha) \leq 1$, define,

$$x(\theta, \gamma(\alpha)) = \frac{u(\theta) - \sqrt{(u(\theta))^2 - 4\gamma(\alpha)\left(\frac{1-F(\theta)}{f(\theta)}\right) u'(\theta)}}{2\gamma(\alpha)}. \tag{30}$$

Given our regularity condition (RC), (condition (21)), it is easy to verify that, for any α such that $0 \leq \gamma(\alpha) \leq 1$, $x(\theta, \gamma(\alpha))$ is well defined. I now state the main result in the paper.

Proposition 1. *In my model, the optimal feasible scheme $\langle Q_a^*, Q_n^*, p_a^*, p_n^* \rangle$ takes the following form: there exist cutoff valuations θ^*, $\bar{\theta}$, such that $\theta_* < \bar{\theta}$, and such that,*

1. *for the risk neutral bidder, n*

$$Q_n^*(\theta) = \begin{cases} 0, & \textit{if } \theta < \theta_* \\ F(\bar{\theta}), & \textit{if } \theta_* \leq \theta < \bar{\theta} \\ F(\theta), & \textit{otherwise.} \end{cases} \tag{31}$$

2. *for the risk averse bidder, a*

$$Q_a^*(\theta) = \begin{cases} 0, & \textit{if } \theta \leq \bar{\theta} \\ F(\theta), & \textit{otherwise.} \end{cases} \tag{32}$$

3. *The cutoffs θ_* and $\bar{\theta}$ are defined as follows:*

   (a) *the cutoff θ_* is such that,*

$$\gamma(\theta_*) = \theta_* - \frac{1 - F(\theta_*)}{f(\theta_*)} = 0. \tag{33}$$

   (b) *the cut-off $\bar{\theta}$ is such that,*

$$\frac{1 - x\left(\theta, \gamma(\bar{\theta})\right)}{1 - u'(\theta)} = F(\bar{\theta}). \tag{34}$$

4. *The payment functions are determined through Q_n^* and Q_a^* as follows:*

$$p_n^*(\theta) = p_n^*(0) + \theta Q_n^*(\theta) - \int_0^\theta Q_n(\alpha)\, d\alpha \tag{35}$$

$$p_a^*(\theta) = p_a^*(0) + \psi(\theta, Q_a^*(\theta)) - \int_0^\theta \psi_1(\alpha, Q_a^*(\alpha))\, d\alpha \tag{36}$$

Proof. As noted above, concentrating on the first order conditions for the Hamiltonian would suffice. For any Q and any y , with $0 \leq Q \leq 1$, note that,

$$\psi_2(y, Q) = \frac{\partial}{\partial Q}\psi(y, Q) = \frac{u(y)}{1 - Q + Qu'(y)}$$

$$\text{and } \psi_{12}(y, Q) = \frac{\partial^2}{\partial Q \partial y}\psi(y, Q) = \frac{u'(y)}{(1 - Q + Qu'(y))^2}.$$

Now, for any fixed θ, μ, Y, and if Q_n, $Q_a > 0$, the Hamiltonian first order conditions are as follows.

$$Q_n : \frac{\partial}{\partial Q_n} H = 0 \Rightarrow \theta - \frac{1 - F(\theta)}{f(\theta)} + \mu(\theta) = 0. \tag{37}$$

$$Q_a : \frac{\partial}{\partial Q_a} H = 0 \Rightarrow \psi_2(\theta, Q_a(\theta)) - \frac{1 - F(\theta)}{f(\theta)} \psi_{12}(\theta, Q_a(\theta)) + \mu(\theta) = 0. \tag{38}$$

From (37), we obtain that if $Q_n > 0$, then at the optimum,

$$\mu(\theta) = \frac{1 - F(\theta)}{f(\theta)} - \theta. \tag{39}$$

Note that $\mu(\theta) > 0$ for $\theta < \theta_*$ and for $\theta > \theta_*$, $\mu(\theta) < 0$.

We first turn to the problem of the risk-averse bidder. Note that in equation (38) $-\mu$ is the opportunity cost to the seller for not selling the object to another bidder. Denote $\gamma = -\mu$. In order to analyse the problem of the risk averse bidder, I begin with the case where there is no opportunity cost i.e., no capacity constraint. In other words, $\gamma = -\mu = 0$. In this case, from the Hamiltonian we obtain

$$Q_a(\theta) = \bar{Q}_a(\theta, \gamma = 0) = \frac{1 - \frac{1-F(\theta)}{f(\theta)} \frac{u'(\theta)}{u(\theta)}}{1 - u'(\theta)}. \tag{40}$$

The above equation (40), provides two cut-off valuations $\underline{\theta}_0$ and $\bar{\theta}_0$ such that, for all $\theta \in (\underline{\theta}_0, \bar{\theta}_0)$ such that, (i) $\bar{Q}_a(\theta, \gamma = 0)$ is strictly increasing in in θ, (ii) $\bar{Q}_a(\underline{\theta}_0, \gamma = 0) = 0$ and (iii) $\bar{Q}_a(\bar{\theta}_0, \gamma = 0) = 1$. Thus for a seller with zero opportunity cost (no capacity constraint), the optimal allocation probability for the risk averse bidder is given by $\bar{Q}_a(\theta, \gamma = 0)$ if $\theta \in (\underline{\theta}_0, \bar{\theta}_0)$. On the other hand the optimal allocation probability is 0 if $\theta < \underline{\theta}_0$ and is 1, if $\theta \geq \bar{\theta}_0$.

Now fix $\gamma = -\mu$ at some positive level, i.e., $\gamma > 0$. We continue to assume that there is no capacity constraint. This may happen, either because, there is only the risk averse bidder a, that there is more than one unit to sell. The interpretation of γ is the following: the seller, with no capacity constraint will incur a cost γ if he sells the object to the risk averse bidder. Assuming

$Q_a > 0$, from the first order condition (38), we obtain,

$$\begin{aligned}\bar{Q}_a(\theta,\gamma) &= \frac{1-\frac{u(\theta)-\sqrt{(u(\theta))^2-4\gamma\left(\frac{1-F(\theta)}{f(\theta)}\right)u'(\theta)}}{2\gamma}}{1-u'(\theta)} \\ &= \frac{1-x(\theta,\gamma)}{1-u'(\theta)}. \end{aligned} \tag{41}$$

The above equation, provides two cutoff valuations $\underline{\theta}_\gamma$ and $\bar{\theta}_\gamma$ such that the optimal allocation probability of the risk averse bidder takes the following form,

$$\bar{Q}_a(\theta,\gamma) = \begin{cases} 0, & \text{if } \theta < \underline{\theta}_\gamma \\ \frac{1-x(\theta,\gamma)}{1-u'(\theta)}, & \text{if } \underline{\theta}_\gamma \leq \theta < \bar{\theta}_\gamma \\ 1, & \text{otherwise.} \end{cases} \tag{42}$$

For each fixed value of γ, $\bar{Q}_a(\theta,\gamma)$ is nondecreasing in θ. Moreover, $\bar{Q}_a(\theta,\gamma)$ is strictly increasing in θ within the range, $\underline{\theta}_\gamma \leq \theta < \bar{\theta}_\gamma$. Also note that $\bar{Q}_a(\theta,\gamma)$ decreases with γ, that is $\gamma_1 > \gamma_2 \Rightarrow \bar{Q}_a(\theta,\gamma_1) \leq \bar{Q}_a(\theta,\gamma_2)$ for any $\theta \in [0,1]$. Recalling that we have defined $\gamma(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$, we now define $\bar{\theta}$ to be such that,

$$\bar{Q}_a(\bar{\theta},\gamma(\bar{\theta})) = F(\bar{\theta}). \tag{43}$$

Set $\bar{\gamma} = \gamma(\bar{\theta})$. The interpretation is that , without any capacity constraint a seller with cost $\gamma(\bar{\theta})$ would assign the risk averse bidder with valuation $\bar{\theta}$, the probability $F(\bar{\theta})$ of obtaining the unit. In other words, in the absence of any capacity constraint, a seller would assign a bidder of type $\bar{\theta}$ the probability of $F(\bar{\theta})$ of obtaining the unit, and if he assigns the object to the risk averse bidder, incurs a cost of $\bar{\gamma} = \gamma(\bar{\theta}) = \bar{\theta} - \frac{1-F(\bar{\theta})}{f(\theta)}$. I define the valuation $\underline{\theta}$ as,

$$\bar{Q}_a(\underline{\theta},\gamma(\bar{\theta})) = 0. \tag{44}$$

We make two observations. First, it is immediate that $\underline{\theta} < \bar{\theta}$. Secondly, since Q is decreasing in γ, and increasing in θ for each fixed γ, a simple fixed point argument reveals that $\bar{\theta}$ exists. Two cases may occur.

Case I: : $\bar{\theta} = 1$.

In this case, note $\gamma(\bar{\theta}) = 1$, and $\bar{Q}(\theta, 1) = 0$. i.e. the cost is too high for the object to be allocated to the risk averse bidder. This is the only case where $\underline{\theta} = \bar{\theta} = 1$ and $\bar{Q}(\theta, 1) = 0$.

CASE II: $\bar{\theta} < 1$.

In this case $\underline{\theta} < \bar{\theta}$ and $\bar{Q}(\theta, \gamma(\bar{\theta}))$ is strictly increasing in θ in the interval $\underline{\theta} < \theta < \bar{\theta}$.

To summarize, we have defined $\bar{Q}_a(\theta, \gamma(\bar{\theta}))$, as follows :

$$\bar{Q}(\theta, \gamma(\bar{\theta})) = \begin{cases} 0, & \text{if } \theta \leq \underline{\theta} \\ \frac{1 - x\left(\theta, \gamma(\bar{\theta})\right)}{1 - u'(\theta)}, & \text{if } \underline{\theta} < \theta < \bar{\theta} \\ F(\theta), & \text{otherwise.} \end{cases} \tag{45}$$

Notice that over the range $\underline{\theta} < \theta < \bar{\theta}$, $\bar{Q}(\theta, \gamma(\bar{\theta}))$ is the optimal allocation probability, as obtained from the first order condition, for a seller with cost $\bar{\gamma} = \gamma(\bar{\theta})$. That is, *in the absence of any capacity constraint*, a seller will assign to a risk averse bidder of type $\theta \in (\underline{\theta}, \bar{\theta})$, a probability $\bar{Q}(\theta, \gamma(\bar{\theta}))$ of getting the object, and would incur a cost of $\bar{\gamma} = \gamma(\bar{\theta})$ in the event the object is assigned to the bidder.

We now turn to the problem for the risk neutral bidder and see how the unit capacity constraint affects the allocation probabilities. From the first order condition (37), we obtain, that if $Q_n > 0$,

$$\gamma(\theta) = -\mu(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

Since $\mu(\theta) \leq 0$ for all θ, the constraint $0 \leq Q_n \leq 1$, immediately implies that for $\theta \leq \theta_*$, the optimal allocation probability for the risk neutral bidder is zero, i.e., $Q_n^\star(\theta) = 0$ for $\theta \leq \theta_*$.

Moreover, it is immediate that $\theta_* < \underline{\theta}_0 \leq \underline{\theta}$. So whenever the type of the risk-averse bidder is less than $\underline{\theta}$ and the type of the risk neutral bidder is $\theta > \theta_*$, the revenue of the seller is maximized by allocating the object to the risk neutral bidder. Thus for $\theta \in (\theta_*, \underline{\theta}]$,

$$Q_n^*(\theta) = F(\underline{\theta}). \tag{46}$$

Now we come to the crucial part of the analysis. Consider a type θ for the risk neutral bidder such that $\underline{\theta} \leq \theta < \bar{\theta}$. From the first order condition for the risk neutral bidder, (37), we obtain that at the optimum, the opportunity cost to the seller of allocating the object to the risk neutral bidder is,

$$\gamma(\theta) = -\mu(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

That is for a risk neutral bidder with type θ, if the seller allocates the object, he incurs an opportunity cost of $\gamma(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$. On the other hand, if the seller allocates the object to the risk averse bidder with positive probability, he incurs a cost of $\bar{\gamma} = \gamma(\bar{\theta})$. Now,

$$\theta < \bar{\theta} \Rightarrow \gamma(\theta) < \gamma(\bar{\theta}) = \bar{\gamma}.$$

This implies, that it is always optimal for the seller to allocate the object to the risk-neural bidder of type θ. Since θ arbitrarily chosen, this implies that for all θ, it is optimal for the seller to allocate the object to the risk neutral bidder. In other words,

$$Q_n^*(\theta) = F(\bar{\theta}) \quad \text{ if, } \theta \leq \bar{\theta}. \tag{47}$$

The above equation in turn implies that,

$$Q_a^*(\theta) = 0 \quad \text{ if, } \theta \leq \bar{\theta}. \tag{48}$$

Now at $\bar{\theta}$, the opportunity costs for the two types of bidders coincide. Consequently, for any $\theta > \bar{\theta}$, the optimal scheme is to allocate the object to the bidder of the higher type. That is for all $i \in \{a, n\}$,

$$Q_i^*(\theta) = F(\theta) \quad \text{if} \quad \theta > \bar{\theta}. \tag{49}$$

Combining the analyses for the risk-averse and the risk neutral bidder we arrive at the optimal allocation probabilities as follows:

1. for the risk averse bidder, a

$$Q_a^*(\theta) = \begin{cases} 0, & \text{if } \theta \leq \bar{\theta} \\ F(\theta), & \text{otherwise.} \end{cases} \tag{50}$$

2. for the risk neutral bidder, n

$$Q_n^*(\theta) = \begin{cases} 0, & \text{if } \theta < \theta_* \\ F(\bar{\theta}), & \text{if } \theta_* \leq \theta < \bar{\theta} \\ F(\theta), & \text{otherwise.} \end{cases} \tag{51}$$

Note that both $Q_a^\star$ and $Q_n^\star$ are non-decreasing in θ.

The payment functions are determined from the expected allocation probabilities as follows:

$$\begin{aligned} p_n^*(\theta) &= p_n^*(0) + \theta Q_n^*(\theta) - \int_0^\theta Q_n(\alpha)\, d\alpha \\ p_a^*(\theta) &= p_a^*(0) + \psi(\theta, Q_a^*(\theta)) - \int_0^\theta \psi_1(\alpha, Q_a^*(\alpha))\, d\alpha. \end{aligned} \tag{52}$$

We now complete the proof. Define $\mu^\star$ as

$$\mu^\star(\theta) = \begin{cases} 0, & \text{if } \theta \leq \bar{\theta} \\ \frac{1-F(\theta)}{f(\theta)}, & \text{otherwise.} \end{cases}$$

The constraints, (20) and (19) are satisfied by construction. Moreover $Y \geq 0$, so the constraint (16) holds. The Hamiltonian, composed of $Y, \mu^\star$, Q_a^* and $Q_n^\star$ is continuous in θ. More over, with respect to Q_n^*, the relevant part of the Hamiltonian is concave in Q_n^*. Likewise, with respect to Q_a^* the relevant part of the Hamiltonian is concave in Q_a^*. Hence $Q_n^\star$, Q_a^* maximize the Hamiltonian. Moreover, whenever, $Y(\theta) > 0$, $\mu^\star(\theta) = 0$. Define -$\eta^\star$ to be the left derivative of $\mu^\star$. Also note whenever, $\mu^\star$ is positive, $Y(\theta) = 0$. Therefore $\eta^\star(\theta)Y(\theta) = 0$, as desired. This completes the proof. □

We still need to show the existence of q_i's that implement the Q_i^*'s. In addition we need to show that the second set of ignored constraints – the IR constraints (14), are satisfied. We establish this in our next result.

Proposition 2. *When there are two bidders, one risk averse and one risk neutral with the identities of the risk averse and risk neutral bidder commonly known, the optimal selling mechanism*

$\langle q_n, q_a, p_a, p_n \rangle$ takes the form of an all pay auction. The bidders submit non-refundable bids. The allocation probabilities, i.e. the $q_i s$, $i \in \{a, n\}$ are determined as follows:

1. *For the risk neutral bidder n, there are two subcases depending on whether the valuation/type of the risk averse bidder $\theta_a < \bar{\theta}$ or $\theta_a > \bar{\theta}$. We list the two cases separately.*

   (a) *Case 1: $\theta_a < \bar{\theta}$: In this case,*

$$q_n^*(\theta_a, \theta_n) = \begin{cases} 0, & \text{if } \theta_n < \theta_* \\ 1, & \text{otherwise.} \end{cases} \tag{53}$$

   (b) *Case 2: $\theta_a > \bar{\theta}$. In this case,*

$$q_n^*(\theta_a, \theta_n) = \begin{cases} 0, & \text{if } \theta_n < \theta_a \\ 1, & \text{otherwise.} \end{cases} \tag{54}$$

2. *For the risk averse, bidder, $q^*(\theta_a, \theta_n) = 0$ if $\theta_a < \bar{\theta}$. On the other hand, if $\theta_a > \bar{\theta}$ then,*

$$q_a^*(\theta_a, \theta_n) = \begin{cases} 0, & \text{if } \theta_a < \theta_n \\ 1, & \text{otherwise.} \end{cases} \tag{55}$$

3. *The payment functions are given by,*

$$p_n^*(\theta) = p_n^*(0) + \theta Q_n^*(\theta) - \int_0^{\theta} Q_n^*(\alpha)\, d\alpha \tag{56}$$

   and

$$p_a^*(\theta) = p_a^*(0) + \psi(\theta, Q_a^*(\theta)) - \int_0^{\theta} \psi_1(\alpha, Q_a^*(\alpha))\, d\alpha \tag{57}$$

   The Q^*a and Q_n^* functions are as defined in equations (50) and (51) respectively. Moreover the above selling mechanism is individually rational.

Proof. First observe that for any $(\theta_a, \theta_n) \in [0, 1]^2$, $0 \leq q_i^*(\theta_a, \theta_n) \leq 1$ for all i. Moreover, whenever

$q_i^*(\theta_a,\theta_n)=1$, $q_j^*(\theta_a,\theta_n)=0$ for $j\neq i$. This establishes that

$$q_a^*(\theta_a,\theta_n)+q_n^*(\theta_a,\theta_n)\leq 1. \tag{58}$$

Moreover, $q_i^\star$, $i\in\{a,n\}$, generate the expected allocation probabilities Q_i^*'s . Hence the optimal selling scheme in Proposition (1) is implementable by my selling mechanism. It remains to be shown now is that the selling mechanism is individually rational.

Establishing individual rationality for the risk neutral bidder is straight forward. Note that for the risk neutral bidder,

$$V_n(\theta)=V_n(0)+\int_0^\theta Q_n^*(x)dx. \tag{59}$$

We have already established that Q_n^* is non decreasing in θ. So IR requires that for any $\theta<\theta_*$, $p_n^\star=0$. This implies $V_n(0)=0$ and since $Q_n^\star$ is non-decreasing, $V_n(\theta)\geq 0$ for all θ. This establishes individual rationality.

Note that if $\theta_*<\theta<\bar{\theta}$, the risk neutral bidder gets the object for sure and his payment is a fixed fee $\theta_*F(\bar{\theta})$. For $\theta>\bar{\theta}$, $p_n^\star(\theta)=\theta_*F(\bar{\theta})+t_n(\theta)$ where $\theta_*F(\bar{\theta})+t_n(\theta)$ satisfy the integral condition, (12).

We now turn to the issue of individual rationality for the risk averse bidder . Note that for the risk averse bidder of type θ,

$$V_a(\theta)=V_a(0)+\int_0^\theta \psi_1(z,Q_a(z))\,dz \tag{60}$$

Since the risk averse bidder is not allocated the object with positive probability if $\theta\leq\bar{\theta}$, IR constraint for the risk averse bidder implies, $p_a^*(\theta)\leq 0$, for $\theta\leq\bar{\theta}$. This forces, $p_a^*(\theta)=0$ for $\theta\leq\bar{\theta}$. Now $0\leq\psi_1(z,Q_a(z))\leq 1$ for all $z\in[0,1]$. Consequently,

$$V_a(\theta)=0+\int_{\bar{\theta}}^\theta \psi_1(z,Q_a(z))\,dz\geq 0. \tag{61}$$

This establishes individual rationality for the risk averse bidder and completes the proof. □

2.5 Conclusion

In this paper, I have studied the optimal mechanism for selling an indivisible object when there is one risk neutral and one risk averse bidder. My results show that there is a "significant region" in the valuation space where the risk averse bidder, in short the presence of the risk averse bidder limits competition. The interesting issue is how the optimal mechanism deals with this. It does so by incorporating features of, both a competitive auction, forcing bidders to compete in the higher range of valuations, and monopoly fixed pricing for the risk neutral bidder in the lower range of valuations where the risk averse bidder is not competitive. The determination of the high threshold, beyond which both bidders are competitive, is subtle and not immediate.

In this paper, I have studied a very simple model where there are two bidders whose attitudes towards risk are different by commonly known. There are many directions in which the model can be advanced. One obvious one is to introduce many bidders with heterogeneous attitude towards risk. A more subtle problem is to consider the same problem as above, but where the risk aversion parameter is not common knowledge. This second problem then becomes a problem of designing a successful selling mechanism where types are multi-dimensional. The difficulty is that the literature on multi-dimensional mechanism design is sparse, and a number of open questions remain. I plan to address some of these issues in my future research.

One important issue that has not been covered in my paper is a comparative static analysis—for example, what happens if one changes the risk aversion parameter for the risk averse agent? I can provide a precise answer to that question. Note that, if the risk aversion parameter goes down, the middle interval in my model will shrink, and in the limit will disappear, and we are back in the setup of Myerson (1981) [30]. However, there is room for a more general comparative static analysis which is of independent interest, and I hope to address some of these questions in future work.

Chapter 3

Preference Aggregation for Couples

3.1 Introduction

Finding jobs has become more complicated due to the increase in the number of couples who are interested in entering the labor market together. Since there used to be many more men than women seeking positions, only a few couples would apply for jobs simultaneously. This is no longer the case today, which has considerable implications for centralized entry-level labor markets such as the medical residency matches in the US, the UK, and Canada. The National Resident Matching Program (NRMP) in the US, the most prominent example of such a labor market, assigns thousands of medical school graduates to hospital intern positions through a central clearing house each year (Roth 1984 [32], Roth and Peranson 1999) [34]. As discussed by Roth and Peranson (1999) [34] and Klaus et al. (2007) [23], among others, if the centralized matching procedure does not accommodate couples' wishes to find jobs in the same geographical area, then they might prefer to apply for positions directly, instead of going through the centralized matching system which may easily assign them jobs that are far away from each other, leaving the couple unhappy.

Beyond the workings of the matching algorithm itself, a more fundamental issue is that a stable matching may not even exist (Roth 1984 [32], Cantala 2004 [7]). Klaus and Klijn (2005) [22] demonstrate that if both couples and hospitals have responsive preferences then there is always a responsive preference extension for which a stable matching exists, and show that responsive preferences constitute a maximal preference domain in this sense to guarantee the existence of a stable matching (see also Klaus et al. 2009 [24]). Responsiveness means that unilateral improvements according to the preference of one partner are beneficial for the couple. Khare et al. (2018) [21] characterize the responsive preferences of couples under which a stable matching always exists. Even under the milder requirements of Klaus and Klijn (2005) [22], existence of a stable matching can only be achieved when couples' preferences are responsive, and responsiveness essentially reduces the couple's joint preferences to two independent individual preferences. This rules out the complementary preferences of couples over jobs that arise due to distance considerations, the most salient characteristic of couples' preferences, which is not surprising since some substitutability condition is typically required for the existence of a stable matching.

Dutta and Massó (1997) [9] studies externalities in preferences and provides possibility results for couples under specific preference restrictions. An alternative approach of relaxing the stability requirement is presented by Jiang and Tian (2013) [17]. Khare and Roy (2018) [20] pursues further the issue of the existence of stable matchings in markets with couples when preferences are not responsive. Delacrétaz (2019) [8] and Sidibé (2020) [37] are other recent studies that are relevant for the couples' matching problem, as they study matching with agents of different sizes (i.e., agents may require multiple items on the other side of the market). Even if a stable matching with couples exists at a given preference profile, it is not guaranteed that an algorithm will be able to identify and choose a stable matching at such a preference profile. Klaus et al. (2007) [23] show that the new NRMP algorithm (see also Roth and Peranson (1999) [34]) may not reach an existing stable matching, even when couples' preferences are responsive. They also demonstrate that the new NRMP algorithm may be manipulable by couples acting as singles.

Centralized labor markets with explicitly recognized couples, such as the NRMP and the Canadian Resident Matching Service (CaRMS) today, require participating couples to report their joint preference orderings over pairs of positions, and the relevant matching theory literature takes these joint preferences to be exogenously given. However, an overlooked issue is that given the two partners' respective preferences over individual positions, it is not necessarily clear what the couple's joint preferences are. It is natural for each partner to be aware of their individual preference ordering over the jobs, but it is not obvious that couples know or understand well their preferences over pairs of positions which reflect the preferences of both partners even without geographical considerations, and especially when it comes to incorporating the complementary nature of the two positions.

The preference aggregation issue for a couple has not been addressed by the social choice theory literature either, which mainly focuses on aggregating preferences over social outcomes. Our problem formally differs from the preference aggregation literature in several aspects. The aggregation for couples has two preference rankings over individual positions as input, and a joint preference ranking over pairs of positions as an output. Standard preference aggregation rules going back to Arrow (1963) [1] take identical inputs and turn them into an output of the same form as the inputs. In the preference aggregation literature social outcomes are typically public in nature and individuals care about all aspects of the outcome. Some work has been done on economic domains assuming selfishness (i.e., individuals care about their own allocation only) which are surveyed in Bossert and Weymark (2008) [5] and Le Breton and Weymark (2011) [27]. Bordes and Le Breton (1990) [4] study Arrow consistency in various matching models. Kalai and Ritz's (1980) [18] setup comes closest to ours, since they study the same preference aggregation model as us, but with n agents. Unlike us, they focus only on Arrow social welfare functions. Clearly, our model also differs from usual models in that we only have two agents, while preference aggregation is typically considered for an arbitrary number of agents. This makes our task simpler, but it also renders tie-breaking more important since ties arise frequently with two agents only. Finally, and importantly, in the labor market context we need to worry about complements in couples' preferences due to geographical considerations, which is absent from the preference aggregation literature and only pertains to aggregating preferences over private assignments with specific restricted externalities arising from a preference for compatibility of the private assignments.

In this paper we study how to form a joint preference ordering by aggregating a couple's respective individual preferences over single jobs. Can we find consistent, efficient, and fair methods to aggregate the two individual preferences? This aggregation is of interest since reaching a consensus, a feasible compromise that reflects the couple's preferences, may be difficult. We believe, furthermore, that a clear preference aggregation method is not only of relevance to couples when submitting their joint preferences, but it could also play a role in the matching procedure itself, if the matching algorithm is modified accordingly. If couples were restricted to submitting joint preferences with a simplified structure, captured by a handful of parameters only which still allow couples the freedom to express their joint preferences, it may become possible to design more effective matching mechanisms, which would take advantage of the clear structure of the preferences submitted by couples. In addition to being able to construct better matching algorithms, a parametric family of a couple's preferences may also help with the evaluation of the performance of the matching algorithm as a function of different parameter values submitted by participating couples. In light of the severe difficulties with stability and incentives in the presence of couples, as demonstrated by the extant literature, our hope is that such an approach may turn out to be useful. Thus, one of the contributions of this paper is that it initiates this new approach to matching with couples, in addition to proposing and analyzing specific families of preference aggregation rules for couples.

We start by considering two interesting pairing rules in this setting, the Lexicographic (serially dictatorial) and the Rank-Based Leximin rules. In our terminology lexicographic means that preferences are lexicographic in the way they prioritize the two partners, as opposed to lexicographically considering different aspects of the respective individual rankings of jobs. In the absence of cardinal utilities which would be difficult to elicit, and if elicited would further escalate the incentive problems for the matching rule, our aggregation rules are based on the ordinal rankings of individual positions by the two partners and rely on a comparison of these rank numbers between them. Comparing rank numbers and differences in rank numbers may be unusual, given that the elicited preferences are ordinal in nature, but such comparisons based on rank numbers are inevitable in this setting if we don't want to restrict ourselves to serial dictatorships exclusively, hence the name "Rank-Based" Leximin rule.

Both aggregation rules (or pairing rules, as we refer to them) are characterized by appealing normative axioms. The characterization of the Lexicographic rule is a classic one and follows from Kalai and Ritz (1980) [18] , while the Rank-Based Leximin rule is characterized by a new set of axioms in our setting (Theorem 1). We then identify a class of pairing rules, the General Lexi-Pairing rules, which includes both the Lexicographic and the Rank-Based Leximin rules, and argue that it is desirable to further narrow down this set of rules when studying couples' preference aggregation choices and their incentives in labor markets, in order to restrict the couple to report a member of a simple but flexible family of preferences. We propose a family of aggregation rules parameterized by k, the k-Lexi-Pairing rules, which yields a ranking of paired positions for a couple prior to taking into account the complementarities in preferences. The parameter k

represents the degree to which one partner is favored, which indicates the degree of "selfishness" for this partner, where the least selfish leximin preference aggregation is at one extreme, and the most selfish lexicographic preference aggregation is at the other extreme. Symmetrically, k also shows the extent of "altruism" of the other partner, with the least altruistic rule being the leximin preference aggregation, and the most altruistic the lexicographic aggregation. Given that the aggregation problem is symmetric in the two partners, we assume that the partners are interchangeable and simplify our exposition by omitting the symmetric case where partner 1 is altruistic and partner 2 is selfish.

We provide an axiomatic characterization of k-Lexi-Pairing rules (Theorem 2), which shows that these are the only rules satisfying a natural efficiency requirement (Strong Pareto) and an axiom requiring a uniform degree of concessions that determines when to take into account the partner's preferences ahead of one's own preferences (k-Compromise), together with a consistent tie-breaking axiom (k-Threshold-Consistency). Furthermore, we introduce a general framework for considering geographic location and proximity, and propose a modification of Lexi-Pairing rules which is based on a togetherness parameter that we introduce, which allows the couple to incorporate their subjective preferences for the proximity of their respective jobs. Preferences for proximity have been considered by Dutta and Massó (1997) [9] and by Khare and Roy (2018) [20] in the simple form of preference for getting positions at the same firm or hospital, while Cantala (2004) [7] and Sethuraman et al. (2018) [36] offer more general but still limited geographic considerations. We introduce two natural axioms for modifying the joint preferences of the couple to reflect their preference for proximity, which are satisfied by our proposed aggregation rules for couples, the Couple-Lexi-Pairing rules, and find that the characterization result of Theorem 2 still holds in essence when we modify the axioms to take into account the togetherness parameter.

In the formal exposition we use the terminology of medical residency matching and call the two partners in a couple doctors, while the jobs are referred to as hospitals. Nonetheless, the analysis is relevant for couples in any centralized labor market which involves matching not only single individuals but also couples to jobs. Moreover, all the results pertain to a general two-agent preference aggregation problem over private alternatives, except for the findings in Section 7 which focuses on geographic considerations, and thus it is specific to couples in labor markets.

3.2 Setup

There is a set of q hospitals H, and two doctors denoted by $i \in \{1, 2\}$. We assume that each hospital has at least two positions, where each position of a hospital $h \in H$ is assumed to be identical. Thus, we regard $H \times H$ as the set of of paired hospital positions, given that positions at each hospital are the same, and $(h_1, h_2) \in H \times H$ indicates a pair of positions where h_1 denotes the position for doctor 1 and h_2 denotes the position for doctor 2. Note that if $h_1 = h$ and $h_2 = h$ for some $h \in H$ then both doctors are matched to a position at the same hospital. Although we refer to "hospital pairs" throughout the paper for simplicity, it is understood that a hospital pair may consist of two positions at the same hospital.

Each doctor has a strict preference ordering over the set of hospitals H, indicating the doctor's individual preferences over the hospitals. The individual preference ordering of doctor $i \in \{1, 2\}$ is expressed by the ranking of each hospital h_i, denoted by $r_i(h_i) \in \{1, \ldots, q\}$, where $r_i(h_i) < r_i(h'_i)$ means that doctor i prefers hospital h_i to h'_i, since h_i has a lower rank number than h'_i. We assume that each hospital position is acceptable to both doctors, since each doctor would rather get a job than remain unmatched. Let R denote the set of individual hospital rankings. For each doctor $i \in \{1, 2\}$, let $r_i \in R$ denote a particular ranking of all hospitals in H by doctor i. Let $\mathcal{P}$ denote the set of strict preference orderings over the ordered pairs of hospitals, that is, the set of aggregated preference orderings of paired hospital positions. Then $P \in \mathcal{P}$ is a strict preference ordering over $H \times H$ and represents the joint preferences of the two doctors.

A **pairing rule** is a preference aggregation function for two doctors, which maps from two strict individual preference orderings of individual hospital positions to one strict preference ordering of paired hospital positions. Formally, a pairing rule is a function $\varphi : R \times R \to \mathcal{P}$, specifying the preference aggregation of the respective individual hospital rankings of the two doctors. We will also use the notation $(h_1, h_2)P(h'_1, h'_2)$ to indicate that (h_1, h_2) is preferred to (h'_1, h'_2) in the joint preferences $P \in \mathcal{P}$.

3.3 The Lexicographic Rule: No Interpersonal Comparisons

We begin with the Strong Pareto axiom, which would naturally be satisfied by the aggregated preferences when there are no complementary preferences over hospital positions. As usual, (h'_1, h'_2)

Pareto-dominates (h_1, h_2) if $r_1(h_1) \geqslant r_1(h_1')$ and $r_2(h_2) \geqslant r_2(h_2')$, with at least one strict inequality.

Strong Pareto. P satisfies Strong Pareto at (r_1, r_2) if, for all (h_1, h_2) and (h_1', h_2') such that (h_1', h_2') Pareto-dominates (h_1, h_2), $(h_1', h_2')P(h_1, h_2)$. A pairing rule φ satisfies Strong Pareto if for all $(r_1, r_2) \in R \times R$, $\varphi(r_1, r_2)$ satisfies Strong Pareto.

Strong Pareto is closely related to the responsiveness notions for doctors used by Klaus and Klijn (2005) [22] and Khare et al. (2018) [21] , although their models are slightly different from ours. We consider Strong Pareto our most basic axiom, since it simply says that whenever there is an agreement between the two partners over two hospital pairs, the aggregation honors their common preferences. The question is how to rank two hospital pairs when the partners disagree over their rankings based on their individual hospital matches, and some of the other axioms directly address this case.

Given that we only have ordinal preferences as input, which is consistent with the matching theory literature, imposing an independence of irrelevant alternatives (IIA) axiom may not necessarily be deemed too restrictive, even though it implies the lack of interpersonal comparisons. When IIA is combined with Strong Pareto, we immediately get Arrow's impossibility result since only a serial dictatorship rule satisfies both axioms, given that the aggregate preferences need to be transitive. We refer to serial dictatorships as lexicographic rules, since they prioritize one partner over the other in a lexicographic manner. Klaus and Klijn (2005) [22] refers to the same as the leader-follower responsive preferences.

Independence of Irrelevant Alternatives (IIA). A pairing rule φ satisfies Independence of Irrelevant Alternatives if the following holds for all (h_1, h_2), $(h_1', h_2') \in H \times H$ and (r_1, r_2), $(\bar{r}_1, \bar{r}_2) \in R \times R$, where $P = \varphi(r_1, r_2)$ and $\bar{P} = \varphi(\bar{r}_1, \bar{r}_2)$. If

(i) both r_1 and $\bar{r}_1$ rank h_1 and h_1' in the same way,

(ii) both r_2 and $\bar{r}_2$ rank h_2 and h_2' in the same way, and

(iii) $(h_1', h_2')P(h_1, h_2)$

then $(h_1', h_2')\bar{P}(h_1, h_2)$.

In the following definition of the Lexicographic rule, we assume without loss of generality that doctor 1 is the first dictator (or the leader), that is, doctor 1 is the partner whose preferences always dominate the other's.

Lexicographic rule

Fix $(r_1, r_2) \in R \times R$ and let P denote the paired preference ordering $\varphi(r_1, r_2)$, where φ is the Lexicographic rule. Then for $(h_1, h_2), (h'_1, h'_2) \in H \times H$, $(h'_1, h'_2) P (h_1, h_2)$ if one of the following two cases holds:

1. $r_1(h_1) > r_1(h'_1)$;

2. $r_1(h_1) = r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$.

This is the Lexicographic rule favoring doctor 1. If doctor 1 prefers h'_1 to h_1, or if $h'_1 = h_1$ and doctor 2 prefers h'_2 to h_2 then the pair (h'_1, h'_2) is preferred to the pair (h_1, h_2) according to the joint preferences.

Proposition 1. (Characterization of the Lexicographic rule)

A pairing rules satisfies Strong Pareto and IIA if and only if it is the Lexicographic rule.

This is Arrow's famous impossibility theorem adopted to our setting. We omit the straightforward proof of the proposition which also follows from Kalai and Ritz (1980) [18] .

Arguably, in our setting a serial dictatorship is not as undesirable as in other contexts. First of all we have private outcomes, so if agents are selfish and only care about their own allocations then having a consensus over ranking two hospital pairs is more likely than in the public outcome setting. Secondly, we only have two agents, and thus favoring one over the other is not nearly as extreme as favoring one agent over all other agents when the number of agents is large. Thirdly, in the couple preference aggregation problem specifically, it may be desirable for the two spouses to favor one of them over the other, since the two partners wish to cooperate with each other and there may be consensus that one partner's preferences should "weigh" more than the other's due to various reasons. For example, if one is likely to face a tougher job market than the other, or if one spouse has a bigger need for a good job placement than the other for any reason, then the other partner may consent to favoring this partner's preferences.

Nonetheless, the Lexicographic rule is an extreme asymmetric pairing rule, and we want to study other pairing rules that treat agents more symmetrically. For this we need to assume that we can make some interpersonal comparisons, which will amount to treating the ordinal rankings of the agents as comparable utility levels, since we do not have information about preference intensities.[1] In the next section we consider axioms that ask for interpersonal comparisons based on the respective preference ranks and aim to ensure some degree of equity between the two partners, together with some utilitarian notions of efficiency.

3.4 Equity and the Rank-Based Leximin Rule

We start by presenting some desirable fairness and efficiency properties of pairing rules based on the comparisons of preference rank numbers. All the axioms from here on are defined for a specific pair of individual rankings (r_1, r_2), and a pairing rule φ satisfies an axiom if for all $(r_1, r_2) \in R \times R$, the paired ranking $\varphi(r_1, r_2)$ satisfies the axiom at (r_1, r_2).

First we introduce a few preliminary notions and terminology. Given $(r_1, r_2) \in R \times R$, (h_1', h_2') **cross-dominates** (h_1, h_2) if $r_1(h_1) \geqslant r_2(h_2')$ and $r_2(h_2) \geqslant r_1(h_1')$, with at least one inequality, but (h_1', h_2') does not Pareto-dominate (h_1, h_2). Moreover, (h_1', h_2') **dominates** (h_1, h_2) if (h_1', h_2') either Pareto-dominates or cross-dominates (h_1, h_2). We will also say that there is no dominance relation between two hospital pairs if neither dominates the other. For a given (h_1, h_2), let $\Sigma = r_1(h_1) + r_2(h_2)$ be the **sum** of the two rankings, and let $g = |r_1(h_1) - r_2(h_2)|$ be the **gap** between the two rankings, i.e., the absolute value of the difference between the two rankings. In the following, let $\Sigma, \Sigma', \tilde{\Sigma}, \ldots$ denote the sum of $(h_1, h_2), (h_1', h_2'), (\tilde{h}_1, \tilde{h}_2), \ldots$ respectively. Similarly, let $g, g', \tilde{g}, \ldots$ denote the corresponding gaps.

Cross-Dominance. P satisfies Cross-Dominance at (r_1, r_2) if, for all (h_1, h_2) and (h_1', h_2') such that (h_1', h_2') cross-dominates (h_1, h_2), $(h_1', h_2') P (h_1, h_2)$.

Dominance. P satisfies Dominance at (r_1, r_2) if, for all (h_1, h_2) and (h_1', h_2') such that (h_1', h_2') dominates (h_1, h_2), $(h_1', h_2') P (h_1, h_2)$.

[1] Bossert and Weymark (2004) [6] provide a comprehensive treatment of the literature on social choice with interpersonal utility comparisons.

Cross-Dominance requires that a hospital pair that cross-dominates another one is preferred to the other hospital pair. Dominance requires that a hospital pair that dominates another one is preferred to the other hospital pair. Note that Dominance is equivalent to the conjunction of Strong Pareto and Cross-Dominance. Dominance is also known as Suppes-Sen dominance in more general settings.

For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\ (r_1(h_1), r_2(h_2));$$
$$\text{Min} \equiv \min\ (r_1(h_1), r_2(h_2)).$$

For $(h'_1, h'_2) \in H \times H$, let

$$\text{Max}' \equiv \max\ (r_1(h'_1), r_2(h'_2));$$
$$\text{Min}' \equiv \min\ (r_1(h'_1), r_2(h'_2)).$$

With this notation in hand, we note that if (h'_1, h'_2) dominates (h_1, h_2) then Max $\geqslant$ Max$'$ and Min $\geqslant$ Min$'$, which can be checked directly.

Limited Equity. P satisfies Limited Equity at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that there is no dominance relation between them, $g > g'$ implies $(h'_1, h'_2)P(h_1, h_2)$.

Equal-Sum Equity. P satisfies Equal-Sum Equity at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that $\Sigma = \Sigma'$, $g > g'$ implies $(h'_1, h'_2)P(h_1, h_2)$.

Observe that if (h'_1, h'_2) dominates (h_1, h_2) then $\Sigma > \Sigma'$. Thus, if the sums of two hospital pair rankings are equal ($\Sigma = \Sigma'$) then there is no dominance relation between them, and therefore Limited Equity implies Equal-Sum Equity. We will show next that the only pairing rule that satisfies Strong Pareto, Cross-Dominance and Limited Equity is a rule which closely resembles the leximin rule in contexts where agents are assumed to have interpersonally comparable utilities. Although we do not have utilities in our model, only ordinal rankings, if we want to make some interpersonal comparisons then the ordinal rank numbers have to be treated as utility levels that can be compared. We will define the Rank-Based Leximin rule in our setup next.

Rank-Based Leximin rule

Fix $(r_1, r_2) \in R \times R$ and let P denote the joint preference ordering $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin rule. Given (r_1, r_2), for (h_1, h_2) and $(h'_1, h'_2) \in H \times H$, let Max, Min, Max$'$ and Min$'$ be defined as above. Then $(h'_1, h'_2)P(h_1, h_2)$ if one of the following three cases holds:

1. Max $>$ Max$'$;
2. Max $=$ Max$'$ and Min $>$ Min$'$;
3. Max $=$ Max$'$, Min $=$ Min$'$ and $r_1(h_1) > r_1(h'_1)$.

The Rank-Based Leximin rule first compares the ranks of the two lower-ranked hospitals respectively in each of two hospital pairs and chooses the hospital pair with the more preferred lower-ranked hospital in terms of the ordinal rank numbers. If this comparison leads to a tie then it compares similarly the two higher-ranked hospitals, and if this also leads to a tie than it chooses on the basis of doctor 1's preferences. This means that the definition follows the convention that when comparing two symmetric pairs with ranks (x, y) and (y, x), where $x \neq y$, what we will refer to as *symmetric opposites* from now on, then the rule always favors doctor 1, that is, if $x < y$ then (x, y) is preferred to (y, x) in the aggregate preferences, since doctor 1 prefers the hospital with rank x to the hospital with rank y. This implies that Max $=$ Max$'$ and Min $=$ Min$'$, and in this case $r_1(h_1) > r_1(h'_1)$ leads to $(h'_1, h'_2)P(h_1, h_2)$. Although a less systematic favoring of one agent over the other would be slightly more equitable, we make this assumption in 3. for ease of exposition.

Our characterization of the Rank-Based Leximin rule is stated next. Note that since all three axioms are compatible with anonymity (the agents' names don't matter), the conjunction of these axioms does not necessarily break the "tie" between two hospital pairs that are symmetric opposites, and we use the same convention in the characterization as for the rule itself, favoring doctor 1 over doctor 2 in such cases. The proof of the theorem is in the Appendix.

Theorem 1. (Characterization of the Rank-Based Leximin rule)
A pairing rule satisfies Strong Pareto, Cross-Dominance and Limited Equity if and only if it is the Rank-Based Leximin rule.

Equivalently, we could also state that a pairing rules satisfies Dominance and Limited Equity if and only if it is the Rank-Based Leximin rule. The combination of the axioms in the theorem gives us a good intuitive idea about the Rank-Based Leximin rule. When there is a dominance relation between two hospital pairs, the rule ranks the dominating pair higher, indicating its efficiency properties. When there is no dominance relation between two hospital pairs, the rule ranks the hospital pair with the lower gap higher, which reflects that the aggregation rule prefers treating the two partners more equitably in these cases in terms of their preference ranks. Note also that Cross-Dominance has a rank-based fairness component as well, in the spirit of Rawls' 'behind a veil of ignorance' concept. It is also worth noting that the requirements of the axioms are compatible with each other in the sense that they lead to a transitive aggregate preference ordering.

Example 1. We provide examples of pairing rules to establish the independence of the three axioms in Theorem 1. The examples are given for $q = 4$, but similar examples can be found for higher numbers of hospitals as well. The orderings in Table 3.1 are in descending order of preference, and since they are indicated in terms of rank numbers, an ordering of all rank pairs entirely describes a pairing rule for all different rankings of hospitals.[2] The first column in Table 3.1 shows the Rank-Based Leximin rule, denoted by P^L, and the difference from this rule is indicated in bold in the other columns. $\bar{P}$ is an example where Limited Equity is not satisfied but Dominance is. This is an additive (or Borda) rule, since any pair with a lower sum is ranked ahead of a pair with a higher sum, which immediately implies that Dominance is satisfied. This rule also satisfies Equal-Sum Equity, which demonstrates that Limited Equity cannot be weakened to Equal-Sum Equity in the characterization in Theorem 1. $\tilde{P}$ is a pairing rule which satisfies Limited Equity and Cross-Dominance but not Strong Pareto, and $\hat{P}$ is a pairing rule which doesn't satisfy Cross-Dominance but satisfies the other two axioms. We note that the axioms in Theorem 1 are independent as long as $q \geq 3$, but for $q = 3$ Limited Equity can be replaced by Equal-Sum Equity. For $q = 2$ the Rank-Based Leximin rule is characterized by Dominance alone.

[2]This is possible because all these rules are neutral, that is, the hospitals' names don't matter.

$\boldsymbol{P^L}$	$\bar{\boldsymbol{P}}$	$\tilde{\boldsymbol{P}}$	$\hat{\boldsymbol{P}}$
$1, 1$	$1, 1$	$1, 1$	$1, 1$
$1, 2$	$1, 2$	$1, 2$	$1, 2$
$2, 1$	$2, 1$	$2, 1$	$2, 1$
$2, 2$	$2, 2$	$2, 2$	$2, 2$
$1, 3$	$1, 3$	$\mathbf{3, 1}$	$1, 3$
$3, 1$	$3, 1$	$\mathbf{2, 3}$	$\mathbf{2, 3}$
$2, 3$	$2, 3$	$\mathbf{1, 3}$	$\mathbf{3, 1}$
$3, 2$	$3, 2$	$3, 2$	$3, 2$
$3, 3$	$\mathbf{1, 4}$	$3, 3$	$3, 3$
$1, 4$	$\mathbf{4, 1}$	$1, 4$	$1, 4$
$4, 1$	$\mathbf{3, 3}$	$4, 1$	$4, 1$
$2, 4$	$2, 4$	$\mathbf{4, 2}$	$2, 4$
$4, 2$	$4, 2$	$\mathbf{3, 4}$	$4, 2$
$3, 4$	$3, 4$	$\mathbf{2, 4}$	$3, 4$
$4, 3$	$4, 3$	$4, 3$	$4, 3$
$4, 4$	$4, 4$	$4, 4$	$4, 4$

Table 3.1: Independence of the axioms in Theorem 1

3.5 Quasi-Equitable Pairing Rules

We are also interested in rules that are less extreme than either the Lexicographic rule or the Rank-Based Leximin rule. Thus, we want to study a class of rules which includes both of these pairing rules, among others. These rules are necessarily lopsided, treating one agent better than the other, and we relax Limited Equity to allow for more asymmetry in the treatment of the two partners. We will follow the convention used before, which favors doctor 1 over doctor 2 when the rule treats the two partners asymmetrically.

Quasi-Equity. P satisfies Quasi-Equity at (r_1, r_2) if, for all (h_1, h_2) and (h'_1, h'_2) such that there is no dominance relation between them, $g > g'$ and $(h_1, h_2)P(h'_1, h'_2)$ imply that $r_1(h_1) < r_1(h'_1)$.

This axiom allows to escape the conclusion of Limited Equity, namely $(h'_1, h'_2)P(h_1, h_2)$, only if doctor 1 prefers h_1 to h'_1. It is easy to see that Limited Equity implies Quasi-Equity, and thus it follows from Theorem 1 that the Rank-Based Leximin rule satisfies Quasi-Equity. The Lexicographic rule (favoring doctor 1) also satisfies Quasi-Equity, which can be verified as follows. If $(h_1, h_2)P(h'_1, h'_2)$ for the Lexicographic rule at some (r_1, r_2) then $r_1(h_1) \leq r_1(h'_1)$ is implied immediately. To rule out $r_1(h_1) = r_1(h'_1)$, note that in this case there is a dominance relation between (h_1, h_2) and (h'_1, h'_2), contrary to the premise of the axiom. Therefore, since both the Rank-Based Leximin and the Lexicographic rules satisfy Strong Pareto, Strong Pareto together with Quasi-Equity captures a family of pairing rules which includes both of these rules. We call this family of pairing rules General Lexi-Pairing rules. We omit the straightforward proof which demonstrates that the General Lexi-Pairing rules, as defined below, are the only rules which satisfy Strong Pareto and Quasi-Equity.

General Lexi-Pairing rules

Fix $(r_1, r_2) \in R \times R$ and let P denote the paired preference ordering $\varphi(r_1, r_2)$, where φ is a General Lexi-Pairing rule. Let (h_1, h_2), $(h'_1, h'_2) \in H \times H$. Then $(h'_1, h'_2)P(h_1, h_2)$ if either one of the following holds:

1. (h'_1, h'_2) Pareto-dominates (h_1, h_2);
2. there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), $g > g'$, and $r_2(h_2) - r_1(h_1) < r_2(h'_2) - r_1(h'_1)$.

Proposition 2. *A pairing rule satisfies Strong Pareto and Quasi-Equity if and only if it is a General Lexi-Pairing rule.*

We can explain the family of General Lexi-Pairing rules intuitively in terms of a directed graph which shows all "immediate" Pareto-dominance relationships, where immediate means that only one doctor's allocation is different between the two hospital pairs, and this doctor's ranking of the two hospitals only differs by one, where the directed edge points toward the less preferred pair (with the higher rank number). This graph is presented in Figure 1 for the case of 4 hospitals, where the hospital pairs are represented by their rank numbers. If a hospital pair (h_1, h_2) can be reached from another hospital pair (h'_1, h'_2) by following a sequence of directed edges, then we call (h'_1, h'_2) a *predecessor* of (h_1, h_2) in the graph. Clearly, if (h'_1, h'_2) is a predecessor of (h_1, h_2) then it Pareto-dominates (h_1, h_2). Furthermore, we can see that the middle "column" (with $g = 0$) contains all pairs of the form (r, r), which are pairs that assign the same-ranked hospital to both partners, and hospital pairs to the left of this column are the hospital pairs which are better for doctor 1, while hospital pairs to the right of this middle column are better for doctor 2 in terms of the respective rank numbers.

The top-ranked pair for all General Lexi-Pairing rules is the pair of first-ranked hospitals for the two doctors respectively: $(1, 1)$. Point 1. in the definition of General Lexi-Pairing rules says that Strong Pareto is satisfied by the rule, and Strong Pareto implies that starting from $(1, 1)$ the pairing rule "traverses" the graph, although not necessarily following directed edges but picking up each vertex (i.e., each hospital pair), eventually ending this procedure with (q, q). This determines an ordering of the hospital pairs and thus identifies a neutral pairing rule. In order to ensure that Strong Pareto is satisfied, whenever the next vertex is picked each predecessor of this vertex in the graph has to have been picked already. Intuitively, the Lexicographic rule "keeps to the left" in the graph when picking vertices, subject to Strong Pareto, given that the left side favors agent 1, while the Rank-Based Leximin rule "keeps to the middle."

In order to find all General Lexi-Pairing rules, the only restriction regarding the order of picking vertices in the graph, in addition to Strong Pareto (i.e., ensuring that all predecessors in the graph had been already picked for each pair), as specified in point 2. of the definition of General Lexi-Pairing rules, is that if there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), the gap is smaller for (h'_1, h'_2) than for (h_1, h_2), and (h'_1, h'_2) is to the left of (h_1, h_2) in the graph, indicating that agent 1 is favored relatively more by (h'_1, h'_2) than by (h_1, h_2), then (h'_1, h'_2) is ranked ahead of (h_1, h_2). Apart from this restriction, any other ordering of hospital pairs satisfying Strong Pareto leads to a General Lexi-Pairing rule. This restriction rules out, for example, the additive rule $\bar{P}$ presented in Example 1 since, for instance, this restriction implies that $(3, 3)$ is ranked ahead of $(4, 1)$ by any General Lexi-Pairing rule.

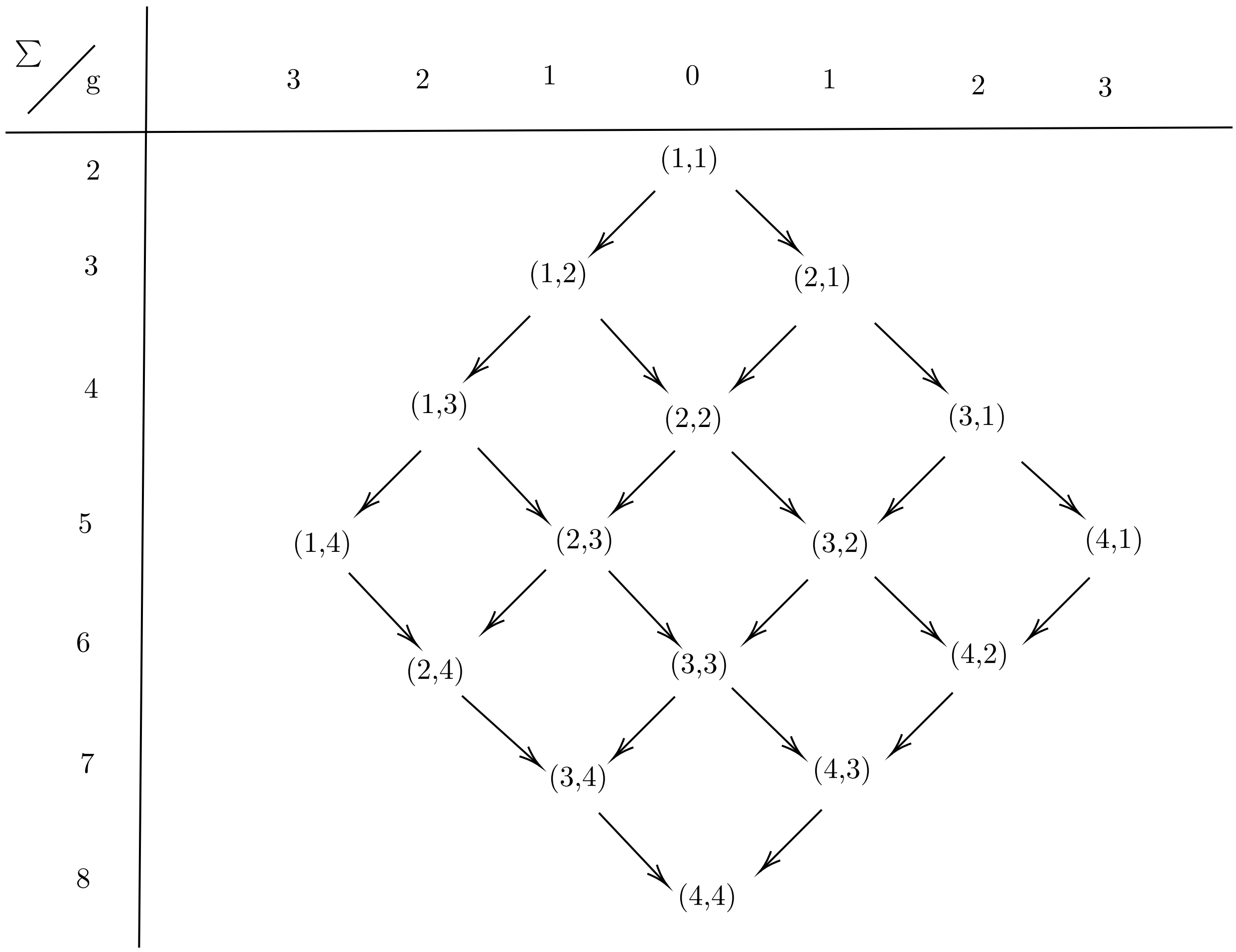

Figure 3.1: Immediate Pareto-dominance graph when $q = 4$

3.6 The Lexi-Pairing Rules

The family of General Lexi-Pairing rules is large and contains members that are not very desirable. One example of such a General Lexi-Pairing rule is the following. Let the aggregated preference ordering start with $(1, 1,), (1, 2), (1, 3), (1, 4)$, as in the Lexicographic rule, and let the rest of the ordering follow the ordering of the pairs in the Rank-Based Leximin rule. This pairing rule satisfies Strong Pareto (use the graph in Figure 1 to easily verify this) and Quasi-Equity. However, this rule is rather inconsistent in its treatment of the two partners. In order to gain some consistency for pairing rules and reduce this family of rules to a subfamily whose members possess a clear structure, we introduce two more (classes of) properties of pairing rules to ensure a consistent compromise between the partners. The extent of equity between the partners is represented by the parameter k.

k-Compromise. Given $k \in \{0, \dots, q-1\}$, P satisfies k-Compromise at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h'_1, h'_2) such that $r_1(h_1) < r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$:

1. $r_1(h'_1) < r_2(h_2) - k$ implies that $(h'_1, h'_2)P(h_1, h_2)$, and
2. $r_1(h'_1) > r_2(h_2) - k$ implies that $(h_1, h_2)P(h'_1, h'_2)$.

Unlike Strong Pareto, this axiom is relevant when there is a disagreement between the two doctors: doctor 1 prefers h_1 to h'_1 and doctor 2 prefers h'_2 to h_2. The parameter k shows the degree to which one partner is willing to compromise and take into account the other partner's preferences over hospital matches ahead of her own preferences and represents the degree of selfishness/altruism. In our exposition doctor 1 is always the selfish or favored partner and doctor 2 is always the altruistic or non-favored partner, but the symmetric case where doctor 1 is altruistic and doctor 2 is selfish applies equally.

The axiom focuses on the rank of the worse alternative for each doctor, h'_1 and h_2 respectively in the definition. When $k = 0$, if h'_1 has a lower rank number than h_2 then $(h'_1, h'_2)P(h_1, h_2)$ and vice versa, and there is as much compromise between the two doctors as possible. When $k = q - 1$ then doctor 1 is always favored and there is no compromise at all. In general, taking into account the selfishness/altrusim level k, if the less preferred alternative of doctor 1 has a relatively lower rank number compared to the rank number of the less preferred alternative of doctor 2 (in the two doctors' respective preferences), then the hospital pair less preferred by doctor 1 is ranked above the other hospital pair by the joint preference ordering P. In sum, this is an axiom that requires consistency regarding how each doctor compromises in favor of her partner, which is based on the ranking of alternatives. One can think of this as a consistent degree of selfishness or altruism, or ”uniform” concession in terms of preference rank differences.

In order to show that k-Compromise is indeed a stronger requirement than Quasi-Equity, we prove the following result.

Proposition 3. *Let $k \in \{0, \dots, q-1\}$. Then k-Compromise implies Quasi-Equity.*

Proof. Let φ satisfy k-Compromise for some $k \in \{0, \dots, q-1\}$. Suppose that φ does not satisfy Quasi-Equity. Specifically, suppose that there exists $(r_1, r_2) \in R \times R$ such that for (h_1, h_2) and

(h_1', h_2'), with no dominance relation between them, we have $g' > g$, $(h_1', h_2')P(h_1, h_2)$ where P denotes $\varphi(r_1, r_2)$, and $r_1(h_1') \geqslant r_1(h_1)$.

Since there is no Pareto-dominance, $r_1(h_1') > r_1(h_1)$, and thus no Pareto-dominance implies $r_2(h_2) > r_2(h_2')$. Then, if $r_1(h_1') > r_2(h_2) - k$, k-Compromise implies that $(h_1, h_2)P(h_1', h_2')$. Therefore, $r_1(h_1') \leqslant r_2(h_2) - k$, and hence no cross-dominance implies that $r_1(h_1') < r_2(h_2)$. In sum,

$$r_2(h_2) > r_1(h_1') > r_1(h_1). \tag{62}$$

Since $r_2(h_2) > r_1(h_1')$, no cross-dominance implies $r_2(h_2') > r_1(h_1)$. Then, in sum,

$$r_2(h_2) > r_2(h_2') > r_1(h_1). \tag{63}$$

Finally, note that (1) and (2) imply that $g' < g$, which is a contradiction. $\square$

The next axiom pertains to the preference ordering of hospital pairs in the special case of a "tie," based on the selfishness/altruism parameter k.

***k*-Threshold-Consistency.** Given $k \in \{0, \ldots, q-1\}$, P satisfies k-Threshold-Consistency at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h_1', h_2') such that $r_1(h_1) < r_1(h_1')$, $r_2(h_2) > r_2(h_2')$ and $r_1(h_1') = r_2(h_2) - k$, $(h_1', h_2')P(h_1, h_2)$ if and only if $r_1(h_1) > r_2(h_2') - k$.

The axiom states that if the less-preferred hospital of doctor 1, h_1', relatively ties in terms of ranking with the less-preferred hospital of doctor 2's hospital, h_2, taking into account k as the degree of selfishness/altruism, then (h_1', h_2') is preferred to (h_1, h_2) if and only if the better option of doctor 1, h_1, has a higher rank number relatively, given k, than the better option of doctor 2, h_2'. For $k = 0$, when the less preferred alternative of each doctor has the same rank number, $(h_1', h_2')P(h_1, h_2)$ if and only if the more preferred alternative of doctor 1, h_1, has a rank number which is higher than the rank number of the more preferred alternative of doctor 2, h_2'.

These two axioms together with Strong Pareto lead to a parametric family of rules which includes both the Lexicographic and the Rank-Based Leximin rules. We call these pairing rules k-Lexi-Pairing rules (or Lexi-Pairing rules, for short).

k-Lexi-Pairing rules ψ^k ($k \in \{0, \ldots, q-1\}$)

Fix $k \in \{0, ..., q-1\}$ and $(r_1, r_2) \in R \times R$ and let P^k denote the paired preference ordering $\psi^k(r_1, r_2)$. For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\,(r_1(h_1), r_2(h_2) - k);$$

$$\text{Min} \equiv \min\,(r_1(h_1), r_2(h_2) - k).$$

For $(h'_1, h'_2) \in H \times H$, let

$$\text{Max}' \equiv \max(r_1(h'_1), r_2(h'_2) - k);$$

$$\text{Min}' \equiv \min\,(r_1(h'_1), r_2(h'_2) - k).$$

Then $(h'_1, h'_2)P^k(h_1, h_2)$ if one of the following three cases holds:

1. $\text{Max} > \text{Max}'$;

2. $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$;

3. $\text{Max} = \text{Max}'$, $\text{Min} = \text{Min}'$ and $r_1(h_1) > r_1(h'_1)$.

It is important to note that the definition of k-Lexi-Pairing rules assigns a strict preference ordering P to every $(r_1, r_2) \in R \times R$ for each rule ψ^k, since the definition always specifies a strictly preferred hospital pair from any two distinct hospital pairs (h_1, h_2) and (h'_1, h'_2): if $\text{Max} \neq \text{Max}'$ then case 1. applies, if $\text{Max} = \text{Max}'$ and $\text{Min} \neq \text{Min}'$ then case 2. applies, and if $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$ then if both $r_1(h_1) = r_1(h'_1)$ and $r_2(h_2) = r_2(h'_2)$ then the two hospital pairs are the same, and if $r_1(h_1) = r_2(h'_2)$, $r_2(h_2) = r_1(h'_1)$, and $r_1(h_1) = r_1(h'_1)$ then again the two hospital pairs are the same. Hence, we must have $r_1(h_1) = r_2(h'_2)$, $r_2(h_2) = r_1(h'_1)$, and $r_1(h_1) \neq r_1(h'_1)$ (what we call symmetric opposites) and then case 3. applies. Moreover, transitivity of the preference relation can also be verified easily, since both the $>$ and the $\geqslant$ relations are transitive on the set of natural numbers.

There are q Lexi-Pairing rules in total, allowing a couple to choose from a range of q pairing rules which connect the seemingly unrelated Rank-Based Leximin and Lexicographic rules. When $k = 0$ the k-Lexi-Pairing rule is the Rank-Based Leximin rule, when $k = q - 1$ it is the Lexicographic rule, and all other Lexi-Pairing rules in-between are less extreme, where k represents the degree of selfishness of doctor 1 (and the degree of altruism of doctor 2). When comparing two hospital pairs in the two doctors' own respective preference rankings, k represents the willingness of doctor 1 to switch the order of the two hospital pairs in question when doctor 1 is relatively better off than doctor 2 in terms of individual hospital rankings in the preferred hospital pair, so that doctor 1 would be worse off and her partner would be better off after switching. As k increases, doctor 1 is less and less willing to make this switch, which makes doctor 1 more selfish and doctor 2 more altruistic. At the more symmetric end of the range, the Rank-Based Leximin rule is as fair between the two partners as possible within this family of pairing rules. There is no completely symmetric pairing rule in general, since in the case of symmetric opposites such as $(1, 3)$ and $(3, 1)$, one hospital pair has to be ranked above the other in a strict joint preference ordering. The Rank-Based Leximin pairing rule, as we defined it, always favors one partner over the other in such cases, but otherwise it treats the two doctors symmetrically in terms of their individual preference rankings and the asymmetric treatment between the two partners is minimal.

We present an example of Lexi-Pairing rules next. This example shows all four Lexi-Pairing rules in terms of the rank numbers when there are four hospitals.

Example 2. Let $q = 4$. Then there are four Lexi-Pairing rules when doctor 1 is selfish, corresponding to $k \in \{0, 1, 2, 3\}$. The top-ranked pair for all Lexi-Pairing rules (as noted before for General Lexi-Pairing rules) is $(1, 1)$ and the last-ranked pair is $(4, 4)$. The second-ranked pair is $(1, 2)$ for each, but the third choice is $(2, 1)$ when $k = 0$ (Rank-Based Leximin rule), and $(1, 3)$ when $k = 3$ (Lexicographic rule). For each parameter k Table 2 displays the joint preference ordering P^k when doctor 1 is selfish and doctor 2 is altruistic.

The family of k-Lexi-Pairing rules consists of efficient and consistent pairing rules, each of which is uniquely described by the axioms of Strong Pareto, k-Compromise, and k-Threshold-Consistency. We will state our main result next, which provides a characterization of each of the Lexi-Pairing rules. The proof of the theorem is relegated to the Appendix.

$\boldsymbol{k=0}$	$\boldsymbol{k=1}$	$\boldsymbol{k=2}$	$\boldsymbol{k=3}$
$\boldsymbol{P^0}$	$\boldsymbol{P^1}$	$\boldsymbol{P^2}$	$\boldsymbol{P^3}$
$1,1$	$1,1$	$1,1$	$1,1$
$1,2$	$1,2$	$1,2$	$1,2$
$2,1$	$2,1$	$1,3$	$1,3$
$2,2$	$1,3$	$2,1$	$1,4$
$1,3$	$2,2$	$2,2$	$2,1$
$3,1$	$2,3$	$1,4$	$2,2$
$2,3$	$3,1$	$2,3$	$2,3$
$3,2$	$1,4$	$2,4$	$2,4$
$3,3$	$3,2$	$3,1$	$3,1$
$1,4$	$2,4$	$3,2$	$3,2$
$4,1$	$3,3$	$3,3$	$3,3$
$2,4$	$3,4$	$3,4$	$3,4$
$4,2$	$4,1$	$4,1$	$4,1$
$3,4$	$4,2$	$4,2$	$4,2$
$4,3$	$4,3$	$4,3$	$4,3$
$4,4$	$4,4$	$4,4$	$4,4$

Table 3.2: Lexi-Pairing rules when $q = 4$

Theorem 2. (Characterizations of Lexi-Pairing rules)
Let $k \in \{0, ..., q-1\}$. A pairing rule satisfies Strong Pareto, k-Compromise, and k-Threshold-Consistency if and only if it is the k-Lexi-Pairing rule.

Now we verify whether the three axioms in the theorem are independent of each other for $k \in \{0, \ldots, q-1\}$. Strong Pareto pertains to the pairwise ranking of hospital pairs over which the two doctors have no disagreement, while the other two axioms pertain to the pairwise ranking of hospital pairs over which the two doctors disagree. Hence, Strong Pareto is independent of the other two axioms, since comparisons when the partners are in agreement always need to be made. Similarly, k-Compromise is independent of the other two axioms, since comparisons when agents are not in agreement and their rank numbers do not tie always need to be made. However, it turns out that k-Threshold-Consistency is not needed when $k = q - 1$. k-Threshold-Consistency is used a lot when k is small, since the more symmetric treatment of the partners results in lots of ties, but as k increases there are fewer instances where such ties occur. For instance, we can see that in Example 1 there are 9 instances where the ranking of adjacent hospital pairs is determined by 0-Threshold-Consistency in the Rank-Based Leximin ordering P^0, there are 6 such instances in P^1 where 1-Threshold-Consistency is invoked, and only 2 instances in P^2 where 2-Threshold-Consistency is needed to pin down the joint preference ordering. Finally, the Lexicographic rule has zero such instances. This is always the case for the Lexicographic rule for an arbitrary number of hospitals q, because $r_1(h'_1) = r_2(h_2) - k$ is never satisfied when $k = q - 1$, which can be seen as follows. Given that $r_1(h_1) < r_1(h'_1)$, it follows that $r_1(h'_1) \geqslant 2$, and since $r_2(h_2) \leqslant q$ we have $r_2(h_2) - (q-1) \leqslant 1$. Thus, $r_1(h'_1) \neq r_2(h_2) - (q-1)$, and $(q-1)$-Threshold-Consistency is satisfied vacuously. It is clear, however, that $r_1(h'_1) = r_2(h_2) - k$ is possible for all $k \in \{1, \ldots, q-2\}$, so k-Threshold-Consistency is only redundant for the Lexicographic rule and is needed for all the other Lexi-Pairing rules, as it is independent of the other two axioms for any k less than $q - 1$. Since preferences over hospital pairs are specified under each of these mutually exclusive scenarios by any well-defined pairing rule for all $k \in \{1, \ldots, q-2\}$, there is no redundant axiom in the corresponding characterizations.

Below we state two corollaries of Theorem 2 which show the characterizations of the Rank-Based Leximin and Lexicographic rules, providing alternative characterizations to Proposition 1 and Theorem 1. Let Compromise be the special case of k-Compromise with $k = 0$, and let Threshold-Consistency be the special case of k-Threshold-Consistency with $k = 0$. These axioms, which are used to characterize the Rank-Based Leximin rule, are particularly simple.

Corollary 3. *A pairing rule satisfies Strong Pareto, Compromise, and Threshold-Consistency if and only if it is the Rank-Based Leximin rule.*

Let Lexicographic-Compromise be the special case of k-Compromise with $k = q - 1$. Note that, as shown above, for $k = q-1$ we have $r_1(h'_1) \geqslant 2$ and $r_2(h_2) - (q-1) \leqslant 1$. Thus, $r_1(h'_1) > r_2(h_2) - k$ always holds and Lexicographic-Compromise (which offers no compromise at all, as we will see) can be stated simply as follows.

Lexicographic-Compromise. P satisfies Lexicographic-Compromise at $(r_1, r_2) \in R \times R$ if for all

(h_1, h_2) and (h'_1, h'_2) such that $r_1(h_1) < r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$, $(h_1, h_2)P(h'_1, h'_2)$.

This axiom together with Strong Pareto renders the characterization of the Lexicographic rule, stated below, straightforward.

Corollary 4. *A pairing rule satisfies Strong Pareto and Lexicographic-Compromise if and only if it is the Lexicographic rule.*

3.7 Geographic Constraints and Preferences for Being Together

A couple would typically prefer hospital positions that are close to each other, and in this section we explore the constraints that couples face when trying to find positions in the same geographic area. The geographic constraints may be given by a partition of the set of hospitals H, where each member represents a geographic area, or more generally we can use a graph, with the hospitals as vertices and the edges representing the compatibility of two hospitals in terms of togetherness for couples. It may be the case that hospitals h and h' are close enough to each other, and hospitals h' and h'' are also close enough to each other, to be acceptable to couples to have positions at h and h' respectively, and also at h' and h'' respectively, but not at h and h'', which would be deemed to be too far from each other. Also, some hospitals may be close enough to an airport with a good connection to some other airports, for example, and thus two hospitals near airports that are well connected would be considered compatible for a couple, but different hospitals further away from the airports in two different cities may not be considered compatible for a couple. We can summarize the hospital compatibility information by a set $G \subset H \times H$ which consists of compatible hospital pairs for couples (a set of ordered pairs of hospitals or, equivalently, a set of edges in the associated directed graph), and denote the set of incompatible hospital pairs by $\bar{G}$, where $G \cap \bar{G} = \emptyset$ and $G \cup \bar{G} = H \times H$. If $(h, h') \in G$ then getting positions at hospitals h and h' respectively for a couple is considered compatible in terms of geographic constraints, while if $(h, h'') \in \bar{G}$ then positions at h and h'' for a couple are not considered compatible. This defines a binary relation over H, which is assumed to be reflective (for all $h \in H$, $(h, h) \in G$) and symmetric (for all $(h, h') \in G$, $(h', h) \in G$ also holds). Note that transitivity, however, does not necessarily hold, as argued above.

Although couples may have their own subjective opinions about which hospital pairs are close enough to be considered geographically compatible, for simplicity we take G to be a primitive of the model, and thus assume that any couple would deem the same hospital pairs compatible. Note that all previous papers that study a couple's preference for togetherness in the matching theory literature assume, at least implicitly, that geographical constraints are exogenously given, and our setup encompasses all different geographic considerations in the literature. One simple way is to assume that couples only find two positions close enough to each other if they are at the same hospital, as seen in Dutta and Massó (1997) [9] and Khare and Roy (2018) [20] . This is a very specific case of our setup, where G is described by a partition of the set of hospitals into geographical areas with a single hospital in each geographical area. Another special case of our setup is explored by Cantala (2004) [7] and Sethuraman et al. (2018) [36] , where hospitals are partitioned into regions and regions are assumed to have a common preference ranking by all couples.

We will introduce next two basic criteria that any paired preference ordering with geographic considerations based on G, denoted by P^G, should satisfy with respect to the paired preference ordering P which does not account for geographic constraints. These axioms together are related to the 'responsiveness violated for togetherness' (RVT) condition of Khare and Roy (2018) [20] , but in their paper togetherness always means that the couple gets placed at the same hospital.

Geographic Invariance. Let P and P^G be two paired preference orderings over $H \times H$. P^G satisfies Geographic Invariance with respect to P if the following two conditions hold:

1. for all $(h_1, h_2) \in G$ and $(h'_1, h'_2) \in G$, $(h_1, h_2)P^G(h'_1, h'_2)$ if and only if $(h_1, h_2)P(h'_1, h'_2)$;

2. for all $(h_1, h_2) \notin G$ and $(h'_1, h'_2) \notin G$, $(h_1, h_2)P^G(h'_1, h'_2)$ if and only if $(h_1, h_2)P(h'_1, h'_2)$.

Given two pairing rules φ and φ^G, φ^G satisfies Geographic Invariance with respect to φ if for all $(r_1, r_2) \in R \times R$, $\varphi^G(r_1, r_2)$ satisfies Geographic Invariance with respect to $\varphi(r_1, r_2)$.

Geographic Invariance expresses that the only valid preference reversals compared to the original paired preference ordering of the couple which does not take into account geographic preferences are ones based on the geographic constraints given by G.

Togetherness. Let P and P^G be two paired preference orderings over $H \times H$. P^G satisfies Togetherness with respect to P if for all $(h_1, h_2) \in G$ and $(h'_1, h'_2) \notin G$, if $(h_1, h_2)P(h'_1, h'_2)$ then $(h_1, h_2)P^G(h'_1, h'_2)$.

Given two pairing rules φ and φ^G, φ^G satisfies Togetherness with respect to φ if for all $(r_1, r_2) \in R \times R$, $\varphi^G(r_1, r_2)$ satisfies Togetherness with respect to $\varphi(r_1, r_2)$.

The axiom of Togetherness requires that if a hospital pair is ranked by P ahead of another hospital pair, and the former pair is geographically compatible while the latter is not, then the former pair be still preferred by P^G to the latter, since the latter, due to the geographic incompatibility of the two hospitals, should only be ranked lower, not higher, when the joint preferences of the couple take into account geographic considerations.

Although G is not subject to a couple's subjective preferences and is assumed to be exogenously given, this doesn't mean that couples have to have the same kind of preferences over the geographic constraints. We allow couples to attribute different levels of importance to togetherness, and to this end introduce a togetherness parameter, denoted by t, which captures the extent to which a couple considers incompatible hospital pairs relatively less desirable when compared to compatible hospital pairs. Specifically, we use modified "rank numbers" based on t, denoted by $\hat{r}_i$ for doctor $i \in \{1, 2\}$, instead of the original rank numbers r_i. Although we still work with individual rank numbers of hospitals $\hat{r}_1$ and $\hat{r}_2$, due to the geographic constraints these are no longer the actual rank numbers of individual hospitals, instead, these are functions of a hospital pair, which allows for taking into account the geographic compatibility of the hospitals. Introducing the togetherness parameter t into k-Lexi-Pairing rules leads to the (k, t)-Couple-Lexi-Pairing rules as defined below. Note that in this more general framework the Lexi-Pairing preference orderings no longer satisfy responsiveness, as defined for couples by Khare et al. (2018), among others.

(k, t)-Couple-Lexi-Pairing rules $\psi^{(k,t)}$

Fix $k, t \in \{0, ..., q - 1\}$ and let $(r_1, r_2) \in R \times R$.

Define $\hat{r}_1$ and $\hat{r}_2$ based on (r_1, r_2) as follows. Given $t \in \{0, \ldots, q - 1\}$, for all $(h_1, h_2) \in H \times H$,

1. if $(h_1, h_2) \in G$ then $\hat{r}_1(h_1, h_2) = r_1(h_1)$ and $\hat{r}_2(h_1, h_2) = r_2(h_2)$;

2. if $(h_1, h_2) \in \bar{G}$ then $\hat{r}_1(h_1, h_2) = r_1(h_1) + t + \epsilon$ and $\hat{r}_2(h_1, h_2) = r_2(h_2) + t + \epsilon$, where $0 < \epsilon < 1$.

For $(h_1, h_2) \in H \times H$, let

$$\text{Max} \equiv \max\ (\hat{r}_1(h_1, h_2), \hat{r}_2(h_1, h_2) - k);$$

$$\text{Min} \equiv \min\ (\hat{r}_1(h_1, h_2), \hat{r}_2(h_1, h_2) - k).$$

For $(h'_1, h'_2) \in H \times H$, let

$$\text{Max}' \equiv \max(\hat{r}_1(h'_1, h'_2), \hat{r}_2(h'_1, h'_2) - k);$$
$$\text{Min}' \equiv \min\ (\hat{r}_1(h'_1, h'_2), \hat{r}_2(h'_1, h'_2) - k),$$

Let $P^{(k,t)}$ denote the paired preference ordering $\psi^{(k,t)}(r_1, r_2)$. Then $(h'_1, h'_2)P^{(k,t)}(h_1, h_2)$ if one of the following three cases holds:

1. $\text{Max} > \text{Max}'$;
2. $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$;
3. $\text{Max} = \text{Max}'$, $\text{Min} = \text{Min}'$, and $\hat{r}_1(h_1, h_2) > \hat{r}_1(h'_1, h'_2)$.

Geographically incompatible hospital pairs are less preferred, since their respective rank numbers are increased by $t + \epsilon$. Adding ϵ is needed to avoid potential ties in the rankings of hospital pairs, and the incompatible hospital pair is defined to be less preferred in case of a tie. For example, if the hospitals with original rank numbers $(1, 3)$ are incompatible (i.e., not in G) and $t = 2$, then without adding ϵ the rankings of these hospitals would become $(3, 5)$, which is the same as a compatible hospital pair (a hospital pair in G) that has the rank number pair $(3, 5)$ originally, and this would make these two hospital pairs indistinguishable when trying to rank them in the joint preferences. As a tie-breaker, adding ϵ causes the geographically compatible hospital pair to be preferred by both doctors to the geographically incompatible pair, but clearly this could be modified easily to reflect a preference for the incompatible hospital pair by subtracting ϵ instead of adding it.

The togetherness parameter $t \in \{0, \ldots, q - 1\}$ expresses the preferences of a couple to obtain compatible positions in terms of geographic constraints. If $t = 0$ then the couple does not care about being together and the paired preference ordering is unchanged (since $\epsilon < 1$), regardless of G. At the other extreme, if $t = q - 1$ then each compatible pair of hospitals is preferred to each non-compatible pair of hospitals, while leaving all the other preference orderings unchanged. There are other cases between these two extremes which may more realistically depict a couple's preferences than either extremes, and the parameter t allows to systematically and consistently reduce the ranking of incompatible hospital pairs, while keeping other preference orderings the same. A higher value of t indicates that the couple finds it more important to find geographically compatible jobs, but note that the preference ordering may not necessarily change when the value of t changes, depending on the original preferences and on G.

It should be clear that $1 \leqslant \hat{r}_i \leqslant q + t + \epsilon$ and need not be natural numbers. While these are unusual "rank numbers," they allow us to simply apply the method of the Lexi-Pairing rules to these modified rank numbers which are based on G and the parameter t, expressing the couple's preference for togetherness.

Example 3. Let $H = \{a, b, c, d\}$. Since $q = 4$, there are four Lexi-Pairing rules when doctor 1 is selfish, as shown in Example 2. Let doctor 1's individual preference ordering over hospitals be (a, b, c, d), where $r_1(a) = 1, r_1(b) = 2$, and so on. Let doctor 2's individual preference ordering over hospitals be (a, c, d, b), where $r_2(a) = 1, r_2(c) = 2$, and so on. Assume also that hospitals a and b are in one geographic area, and hospitals c and d are in a different geographic area. Therefore, $G = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$, where the first hospital is doctor 1's assigned hospital, and the second hospital is doctor 2's assigned hospital. Let the togetherness parameter be $t = 1$. Table 3 shows the k-Lexi-Pairing ordering for the given preferences favoring doctor 1 in the first column, denoted by P^k, and the $(k, 1)$-Couple-Lexi-Pairing ordering for the given preferences favoring doctor 1 in the second column, denoted by $P^{(k,1)}$. The geographically incompatible hospital pairs are indicated in bold letters.

$\boldsymbol{k=0}$		$\boldsymbol{k=1}$		$\boldsymbol{k=2}$		$\boldsymbol{k=3}$	
$\boldsymbol{P^0}$	$\boldsymbol{P^{(0,1)}}$	$\boldsymbol{P^1}$	$\boldsymbol{P^{(1,1)}}$	$\boldsymbol{P^2}$	$\boldsymbol{P^{(2,1)}}$	$\boldsymbol{P^3}$	$\boldsymbol{P^{(3,1)}}$
a, a	a, a	a, a	a, a	a, a	a, a	a, a	a, a
$\boldsymbol{a, c}$	b, a	$\boldsymbol{a, c}$	b, a	$\boldsymbol{a, c}$	b, a	$\boldsymbol{a, c}$	a, b
b, a	$\boldsymbol{a, c}$	b, a	$\boldsymbol{a, c}$	$\boldsymbol{a, d}$	a, b	$\boldsymbol{a, d}$	b, a
$\boldsymbol{b, c}$	c, c	$\boldsymbol{a, d}$	a, b	b, a	$\boldsymbol{a, c}$	a, b	$\boldsymbol{a, c}$
$\boldsymbol{a, d}$	c, d	$\boldsymbol{b, c}$	c, c	$\boldsymbol{b, c}$	b, b	b, a	b, b
$\boldsymbol{c, a}$	$\boldsymbol{b, c}$	$\boldsymbol{b, d}$	b, b	a, b	$\boldsymbol{a, d}$	$\boldsymbol{b, c}$	$\boldsymbol{a, d}$
$\boldsymbol{b, d}$	a, b	$\boldsymbol{c, a}$	$\boldsymbol{a, d}$	$\boldsymbol{b, d}$	c, c	$\boldsymbol{b, d}$	c, c
c, c	b, b	a, b	c, d	b, b	c, d	b, b	c, d
c, d	$\boldsymbol{a, d}$	c, c	$\boldsymbol{b, c}$	$\boldsymbol{c, a}$	$\boldsymbol{b, c}$	$\boldsymbol{c, a}$	$\boldsymbol{b, c}$
a, b	d, c	b, b	$\boldsymbol{b, d}$	c, c	$\boldsymbol{b, d}$	c, c	$\boldsymbol{b, d}$
$\boldsymbol{d, a}$	$\boldsymbol{c, a}$	c, d	d, c	c, d	d, c	c, d	d, c
b, b	$\boldsymbol{b, d}$	$\boldsymbol{c, b}$	$\boldsymbol{c, a}$	$\boldsymbol{c, b}$	$\boldsymbol{c, a}$	$\boldsymbol{c, b}$	$\boldsymbol{c, a}$
d, c	d, d	$\boldsymbol{d, a}$	d, d	$\boldsymbol{d, a}$	d, d	$\boldsymbol{d, a}$	d, d
$\boldsymbol{c, b}$	$\boldsymbol{d, a}$	d, c	$\boldsymbol{c, b}$	d, c	$\boldsymbol{c, b}$	d, c	$\boldsymbol{c, b}$
d, d	$\boldsymbol{c, b}$	d, d	$\boldsymbol{d, a}$	d, d	$\boldsymbol{d, a}$	d, d	$\boldsymbol{d, a}$
$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$	$\boldsymbol{d, b}$

Table 3.3: Couple-Lexi-Pairing orderings when $q = 4$ and $t = 1$

Now we modify the axioms that characterize the Lexi-Pairing rules, so that they reflect the preferences of couples for being together, given the geographic constraints represented by G. For each of the three axioms below and for each $(r_1, r_2) \in R \times R$, define $\hat{r}_1^t$ and $\hat{r}_2^t$ as a function of t, as before. Thus, for all $t \in \{0, \ldots, q-1\}$:

1. if $(h_1, h_2) \in G$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1)$ and $\hat{r}_2^t(h_1, h_2) = r_2(h_2)$;

2. if $(h_1, h_2) \in \bar{G}$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1) + t + \epsilon$ and $\hat{r}_2^t(h_1, h_2) = r_2(h_2) + t + \epsilon$, where $0 < \epsilon < 1$.

***t*-Strong Pareto** (Strong Pareto with t-togetherness). Given $t \in \{0, \ldots, q-1\}$, P satisfies t-Strong Pareto at $(r_1, r_2) \in R \times R$ if, for all (h_1, h_2) and (h_1', h_2') such that $\hat{r}_1^t(h_1, h_2) \geqslant \hat{r}_1^t(h_1', h_2')$

and $\hat{r}_2^t(h_1, h_2) \geqslant \hat{r}_2^t(h'_1, h'_2)$ with at least one strict inequality, $(h'_1, h'_2)P(h_1, h_2)$.

(k,t)-Compromise (k-Compromise with t-togetherness). Given $k, t \in \{0, \ldots, q-1\}$, P satisfies (k,t)-Compromise at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h'_1, h'_2) such that $\hat{r}_1^t(h_1, h_2) < \hat{r}_1^t(h'_1, h'_2)$ and $\hat{r}_2^t(h_1, h_2) > \hat{r}_2^t(h'_1, h'_2)$:

1. $\hat{r}_2^t(h'_1, h'_2) < \hat{r}_1^t(h_1, h_2) - k$ implies that $(h'_1, h'_2)P(h_1, h_2)$, and
2. $\hat{r}_2^t(h'_1, h'_2) > \hat{r}_1^t(h_1, h_2) - k$ implies that $(h_1, h_2)P(h'_1, h'_2)$.

(k,t)-Threshold-Consistency (k-Threshold-Consistency with t-togetherness). Given $k, t \in \{0, \ldots, q-1\}$, P satisfies (k,t)-Threshold-Consistency at $(r_1, r_2) \in R \times R$ if for all (h_1, h_2) and (h'_1, h'_2) such that $\hat{r}_1^t(h_1, h_2) < \hat{r}_1^t(h'_1, h'_2)$, $\hat{r}_2^t(h_1, h_2) > \hat{r}_2^t(h'_1, h'_2)$ and $\hat{r}_1^t(h'_1, h'_2) = \hat{r}_2^t(h_1, h_2) - k$, $(h'_1, h'_2)P(h_1, h_2)$ if and only if $\hat{r}_1^t(h_1, h_2) > \hat{r}_2^t(h'_1, h'_2) - k$.

Each of the above axioms, given a fixed togetherness parameter $t \in \{1, \ldots, q-1\}$ and G, is defined for a specific pair of individual rankings (r_1, r_2), and we will say that a pairing rule φ satisfies an axiom if for all $(r_1, r_2) \in R \times R$, the paired ranking $\varphi(r_1, r_2)$ satisfies the axiom at (r_1, r_2).

Proposition 4. (Properties of Couple-Lexi-Pairing rules)
Let $k \in \{0, \ldots, q-1\}$.

1. *For all $t \in \{0, \ldots, q-1\}$, a pairing rule satisfies t-Strong Pareto, (k,t)-Compromise, and (k,t)-Threshold-Consistency if and only if it is the (k,t)-Couple-Lexi-Pairing rule.*
2. *For all $t, \bar{t} \in \{0, \ldots, q-1\}$, the (k,t)-Couple-Lexi-Pairing rule satisfies Geographic Invariance with respect to the $(k,\bar{t})$-Couple-Lexi-Pairing rule.*
3. *For all $t, \bar{t} \in \{0, \ldots, q-1\}$ such that $t < \bar{t}$, the $(k,\bar{t})$-Couple-Lexi-Pairing rule satisfies Togetherness with respect to the (k,t)-Couple-Lexi-Pairing rule.*

Proof.

1. *Characterization.* This is a straightforward extension of the characterization in Theorem 2, since it can be seen easily that the proof of Theorem 2 holds for any pair of rank numbers

associated with paired hospital positions, and need not be the rank numbers of individual hospital positions, as in Theorem 2. Although the modified $\hat{r}_i(h_1, h_2)$ rank numbers for $i \in \{1, 2\}$ may not be natural numbers, the proof of Theorem 2 still holds with the modified rank numbers as long as no two distinct hospital pairs have identical rank numbers. This condition is automatically satisfied when the rank numbers are simply the individual hospital ranks in the two respective individual preference orderings, but when more general rank numbers are allowed such a tie may occur, which would make it impossible to distinguish between the two hospital pairs with the same rank numbers for both hospital positions. However, given that such ties cannot exist if the original individual rank numbers $r_i(h_i)$ of hospitals are used, a tie could only occur between a hospital pair $(h_1, h_2) \in G$ and $(h_1', h_2') \notin G$, and only when a given fixed $t \in \{0, \ldots, q-1\}$ is added to the ranking of the incompatible hospitals h_1' and h_2'. Therefore, ties are prevented by the addition of ϵ to the rank numbers of h_1' and h_2' and we can apply the proof of Theorem 2 to obtain this characterization result.

2. *Geographic Invariance.* Fix $(r_1, r_2) \in R \times R$ and let $P^{(k,t)}$ denote $\psi^{(k,t)}(r_1, r_2)$, where $\psi^{(k,t)}$ is the (k,t)-Couple-Lexi-Pairing rule, and let $P^{(k,\bar{t})}$ denote $\psi^{(k,\bar{t})}(r_1, r_2)$, where $\psi^{(k,\bar{t})}$ is the $(k,\bar{t})$-Couple-Lexi-Pairing rule. If $(h_1, h_2) \in G$ and $(h_1', h_2') \in G$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1)$, $\hat{r}_2^t(h_1, h_2) = r_2(h_2)$, $\hat{r}_1^t(h_1', h_2') = r_1(h_1')$, $\hat{r}_2^t(h_1', h_2') = r_2(h_2')$, $\hat{r}_1^{\bar{t}}(h_1, h_2) = r_1(h_1)$, $\hat{r}_2^{\bar{t}}(h_1, h_2) = r_2(h_2)$, $\hat{r}_1^{\bar{t}}(h_1', h_2') = r_1(h_1')$, and $\hat{r}_2^{\bar{t}}(h_1', h_2') = r_2(h_2')$. Thus, $(h_1, h_2)P^{(k,\bar{t})}(h_1', h_2')$ if and only if $(h_1, h_2)P^{(k,t)}(h_1', h_2')$. If $(h_1, h_2) \notin G$ and $(h_1', h_2') \notin G$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1) + t + \epsilon$, $\hat{r}_2^t(h_1, h_2) = r_2(h_2) + t + \epsilon$, $\hat{r}_1^t(h_1', h_2') = r_1(h_1') + t + \epsilon$, $\hat{r}_2^t(h_1', h_2') = r_2(h_2') + t + \epsilon$, $\hat{r}_1^{\bar{t}}(h_1, h_2) = r_1(h_1) + \bar{t} + \epsilon$, $\hat{r}_2^{\bar{t}}(h_1, h_2) = r_2(h_2) + \bar{t} + \epsilon$, $\hat{r}_1^{\bar{t}}(h_1', h_2') = r_1(h_1') + \bar{t} + \epsilon$, and $\hat{r}_2^{\bar{t}}(h_1', h_2') = r_2(h_2') + \bar{t} + \epsilon$, where $0 < \epsilon < 1$. Since adding a constant preserves the Max, the Min and the rank comparisons, $(h_1, h_2)P^{(k,\bar{t})}(h_1', h_2')$ if and only if $(h_1, h_2)P^{(k,t)}(h_1', h_2')$.

3. *Togetherness.* Fix $(r_1, r_2) \in R \times R$ and let $P^{(k,t)}$ denote $\psi^{(k,t)}(r_1, r_2)$, where $\psi^{(k,t)}$ is the (k,t)-Couple-Lexi-Pairing rule, and let let $P^{(k,\bar{t})}$ denote $\psi^{(k,\bar{t})}(r_1, r_2)$, where $\psi^{(k,\bar{t})}$ is the $(k,\bar{t})$-Couple-Lexi-Pairing rule and $t < \bar{t}$. If $(h_1, h_2) \in G$ and $(h_1', h_2') \notin G$ then $\hat{r}_1^t(h_1, h_2) = r_1(h_1)$, $\hat{r}_2^t(h_1, h_2) = r_2(h_2)$, $\hat{r}_1^t(h_1', h_2') = r_1(h_1') + t + \epsilon$ and $\hat{r}_2^t(h_1', h_2') = r_2(h_2') + t + \epsilon$, $\hat{r}_1^{\bar{t}}(h_1, h_2) = r_1(h_1)$, $\hat{r}_2^{\bar{t}}(h_1, h_2) = r_2(h_2)$, $\hat{r}_1^{\bar{t}}(h_1', h_2') = r_1(h_1') + \bar{t} + \epsilon$ and $\hat{r}_2^{\bar{t}}(h_1', h_2') = r_2(h_2') + \bar{t} + \epsilon$, where $0 < \epsilon < 1$. Thus, given the definition of (k,t)-Couple-Lexi-Pairing rules, it is straightforward

to check that if $(h_1, h_2)P^{(k,t)}(h'_1, h'_2)$ then $(h_1, h_2)P^{(k,\bar{t})}(h'_1, h'_2)$. □

3.8 Concluding Remarks

Apart from (k, t)-Couple-Lexi-Pairing rules, there are other natural ways to define preference aggregation rules for couples who take into account geographic constraints. We propose the Lexi-Pairing rules which have appealing efficiency and fairness properties and are very consistent in terms of the compromises made between the two partners. Geographic preferences may also be defined differently. For example, we could let couples leave top-ranked hospital pairs where they originally are in the joint preference ordering and only reduce the joint ranking of individually lower-ranked hospital pairs, expressing that a couple is less willing to sacrifice their individual hospital choices if they can get very highly preferred hospitals, even when these hospitals are not in the same geographic area. Our setup could also allow the two partners to use two different t-parameters which are added to the rank numbers of incompatible hospital pairs, expressing that one partner may value togetherness more than the other (although this could lead to divorce). Thus, if $t_1 > t_2$ then doctor 1 values togetherness more than doctor 2, and vice versa.

While there are many ways to aggregate couples' individual preferences over pairs of jobs, our intention was to propose and study specific intuitively appealing aggregation methods. According to our proposed family of rules, if a couple is not sure about how to rank the job pairs but know their individual rankings over jobs, the two partners would only need to negotiate about the compromise parameter k and agree on their preferences over the level of togetherness to determine parameter t. While this imposes constraints on the couple's joint preference choices, the proposed Couple-Lexi-Pairing rules offer a simple way to generate systematically aggregated paired preference rankings, and the preference aggregation choices are clarified by their properties that are shown in this paper.

It is also important to note that the proposed (k, t)-Couple-Lexi-Pairing rules are informationally simple when considering a couple's reported preferences as an input to a centralized matching system. When submitting the preferences of the couple to a clearing house such as the NRMP, a couple would not need to report an entire preference ordering over hospital pairs, which may be cumbersome and could lead to listing fewer hospital pairs than acceptable to the couple. Rather, they would report their individual rankings over hospitals, just like any single applicant, and in addition they would only need to report two parameters: k, to specify their joint compromise over individual rankings, and t, to indicate their joint preference for geographic proximity. Furthermore, the design of the matching mechanism may be able to exploit the simple structure and clarity of couples' preferences and produce more desirable outcomes for matching markets with couples.

Appendix

Proof of Theorem 1

Claim 1.1. *The Rank-Based Leximin rule satisfies Limited Equity.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin Rule. Let (h_1, h_2) and (h'_1, h'_2) be such that there is no dominance relation between them and $g > g'$. We will show that then $(h'_1, h'_2)P(h_1, h_2)$.

Case 1: $\text{Max} > \text{Max}'$.

Then $(h'_1, h'_2)P(h_1, h_2)$.

Case 2: $\text{Max} < \text{Max}'$

Since $g > g'$, we must have $\text{Min} < \text{Min}'$. This implies that (h_1, h_2) dominates (h'_1, h'_2), which is a contradiction.

Case 3: $\text{Max} = \text{Max}'$

Then if $\text{Min}' \neq \text{Min}$ there is a dominance relation between (h_1, h_2) and (h'_1, h'_2), and thus $\text{Min} = \text{Min}'$. This implies that $g = g'$, which is a contradiction. $\square$

Claim 1.2. *The Rank-Based Leximin rule satisfies Cross-Dominance.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ is the Rank-Based Leximin Rule. Let (h_1, h_2) and (h'_1, h'_2) be such that $r_1(h_1) \geqslant r_2(h'_2)$ and $r_2(h_2) \geqslant r_1(h'_1)$ with at least on inequality. We will show that then $(h'_1, h'_2)P(h_1, h_2)$.

Case 1: $\text{Max} > \text{Max}'$

Then $(h'_1, h'_2)P(h_1, h_2)$.

Case 2: $\text{Max} < \text{Max}'$

Subcase 2.1: If $\text{Max}' = r_1(h'_1)$ then, since $r_2(h_2) \geqslant r_1(h'_1)$, $Max = r_1(h_1)$ and $r_2(h_2) \geqslant r_1(h'_1) >$

$r_1(h_1)$. Thus, Max $= r_2(h_2)$, which is a contradiction.

Subcase 2.2: If Max$' = r_2(h_2')$ then, since $r_1(h_1) \geqslant r_2(h_2')$, Max $= r_2(h_2)$ and $r_1(h_1) \geqslant r_2(h_2') > r_2(h_2)$. Thus, Max $= r_1(h_1)$, which is a contradiction.

Case 3: Max $=$ Max$'$

If Min $>$ Min$'$ then $(h_1', h_2')P(h_1, h_2)$. Assume that Min $\leqslant$ Min$'$.

Subcase 3.1: If Min $= r_1(h_1)$ then either a) $r_1(h_1) = r_2(h_2')$ and Min$' = r_2(h_2')$ or b) Min$' = r_1(h_1')$. If a) holds then $r_2(h_2) > r_1(h_1')$ and then Max$' \geqslant$ Min$'$ implies $r_2(h_2) > r_1(h_1') \geqslant r_2(h_2') = r_1(h_1)$. Then Max $= r_2(h_2) \neq$ Max$'$, which is a contradiction. Thus, b) holds. Therefore, Min$' = r_1(h_1')$ and $r_1(h_1') \geqslant r_1(h_1) \geqslant r_2(h_2')$. Thus, Min$' = r_2(h_2')$. Since Min$' = r_1(h_1')$, $r_2(h_2) \geqslant r_1(h_1') = r_1(h_1) = r_2(h_2')$. Then $r_2(h_2) > r_1(h_1')$. However, this contradicts Max $=$ Max$'$.

Subcase 3.2: If Min $= r_2(h_2)$ then either a) $r_2(h_2) = r_1(h_1')$ and Min$' = r_1(h_1')$ or b) Min$' = r_2(h_2')$. If a) holds then $r_1(h_1) > r_2(h_2')$ and then Max$' \geqslant$ Min$'$ implies $r_1(h_1) > r_2(h_2') \geqslant r_1(h_1') = r_2(h_2)$. Then Max $= r_1(h_1) \neq$ Max$'$, which is a contradiction. Thus, b) holds. Therefore, Min$' = r_2(h_2')$ and $r_2(h_2') \geqslant r_2(h_2) \geqslant r_1(h_1')$. Thus, Min$' = r_1(h_1')$. Since Min$' = r_2(h_2')$, $r_1(h_1) \geqslant r_2(h_2') = r_2(h_2) = r_1(h_1')$. Then $r_1(h_1) > r_2(h_2')$. However, this contradicts Max $=$ Max$'$. □

Claim 1.3. *If a pairing rule satisfies Strong Pareto, Cross-Dominance and Limited Equity, then it is the Rank-Based Leximin rule.*

Proof. Fix $(r_1, r_2) \in R \times R$ and let P denote $\varphi(r_1, r_2)$, where φ satisfies Strong Pareto, Cross-Dominance and Limited Equity. Let (h_1, h_2) and (h_1', h_2') be two distinct hospital pairs.

If Max $=$ Max$'$ and Min $>$ Min$'$, (h_1', h_2') dominates (h_1, h_2). Then Dominance implies that $(h_1', h_2')P(h_1, h_2)$.

If Max $=$ Max$'$ and Min $=$ Min$'$ then, given that $(h_1, h_2) \neq (h_1', h_2')$, we have symmetric opposites: $r_1(h_1) = r_2(h_2')$, $r_2(h_2) = r_1(h_1')$ and $r_1(h_1) \neq r_1(h_1')$. Then, if $r_1(h_1) > r_1(h_1')$ then $(h_1', h_2')P(h_1, h_2)$ by convention.

If Max $>$ Max$'$, we consider three cases regarding $\sum$ and $\sum'$. Note first that Max $=$ Min $+g = \sum -$ Min, and hence 2Max $= \sum + g$. Similarly, 2Max$' = \sum' + g'$. Thus Max $>$ Max$'$ if and only if $\sum + g > \sum' + g'$.

Case 1: $\sum = \sum'$

Then $\sum + g > \sum' + g'$ implies $g > g'$, and thus $(h_1', h_2')P(h_1, h_2)$ by Equal-Sum Equity.

Case 2: $\sum > \sum'$

Let $(\tilde{h}_1, \tilde{h}_2)$ be such that $\tilde{\text{Max}} = \text{Max}' + 1$ and $\tilde{\text{Min}} = \sum - \text{Max}' - 1$. Note that $\tilde{\text{Min}}$ is feasible, since $\text{Min}' \geqslant 1$, $\sum' \geqslant \text{Max}' + 1$ and thus $\sum > \sum'$ implies that $\sum > \text{Max}' + 1$, and hence $\tilde{\text{Min}} > 0$. Observe that $\tilde{\sum} = \sum$. We will show next that $\tilde{g} \leqslant g$. Given that $\text{Max}' < \text{Max}$, $\text{Max}' + 1 \leqslant \text{Max}$. Then $2\text{Max}' + 2 \leqslant 2\text{Max} = \text{Max} - \text{Min} + \text{Max} + \text{Min}$. This means that $2\text{Max}' + 2 - \sum \leqslant \text{Max} - \text{Min}$, which is equivalent to $\tilde{g} \leqslant g$, since $\tilde{g} = \tilde{\text{Max}} - \tilde{\text{Min}} = \text{Max}' + 1 - \sum + \text{Max}' + 1 = 2\text{Max}' + 2 - \sum$ and $g = \text{Max} - \text{Min}$.

Now note that $\text{Max}' < \tilde{\text{Max}}$. We will show that $\text{Min}' \leqslant \tilde{\text{Min}}$. Since $\sum > \sum'$, $\sum' + 1 \leqslant \sum$. Thus, $\text{Min}' + \text{Max}' + 1 \leqslant \sum$ and $\text{Min}' \leqslant \sum - \text{Max}' - 1 = \tilde{\text{Min}}$. Therefore, (h'_1, h'_2) dominates $(\tilde{h}_1, \tilde{h}_2)$ and Dominance implies that $(h'_1, h'_2)P(\tilde{h}_1, \tilde{h}_2)$.

If $\tilde{g} < g$ then, given that $\tilde{\sum} = \sum$, Equal-Sum Equity implies $(\tilde{h}_1, \tilde{h}_2)P(h_1, h_2)$. If $\tilde{g} = g$, either $(\tilde{h}_1, \tilde{h}_2) = (h_1, h_2)$ or $(\tilde{h}_1, \tilde{h}_2)$ and (h_1, h_2) are symmetric opposites. Since (h'_1, h'_2) dominates $(\tilde{h}_1, \tilde{h}_2)$, it follows that in both cases (h'_1, h'_2) dominates (h_1, h_2), and therefore $(h'_1, h'_2)P(h_1, h_2)$ by Dominance.

Case 3: $\sum < \sum'$

Given that $\sum < \sum'$, (h'_1, h'_2) does not dominate (h_1, h_2). Since $\text{Max} > \text{Max}'$, (h_1, h_2) does not dominate (h'_1, h'_2). Now note that $\sum + g > \sum' + g'$ implies that $g > g'$. Thus, given that there is no dominance relation between (h_1, h_2) and (h'_1, h'_2), Limited Equity implies that $(h'_1, h'_2)P(h_1, h_2)$. □

Finally, note that it follows from Claim 2.1 below that the Rank-Based Leximin rule satisfies Strong Pareto (the $k = 0$ case in Claim 2.1). Together with this result, Claims 1.1, 1.2, and 1.3 prove Theorem 1.

Proof of Theorem 2

Claim 2.1. *For all $k \in \{0, ..., q-1\}$ the k-Lexi-Pairing rule ψ^k satisfies Strong Pareto.*

Proof. Let $k \in \{0, ..., q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h'_1, h'_2) satisfy $r_1(h_1) \geqslant r_1(h'_1)$ and $r_2(h_2) \geqslant r_2(h'_2)$, with at least one strict inequality. We need to show that $(h'_1, h'_2)P^k(h_1, h_2)$.

Case 1. $r_1(h_1) > r_1(h'_1)$ and $r_2(h_2) > r_2(h'_2)$

Subcase 1.1: If $r_1(h_1) \geqslant r_2(h_2) - k$ then $\text{Max} = r_1(h_1) > r_1(h'_1)$, and $\text{Max} > r_2(h'_2) - k$. Thus, Max > Max$'$.

Subcase 1.2: If $r_1(h_1) < r_2(h_2) - k$ then Max $= r_2(h_2) - k > r_1(h_1) > r_1(h_1')$, and Max$> r_2(h_2') - k$. Thus, Max > Max$'$.

Case 2. $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$

Subcase 2.1: If $r_1(h_1) \geqslant r_2(h_2) - k$, then Max $= r_1(h_1) = r_1(h_1')$ and Max $> r_2(h_2') - k$. Thus, Max$'$ $= r_1(h_1')$ and Max = Max$'$. Furthermore, Min $= r_2(h_2) - k$, Min $> r_2(h_2') - k$, and since Max$'$ $= r_1(h_1')$, Min$'$ $= r_2(h_2') - k$. Thus, Min > Min$'$.

Subcase 2.2: If $r_1(h_1) < r_2(h_2) - k$ then Max $= r_2(h_2) - k$, Max $> r_2(h_2') - k > r_1(h_1) = r_1(h_1')$. Thus, Max > Max$'$.

Case 3. $r_1(h_1) > r_1(h_1')$ and $r_2(h_2) = r_2(h_2')$

Subcase 3.1: If $r_1(h_1) > r_2(h_2) - k$ then Max $= r_1(h_1) > r_1(h_1')$, and Max $> r_2(h_2') - k$. Thus, Max > Max$'$.

Subcase 3.2: If $r_1(h_1) \leqslant r_2(h_2) - k$ then Max $= r_2(h_2) - k = r_2(h_2') - k \geqslant r_1(h_1) > r_1(h_1')$. Thus, Max = Max$'$. Then, Min $= r_1(h_1)$ and Min $> r_1(h_1')$, which implies that Min > Min$'$.

Therefore, $(h_1', h_2')P^k(h_1, h_2)$ in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies Strong Pareto. □

Claim 2.2. *For all $k \in \{0, \ldots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Compromise.*

Proof. Let $k \in \{0, \ldots, q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h_1', h_2') satisfy $r_1(h_1) < r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$.

Case 1: We will show that $r_1(h_1') < r_2(h_2) - k$ implies $(h_1', h_2')P^k(h_1, h_2)$.

Note first that $r_2(h_2) - k > r_1(h_1') > r_1(h_1)$ and thus Max $= r_2(h_2) - k$.

Subcase 2.1: If Max$'$ $= r_1(h_1')$ then $r_2(h_2) - k > r_1(h_1')$ implies Max > Max$'$.

Subcase 2.2: If Max$'$ $= r_2(h_2') - k$ then $r_2(h_2) > r_2(h_2')$ implies Max > Max$'$.

Hence, $(h_1', h_2')P^k(h_1, h_2)$ in both subcases.

Case 2: We will show that $r_1(h_1') > r_2(h_2) - k$ implies $(h_1, h_2)P^k(h_1', h_2')$.

Note first that $r_1(h_1') + k > r_2(h_2) > r_2(h_2')$ and thus Max$'$ $= r_1(h_1')$.

Subcase 1.1: If Max $= r_1(h_1)$ then $r_1(h_1) < r_1(h_1')$ implies Max < Max$'$.

Subcase 1.2: If Max $= r_2(h_2) - k$ then $r_2(h_2) - k < r_1(h_1')$ implies Max < Max$'$.

Hence, $(h_1, h_2)P^k(h_1', h_2')$ in both subcases.

Therefore, k-Compromise is satisfied in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \dots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Compromise. $\square$

Claim 2.3. *For all $k \in \{0, \dots, q-1\}$, the k-Lexi Pairing rule satisfies k-Threshold-Consistency.*

Proof. Let $k \in \{0, ..., q-1\}$. Fix $(r_1, r_2) \in R \times R$ and let P^k denote $\psi^k(r_1, r_2)$. Let (h_1, h_2) and (h_1', h_2') satisfy $r_1(h_1) < r_1(h_1')$, $r_2(h_2) > r_2(h_2')$ and $r_1(h_1') = r_2(h_2) - k$.

Then $r_1(h_1) < r_1(h_1') = r_2(h_2) - k$ and thus $\text{Max} = r_2(h_2) - k$. Also, $r_1(h_1') = r_2(h_2) - k > r_2(h_2') - k$ and thus $\text{Max}' = r_1(h_1')$. Therefore, $\text{Max} = \text{Max}'$. Moreover, $\text{Min} = r_1(h_1)$ and $\text{Min}' = r_2(h_2') - k$. .

Case 1: $r_1(h_1) > r_2(h_2') - k$

Then $\text{Min} > \text{Min}'$ and $(h_1', h_2') P^k (h_1, h_2)$.

Case 2: $r_1(h_1) < r_2(h_2') - k$

Then $\text{Min} < \text{Min}'$ and $(h_1, h_2) P^k (h_1', h_2')$.

Case 3: $r_1(h_1) = r_2(h_2') - k$

Then $\text{Min} = \text{Min}'$ and, since $r_1(h_1) < r_1(h_1')$, $(h_1, h_2) P^k (h_1', h_2')$.

Therefore, k-Threshold-Consistency is satisfied in each case. Since this holds for all $(r_1, r_2) \in R \times R$, for all $k \in \{0, \dots, q-1\}$, the k-Lexi-Pairing rule satisfies k-Threshold-Consistency. $\square$

Claim 2.4. *Let $k \in \{0, \dots, q-1\}$. If a pairing rule satisfies Strong Pareto, k-Compromise, and k-Threshold-Consistency then it is the k-Lexi-Pairing rule.*

Proof. Let $k \in \{0, \dots, q-1\}$ and let $(h_1, h_2), (h_1', h_2') \in H \times H$ such that $(h_1, h_2) \neq (h_1', h_2')$. Let Max, Min, Max' and Min' be defined as before. We consider four scenarios (I-IV) depending on the values these take.

I. $\text{Max} = r_1(h_1)$, $\text{Min} = r_2(h_2) - k$, $\text{Max}' = r_1(h_1')$, $\text{Min}' = r_2(h_2') - k$

Case 1: Let $\text{Max} > \text{Max}'$. Then $r_1(h_1) > r_1(h_1')$.

Subcase 1.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2') P_1 (h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_2(h_2) < r_2(h_2')$ then, since $\text{Max}' \geqslant \text{Min}'$ implies $r_1(h_1') \geqslant r_2(h_2') - k$, we have $r_1(h_1) > r_2(h_2') - k$. Thus, $(h_1', h_2') P_1 (h_1, h_2)$ by k-Compromise.

Case 2: Let $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$. Then $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) > r_2(h_2')$. Then $(h_1', h_2') P_1 (h_1, h_2)$ by Strong Pareto.

Case 3: Let Max = Max$'$ and Min = Min$'$. Then $r_1(h_1) = r_1(h_1')$ and $r_2(h_2) = r_2(h_2')$. Thus, $(h_1, h_2) = (h_1', h_2')$, which is ruled out.

II. Max $= r_2(h_2) - k$, Min $= r_1(h_1)$, Max$'$ $= r_2(h_2') - k$, Min$'$ $= r_1(h_1')$

Case 1: Let Max > Max$'$. Then $r_2(h_2) > r_2(h_2')$.

Subcase 1.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_1(h_1) < r_1(h_1')$ then, since Max$'$ $\geqslant$ Min$'$ implies $r_2(h_2') - k \geqslant r_1(h_1')$, which means $r_2(h_2') \geqslant r_1(h_1') + k$, we have $r_2(h_2) > r_1(h_1') + k$. Thus, $r_1(h_1') < r_2(h_2) - k$ and $(h_1', h_2')P_1(h_1, h_2)$ by k-Compromise.

Case 2: Let Max = Max$'$ and Min > Min$'$. Then $r_2(h_2) = r_2(h_2')$ and $r_1(h_1) > r_1(h_1')$. Then $(h_1', h_2')P_1(h_1, h_2)$ by Strong Pareto.

Case 3: Let Max = Max$'$ and Min = Min$'$. Then $r_2(h_2) = r_2(h_2')$ and $r_1(h_1) = r_1(h_1')$. Thus, $(h_1, h_2) = (h_1', h_2')$, which is ruled out.

III. Max $= r_2(h_2) - k$, Min $= r_1(h_1)$, Max$'$ $= r_1(h_1')$, Min$'$ $= r_2(h_2') - k$

Case 1: Let Max > Max$'$. Then Max > Max$'$ $\geqslant$ Min$'$ and thus $r_2(h_2) > r_2(h_2')$.

Subcase 1.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_1(h_1) < r_1(h_1')$ then, since Max > Max$'$, $r_2(h_2) - k > r_1(h_1')$ and thus $(h_1', h_2')P(h_1, h_2)$ by k-Compromise.

Case 2: Let Max = Max$'$ and Min > Min$'$. Then Max $\geqslant$ Min >Min$'$ and thus $r_2(h_2) > r_2(h_2')$.

Subcase 2.1: If $r_1(h_1) \geqslant r_1(h_1')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 2.2: If $r_1(h_1) < r_1(h_1')$, Max = Max$'$ implies $r_2(h_2) - k = r_1(h_1')$, and Min > Min$'$ implies $r_1(h_1) > r_2(h_2') - k$, and thus $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency.

Case 3: Let Max = Max$'$ and Min = Min$'$. Suppose that Max$'$ = Min$'$. Then $(h_1, h_2) = (h_1', h_2')$, which is ruled out. Thus Max$'$ > Min$'$ = Min and $r_1(h_1') > r_1(h_1)$. Also, Max > Min$'$ and thus $r_2(h_2) > r_2(h_2')$. Moreover, Max = Max$'$ implies $r_1(h_1') = r_2(h_2) - k$ and Min = Min$'$ implies $r_1(h_1) = r_2(h_2') - k$. Thus, $(h_1, h_2)P(h_1', h_2')$ by k-Threshold-Consistency.

IV. Max $= r_1(h_1)$, Min $= r_2(h_2) - k$, Max$'$ $= r_2(h_2') - k$, Min$'$ $= r_1(h_1')$

Case 1: Let Max > Max$'$. Then Max > Min$'$ and thus $r_1(h_1) > r_1(h_1')$.

Subcase 1.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 1.2: If $r_2(h_2) < r_2(h_2')$ then, since $\text{Max} > \text{Max}'$, $r_1(h_1) > r_2(h_2') - k$ and thus $(h_1', h_2')P(h_1, h_2)$ by k-Compromise.

Case 2: Let $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$. Then $\text{Max} > \text{Min}'$ and thus $r_1(h_1) > r_1(h_1')$.

Subcase 2.1: If $r_2(h_2) \geqslant r_2(h_2')$ then $(h_1', h_2')P(h_1, h_2)$ by Strong Pareto.

Subcase 2.2: If $r_2(h_2) < r_2(h_2)$, $\text{Max} = \text{Max}'$ implies $r_1(h_1) = r_2(h_2') - k$, and $\text{Min} > \text{Min}'$ implies $r_2(h_2) - k > r_1(h_1')$, and thus $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency.

Case 3: Let $\text{Max} = \text{Max}'$ and $\text{Min} = \text{Min}'$. Suppose that $\text{Max}' = \text{Min}'$. Then $(h_1, h_2) = (h_1', h_2')$, which is ruled out. Thus $\text{Max}' > \text{Min}' = \text{Min}$ and $r_2(h_2') > r_2(h_2)$. Also, $\text{Max} > \text{Min}'$ and thus $r_1(h_1) > r_1(h_1')$. Moreover, $\text{Max} = \text{Max}'$ implies $r_1(h_1) = r_2(h_2') - k$ and $\text{Min} = \text{Min}'$ implies $r_1(h_1') = r_2(h_2) - k$. Thus, $(h_1', h_2')P(h_1, h_2)$ by k-Threshold-Consistency. $\square$

Claims 2.1, 2.2, 2.3 and 2.4 together prove Theorem 2.

Chapter 4

Matching Couples in the Labour Market

4.1 Introduction

After finishing medical school, doctors in the United States, Canada and the UK, among many other countries, are required to take up medical residency positions in hospitals. The NRMP (National Resident Matching Program) in the Unites States has been much studied as one of the applications of matching theory, and other examples of matching problems of university graduates to entry-level positions in labour markets are also well-known, such as the assignment of law clerks and the allocation of traineeships to teachers in different countries. The matching problem in such centralized labour markets is further complicated by the presence of couples in the same market. Couples often wish to coordinate their assigned positions in order to stay in the same geographical area, which means that their preferences exhibit complementarities.

The algorithm that was successfully used by the NRMP since 1952 suffered a crisis of confidence in the 1990's, since it used the hospital-proposing Deferred Acceptance algorithm (Gale and Shapley, 1962 [11]) which favored hospitals at the expense of the medical doctors and was manipulable by doctors. Moreover, with the increase of women in the medical profession, the quickly growing number of couples in the NRMP match also contributed to this breakdown of the centralized system, as the existence of stable matchings in a market with couples cannot be guaranteed (Roth, 1984

[32] ; Cantala, 2004 [7]; Klaus and Klijn, 2005 [22] ; Hatfield and Kominers, 2017 [14]), and couples were forced to find jobs outside the centralized residency match. Subsequently, Roth and Peranson redesigned the NRMP matching algorithm in the mid 1990's and the new mechanism, the Applicant Proposing Algorithm which is still used today, enables couples to submit their preferences jointly Roth and Peranson (1999) [34]. The new design also involved switching to the doctor-proposing Deferred Acceptance algorithm, and it tried to reduce the effects of the stability and incentive issues that seem to be inevitable in markets with couples. Although Roth and Peranson (1999) [34] shows, by way of simulations, that their mechanism works well in practice and frequently finds a stable matching in the NRMP, Klaus et al. (2007) [23] demonstrates that the redesigned NRMP algorithm nonetheless suffers from severe stability and incentive issues, at least in theory. Specifically, they show that the algorithm may not select a stable matching even when preferences are responsive and hence a stable matching exists, and they also demonstrate that there are preference profiles at which a couple can manipulate by pretending to be singles. Despite these drawbacks, the successful practical use of the new NRMP algorithm is justified by an increasing number of papers studying large markets with complementarities or couples (see, e.g., Kojima et al. 2013 [25] and Ashlagi et al. 2014 [2]).

Klaus and Klijn (2005) [22] shows that for couples markets with strictly unemployment averse couples the domain of responsive preferences where all hospitals are considered acceptable in the associated individual preferences is a maximal domain for the existence of stable matchings. This is a negative result because the typical couple's preferences with complementarities rules out responsiveness. Responsive preferences imply that the couple prefers joint assignments that are reached by a unilateral improvement for one of the two partners, but this is exactly ruled out when the joint preferences of a couple reflect a preference for geographical proximity for their positions. Khare et al. (2018) [21] builds on these previous results and characterize the exact responsive preferences of couples under which a stable matching always exists, while Khare and Roy (2018) [20] studies the existence of stable matchings in couples markets when preferences do not satisfy responsiveness. Dutta and Massó (1997) [9] imposes very specific preference restrictions and show the existense of stable matchings, and Jiang and Tian (2013) [17] studies a relaxation of the stability concept to obtain positive results.

In this paper we propose a new mechanism for matching couples in the labour market using the (k, t)-Lexi-Pairing rules introduced in Chapter 3. Thus, couples form their paired preference ordering based on their individual preference orderings, using only two criteria. One criterion shows the level of willingness to sacrifice a good position in order for the worse-off partner to get a better position, and the other one reflects the level of eagerness for the couple to stay in the same geographical area, even if this means sacrificing some of the individually higher-ranked options in order to stay together. This means that the input of the couples is restricted to two parameters in addition to their individual preference orderings: parameter k shows the degree to which one partner is favored over the other, and parameter t indicates the extent to which the couple prioritizes the proximity of their jobs.

Even though some of the information about the actual preferences may be lost with these restrictions, it is nonetheless desirable to restrict the couples' preference input in this manner for multiple reasons. First, couples may not know how to aggregate their individual preferences, and this offers a clear method to identify their joint preferences. Second, this restriction means informational simplicity and efficiency when submitting the preferences, which generates the entire list of paired preference choices, which is likely to be longer than what couples would submit if they were asked to rank hospital pairs (which is the current practice for the NRMP and other similar matches). This could increase the efficiency of the matching. Third, and this is the most important advantage of this method in the current context, these restrictions ensure a clear and simplified structure of the couples' preferences, which may not only be conducive to designing a transparent mechanism, but can also be used directly by the mechanism. This is indeed the case for the proposed Lexi Couples mechanism, and consequently the main novelty of our contribution is that this is the first time that the desirable level of togetherness in the joint preference orderings of couples is directly utilized in the algorithm in order to mitigate the occurrence of cycles and reach stability. We show that the Lexi Couples mechanism satisfies responsive-stability, which means that a stable matching is selected at every responsive preference profile, and we also establish that the mechanism is non-wasteful, which is a non-trivial property of algorithms for matching markets with couples.

4.2 Model and Definitions

A set of doctors is matched to a set of hospitals in a many-to-one matching. The set of hospitals is denoted by H. Each hospital $h \in H$ has a finite capacity $q_h \geq 2$, that is, each hospital has at least two positions. Multiple positions at each hospital are identical. We denote the set of doctors by D and let $D = F \cup M \cup S$, where F, M and S are pairwise disjoint sets with $|F| = |M|$. A couple is $c = (f, m) \in (F \times M)$ such that the set of couples C is given by a bijection between F and M. The doctors in set S are single doctors. We refer to members of a couple as partners.

4.2.1 Preferences

We introduce the preferences of doctors and hospitals next. For a set X, let $\mathbb{L}(X)$ be the set of linear orders, i.e., complete, transitive, and antisymmetric binary relations over X. An element $P \in \mathbb{L}(X)$ is called a preference over X, and R denotes the weak part of P.

Hospitals' preferences

For each hospital $h \in H$, a preference of h over individual doctors, denoted by $P(h)$, is defined as an element of $\mathbb{L}(D)$, and we refer to this as the strict preference ordering of the hospital over individual doctors. We assume that each hospital prefers to have any doctor to a vacant position. For each hospital $h \in H$, the feasible sets of doctors are given by $\{D^{'} \subseteq D : |D^{'}| \leq q_h\}$. A preference over feasible sets of doctors for hospital h, denoted by $\tilde{P}(h)$, is given by a strict preference over the feasible sets of doctors, and thus it is an element in $\mathbb{L}(\{D^{'} \subseteq D : |D^{'}| \leq q_h\})$.

Responsive preferences for hospitals

Now we introduce the responsive extension of a hospital's preference over individual doctors to feasible sets of doctors. This is a standard notion used in the literature (see, for example, Roth (1985) [33]). Let $h \in H$ and let P_h be a preference of h over individual doctors. Then, a preference $\tilde{P}_h$ of h over feasible sets of doctors satisfies responsiveness with respect to $P(h)$ if

1. the restriction of $\tilde{P}_h$ to individual doctors coincides with P_h, that is, for all $d, d' \in D$, $d\tilde{P}_h d'$ if and only if $dP_h d'$, and

2. for all $D' \subset D$ and all $d_1, d_2 \in D \setminus D'$ such that $|D'| < q_h$, we have $(D' \cup d_1)\tilde{P}_h(D' \cup d_2)$ if and only if $d_1 P_h d_2$.

We assume that each hospital $h \in H$ has responsive preferences over feasible sets of doctors, given the hospital's preference over individual doctors. Let $P_H \equiv (P_h)_{h \in H}$.

Doctors' preferences

For each doctor $d \in D$, a preference of d over hospitals, denoted by P_d, is defined as an element of $\mathbb{L}(H)$, and we refer to this as the strict preference ordering of the doctors over hospitals. We assume that each doctor finds each hospital acceptable, that is, prefers to be matched to any hospital rather than remain unmatched. Let $P_D \equiv (P_d)_{d \in D}$.

Couples' preferences

We have already defined the preferences of individual doctors, regardless of whether it is a single doctor or a member of a couple. Now we define the preferences of couples $c \in C$, which are assumed to be restricted to the Lexi-Pairing preferences studied in Chapter 3. We make this assumption in order to restrict the input that can be submitted by couples to run the algorithm, which is similar to restricting the preferences of hospitals to responsive preferences, allowing for a simple input and informational efficiency.

Pairing rules

$H \times H$ is the set of paired hospital positions. Given that multiple positions at each hospital are the same, $(h_f, h_m) \in H \times H$ indicates a pair of positions where h_f denotes the position for doctor f and h_m denotes the position for doctor m. Note that since we assume that each hospital has at least two positions, it is possible that $h_f = h_m = h$ for all $h \in H$, and thus both doctors are matched to a position at the same hospital.

Given that each doctor has individual preferences over the hospitals, as defined above, we can express the individual preference ordering of doctor $i \in D$ by the ranking of each hospital h, denoted by $r_i(h) \in \{1, \ldots, m\}$, where $r_i(h) < r_i(h')$ means that doctor i prefers hospital h to h', since h_i has a lower rank number than h'. We assume that each hospital position is acceptable to both doctors, that is, each doctor would rather get a job than remain unmatched. Let $\mathcal{R}$ denote the set of individual hospital rankings. For each doctor $i \in \{f, m\}$, let $r_i \in \mathcal{R}$ denote a particular ranking of all hospitals in H by doctor i. Let $\hat{\mathcal{P}}$ denote the set of strict preference orderings over the ordered pairs of hospitals, that is, the set of aggregated preference orderings of paired hospital positions. Then $P \in \hat{\mathcal{P}}$ is a strict preference ordering over $H \times H$ and represents the joint preferences of the two doctors.

A *pairing rule* is a preference aggregation function for two doctors, which maps from two strict individual preference orderings of individual hospital positions to one strict preference ordering of paired hospital positions. Formally, a pairing rule is a function $\varphi : (\mathcal{R} \times \mathcal{R}) \to \hat{\mathcal{P}}$, specifying the preference aggregation of the respective individual hospital rankings of the two doctors. We will also use the notation $(h_f, h_m)\hat{P}(h'_f, h'_m)$ to indicate that (h_f, h_m) is preferred to (h'_f, h'_m) in the joint preferences $\hat{P} \in \hat{\mathcal{P}}$.

k-Lexi-Pairing rules $(k \in \{0, \ldots, m-1\})$

Fix $k \in \{0, ..., m-1\}$ and $(r_f, r_m) \in \mathcal{R} \times \mathcal{R}$ and let $\hat{P}^k$ denote the paired preference ordering for k. For $(h_f, h_m) \in H \times H$, let

$$\text{Max} \equiv \max\ (r_f(h_f), r_m(h_m) - k);$$

$$\text{Min} \equiv \min\ (r_f(h_f), r_m(h_m) - k).$$

For $(h'_f, h'_m) \in H \times H$, let

$$\text{Max}' \equiv \max(r_f(h'_f), r_m(h'_m) - k);$$

$$\text{Min}' \equiv \min\ (r_f(h'_f), r_m(h'_m) - k).$$

Then $(h'_f, h'_m)P^k(h_f, h_m)$ if one of the following three cases holds:

1. $\text{Max} > \text{Max}'$;

2. $\text{Max} = \text{Max}'$ and $\text{Min} > \text{Min}'$;

3. $\text{Max} = \text{Max}'$, $\text{Min} = \text{Min}'$ and $r_f(h_f) > r_m(h'_f)$.

Before defining the couples' preferences that take into account the geographical proximity of the two hospital positions, let us mention that we also use the concept of *responsive* preferences over hospital pairs by couples. Intuitively, a couple's preferences are responsive if the unilateral improvement of one partner's job is seen is beneficial for the couple as well. Formally, responsive preferences for couples are defined identically to the responsive preferences of hospitals, so we skip the formal definition. Note that if a couple has Lexi-Pairing preferences then the couples' preferences are responsive (see Chapter 3).

Geographical compatibility

The geographical compatibility information is given by a set $G \subset H \times H$ which consists of compatible hospital pairs for all couples. These are ordered pairs of hospitals or, equivalently, a set of edges in a compatibility graph, and are given exogenously (that is, not couple-specific). We denote the set of incompatible hospital pairs by $\bar{G}$, where $G \cap \bar{G} = \emptyset$ and $G \cup \bar{G} = H \times H$ (i.e., $\bar{G}$ is the complement of G). If $(h, h') \in G$ then getting positions at hospitals h and h' respectively for a couple is considered compatible in terms of geographic constraints, while if $(h, h') \in \bar{G}$ then positions at h and h' for a couple are not considered compatible. We assume that for all $(h, h') \in G$, $(h', h) \in G$ also holds. Naturally, for all $h \in H$, $(h, h) \in G$.

Although G is not subject to a couple's subjective preferences, it doesn't follow that couples have to have the same kind of preferences over the geographic constraints. We allow couples to attribute different levels of importance to geographical compatibility, and to this end introduce a *togetherness parameter,* denoted by t, which captures the extent to which a couple considers incompatible hospital pairs relatively less desirable when compared to compatible hospital pairs.

(k, t)*-Couple-Lexi-Pairing rules*

Fix $k, t \in \{0, ..., m-1\}$ and let $(r_f, r_m) \in \mathcal{R} \times \mathcal{R}$.

Define $\hat{r}_f$ and $\hat{r}_m$ based on (r_f, r_m) as follows. Given $t \in \{0, \ldots, m-1\}$, for all $(h_f, h_m) \in H \times H$,

1. if $(h_f, h_m) \in G$ then $\hat{r}_f(h_f, h_m) = r_f(h_f)$ and $\hat{r}_m(h_f, h_m) = r_m(h_m)$;
2. if $(h_f, h_m) \in \bar{G}$ then $\hat{r}_f(h_f, h_m) = r_f(h_f) + t + \epsilon$ and $\hat{r}_m(h_f, h_m) = r_m(h_m) + t + \epsilon$, where $0 < \epsilon < 1$.

For $(h_f, h_m) \in H \times H$, let

$$\text{Max} \equiv \max\,(\hat{r}_f(h_f, h_m), \hat{r}_m(h_f, h_m) - k);$$
$$\text{Min} \equiv \min\,(\hat{r}_f(h_f, h_m), \hat{r}_m(h_f, h_m) - k).$$

For $(h'_f, h'_m) \in H \times H$, let

$$\text{Max}' \equiv \max(\hat{r}_f(h'_f, h'_m), \hat{r}_m(h'_f, h'_m) - k);$$
$$\text{Min}' \equiv \min\,(\hat{r}_f(h'_f, h'_m), \hat{r}_m(h'_f, h'_m) - k),$$

Let $P^{(k,t)}$ denote the paired preference ordering. Then $(h'_f, h'_m) P^{(k,t)} (h_f, h_m)$ if one of the following three cases holds:

1. Max $>$ Max$'$;

2. Max $=$ Max$'$ and Min $>$ Min$'$;

3. Max $=$ Max$'$, Min $=$ Min$'$, and $\hat{r}_f(h_f, h_m) > \hat{r}_f(h'_f, h'_m)$.

Geographically incompatible hospital pairs are less preferred, since their respective rank numbers are increased by $t + \epsilon$. Adding ϵ is needed to avoid potential ties in the rankings of hospital pairs, and the incompatible hospital pair is defined to be less preferred in case of a tie.

The togetherness parameter $t \in \{0, \ldots, m-1\}$ expresses the preferences of a couple to obtain compatible positions in terms of geographic constraints. If $t = 0$ then the couple does not care about being together and the paired preference ordering is unchanged (since $\epsilon < 1$), regardless of G. At the other extreme, if $t = m - 1$ then each compatible pair of hospitals is preferred to each non-compatible pair of hospitals, while leaving all the other preference orderings unchanged. There are other cases between these two extremes which may more realistically depict a couple's preferences than either extremes, and the parameter t allows to systematically and consistently reduce the ranking of incompatible hospital pairs, while keeping other preference orderings the same. A higher value of t indicates that the couple finds it more important to find geographically compatible positions, but note that the preference ordering may not necessarily change when the value of t changes, depending on the original preferences and on G.

We assume that each couple $(f, m) \in C$ has (k, t)-Couple-Lexi-Pairing preferences over pairs of hospitals. Let $P_C \equiv (P_{(f,m)})_{(f,m)\in C}$.

Preference profile

A preference profile is a triple $((P_h)_{h\in H}), (P_d)_{d\in D}, (k_c, t_c)_{c\in C})$ consisting of the hospitals' preference profile, the doctors' preference profile, and for each couple c the parameters k_c and t_c.

4.2.2 Basic Notions

Matching

This is a many-to-one two-sided matching market which consists of hospitals and doctors. Each hospital is matched to at most as many doctors as its quota. Doctors maybe single or form couples.

Each single doctor is matched to a hospital or remains single, and each couple is matched to a pair of hospitals, or one or both partners remain single. We will define matchings below formally, where we don't distinguish between single doctors and doctors who are members of a couple.

We will say that a doctor remains unmatched if the doctor is matched to $\emptyset$. A doctor may be matched either to a hospital $h \in H$ or to $\emptyset$. A matching μ is a mapping from $H \cup D$ to $H \cup D \cup \emptyset$ such that:

- for all $h \in H$, $\mu(h) \subseteq D$ with $|\mu(h)| \leq q_h$,
- for all $d \in D$, $\mu(d) \in H \cup \emptyset$,
- for all $d \in D$ and all $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

Mechanism

A mechanism assigns a matching to every preference profile.

4.2.3 Axioms

We define next the relevant axioms for mechanisms.

Individual rationality. A matching μ is *individually rational* if all single doctors and couples weakly prefer their assignment in μ to being unassigned. Formally, for all doctors $d \in D$, $\mu(d) R_d \emptyset$ and for all $c = (f, m))$, $(\mu(f), \mu(m)) R_c (\emptyset, \emptyset)$.

Blocking coalitions. Fix a preference profile. Given a single doctor a, a hospital h, and a matching μ, (h, a) *blocks* μ if h and a prefer each other compared to their assignments at μ. Then (h, a) is a *blocking coalition* for μ.

We will say that hospital $h \in H$ would prefer to be assigned a doctor $d \in D$ compared to its assignment at μ if it prefers replacing a doctor in the set of doctors $\mu(h)$ by d or if $|\mu(h)| < q_h$. We will also say that hospital $h \in H$ would prefer to be assigned a pair of doctors $\{d, d'\} \subset D$ compared to its assignment at μ if it prefers replacing a pair of doctors in the set of doctors $\mu(h)$ by d and d' or, if $|\mu(h)| < q_h$, it prefers replacing a doctor in $\mu(h)$ by either d or d', or if $|\mu(h)| < q_h - 1$.

Given a couple $c = (f, m)$, a pair of distinct hospitals (h_f, h_m), and a matching μ, $((h_f, h_m), c)$ *blocks* μ if c prefers (h_f, h_m) compared to its assignment at μ, and

1. if $\mu(x) \neq h_x$ for both $x \in \{f, m\}$, then h_f would prefer to be assigned f and h_m would prefer to be assigned m compared to its assignment at μ,

2. if $\mu(x) = h_x$ and $\mu(y) \neq h_y$ for $x, y \in \{f, m\}$, then h_y would prefer to be assigned y compared to her assignment at μ.

Then $((h_f, h_m), c)$ is a *blocking coalition* for μ.

Given a couple $c = (f, m)$, a hospital h, and a matching μ, (h, c) *blocks* μ if c prefers (h, h) compared to its assignment at μ, and h would prefer to be assigned $\{f, m\}$ compared to its assignment at μ. Then (h, c) is a *blocking coalition* for μ.

Stability. A matching is stable at a given preference profile if it satisfies individual rationality and has no blocking coalition. A matching mechanism is stable if it assigns a stable matching to all preference profiles.

Klaus and Klijn (2005) [22] shows that for couples markets with strictly unemployment averse couples, the domain of responsive preferences for couples where all hospitals are considered acceptable in the associated individual preferences is a maximal domain for the existence of stable matchings. Since we also assume that couples are strictly unemployment averse, the existence of a stable matching can only be guaranteed in our model if couples' preferences are responsive. Therefore, we cannot require that the mechanism satisfy the standard axiom of stability, and we weaken stability to the following.

Responsive-stability. A matching mechanism is responsive-stable if it assigns a stable matching to all preference profiles where each couple's preferences are responsive.

A standard basic efficiency axiom is non-wastefulness.

Non-wastefulness. A matching is non-wasteful at a given preference profile if

1. there is no unfilled hospital position which is preferred by a single doctor to her assignment at this matching, and

2. there is no unfilled hospital position which is preferred by a couple in the sense that if one partner in the couple was assigned this hospital position while the other partner kept his/her current assignment then the couple would prefer the resulting hospital pair to their joint assignment at this matching, given their joint preferences.

A matching mechanism is non-wasteful if it assigns a non-wasteful matching to all preference profiles.

Finally, we define informational efficiency. Informational efficiency is a practical feature of mechanisms for couples markets, since couples' preferences could be difficult to both establish and to transmit, if an entire ranking over hospital pairs is required, which is typical for most mechanisms accommodating couples in entry-level labour markets. We restrict the number of parameters to two, to be concrete, but it should be clear that this number could be higher as long as the number of parameters remains "small," where the specification of "small" depends on the context.

Informational efficiency. A matching mechanism is informationally efficient if the reported preferences by each couple consist of the two respective individual rankings of hospitals by the two partners, and in addition only $x \leq 2$ parameters need to be reported by each couple.

4.3 The Lexi Couples Mechanism

In this section we introduce a new mechanism, our proposed solution to the couples' matching problem. This mechanism, which we refer to as the Lexi Couples Mechanism, relies on the $(k, t,)$-Couple-Lexi-Pairing preferences submitted by couples. It is based on the Deferred Acceptance mechanism of Gale and Shapley (1962) [11] , and we refer the reader to this paper for a definition of the Deferred Acceptance mechanism (we skip the well-known definition here, since it appears in countless other papers).

Lexi Couples Mechanism:

The mechanism consists of iterative steps. First we need the following definitions to introduce the mechanism.

Eligibility. A doctor is *eligible* for a hospital in a given step if either the doctor has not yet been rejected by the hospital in any previous step or if the doctor is declared eligible again after the last previous rejection by this hospital. A couple is *eligible* for a hospital pair in a given step if both partners are eligible for their respective hospitals in this step.

Cycle. There is a *cycle* if there exists a doctor who leaves a hospital position without being rejected by the hospital in a particular step, and in a later step this doctor proposes to this hospital again.

This later step will be called the step in which the cycle is *completed.* Note that only a partner in a couple leaves a hospital position initially without being rejected by the hospital, although this can trigger a single doctor to do the same in a later step, due to updating the eligibility status of doctors, as we will see in the description of the mechanism. Thus, a cycle is always initiated by a couple. We will refer to this couple as the couple that is *responsible for the cycle.* The responsible couple for each cycle is unique.

Fix a preference profile $((P_h)_{h\in H}, (P_d)_{d\in D}, (k_c, t_c)_{c\in C})$.

Step 1: Each single doctor applies to her highest-ranked hospital, and each couple applies to their jointly highest-ranked hospital pair. Hospitals accept applicants one at a time, filling positions up to their capacity, based on their preference ordering over individual doctors. Acceptances are temporary and matched applicants can be replaced by new applicants in later steps.

Step $\boldsymbol{k}$: Each single doctor who was rejected in step $k-1$ applies to her highest-ranked *eligible* hospital, and each couple with a partner who was rejected in step $k-1$ applies to their jointly highest-ranked *eligible* hospital pair. Hospitals consider both temporarily matched doctors and new applicants, and accept them one at a time, filling positions up to their capacity, based on their preference ordering over individual doctors. Acceptances are temporary and matched applicants can be replaced by new applicants in later steps.

Note that this implies that the rejection of a partner may lead to a vacant position left behind by the other partner who was not rejected by the hospital he/she was temporarily matched to. If there is such a vacant hospital position, left behind by a doctor who was not rejected by this hospital, then the most preferred *suitable* doctor who has been rejected by this hospital is considered eligible by this hospital in the next step. Here any single doctor is *suitable*, and any partner is *suitable* if this partner was assigned this vacant hospital position then together with the other partner's current assignment in Step k the resulting hospital pair would be preferred by the couple to their joint assignment in Step k, given their joint preferences.

If there is a step in which at least one cycle is completed, reduce the parameter t by 1 for the responsible couple for each cycle, and go back to Step 1 and run the algorithm again with this modified preference profile in which all the preferences are the same as before, except for the reduction in t for the relevant couple(s).

Keep iterating these steps.

End: If no proposals are rejected in a step, then each doctor is matched to the hospital they are currently matched to, which determines the final matching. Note that, depending on the number of hospital positions and doctors, it is possible that some doctors remain unmatched or some hospital positions remain vacant. $\diamond$

Now we need to verify that the mechanism is well-defined. Specifically, we will show that the reduction in the parameter t that is called for in case of a cycle is always feasible. Note that a couple that is responsible for a cycle does not have responsive preferences, since one of the partners in this couple leaves behind a vacant position even though this partner was not rejected by the hospital in question. Suppose that a couple (f, m) that is responsible for a cycle is temporarily matched to (h_f, h_m) when h_m rejects partner m, while h_f accepts f. Given that m's next choice where m is accepted is h', we need to show that the couple prefers (h_f, h') to (h'', h') for any hospital $h'' \neq h_f$ such that h'' has not rejected f yet, given that the couple has responsive preferences. Suppose that the couple (f, m) prefers (h'', h') to (h_f, h') and has responsive preferences. Then, by responsiveness, f prefers h'' to h_f. This is a contradiction, since h'' hasn't rejected f yet and f is matched to h_f. Therefore, if a couple is responsible for a cycle then this couple cannot have responsive preferences in the current step of the procedure. This means that $t > 1$ for this couple, since $t = 0$ implies that the couple has responsive preferences. A symmetric argument applies to the cases where h_f rejects partner f. Therefore, the required reduction of the t parameter is always feasible and the Lexi Couples mechanism is well-defined.

Next, we provide an example to demonstrate how the Lexi Couples mechanism works.

Example 4. There are four hospitals, h_1, h_2, h_3, h_4, each with two positions to fill. There are four single doctors, d_1, d_2, d_3, d_4, and one couple, (f_1, m_1), with parameters $k = 0$ and $t = 2$. Hospitals h_1 and h_2 are in one region, and h_3 and h_4 are in another region.

The preferences of individual doctors over hospitals:

f_1: h_1, h_2, h_3, h_4 $\quad$ m_1: h_3, h_1, h_4, h_2

d_1: $h_2, \ldots$ $\quad$ d_2: $h_1, \ldots$ $\quad$ d_3: $h_2, h_1, \ldots$ $\quad$ d_4: $h_2, h_3, \ldots$

Hospitals' preferences over doctors:

h_1: $d_3, d_2, m_1, f_1, \ldots$ $\quad$ h_2: $d_1, f_1, d_3, d_4, \ldots$ $\quad$ h_3: $d_4, m_1, f_1, \ldots$ $\quad$ $h_4 : \ldots$

Applying the $(0,2)$-Lexi-Pairing rule, the couple's paired preference ordering is

$(f_1, m_1) : (h_1, h_1), (h_2, h_1), (h_3, h_3), (h_3, h_4), (h_1, h_3), (h_1, h_2), (h_4, h_3), (h_2, h_2), (h_4, h_4), (h_2, h_3), \ldots$

Round 1: $t = 2$

Step	h_1	h_2	h_3	h_4	
1	$\boldsymbol{f_1}$, m_1, d_2	d_1, d_3, $\boldsymbol{d_4}$			h_1 rejects f_1 and h_2 rejects d_4
2	m_1, d_2	f_1, d_1, $\boldsymbol{d_3}$	d_4		h_2 rejects d_3
3	$\boldsymbol{m_1}$, d_2, d_3	f_1, d_1	d_4		h_1 rejects m_1
4	d_2, d_3	d_1	$\boldsymbol{f_1}$, m_1, d_4		h_3 rejects f_1
5	d_2	d_1, d_3	f_1, d_4	m_1	d_3 is eligible for h_2
6	m_1, d_2	f_1, d_1, $\boldsymbol{d_3}$	d_4		m_1 is eligible for h_1; h_2 rejects d_3 : **Cycle**

At $t = 1$, $(f_1, m_1) : (h_1, h_1), (h_2, h_1), (h_1, h_3), (h_3, h_3), (h_2, h_3), \ldots$

Round 2: $t = 1$

Step	h_1	h_2	h_3	h_4	
1	$\boldsymbol{f_1}$, m_1, d_2	d_1, d_3, $\boldsymbol{d_4}$			h_1 rejects f_1 and h_2 rejects d_4
2	m_1, d_2	f_1, d_1, $\boldsymbol{d_3}$	d_4		h_2 rejects d_3
3	$\boldsymbol{m_1}$, d_2, d_3	f_1, d_1	d_4		h_1 rejects m_1
4	d_2, d_3	d_1	$\boldsymbol{f_1}$, m_1, d_4		h_3 rejects f_1
5	d_2	f_1, d_1, $\boldsymbol{d_3}$	m_1 , d_4		d_3 is eligible for h_2, h_2 rejects d_3
6	$\boldsymbol{m_1}$, d_2, d_3	f_1, d_1	d_4		m_1 is eligible for h_1; h_1 rejects m_1 : **Cycle**

At $t = 0$, $(f_1, m_1) : (h_1, h_3), \ldots$

Round 3: $t = 0$

Step	h_1	h_2	h_3	h_4	
1	f_1, d_2	d_1, d_3, $\boldsymbol{d_4}$	m_1		h_2 rejects d_4
2	f_1, d_2	d_1, d_3	m_1, d_4		**Final Matching**

Since $t = 0$, this is the DA outcome, equivalent to having two single doctors instead of the couple.

Next, we consider new preferences for d_3, namely d_3: $h_2, h_4, \ldots$

Round 1: $\boldsymbol{t = 2}$

Step	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\boldsymbol{h_3}$	$\boldsymbol{h_4}$	
1	$\boldsymbol{f_1}$, m_1, d_2	d_1, d_3, $\boldsymbol{d_4}$			h_1 rejects f_1 and h_2 rejects d_4
2	m_1, d_2	f_1, d_1, $\boldsymbol{d_3}$	d_4		h_2 rejects d_3
3	m_1, d_2	f_1, d_1	d_4	d_3	**Final Matching**

Here the couple is assigned the compatible hospital pair (h_2, h_1), which is ranked higher in the couple's joint preference ordering than their DA assignment (h_1, h_3), which is not compatible.

4.4 Properties of the Lexi Couples Mechanism

First we note that since all hospitals are acceptable to all the doctors, and all the doctors are acceptable to all the hospitals, the Lexi Couples mechanism is trivially individually rational. It is also informationally efficient, as stated below.

Theorem 3. (Informational Efficiency)
The Lexi Couples Mechanism is informationally efficient.

Proof. Since couples are required to submit (k, t)-Couple-Lexi-Pairing preferences to participate in the centralized matching using the Lexi Couples mechanism, it is immediate that the mechanism is informationally efficient, since the (k, t)-Couple-Lexi-Pairing preferences of a couple are completely described by the two respective individual preference orderings of the partners over hospitals and by the two parameter values k and t. □

We will show next that the Lexi Couples mechanism is responsive-stable.

Theorem 4. (Responsive-Stability)
The Lexi Couples mechanism is responsive-stable.

Proof. If each couple has responsive preferences then none of the doctors leave a hospital behind without being rejected by this hospital, just like in the Deferred Acceptance mechanism. Therefore, there are no cycles in the mechanism, as shown above. Furthermore, note that the responsive preferences of couples imply that couples apply to hospitals exactly as if the two partners were single doctors. Thus, the mechanism is equivalent at such a preference profile to the doctor-proposing Deferred Acceptance mechanism and selects the doctor-optimal matching. Since the doctor-optimal matching is stable (Gale and Shapley, 1962 [11]), the Lexi Couples Mechanism is responsive-stable. □

The Lexi Couples mechanism produces the same matching as the Deferred Acceptance mechanism when all couples have responsive preferences. This means that the matching is non-wasteful at these profiles since the Deferred Acceptance mechanism is stable and stability implies non-wastefulness. However, if some couples have non-responsive preferences, not only a stable matching may not exist, but also the selected matching may not be non-wasteful, given that some doctors may leave a hospital behind in the procedure without being rejected by this hospital, and thus it is not clear whether the mechanism is non-wasteful. We prove next that it is.

Theorem 5. (Non-Wastefulness)

The Lexi Couples mechanism is non-wasteful.

Proof. Suppose that the Lexi Couples mechanism is wasteful. Then there exists a preference profile at which a hospital position is unfilled in matching μ that is assigned to this preference profile by the Lexi Couples mechanism, and either

1. there is a single doctor who prefers this hospital position to her assignment in μ, or
2. there is a couple who prefers this hospital position in the sense that if one partner in the couple was assigned this hospital position while the other partner kept his/her current assignment then the couple would prefer the resulting hospital pair to their joint assignment in μ, given their joint preferences.

Let this hospital be h. Then there exists a step in the procedure at this preference profile such that this hospital position at h is left behind by a doctor without being rejected by hospital h, since otherwise the position at h would not be unfilled in μ, given that there is a doctor who desires this open hospital position, whether it is a single doctor or a partner in a couple in one of the above listed two scenarios. Assume without loss of generality that this hospital position at h is not filled in any step after this step, which means that this is the last step in the procedure where a doctor leaves behind a position at h without being rejected by h. Now note that in the Lexi Couples mechanism the most preferred suitable doctor who has been rejected by h becomes eligible for h, and this doctor, say d, applies to h in the next step. This means that either d or a different doctor who is preferred by h to d is accepted by h for this open position in the next step of the procedure, contradicting our assumption that this hospital position is not filled after this step. Therefore, the Lexi Couples mechanism is non-wasteful. □

4.5 Conclusion: Extensions and Open Questions

We proposed a new mechanism for use in entry-level labour markets where couples are present. The novelty of this new mechanism is that it does not require couples to submit entire preference orderings over pairs of hospitals positions, but instead it asks for the submission of only two parameters from each couple in addition to their individual preference rankings over hospital positions. This feature does not only ensure the informational efficiency of the mechanism, both in terms of formulating the couples' preferences and information transmission, but it is also used in the construction of the mechanism, leading to further desirable properties of the proposed Lexi Couples mechanism. In particular, this mechanism satisfies the stability property of responsive-stability and it is also non-wasteful. Stability is notoriously difficult to establish for markets with complementarities, and the presence of couples is a prominent case of complementarities in the preferences due to geographical considerations, so even the weak stability property of responsive-stability is difficult to achieve. Most notably, the redesigned NRMP algorithm Roth and Peranson (1999) [34] does not satisfy responsive-stability, which is demonstrated by Klaus et al. (2007) [23]. Non-wastefulness is also a basic requirement, but it is well-known that in matching markets with distributional restrictions it is often difficult to satisfy it together with other basic criteria (see, for example, Ehlers et al., 2014 [10]). In couples markets it is also a non-trivial requirement, because it is natural for the algorithm to require that one member of a couple leave a position behind to coordinate with the other member, a feature which not only tends to induce cycles but also may easily lead to vacant positions that somebody else might desire. If these issues are not resolved satisfactorily then the mechanism may not be well-defined, or it may allow for wasteful outcomes. The Lexi Couples mechanisms resolves both of these issues successfully, even for small markets.

Moreover, we can extend the results presented here in a straightforward manner to other informationally efficient mechanisms which use pairing rules for the couples' preference aggregation that are different from the Lexi-Pairing rules. One example is an additive pairing rule which takes the sum of the rankings of the two-hospitals to determine the joint rankings: the higher the sum of the rank numbers, the lower the pair is in the preference ordering. With adequate tie-breaking and with the additional transformation of the preferences according to a togetherness parameter which would be applied exactly the same way as done for Lexi-Pairing rules, we get a different mechanism. Such mechanisms are still informationally efficient, and they can be shown to be responsive-stable and non-wasteful as well, similarly to the Lexi Couples mechanism, as proved by Theorems 2 and 3. Moreover, we could extend Theorems 2 and 3 further by using an arbitrary responsive pairing rule for the couples' preference aggregation prior to applying the togetherness parameter. This is a more flexible mechanism in terms of the allowable preferences for couples while it retains the properties of responsive-stability and non-wastefulness, but it is no longer informationally efficient, since the entire preference ordering over pairs of positions would need to be reported by the couples in addition to the togetherness parameter value, and these preferences would still need to be restricted to responsive preferences.

Remaining open questions concern the incentive properties of the mechanism. We conjecture that couples cannot benefit from pretending to be singles in the Lexi Couples mechanism, which is an incentive property that does not hold for the algorithm used by the NRMP, as shown by Klaus et al. (2007) [23]. Since in our proposed mechanism this amounts to a couple reporting $t = 0$ for the togetherness parameter instead of their true positive parameter value, and it appears intuitively that lowering the value of the togetherness parameter is unlikely to improve a couple's assignments, we believe that a positive result in this direction can be established in future work. However, unlike the normative properties which are somewhat robust with respect to how the couples' aggregate their preferences, as argued above, it seems more likely that the attributes of the pairing rules used by couples will impact the incentive properties of the resulting mechanism.

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