Relation Between Geodesic Polar and Isothermal Coordinates

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the Degree of

Master of Science (Mathematics) at

Concordia University

Montréal, Québec, Canada

May 2022

CONCORDIA UNIVERSITY
School of Graduate Studies

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Entitled: **Relation Between Geodesic Polar and Isothermal Coordinates**

and submitted in partial fulfillment of the requirements for the degree of

Master of Science (Mathematics)

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Abstract

Relation Between Geodesic Polar and Isothermal Coordinates

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The main result of this thesis is an asymptotic formula establishing a correspondence between two at first sight unrelated systems of local coordinates on a two-dimensional real analytic Riemannian manifold: the so-called isothermal local parameter and the Riemann normal coordinates. This formula was stated without proof in 1992 Fay's memoir on analytic torsion, a weaker statement was proved in Walsh's 2012 PhD thesis. A survey of basic facts from differential geometry used in our proof is also given.

Acknowledgments

I would like to express my gratitude to my supervisor Dr. Kokotov for his guidance and all the useful discussions; always making sure I understood everything to the hilt. I am also thankful to my other professors and department staff for all their support, recommendations, inspirations and guidance. Last but not least, thanks to Pierre for his support and insights during the hard times of COVID; as well as my cousin Ramin for providing me the laptop to do my teaching and thesis related duties.

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Chapter 1

Introduction

Let M be a real 2-dimensional Riemannian C^∞ manifold with metric g. Let $p \in M$ and (x, y) be local coordinates near p and the metric in the vicinity of p have the form $g = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$. We are going to question whether it is possible to reduce this metric to diagonal form, i.e, $g = \rho^{-2}(u, v)\,(du^2 + dv^2)$ via an appropriate change of variables: $u = u(x, y), v = v(x, y)$. The corresponding statement has been proven under various conditions on regularity of metric coefficients by Gauss [2]; then Korn and Lichtenstein [3], and Chern [7]. Modern presentation could be found in Spivak [4] and Taylor [5]. Some hints for other variants of proof are given in Kazdan [6] and a relatively recent article in Wikipedia. In the next chapter, we shall discuss modern version of Gauss proof where the coefficients are real analytic. In chapter three, we then give the proof of the relationship between isothermal and geodesic polar coordinates. Finally in appendices section, we will discuss the proof of existence of solutions for complex ODE's, an ingredient in Gauss proof; we will also formulate the statements of Taylor and Chern, and discuss preliminary lemmas leading to their proofs.

Chapter 2

Existence of Conformal coordinates

2.1 Gauss Formulation

In this section we will prove the following theorem:

Theorem 2.1.0.1 (Gauss). *Let the coefficients E, F, and G in the formula metric be real analytic functions of x and y, in a vicinity of a point p. Then g could be made diagonal, in this vicinity, via an appropriate real analytic change of variables:*

$$g = \lambda(u,v)(du^2 + dv^2) = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2 \tag{2.1}$$

Reminder:

Definition 2.1.0.1. *Function $f(x,y)$ is called real analytic in the domain $\Omega \subset \mathbb{R}^2$, if there exists a complex analytic function $\tilde{f}(z,\xi)$, of two complex variables z and ξ, which is holomorphic in some vicinity of $\tilde{\Omega} = \{\,(z,\xi) \mid z = x + i0, \xi = y + i0, (x,y) \in \Omega\,\} \subset \mathbb{C}^2$ in $\mathbb{C}^2$ such that $f = \tilde{f}\Big|_{\Omega}$.*

Definition 2.1.0.2. *(Equivalent to 2.1.01) Function f(x,y) is called real analytic in the domain $\Omega \subset \mathbb{R}^2$, if for any $(x_0,y_0) \in \Omega$, $f(x,y) = \sum_{n,m=0}^{\infty} a_{nm}(x-x_0)^n(y-y_0)^m$ in some open neighbourhood of (x_0,y_0).*

Our plan of proof of Theorem 2.1.0.1 is as follows. First, recall since $E > 0$, $EG - F^2 > 0$, g could be formally factorized as $g = [\alpha(x,y)dx + \beta_+(x,y)dy][\alpha(x,y)dx+$

$\beta_-(x,y)dy]$, where $\alpha = \sqrt{E}$ and $\beta_\pm = \frac{F \pm i(EG-F^2)}{\sqrt{E}}$. Clearly α is real analytic and $\beta_\pm$ has real analytic real and imaginary parts. We are going to prove that the complex 1-form $\alpha(z,\xi)dz + \beta_+(z,\xi)d\xi$, where $\alpha(z,\xi)$ and $\beta_+(z,\xi)$ are holomorphic extensions of $\alpha(x,y)$ and $\beta_+(x,y)$ (see definition 2.1.0.1), could be made exact via multiplication by an appropriate integrating factor Λ. Then we shall have $\Lambda(\alpha dz + \beta_+ d\xi) = dh$, and $\bar{\Lambda}(\bar{\alpha}d\bar{z} + \bar{\beta_+}d\bar{\xi}) = d\bar{h}$. Hence, $|\Lambda|^2|_{\tilde{\Omega}}\, g = |dh|^2|_{\tilde{\Omega}} = (du)^2 + (dv)^2$ where $u = \Re(h)|_{\tilde{\Omega}}, v = \Im(h)|_{\tilde{\Omega}}$.

Remark. *Existence of integrating factor of a real 1-form in two variables easily follows from the main theorem of ODE. (For forms with more than two variables, integrating factor generally does not exist and criterion is given by Frobenius theorem.)*

The main difficulty of this Gauss theorem is that the coefficients are complex and the theory of ODE with real argument is not applicable; we have to apply the result from the theory of analytic differential equations.

Proof of theorem 2.1.0.1: We will be using the following holomorphic analogue of the main theorem of ODE: consider the initial value problem $\frac{dY(z)}{dz} = H(z, Y(z))$, $Y(z_0) = Y_0$; where H is a holomorphic function of two variables in the vicinity of (z_0, Y_0). This problem has a unique solution defined in $\{\, z \mid |z - z_0| < \epsilon \,\} \subset \mathbb{C}$, for some $\epsilon > 0$. (Remark: standard way to prove this is to derive it from Frobenius theorem; but there is a simpler way using Picard contraction mapping. We will discuss both proofs in an appendix at the end.) In our case, $H(z, Y(z)) = \frac{-\alpha(z,Y(z))}{\beta_+(z,Y(z))}$ and since $EG - F^2 > 0$, β_+ never vanishes (our H is holomorphic) and we are in condition of Theorem A.0.0.1. Due to Theorem A.0.0.1, there exists a vicinity $U \subset \mathbb{C} \times \mathbb{C}$ of a point $p_0(z_0, Y_0)$ such that through each point (z, Y) of this vicinity passes an integral curve $Y = Y(z)$ and z_0 belongs to the domain of definition of Y. Now, consider the function $h : U \to \mathbb{C}$ defined by $h(z, Y) = Y(z_0)$, where the function $Y(\eta)$ is the solution to the initial value problem

$$\frac{dY(\eta)}{d\eta} = \frac{-\alpha(\eta, Y(\eta))}{\beta_+(\eta, Y(\eta))}, \quad Y = Y(z). \tag{2.2}$$

The integral curves in U are therefore level curves for function h. We have that $dh = \frac{\partial h}{\partial \eta}d\eta + \frac{\partial h}{\partial Y}dY + \frac{\partial h}{\partial \bar{\eta}}d\bar{\eta} + \frac{\partial h}{\partial \bar{Y}}d\bar{Y}$, where the last two terms are absent due to holomorphicity of h.

Proposition 2.1.0.1. *Let $\vec{v}$ be a complex tangent vector to the complex integral curve of equation (2.2) at some point $p \in U$. Then, $dh(p)(\vec{v}) = 0$.*

Proof. Vectors $\frac{\partial}{\partial Y}, \frac{\partial}{\partial \eta}$ generate the tangent space to $\mathbb{C}^2$ at p. Tangent space to the integral curve $Y = Y(\eta)$ is generated by the vector $\left(\frac{\partial}{\partial \eta} + \left.\frac{dY}{d\eta}\right|_p \frac{\partial}{\partial Y}\right)$[1]. On the other hand $dh(p) = \left.\frac{\partial h}{\partial \eta}\right|_p d\eta + \left.\frac{\partial h}{\partial Y}\right|_p dY$; thus, $dh(p)(\vec{v}) = \left.\frac{\partial h}{\partial \eta}\right|_p + \left.\frac{\partial h}{\partial Y}\right|_p \left.\frac{dY}{d\eta}\right|_p = \left.\frac{d}{d\eta} h(\eta, Y(\eta))\right|_p = \left.\frac{d}{d\eta} (Y(z_0))\right|_p = 0$. ☐

Notice that the 1-form $\omega(p) = \alpha(p)d\eta + \beta_+(p)dY$ also annihilates the complex tangent vector $\vec{v}$: $\omega(p)(\vec{v}) = \alpha(p) + \beta_+(p) \left.\frac{dY}{d\eta}\right|_p = \alpha(p) + \beta_+(p)\left(\frac{-\alpha(p)}{\beta_+(p)}\right) = 0$. Hence, for any $p \in U$, $\omega(p)$ and $dh(p)$ both have the same kernel and therefore $\omega(p) = \lambda(p)dh(p)$, where λ is a complex valued function on U. (Remark: for two complex linear functionals f_1 and f_2 on $\mathbb{C}^2$, the relation $\ker f_1 = \ker f_2$ implies $f_1 = \alpha f_2$ for some $\alpha \in \mathbb{C}$.) Thus, $g = |\lambda|^2|_{\tilde{\Omega}} \, |dh|^2|_{\tilde{\Omega}} = |\lambda(x, y)|^2(du^2 + dv^2)$.

Remark:

The Jacobian of the map $(x, y) \to (u(x, y), v(x, y))$ is nonzero. Otherwise, the linear system of equations $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ at a point p would have a nonzero solution $\vec{s} = \begin{pmatrix} s^1 \\ s^2 \end{pmatrix}$. This would then imply

$$g(\vec{s}, \vec{s}) = |\lambda(p)|^2 \left[\left(\left.u_x\right|_p s^1 + \left.u_y\right|_p s^2 \right)^2 + \left(\left.v_x\right|_p s^1 + \left.v_y\right|_p s^2 \right)^2 \right] = 0,$$

a contradiction!

By inverse function theorem, there exists a vicinity of point p_0 in which the metric can be written $g = |\lambda(u, v)|^2(du^2 + dv^2)$. Gauss proof completed. ☐

[1] $\frac{d}{d\eta} f(\eta, Y(\eta)) = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial Y} \frac{dY}{d\eta}$

Chapter 3

Geodesic Polar and Conformal Coordinates

3.1 Preliminaries Concerning differential geometry

Let v be a vector field on our Riemannian manifold M, $x \in M$, and $w \in T_xM$. Define the mapping: $v, w \mapsto \nabla_w v \in T_xM$, with properties:
i) ∇ linear in both its arguments;
ii) if f is a smooth function, $\nabla_{fw}v = f\nabla_w v$, and $\nabla_w fv = (\partial_w f)v + f\nabla_w v$.
The value $\nabla_w v$ is called the *covariant derivative* of v in the direction w. In local coordinates u^i, let $\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k$; the Γ^k_{ij} are called the *Christoffel Symbols*. Now, let $\gamma : (a, b) \to M$ be a curve, a vector field v is *parallel* along γ if $\nabla_{\dot\gamma}v = 0$ everywhere along γ. A curve γ with the property $\nabla_{\dot\gamma}\dot\gamma = 0$ is called a *geodesic*. Under natural assumption, we can say $\nabla_w(u, v) = (\nabla_w u, v) + (u, \nabla_w v)$. From this, one can deduce the expressions for Christoffel symbols:

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2}g^{kl}\Big(\frac{\partial g_{lj}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\Big) \\ \Gamma_{ij,p} = g_{kp}\Gamma^k_{ij} &= \frac{1}{2}\Big(\frac{\partial g_{pj}}{\partial u^i} + \frac{\partial g_{ip}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p}\Big). \end{aligned} \tag{3.1}$$

The equation for the geodesic curve becomes:

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0. \tag{3.2}$$

Theorem 3.1.0.1. *Let $x \in M$. Then there exists $\epsilon > 0$ and a neighbourhood U of x such that $\forall y \in U$, $\forall v \in T_yM$ with $|v| < \epsilon$ there exists a geodesic $\gamma : (-1, 1) \to M$ with*

$$\gamma(0) = y \qquad \dot{\gamma}(0) = v. \tag{3.3}$$

Proof. By the existence and uniqueness theorem of ordinary differential equations, there exists a neighbourhood $W(x, 0) \subset TM$ and $\delta > 0$ such that $\forall (y, v) \in W$, there exists a unique geodesic $\gamma : (-\delta, \delta) \to M$ satisfying (3.3). Let a neighbourhood U of x and $\epsilon_1 > 0$ be such that $U \times B_{\epsilon_1}(0) \subset W$. Now, if $\gamma(t)$ satisfies (3.2) so does $\tilde{\gamma}(t) = \gamma(\delta t)$ for $t \in (-1, 1)$. Letting $\epsilon = \delta\epsilon_1$, $\tilde{\gamma}(t)$ is the desired geodesic. □

3.2 Riemannian Normal Coordinate System

Let M be our manifold and $p \in M$. By Theorem 3.1.0.1, there exists ϵ_p such that $B_{\epsilon_p}(p) \subset T_pM$ is mapped uniquely onto a neighbourhood $W \subset M$ (containing p) via exponential map: $\exp_p(v) = \gamma_v(1)$, where $\gamma_v(t)$ is the geodesic curve originating at p and with speed v.

Lemma 3.2.0.1. *The exponential map $\exp_p : T_pM \supset V \to M$ has nonzero Jacobian at p.*

Proof. In local coordinates x^i, Taylor expanding $x^k(t)$ near $t = 0$, for a geodesic originating at p with speed v, we have:

$$x^k(t) = x^k(0) + \dot{x}^k(0)t + \frac{1}{2}\ddot{x}^k(0)t^2 + O(t^3) = x^k(0) + v^k t + \frac{1}{2}\Gamma^k_{ij}v^iv^jt^2 + O(t^3).$$

This gives $[\exp_p(v)]^k = x^k(p) + v^k + \frac{1}{2}\Gamma^k_{ij}v^iv^j + O(|v|^3)$; hence, $\left.\frac{\partial[\exp_p(v)]^k}{\partial v^j}\right|_{v=0} = \delta^k_j$. □

Introducing orthonormal local coordinates u^1, u^2 in $B_{\epsilon_p}(p)$, the chart $(W, \exp_p^{-1}; u^i)$

will be Riemannian normal coordinate system in neighbourhood of p. Then, we have

$$g_{ij}(0,0) = \delta_{ij} \quad i,j = 1,2.$$

The curves $(u^1,u^2) = (\alpha^1 t, \alpha^2 t)$ will be geodesic and one has (based on (3.2)):

$$\Gamma_{jk,p}(\alpha^1 t, \alpha^2 t)\alpha^j \alpha^k = 0 \quad p = 1,2. \tag{3.4}$$

At $t = 0$ we get, $\Gamma_{jk,p}(0,0) = 0$; differentiating (3.4) with respect to t and setting $t = 0$, denoting $\frac{\partial \Gamma_{jk,p}}{\partial u^i} := \frac{\partial \Gamma_{jk,p}}{\partial u^i}(0,0)$ we get:

$$\frac{\partial \Gamma_{jk,p}}{\partial u^i}\alpha^i \alpha^j \alpha^k = 0. \tag{3.5}$$

Expanding out (3.5), we get:

$$\begin{aligned}\frac{\partial \Gamma_{11,p}}{\partial u^1}(\alpha^1)^3 + \Big(2\frac{\partial \Gamma_{12,p}}{\partial u^1} + \frac{\partial \Gamma_{11,p}}{\partial u^2}\Big)(\alpha^1)^2\alpha^2 + \Big(2\frac{\partial \Gamma_{12,p}}{\partial u^2} + \frac{\partial \Gamma_{22,p}}{\partial u^1}\Big)(\alpha^2)^2\alpha^1 \\ + \frac{\partial \Gamma_{22,p}}{\partial u^2}(\alpha^2)^3 = 0, \quad p = 1,2.\end{aligned} \tag{3.6}$$

This gives rise to the following equations, using (3.1) (as before, $\frac{\partial^2 g_{ij}}{\partial u_k \partial u_l} := \frac{\partial^2 g_{ij}}{\partial u_k \partial u_l}(0,0)$):

$$\begin{aligned}\frac{\partial \Gamma_{11,p}}{\partial u^1} = 0 &\Longrightarrow \frac{1}{2}\left(\frac{\partial^2 g_{p1}}{(\partial u^1)^2} + \frac{\partial^2 g_{1p}}{(\partial u^1)^2} - \frac{\partial^2 g_{11}}{\partial u^p \partial u^1}\right) = 0 \\ &\Longrightarrow \frac{\partial^2 g_{1p}}{(\partial u^1)^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^1 \partial u^p} = 0,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&2\frac{\partial \Gamma_{12,p}}{\partial u^1} + \frac{\partial \Gamma_{11,p}}{\partial u^2} = 0 \Longrightarrow \\ &\frac{\partial^2 g_{p2}}{(\partial u^1)^2} + \frac{\partial^2 g_{1p}}{\partial u^1 \partial u^2} - \frac{\partial^2 g_{12}}{\partial u^p \partial u^1} + \frac{1}{2}\left(\frac{\partial^2 g_{p1}}{\partial u^2 \partial u^1} + \frac{\partial^2 g_{1p}}{\partial u^2 \partial u^1} - \frac{\partial^2 g_{11}}{\partial u^2 \partial u^p}\right) = 0 \\ &\Longrightarrow 2\frac{\partial^2 g_{1p}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^2 \partial u^p} + \frac{\partial^2 g_{2p}}{(\partial u^1)^2} - \frac{\partial^2 g_{12}}{\partial u^1 \partial u^p} = 0,\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&2\frac{\partial \Gamma_{12,p}}{\partial u^2} + \frac{\partial \Gamma_{22,p}}{\partial u^1} = 0 \Longrightarrow \\
&\frac{\partial^2 g_{p2}}{\partial u^2 \partial u^1} + \frac{\partial^2 g_{1p}}{(\partial u^2)^2} - \frac{\partial^2 g_{12}}{\partial u^2 \partial u^p} + \frac{1}{2}\left(\frac{\partial^2 g_{p2}}{\partial u^1 \partial u^2} + \frac{\partial^2 g_{2p}}{\partial u^1 \partial u^2} - \frac{\partial^2 g_{22}}{\partial u^1 \partial u^p}\right) = 0 \\
&\Longrightarrow 2\frac{\partial^2 g_{2p}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^p} + \frac{\partial^2 g_{1p}}{(\partial u^2)^2} - \frac{\partial^2 g_{12}}{\partial u^2 \partial u^p} = 0,
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
\frac{\partial \Gamma_{22,p}}{\partial u^2} = 0 &\Longrightarrow \frac{1}{2}\left(\frac{\partial^2 g_{p2}}{(\partial u^2)^2} + \frac{\partial^2 g_{2p}}{(\partial u^2)^2} - \frac{\partial^2 g_{22}}{\partial u^p \partial u^2}\right) = 0 \\
&\Longrightarrow \frac{\partial^2 g_{2p}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^2 \partial u^p} = 0.
\end{aligned} \tag{3.10}
$$

Putting $p = 1$ in (3.7) and $p = 2$ in (3.10):

$$
\left.\begin{aligned}
\frac{\partial^2 g_{11}}{(\partial u^1)^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^1)^2} = 0 \\
\frac{\partial^2 g_{22}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^2)^2} = 0
\end{aligned}\right\} \Rightarrow \frac{\partial^2 g_{11}}{(\partial u^1)^2}(0,0) = \frac{\partial^2 g_{22}}{(\partial u^2)^2}(0,0) = 0. \tag{3.11}
$$

Putting $p = 2$ in (3.7) and $p = 1$ in (3.8):

$$
\begin{aligned}
&\left.\begin{aligned}
\frac{\partial^2 g_{12}}{(\partial u^1)^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^1 \partial u^2} = 0 \\
2\frac{\partial^2 g_{11}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^2 \partial u^1} + \frac{\partial^2 g_{21}}{(\partial u^1)^2} - \frac{\partial^2 g_{12}}{(\partial u^1)^2} = 0
\end{aligned}\right\} \\
&\Rightarrow \frac{\partial^2 g_{12}}{(\partial u^1)^2} = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial u^1 \partial u^2} \qquad ; \quad \frac{3}{2}\frac{\partial^2 g_{11}}{\partial u^1 \partial u^2} = 0 \\
&\Rightarrow \frac{\partial^2 g_{11}}{\partial u^1 \partial u^2}(0,0) = \frac{\partial^2 g_{12}}{(\partial u^1)^2}(0,0) = 0.
\end{aligned} \tag{3.12}
$$

With $p = 1$ in (3.10) and $p = 2$ in (3.9):

$$\left.\begin{aligned} \frac{\partial^2 g_{21}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^2 \partial u^1} &= 0 \\ 2\frac{\partial^2 g_{22}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^2} + \frac{\partial^2 g_{12}}{(\partial u^2)^2} - \frac{\partial^2 g_{12}}{(\partial u^2)^2} &= 0 \end{aligned}\right\}$$

$$\Rightarrow \frac{\partial^2 g_{21}}{(\partial u^2)^2} = \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^2} \qquad ; \qquad \frac{3}{2}\frac{\partial^2 g_{22}}{\partial u^1 \partial u^2} = 0$$

$$\Rightarrow \frac{\partial^2 g_{22}}{\partial u^1 \partial u^2}(0,0) = \frac{\partial^2 g_{12}}{(\partial u^2)^2}(0,0) = 0. \tag{3.13}$$

Put $p = 2$ in (3.8) and $p = 1$ in (3.9):

$$\begin{aligned} \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2} + \frac{\partial^2 g_{22}}{(\partial u^1)^2} &= 0 \\ \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} + \frac{\partial^2 g_{11}}{(\partial u^2)^2} &= 0. \end{aligned}$$

Rearranging,

$$\begin{aligned} \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} &= -\frac{3}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} \\ \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2} &= -\frac{3}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2}. \end{aligned} \tag{3.14}$$

Using expression for Gaussian Curvature

$$K = \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2}\frac{\partial^2 g_{11}}{(\partial u^2)^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{(\partial u^1)^2},$$

we obtain

$$\begin{aligned} &\frac{\partial^2 g_{22}}{(\partial u^1)^2}(0,0) = \frac{\partial^2 g_{11}}{(\partial u^2)^2}(0,0) = -\frac{2}{3}K(0,0) \\ &\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2}(0,0) = K(0,0) - \frac{1}{3}K(0,0) - \frac{1}{3}K(0,0) = \frac{1}{3}K(0,0). \end{aligned} \tag{3.15}$$

It then follows that:

$$\begin{aligned} g_{11}(u^1,u^2) &= 1 - \frac{1}{3}K(0,0)(u^2)^2 + o((u^1)^2 + (u^2)^2) \\ g_{12}(u^1,u^2) &= \frac{1}{3}K(0,0)u^1u^2 + o((u^1)^2 + (u^2)^2) \\ g_{22}(u^1,u^2) &= 1 - \frac{1}{3}K(0,0)(u^1)^2 + o((u^1)^2 + (u^2)^2). \end{aligned} \tag{3.16}$$

3.3 Relation Between Geodesic Polar and Conformal Coordinates

Let (α, β) be Riemann Normal coordinates in a vicinity of a point $p \in M$. Introducing the complex parameter $\zeta = \alpha + i\beta = re^{i\theta}$, the (r, θ) are called *geodesic polar coordinates*. In terms of conformal parameter z, the conformal metric takes the form:

$$g = \rho^{-2}(z, \bar{z})|dz|^2, \tag{3.17}$$

whereas in normal coordinates we have:

$$g = E(d\alpha)^2 + 2Fd\alpha\beta + G(d\beta)^2, \tag{3.18}$$

where, (using (3.16))

$$\begin{aligned} E &= 1 - \frac{1}{3}K(0,0)\beta^2 + o(\alpha^2 + \beta^2) \\ F &= \frac{1}{3}K(0,0)\alpha\beta + o(\alpha^2 + \beta^2) \\ G &= 1 - \frac{1}{3}K(0,0)\alpha^2 + o(\alpha^2 + \beta^2). \end{aligned} \tag{3.19}$$

In terms of complex parameter ζ, let the metric in geodesic polar coordinates has the form:

$$g = \lambda(\zeta, \bar{\zeta})|d\zeta + \mu d\bar{\zeta}|^2. \tag{3.20}$$

Let us now express μ and λ in terms of E, F and G:

$$\begin{aligned}\lambda(d\zeta+\mu d\bar{\zeta})(d\bar{\zeta}+\bar{\mu}d\zeta) &= \lambda|d\zeta|^2+\lambda\bar{\mu}d\zeta^2+\lambda\mu d\bar{\zeta}^2+\lambda|\mu|^2|d\zeta|^2\\ &= \lambda(d\alpha^2+d\beta^2)+\lambda\bar{\mu}(d\alpha^2+2id\alpha d\beta-d\beta^2)\\ &+\lambda\mu(d\alpha^2-2id\alpha d\beta-d\beta^2)+\lambda|\mu|^2(d\alpha^2+d\beta^2); \end{aligned} \tag{3.21}$$

comparison with (3.18) gives:

$$\lambda+\lambda\bar{\mu}+\lambda\mu+\lambda|\mu|^2=E \tag{3.22}$$

$$\lambda-\lambda\bar{\mu}-\lambda\mu+\lambda|\mu|^2=G \tag{3.23}$$

$$\lambda\bar{\mu}-\lambda\mu=-iF. \tag{3.24}$$

Subtracting (3.23) from (3.22) and then dividing by 2 we get:

$$\lambda\bar{\mu}+\lambda\mu=\frac{E-G}{2}; \tag{3.25}$$

adding (3.25) to (3.24):

$$2\lambda\bar{\mu}=\frac{E-G}{2}-iF \Longrightarrow \qquad \bar{\mu}=\frac{E-G-2iF}{4\lambda}. \tag{3.26}$$

Putting the expression for μ into (3.22) we have:

$$\begin{aligned}\lambda+\frac{E-G-2iF}{4}+\frac{E-G+2iF}{4}+\frac{(E-G)^2+4F^2}{16\lambda}&=E\\ \Rightarrow \lambda^2-\left(\frac{E+G}{2}\right)\lambda+\frac{(E-G)^2+4F^2}{16}&=0\\ \Rightarrow \lambda=\frac{\left(\frac{E+G}{2}\right)+\sqrt{\frac{(E+G)^2}{4}-\frac{(E-G)^2+4F^2}{4}}}{2}=\frac{E+G+2\sqrt{EG-F^2}}{4}&. \end{aligned} \tag{3.27}$$

To express μ and λ in terms of ζ, we use (3.19), (3.26), and (3.27):

$$\begin{aligned}
\lambda(\zeta,\bar\zeta) &= \frac{1}{4}\Bigg(2 - \frac{1}{3}K(0,0)(\alpha^2+\beta^2) + o(\alpha^2+\beta^2) \\
&\quad + 2\sqrt{1 - \frac{1}{3}K(0,0)(\alpha^2+\beta^2) + \frac{1}{9}K(0,0)\alpha^2\beta^2 - \frac{1}{9}K(0,0)\alpha^2\beta^2 + o(\alpha^2+\beta^2)}\Bigg) \\
&= \frac{1}{4}\left[2 - \frac{1}{3}K(0,0)(\alpha^2+\beta^2) + 2\left(1 - \frac{1}{6}K(0,0)(\alpha^2+\beta^2)\right) + o(\alpha^2+\beta^2)\right] \\
&= \frac{1}{4}\left[4 - \frac{2}{3}K(0,0)(\alpha^2+\beta^2) + o(\alpha^2+\beta^2)\right] \\
&= 1 - \frac{1}{6}K(0,0)(\alpha^2+\beta^2) + o(\alpha^2+\beta^2) \\
&= 1 - \frac{1}{6}K(0,0)\zeta\bar\zeta + o(|\zeta|^2) \qquad (3.28)
\end{aligned}$$

$$\begin{aligned}
\mu(\zeta,\bar\zeta) &= \frac{\frac{1}{4}\left(\frac{1}{3}K(0,0)(\alpha^2-\beta^2+2i\alpha\beta) + o(\alpha^2+\beta^2)\right)}{\left(1 - \frac{1}{6}K(0,0)(\alpha^2+\beta^2) + o(\alpha^2+\beta^2)\right)} \\
&= \frac{1}{4}\left(\frac{1}{3}K(0,0)(\alpha^2-\beta^2+2i\alpha\beta) + o(\alpha^2+\beta^2)\right)\left(1 + o(\alpha^2+\beta^2)\right) \\
&= \frac{1}{12}K(0,0)(\alpha^2-\beta^2+2i\alpha\beta) + o(\alpha^2+\beta^2) \\
&= \frac{1}{12}K(0,0)\zeta^2 + o(|\zeta|^2) \qquad (3.29)
\end{aligned}$$

To find relation between conformal and geodesic polar coordinates, we have to relate equation (3.20) with (3.17); to this end, let us express (3.20) in terms of z and $\bar{z}$:

$$\begin{aligned}
g &= \lambda(z,\bar{z})(d\zeta + \mu d\bar{\zeta})(d\bar{\zeta} + \bar{\mu} d\zeta) \\
&= \lambda(z,\bar{z})\left[\frac{\partial \zeta}{\partial z}dz + \frac{\partial \zeta}{\partial \bar{z}}d\bar{z} + \mu(z,\bar{z})\left(\frac{\partial \bar{\zeta}}{\partial z}dz + \frac{\partial \bar{\zeta}}{\partial \bar{z}}d\bar{z}\right)\right] \\
&\left[\frac{\partial \bar{\zeta}}{\partial z}dz + \frac{\partial \bar{\zeta}}{\partial \bar{z}}d\bar{z} + \bar{\mu}(z,\bar{z})\left(\frac{\partial \zeta}{\partial z}dz + \frac{\partial \zeta}{\partial \bar{z}}d\bar{z}\right)\right] \\
&= \lambda(z,\bar{z})\left(\frac{\partial \zeta}{\partial z} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \bar{\zeta}}{\partial z} + \bar{\mu}(z,\bar{z})\frac{\partial \zeta}{\partial z}\right)(dz)^2 \\
&+ \lambda(z,\bar{z})\left(\frac{\partial \zeta}{\partial z} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \bar{\zeta}}{\partial \bar{z}} + \bar{\mu}(z,\bar{z})\frac{\partial \zeta}{\partial \bar{z}}\right)dz d\bar{z} \\
&+ \lambda(z,\bar{z})\left(\frac{\partial \zeta}{\partial \bar{z}} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)\left(\frac{\partial \bar{\zeta}}{\partial z} + \bar{\mu}(z,\bar{z})\frac{\partial \zeta}{\partial z}\right)d\bar{z} dz \\
&+ \lambda(z,\bar{z})\left(\frac{\partial \zeta}{\partial \bar{z}} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)\left(\frac{\partial \bar{\zeta}}{\partial \bar{z}} + \bar{\mu}(z,\bar{z})\frac{\partial \zeta}{\partial \bar{z}}\right)(d\bar{z})^2 = \rho^{-2}(z,\bar{z})|dz|^2. \qquad (3.30)
\end{aligned}$$

Hence, we obtain:

$$\left(\frac{\partial \zeta}{\partial z} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial z}\right)\left(\frac{\partial \bar{\zeta}}{\partial z} + \bar{\mu}(z,\bar{z})\frac{\partial \zeta}{\partial z}\right) = 0 \qquad (3.31)$$

$$\left|\frac{\partial \zeta}{\partial z} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial z}\right|^2 + \left|\frac{\partial \zeta}{\partial \bar{z}} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right|^2 = \frac{\rho^{-2}}{\lambda}(z,\bar{z}). \qquad (3.32)$$

Proposition 3.3.0.1. *In a vicinity of P, normal coordinates could be chosen in such a way that $\frac{\partial \zeta}{\partial z}$ is real and nonzero and we have the relations:*

$$\frac{\partial \zeta}{\partial \bar{z}} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial \bar{z}} = 0 \qquad (3.33)$$

$$\left|\frac{\partial \zeta}{\partial z} + \mu(z,\bar{z})\frac{\partial \bar{\zeta}}{\partial z}\right|^2 = \frac{\rho^{-2}}{\lambda}(z,\bar{z}). \qquad (3.34)$$

Proof. Since the transformation $(x,y) \mapsto (\alpha(x,y), \beta(x,y))$ is orientation preserving, $\alpha_x\beta_y - \alpha_y\beta_x > 0$. Hence, $\frac{\partial \zeta}{\partial z}\big|_P \neq 0$; otherwise at point P we would have $\alpha_x = -\beta_y$, $\alpha_y = \beta_x$, and thus $\begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} = -\beta_y^2 - \beta_x^2 \leq 0$, a contradition! Now, let us choose our coordinate axes in such a way that the curves $\theta = 0$ and $\Im z = 0$ have the same unit tangent vector at point P. Then we shall have $\alpha_y|_P = \beta_x|_P = 0$; $\alpha_x|_P$ and $\beta_y|_P$ both nonzero and having the same sign. This would then imply $\frac{\partial\zeta}{\partial z}(P)$ is also real and, in a vicinity of P the second factor in (3.31) is zero. □

Based on our choice of coordinate axes in previous proposition, we have:

Theorem 3.3.0.1. *Given a point P' in a vicinity of P, let $z(P) = z$ and $z(P') = z'$. In terms of geodesic polar coordinates, let $\zeta(P) = 0$ and $\zeta(P') = re^{i\theta}$. Then there is the following relation as $P' \to P$:*

$$\begin{aligned}\rho(z,\bar{z})re^{i\theta} &= (z'-z) - (\partial_z \log\rho)(z'-z)^2 + \left[(\partial_z \log\rho)^2 - \frac{1}{3}\rho^{-1}\partial^2_{zz}\rho\right](z'-z)^3 \\ &\quad - \left[\frac{1}{3}\partial^2_{z\bar{z}}\log\rho\right](z'-z)^2\overline{(z'-z)} + O(r^4). \end{aligned}\tag{3.35}$$

Proof. For simplicity, let $z = 0$ represent the coordinate of P and z that of P'. Consider the Taylor expansion

$$\begin{aligned}\zeta(z,\bar{z}) &= 0 + \zeta_{\bar{z}}(0)\bar{z} + \zeta_{z\bar{z}}(0)z\bar{z} + \frac{1}{2}\zeta_{\bar{z}\bar{z}}(0)\bar{z}^2 + \frac{1}{6}\zeta_{\bar{z}\bar{z}\bar{z}}(0)\bar{z}^3 + \frac{1}{2}\zeta_{\bar{z}\bar{z}z}(0)\bar{z}^2 z \\ &\quad + \zeta_z(0)z + \frac{1}{2}\zeta_{zz}(0)z^2 + \frac{1}{2}\zeta_{zz\bar{z}}(0)z^2\bar{z} + \frac{1}{6}\zeta_{zzz}(0)z^3 + O(r^4).\end{aligned}\tag{3.36}$$

We will show that all terms in the first line of (3.36) are zero. Using (3.33), we have $\zeta_{\bar{z}}(0) = -\mu(0)\bar{\zeta}_{\bar{z}}(0)$ and hence, using (3.29), we get $\zeta_{\bar{z}}(0) = 0$. Differentiating (3.33) with respect to z, we get:

$$\zeta_{\bar{z}z} = -\mu_z\bar{\zeta}_{\bar{z}} - \mu\bar{\zeta}_{\bar{z}z}.$$

Using (3.29) and (3.36), we conclude

$$\begin{aligned}\mu(z,\bar{z}) &= \frac{1}{12}K(0,0)\big(\zeta^2_{\bar{z}}(0)\bar{z}^2 + 2\zeta_{\bar{z}}(0)\zeta_z(0)\bar{z}z + \zeta^2_z(0)z^2 + \dots\big) + o\big(|z|^2\big) \\ &\equiv Az^2 + Bz\bar{z} + C\bar{z}^2 + \text{higher order terms}.\end{aligned}\tag{3.37}$$

Hence, $\zeta_{\bar{z}z}(0) = 0$. Differentiating (3.33) with respect to $\bar{z}$, we get:

$$\zeta_{\bar{z}\bar{z}} = -\mu_{\bar{z}}\bar{\zeta}_{\bar{z}} - \mu\bar{\zeta}_{\bar{z}\bar{z}}.$$

Again, using (3.37) $\zeta_{\bar{z}\bar{z}}(0) = 0$. Differentiating further,

$$\zeta_{z\bar{z}\bar{z}} = -\mu_{z\bar{z}}\bar{\zeta}_{\bar{z}} - \mu_{\bar{z}}\bar{\zeta}_{z\bar{z}} - \mu_{\bar{z}}\bar{\zeta}_{\bar{z}z} - \mu\bar{\zeta}_{z\bar{z}\bar{z}}$$
$$\zeta_{\bar{z}\bar{z}\bar{z}} = -\mu_{\bar{z}\bar{z}}\bar{\zeta}_{\bar{z}} - 2\mu_{\bar{z}}\bar{\zeta}_{\bar{z}\bar{z}} - \mu\bar{\zeta}_{\bar{z}\bar{z}\bar{z}}.$$

It follows from differentiation of (3.29):

$$\mu_{z\bar{z}} = \text{const.}(\zeta_z\zeta_{\bar{z}} + \zeta\zeta_{\bar{z}z}) + \text{terms vanishing at } z = 0 \tag{3.38}$$
$$\mu_{\bar{z}\bar{z}} = \text{const.}(\zeta_{\bar{z}\bar{z}}\zeta + \zeta_{\bar{z}}^2) + \text{terms vanishing at } z = 0. \tag{3.39}$$

Hence,

$$\mu_{z\bar{z}}(0) = \mu_{\bar{z}\bar{z}}(0) = 0 \quad \Rightarrow \quad \zeta_{z\bar{z}\bar{z}}(0) = \zeta_{\bar{z}\bar{z}\bar{z}}(0) = 0$$

Next, we will show that the terms in the second line of (3.36) are nonzero. Plugging $z = 0$ into (3.34), along with the fact that $\lambda(0) = 1$ (using (3.28)), we get:

$$\zeta_z^2(0) = \rho^{-2}(0) \quad \Rightarrow \quad \zeta_z(0) = \rho^{-1}(0). \tag{3.40}$$

Differentiating (3.34) with respect to z, we get:

$$\begin{aligned}&(\zeta_{zz} + \mu_z\bar{\zeta}_z + \mu\bar{\zeta}_{zz})(\bar{\zeta}_{\bar{z}} + \bar{\mu}\zeta_{\bar{z}}) + (\zeta_z + \mu\bar{\zeta}_z)(\bar{\zeta}_{\bar{z}z} + \bar{\mu}_z\zeta_{\bar{z}} + \bar{\mu}\zeta_{z\bar{z}})\\ &= \frac{-2\rho^{-3}\rho_z\lambda - \lambda_z\rho^{-2}}{\lambda^2}.\end{aligned} \tag{3.41}$$

Differentiating (3.28) we get

$$\lambda_z = \text{const.}(\zeta_z\bar{\zeta} + \zeta\bar{\zeta}_z) + \text{terms vanishing at } z = 0. \tag{3.42}$$

Using

$$\begin{aligned}&\bar{\zeta}_z(0) = \zeta_{\bar{z}}(0) = 0 \qquad \bar{\zeta}_{\bar{z}}(0) = \zeta_z(0) = \rho^{-1}(0) \qquad \bar{\mu}(0) = \mu(0) = 0\\ &\bar{\zeta}_{\bar{z}z}(0) = \zeta_{\bar{z}z}(0) = 0,\end{aligned} \tag{3.43}$$

we have,

$$\zeta_{zz}(0) = -2\rho^{-2}(0)\rho_z(0). \tag{3.44}$$

Applying ∂_z to (3.41), we get:

$$(\zeta_{zzz} + \mu_{zz}\bar{\zeta}_z + 2\mu_z\bar{\zeta}_{zz} + \mu\bar{\zeta}_{zzz})(\bar{\zeta}_{\bar{z}} + \bar{\mu}\zeta_{\bar{z}}) + (\bar{\zeta}_{\bar{z}z} + \bar{\mu}_z\zeta_{\bar{z}} + \bar{\mu}\zeta_{\bar{z}z})$$
$$(\zeta_{zz} + \mu_z\bar{\zeta}_z + \mu\bar{\zeta}_{zz}) + (\zeta_{zz} + \mu_z\bar{\zeta}_z + \mu\bar{\zeta}_{zz})(\bar{\zeta}_{\bar{z}z} + \bar{\mu}_z\zeta_{\bar{z}} + \bar{\mu}\zeta_{z\bar{z}}) + (\zeta_z + \mu\bar{\zeta}_z)$$
$$(\bar{\zeta}_{\bar{z}zz} + \bar{\mu}_{zz}\zeta_{\bar{z}} + 2\bar{\mu}_z\zeta_{z\bar{z}} + \bar{\mu}\zeta_{zz\bar{z}}) = \frac{-\lambda_{zz}\rho^{-2} + 2\lambda_z\rho^{-3}\rho_z}{\lambda^2} + \frac{2\lambda\lambda_z^2\rho^{-2}}{\lambda^4} +$$
$$2\lambda^{-2}\lambda_z\rho^{-3}\rho_z + \left(\frac{2}{\lambda}\right)(3\rho^{-4}\rho_z^2 - \rho^{-3}\rho_{zz}). \tag{3.45}$$

Applying ∂_{zz} to (3.28), we get:

$$\lambda_{zz} = \text{const.}\big(\zeta_{zz}\bar{\zeta} + 2\zeta_z\bar{\zeta}_z + \zeta\bar{\zeta}_{zz}\big) + \text{ terms vanishing at } z = 0$$
$$\Rightarrow \quad \lambda_{zz}(0) = 0.$$

Using (3.43) along with

$$\bar{\zeta}_{zz}(0) = \zeta_{\bar{z}\bar{z}}(0) = 0 \qquad \bar{\zeta}_{\bar{z}zz}(0) = \zeta_{z\bar{z}\bar{z}}(0) = 0, \tag{3.46}$$

plugging $z = 0$ into (3.45), one obtains:

$$\zeta_{zzz}(0)\rho^{-1}(0) = 2(3\rho^{-4}(0)\rho_z^2(0) - \rho^{-3}(0)\rho_{zz}(0))$$
$$\Longrightarrow \zeta_{zzz}(0) = 6\rho^{-3}(0)\rho_z^2(0) - 2\rho^{-2}(0)\rho_{zz}(0). \tag{3.47}$$

Applying ∂_{zz} to (3.33) we get:

$$\zeta_{\bar{z}zz} = -\mu_{zz}\bar{\zeta}_{\bar{z}} - 2\mu_z\bar{\zeta}_{\bar{z}z} - \mu\bar{\zeta}_{\bar{z}zz}$$
$$\Longrightarrow \zeta_{\bar{z}zz}(0) = -\mu_{zz}(0)\bar{\zeta}_{\bar{z}}(0) = -\mu_{zz}(0)\rho^{-1}(0). \tag{3.48}$$

Applying ∂_{zz} to 3.29, we get:

$$\mu_{zz} = \frac{1}{6}K(0,0)(\zeta_z^2 + \zeta\zeta_{zz}) + \text{terms vanishing at } z = 0. \tag{3.49}$$

Using expression for curvature

$$K = 4\rho^2 \partial^2_{z\bar{z}} \log \rho, \tag{3.50}$$

we have,

$$\zeta_{\bar{z}zz}(0) = (\frac{-1}{6}) 4\rho(0)^2\, \partial^2_{z\bar{z}} \log \rho\big|_{(0,0)}\, \rho(0)^{-2}\rho(0)^{-1} = \frac{-2}{3}\rho(0)^{-1}\, \partial_{z\bar{z}} \log \rho|_{(0,0)}\,. \tag{3.51}$$

It then follows that

$$\begin{aligned}
&\rho(0)\zeta_z(0) = 1\\
&\frac{1}{2}\rho(0)\zeta_{zz}(0) = -\, \partial_z \log \rho|_{(0,0)}\\
&\frac{1}{6}\rho(0)\zeta_{zzz}(0) = \rho(0)^{-2}\rho_z(0)^2 - \frac{1}{3}\rho(0)^{-1}\rho_{zz}(0) = \big(\, \partial_z \log \rho|_{(0,0)}\,\big)^2 - \frac{1}{3}\rho(0)^{-1}\, \partial_{zz}\rho|_{(0,0)}\\
&\frac{1}{2}\rho(0)\zeta_{\bar{z}zz}(0) = -\frac{1}{3}\, \partial_{z\bar{z}} \log \rho|_{(0,0)}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\rho(0)\zeta(z,\bar{z}) = z - \partial_z \log \rho|_{(0,0)}\, z^2 + \left[\big(\, \partial_z \log \rho|_{(0,0)}\,\big)^2 - \frac{1}{3}\rho(0)^{-1}\, \partial_{zz}\rho|_{(0,0)}\right] z^3\\
- \left(\frac{1}{3}\, \partial_{z\bar{z}} \log \rho|_{(0,0)}\right) z^2\bar{z} + O\big(|z|^4\big).
\end{aligned}$$

□

Appendices

Appendix A

Two Proofs for Existence of Complex ODE

Theorem A.0.0.1. *Let $f : \mathbb{C}\times\mathbb{C} \supset \mathcal{U} \to \mathbb{C}$ be holomorphic. Then, in a neighbourhood $\mathcal{W}$ of any point z_0, there exists a holomorphic function $\phi : \mathcal{W} \to \mathbb{C}$ such that*

$$\phi'(z) = f(z, \phi(z)) \quad \phi(z_0) = w_0. \tag{A.1}$$

Proof. There are two ways to prove this theorem: Frobenius method and fixed point iterations; although the latter is simpler. We will present both cases here:

- Frobenius method:
  We need the following Lemma:

  Lemma A.0.0.1. *Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ (open) ; $s_0 \in U$; $f_j \in C^\infty(U \times V; \mathbb{R}^n)$; $j = 1, \ldots, m$. Then, there exists a unique solution $\alpha : W \subset \mathbb{R}^m \to \mathbb{R}^n$, defined in a neighbourhood W of s_0, to the initial value problem:*

  $$\frac{\partial \alpha}{\partial t^j} = f_j(t_1, \ldots, t_m, \alpha(t)) \quad \alpha(s_0) = x, \tag{A.2}$$

  if and only if in a neighbourhood of $(s_0, x) \in U \times V$ the following condition is *satisfied:*

$$\frac{\partial f_j}{\partial t^i} - \frac{\partial f_i}{\partial t^j} + \sum_{h=1}^{n} \frac{\partial f_j}{\partial x^h} f_i^h - \sum_{h=1}^{n} \frac{\partial f_i}{\partial x^h} f_j^h = 0 \quad i, j = 1, \ldots, m. \tag{A.3}$$

Proof. $\Longrightarrow$: Consider the distribution Δ of m-planes in $\mathbb{R}^{m+n}$ as follows: Let $v_j = \left(\frac{\partial}{\partial t^j} + \sum_{k=1}^{n} f_j^k \frac{\partial}{\partial x^k}\right)^1$. Then $\Delta_p =$ l.h. at point p of $\{v_j\}_{j=1}^{m}$. We claim this this distribution is integrable if condition (A.3) is satisfied. Indeed by simple computation (using Einstein notation):

$$\begin{aligned}
[v_i, v_j] &= \left(\frac{\partial}{\partial t^i} + f_i^k \frac{\partial}{\partial x^k}\right)\left(\frac{\partial}{\partial t^j} + f_j^h \frac{\partial}{\partial x^h}\right) - \left(\frac{\partial}{\partial t^j} + f_j^h \frac{\partial}{\partial x^h}\right)\left(\frac{\partial}{\partial t^i} + f_i^k \frac{\partial}{\partial x^k}\right) \\
&= \frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j} + \frac{\partial f_j^h}{\partial t^i}\frac{\partial}{\partial x^h} + f_j^h \frac{\partial}{\partial t^i}\frac{\partial}{\partial x^h} + f_i^k \frac{\partial}{\partial x^k}\frac{\partial}{\partial t^j} + f_i^k \frac{\partial f_j^k}{\partial x^k}\frac{\partial}{\partial x^h} \\
&+ f_i^k f_j^h \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^h} - \frac{\partial}{\partial t^j}\frac{\partial}{\partial t^i} - \frac{\partial f_i^k}{\partial t^j}\frac{\partial}{\partial x^k} - f_i^k \frac{\partial}{\partial t^j}\frac{\partial}{\partial x^k} - f_j^h \frac{\partial}{\partial x^h}\frac{\partial}{\partial t^i} \\
&- f_j^h \frac{\partial f_i^k}{\partial x^h}\frac{\partial}{\partial x^k} - f_j^h f_i^k \frac{\partial}{\partial x^h}\frac{\partial}{\partial x^k} = \frac{\partial f_j^h}{\partial t^i}\frac{\partial}{\partial x^h} - \frac{\partial f_i^k}{\partial t^j}\frac{\partial}{\partial x^k} + f_i^k \frac{\partial f_j^h}{\partial x^k}\frac{\partial}{\partial x^h} \\
&- f_j^h \frac{\partial f_i^k}{\partial x^h}\frac{\partial}{\partial x^k} = \left(\frac{\partial f_j^k}{\partial t^i} - \frac{\partial f_i^k}{\partial t^j}\right)\frac{\partial}{\partial x^k} + \left(f_i^h \frac{\partial f_j^k}{\partial x^h} - f_j^h \frac{\partial f_i^k}{\partial x^h}\right)\frac{\partial}{\partial x^k} \\
&\equiv \sum_{k=1}^{n}\left(\frac{\partial f_j^k}{\partial t^i} - \frac{\partial f_i^k}{\partial t^j} + \sum_{h=1}^{n} f_i^h \frac{\partial f_j^k}{\partial x^h} - \sum_{h=1}^{n} f_j^h \frac{\partial f_i^k}{\partial x^h}\right)\frac{\partial}{\partial x^k}.
\end{aligned}$$

According to Frobenius Criterion (see Appendix B) we must have $[v_i, v_j] = \sum_\alpha C_{ij}^\alpha v_\alpha$, which will then be satisfied.
$\Longleftarrow$: If α exists then, $\left[\frac{d}{dt^i}, \frac{d}{dt^j}\right]\alpha = 0$; hence, $\frac{df_j}{dt^i} - \frac{df_i}{dt^j} = 0$. Now,

$$\begin{aligned}
\frac{df_j}{dt^i} &= \frac{\partial f_j}{\partial t^i} + \sum_{k=1}^{n} \frac{\partial f_j}{\partial x^k}\frac{\partial \alpha^k(t)}{\partial t^i} = \frac{\partial f_j}{\partial t^i} + \sum_{k=1}^{n} \frac{\partial f_j}{\partial x^k} f_i^k \\
\frac{df_i}{dt^j} &= \frac{\partial f_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k}\frac{\partial \alpha^k(t)}{\partial t^j} = \frac{\partial f_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k} f_j^k;
\end{aligned}$$

[1] If $\psi : \mathbb{R}^m \to \mathbb{R}^{m+n}$ is defined by $\psi(t) = (t, A(t))$, then $d\psi\left(\frac{\partial}{\partial t^i}\big|_x\right) = \frac{\partial}{\partial t^i}\big|_{\psi(x)} + \sum_{k=1}^{n} \frac{\partial A^k}{\partial t^i}\frac{\partial}{\partial x^k}\big|_{\psi(x)}$.

so we arrive at (A.3). To show uniqueness, suppose $\alpha(t)$ and $\tilde{\alpha}(t)$ satisfy (A.2). Then,

$$\frac{\partial}{\partial t^j}\big(\alpha(t)-\tilde{\alpha}(t)\big)=0 \Rightarrow \quad \alpha(t)-\tilde{\alpha}(t)=\text{Const.}\in\mathbb{R}^n$$
$$\alpha(s_0)-\tilde{\alpha}(s_0)=x-x=0 \Rightarrow \quad \text{Const.}=0 \Rightarrow \quad \alpha(t)=\tilde{\alpha}(t).$$

$\square$

Back to the proof of the theorem, denote $z_1=x+iy, z_2=x'+iy'$; let us write $f(z_1,z_2)$ as $f(x,y,x',y')=u(x,y,x',y')+iv(x,y,x',y')$. We are looking for a holomorphic function $\phi(z)=\phi^1(x,y)+i\phi^2(x,y)$ such that $\phi'(z)=\phi^1_x(x,y)+i\phi^2_x(x,y)=u(x,y,\phi^1(x,y),\phi^2(x,y))+iv(x,y,\phi^1(x,y),\phi^2(x,y))$. This gives rise to the following sets of equations:

$$\begin{aligned}\frac{\partial\phi^1}{\partial x} &= u(x,y,\phi^1(x,y),\phi^2(x,y)) &= \frac{\partial\phi^2}{\partial y}\\ -\frac{\partial\phi^1}{\partial y} &= v(x,y,\phi^1(x,y),\phi^2(x,y)) &= \frac{\partial\phi^2}{\partial x}\end{aligned}$$

By simple computation using $n=m=2$ in Lemma A.0.0.1; $x:=t^1, y:=t^2$; $f_1:=\begin{pmatrix}u(x,y,\phi^1(x,y),\phi^2(x,y))\\ v(x,y,\phi^1(x,y),\phi^2(x,y))\end{pmatrix}$, $f_2:=\begin{pmatrix}-v(x,y,\phi^1(x,y),\phi^2(x,y))\\ u(x,y,\phi^1(x,y),\phi^2(x,y))\end{pmatrix}$ we see this set satisfies condition (A.3):

$$\begin{aligned} i=2, j=1: \quad & \frac{\partial f_1}{\partial y}-\frac{\partial f_2}{\partial x}+\frac{\partial f_1}{\partial x'}f_2^{(1)}+\frac{\partial f_1}{\partial y'}f_2^{(2)}-\frac{\partial f_2}{\partial x'}f_1^{(1)}-\frac{\partial f_2}{\partial y'}f_1^{(2)}= \\ & \begin{pmatrix}u_y\\ v_y\end{pmatrix}-\begin{pmatrix}-v_x\\ u_x\end{pmatrix}+\begin{pmatrix}u_{x'}(-v)\\ v_{x'}(-v)\end{pmatrix}+\begin{pmatrix}u_{y'}u\\ v_{y'}u\end{pmatrix}-\begin{pmatrix}-v_{x'}u\\ u_{x'}u\end{pmatrix} \\ & -\begin{pmatrix}-v_{y'}v\\ u_{y'}v\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

$$i = 1, j = 2: \quad \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x'} f_1^{(1)} + \frac{\partial f_2}{\partial y'} f_1^{(2)} - \frac{\partial f_1}{\partial x'} f_2^{(1)} - \frac{\partial f_1}{\partial y'} f_2^{(2)} =$$

$$\begin{pmatrix} -v_x \\ u_x \end{pmatrix} - \begin{pmatrix} u_y \\ v_y \end{pmatrix} + \begin{pmatrix} -v_{x'} u \\ u_{x'} u \end{pmatrix} + \begin{pmatrix} -v_{y'} v \\ u_{y'} v \end{pmatrix} - \begin{pmatrix} u_{x'}(-v) \\ v_{x'}(-v) \end{pmatrix}$$

$$- \begin{pmatrix} u_{y'} u \\ v_{y'} u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This completes the proof using Frobenius method.

- Fixed point method: For this method first we have to identify our Banach space, on which to perform contraction mapping. Consider the polydisc $D_\rho = \{\, z_i \mid |z_i - a_i| < \rho, i = 1, 2 \,\}$ and let $\mathcal{A}(D_\rho) = \mathcal{H}(D_\rho) \cap C(\bar{D}_\rho)$, equipped with the Sup norm: $\|f\|_\rho = \sup\limits_{z \in D_\rho} |f(z)|$.

Lemma A.0.0.2. *$\mathcal{A}(D_\rho)$ is a Banach space.*

Proof. Any $g \in \mathcal{A}(D_\rho)$ has the integral representation:

$$g(z_1, z_2) = \frac{-1}{4\pi^2} \int_{|\zeta_1 - a_1| = \rho} d\zeta_1 \int_{|\zeta_2 - a_2| = \rho} d\zeta_2 \frac{g(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}. \tag{A.4}$$

Let $\{g_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{A}(D_\rho)$. Then this sequence converges uniformly in $\bar{D}_\rho$ to $g^* \in C(\bar{D}_\rho)$. Hence, with the integral representation one can pass the limit inside the integral, so that g^* will have the integral representation (A.4). The right hand side is obviously holomorphic with respect to z_1, z_2 and therefore $g^* \in \mathcal{A}(D_\rho)$. $\square$

Now consider D_ϵ centered at (z_0, w_0); let $H \Subset \mathcal{U}$, $(z_0, w_0) \in H$, $L_1 = \sup\limits_{\substack{(\xi, z_1) \in H \\ (\xi, z_2) \in H \\ z_1 \neq z_2}} \frac{|f(\xi, z_1) - f(\xi, z_2)|}{|z_1 - z_2|}$; and define $M = \{\, \psi(z, v) \in \mathcal{A}(D_\epsilon) \mid \forall (z, v) \in D_\epsilon, |\psi(z, v) - v| \leq L_0 \epsilon \,\}$, where $L_0 = \sup\limits_{(z, v) \in H} |f(z, v)|$. ($M$ is a closed subset of $\mathcal{A}(D_\epsilon)$ and hence complete.)

Proposition A.0.0.1. *For ϵ sufficiently small, 1)$M \neq \emptyset$; 2) The Picard map $\psi(z,v) \mapsto [\mathcal{P}\psi](z,v) = v + \int_{z_0}^{z} f(\xi, \psi(\xi, v))d\xi$: i) is well defined for $\psi \in M$ ii) $\mathcal{P}(M) \subset M$ iii) $\mathcal{P} : M \to M$ is contracting.*

Proof. There exists r_1, r_2 such that $D(z_0, r_1) \times D(w_0, r_2) \subset H$. Choose ϵ such that $\epsilon < \min\{\ r_1, r_2, \frac{1}{L_1}\}$ and for all $v \in D(w_0, \epsilon)$, $\overline{D(v, L_0\epsilon)} \subset D(w_0, r_2)$. 1)Obvious! (pick $\psi(z,v) = v$.) 2) i) Follows easily from the bound on ϵ. ii) This follows from the estimate:

$$|[\mathcal{P}\psi](z,v) - v| = \left| \int_{z_0}^{z} f(\xi, \psi(\xi, v))d\xi \right| \leq L_0|z - z_0| < L_0\epsilon.$$

iii) Again,

$$\begin{aligned} |[\mathcal{P}\psi_1](z,v) - [\mathcal{P}\psi_2](z,v)| &= \left| \int_{z_0}^{z} f(\xi, \psi_1(\xi, v)) - f(\xi, \psi_2(\xi, v))d\xi \right| \\ &\leq |z - z_0| L_1 \sup_{(\xi,v)\in D_\epsilon} |\psi_1(\xi, v) - \psi_2(\xi, v)| \\ &< \epsilon L_1 \left\| \psi_1 - \psi_2 \right\|_\epsilon . \end{aligned}$$

□

Then, the Picard map has a fixed point: $\psi(z,v) = v + \int_{z_0}^{z} f(\xi, \psi(\xi, v))d\xi$. For fixed w_0, $\phi(z) = \psi(z, w_0)$ is the solution to the initial value problem (A.1), which depends holomorphically on w_0. This completes Picard proof.

□

Appendix B

Proof of Frobenius Theorem

Theorem B.0.0.1 (Frobenius)**.** *Let $v_1, v_2, \ldots, v_r$ be C^∞ vector fields, in a vicinity of 0, in $\mathbb{R}^n$. There exists a coordinate chart $(U; y^k)$ in a neighbourhood of 0 such that $\frac{\partial}{\partial y^i} = \sum_{j=1}^{r} a_{ij} v_j$, i=1,$\ldots$,r (a_{ij} an invertible matrix), if and only if $v_1(0), v_2(0), \ldots, v_r(0)$ are linearly independent and $[v_i, v_j] = \sum_\alpha c_{ij}^\alpha v_\alpha$, $i, j = 1, \ldots, r$.*

Proof. $\Rightarrow$: We proceed by induction on n. Case $n = 1$ is obvious: the l.i. set contains one vector field v; simply set $\frac{\partial}{\partial y^1} = v$. Now assume the statement holds for dimensions less than n. Choose a coordinate system y^i such that $v_1 = \frac{\partial}{\partial y^1}$ (see [11]), and after subtracting by a multiple of $\frac{\partial}{\partial y^1}$, we may assume $v_i = \sum_{j=2}^{n} a_{ij} \frac{\partial}{\partial y^j}$, $\quad i = 2, \ldots, r$. When $y^1 = 0$, v_i's lie in $\mathbb{R}^{n-1}$. By induction hypothesis, let's choose our coordinates $y^i (i = 2, \ldots, n)$ such that $a_{ij} = 0$ for $i = 2, \ldots, r$ and $j > r$, when $y^1 = 0$. Now,

$$\frac{\partial a_{il}}{\partial y^1} = v_1 v_i y^l = \big[v_1, v_i\big] y^l = \sum_{\alpha=2}^{r} C_{1i}^{\alpha} v_\alpha y^l = \sum_{\alpha=2}^{r} C_{1i}^{\alpha} a_{\alpha l} \qquad i = 2, \ldots, r \quad l = 2, \ldots, n.$$

For a fixed $l > r$, applying the ODE theorem with initial conditions, this system has a unique solution $a_{il} = 0$ in a neighbourhood of 0.

$\Leftarrow$:

$$\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0 \Rightarrow \left[\sum_k a_{ik} v_k, \sum_l a_{jl} v_l\right] = 0 \Rightarrow \sum_k \sum_l \left[a_{ik} v_k, a_{jl} v_l\right] = 0$$

$$\Rightarrow \sum_k \sum_l a_{ik}(v_k a_{jl}) v_l + a_{ik} a_{jl} v_k v_l - a_{jl}(v_l a_{ik}) v_k - a_{jl} a_{ik} v_l v_k$$

$$= \sum_k \sum_l a_{ik} a_{jl} \left[v_k, v_l\right] + \sum_k \sum_l \big(a_{il}(v_l a_{jk}) - a_{jl}(v_l a_{ik})\big) v_k = 0$$

$$\Rightarrow \left[v_k, v_l\right] = \sum_\alpha C^\alpha_{kl} v_\alpha.$$

To check for linear independence of v_i's in a neighbourhood of 0, suppose $\sum_i b_i v_i = 0$.

Since $v_i = \sum_k \tilde{a}_{ik} \frac{\partial}{\partial y^k}$, we have $\tilde{\mathbf{a}}^T \mathbf{b} = 0$. Now, $\det \tilde{a}_{ik} \neq 0 \Rightarrow \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix} = 0.$ $\square$

Appendix C

Formulation of Statements of Taylor and Chern

In this chapter, we formulate the theorems of Taylor and Chern, and discuss preliminary lemmas that lead to their proofs.

C.1 Taylor Formulation

Theorem C.1.0.1 (Taylor). *Let M be a 2-D Riemannian manifold with metric $g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$. Suppose E, F, and G are C^∞ functions of their arguments. Then in a vicinity of a point $p \in M$, there exist C^∞ isothermal coordinates $u(x,y)$ and $v(x,y)$, in which the metric takes the form $g = \lambda(u,v)\big(du^2 + dv^2\big)$, for some $\lambda \in C^\infty$.*

The following lemmas are needed to construct the proof.

Lemma C.1.0.1. *On a 2-D Riemannian Manifold with metric $g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$, in a vicinity of a point p, suppose there exists local coordinate transformation $(x,y) \to (u(x,y), v(x,y))$ such that $\lambda(u,v)(du^2 + dv^2) =$*

$E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$. Then, the following equations hold:

$$\begin{aligned} u_x &= \frac{1}{\sqrt{EG-F^2}}(Ev_y - Fv_x) \\ u_y &= \frac{1}{\sqrt{EG-F^2}}(Fv_y - Gv_x) \end{aligned} \tag{C.1}$$

$$\begin{aligned} v_x &= \frac{1}{\sqrt{EG-F^2}}(Fu_x - Eu_y) \\ v_y &= \frac{1}{\sqrt{EG-F^2}}(Gu_x - Fu_y) \end{aligned} \tag{C.2}$$

Proof. With the law of transformation of g between local coordinates $\{x^i_{(\alpha)}\}$ and $\{x^{i'}_{(\beta)}\}$: $g^{(\beta)}_{i'j'} = g^{(\alpha)}_{ij} \frac{\partial x^i_{(\alpha)}}{\partial x^{i'}_{(\beta)}} \frac{\partial x^j_{(\alpha)}}{\partial x^{j'}_{(\beta)}}$, we have

$$A^T \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} A = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where $A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. Hence,

$$\begin{aligned} &\lambda^2(\det A)^2 = EG - F^2 \Longrightarrow \det A = \frac{\sqrt{EG-F^2}}{\lambda}; \\ &A^T = \begin{pmatrix} E & F \\ F & G \end{pmatrix} A^{-1} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} = \frac{1}{\sqrt{EG-F^2}} \begin{pmatrix} Ev_y - Fv_x & Fu_x - Eu_y \\ Fv_y - Gv_x & Gu_x - Fu_y \end{pmatrix} \end{aligned}$$

and we have our sets of equations. Note A is the Jacobian matrix of transformation whose determinant has to be positive (to preserve orientation). □

Denote $x := u^1$, $y := u^2$; and $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$; and $\det g = EG - F^2$. Introducing the operator Δ:

$$\Delta\phi = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial u^j} \left(\sqrt{\det g} g^{jk} \frac{\partial \phi}{\partial u^k} \right).$$

Lemma C.1.0.2. *Suppose u and v satisfy equations (C.1) and (C.2); and E, G, F $\in C^2$. Then, $\Delta u = \Delta v = 0$.*

Proof.

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0 \Rightarrow \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\det g}}\big(Fv_y - Gv_x\big)\right] - \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{\det g}}\big(Ev_y - Fv_x\big)\right] = 0;$$

$$g^{jk} = \frac{1}{\det g}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \Rightarrow \Delta v = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x}\left[\frac{-1}{\sqrt{\det g}}\big(Fv_y - Gv_x\big)\right] + \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial y}$$
$$\left[\frac{1}{\sqrt{\det g}}\big(Ev_y - Fv_x\big)\right].$$

So we have $\big(-\sqrt{\det g}\big)\Delta v = 0 \Rightarrow \Delta v = 0$. Similar calculation shows $\Delta u = 0$. □

The next lemma is to ensure the positivity of Jacobian:

Lemma C.1.0.3. *Suppose u and v satisfy (C.1) and (C.2). The Jacobian of A will be strictly positive at a point (x_0, y_0) if and only if $\nabla\, v|_{(x_0,y_0)} = (v_x(x_0, y_0), v_y(x_0, y_0)) \neq \vec{0}$.*

Proof. Denote $(x, y) \coloneqq (x^1, x^2)$. Then,

$$\begin{aligned}
\det A = u_x v_y - u_y v_x &= \frac{1}{\sqrt{EG - F^2}}\big(Ev_y - Fv_x\big)v_y - \frac{1}{\sqrt{EG - F^2}}\big(Fv_y - Gv_x\big)v_x \\
&= \frac{1}{\sqrt{EG - F^2}}\big(Gv_x^2 - 2Fv_x v_y + Ev_y^2\big) \\
&= \sqrt{EG - F^2}\, g^{jk} v_{x^j} v_{x^k} > 0 \text{ if and only if } \nabla v \neq 0.
\end{aligned}$$

□

From these lemmas, we see to construct isothermal coordinates in a vicinity of p one has to find a function v such that

$$\Delta v = 0 \text{ and } \nabla v \neq \vec{0} \text{ in a vicinity of p.} \tag{C.3}$$

Once v is constructed, u can also be determined using equations (C.1). The existence of v that satisfies (C.3) is discussed in Taylor.

C.2 Chern Formulation

Definition C.2.0.1. *A function $f : \mathbb{R}^2 \supset D \to \mathbb{R}$ is Hölder continuous of order γ ($0 < \gamma < 1$) if there exists a constant K such that $|f(x_1) - f(x_2)| < K|x_1 - x_2|^\gamma$ $\forall x_1, x_2 \in D$. Such a function is called a C^γ function. If, in addition, all its n^{th} derivatives are C^γ, it is called a $C^{n+\gamma}$ function.*

Theorem C.2.0.1 (Chern)**.** *Let M be a 2-D Riemannian manifold with the metric $g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$, whose coefficients are C^γ. Then, in a vicinity of a point p, there exist $C^{1+\gamma}$ isothermal coordinates $u(x,y)$ and $v(x,y)$ in which the metric takes the form $g = \lambda(u,v)\big(du^2 + dv^2\big)$, for some $\lambda \in C^\gamma$.*

To carry out the proof, let us denote $w = u(x,y) + iv(x,y)$. Then, using $\partial_z = \frac{1}{2}\big(\partial_x - i\partial_y\big)$ and $\partial_{\bar{z}} = \frac{1}{2}\big(\partial_x + i\partial_y\big)$, we have:

$$\begin{aligned} w_z &= \frac{u_x + v_y}{2} + \frac{i(v_x - u_y)}{2} \\ w_{\bar{z}} &= \frac{u_x - v_y}{2} + \frac{i(v_x + u_y)}{2}. \end{aligned} \tag{C.4}$$

Using (C.1) (to make substitution for u) we get:

$$\begin{aligned} 2w_{\bar{z}}\sqrt{EG - F^2} &= \big[-F + i\big(\sqrt{EG - F^2} - G\big)\big]v_x + \big(E - \sqrt{EG - F^2} + iF\big)v_y; \\ 2w_z\sqrt{EG - F^2} &= \big[-F + i\big(\sqrt{EG - F^2} + G\big)\big]v_x + \big(E + \sqrt{EG - F^2} - iF\big)v_y. \end{aligned}$$

Calculation then shows

$$\frac{w_{\bar{z}}}{w_z} = \sigma, \text{ where } \sigma = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}; \tag{C.5}$$

and (using (C.4) and (C.5)) the Jacobian is

$$\det A = u_x v_y - u_y v_x = |w_z|^2 - |w_{\bar{z}}|^2 = |w_z|^2\big(1 - |\sigma|^2\big). \tag{C.6}$$

Now, since

$$\begin{aligned}
|\sigma|^2 &= \frac{(E-G)^2+4F^2}{E^2+G^2+6EG+4(E+G)\sqrt{EG-F^2}-4F^2} \\
&= \frac{(E-G)^2+4F^2}{(E-G)^2+4F^2+8EG-8F^2+4(E+G)\sqrt{EG-F^2}} \\
&= \frac{\mu}{\mu+\nu} < 1,
\end{aligned}$$

where $\mu = (E-G)^2+4F^2,\ \nu = 8(EG-F^2)+4(E+G)\sqrt{EG-F^2}$; the Jacobian is strictly positive at a point p if $w_z(p) \neq 0$.

To prove the existence of isothermal coordinates, one therefore has to prove the existence of a function w that satisfies (C.5) (with $|\sigma| < 1$) such that $w_z \neq 0$ in a vicinity of p. Then we shall have $g = \lambda(w,\bar{w})dwd\bar{w} = \lambda(u,v)(du^2+dv^2)$, where

$$\lambda = \frac{\sqrt{EG-F^2}}{|w_z|^2\big(1-|\sigma|^2\big)}.$$

This proof is explained at length in [4].

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