Arithmetic aspects of GSp_{2g}

p-adic families of Siegel modular forms, eigenvarieties, and families of Galois representations

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ABSTRACT

Arithmetic aspects of GSp_{2g}: p-adic families of Siegel modular forms, eigenvarieties and families of Galois representations

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This thesis reports the three articles [Wu21; DRW22; Wu22] written by the author and his collaborators. These three papers concern various arithmetic aspects of the algebraic group GSp_{2g}, which are interrelated under the theme of eigenvarieties.

We first present a construction of sheaves of overconvergnet Siegel modular forms by using the perfectoid method, originally introduced by Chojecki–Hansen–Johansson for automorphic forms on compact Shimura curves over $\mathbf{Q}$. These sheaves are then proven to be isomorphic to the ones constructed by Andreatta–Iovita–Pilloni. Using perfectoid methods, we establish an overconvergent Eichler–Shimura morphism for Siegel modular forms, generalising the result of Andreatta–Iovita–Stevens for elliptic modular forms. More precisely, we establish a Hecke- and Galois-equivariant morphism from the overconvergent cohomology groups associated with GSp_{2g} to the space of overconvergent Siegel modular forms.

It was asked by Andreatta–Iovita–Pilloni whether the classical points of the eigenvariety parametrising the finite-slope cuspidal Siegel eigenforms are étale over the weight space. Inspired by Kim's pairing presented in the book of Bellaïche, which allows one to study the ramification locus of the eigencurve, we generalise Kim's pairing to study the ramification locus of the cuspidal eigenvariety for GSp_{2g}, providing some partial answer to the question asked by Andreatta–Iovita–Pilloni.

Finally, it is expected that such a pairing not only allows one to study the geometry of the eigenvariety but also carries interesting arithmetic information. Inspired by the book of Bellaïche–Chenevier, we study families of Galois representations over the cuspidal eigenvariety for GSp_{2g}. Under some reasonable hypotheses as well as some conditions, we deduce the vanishing of the adjoint Selmer group associated with the Galois representation attached to a cuspidal eigenclass in the cohomology of the Siegel modular variety.

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To Meng-Shan

'Let me stand up like a Taiwanese!'
– Wen-Shiung Huang （黃文雄）

Contribution of authors

I, Ju-Feng Wu, hereby confirm that the results presenting in this thesis are due to myself or the joint work with my collaborators. More precisely, the contents in Chapter 2, Chapter 3, Chapter 6 and Appendix A are taken and slightly modified from the collaboration with Hansheng Diao and Giovanni Rosso ([DRW22]). Remark that [§A.2, *op. cit.*] is not included in this thesis since I did not contribute to that part. Where information has been derived from other sources, I confirm that this has been indicated in this thesis.

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Chapter 1

Introduction

1.1 An overview on eigenvarieties

1.1.1. After its initiation in [Ser73; Kat73], the theory of p-adic modular forms has been explored further by mathematicians and is now playing an important role in modern studies of algebraic number theory and arithmetic geometry. One of the most inspiring development in the theory of p-adic modular forms is the notion of *p-adic families of modular forms*. In [Hid86], H. Hida established families of *ordinary modular forms*, which are now known as *Hida families*. Later, R. Coleman introduced *overconvergent modular forms* in [Col95; Col97]. He and B. Mazur then discovered in [CM98] that overconvergent modular eigenforms can be parametrised by a rigid analytic curve, now known as the *eigencurve*.

It is well-known that the geometry of the eigencurve is quite mysterious. For example, it is still unknown whether the eigencurve admits finite or infinite irreducible components. On the other hand, the geometry of the eigencurve is known to encode interesting arithmetic information. For example, the information about the adjoint L-value and the adjoint Bloch–Kato Slemer groups, which shall be discussed in more details.

It is a natural question to ask whether the notion of p-adic families of modular forms can be generalised to other automorphic forms. The generalisation of Hida families are given by Hida himself in [Hid02] for automorphic forms over Shimura varieties of PEL-type. The case for Siegel modular forms is rewritten in more details in [Pil12].

On the other hand, the work of V. Pilloni ([Pil13]) and the work of F. Andreatta, A. Iovita and G. Stevens ([AIS10; AIS14]) provided a 'geometrisation' of Coleman theory; more precisely, they constructed *sheaves of overconvergent modular forms*. Such an idea has then been established further by Andreatta–Iovita–Pilloni in [AIP15] where they constructed *sheaves of overconvergent Siegel modular forms*. Consequently, they produced an eiquidimensional reduced *eigenvariety* $\mathcal{E}_0^{\mathrm{AIP}}$ that parametrises overconvergent cuspidal Siegel eigenforms. They then raised the following question regarding the geometry of the eigenvariety $\mathcal{E}_0^{\mathrm{AIP}}$.

Question 1.1.2 ([AIP15, Open problem 1])**.** *Are the classical points in $\mathcal{E}_0^{\mathrm{AIP}}$ unramified over the weight space?*

1.1.3. In another direction, Stevens introduced the notion of *overconvergent modular symbols* in [Ste94] as a new tool to study the eigencurve, method of study which was taken over by other authors (for example, [Par10; Bel12; Bel21]).

The idea of overconvergent modular symbols turns out to be a powerful tool for generalisations. A. Ash and Stevens's study of *overconvergent cohomology groups* in [AS08] was then applied to the construction of eigenvarieties for general reductive groups in E. Urban's paper [Urb11] and D. Hansen's article [Han17]. Recently, C. Johansson and J. Newton further carried out details of such a formalism in the language of adic spaces in [JN19], which consequently allows one to read information of the $p = 0$ loci of the eigenvaireties.

1.1.4. One sees from above that the notion of eigenvarieties can be either constructed from families of overconvergent automorphic forms or from families of overconvergent cohomology groups. One would expect the existence of a comparison between these two constructions. In other words, one would expect a comparison morphism between families of overconvergent cohomology groups and families of overconvergent automorphic forms, which is Hecke- (and Galois-) equivariant, *i.e.*, an *overconvergent Eichler–Shimura morphism.*

Such a comparison for $\mathrm{GL}_{2/\mathbf{Q}}$ is first discovered by Andreatta–Iovita–Stevens in [AIP15], where they established a Hecke- and Galois-equivariant morphism from the space of overconvergent modular symbols to the space of overconvergent modular forms. Their method are taken to study such a comparison for automorphic forms over other Shimura curves by D. Barrera and S. Gao in [BG17; BG21].

In the article of P. Chojecki, Hansen and Johansson ([CHJ17]), they re-established such a comparison for Shimura curves over $\mathbf{Q}$ via perfectoid methods. Roughly speaking, the overconvergent Eichler–Shimura morphism of *op. cit.* follows from the following steps:

(I) Use the perfectoid Shimura variety introduced by P. Scholze in [Sch15] to construct the sheaves over overconvergent automorphic forms. These sheaves are then proven to be isomorphic to the aforementioned ones constructed by Pilloni and Andreatta–Iovita–Stevens.

(II) Compute the overconvergent cohomology groups via the language of pro-étale sites. In particular, there exist sheaves on the proétale site of the Shimura curve that compute the overconvergent cohomology groups.

(III) Establish a Hecke-equivariant morphism from the sheaves that computes the overconvergent cohomology groups to the sheaves of overconvergent automorphic forms on the proétale site of the Shimura curve. The desired Hecke- and Galois-equivariant overconvergent Eichler–Shimura morphisms are then given by these morphisms on sheaves after taking cohomology.

1.2 An overview on Bloch–Kato conjecture and the adjoint L-values

1.2.1. Fix a (rational) prime number p and let $\mathtt{S}_{\mathrm{bad}}$ be a finite set of (rational) prime numbers such that $p \in \mathtt{S}_{\mathrm{bad}}$. Let $\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}$ be the Galois group of the maximal extension of $\mathbf{Q}$ that is unramified outside $\mathtt{S}_{\mathrm{bad}}$. Let F be a finite extension of $\mathbf{Q}_p$ and suppose we are given a representation

$$\rho : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \mathrm{GL}_n(F)$$

which is de Rham at p. Denote by $\mathrm{ad}\,\rho$ the adjoint representation associated with ρ, *i.e.*, the underlying space of $\mathrm{ad}\,\rho$ is the Lie algebra $\mathfrak{gl}_n(F)$ equipped with the adjoint action of $\mathrm{Gal}_{\mathbf{Q},S_{\mathrm{bad}}}$ induced by ρ. Let $\mathrm{ad}^0\,\rho$ be the subrepresentation of trace-zero part in $\mathrm{ad}\,\rho$. S. Bloch and K. Kato defined in [BK07] a certain subspace

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho) \subset H^1(\mathrm{Gal}_{\mathbf{Q},S_{\mathrm{bad}}}, \mathrm{ad}^0\,\rho)$$

in the continuous Galois cohomology group $H^1(\mathrm{Gal}_{\mathbf{Q},S_{\mathrm{bad}}}, \mathrm{ad}\,\rho)$, which is now known as the *adjoint Bloch–Kato Selmer group.*

Conjecture 1.2.2 (Bloch–Kato)**.** *Suppose ρ is absolute irreducible and denote by $L(\mathrm{ad}^0\,\rho, s)$ the L-function attached to the Galois representation $\mathrm{ad}^0\,\rho$. Then, we expect the following:*

(i) We have $\mathrm{ord}_{s=1} L(\mathrm{ad}^0\,\rho, s) = \dim_F H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho)$.

(ii) The adjoint Bloch–Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho)$ vanishes.

1.2.3. When $n = 1$, the conjecture is a theorem of C. Soulé [Sou81] and when $n = 2$, the conjecture is a consequence of the '$R = T$ theorem' obtained by the Taylor–Wiles patching method (see, for example, [Hid16]). More precisely, it is shown in *loc. cit.* that if ρ is the Galois representation attached to a cuspidal (elliptic) eigenform f, the adjoint L-value of $L(\mathrm{ad}^0\,\rho, s)$ can be written as a formula related to the Petersson inner product of f with itself ([*op.cit.*, Theorem 5.1 & Theorem 5.2]). However, little do we know about other cases.

The idea of using pairing can be seen in the work of A. Genestier and J. Tilouine ([GT05]). More precisely, consider the Siegel modular variety X parametrising abelian surfaces, the author of *loc. cit.* considered a pairing on the (étale) cohomology group of X and related the value of this pairing with the length of certain Slemer group as an application of the '$R = T$ theorem' coming from the Taylor–Wiles patching method ([*op. cit.*, §12]). However, it is unknown whether the value of this pairing can be directly related to the adjoint L-value.

1.2.4. On the other hand, there is a p-adic variant of the philosophy above. In his Ph.D. thesis ([Kim06]), W. Kim produced a pairing on the overconvergent cuspidal modular symbols and proved that this pairing p-adically interpolates the adjoint L-values. He moreover used this pairing to study the ramification locus of the cuspidal eigencurve over the weight space. Such a result consequently suggests that one can read information about the geometry of the cuspidal eigencurve over the weight space via some information of the Bloch–Kato Selmer group attached to the adjoint Galois representation of a cuspidal eigenform and vice versa. We remark that Kim's pairing and results are rewritten in a more conceptual way by J. Bellaïche in [Bel21].

Kim's pairing and the result of Genestier–Tilouine suggest a direction for the generalisation to higher dimensional cases. More precisely, one would expect the existence of a pairing on the overconvergent parabolic cohomology groups of GSp_{2g} such that

- its value allows one to detect some geometric information of the cuspidal eigenvariety over the weight space;
- its value also provides information about the adjoint Selmer group (and so some (conjectural) information of the adjoint L-value).

1.3 Motivations and main results

1.3.1. The present thesis is motivated by the aforementioned problems. That is, on one hand, we would like to study the comparison between the overconvergent cohomology groups and the overconvergent automorphic forms; on the other hand, we would like to generalise Kim's pairing to GSp_{2g}, providing a way to study the geometry of the cuspidal eigenvarieties as well as a way to study the arithmetic information of the adjoint Bloch–Kato Selmer groups. More precisely, we ask the following questions:

Question 1.3.2. *(i) Can the perfectoid method used in [CHJ17] be generalised to the Siegel case? That is, can one use perfectoid method to construct sheaves of overconvergent Siegel modular forms and establish overconvergent Eichler–Shimura morphisms in this situation?*

(ii) Can one generalise Kim's pairing to GSp_{2g}*, providing a way to answer Andreatta–Iovita–Pilloni's question (Question 1.1.2) as well as showing a relation between the adjoint Bloch–Kato Selmer groups and the geometry of the eigenvarieties in this case?*

1.3.3. The answers to the questions are positive. More precisely, (i) is the result of the joint work with H. Diao and G. Rosso ([DRW22]) while the pairing we wished to obtain in (ii) is established in [Wu21] and its relation with the adjoint Bloch–Kato Selmer group is the main theme of [Wu22].

The rest of this section is devoted to explain in more details about how we attempt to answer the motivating questions. Our main results are then stated along the explanation.

1.3.4. Let g be a positive integer and p be an odd prime number. The main geometric objects in our consideration are the adic spaces

$$\overline{\mathcal{X}}, \quad \overline{\mathcal{X}}_{\mathrm{Iw}^+} \quad \text{and} \quad \overline{\mathcal{X}}_{\Gamma(p^\infty)}$$

over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. They are toroidally compactified genus-g Siegel modular varieties of tame level, of *strict* Iwahori level at p and of infinite level at p respectively. Remark that

- $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ is a deeper level variety compared with the usual Siegel modular variety of Iwahori level;
- $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is moreover a perfectoid space by [PS16].

Moreover, by employing the language of *log adic spaces* developed in [DLLZ19], the natural maps

$$h_{\mathrm{Iw}^+} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+} \quad \text{and} \quad h : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}$$

are pro-Kummer étale of Galois group $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ and $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$ respectively. Here, $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ is the *strict* Iwahori subgroup of $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$ defined in §2.1.

The perfectoid space $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is equipped with a well-known *Hodge–Tate period map*

$$\pi_{\mathrm{HT}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \mathcal{F}\ell,$$

where $\mathcal{F}\ell$ is the flag variety parameterises maximal lagrangian subspaces of a fixed symplectic space of rank $2g$. Using π_{HT}, for any $w \in \mathbf{Q}_{>0}$, we consider the *w-ordinary loci*

$$\overline{\mathcal{X}}_w, \quad \overline{\mathcal{X}}_{\mathrm{Iw}^+,w} \quad \text{and} \quad \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$$

of $\overline{\mathcal{X}}$, $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ and $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ respectively.

Let $(\kappa_{\mathcal{U}}, R_{\mathcal{U}})$ be weight and suppose $w > 1 + r_{\mathcal{U}}$,[1] we construct a sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ over $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ by first defining a sheaf on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ and descend to $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ by using the Galois group $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. Roughly speaking, sections of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ consist of functions f on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ which

- take value in a certain weight-$\kappa_{\mathcal{U}}$ analytic representation $C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}})$ of the Iwahori subgroup of $\mathrm{GL}_g(\mathbf{Z}_p)$, and
- satisfy the following formula regarding the natural action of the strict Iwahori subgroup $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ of $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$:

$$\boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\, \boldsymbol{\gamma}_c)^{-1} f \qquad \text{for any} \qquad \boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}},$$

  where $\mathfrak{z}$ stands for the pullback of the coordinate function on the flag variety and $\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\, \boldsymbol{\gamma}_c)$ stands for a certain automorphism on $C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}})$.

The sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ is called the sheaf of *w-overconvergent Siegel modular forms of strict Iwahori level.*

1.3.5. On the other hand, let $\mathcal{X}_{\mathrm{Iw}^+}$ be the locus of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ away from boundary. By fixing an isomorphism $\mathbf{C} \simeq \mathbf{C}_p$, and consider the algebraic model X_{Iw^+} of $\mathcal{X}_{\mathrm{Iw}^+}$, it is well-known that the $\mathbf{C}$-valued point of X_{Iw^+} can be identified with the locally symmetric space

$$X_{\mathrm{Iw}^+}(\mathbf{C}) := \mathrm{GSp}_{2g}(\mathbf{Q}) \backslash \mathbb{H}_g \times \mathrm{GSp}_{2g}(\mathbf{A}_f)/\Gamma^{(p)}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}},$$

where

- $\mathbb{H}_g$ is the disjoint union of the Siegel upper- and lower-half spaces,
- $\mathbf{A}_f$ is the ring of finite adèles of $\mathbf{Q}$, and
- $\Gamma^{(p)}$ is the tame level of $\mathcal{X}_{\mathrm{Iw}^+}$.

Let $(\kappa_{\mathcal{U}}, R_{\mathcal{U}})$ be a weight and $r > 1 + r_{\mathcal{U}}$, we consider the r-analytic distribution $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ by following the idea of [AS08] (see also [Urb11; Han17; JN19]). This module turns out to define a local system on $X_{\mathrm{Iw}^+}(\mathbf{C})$ and hence one can consider the Betti cohomology groups

$$H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$$

for any $t \in \mathbf{Z}$.

The following theorem summarises our attempt to answer Question 1.3.2 (i):

[1] For the definition of weights and $r_{\mathcal{U}}$, see Definition 3.1.2 and Definition 3.1.10.

Theorem 1.3.6 (Theorem 3.7.2, Proposition 6.2.8, Theorem 6.3.6, Theorem 6.4.4)**.** *Keep the notation as above. We have the following.*

(i) When $p > 2g$, the sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ is isomorphic to the sheaf of overconvergent Siegel modular forms constructed in [AIP15].

(ii) If $(\kappa_{\mathcal{U}}, R_{\mathcal{U}})$ is a small weight (see Definition 3.1.2) and $r \geq w > 1 + r_{\mathcal{U}}$, there is a Hecke- and Galois-equivariant morphism

$$\mathrm{ES}_{\kappa_{\mathcal{U}}} : H^{n_0}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}+g+1})(-n_0),$$

where $n_0 = \dim_{\mathbf{C}_p} \mathcal{X}_{\mathrm{Iw}^+}$. The morphism $\mathrm{ES}_{\kappa_{\mathcal{U}}}$ is called the overconvergent Eichler–Shimura morphism of Siegel modular forms of weight $\kappa_{\mathcal{U}}$.

(iii) At a dominant classical weight $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{\geq 0}$, the image of ES_k is contained in the space of classical Siegel modular forms.

(iv) Finally, the Eichler–Shimura morphism can be promoted to a morphism between coherent sheaves on the equidimensional cuspidal eigenvariety $\mathcal{E}_0$, paramterising finite slope cuspidal Siegel eigenforms.

1.3.7. Now, we turn our attention to Question 1.3.2 (ii). Our first observation is that there exists a pairing

$$[\cdot, \cdot]^\circ_{\kappa_{\mathcal{U}}} : D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \times D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \to R_{\mathcal{U}}$$

on the r-analytic distribution. Together with the cup product, this pairing then induces a pairing on the cohomology

$$H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \times H^{2n_0-t}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to R_{\mathcal{U}}$$

for any $0 \leq t \leq 2n_0$, where $H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ is the compactly supported cohomology group. Note that there is a natural morphism

$$H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

Thus, by writing

$$H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) := \mathrm{image}\left(H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))\right),$$

the pairing above then induces a pairing

$$[\cdot, \cdot]_\kappa : H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \times H^{2n_0-t}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to R_{\mathcal{U}} \tag{1.1}$$

for any $0 \leq t \leq 2n_0$. Consequently, inspired by [Bel21, Chapter VIII], we use this pairing to deduce the following result:

Theorem 1.3.8 (Corollary 7.3.16)**.** *Let $\mathcal{E}_0^{\mathrm{oc}}$ be the equidimensional cuspidal eigenvariety constructed in §5.2 and let $\mathcal{E}_0^{\mathrm{oc,fl}}$ be the flat locus of $\mathcal{E}_0^{\mathrm{oc}}$ with respect to the weight map* $\mathrm{wt} : \mathcal{E}_0^{\mathrm{oc}} \to \mathcal{W}$.

(i) *Suppose $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ is a good point (see Definition 7.3.2), then there is a function L^{adj} on a small neighbourhood of $\boldsymbol{x}$, which is defined uniquely (up to a unit in the local eigenalgebra at $\boldsymbol{x}$) by the pairing (1.1).*

(ii) *If $\boldsymbol{x}$ is a good point whose weight is a dominant algebraic weight and whose slope is small enough, then the following hold.*

- *The function L^{adj} vanishes at $\boldsymbol{x}$ if and only if the weight map* wt *is ramified at $\boldsymbol{x}$.*
- *If $\boldsymbol{x}$ is further a smooth point, then the order of vanishing of L^{adj} at $\boldsymbol{x}$ is equal to the quantity $e(\boldsymbol{x})$ defined in Theorem 7.3.11.*

1.3.9. The function L^{adj} in the theorem above is called the *adjoint p-adic L-function* (on a small enough neighbourhood of $\boldsymbol{x}$). This terminology is given by the terminology used in [Bel21]. However, the author of *loc. cit.* justified such a terminology by computing the special value of the adjoint L-function associated with a elliptic newform in [*loc. cit.*, §VIII.5.2] while we did not know what is the link between our L^{adj} with the adjoint L-function associated with cuspidal Siegel eigenforms.

Inspired by the conjectural link between the pairing considered in [GT05, §12] and the adjoint L-value of cuspdial Siegel eigenform, we expect a natural relation between L^{adj} and the adjoint Bloch–Kato Selmer group associated with the Galois representation attached to cuspidal Siegel eigenforms. (Hence, the Bloch–Kato conjecture allows us to conjecturally justify the name of L^{adj}.) Hence, in Chapter 8, we study the adjoint Bloch–Kato Selmer groups associated with these Galois representations. However, such Galois representations are not well-established at the current stage. Therefore, we need the following hypotheses, which are reasonable (but might be difficult to verify):

- Hypothesis 1: Roughly speaking, this hypothesis states that one can attach a GSpin_{2g+1}-valued Galois representation $\rho_{\boldsymbol{x}}^{\mathrm{spin}}$ of $\mathrm{Gal}_{\mathbf{Q}}$ to any classical point $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc}}$, where $\mathrm{Gal}_{\mathbf{Q}}$ denotes the absolute Galois group of $\mathbf{Q}$.

- Hypothesis 2: Roughly speaking, this hypothesis ensures that there exists a real finite extension L of $\mathbf{Q}$ and a generic cuspidal automorphic representation $\mathrm{GL}_{2^g}(\mathbf{A}_L)$ whose associated Galois representation coincide with $\rho_{\boldsymbol{x}}^{\mathrm{spin}}|_{\mathrm{Gal}_L}$, where $\mathbf{A}_L$ is the ring of adèles of L and Gal_L is the absolute Galois group of L.

- Hypothesis 3: This a technical hypothesis on the Hecke algebra due to our lack of knowledge on the Hecke algebra of strict Iwahori level.

- Hypothesis 4: This is a technical hypothesis, which ensures us to obtain a GSpin_{2g+1}-valued Galois representation with coefficients in the local eigenalgebra of $\boldsymbol{x}$ and that the chosen tame $\Gamma^{(p)}$ implies a particular ramification type of this Galois representation at bad primes.

Theorem 1.3.10 (Corollary 8.5.7)**.** *Let $\boldsymbol{x} \in \mathcal{E}_0$ whose weight is a dominant algebraic weight and whose slope is small enough. Suppose the following assumptions hold:*

(I) *Standard assumptions:*

- *The point $\boldsymbol{x}$ corresponds to a p-stabilisation of an eigenclass of tame level (see §8.2 and §8.3 for more discussion).*
- *Hypothesis 1 holds so that we get a GSpin_{2g+1}-valued Galois representation $\rho_{\boldsymbol{x}}^{\mathrm{spin}}$ attached to $\boldsymbol{x}$. We write $\rho_{\boldsymbol{x}} := \mathrm{spin} \circ \rho_{\boldsymbol{x}}^{\mathrm{spin}}$ be the associated GL_{2^g}-valued Galois representation.*

(II) Technical assumption: Hypothesis 4 hold.

(III) Assumptions used in the strategy of [BC09]:

- *The Galois representation $\rho_{\boldsymbol{x}}$ admits a refinement $\mathbb{F}_{\bullet}^{\boldsymbol{x}}$ that satisfies (REG) and (NCR) (see §8.1 for definitions of $\mathbb{F}_{\bullet}^{\boldsymbol{x}}$, (REG) and (NCR)).*
- *The restriction $\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is not isomorphic to its twist by the p-adic cyclotomic character.*

(IV) Assumptions to apply [NT20]:

- *Hypothesis 2 holds.*
- *The cuspidal automorphic representation $\pi_{\boldsymbol{x}}$ of $\mathrm{GL}_{2^g}(\mathbf{A}_L)$ ensured by Hypothesis 2 is regular algebraic and polarised (see, for example, [BLGGT14, §2.1]).*
- *The image $\rho_{\boldsymbol{x}}(\mathrm{Gal}_{L(\zeta_{p^\infty})})$ is enormous (see [NT20, Definition 2.27]).*

Then

(i) The adjoint Bloch–Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho_{\boldsymbol{x}}^{\mathrm{spin}})$ associated with $\rho_{\boldsymbol{x}}^{\mathrm{spin}}$ vanishes.

(ii) There is an 'infinitesimal $R = T$ theorem' locally at $\boldsymbol{x}$.

1.3.11. There is another situation that one can also deduce the vanishing of the adjoint Bloch–Kato Selmer group. It is in this situation we obtain the link between L^{adj} and the adjoint Bloch–Kato Selmer group. Consequently, in light of the Bloch–Kato conjecture, such a link (conjecturally) justifies the name for L^{adj}.

Theorem 1.3.12 (Corollary 8.5.9)**.** *Let $\boldsymbol{x}$ be a good point whose weight is a dominant algebraic weight. Suppose (I), (II), and (III) in Theorem 1.3.10 hold for $\boldsymbol{x}$. Assume that the weight map* wt *is étale at $\boldsymbol{x}$ and the 'infinitesimal $R = T$ theorem' holds locally at $\boldsymbol{x}$. Then,*

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho_{\boldsymbol{x}}^{\mathrm{spin}}) = 0.$$

In particular, we have

$$\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} = \dim_{k_{\boldsymbol{x}}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho_{\boldsymbol{x}}^{\mathrm{spin}}).$$

Remark 1.3.13. Finally, we remark that the results presented in this thesis have their Hilbert-modular analogues:

(i) In [BHW19], C. Birkbeck, B. Heuer and C. Williams gave a perfectoid construction of sheaves of overconvergent Hilbert modular forms. They are then using such construction to establish the overconvergent Eichler–Shimura morphism for Hilbert modular forms in their forthcoming work.

(ii) B. Balasubramanyam and M. Longo generalised Kim's pairing to the Hilbert modular case in [BL20]. Unlike to the case of GSp_{2g} that we can only establish a conjectural link between the adjoint p-adic L-function and the adjoint L-function attached to the associated Galois representation, authors of *loc. cit.* provided a direct link between their pairing and the adjoint L-function associated with a Hilbert modular form.

1.4 Miscellaneous: structure of the thesis and conventions

1.4.1. We briefly discuss the organisation of this thesis.

In Chapter 2, we introduce the geometric objects in our interests more carefully. In particular, we recall the construction of the toroidally compactified Siegel modular variety at infinite level $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ by following [PS16]. We then study the flag variety $\mathcal{F}\ell$ and introduce the Hodge–Tate period map π_{HT} in §2.4 and §2.5.

In Chapter 3, we use perfectoid method to construct the sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$. We provide a thorough local description of this sheaf in §3.2 and justify the name by showing that classical Siegel modular forms is contained in the global sections of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ in §3.4. We briefly review the construction of Andreatta–Iovita–Pilloni in §3.5 and devote §3.6 and §3.7 to compare the two constructions under the condition $p > 2g$ (due to some technicality).

In Chapter 4, we introduce the overconvergent cohomology groups. These cohomology groups are inspired by [AS08; AIS15; Han17]. We shall also discuss their algebraic counterpart in §4.4.

Chapter 5 is devoted to construct the eigenvarieties that considered in this thesis. In fact, we construct two equidimensional eigenvarieties $\mathcal{E}_0^{\mathrm{oc}}$ and $\mathcal{E}_0$. The former is the one constructed by considering overconvergent cohomology groups (§5.2) while the latter is constructed with respect to the overconvergent Siegel cuspforms (§5.3). It turns out that if we base change $\mathcal{E}_0^{\mathrm{oc}}$ to $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, then there is a closed immersion $\mathcal{E}_0 \hookrightarrow \mathcal{E}_0^{\mathrm{oc}}$ (Proposition 5.3.6).

The overconvergent Eichler–Shimura morphism for Siegel modular forms is the main result in Chapter 6. Such a morphism is deduced from a morphism between sheaves on the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}$ in §6.2. Thus, as a preparation, we briefly discuss the pro-Kummer étale cohomology groups in our interests in §6.1. We study the image of the overconvergent Eichler–Shimura morphism in §6.3 and then promote it to a morphism between coherent sheaves on $\mathcal{E}_0$ in §6.4.

We discuss the generalisation of Kim's pairing in Chapter 7. More precisely, we first construct the pairing on the cohomology groups in §7.1. After reviewing some material of commutative algebra in §7.2 (by following [Bel21, Chapter VIII]), we study the ramification locus of the eigenvariety $\mathcal{E}_0^{\mathrm{oc}}$ in §7.3.

The relation between the pairing studied in Chapter 7 and the adjoint Bloch–Kato Selmer group is the main theme of Chapter 8. We shall recall some terminology of families of Galois representation in §8.1 by following [BC09]. We shall also discuss some hypotheses that we will assume for Galois representations attached to automorphic representations of GSp_{2g} in §8.2. We follow the strategy of [BC09] to construct families of Galois representations over a sublocus of $\mathcal{E}_0^{\mathrm{oc}}$ in §8.3 and study some Galois deformation problems in §8.4. The main results of this chapter are then stated and proved in §8.5.

Finally, Appendix A is about log adic spaces. In particular, we briefly review log adic spaces by following [DLLZ19] in §A.1. We introduce the notion of Banach sheaves and prove a (generalised projection formula) in §A.2, which is essential to our construction of the overconvergent Eichler–Shimura morphism.

1.4.2. Through out this thesis, we fix the following notations and conventions:

- $g \in \mathbf{Z}_{\geq 1}$.
- For any prime number ℓ, we fix once and forever an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ and an algebraic isomorphism $\mathbf{C}_\ell \simeq \mathbf{C}$, where $\mathbf{C}_\ell$ is the ℓ-adic completion of $\overline{\mathbf{Q}}_\ell$. We write $\mathrm{Gal}_{\mathbf{Q}_\ell}$ for the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$. We also fix the ℓ-adic absolute value on $\mathbf{C}_\ell$ so that $|\ell| = \ell^{-1}$.
- We fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and embeddings $\overline{\mathbf{Q}}_\ell \hookleftarrow \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, which is compatible with the chosen isomorphisms $\mathbf{C}_\ell \simeq \mathbf{C}$. We analogously write $\mathrm{Gal}_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and identify $\mathrm{Gal}_{\mathbf{Q}_\ell}$ as a (decomposition) subgroup of $\mathrm{Gal}_{\mathbf{Q}}$.
- We fix an odd prime number $p \in \mathbf{Z}_{>0}$. Due to certain technicality, we will have to assume $p > 2g$ at some places. Such an assumption shall be clear in the context.
- For any $w \in \mathbf{Q}_{>0}$, we denote by 'p^w' an element in $\mathbf{C}_p$ with absolute value p^{-w}. All constructions in the thesis will not depend on such choices.
- We adopt the language of almost mathematics. In particular, for an $\mathcal{O}_{\mathbf{C}_p}$-module M, we denote by M^a for the associated almost $\mathcal{O}_{\mathbf{C}_p}$-module.
- For $n \in \mathbf{Z}_{\geq 1}$ and any set R, we denote by $M_n(R)$ the set of n by n matrices with coefficients in R.
- The transpose of a matrix $\boldsymbol{\alpha}$ is denoted by ${}^{\mathtt{t}}\boldsymbol{\alpha}$.
- For any $n \in \mathbf{Z}_{\geq 1}$, we denote by $\mathbb{1}_n$ the $n \times n$ identity matrix and denote by $\breve{\mathbb{1}}_n$ the $n \times n$ anti-diagonal matrix whose non-zero entries are 1; *i.e.,*

$$\mathbb{1}_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \breve{\mathbb{1}}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

- In principle, symbols in Gothic font (*e.g.*, $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$) stand for formal schemes; symbols in calligraphic font (*e.g.*, $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$) stand for adic spaces; and symbols in script font (*e.g.*, $\mathscr{O}, \mathscr{F}, \mathscr{E}$) stand for sheaves (over various geometric objects).

Chapter 2

The geometry of Siegel modular varieties

In this chapter, we introduce and study the main geometric objects concerned in this thesis, *i.e.*, the Siegel modular varieties. In the §2.1, we fix the notations of the algebraic and p-adic groups that will appear in our studies. The Siegel modular varieties and their toroidal compactifications are recalled in §2.2. We then follow [PS16, Appendice: Compactifications] and discuss the perfectoid toroidal compactification of the Siegel modular variety at the infinite level in §2.3. The perfectoid toroidally compactified Siegel modular variety at the infinite level is equipped with the so-called 'Hodge–Tate period map'. Such a map will be described in details in §2.5 after the study of the flag varieties in §2.4.

2.1 Algebraic and p-adic groups

2.1.1. In this section, we setup the algebraic and p-adic groups that we shall be studying in this thesis. We start with the definition of the general symplectic group GSp_{2g}.

Let $\mathbf{V} = \mathbf{V}_{\mathbf{Z}} := \mathbf{Z}^{2g}$ and we equip it with an alternative pairing

$$\langle \cdot, \cdot \rangle : \mathbf{V}_{\mathbf{Z}} \times \mathbf{V}_{\mathbf{Z}} \to \mathbf{Z}, \quad (\boldsymbol{v}, \boldsymbol{v}') \mapsto {}^{\mathtt{t}}\boldsymbol{v} \begin{pmatrix} & -\breve{\mathbb{1}}_g \\ \breve{\mathbb{1}}_g & \end{pmatrix} \boldsymbol{v}', \tag{2.1}$$

where we view elements in $\mathbf{V}_{\mathbf{Z}}$ as column vectors. In particular, if $e_1, ..., e_{2g}$ is the standard basis for $\mathbf{V}_{\mathbf{Z}}$, then

$$\langle e_i, e_j \rangle = \begin{cases} -1 & \text{if } i < j \text{ and } j = 2g+1-i \\ 1 & \text{if } i > j \text{ and } j = 2g+1-i \\ 0 & \text{else} \end{cases}.$$

We then define the algebraic group GSp_{2g} (over $\mathbf{Z}$) to be

$$\mathrm{GSp}_{2g} := \left\{ \boldsymbol{\gamma} \in \mathrm{GL}_{2g} : {}^{\mathtt{t}}\boldsymbol{\gamma} \begin{pmatrix} & -\breve{\mathbb{1}}_g \\ \breve{\mathbb{1}}_g & \end{pmatrix} \boldsymbol{\gamma} = \varsigma(\boldsymbol{\gamma}) \begin{pmatrix} & -\breve{\mathbb{1}}_g \\ \breve{\mathbb{1}}_g & \end{pmatrix} \text{ for some } \varsigma(\boldsymbol{\gamma}) \in \mathbb{G}_m \right\}.$$

Equivalently, for any $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{GL}_{2g}$, $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}$ if and only if

$$ {}^{\mathtt{t}}\boldsymbol{\gamma}_a \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_c = {}^{\mathtt{t}}\boldsymbol{\gamma}_c \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_a, \quad {}^{\mathtt{t}}\boldsymbol{\gamma}_b \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_d = {}^{\mathtt{t}}\boldsymbol{\gamma}_d \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_b, \text{ and } {}^{\mathtt{t}}\boldsymbol{\gamma}_a \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_d - {}^{\mathtt{t}}\boldsymbol{\gamma}_c \, \breve{\mathbb{1}}_g \, \boldsymbol{\gamma}_b = \varsigma(\boldsymbol{\gamma}) \, \breve{\mathbb{1}}_g $$

for some $\varsigma(\boldsymbol{\gamma}) \in \mathbb{G}_m$. One can easily check that GSp_{2g} is stable under transpose. Thus, the above conditions are also equivalent to

$$ \boldsymbol{\gamma}_a \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_b = \boldsymbol{\gamma}_b \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_a, \quad \boldsymbol{\gamma}_c \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_d = \boldsymbol{\gamma}_d \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_c, \quad \text{and } \boldsymbol{\gamma}_a \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_d - \boldsymbol{\gamma}_b \, \breve{\mathbb{1}}_g \, {}^{\mathtt{t}}\boldsymbol{\gamma}_c = \varsigma(\boldsymbol{\gamma}) \, \breve{\mathbb{1}}_g $$

for some $\varsigma(\boldsymbol{\gamma}) \in \mathbb{G}_m$.

2.1.2. We consider the upper triangular Borel subgroups

$$ \begin{aligned} B_{\mathrm{GL}_g} &:= \text{the Borel subgroup of upper triangular matrices in } \mathrm{GL}_g \\ B_{\mathrm{GSp}_{2g}} &:= \text{the Borel subgroup of upper triangular matrices in } \mathrm{GSp}_{2g}. \end{aligned} $$

The reason why we are able to consider the upper triangular Borel subgroup for GSp_{2g} is because of the choice of the pairing in (2.1).

The corresponding unipotent radicals are

$$ \begin{aligned} U_{\mathrm{GL}_g} &:= \text{ the upper triangular } g \times g \text{ matrices whose diagonal entries are all } 1 \\ U_{\mathrm{GSp}_{2g}} &:= \text{ the upper triangular } 2g \times 2g \text{ matrices in } \mathrm{GSp}_{2g} \text{ whose diagonal entries are all } 1. \end{aligned} $$

Consequently, the maximal tori for both algebraic groups are the tori of diagonal matrices, which are denoted by T_{GL_g} and $T_{\mathrm{GSp}_{2g}}$ respectively. The Levi decomposition then yields

$$ B_{\mathrm{GL}_g} = U_{\mathrm{GL}_g} T_{\mathrm{GL}_g} \quad \text{and} \quad B_{\mathrm{GSp}_{2g}} = U_{\mathrm{GSp}_{2g}} T_{\mathrm{GSp}_{2g}}. $$

Moreover, we denote by $U_{\mathrm{GL}_g}^{\mathrm{opp}}$ and $U_{\mathrm{GSp}_{2g}}^{\mathrm{opp}}$ the opposite unipotent radical of U_{GL_g} and $U_{\mathrm{GSp}_{2g}}$ respectively.

To simplify the notation, for any $s \in \mathbf{Z}_{\geq 0}$, we write

$$ \begin{aligned} T_{\mathrm{GL}_g, s} &:= \begin{cases} T_{\mathrm{GL}_g}(\mathbf{Z}_p), & s = 0 \\ \ker(T_{\mathrm{GL}_g}(\mathbf{Z}_p) \to T_{\mathrm{GL}_g}(\mathbf{Z}/p^s\mathbf{Z})), & s > 0 \end{cases} \\ U_{\mathrm{GL}_g, s} &:= \begin{cases} U_{\mathrm{GL}_g}(\mathbf{Z}_p), & s = 0 \\ \ker(U_{\mathrm{GL}_g}(\mathbf{Z}_p) \to U_{\mathrm{GL}_g}(\mathbf{Z}/p^s\mathbf{Z})), & s > 0 \end{cases} \\ T_{\mathrm{GSp}_{2g}, s} &:= \begin{cases} T_{\mathrm{GSp}_{2g}}(\mathbf{Z}_p), & s = 0 \\ \ker(T_{\mathrm{GSp}_{2g}}(\mathbf{Z}_p) \to T_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p^s\mathbf{Z})), & s > 0 \end{cases} \\ U_{\mathrm{GSp}_{2g}, s} &:= \begin{cases} U_{\mathrm{GSp}_{2g}}(\mathbf{Z}_p), & s = 0 \\ \ker(U_{\mathrm{GSp}_{2g}}(\mathbf{Z}_p) \to U_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p^s\mathbf{Z})), & s > 0 \end{cases} \end{aligned} $$

The maps above are all reduction maps.

2.1.3. The Iwahori subgroups are defined to be

$$\begin{aligned}\mathrm{Iw}_{\mathrm{GL}_g} &:= \text{ the preimage of } B_{\mathrm{GL}_g}(\mathbf{F}_p) \text{ under the reduction map } \mathrm{GL}_g(\mathbf{Z}_p) \to \mathrm{GL}_g(\mathbf{F}_p)\\ \mathrm{Iw}_{\mathrm{GSp}_{2g}} &:= \text{ the preimage of } B_{\mathrm{GSp}_{2g}}(\mathbf{F}_p) \text{ under the reduction map } \mathrm{GSp}_{2g}(\mathbf{Z}_p) \to \mathrm{GSp}_{2g}(\mathbf{F}_p).\end{aligned}$$

The Iwahori decomposition yields that

$$\mathrm{Iw}_{\mathrm{GL}_g} = U^{\mathrm{opp}}_{\mathrm{GL}_g,1} T_{\mathrm{GL}_g,0} U_{\mathrm{GL}_g,0} \quad \text{and} \quad \mathrm{Iw}_{\mathrm{GSp}_{2g}} = U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1} T_{\mathrm{GSp}_{2g},0} U_{\mathrm{GSp}_{2g},0}.$$

We also introduce the *strict* Iwahori subgroups

$$\begin{aligned}\mathrm{Iw}^+_{\mathrm{GL}_g} &:= \text{ the preimage of } T_{\mathrm{GL}_g}(\mathbf{F}_p) \text{ under the reduction map } \mathrm{GL}_g(\mathbf{Z}_p) \to \mathrm{GL}_g(\mathbf{F}_p)\\ \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} &:= \left\{ \boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Z}_p) : \boldsymbol{\gamma} \equiv \left(\begin{array}{ccc|ccc} * & & & * & \cdots & * \\ & \ddots & & \vdots & & \vdots \\ & & * & * & \cdots & * \\ \hline & & & * & & \\ & & & & \ddots & \\ & & & & & * \end{array}\right) \mod p \right\}\end{aligned}$$

The Iwahori decompositions for the Iwahori subgroups then induce the decompositions

$$\mathrm{Iw}^+_{\mathrm{GL}_g} = U^{\mathrm{opp}}_{\mathrm{GL}_g,1} T_{\mathrm{GL}_g,0} U_{\mathrm{GL}_g,1} \quad \text{and} \quad \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} = U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1} T_{\mathrm{GSp}_{2g},0} U^+_{\mathrm{GSp}_{2g},0},$$

where $U^+_{\mathrm{GSp}_{2g},0} = \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \cap U_{\mathrm{GSp}_{2g},0}$.

2.1.4. We introduce the notion of 'w-neighbourhood' of some aforementioned p-adic groups. For any $w \in \mathbf{Q}_{>0}$ and $s \in \mathbf{Z}_{\geq 0}$, define

$$\begin{aligned}T^{(w)}_{\mathrm{GL}_g,s} &:= \left\{ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij})_{i,j} \in T_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) : |\, \boldsymbol{\lambda}_{ij} - \boldsymbol{\lambda}'_{ij} \,| \leq p^{-w} \text{ for some } \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_{ij})_{i,j} \in T_{\mathrm{GL}_g,s} \right\},\\ U^{(w)}_{\mathrm{GL}_g,s} &:= \left\{ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij})_{i,j} \in U_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) : |\, \boldsymbol{\lambda}_{ij} - \boldsymbol{\lambda}'_{ij} \,| \leq p^{-w} \text{ for some } \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_{ij})_{i,j} \in U_{\mathrm{GL}_g,s} \right\},\\ B^{(w)}_{\mathrm{GL}_g,s} &:= \left\{ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij})_{i,j} \in B_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) : |\, \boldsymbol{\lambda}_{ij} - \boldsymbol{\lambda}'_{ij} \,| \leq p^{-w} \text{ for some } \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_{ij})_{i,j} \in B_{\mathrm{GL}_g,s} \right\}.\end{aligned}$$

The groups $U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,s}$ and $B^{\mathrm{opp},(w)}_{\mathrm{GL}_g,s}$ are defined similarly.

Similarly, define

$$\begin{aligned}\mathrm{Iw}^{(w)}_{\mathrm{GL}_g} &:= \left\{ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij})_{i,j} \in \mathrm{GL}_g(\mathcal{O}_{\mathbf{C}_p}) : |\, \boldsymbol{\lambda}_{ij} - \boldsymbol{\lambda}'_{ij} \,| \leq p^{-w} \text{ for some } \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_{ij})_{i,j} \in \mathrm{Iw}_{\mathrm{GL}_g} \right\}\\ \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g} &:= \left\{ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij})_{i,j} \in \mathrm{GL}_g(\mathcal{O}_{\mathbf{C}_p}) : |\, \boldsymbol{\lambda}_{ij} - \boldsymbol{\lambda}'_{ij} \,| \leq p^{-w} \text{ for some } \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_{ij})_{i,j} \in \mathrm{Iw}^+_{\mathrm{GL}_g} \right\}.\end{aligned}$$

Then, the Iwahori decomposition induces

$$\mathrm{Iw}^{(w)}_{\mathrm{GL}_g} = U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} T^{(w)}_{\mathrm{GL}_g,0} U^{(w)}_{\mathrm{GL}_g,0} \quad \text{and} \quad \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g} = U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} T^{(w)}_{\mathrm{GL}_g,0} U^{(w)}_{\mathrm{GL}_g,1}.$$

We also write

$$\begin{aligned}
T_w &= \ker(T_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) \to T_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}/p^w)),\\
U_w &= \ker(U_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) \to U_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}/p^w)),\\
B_w &= \ker(B_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}) \to B_{\mathrm{GL}_g}(\mathcal{O}_{\mathbf{C}_p}/p^w)).
\end{aligned}$$

The groups U_w^{opp} and B_w^{opp} are defined similarly. Then we have

$$T_{\mathrm{GL}_g,0}^{(w)} = T_{\mathrm{GL}_g,0}T_w, \quad U_{\mathrm{GL}_g,0}^{(w)} = U_{\mathrm{GL}_g,0}U_w, \quad B_{\mathrm{GL}_g,0}^{(w)} = B_{\mathrm{GL}_g,0}B_w.$$

There are similarly identities for $U_{\mathrm{GL}_g,0}^{\mathrm{opp},(w)}$ and $B_{\mathrm{GL}_g,0}^{\mathrm{opp},(w)}$.

2.1.5. In this paragraph, we recall the Weyl groups and the root systems for GSp_{2g} and $H := \mathrm{GL}_g \times \mathbb{G}_m$ from [FC90, Chapter VI, §5]. Here, we view H as an algebraic subgroup of GSp_{2g} via the embedding

$$H = \mathrm{GL}_g \times \mathbb{G}_m \hookrightarrow \mathrm{GSp}_{2g}, \quad (\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \mapsto \begin{pmatrix} \boldsymbol{\gamma} & \\ & \boldsymbol{\upsilon}\, \breve{\mathbb{1}}_g\, {}^{\mathsf{t}}\boldsymbol{\gamma}^{-1}\, \breve{\mathbb{1}}_g \end{pmatrix}.$$

Consider the character group $\mathbb{X} = \mathrm{Hom}(T_{\mathrm{GSp}_{2g}}, \mathbb{G}_m)$. We have the following isomorphism

$$\mathbf{Z}^{g+1} \xrightarrow{\sim} \mathbb{X}, \quad (k_1, ..., k_g; k_0) \mapsto \left(\mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g, \boldsymbol{\tau}_0\, \boldsymbol{\tau}_g^{-1}, ..., \boldsymbol{\tau}_0\, \boldsymbol{\tau}_1^{-1}) \mapsto \prod_{i=0}^{g} \boldsymbol{\tau}_i^{k_i} \right).$$

Let $x_1, ..., x_g, x_0$ be the basis of $\mathbb{X}$ that corresponds to the standard basis on $\mathbf{Z}^{g+1}$. Note that $\mathbb{X}$ can also be viewed as the character group of the maximal torus $T_H = T_{\mathrm{GL}_g} \times \mathbb{G}_m$ of H via the isomorphisms $T_{\mathrm{GSp}_{2g}} \simeq \mathbb{G}_m^{g+1} \simeq T_{\mathrm{GL}_g} \times \mathbb{G}_m = T_H$.

Under the above choices of the maximal tori, we can describe the root systems of GSp_{2g} and H explicitly

$$\begin{aligned}
\boldsymbol{\Phi}_{\mathrm{GSp}_{2g}} &= \{\pm(x_i - x_j),\ \pm(x_i + x_j - x_0),\ \pm(2x_t - x_0) : 1 \leq i < j \leq g, 1 \leq t \leq g\}\\
\boldsymbol{\Phi}_H &= \{\pm(x_i - x_j),\ \pm x_g,\ \pm x_0 : 1 \leq i < j \leq g\}.
\end{aligned}$$

Moreover, the choices of the Borel subgroups yields the description of the positive roots

$$\begin{aligned}
\boldsymbol{\Phi}^+_{\mathrm{GSp}_{2g}} &= \{x_i - x_j,\ x_i + x_j - x_0,\ 2x_t - x_0 : 1 \leq i < j \leq g, 1 \leq t \leq g\}\\
\boldsymbol{\Phi}^+_H &= \{x_i - x_j : 1 \leq i < j \leq g\} (= \Phi_H \cap \Phi^+_{\mathrm{GSp}_{2g}}).
\end{aligned}$$

The Weyl groups of GSp_{2g} and H are defined as

$$\mathrm{Weyl}_{\mathrm{GSp}_{2g}} := N_{\mathrm{GSp}_{2g}}(T_{\mathrm{GSp}_{2g}})/T_{\mathrm{GSp}_{2g}} \quad \text{and} \quad \mathrm{Weyl}_H := N_H(T_H)/T_H,$$

where $N_{\mathrm{GSp}_{2g}}(T_{\mathrm{GSp}_{2g}})$ (resp., $N_H(T_H)$) is the group of normalisers of $T_{\mathrm{GSp}_{2g}}$ (resp., T_H) in GSp_{2g} (resp., H). They can also be described explicitly as follows.

- We can identify $\mathrm{Weyl}_{\mathrm{GSp}_{2g}}$ with $\boldsymbol{\Sigma}_g \ltimes (\mathbf{Z}/2\mathbf{Z})^g$, where $\boldsymbol{\Sigma}_g$ denotes the permutation group on g letters. For any $\boldsymbol{\tau} = \mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g, \boldsymbol{\tau}_0\, \boldsymbol{\tau}_g^{-1}, ..., \boldsymbol{\tau}_0\, \boldsymbol{\tau}_1^{-1}) \in T_{\mathrm{GSp}_{2g}}$, the actions of $\boldsymbol{\Sigma}_g$ and $(\mathbf{Z}/2\,\mathbf{Z})^g$ are given as

(i) $\boldsymbol{\Sigma}_g$ permutes $\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g$,

(ii) the element $(\underbrace{0, ..., 0}_{i-1}, 1, 0, ..., 0) \in (\mathbf{Z}/2\,\mathbf{Z})^g$ maps $\boldsymbol{\tau}$ to

$$\mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_{i-1}, \boldsymbol{\tau}_0\, \boldsymbol{\tau}_i^{-1}, \boldsymbol{\tau}_{i+1}, ..., \boldsymbol{\tau}_g, \boldsymbol{\tau}_0\, \boldsymbol{\tau}_g^{-1}, ..., \boldsymbol{\tau}_0\, \boldsymbol{\tau}_{i+1}^{-1}, \boldsymbol{\tau}_i, \boldsymbol{\tau}_0\, \boldsymbol{\tau}_{i-1}^{-1}, ..., \boldsymbol{\tau}_0\, \boldsymbol{\tau}_1^{-1}).$$

- We can identify Weyl_H with $\boldsymbol{\Sigma}_g$, whose action on T_H is defined as the action of $\boldsymbol{\Sigma}_g$ on $T_{\mathrm{GSp}_{2g}}$.

The actions of the Weyl groups on the maximal tori then induce actions on the root systems $\boldsymbol{\Phi}_{\mathrm{GSp}_{2g}}$ and $\boldsymbol{\Phi}_H$. Following [FC90, Chapter VI, §5], let

$$\mathrm{Weyl}^H := \{w \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}} : w(\Phi^+_{\mathrm{GSp}_{2g}}) \supset \Phi^+_H\} \subset \mathrm{Weyl}_{\mathrm{GSp}_{2g}}.$$

It turns out that Weyl^H is a system of representatives of the quotient $\mathrm{Weyl}_H \setminus \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$.

2.2 Siegel modular varieties

2.2.1. The Siegel upper- and lower-half spaces $\mathbb{H}_g^{\pm}$ (of genus g) are defined as follows

$$\mathbb{H}_g^+ := \left\{ \boldsymbol{\alpha} \in M_g(\mathbf{C}) : \begin{array}{l} \boldsymbol{\alpha} \text{ is symmetric w.r.t the antidiagonal} \\ \Im\boldsymbol{\alpha} \text{ is positive definite} \end{array} \right\}$$
$$\mathbb{H}_g^- := \left\{ \boldsymbol{\alpha} \in M_g(\mathbf{C}) : \begin{array}{l} \boldsymbol{\alpha} \text{ is symmetric w.r.t the antidiagonal} \\ \Im\boldsymbol{\alpha} \text{ is negative definite} \end{array} \right\},$$

where $\Im\boldsymbol{\alpha}$ stands for the imaginary part of $\boldsymbol{\alpha}$. We denote by $\mathbb{H}_g$ the disjoint union of $\mathbb{H}_g^+$ and $\mathbb{H}_g^-$. There is a $\mathrm{GSp}_{2g}(\mathbf{R})$-action on $\mathbb{H}_g$ given by the formula

$$\begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \cdot \boldsymbol{\alpha} = (\boldsymbol{\gamma}_a\, \boldsymbol{\alpha} + \boldsymbol{\gamma}_b)(\boldsymbol{\gamma}_c\, \boldsymbol{\alpha} + \boldsymbol{\gamma}_d)^{-1}$$

for any $\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbf{R})$ and any $\boldsymbol{\alpha} \in \mathbb{H}_g$.

For any congruence subgroup $\Gamma^{(p)} \subset \mathrm{GSp}_{2g}(\widehat{\mathbf{Z}})$, *i.e.*, $\Gamma^{(p)}$ contains

$$\Gamma(M) := \left\{ \boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\widehat{\mathbf{Z}}) : \boldsymbol{\gamma} \equiv \mathbb{1}_{2g} \mod M \right\}$$

for some $M \in \mathbf{Z}_{>0}$. We shall assume

$$\Gamma^{(p)} = \prod_{\ell\text{: prime}} \Gamma_\ell^{(p)}$$

for $\Gamma_\ell^{(p)} \subset \mathrm{GSp}_{2g}(\mathbf{Z}_\ell)$ such that $\Gamma_\ell^{(p)} = \mathrm{GSp}_{2g}(\mathbf{Z}_\ell)$ for almost all prime number ℓ. We denote by

$$\mathtt{S}_{\mathrm{bad}} = \{\ell : \Gamma_\ell^{(p)} \subsetneq \mathrm{GSp}_{2g}(\mathbf{Z}_\ell)\}$$

and write

$$N := \max\{M \in \mathbf{Z}_{\geq 0} : \Gamma(M) \subset \Gamma^{(p)}\}.$$

Given a congruence subgroup $\Gamma^{(p)} \subset \mathrm{GSp}_{2g}(\widehat{\mathbf{Z}})$, one can consider the locally symmetric space

$$X(\mathbf{C}) := \mathrm{GSp}_{2g}(\mathbf{Q})\backslash \mathbb{H}_g \times \mathrm{GSp}_{2g}(\mathbf{A}_f)/\Gamma^{(p)}\,\mathrm{GSp}_{2g}(\mathbf{Z}_p),$$

where

- $\mathbf{A}_f$ is the ring of finite adèles of $\mathbf{Q}$,
- $\mathrm{GSp}_{2g}(\mathbf{Q})$ acts on $\mathrm{GSp}_{2g}(\mathbf{A}_f)$ via the left multiplication and acts diagonally on $\mathbb{H}_g \times \mathrm{GSp}_{2g}(\mathbf{A}_f)$.

We assume further that

(i) $p \nmid N$ (so $p \notin \mathtt{S}_{\mathrm{bad}}$)

(ii) $\Gamma^{(p)}$ is chosen so that $X(\mathbf{C})$ is a smooth manifold.

2.2.2. Given $\Gamma^{(p)}$ as above, we fix a primitive N-th root of unity $\zeta_N \in \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$ and let $\mathbf{Sch}_{\mathbf{Z}_p[\zeta_N]}$ be the category of locally noetherian schemes over $\mathbf{Z}_p[\zeta_N]$. Then, the functor

$$\mathbf{Sch}_{\mathbf{Z}_p[\zeta_N]} \to \mathbf{Sets},$$
$$S \mapsto \left\{ (A_{/S}, \lambda, \psi_N) : \begin{array}{l} A \text{ is a principally polarised abelian scheme over } S \\ \lambda \text{ is a principal polarisation on } A \\ \psi_N \text{ is a level structure defined by } \Gamma^{(p)} \end{array} \right\} / \simeq .$$

is represented by a scheme $X_{\mathbf{Z}_p[\zeta_N]}$. It is well-known that the $\mathbf{C}$-point of $X_{\mathbf{Z}_p[\zeta_N]}$ can be identified with $X(\mathbf{C})$. Here, $\mathbf{C}$ is viewed as a $\mathbf{Z}_p[\zeta_N]$-algebra via the chosen isomorphism $\mathbf{C}_p \simeq \mathbf{C}$. For any $\mathbf{Z}_p[\zeta_N]$-algebra R, we write X_R for the base change

$$X_R := X_{\mathbf{Z}_p[\zeta_N]} \times_{\mathrm{Spec}\,\mathbf{Z}_p[\zeta_N]} \mathrm{Spec}\, R.$$

We refer X_R as the ***Siegel modular scheme of tame level*** $\Gamma^{(p)}$.

Example 2.2.3. Suppose $\Gamma^{(p)} = \Gamma(N) := \ker(\mathrm{GSp}_{2g}(\widehat{\mathbf{Z}}) \to \mathrm{GSp}_{2g}(\mathbf{Z}/N\mathbf{Z}))$ for N large enough, then $\Gamma(N)$ defines the level structure asking for symplectic isomorphisms,

$$\psi_N : A[N] \xrightarrow{\simeq} (\mathbf{Z}/N\mathbf{Z})^{2g},$$

i.e., isomorphisms that preserve the symplectic pairings on both sides up to units, where we consider the Weil pairing on the left-hand side and the symplectic pairing induced by (2.1) on the right-hand side.

2.2.4. We would like to consider Siegel modular varieties with level structures at p. Before defining these varieties, we fix a compatible system of p-power roots of unities $(\zeta_{p^m})_{m\in\mathbf{Z}_{>0}}$ in $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$.

- For each $m \in \mathbf{Z}_{>0}$, the ***Siegel modular variety of principal*** p^m***-level*** over $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$ is the algebraic variety $X_{\Gamma(p^m),\mathbf{Q}_p(\zeta_N,\zeta_{p^m})}$ over $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$ that represents the functor

$$\mathbf{SCH}_{\mathbf{Q}_p(\zeta_N,\zeta_{p^m})} \to \mathbf{SETS},$$
$$S \mapsto \left\{ (A, \lambda, \psi_N, \psi_{p^m}) : \begin{array}{c} (A, \lambda, \psi_N) \in X_{\mathbf{Z}_p[\zeta_N]}(S) \\ \psi_{p^m} : A[p^m] \xrightarrow{\sim} (\mathbf{Z}/p^m\mathbf{Z})^{2g} \\ \text{is a symplectic isomorphism} \end{array} \right\} / \simeq .$$

  Here, $\mathbf{SCH}_{\mathbf{Q}_p(\zeta_N,\zeta_{p^m})}$ is the category of locally noetherian schemes over $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$ and the symplectic isomorphism is taken with respect to the Weil pairing on the left-hand side and the symplectic pairing induced by (2.1) on the right-hand side. Again, for any $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$-algebra R, we denote by $X_{\Gamma(p^m),R}$ the base change of $X_{\Gamma(p^m),\mathbf{Q}_p(\zeta_N,\zeta_{p^m})}$ to R.

- The ***Siegel modular variety of Iwahori level*** over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ is the algebraic variety $X_{\mathrm{Iw},\mathbf{Q}_p(\zeta_N,\zeta_p)}$ over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ that represent the functor that assign each locally noetherian scheme S over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ to the set of tuples

$$(A_{/S}, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p]),$$

  where

  - $(A_{/S}, \lambda, \psi_N) \in X_{\mathbf{Z}_p[\zeta_N]}(S)$
  - $\mathrm{Fil}_\bullet A[p]$ is the full flag of $A[p]$ such that

$$(\mathrm{Fil}_\bullet A[p])^\perp \simeq \mathrm{Fil}_{2g-\bullet} A[p]$$

    with respect to the Weil pairing.

  Again, for any $\mathbf{Q}_p(\zeta_N, \zeta_p)$-algebra R, we denote by $X_{\mathrm{Iw},R}$ the base change of $X_{\mathrm{Iw},\mathbf{Q}_p(\zeta_N,\zeta_p)}$ to R.

- The ***Siegel modular variety of strict Iwahori level*** over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ is the algebraic variety $X_{\mathrm{Iw}^+,\mathbf{Q}_p(\zeta_N,\zeta_p)}$ over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ that represents the functor that assign each locally noetherian scheme S over $\mathbf{Q}_p(\zeta_N, \zeta_p)$ to the set of tuples

$$(A_{/S}, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], \{C_i : i = 1, ..., g\}),$$

  where

  - $(A_{/S}, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p]) \in X_{\mathrm{Iw},\mathbf{Q}_p(\zeta_N,\zeta_p)}(S)$
  - $\{C_i : i = 1, ..., g\}$ is a collection of subgroups $C_i \subset A[p]$ of order p such that

$$\mathrm{Fil}_i A[p] = \langle C_1, \dots, C_i \rangle$$

    for all $i = 1, \dots, g$. Again, for any $\mathbf{Q}_p(\zeta_N, \zeta_p)$-algebra R, we denote by $X_{\mathrm{Iw}^+,R}$ the base change of $X_{\mathrm{Iw}^+,\mathbf{Q}_p(\zeta_N,\zeta_p)}$ to R.

2.2.5. For any $m \in \mathbf{Z}_{>0}$ and any complete field $K \subset \mathbf{C}_p$ containing $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$, there are natural forgetful maps

$$X_{\Gamma(p^m),K} \to X_{\Gamma(p),K} \to X_{\mathrm{Iw}^+,K} \to X_{\mathrm{Iw},K} \to X_K,$$

where

- The first arrow sends $(A, \lambda, \psi_N, \psi_{p^n})$ to $(A, \lambda, \psi_N, p^{n-1}\psi_{p^n})$.
- The second arrow sends $(A, \lambda, \psi_N, \psi_p)$ to $(A, \lambda, \psi_N, \mathrm{Fil}^{\psi_p}_{\bullet} A[p], \{\langle \psi_p(e_i)\rangle : i = 1, ..., g\})$ where $\mathrm{Fil}^{\psi_p}_{\bullet} A[p]$ stands for the full flag

$$0 \subset \langle \psi_p(e_1)\rangle \subset \langle \psi_p(e_1), \psi_p(e_2)\rangle \subset \cdots \subset \langle \psi_p(e_1), \dots, \psi_p(e_{2g})\rangle.$$

- The third arrow sends $(A, \lambda, \psi_N, \mathrm{Fil}_{\bullet} A[p], \{C_i : i = 1, ..., g\})$ to $(A, \lambda, \psi_N, \mathrm{Fil}_{\bullet} A[p])$.
- The fourth arrow sends $(A, \lambda, \psi_N, \mathrm{Fil}_{\bullet} A[p])$ to (A, λ, ψ_N).

From the construction, we know that the morphisms

$X_{\Gamma(p^n),K} \to X_K$ is Galois with Galois group $\mathrm{GSp}_{2g}(\mathbf{Z}/p^n\mathbf{Z})$;

$X_{\Gamma(p),K} \to X_{\mathrm{Iw},K}$ is Galois with Galois group $B_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z})$;

$X_{\Gamma(p),K} \to X_{\mathrm{Iw}^+,K}$ is Galois with Galois group $B^+_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z})$.

Here,

$$B^+_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z}) := \left\{ \begin{pmatrix} \gamma_a & \gamma_b \\ & \gamma_d \end{pmatrix} \in B_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z}) : \boldsymbol{\gamma}_a \text{ is diagonal} \right\}.$$

Moreover, for $\Gamma \in \{\Gamma(p^m), \mathrm{Iw}^+ = \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, \mathrm{Iw} = \mathrm{Iw}_{\mathrm{GSp}_{2g}}\}$, the $\mathbf{C}$-point of $X_{\Gamma,K}$ can be similarly identified with the locally symmetric space

$$X_{\Gamma,K}(\mathbf{C}) = \mathrm{GSp}_{2g}(\mathbf{Q})\backslash \mathbb{H}_g \times \mathrm{GSp}_{2g}(\mathbf{A}_f)/\Gamma^{(p)}\Gamma.$$

Since this identification is independent to the choice of K, we will, from now on, use the symbol $X_\Gamma(\mathbf{C})$ to denote the locally symmetric space.

2.2.6. For any $m \in \mathbf{Z}_{>0}$ and any complete field extension K over $\mathbf{Q}_p$ containing $\mathbf{Q}_p(\zeta_N, \zeta_{p^m})$, our next goal is to construct *toroidal compactifications* $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ for each $X_{\Gamma,K}$ for $\Gamma \in \{\Gamma(p^m), \mathrm{Iw}^+, \mathrm{Iw}, \varnothing\}$ with the following properties

(Tor1) $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ is finite Kummer étale over $\overline{X}^{\mathrm{tor}}_{K}$;

(Tor2) There is a cartesian diagram

$$\begin{tikzcd} X_{\Gamma,K} \arrow[r, hook] \arrow[d] & \overline{X}^{\mathrm{tor}}_{\Gamma,K} \arrow[d] \\ X_K \arrow[r, hook] & \overline{X}^{\mathrm{tor}}_{K} \end{tikzcd}$$

and that the log structure on $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ is the divisorial log structure defined by the divisor $Z_{\Gamma,K} := \overline{X}^{\mathrm{tor}}_{\Gamma,K} \smallsetminus X_{\Gamma,K}$;

(Tor3) (i) If $\Gamma = \Gamma(p^n)$, then

$$\overline{X}^{\mathrm{tor}}_{\Gamma,K} \to \overline{X}^{\mathrm{tor}}_{K}$$

is Galois with Galois group $\mathrm{GSp}_{2g}(\mathbf{Z}/p^n\mathbf{Z})$.

(ii) If $\Gamma = \mathrm{Iw}$, then

$$\overline{X}^{\mathrm{tor}}_{\Gamma(p),K} \to \overline{X}^{\mathrm{tor}}_{\mathrm{Iw},K}$$

is Galois with Galois group $B_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z})$.

(iii) If $\Gamma = \mathrm{Iw}^+$, then

$$\overline{X}^{\mathrm{tor}}_{\Gamma(p),K} \to \overline{X}^{\mathrm{tor}}_{\mathrm{Iw}^+,K}$$

is Galois with Galois group $B^+_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\mathbf{Z})$.

2.2.7. In order to construct the toroidal compactifications, we set the following notations.

- We denote by $\mathfrak{C}$ the collection of all totally isotropic direct summands of $\mathbf{V}$.
- For any totally isotropic direct summand $\mathbf{V}' \subset \mathbf{V}$, let $C(\mathbf{V}/\mathbf{V}'^{\perp})$ denote the cone of symmetric bilinear forms on $(\mathbf{V}/\mathbf{V}'^{\perp})\otimes_{\mathbf{Z}}\mathbf{R}$ which are positive semi-definite and whose kernel is defined over $\mathbf{Q}$.
- Observe that if $\mathbf{V}', \mathbf{V}'' \in \mathfrak{C}$ such that $\mathbf{V}' \subset \mathbf{V}''$, there is a natural inclusion $C(\mathbf{V}/\mathbf{V}'^{\perp}) \subset C(\mathbf{V}/\mathbf{V}''^{\perp})$. We define

$$\mathcal{C} := \Big(\bigsqcup_{\mathbf{V}'\in\mathfrak{C}} C(\mathbf{V}/\mathbf{V}'^{\perp})\Big)/\sim$$

where the equivalence relation is given by the aforementioned inclusions.

- Let $\mathfrak{S}$ be a fixed $\mathrm{GSp}_{2g}(\mathbf{Z})$-admissible smooth rational polyhedral cone decomposition of $\mathcal{C}$ (see [Str10, Definition 3.2.3.1]). This means $\mathfrak{S}$ consists of a smooth rational polyhedral cone decomposition of $C(\mathbf{V}/\mathbf{V}'^{\perp})$ (in the sense of [FC90, Chapter IV, §2]) for every $\mathbf{V}' \in \mathfrak{C}$ such that

  (i) The decomposition of $C(\mathbf{V}/\mathbf{V}'^{\perp})$ coincides with the restriction of the decomposition of $C(\mathbf{V}/\mathbf{V}''^{\perp})$ whenever $\mathbf{V}' \subset \mathbf{V}''$, and

  (ii) $\mathfrak{S}$ is $\mathrm{GSp}_{2g}(\mathbf{Z})$-invariant and $\mathfrak{S}/\mathrm{GSp}_{2g}(\mathbf{Z})$ is a finite set.

- We will use the convention that $\Gamma(p^0) = \mathrm{GSp}_{2g}(\mathbf{Z}_p)$ and $X_{\mathrm{GSp}_{2g}(\mathbf{Z}_p),K} = X_K$. Moreover, for any $\Gamma \in \{\Gamma(p^m), \mathrm{Iw}^+, \mathrm{Iw}\}$, we write

$$\widetilde{\Gamma} := \mathrm{GSp}_{2g}(\mathbf{Z}) \cap \Gamma.$$

Given such data, we have a toroidal compactification $\overline{X}^{\mathrm{tor}}_{\mathbf{Z}_p[\zeta_N]}$ for $X_{\mathbf{Z}_p[\zeta_N]}$ (see [FC90; Pin88; Lan13]). Consequently, we have a toroidal compactification $\overline{X}^{\mathrm{tor}}_{K}$ for X_K by base change.

2.2.8. The construction of the toroidal compactification in the case $\Gamma = \Gamma(p^m)$ is well-known. We briefly review the construction of $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)}$ following [PS16]. In order to simplify the notation, we follow the strategy of *loc. cit.*, assuming $\Gamma^{(p)} = \mathrm{GSp}_{2g}(\mathbf{A}_f^p)$, where $\mathbf{A}_f^p$ is the ring of finite adèles away from p.

Notice that every $\sigma \in \mathfrak{S}$ necessarily lives in the interior of $C(\mathbf{V} \,/\, \mathbf{V}'^{\perp})$ for a unique $\mathbf{V}' \in \mathfrak{C}$ of some rank $r \leq g$. We have the following diagram from [PS16, 4.1.A]:

$$\begin{tikzcd} M_{\mathbf{V}',m} \arrow[r] \arrow[dr] & M_{\mathbf{V}',m,\sigma} \arrow[r] \arrow[d] & M_{\mathbf{V}',m,\mathfrak{S}} \arrow[dl] \\ & B_{\mathbf{V}',m} \arrow[d] & \\ & X_{\mathbf{V}',m} & \end{tikzcd}.$$

We briefly describe the objects in the diagram and refer the readers to [PS16, Appendice A] for details:

- Let $X_{0,\mathbf{V}'}$ be the moduli scheme parameterising principally polarised abelian schemes over $\mathcal{O}_K$ of dimension $g-r$, where $\mathcal{O}_K$ is the ring of integers of K. Let $X_{\mathbf{V}'}$ denote the base change of $X_{0,\mathbf{V}'}$ to K.

- Let $X_{\mathbf{V}',n}$ be the finite étale cover of $X_{\mathbf{V}'}$ parameterising principal p^m-level structures. Over $X_{\mathbf{V}',m}$, there is a universal abelian variety $A_{\mathbf{V}'}$.

- Roughly speaking, the algebraic variety $B_{\mathbf{V}',m}$ over $X_{\mathbf{V}',m}$ parameterises semiabelian varieties with '$\Gamma^{(p)}$- and p^m-level structures' where the semiabelian variety is an extension of $A_{\mathbf{V}'}$ by the torus $T_{\mathbf{V}'} := \mathbf{V}' \otimes_{\mathbf{Z}} \mathbb{G}_m$. In particular, over $B_{\mathbf{V}',m}$, there is a universal semiabelian variety
$$0 \to T_{\mathbf{V}'} \to G_{\mathbf{V}'} \to A_{\mathbf{V}'} \to 0$$
together with a universal isogeny of semiabelian varieties
$$\begin{tikzcd} T_{\mathbf{V}'} \arrow[r] \arrow[d, "\mathrm{id}"] & G_{\mathbf{V}'} \arrow[r] \arrow[d] & A_{\mathbf{V}'} \arrow[d, "p^m"] \\ T_{\mathbf{V}'} \arrow[r] & G_{\mathbf{V}'} \arrow[r] & A_{\mathbf{V}'} \end{tikzcd}$$
whose kernel induces a natural inclusion $A_{\mathbf{V}'}[p^m] \subset G_{\mathbf{V}'}[p^m]$. This yields a decomposition
$$G_{\mathbf{V}'}[p^m] \simeq (\mathbf{V}' \,/ p^m\, \mathbf{V}' \otimes \mu_{p^m}) \oplus A_{\mathbf{V}'}[p^m].$$

- Roughly speaking, the algebraic variety $M_{\mathbf{V}',m}$ over $B_{\mathbf{V}',m}$ parameterises principally polarised 1-motives of type $[\mathbf{V} \,/\, \mathbf{V}'^{\perp} \to G_{\mathbf{V}'}]$ together with a 'principal p^m-level structure'. In particular, over $M_{\mathbf{V}',m}$, there is a universal 1-motive

$$\widetilde{M}_{\mathbf{V}'} = [\mathbf{V} \,/\, \mathbf{V}'^{\perp} \to G_{\mathbf{V}'}]$$

together with a universal decomposition

$$\widetilde{M}_{\mathbf{V}'}[p^m] \simeq (\mathbf{V}' \,/ p^n\, \mathbf{V}' \otimes \mu_{p^m}) \oplus A_{\mathbf{V}'}[p^m] \oplus (\mathbf{V} \,/\, \mathbf{V}'^{\perp} \otimes \mathbf{Z} \,/ p^m\, \mathbf{Z}).$$

It turns out $M_{\mathbf{V}',m}$ is a torus over $B_{\mathbf{V}',m}$ with the torus

$$\mathrm{Hom}\left(\frac{1}{Np^m}\,\mathrm{Sym}^2(\mathbf{V} \,/\, \mathbf{V}'^{\perp}), \mathbb{G}_m\right).$$

- The morphism $M_{\mathbf{V}',m} \to M_{\mathbf{V}',m,\sigma}$ is the affine toroidal embedding attached to the cone $\sigma \in C(\mathbf{V} \,/\, \mathbf{V}'^{\perp})$. Let $Z_{\mathbf{V}',m,\sigma} := M_{\mathbf{V}',m,\sigma} \smallsetminus M_{\mathbf{V}',m}$ denote the closed stratum of $M_{\mathbf{V}',m,\sigma}$. Since σ uniquely determines $\mathbf{V}'$, we might simply write $Z_{m,\sigma}$.

- The morphism $M_{\mathbf{V}',m} \to M_{\mathbf{V}',m,\mathfrak{S}}$ is the toroidal embedding attached to the polyhedral decomposition $\mathfrak{S}$. Let $Z_{\mathbf{V}',m,\mathfrak{S}} := M_{\mathbf{V}',m,\mathfrak{S}} \smallsetminus M_{\mathbf{V}',m}$ denote the closed stratum of $M_{\mathbf{V}',m,\mathfrak{S}}$.

Theorem 2.2.9 ([PS16, Théorème 4.1])**.** *We have*

(i) The toroidal compactification $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)}$ admits a stratification indexed by the finite set $\mathfrak{S} \,/ \widetilde{\Gamma}(p^m)$. For any $\sigma \in \mathfrak{S}$, the corresponding stratum in $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)}$ is isomorphic to $Z_{V',m,\sigma}$.

(ii) The boundary $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)} \smallsetminus X_{\Gamma(p^m)}$ is given by a normal crossing divisor. The codimension-one strata $Z_{V',m,\sigma}$ are in bijection with the irreducible components of the normal crossing divisor. Such V' necessarily has rank 1.

(iii) The toroidal compactification is compatible with change of levels. In particular, there are natural finite morphisms $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)} \to \overline{X}^{\mathrm{tor}}_{\Gamma(p^n)}$ for $m \geq n$.

(iv) There is a natural action of $\mathrm{GSp}_{2g}(\mathbf{Z}_p)/\Gamma(p^m)$ on $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m)}$. It permutes the boundary strata accordingly.

2.2.10. We remark that the case for $\Gamma = \mathrm{Iw}$ is carefully studied in [Str10]. However, instead of following *loc. cit.*, we propose an alternative way to obtain $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ with the desired properties (Tor1), (Tor2) and (Tor3). To this end, we recall a theorem of K. Fujiwara and K. Kato ([Ill02, Theorem 7.6]):

Theorem 2.2.11 (Fujiwara–Kato)**.** *Let Y be a regular scheme, D an effective divisor of Y with normal crossing and $U := Y \smallsetminus D$. Equip Y with the divisorial log structure defined by D. Then, the restriction functor*

$$\left[\begin{array}{c}\textit{finite Kummer étale}\\ \textit{cover over } Y\end{array}\right] \to \left[\begin{array}{c}\textit{finite étale}\\ \textit{cover over } U\end{array}\right], \quad T \mapsto T \times_Y U$$

if fully faithful. The essential image of this functor consisting of those finite étale covers over U which are tamely ramified along D.

2.2.12. In particular, when Y is further a variety over a field of characteristic 0, every finite étale cover over U is tamely ramified along D. That is, one obtains an isomorphism between the finite Kummer étale site $Y_{\mathrm{fkét}}$ and the finite étale site $U_{\mathrm{fét}}$.

Proposition 2.2.13. *Let Γ denote either $\Gamma(p^m)$ (for some $m > 0$), Iw, or Iw^+. There exists a unique fs log scheme $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ over $\overline{X}^{\mathrm{tor}}_{K}$ satisfying (Tor1), (Tor2), and (Tor3).*

Proof. Recall that $\overline{X}^{\mathrm{tor}}_{K}$ is equipped with the divisorial log structure given by the boundary divisor $Z_K = \overline{X}^{\mathrm{tor}}_{K} \smallsetminus X_K$ of normal crossing (by [FC90, Chapter IV, Theorem 6.7 (1)]). Theorem 2.2.11 yields a unique log scheme $\overline{X}_{\Gamma,K}$, which is finite Kummer étale over $\overline{X}^{\mathrm{tor}}_{K}$, extending the finite étale morphism $X_{\Gamma,K} \to X_K$. This shows that $\overline{X}_{\Gamma,K}$ satisfies (Tor1) and (Tor2). By Abhyankar's lemma (see, for example, [SGA1, Proposition 5.2], [Stacks, Tag 0EYG], [Stacks, Tag 0EYH], [DLLZ19, Proposition 4.2.1]), the inverse image $Z_{\Gamma,K}$ of Z_K in $\overline{X}_{\Gamma,K}$ is a divisor with normal crossing. Hence, by applying a scheme-theoretic version of Lemma A.1.12, we conclude that $\overline{X}_{\Gamma,K}$ also satisfies (Tor3). □

Remark 2.2.14. When $\Gamma \in \{\Gamma(p^m), \mathrm{Iw}\}$, one should ask whether our construction of $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ coincides with the ones constructed in [PS16] and [Str10]. The answer to this question is affirmative. Indeed, when $\Gamma = \Gamma(p^m)$, [FC90, Chapter IV, Theorem 6.7(6)] implies that $\overline{X}^{\mathrm{tor}}_{\Gamma(p^m),K}$ is finite Kummer étale over $\overline{X}^{\mathrm{tor}}_{K}$ with Galois group $\mathrm{GSp}_{2g}(\mathbf{Z}/p^n\,\mathbf{Z})$. The uniqueness of $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$ then yields the identification. For $\Gamma = \mathrm{Iw}$, it follows similarly by applying [Str10, Théorème 3.2.7.1].

2.2.15. To wrap this section, we pass to the realm of adic spaces. For $\Gamma \in \{\Gamma(p^m), \mathrm{Iw}^+, \mathrm{Iw}\}$, let $\mathcal{X}_{\Gamma,K}$ (resp., $\overline{\mathcal{X}}_{\Gamma,K}$) denote the adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ associated with $X_{\Gamma,K}$ (resp., $\overline{X}^{\mathrm{tor}}_{\Gamma,K}$). In particular, we refer $\overline{\mathcal{X}}_{\Gamma,K}$ as the *toroidal compactification* of $\mathcal{X}_{\Gamma,K}$ determined by the fixed polyheral decomposition $\mathfrak{S}$. It satisfies the following analogues of (Tor1), (Tor2), and (Tor3):

(Tor1') The log adic space $\overline{\mathcal{X}}_{\Gamma,K}$, equipped with the divisorial log structure given by the boundary divisor $\mathcal{Z}_{\Gamma,K} = \overline{\mathcal{X}}_{\Gamma,K} \smallsetminus \mathcal{X}_{\Gamma,K}$, is finite Kummer étale over $\overline{\mathcal{X}}_{K}$;

(Tor2') There is a cartesian diagram

$$\begin{tikzcd} \mathcal{X}_{\Gamma,K} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\Gamma,K} \arrow[d] \\ \mathcal{X}_{K} \arrow[r, hook] & \overline{\mathcal{X}}_{K} \end{tikzcd}$$

(Tor3') (i) If $\Gamma = \Gamma(p^n)$, then

$$\overline{\mathcal{X}}_{\Gamma,K} \to \overline{\mathcal{X}}_{K}$$

is Galois with Galois group $\mathrm{GSp}_{2g}(\mathbf{Z}/p^n\,\mathbf{Z})$.

(ii) If $\Gamma = \mathrm{Iw}$, then
$$\overline{\mathcal{X}}_{\Gamma(p),K} \to \overline{\mathcal{X}}_{\mathrm{Iw},K}$$
is Galois with Galois group $B_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\,\mathbf{Z})$.

(iii) If $\Gamma = \mathrm{Iw}^+$, then
$$\overline{\mathcal{X}}_{\Gamma(p),K} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,K}$$
is Galois with Galois group $B^+_{\mathrm{GSp}_{2g}}(\mathbf{Z}/p\,\mathbf{Z})$.

2.3 The toroidally compactified Siegel modular variety at infinite level

2.3.1. Let $\mathfrak{X}$ (resp., $\overline{\mathfrak{X}}$) be the completion of $X_{\mathcal{O}_{\mathbf{C}_p}}$ (resp., $\overline{X}^{\mathrm{tor}}_{\mathcal{O}_{\mathbf{C}_p}}$) along its special fibre. For any $m \in \mathbf{Z}_{>0}$, we denote by $\mathfrak{X}_{\Gamma(p^m)}$ (resp., $\overline{\mathfrak{X}}_{\Gamma(p^m)}$) the normalisation of $\mathfrak{X}$ (resp., $\overline{\mathfrak{X}}$) inside $\mathcal{X}_{\Gamma(p^m)}$ (resp., $\overline{\mathcal{X}}_{\Gamma(p^m)}$). In order to work with the toroidal compactification at the infinite level, the authors of [PS16] consider modified versions $\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)}$ of the formal schemes $\overline{\mathfrak{X}}_{\Gamma(p^m)}$, which we briefly recall.

Let $m \in \mathbf{Z}_{\geq 0}$ and let $\mathfrak{G}$ be the tautological semiabelian scheme over $\overline{\mathfrak{X}}_{\Gamma(p^m)}$, *i.e.*, the pullback of the tautological semiabelian scheme over $\overline{\mathfrak{X}}$. Let
$$\pi : \mathfrak{G} \to \overline{\mathfrak{X}}_{\Gamma(p^m)}$$
be the natural projection and let
$$\underline{\Omega}_{\Gamma(p^m)} := \pi_* \Omega^1_{\mathfrak{G}/\overline{\mathfrak{X}}_{\Gamma(p^m)}}.$$
Over $\mathfrak{X}_{\Gamma(p^m)}$, composing the universal trivialisation
$$\psi_{p^m} : \mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^m\,\mathbf{Z}) \to \mathfrak{G}[p^m]$$
(which becomes isomorphism on the rigid generic fibre), the duality $\mathfrak{G}[p^m] \simeq \mathfrak{G}[p^m]^\vee$, and the Hodge–Tate map
$$\mathfrak{G}[p^m]^\vee \to \underline{\Omega}_{\Gamma(p^m)}/p^m \underline{\Omega}_{\Gamma(p^m)}$$
we obtain
$$\mathrm{HT}_{\Gamma(p^m)} : \mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^m\,\mathbf{Z}) \to \underline{\Omega}_{\Gamma(p^m)}/p^m \underline{\Omega}_{\Gamma(p^m)}$$
which induces
$$\mathrm{HT}_{\Gamma(p^m)} \otimes \mathrm{id} : \big(\mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^m\,\mathbf{Z})\big) \otimes_{\mathbf{Z}} \mathscr{O}_{\mathfrak{X}_{\Gamma(p^m)}} \to \underline{\Omega}_{\Gamma(p^m)}/p^m \underline{\Omega}_{\Gamma(p^m)}.$$
According to [PS16, Proposition 1.2], this map extends to the toroidal compactification:
$$\mathrm{HT}_{\Gamma(p^m)} \otimes \mathrm{id} : \big(\mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^m\,\mathbf{Z})\big) \otimes_{\mathbf{Z}} \mathscr{O}_{\overline{\mathfrak{X}}_{\Gamma(p^m)}} \to \underline{\Omega}_{\Gamma(p^m)}/p^m \underline{\Omega}_{\Gamma(p^m)}. \tag{2.2}$$
More precisely, in terms of the explicit description in 2.2.8, étale locally at the boundary

stratum, there is a universal semiabelian scheme $G_{\mathbf{V}'}$ with constant toric rank sitting in an exact sequence

$$0 \to T_{\mathbf{V}'} \to G_{\mathbf{V}'} \to A_{\mathbf{V}'} \to 0$$

as well as a principally polarised 1-motive $\widetilde{M}_{\mathbf{V}'} = [\mathbf{V}'^{\perp} / \mathbf{V}' \to G_{\mathbf{V}'}]$. We consider the composition

$$\widetilde{M}_{\mathbf{V}'}[p^m] \simeq \widetilde{M}_{\mathbf{V}'}[p^m]^{\vee} \twoheadrightarrow G_{\mathbf{V}'}[p^m]^{\vee} \xrightarrow{\mathrm{HT}_{G_{\mathbf{V}'}[p^m]^{\vee}}} \underline{\omega}_{G_{\mathbf{V}'}}/p^m,$$

where the first isomorphism is given by the principal polarisation on $\widetilde{M}_{\mathbf{V}'}$. Composing with the universal trivialisation of $\widetilde{M}_{\mathbf{V}'}[p^m]$ and tensoring with the structure sheaf, we arrive at the desired morphism (2.2).

Consider the image of $\mathrm{HT}_{\Gamma(p^m)} \otimes \mathrm{id}$ and then consider its preimage inside $\underline{\Omega}_{\Gamma(p^m)}$. This yields a subsheaf $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^m)} \subset \underline{\Omega}_{\Gamma(p^m)}$. In fact, $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^m)}$ does not depend on m; *i.e.*, if $n \geq m$ and $\overline{\mathfrak{X}}_{\Gamma(p^n)} \to \overline{\mathfrak{X}}_{\Gamma(p^m)}$ is the natural projection, then the pullback of $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^m)}$ coincides with $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^n)}$.

Now, let m be any positive integer greater than $\frac{g}{p-1}$. Consider ideals $\mathscr{I}_1, \ldots, \mathscr{I}_g \subset \mathscr{O}_{\overline{\mathfrak{X}}_{\Gamma(p^m)}}$ generated by the lifts of the determinants of the minors of rank $g, \ldots, 1$ of the map

$$\mathrm{HT}_{\Gamma(p^m)} \otimes \mathrm{id} : \big(V \otimes_{\mathbf{Z}} (\mathbf{Z} / p^m \, \mathbf{Z})\big) \otimes_{\mathbf{Z}} \mathscr{O}_{\overline{\mathfrak{X}}_{\Gamma(p^m)}} \to \underline{\Omega}_{\Gamma(p^m)}/p^m \underline{\Omega}_{\Gamma(p^m)}.$$

Notice that these ideals are invertible on the rigid generic fibre. Let $\widetilde{\mathfrak{X}}_{\Gamma(p^m)}$ be the formal scheme obtained by consecutive formal blowups of $\overline{\mathfrak{X}}_{\Gamma(p^m)}$ along these ideals. In particular, $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^m)}$ becomes locally free over $\widetilde{\mathfrak{X}}_{\Gamma(p^m)}$.

Finally, let $\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)}$ be the normalisation of $\widetilde{\mathfrak{X}}_{\Gamma(p^m)}$ inside its adic generic fibre. We remark that the adic generic fibre of $\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)}$ coincides with the one of $\overline{\mathfrak{X}}_{\Gamma(p^m)}$. For any $m \geq n > \frac{g}{p-1}$, there is a natural finite morphism

$$\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)} \to \overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^n)}.$$

2.3.2. As the adic generic fibre of $\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)}$ coincides with $\overline{\mathcal{X}}_{\Gamma(p^m),\mathbf{C}_p}$, the locally free sheaf $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^m),\mathbf{C}_p}$ gives rise to a locally free $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^m),\mathbf{C}_p}}$-module $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^m)}$ on $\overline{\mathcal{X}}_{\Gamma(p^m),\mathbf{C}_p}$. Inverting p, we obtain the locally free $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^m),\mathbf{C}_p}}$-module $\underline{\omega}_{\Gamma(p^m)}$. Notice that $\underline{\omega}_{\Gamma(p^m)}$ is just the usual sheaf of invariant differentials defined using the universal semiabelian varieties.

Consider the projective limit

$$\overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^\infty)} := \varprojlim \overline{\mathfrak{X}}^{\mathrm{mod}}_{\Gamma(p^m)}$$

in the category of p-adic formal schemes. Let $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ be its adic generic fibre in the sense of [SW13]. The following proposition shows that $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is the desired perfectoid Siegel modular variety at the infinite level.

Proposition 2.3.3 ([PS16, Proposition 4.9 & Corollaire 4.14]). *We have*

(i) The adic generic fibre $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is a perfectoid space such that

$$\overline{\mathcal{X}}_{\Gamma(p^\infty)} \sim \varprojlim_n \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}$$

in the sense of [SW13, Definition 2.4.1].

(ii) For every $n \in \mathbf{Z}_{\geq 0}$, the natural morphism

$$\overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}$$

is a pro-Kummer étale Galois cover with Galois group $\Gamma(p^n)$. (Here we have abused the notation and identify $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ with the object $\varprojlim_n \overline{\mathcal{X}}_{\Gamma(p^n)}$ in the pro-Kummer étale site.) Simiarly, the natural morphism

$$\overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p} \quad (\textit{resp.}, \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p})$$

is a pro-Kummer étale Galois cover with Galois group $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$ (resp., $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$).

Remark 2.3.4. Induced from the stratification on the finite levels, the perfectoid Siegel modular variety $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ admits a stratification by the profinite set

$$\hat{\mathfrak{S}} := \varprojlim_n \mathfrak{S} / \widetilde{\Gamma}(p^n).$$

For each $\hat{\sigma} = (\sigma_n)_{n\geq 0} \in \hat{\mathfrak{S}}$, the $\hat{\sigma}$-stratum is canonically isomorphic to

$$\mathcal{Z}_{\infty,\hat{\sigma}} := \varprojlim_n \mathcal{Z}_{n,\sigma_n}$$

where $\mathcal{Z}_{n,\sigma_n}$ is the adic spaces given by the analytification of Z_{n,σ_n} over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$

2.3.5. We summarise the discussion above in the following commutative diagram

$$\begin{tikzcd}
\mathcal{X}_{\Gamma(p^\infty)} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\Gamma(p^\infty)} \arrow[d] \\
\mathcal{X}_{\Gamma(p^n),\mathbf{C}_p} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p} \arrow[d] \\
\mathcal{X}_{\Gamma(p),\mathbf{C}_p} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\Gamma(p),\mathbf{C}_p} \arrow[d] \\
\mathcal{X}_{\mathrm{Iw}^+,\mathbf{C}_p} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p} \arrow[d] \\
\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p} \arrow[d] \\
\mathcal{X}_{\mathbf{C}_p} \arrow[r, hook] & \overline{\mathcal{X}}_{\mathbf{C}_p}
\end{tikzcd} .$$

where $\mathcal{X}_{\Gamma(p^\infty)}$ is the part of $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ away from the boundary.

There is a natural $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$-action on $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ permuting the p-power level structures. In particular, the chain of natural projections

$$\overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p} \to \overline{\mathcal{X}}_{\Gamma(p),\mathbf{C}_p} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p} \to \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p} \to \overline{\mathcal{X}}_{\mathbf{C}_p}$$

is $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$-equivariant. We name the natural projections

$$h_{\Gamma(p^n)} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}, \quad h_{\mathrm{Iw}^+} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p} \quad \text{and} \quad h_{\mathrm{Iw}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}$$

By Proposition 2.3.3, the following lemma is expected.

Lemma 2.3.6. *We have the following identities of sheaves*

$$\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}} = \left(h_{\mathrm{Iw},*}\,\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}, \quad \mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}} = \left(h_{\mathrm{Iw},*}\,\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}$$
$$\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p}} = \left(h_{\mathrm{Iw}^+,*}\,\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}\right)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}, \quad \mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p}} = \left(h_{\mathrm{Iw},*}\,\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}\right)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}.$$

Proof. We give the proof of the first pair of identities. The second pair can be proven by the same argument.

It suffices to prove the first identity. Let $\mathcal{V} \subset \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}$ be an affinoid open subsapce. For every $n \in \mathbf{Z}_{\geq 1}$, let $\mathcal{V}_n$ be the preimage of $\mathcal{V}$ in $\overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}$ and consider the object $\widetilde{\mathcal{V}}_\infty := \varprojlim_n \mathcal{V}_n$ in the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}$. By Lemma A.1.12, each $\mathcal{V}_n$ is finite Kummer étale over $\mathcal{V}$ with Galois group $G_n := \mathrm{Iw}_{\mathrm{GSp}_{2g}}/\Gamma(p^n)$. Thus,

$$\mathscr{O}^+_{\mathcal{X}_{\Gamma(p^\infty)}}(\mathcal{V}_\infty) = \left(\varinjlim_n \mathscr{O}^+_{\mathcal{X}_{\Gamma(p^n),\mathbf{C}_p}}(\mathcal{V}_n)\right)^\wedge = \left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)\right)^\wedge,$$

where '$\wedge$' stands for the p-adic completion. By [DLLZ19, Lemma 4.1.7 & Corollary 4.4.13], we know

$$\left(\mathscr{O}^+_{\mathcal{X}_{\Gamma(p^n),\mathbf{C}_p}}(\mathcal{V}_n)/p^m\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} = \left(\mathscr{O}^+_{\mathcal{X}_{\Gamma(p^n),\mathbf{C}_p}}(\mathcal{V}_n)/p^m\right)^{G_n} = \mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p}}(\mathcal{V})/p^m$$

for every $m \in \mathbf{Z}_{\geq 1}$. This implies

$$\left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} = \mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p}}(\mathcal{V})/p^m.$$

Note that

$$\left(\varprojlim_m \left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} = \varprojlim_m \left(\left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}\right).$$

Indeed, the inclusion from the left-hand side to the right-hand side is clear. To show the other inclusion, take any $(x_m)_m \in \left(\varprojlim_m \left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}$, then for any $\boldsymbol{\gamma} \in \mathrm{Iw}_{\mathrm{GSp}_{2g}}$,

$$(x_m)_m = \boldsymbol{\gamma}^*(x_m)_m = (\boldsymbol{\gamma}^* x_m)_m.$$

Here, the last equation follows from that each $\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m$ is equipped with the discrete topology and $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$ acts continuously on the projective limit (with respect to the p-adic topology). Projecting back to each $\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m$, we see that $x_m \in \left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}$. Consequently, we have

$$\begin{aligned}
\left(\mathscr{O}^+_{\mathcal{X}_{\Gamma(p^\infty)}}(\mathcal{V}_\infty)\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} &= \left(\left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)\right)^\wedge\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} \\
&= \left(\varprojlim_m \left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}} \\
&= \varprojlim_m \left(\left(\mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p,\mathrm{prok\acute{e}t}}}(\widetilde{\mathcal{V}}_\infty)/p^m\right)^{\mathrm{Iw}_{\mathrm{GSp}_{2g}}}\right) \\
&= \varprojlim_m \mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p}}(\mathcal{V})/p^m \\
&= \mathscr{O}^+_{\mathcal{X}_{\mathrm{Iw},\mathbf{C}_p}}(\mathcal{V}).
\end{aligned}$$

$\square$

2.4 Flag varieties

2.4.1. One of the important feature of the perfectoid toroidally compactified Siegel modular variety at infinite level is the so-called *Hodge–Tate period map*. Before defining this map, we first describe the target flag variety (and its variants) carefully.

Recall that $\mathbf{V}_p = V \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is the standard symplectic space of rank $2g$ over $\mathbf{Z}_p$. Let P_{Siegel} be the (opposite) Siegel parabolic subgroup of GSp_{2g} defined by

$$P_{\mathrm{Siegel}} := \begin{pmatrix} \mathrm{GL}_g & \\ M_g & \mathrm{GL}_g \end{pmatrix} \cap \mathrm{GSp}_{2g}\,.$$

Let $\mathrm{Fl} := P_{\mathrm{Siegel}} \backslash \mathrm{GSp}_{2g}$ be the flag variety over $\mathbf{Z}_p$, parameterising the maximal lagrangians $W \subset \mathbf{V}_p$.[1] There is a natural action of GSp_{2g} on Fl by right multiplication. Let $\mathscr{F}\!\ell$ be the associated adic space of Fl over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, equipped with the induced right action of $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$. Hence, for any p-adically complete sheafy $(\mathbf{Q}_p, \mathbf{Z}_p)$-algebra (R, R^+), $\mathscr{F}\!\ell(R, R^+)$ parameterises maximal lagrangians $W \subset \mathbf{V}_p \otimes_{\mathbf{Z}_p} R$.

2.4.2. Consider the open subset $\mathscr{F}\!\ell^{\times} \subset \mathscr{F}\!\ell$ whose (R, R^+)-points are

$$\mathscr{F}\!\ell^{\times}(R, R^+) = \left\{ (W \subset \mathbf{V}_p \otimes_{\mathbf{Z}_p} R) \in \mathscr{F}\!\ell(R, R^+) : \begin{array}{l} \text{there exists a basis } \{w_i\} \text{ of } W \text{ such that} \\ \text{the matrix } (\langle\, w_i, e_{2g+1-j} \,\rangle)_{1\leq i,j\leq g} \text{ is invertible} \end{array} \right\}.$$

For any $\boldsymbol{x}_W = (W \subset \mathbf{V}_p \otimes_{\mathbf{Z}_p} R) \in \mathscr{F}\!\ell^{\times}(R, R^+)$, there exists a unique basis $\{w_i^{\square}\}$ of W such that

$$(\langle\, w_i^{\square}, e_{2g+1-j} \,\rangle)_{1\leq i,j\leq g} = \mathbb{1}_g\,.$$

Therefore, there exist global sections $\boldsymbol{z}_{i,j} \in \mathscr{O}_{\mathscr{F}\!\ell^{\times}}(\mathscr{F}\!\ell^{\times})$ such that for any $\boldsymbol{x}_W \in \mathscr{F}\!\ell^{\times}(R, R^+)$,

$$w_i^{\square} = e_i + \sum_{j=1}^{g} \boldsymbol{z}_{i,j}(\boldsymbol{x}_W) e_{g+j}.$$

Since $\langle\, w_i^{\square}, w_j^{\square} \,\rangle = 0$, we have

$$\begin{aligned} 0 &= \langle\, w_i^{\square}, w_j^{\square} \,\rangle \\ &= \langle\, e_i, \sum_{k=1}^{g} \boldsymbol{z}_{j,k}(\boldsymbol{x}_W) e_{g+k} \,\rangle + \langle\, \sum_{k=1}^{g} \boldsymbol{z}_{i,k}(\boldsymbol{x}_W) e_{g+k}, e_j \,\rangle \\ &= \boldsymbol{z}_{j,g+1-i}(\boldsymbol{x}_W) - \boldsymbol{z}_{i,g+1-j}(\boldsymbol{x}_W). \end{aligned}$$

[1] In fact, we should use $P_{\mathrm{Siegel}}^{\mathrm{opp}} := \begin{pmatrix} \mathrm{GL}_g & M_g \\ & \mathrm{GL}_g \end{pmatrix} \cap \mathrm{GSp}_{2g}$ to define the flag variety. We apologise for using P_{Siegel} instead of $P_{\mathrm{Siegel}}^{\mathrm{opp}}$ due to some computational convenience in the later context. Here, we identify $P_{\mathrm{Siegel}}^{\mathrm{opp}}$ with P_{Siegel} via

$$P_{\mathrm{Siegel}}^{\mathrm{opp}} \to P_{\mathrm{Siegel}}, \quad \gamma \mapsto {}^{\mathrm{t}}\gamma^{-1}\,.$$

That is, the matrix

$$\boldsymbol{z} := \begin{pmatrix} \boldsymbol{z}_{1,1} & \cdots & \boldsymbol{z}_{1,g} \\ \vdots & & \vdots \\ \boldsymbol{z}_{g,1} & \cdots & \boldsymbol{z}_{g,g} \end{pmatrix}$$

is symmetric with respect to the anti-diagonal. Moreover, we may use the matrix $\begin{pmatrix} \mathbb{1}_g & \boldsymbol{z}(\boldsymbol{x}_W) \end{pmatrix}$ (or just the matrix $\boldsymbol{z}(\boldsymbol{x}_W)$) to represent the element $\boldsymbol{x}_W \in \mathcal{F}\ell^\times(R, R^+)$ because the basis $\{w_i^\square\}$ is represented by the matrix

$$\begin{pmatrix} 1 & & & \boldsymbol{z}_{1,1}(\boldsymbol{x}_W) & \cdots & \boldsymbol{z}_{1,g}(\boldsymbol{x}_W) \\ & \ddots & & \vdots & & \vdots \\ & & 1 & \boldsymbol{z}_{g,1}(\boldsymbol{x}_W) & \cdots & \boldsymbol{z}_{g,g}(\boldsymbol{x}_W) \end{pmatrix} = \begin{pmatrix} \mathbb{1}_g & \boldsymbol{z}(\boldsymbol{x}_W) \end{pmatrix}$$

with respect to the standard basis $e_1, \dots, e_{2g}$ of $\mathbf{V}_p$.

2.4.3. In the rest of the thesis, we take base change of the adic spaces $\mathcal{F}\ell$ and $\mathcal{F}\ell^\times$ to $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$.

For every $w \in \mathbf{Q}_{>0}$, consider an open adic subspace $\mathcal{F}\ell^\times_w \subset \mathcal{F}\ell^\times$ defined by

$$\mathcal{F}\ell^\times_w := \left\{ \boldsymbol{x} \in \mathcal{F}\ell^\times : \max_{i,j} \inf_{h \in \mathbf{Z}_p} \{ |\, \boldsymbol{z}_{i,j}(\boldsymbol{x}) - h| \} \leq p^{-w} \right\}.$$

For any algebraically closed complete nonarchimedean field C containing $\mathbf{Q}_p$, let

$$\mathrm{GSp}_{2g,w}(C) := \left\{ \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{GSp}_{2g}(C) : \begin{array}{l} \boldsymbol{\gamma}_a \in \mathrm{GL}_g(C), \text{ and} \\ \max_{i,j} \inf_{h \in \mathbf{Z}_p} \{ |(\boldsymbol{\gamma}_a^{-1}\boldsymbol{\gamma}_b)_{ij} - h| \} \leq p^{-w} \end{array} \right\}$$

where $(\boldsymbol{\gamma}_a^{-1}\boldsymbol{\gamma}_b)_{ij}$ is the (i,j)-th entry of the matrix $\boldsymbol{\gamma}_a^{-1}\boldsymbol{\gamma}_b$. Then the $(C, \mathcal{O}_C)$-points of $\mathcal{F}\ell^\times_w$ can be identified with the quotient

$$\mathcal{F}\ell^\times_w(C, \mathcal{O}_C) = P_{\mathrm{Siegel}}(C) \backslash \mathrm{GSp}_{2g,w}(C)$$

so that the natural inclusion $\mathcal{F}\ell^\times_w(C, \mathcal{O}_C) \subset \mathcal{F}\ell(C, \mathcal{O}_C)$ is induced by

$$\mathcal{F}\ell^\times_w(C, \mathcal{O}_C) = P_{\mathrm{Siegel}}(C) \backslash \mathrm{GSp}_{2g,w}(C) \hookrightarrow P_{\mathrm{Siegel}}(C) \backslash \mathrm{GSp}_{2g}(C) = \mathcal{F}\ell(C, \mathcal{O}_C).$$

Recall that there is a natural right action of $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$ on $\mathcal{F}\ell$. The following lemma shows that $\mathcal{F}\ell^\times_w$ is stable under the action of the subgroup $\mathrm{Iw}_{\mathrm{GSp}_{2g}} \subset \mathrm{GSp}_{2g}(\mathbf{Q}_p)$.

Lemma 2.4.4. *The adic space $\mathcal{F}\ell^\times_w$ is stable under the right action of $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$. Coordinate-wise, the action is given by*

$$\mathcal{F}\ell^\times_w \times \mathrm{Iw}_{\mathrm{GSp}_{2g}} \to \mathcal{F}\ell^\times_w, \quad \left(\boldsymbol{z}, \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \right) \mapsto (\boldsymbol{\gamma}_a + \boldsymbol{z}\boldsymbol{\gamma}_c)^{-1}(\boldsymbol{\gamma}_b + \boldsymbol{z}\boldsymbol{\gamma}_d).$$

In particular, $\mathcal{F}\ell^\times_w$ is also stable under the right action of the subgroup $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$.

Proof. It follows from the definition that the right action of $\boldsymbol{\gamma} \in \mathrm{Iw}_{\mathrm{GSp}_{2g}}$ indeed sends

$\begin{pmatrix}\mathbb{1}_g & \boldsymbol{z}(\boldsymbol{x}_W)\end{pmatrix}$ to

$$\begin{pmatrix}\mathbb{1}_g & (\boldsymbol{\gamma}_a+\boldsymbol{z}(\boldsymbol{x}_W)\,\boldsymbol{\gamma}_c)^{-1}(\boldsymbol{\gamma}_b+\boldsymbol{z}(\boldsymbol{x}_W)\,\boldsymbol{\gamma}_d)\end{pmatrix} = (\boldsymbol{\gamma}_a+\boldsymbol{z}(\boldsymbol{x}_W)\,\boldsymbol{\gamma}_c)^{-1}\begin{pmatrix}\mathbb{1}_g & \boldsymbol{z}(\boldsymbol{x}_W)\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix}.$$

It remains to show that, for every $\boldsymbol{x}_W \in \mathcal{F}\ell_w^\times$, the matrix $(\boldsymbol{\gamma}_a+\boldsymbol{z}(\boldsymbol{x}_W)\,\boldsymbol{\gamma}_c)^{-1}(\boldsymbol{\gamma}_b+\boldsymbol{z}(\boldsymbol{x}_W)\,\boldsymbol{\gamma}_d)$ lands in $\mathcal{F}\ell_w^\times$. But this is straightforward. □

2.4.5. For latter usage, we would like to understand certain vector bundle on $\mathcal{F}\ell$ and $\mathcal{F}\ell_w^\times$. To this end, let $\mathscr{W}_{\mathrm{Fl}} \subset \mathscr{O}_{\mathrm{Fl}}^{2g}$ be the universal maximal lagrangian over Fl. The total space of $\mathscr{W}_{\mathrm{Fl}}$ can be naturally identified with

$$\mathscr{W}_{\mathrm{Fl}} \simeq P_{\mathrm{Siegel}}\backslash(\mathbb{A}^g \times \mathrm{GSp}_{2g})$$

where

- by viewing elements $\vec{v} \in \mathbb{A}^g$ as column vectors, P_{Siegel} acts on $\mathbb{A}^g$ from the left via $\boldsymbol{\gamma} * \vec{v} = {}^{\mathtt{t}}\boldsymbol{\gamma}_a^{-1}\,\vec{v}$, for any $\boldsymbol{\gamma} = \begin{pmatrix}\gamma_a & \gamma_b\\ \gamma_c & \gamma_d\end{pmatrix} \in P_{\mathrm{Siegel}}$;
- P_{Siegel} acts on GSp_{2g} via the left multiplication.

Similarly, consider the linear dual $\mathscr{W}_{\mathrm{Fl}}^\vee$ of $\mathscr{W}_{\mathrm{Fl}}$. Then the total space of $\mathscr{W}_{\mathrm{Fl}}^\vee$ can be naturally identified with

$$\mathscr{W}_{\mathrm{Fl}}^\vee \simeq P_{\mathrm{Siegel}}\backslash(\mathbb{A}^g \times \mathrm{GSp}_{2g})$$

where, by viewing elements $\vec{v} \in \mathbb{A}^g$ as row vectors, P_{Siegel} acts on $\mathbb{A}^g$ from the left via $\boldsymbol{\gamma} * \vec{v} = \vec{v}\,{}^{\mathtt{t}}\boldsymbol{\gamma}_a$, for any $\boldsymbol{\gamma} = \begin{pmatrix}\gamma_a & \gamma_b\\ \gamma_c & \gamma_d\end{pmatrix} \in P_{\mathrm{Siegel}}$. Under this identification, global sections of $\mathscr{W}_{\mathrm{Fl}}^\vee$ are identified with

$$\left\{\text{algebraic functions } \phi : \mathrm{GSp}_{2g} \to \mathbb{A}^g : \phi(\boldsymbol{\gamma}\,\boldsymbol{\alpha}) = \phi(\boldsymbol{\alpha})\cdot {}^{\mathtt{t}}\boldsymbol{\gamma}_a,\ \forall\,\boldsymbol{\gamma} \in P_{\mathrm{Siegel}},\ \boldsymbol{\alpha} \in \mathrm{GSp}_{2g}\right\}.$$

For every $i = 1, \dots, g$, we consider a global section $\boldsymbol{s}_i$ of $\mathscr{W}_{\mathrm{Fl}}^\vee$ defined by

$$\boldsymbol{s}_i(\boldsymbol{\alpha}) := \text{the } i\text{-th row of } {}^{\mathtt{t}}\boldsymbol{\alpha}_a$$

for all $\boldsymbol{\alpha} = \begin{pmatrix}\boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b\\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d\end{pmatrix} \in \mathrm{GSp}_{2g}$. If we write

$$\boldsymbol{s} := \begin{pmatrix}\boldsymbol{s}_1\\ \vdots\\ \boldsymbol{s}_g\end{pmatrix} \in (\mathscr{W}_{\mathrm{Fl}}^\vee)^g$$

then we have $\boldsymbol{s}(\boldsymbol{\alpha}) = {}^{\mathtt{t}}\boldsymbol{\alpha}_a$.

By passing to the adic space $\mathcal{F}\ell$ and restricting to $\mathcal{F}\ell_w^\times$, the (algebraic) sheaves $\mathscr{W}_{\mathrm{Fl}}$ and $\mathscr{W}_{\mathrm{Fl}}^\vee$ yield (analytic) sheaves $\mathscr{W}_{\mathcal{F}\ell_w^\times}$ and $\mathscr{W}_{\mathcal{F}\ell_w^\times}^\vee$ on $\mathcal{F}\ell_w^\times$. We still use $\boldsymbol{s}_i$'s to denote the restrictions on $\mathcal{F}\ell_w^\times$ of the corresponding algebraic sections. By definition, the sections $\boldsymbol{s}_i$'s are non-vanishing on $\mathcal{F}\ell_w^\times$ and hence $\boldsymbol{s}_i^\vee$'s are well-defined sections on $\mathscr{W}_{\mathcal{F}\ell_w^\times}$. We similarly write

$$\boldsymbol{s}^\vee := \begin{pmatrix} \boldsymbol{s}_1^\vee & \cdots & \boldsymbol{s}_g^\vee \end{pmatrix} \in (\mathscr{W}_{\mathcal{F}\ell_w^\times})^g.$$

Moreover, the right action of $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$ on $\mathcal{F}\ell_w^\times$ induces a right action of $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$ on $\mathscr{W}_{\mathcal{F}\ell_w^\times}$.

Lemma 2.4.6. *For any* $\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}_{\mathrm{GSp}_{2g}}$, *we have*

$$\boldsymbol{\gamma}^*(\boldsymbol{s}^\vee) = \boldsymbol{s}^\vee \cdot {}^{\mathtt{t}}(\boldsymbol{\gamma}_a + \boldsymbol{z}\boldsymbol{\gamma}_c)^{-1}.$$

The right-hand side means the right multiplication of matrices, where we view $\boldsymbol{s}^\vee$ *as a* $(1\times g)$*-matrix with entries* $\boldsymbol{s}_i^\vee$.

Proof. To prove the identity, it suffices to check on the level of $(C, \mathcal{O}_C)$-points. Using the identification

$$\mathcal{F}\ell_w^\times(C, \mathcal{O}_C) = P_{\mathrm{Siegel}}(C)\backslash \mathrm{GSp}_{2g,w}(C),$$

the sections of $\mathscr{W}_{\mathcal{F}\ell_w^\times}$ can be identified with

$$\left\{\text{analytic functions } \phi : \mathrm{GSp}_{2g,w} \to C^g : \phi(\boldsymbol{\gamma}\,\boldsymbol{\alpha}) = {}^{\mathtt{t}}\boldsymbol{\gamma}_a^{-1}\cdot\phi(\boldsymbol{\alpha}),\ \forall\, \boldsymbol{\gamma} \in P_{\mathrm{Siegel}}(C),\ \boldsymbol{\alpha} \in \mathrm{GSp}_{2g,w}(C)\right\}.$$

Here, elements in C^g are viewed as column vectors. Under this identification, $\boldsymbol{s}^\vee$ sends $\boldsymbol{\alpha} \in \mathrm{GSp}_{2g,w}(C)$ to ${}^{\mathtt{t}}\boldsymbol{\alpha}_a^{-1}$. Notice that a section $\phi : \mathrm{GSp}_{2g,w}(C) \to C^g$ of $\mathscr{W}_{\mathcal{F}\ell_w^\times}$ is determined by its restriction on

$$\left\{ \begin{pmatrix} \mathbb{1}_g & \boldsymbol{z} \\ & \mathbb{1}_g \end{pmatrix} : {}^{\mathtt{t}}\boldsymbol{z}\,\breve{\mathbb{1}}_g = \breve{\mathbb{1}}_g\,\boldsymbol{z},\ \max_{i,j}\inf_{h\in\mathbf{Z}_p}\{|\,\boldsymbol{z}_{i,j}(\boldsymbol{x}) - h|\} \leq p^{-w} \right\}.$$

Let $\boldsymbol{\alpha} = \begin{pmatrix} \mathbb{1}_g & \boldsymbol{z} \\ & \mathbb{1}_g \end{pmatrix}$. Then $\boldsymbol{s}^\vee(\boldsymbol{\alpha}) = \mathbb{1}_g$ and

$$(\boldsymbol{\gamma}^*(\boldsymbol{s}^\vee))(\boldsymbol{\alpha}) = \boldsymbol{s}^\vee(\boldsymbol{\alpha}\,\boldsymbol{\gamma}) = \boldsymbol{s}^\vee\left(\begin{pmatrix} \boldsymbol{\gamma}_a + \boldsymbol{z}\,\boldsymbol{\gamma}_c & \boldsymbol{\gamma}_b + \boldsymbol{z}\,\boldsymbol{\gamma}_d \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix}\right) = {}^{\mathtt{t}}(\boldsymbol{\gamma}_a + \boldsymbol{z}\,\boldsymbol{\gamma}_c)^{-1} = \boldsymbol{s}^\vee(\boldsymbol{\alpha})\cdot{}^{\mathtt{t}}(\boldsymbol{\gamma}_a + \boldsymbol{z}\,\boldsymbol{\gamma}_c)^{-1}$$

as desired. □

Corollary 2.4.7. *For any* $\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}_{\mathrm{GSp}_{2g}}$, *we have*

$$\boldsymbol{\gamma}^*(\boldsymbol{s}) = {}^{\mathtt{t}}(\boldsymbol{\gamma}_a + \boldsymbol{z}\boldsymbol{\gamma}_c)\cdot \boldsymbol{s}\,.$$

Proof. This follows immediately from Lemma 2.4.6. □

2.5 The Hodge–Tate period map and w-ordinary loci

2.5.1. The perfectoid toroidally compactified Siegel modular variety $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is equipped with the so-called *Hodge–Tate period map*. Let us briefly discuss its construction.

By definition of $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}$ (for sufficiently large n), the Hodge–Tate map $\mathrm{HT}_{\Gamma(p^n)}$ induces a map (which we abuse the notation and still denote by $\mathrm{HT}_{\Gamma(p^n)}$)

$$\mathrm{HT}_{\Gamma(p^n)} : \mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z} / p^n \mathbf{Z}) \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)} / p^n \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}.$$

Let $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty)}$ and $\underline{\omega}_{\Gamma(p^\infty)}$ denote the pullbacks of $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}$ and $\underline{\omega}_{\Gamma(p^n)}$, respectively, to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$. Pulling back $\mathrm{HT}_{\Gamma(p^n)}$ to the infinite level and taking inverse limit, we obtain

$$\mathrm{HT}_{\Gamma(p^\infty)} : \mathbf{V}_p := \mathbf{V} \otimes_{\mathbf{Z}} \mathbf{Z}_p \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty)}$$

which induces a surjection

$$\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id} : \mathbf{V}_p \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}} \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty)}.$$

Finally, inverting p, the surjection

$$\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id} : \mathbf{V}_p \otimes_{\mathbf{Z}_p} \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}} \to \underline{\omega}_{\Gamma(p^\infty)}$$

induces the ***Hodge–Tate period map***

$$\pi_{\mathrm{HT}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \mathscr{F}\!\ell,$$

where $\mathscr{F}\!\ell$ is the (adic) flag variety parameterising the maximal lagrangians of $\mathbf{V}_p$. According to [PS16, §1], we know that π_{HT} is a morphism of adic spaces.

2.5.2. Let us describe the Hodge–Tate period map π_{HT} more explicitly on the level of points. Suppose C is an algebraically closed and complete extension of $\mathbf{Q}_p$ and (A, λ) is a principally polarised abelian variety over C. The Hodge–Tate sequence of A is

$$0 \to \mathrm{Lie}\, A \to T_pA \otimes_{\mathbf{Z}_p} C \to \omega_{A^\vee} \to 0,$$

where $\omega_{A^\vee}$ is the dual of the Lie algebra of the dual abelian variety $A^\vee$ and the second last map is induced from the Hodge–Tate map $\mathrm{HT}_A : T_pA \to \omega_{A^\vee}$. Here, we ignore the Tate twist by the fixed compatible system of p-power roots of unity $(\zeta_{p^n})_{n \in \mathbf{Z}_{\geq 1}}$ in $\mathbf{C}_p$. Notice that every point $\boldsymbol{x} \in \mathcal{X}_{\Gamma(p^\infty)}(C, \mathcal{O}_C)$ corresponds to a quadruple $(A, \lambda, \psi_N, \psi)$ where (A, λ, ψ_N) is a principally polarised abelian variety over C with a $\Gamma^{(p)}$-level structure and ψ is a symplectic isomorphism $\psi : V_p \simeq T_pA$. Then π_{HT} sends $\boldsymbol{x}$ to the maximal lagrangian

$$\mathrm{Lie}\, A \subset T_pA \otimes_{\mathbf{Z}_p} C \overset{\psi^{-1}}{\simeq} V_p \otimes_{\mathbf{Z}_p} C.$$

One can extend such an explicit description to the boundary points as well using the language of 1-motives. The details are left to the interested readers.

Remark 2.5.3. There are right $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-actions on both sides of the Hodge–Tate map. The $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-action on $\mathscr{F}\!\ell$ is given in §2.4. As for the $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-action on $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$, here we only describe the action away from the boundary. (A similar description applies to the boundary points as well, using the language of 1-motives.) Let $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_p)$ and let

$m \in \mathbf{Z}$ such that $p^m \boldsymbol{\gamma}^{-1} \in M_{2g}(\mathbf{Z}_p)$ and $p^{m-1} \boldsymbol{\gamma}^{-1} \notin M_{2g}(\mathbf{Z}_p)$. Choose $k \in \mathbf{Z}_{\geq 0}$ sufficiently large such that the kernel of $p^m \boldsymbol{\gamma}^{-1} : A[p^k] \to A[p^k]$ stabilises. Let $H \subset A[p^k]$ denote the corresponding kernel. Then $\boldsymbol{\gamma}^{-1}$ sends $(A, \lambda, \psi_N, \psi)$ to $(A' = A/H, \lambda', \psi'_N, \psi')$ where

- λ' is the induced polarisation on A';
- ψ'_N is induced from ψ_N via the isomorphism $A[N] \simeq A'[N]$;
- ψ' is given by the composition

  $$V_p \to V_p \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\psi} T_p A \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to T_p A' \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

  with the first map $V_p \to V_p \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ sending $\vec{v}$ to $(p^m \boldsymbol{\gamma}^{-1})^{-1} \vec{v}$.

One checks that π_{HT} is equivariant with respect to these $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-actions.

Proposition 2.5.4. *There is a natural isomorphism*

$$\pi_{\mathrm{HT}}^* \mathscr{W}_{\mathscr{F}\!\ell}^{\vee} \simeq \underline{\omega}_{\Gamma(p^\infty)}.$$

Proof. Let $\mathcal{A}^{\mathrm{univ}}_{\Gamma(p^\infty)}$ be the pullback of the universal abelian variety $\mathcal{A}^{\mathrm{univ}}_{\mathbf{C}_p}$ over $\mathcal{X}_{\mathbf{C}_p}$ to $\mathcal{X}_{\Gamma(p^\infty)}$. Away from the boundary, we have a universal trivialisation $\psi^{\mathrm{univ}} : \mathbf{V}_p \simeq T_p \mathcal{A}^{\mathrm{univ}}_{\Gamma(p^\infty)}$. Let $\psi^{\mathrm{univ},\vee} : \mathbf{V}_p^{\vee} \simeq T_p \mathcal{A}^{\mathrm{univ},\vee}_{\Gamma(p^\infty)}$ be the dual trivialisation. The Hodge–Tate map on the universal abelian variety $\mathcal{A}^{\mathrm{univ}}_{\Gamma(p^\infty)}$ induces a map

$$\mathrm{HT}_{\Gamma(p^\infty)} : \mathbf{V}_p^{\vee} \overset{\psi^{\mathrm{univ},\vee}}{\simeq} T_p \mathcal{A}^{\mathrm{univ},\vee}_{\Gamma(p^\infty)} \to \underline{\omega}_{\Gamma(p^\infty)}|_{\mathcal{X}_{\Gamma(p^\infty)}}$$

which induces a surjection

$$\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id} : \mathbf{V}_p^{\vee} \otimes_{\mathbf{Z}_p} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty)}} \twoheadrightarrow \underline{\omega}_{\Gamma(p^\infty)}|_{\mathcal{X}_{\Gamma(p^\infty)}}.$$

According to 2.5.1, this surjection extends to a surjection [2]

$$\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id} : \mathbf{V}_p^{\vee} \otimes_{\mathbf{Z}_p} \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}} \twoheadrightarrow \underline{\omega}_{\Gamma(p^\infty)}$$

on the entire perfectoid Siegel modular variety.

Consequently, the sheaf $\pi_{\mathrm{HT}}^* \mathscr{W}_{\mathscr{F}\!\ell}^{\vee}$, being the universal maximal Lagrangian quotient of $\mathbf{V}_p^{\vee} \otimes_{\mathbf{Z}_p} \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}$, coincides with $\underline{\omega}_{\Gamma(p^\infty)}$. ☐

2.5.5. Recall the sections $\boldsymbol{s}_i$ of $\mathscr{W}_{\mathscr{F}\!\ell}^{\vee}$ defined in 2.4.5. We define sections $\mathfrak{s}_i \in \underline{\omega}_{\Gamma(p^\infty)}$ by

$$\mathfrak{s}_i := \pi_{\mathrm{HT}}^* \boldsymbol{s}_i \,. \tag{2.3}$$

[2] The map $\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id}$ here coincides with the map $\mathrm{HT}_{\Gamma(p^\infty)} \otimes \mathrm{id} : \mathbf{V}_p \otimes_{\mathbf{Z}_p} \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}} \to \underline{\omega}_{\Gamma(p^\infty)}$ in 2.5.1 via the symplectic isomorphism $\mathbf{V}_p \simeq \mathbf{V}_p^{\vee}$ sending e_i to $-e^{\vee}_{2g+1-i}$, for $i = 1, \dots, g$, and sending e_i to $e^{\vee}_{2g+1-i}$, for $i = g+1, \dots, 2g$.

From the construction, one sees that

$$\mathfrak{s}_i = \mathrm{HT}_{\Gamma(p^\infty)}(e_i^\vee) \tag{2.4}$$

for all $i = 1, \ldots, g$. These $\mathfrak{s}_i$'s are examples of *fake Hasse invariants* studied in [Sch15]. We also write

$$\mathfrak{s} := \begin{pmatrix} \mathfrak{s}_1 \\ \vdots \\ \mathfrak{s}_g \end{pmatrix} = \pi_{\mathrm{HT}}^* \, \boldsymbol{s} \,.$$

2.5.6. To wrap up the section, we introduce the notion of 'w-ordinary locus' of the perfectoid Siegel modular variety. In particular, it is an open subset of $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ which contains the usual ordinary locus.

For every $w \in \mathbf{Q}_{>0}$, define

$$\overline{\mathcal{X}}_{\Gamma(p^\infty),w} := \pi_{\mathrm{HT}}^{-1}(\mathcal{F}\ell_w^\times).$$

We also define

$$\begin{aligned} \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p,w} &:= h_{\Gamma(p^n)}(\overline{\mathcal{X}}_{\Gamma(p^\infty),w}), & \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p,w} &:= h_{\mathrm{Iw}^+}(\overline{\mathcal{X}}_{\Gamma(p^\infty),w}), \\ \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p,w} &:= h_{\mathrm{Iw}}(\overline{\mathcal{X}}_{\Gamma(p^\infty),w}), & \overline{\mathcal{X}}_{\mathbf{C}_p,w} &:= h(\overline{\mathcal{X}}_{\Gamma(p^\infty),w}), \end{aligned}$$

where $h_{\Gamma(p^n)} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}$, $h_{\mathrm{Iw}^+} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p}$, $h_{\mathrm{Iw}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}$, and $h : \overline{\mathcal{X}}_{\Gamma(p^\infty)} \to \overline{\mathcal{X}}_{\mathbf{C}_p}$ are the natural projections. The subspaces $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$, $\overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p,w}$, $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p,w}$, $\overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p,w}$, and $\overline{\mathcal{X}}_{\mathbf{C}_p,w}$ are called the ***w-ordinary loci*** of $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$, $\overline{\mathcal{X}}_{\Gamma(p^n),\mathbf{C}_p}$, $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathbf{C}_p}$, $\overline{\mathcal{X}}_{\mathrm{Iw},\mathbf{C}_p}$, and $\overline{\mathcal{X}}_{\mathbf{C}_p}$, respectively.

We still denote by

$$\pi_{\mathrm{HT}} : \overline{\mathcal{X}}_{\Gamma(p^\infty),w} \to \mathcal{F}\ell_w^\times$$

the restriction of the Hodge–Tate period map on the w-ordinary locus. It is equivariant under the right $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$-actions on both sides.

Denote by $\mathfrak{z}_{ij} := \pi_{\mathrm{HT}}^* \, \boldsymbol{z}_{ij}$ and $\mathfrak{z} := (\mathfrak{z}_{i,j})_{1 \le i,j \le g} = \pi_{\mathrm{HT}}^* \, \boldsymbol{z}$. Let $\mathfrak{s}_i^\vee := \pi_{\mathrm{HT}}^*(\boldsymbol{s}_i^\vee)$ and

$$\mathfrak{s}^\vee := \begin{pmatrix} \mathfrak{s}_1^\vee & \cdots & \mathfrak{s}_g^\vee \end{pmatrix} = \pi_{\mathrm{HT}}^*(\boldsymbol{s}^\vee).$$

By Lemma 2.4.6 and Corollary 2.4.7, we have

$$\boldsymbol{\gamma}^*(\mathfrak{s}^\vee) = \mathfrak{s}^\vee \cdot {}^{\mathfrak{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\, \boldsymbol{\gamma}_c)^{-1}$$

and

$$\boldsymbol{\gamma}^* \, \mathfrak{s} = {}^{\mathfrak{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\, \boldsymbol{\gamma}_c) \cdot \mathfrak{s} \tag{2.5}$$

for all $\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}_{\mathrm{GSp}_{2g}}$. We will need these sections $\mathfrak{s}_i$'s and $\mathfrak{s}_i^\vee$'s in Chapter 3.

Chapter 3

Overconvergent automorphic sheaves

In this chapter, we use perfectoid method to construct families of sheaves of overconvergent Siegel modular forms. In particular, we give an answer to the first part of Question 1.3.2 (i). The idea of such sheaves are taken from [CHJ17], where the authors of *loc. cit.* used perfectoid methods to construct families of sheaves of overconvergent automorphic forms over the compact Shimura curve over $\mathbf{Q}$. We remark that the materials presented in this chapter are entirely taken from [DRW22, §3].

This chapter is organised as follows. The perfectoid construction of the sheaves are given in §3.1 and the local descriptions of these sheaves are discussed in §3.2. In §3.3, we define the Hecke operators acting on the space of overconvergent Siegel modular forms. We justify our sheaves in §3.4; that is, we prove that our space of overconvergent Siegel modular forms does contain the space of classical (algebraic) Siegel modular forms. The last three sections §3.5, §3.6 and §3.7 are dedicated to the relation between our construction and the construction in [AIP15].

Convention. Starting from this chapter, we denote by $\overline{\mathcal{X}}_{\Gamma}$ (resp., $\overline{\mathcal{X}}_{\Gamma,w}$; resp., $\mathcal{X}_{\Gamma}$) the adic space $\overline{\mathcal{X}}_{\Gamma,\mathbf{C}_p}$ (resp., $\overline{\mathcal{X}}_{\Gamma,\mathbf{C}_p,w}$; resp., $\mathcal{X}_{\Gamma,\mathbf{C}_p}$) over $\mathrm{Spa}(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})$ for any $\Gamma\in\{\Gamma(p^m),\mathrm{Iw}^+,\mathrm{Iw},\varnothing\}$. Similarly, we also write $\overline{X}_{\Gamma}^{\mathrm{tor}}$ (resp., X_{Γ}) for the algebraic variety $\overline{X}_{\Gamma,\mathbf{C}_p}$ (resp., $X_{\Gamma,\mathbf{C}_p}$) over $\mathbf{C}_p$.

3.1 The perfectoid construction

3.1.1. Let $\mathbf{ALG}_{(\mathbf{Z}_p,\mathbf{Z}_p)}$ be the category of complete sheafy $(\mathbf{Z}_p,\mathbf{Z}_p)$-algebras. We consider the functor

$$\mathbf{ALG}_{(\mathbf{Z}_p,\mathbf{Z}_p)}\to\mathbf{SETS},\quad (R,R^+)\mapsto \mathrm{Hom}^{\mathrm{cts}}_{\mathbf{GROUP}}(T_{\mathrm{GL}_g,0},R^\times),$$

which is represented by the $(\mathbf{Z}_p,\mathbf{Z}_p)$-algebra $(\mathbf{Z}_p[\![T_{\mathrm{GL}_g,0}]\!],\mathbf{Z}_p[\![T_{\mathrm{GL}_g,0}]\!])$. The ***weight space*** is

$$\mathcal{W}:=\mathrm{Spa}(\mathbf{Z}_p[\![T_{\mathrm{GL}_g,0}]\!],\mathbf{Z}_p[\![T_{\mathrm{GL}_g,0}]\!])^{\mathrm{rig}},$$

where the superscript 'rig' stands for taking the generic fibre. Every continuous group homomorphism $\kappa:T_{\mathrm{GL}_g,0}\to R^\times$ can be expressed as $\kappa=(\kappa_1,...,\kappa_g)$ where each $\kappa_i:\mathbf{Z}_p^\times\to$

$R^\times$ is a continuous group homomorphism. We write $\kappa^\vee := (-\kappa_g, ..., -\kappa_1)$ where $-\kappa_i$ is the inverse of κ_i.

We are interested in some special *weights*, namely 'small weights' and 'affinoid weights'. These terminologies are adapted from the ones introduced in [CHJ17].

Definition 3.1.2. *(i) A* ***small*** $\mathbf{Z}_p$***-algebra*** *is a p-torsion free reduced ring which is also a finite* $\mathbf{Z}_p[\![T_1, ..., T_d]\!]$*-algebra for some* $d \in \mathbf{Z}_{\geq 0}$*. In particular, a small* $\mathbf{Z}_p$*-algebra is equipped with a canonical adic profinite topology and is complete with respect to the p-adic topology.*

(ii) A ***small weight*** *is a pair* $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ *where* $R_{\mathcal{U}}$ *is a small* $\mathbf{Z}_p$*-algebra and* $\kappa_{\mathcal{U}} : T_{\mathrm{GL}_g,0} \to R_{\mathcal{U}}^\times$ *is a continuous group homomorphism such that* $\kappa_{\mathcal{U}}((1+p)\,\mathbb{1}_g) - 1$ *is topologically nilpotent in* $R_{\mathcal{U}}$ *with respect to the p-adic topology. By the universal property of the weight space, we obtain a natural morphism*

$$\mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}})^{\mathrm{rig}} \to \mathcal{W}\,.$$

Occasionally, we abuse the terminology and call $\mathcal{U} := \mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}})$ *a small weight. For later use, we write* $R_{\mathcal{U}}^+ := R_{\mathcal{U}}$ *in this situation.*

(iii) An ***affinoid weight*** *is a pair* $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ *where* $R_{\mathcal{U}}$ *is a reduced Tate algebra topologically of finite type over* $\mathbf{Q}_p$ *and* $\kappa_{\mathcal{U}} : T_{\mathrm{GL}_g,0} \to R_{\mathcal{U}}^\times$ *is a continuous group homomorphism. By the universal property of weight space, we obtain a natural morphism*

$$\mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}}^\circ) \to \mathcal{W}\,.$$

Ocassionally, we abuse the terminology and call $\mathcal{U} := \mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}}^\circ)$ *an affinoid weight. For later use, we write* $R_{\mathcal{U}}^+ = R_{\mathcal{U}}^\circ$ *in this situation.*

(iv) By a ***weight****, we shall mean either a small weight or an affinoid weight.*

Remark 3.1.3. For any $n \in \mathbf{Z}_{\geq 0}$, we view n as a weight by identifying it with the character

$$T_{\mathrm{GL}_g,0} \to \mathbf{Z}_p^\times, \quad \mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g) \mapsto \prod_{i=1}^g \boldsymbol{\tau}_i^n\,.$$

Moreover, for any weight $\kappa = (\kappa_1, ..., \kappa_g)$, we write $\kappa + n$ for the weight $(\kappa_1 + n, ..., \kappa_g + n)$ defined by

$$\mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g) \mapsto \prod_{i=1}^g \kappa_i(\boldsymbol{\tau}_i)\,\boldsymbol{\tau}_i^n\,.$$

3.1.4. Because of the notion of small weights, we have to work with the 'mixed completed tensor' as in [CHJ17]. We recall the definition.

Definition 3.1.5. *Let* R *be a small* $\mathbf{Z}_p$*-algebra.*

(i) *For any $\mathbf{Z}_p$-module M, we define* [1]

$$M\widehat{\otimes}'R := \varprojlim_{j\in J}(M\otimes_{\mathbf{Z}_p} R/I_j)$$

where $\{I_j : j\in J\}$ runs through a cofinal system of neighborhood of 0 consisting of $\mathbf{Z}_p$-submodules of R. If, in addition, M is a $\mathbf{Z}_p$-algebra, then $M\widehat{\otimes}'R$ is also a $\mathbf{Z}_p$-algebra.

(ii) *Let B be a $\mathbf{Q}_p$-Banach space and let B_0 be an open and bounded $\mathbf{Z}_p$-submodule. We define the **mixed completed tensor***

$$B\widehat{\otimes}R := (B_0\widehat{\otimes}'R)[\frac{1}{p}].$$

which is in fact independent of the choice of B_0.

3.1.6. Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a weight, we will use the following conventions:

(i) For any $\mathbf{Z}_p$-module M, the term $M\widehat{\otimes}R_{\mathcal{U}}^+$ will either stand for $M\widehat{\otimes}'R_{\mathcal{U}}$ in the case of small weights (notice that $R_{\mathcal{U}} = R_{\mathcal{U}}^+$ in this case), or stand for the p-adically completed tensor over $\mathbf{Z}_p$ in the case of affinoid weights.

(ii) For any $\mathbf{Q}_p$-Banach space B, the term $B\widehat{\otimes}R_{\mathcal{U}}$ will either stand for the mixed completed tensor in the case of small weights, or stand for the usual p-adically completed tensor over $\mathbf{Q}_p$ in the case of affinoid weights.

Notice that, given a uniform Banach $\mathbf{Q}_p$-algebra B and any weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$, the tensor product $B\widehat{\otimes}R_{\mathcal{U}}$ also admits a structure of uniform $\mathbf{Q}_p$-Banach algebra. In particular, if B° is the unit ball of B (with respect to the unique power-multiplicative Banach algebra norm), then the unit ball in $B\widehat{\otimes}R_{\mathcal{U}} = B\widehat{\otimes}_{\mathbf{Q}_p}R_{\mathcal{U}}^+[1/p]$ is given by $B^\circ\widehat{\otimes}R_{\mathcal{U}}^+$. Here, note that $R_{\mathcal{U}}^+[1/p]$ has a structure of a uniform Banach $\mathbf{Q}_p$-algebra given by the corresponding spectral norm (see [CHJ17, pp. 202]).

3.1.7. Next, we introduce the notion of 'r-analytic functions' by following [Han17]. Such a notion is not only important in the perfectoid construction of sheaves of overconvergent Siegel modular forms, but also plays an essential role when discussing overconvergent cohomology groups in Chapter 4.

Definition 3.1.8. *Let $r\in\mathbf{Q}_{>0}$ and $n\in\mathbf{Z}_{\geq 1}$. Let B be a uniform $\mathbf{C}_p$-Banach algebra and let B° be the corresponding unit ball.*

(i) *A function $f:\mathbf{Z}_p^n\to B$ (resp., a function $f:(\mathbf{Z}_p^\times)^n\to B$) is called r-**analytic** if for every $\underline{a}=(a_1,\dots,a_n)\in\mathbf{Z}_p^n$ (resp., every $\underline{a}=(a_1,\dots,a_n)\in(\mathbf{Z}_p^\times)^n$), there exists a power series $f_{\underline{a}}\in B[\![T_1,\dots,T_n]\!]$ which converges on the n-dimensional closed unit ball $\mathbf{B}^n(0,p^{-r})\subset\mathbf{C}_p^n$ of radius p^{-r} such that*

$$f(x_1+a_1,\dots,x_n+a_n) = f_{\underline{a}}(x_1,\dots,x_n)$$

[1]Our notation $\widehat{\otimes}'$ corresponds to the notation $\widehat{\otimes}$ in [CHJ17, Definition 6.3]. We make this change to distinguish from the one in Definition 3.1.5 (ii).

for all $x_i \in p^{\lceil r\rceil}\mathbf{Z}_p$, $i=1,\dots,n$. Here $\lceil r\rceil$ stands for the smallest integer that is greater or equal to r.

(ii) Let $C^{r-\mathrm{an}}(\mathbf{Z}_p^n, B)$ (resp., $C^{r-\mathrm{an}}((\mathbf{Z}_p^\times)^n, B)$) denote the set of r-analytic functions from $\mathbf{Z}_p^n$ (resp., $(\mathbf{Z}_p^\times)^n$) to B.

(iii) Let $C^{r-\mathrm{an}}(\mathbf{Z}_p^n, B^\circ)$ (resp., $C^{r-\mathrm{an}}((\mathbf{Z}_p^\times)^n, B^\circ)$) denote the subset of $C^{r-\mathrm{an}}(\mathbf{Z}_p^n, B)$ (resp., $C^{r-\mathrm{an}}((\mathbf{Z}_p^\times)^n, B)$) consisting of those functions with value in B°.

3.1.9. Given a uniform $\mathbf{C}_p$-Banach algebra, we claim that $C^{r-\mathrm{an}}(\mathbf{Z}_p^n, B)$ (resp., $C^{r-\mathrm{an}}((\mathbf{Z}_p^\times)^n, B)$) admits a natural structure of uniform $\mathbf{C}_p$-Banach algebra. Indeed, one first expresses $\mathbf{Z}_p^n$ as the disjoint union of $p^{n\lceil r\rceil}$ closed balls of radius $p^{\lceil r\rceil}$, labelled by an index set A of size $p^{n\lceil r\rceil}$. Then, for every $f \in \mathcal{C}^{r-\mathrm{an}}(\mathbf{Z}_p^n, B)$, the restriction of f on each closed ball (with label $a \in A$) is given by a power series

$$f_a \in B\langle \frac{T_1}{p^r}, \dots, \frac{T_n}{p^r}\rangle$$

where $B\langle \frac{T_1}{p^r}, \dots, \frac{T_n}{p^r}\rangle$ stands for the subset of $B[\![T_1,\dots,T_n]\!]$ which converges on the n-dimensional closed unit ball $\mathbf{B}^n(0,p^{-r}) \subset \mathbf{C}_p^n$. Let $|\bullet|_B$ be the unique power-multipicative norm on B. We can equip $B\langle \frac{T_1}{p^r}, \dots, \frac{T_n}{p^r}\rangle$ with the following norm: for every $f' = \sum_{\nu\in\mathbf{Z}_{\geq 0}^n} b_\nu T^\nu$, we put

$$|f'| := \sup_{\nu\in\mathbf{Z}_{\geq 0}^n} |b_\nu|_B \cdot p^{-r|\nu|}.$$

Finally, if $f \in C^{r-\mathrm{an}}(\mathbf{Z}_p^n, B)$ is represented by $\{f_a\}_{a\in A}$, we put $|f| := \sup_{a\in A}|f_a|$. This is indeed a uniform Banach norm with unit ball $\mathcal{C}^{r-\mathrm{an}}(\mathbf{Z}_p^n, B^\circ)$.

Definition 3.1.10. *(i) A weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is called r-**analytic** if it is r-analytic when viewed as a function*

$$\kappa_{\mathcal{U}} : (\mathbf{Z}_p^\times)^g \to R_{\mathcal{U}}^\times \subset \mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}}$$

via the identification $T_{\mathrm{GL}_g,0} \simeq (\mathbf{Z}_p^\times)^g$.

(ii) For a weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$, we write $r_{\mathcal{U}}$ for the smallest positive integer r such that the weight is r-analytic.

Remark 3.1.11. (i) It is well-known that every continuous character $\mathbf{Z}_p^\times \to R_{\mathcal{U}}^\times$ is r-analytic for sufficiently large r. Moreover, if such a character is r-analytic, it necessary extends to a character

$$\mathbf{Z}_p^\times(1+p^{r+1}\mathcal{O}_{\mathbf{C}_p}) \to (\mathcal{O}_{\mathbf{C}_p}\widehat{\otimes} R_{\mathcal{U}}^+)^\times \subset \mathbf{C}_p\widehat{\otimes} R_{\mathcal{U}}.$$

See, for example, [CHJ17, Proposition 2.6].

(ii) If we write $\kappa_{\mathcal{U}} = (\kappa_{\mathcal{U},1}\dots,\kappa_{\mathcal{U},g})$ with components $\kappa_{\mathcal{U},i} : \mathbf{Z}_p^\times \to R_{\mathcal{U}}^\times$, then $\kappa_{\mathcal{U}}$ is r-analytic if and only if all $\kappa_{\mathcal{U},i}$'s are r-analytic. In this case, for any $w \in \mathbf{Q}_{>0}$ with $w > 1 + r_{\mathcal{U}}$, $\kappa_{\mathcal{U}}$ extends to a character

$$\kappa_{\mathcal{U}} : T_{\mathrm{GL}_g,0}^{(w)} \to (\mathcal{O}_{\mathbf{C}_p}\widehat{\otimes} R_{\mathcal{U}}^+)^\times \subset \mathbf{C}_p\widehat{\otimes} R_{\mathcal{U}}.$$

3.1.12. In the perfectoid construction, we would like to consider 'r-analytic functions on $\mathrm{Iw}_{\mathrm{GL}_g}$'. This notion is made explicitly by the following.

Let B be a uniform $\mathbf{C}_p$-Banach algebra. Note that there is a (topological) identification

$$U_{\mathrm{GL}_g,1}^{\mathrm{opp}} = \begin{pmatrix} 1 & & & \\ p\,\mathbf{Z}_p & 1 & & \\ \vdots & & \ddots & \\ p\,\mathbf{Z}_p & \dots & p\,\mathbf{Z}_p & 1 \end{pmatrix} \simeq \mathbf{Z}_p^{\frac{(g-1)g}{2}}.$$

We say a function $\psi : U_{\mathrm{GL}_g,1}^{\mathrm{opp}} \to B$ is called ***r*-*analytic*** if, under the identification above, the function

$$\psi : \mathbf{Z}_p^{\frac{(g-1)g}{2}} \to B$$

is r-analytic. The space of such functions are denoted by $C^{r-\mathrm{an}}(U_{\mathrm{GL}_g,1}^{\mathrm{opp}}, B)$.

Now, let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be an r-analytic weight. Using the decomposition $B_{\mathrm{GL}_g,0} = T_{\mathrm{GL}_g,0}U_{\mathrm{GL}_g,0}$, we extend $\kappa_{\mathcal{U}}$ to a group homomorphism $\kappa_{\mathcal{U}} : B_{\mathrm{GL}_g,0} \to R_{\mathcal{U}}^{\times}$ by setting $\kappa_{\mathcal{U}}|_{U_{\mathrm{GL}_g,0}} = 1$. We define

$$C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B) := \left\{ f : \mathrm{Iw}_{\mathrm{GL}_g} \to B : \begin{array}{l} f(\boldsymbol{\gamma}\,\boldsymbol{\beta}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta})f(\boldsymbol{\gamma}),\ \forall \boldsymbol{\beta} \in B_{\mathrm{GL}_g,0},\ \boldsymbol{\gamma} \in \mathrm{Iw}_{\mathrm{GL}_g} \\ f|_{U_{\mathrm{GL}_g,1}^{\mathrm{opp}}} \text{ is } r\text{-analytic} \end{array} \right\}.$$

Finally, we write $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^{\circ})$ for the subset of $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ consisting of those functions with value in B°.

Consequently, by 3.1.9, $C^{r-\mathrm{an}}(U_{\mathrm{GL}_g,1}^{\mathrm{opp}}, B)$ admits a structure of uniform $\mathbf{C}_p$-Banach algebra. Notice that an element in $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ is determined by its restriction on $U_{\mathrm{GL}_g,1}^{\mathrm{opp}}$. Thus, $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ admits a structure of uniform $\mathbf{C}_p$-Banach algebra via the identification

$$C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B) \simeq C^{r-\mathrm{an}}(U_{\mathrm{GL}_g,1}^{\mathrm{opp}}, B).$$

In particular, $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^{\circ})$ is the corresponding unit ball in $C_{\kappa_{\mathcal{U}}}^{r-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$.

Remark 3.1.13. Let $\kappa_{\mathcal{U}}$ be a weight and let $w \in \mathbf{Q}_{>0}$ with $w > r_{\mathcal{U}} + 1$. Recall that we have a decomposition $B_{\mathrm{GL}_g,0}^{(w)} = T_{\mathrm{GL}_g,0}^{(w)}U_{\mathrm{GL}_g,0}^{(w)}$. Since $w > 1 + r_{\mathcal{U}}$, $\kappa_{\mathcal{U}}$ extends to a character on $T_{\mathrm{GL}_g,0}^{(w)}$, and hence to a character on $B_{\mathrm{GL}_g,0}^{(w)}$ by setting $\kappa_{\mathcal{U}}|_{U_{\mathrm{GL}_g,0}^{(w)}} = 0$.

We claim that every element f in $C_{\kappa_{\mathcal{U}}}^{w-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ (resp., $C_{\kappa_{\mathcal{U}}}^{w-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^{\circ})$) naturally extends to a function

$$f : \mathrm{Iw}_{\mathrm{GL}_g}^{(w)} \to B \quad (\text{resp., } \ f : \mathrm{Iw}_{\mathrm{GL}_g}^{(w)} \to B^{\circ})$$

such that $f(\boldsymbol{\gamma}\,\boldsymbol{\beta}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta})f(\boldsymbol{\gamma})$ for all $\boldsymbol{\beta} \in B_{\mathrm{GL}_g,0}^{(w)}$ and $\boldsymbol{\gamma} \in \mathrm{Iw}_{\mathrm{GL}_g}^{(w)}$. Indeed, we have decomposition

$$\mathrm{Iw}_{\mathrm{GL}_g}^{(w)} = U_{\mathrm{GL}_g,1}^{\mathrm{opp},(w)}T_{\mathrm{GL}_g,0}^{(w)}U_{\mathrm{GL}_g,0}^{(w)}.$$

Then for every $\boldsymbol{\nu} \in U_{\mathrm{GL}_g,1}^{\mathrm{opp},(w)}$, $\boldsymbol{\tau} \in T_{\mathrm{GL}_g,0}^{(w)}$, and $\boldsymbol{\nu}' \in U_{\mathrm{GL}_g,0}^{(w)}$, we put

$$f(\boldsymbol{\nu}\,\boldsymbol{\tau}\,\boldsymbol{\nu}') = \kappa_{\mathcal{U}}(\boldsymbol{\tau})f(\boldsymbol{\nu}).$$

It is straightforward to check that f is well-defined and satisfies the required condition.

3.1.14. As a consequence of Remark 3.1.13, given $w \in \mathbf{Q}_{>0}$ with $w > 1 + r_{\mathcal{U}}$, there is a natural left action of $\mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g}$ on $C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ and $C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^\circ)$ (hence also a left action of $\mathrm{Iw}^{+}_{\mathrm{GL}_g}$) given by

$$(\boldsymbol{\gamma} \cdot f)(\boldsymbol{\gamma}') = f({}^{\mathrm{t}}\boldsymbol{\gamma}\,\boldsymbol{\gamma}')$$

for all $\boldsymbol{\gamma} \in \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g}$, $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, and $f \in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B)$ (resp., $C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^\circ)$). This left action is denoted by $\rho_{\kappa_{\mathcal{U}}} : \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g} \to \mathrm{Aut}(C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B))$ (resp., $\rho_{\kappa_{\mathcal{U}}} : \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g} \to \mathrm{Aut}(C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, B^\circ))$).

3.1.15. Before defining the sheaves of overconvergent Siegel modular forms, we introduce the following sheaves for any weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and any $w \in \mathbf{Q}_{>0}$ with $w > 1 + r_{\mathcal{U}}$:

(i) Let $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}$ be the sheaf on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ given by

$$\mathcal{Y} \mapsto \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{Y}) \widehat{\otimes} R_{\mathcal{U}}$$

for every affinoid open subset $\mathcal{Y} \subset \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$. This is in fact a sheaf of uniform $\mathbf{C}_p$-Banach algebra; *i.e.*, $(\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})(\mathcal{Y})$ is a uniform $\mathbf{C}_p$-Banach algebra for every affinoid open $\mathcal{Y}$.

Similarly, let $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R^+_{\mathcal{U}}$ be the sheaf on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ given by

$$\mathcal{Y} \mapsto \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{Y}) \widehat{\otimes} R^+_{\mathcal{U}}$$

for every affinoid open subset $\mathcal{Y} \subset \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$.

(ii) For any $r \in \mathbf{Q}_{>0}$ with $r > 1 + r_{\mathcal{U}}$, let $\mathscr{C}^{r-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ denote the sheaf on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ given by

$$\mathcal{Y} \mapsto C^{r-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{Y}) \widehat{\otimes} R_{\mathcal{U}})$$

for every affinoid open subset $\mathcal{Y} \subset \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$. This is also a sheaf of uniform $\mathbf{C}_p$-Banach algebra.

The sheaf $\mathscr{C}^{r-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R^+_{\mathcal{U}})$ is defined in the same way.

Definition 3.1.16. *Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a weight and let $w \in \mathbf{Q}_{>0}$ such that $w > 1 + r_{\mathcal{U}}$.*

(i) The ***sheaf of w-overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$*** *is a subsheaf $\underline{\omega}^{\kappa_{\mathcal{U}}}_w$ of $h_{\mathrm{Iw}^+,*}\mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ defined as follows. For every affinoid open subset $\mathcal{V} \subset \overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ with $\mathcal{V}_\infty = h^{-1}_{\mathrm{Iw}^+}(\mathcal{V})$, we put*

$$\underline{\omega}^{\kappa_{\mathcal{U}}}_w(\mathcal{V}) := \left\{ f \in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty) \widehat{\otimes} R_{\mathcal{U}}) : \begin{array}{l} \boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f, \\ \forall\, \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \end{array} \right\}.$$

Here, $\boldsymbol{\gamma}^* f$ stands for the left action of $\boldsymbol{\gamma}$ on $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}$ induced by the natural right $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-action on $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ defined in §2.5.

*Similarly, the **sheaf of integral w-overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is a subsheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}},+}$ of $h_{\mathrm{Iw}^+,*}\mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}\widehat{\otimes}R^+_{\mathcal{U}})$ defined as follows. For every affinoid open subset $\mathcal{V}\subset\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ with $\mathcal{V}_\infty = h^{-1}_{\mathrm{Iw}^+}(\mathcal{V})$, we put*

$$\underline{\omega}_w^{\kappa_{\mathcal{U}},+}(\mathcal{V}) := \left\{ f\in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty)\widehat{\otimes}R^+_{\mathcal{U}}) : \begin{array}{l} \boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a+\mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1}f, \\ \forall\, \boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix}\in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}\end{array}\right\}.$$

*(ii) The **space of w-overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is defined to be*

$$M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\, \underline{\omega}_w^{\kappa_{\mathcal{U}}}).$$

*We similarly define the **space of integral w-overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** to be*

$$M^{w,+}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\, \underline{\omega}_w^{\kappa_{\mathcal{U}},+}).$$

*(iii) Taking limit with respect to w, the **space of overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is*

$$M^\dagger_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := \lim_{w\to\infty} M^w_{\mathrm{Iw},\kappa_{\mathcal{U}}}.$$

*Similarly, the **space of integral overconvergent Siegel modular forms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is*

$$M^{\dagger,+}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := \lim_{w\to\infty} M^{w,+}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

*(iv) Recall that $\mathcal{Z}_{\mathrm{Iw}^+} = \overline{\mathcal{X}}_{\mathrm{Iw}^+}\smallsetminus\mathcal{X}_{\mathrm{Iw}^+}$ is the boundary divisor. The **sheaf of w-overconvergent Siegel cuspforms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is defined to be the subsheaf $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{cusp}} = \underline{\omega}^{\kappa_{\mathcal{U}}}_w(-\mathcal{Z}_{\mathrm{Iw}^+})$ of $\underline{\omega}^{\kappa_{\mathcal{U}}}_w$ consisting of sections that vanish along $\mathcal{Z}_{\mathrm{Iw}^+}$.*

*A w-overconvergent Siegel modular form of strict Iwahori level and weight $\kappa_{\mathcal{U}}$ is called **cuspidal** if it is an element of*

$$S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\, \underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{cusp}}).$$

*Moreover, by taking limit with respect to w, the **space of overconvergent Siegel cuspforms of strict Iwahori level and weight $\kappa_{\mathcal{U}}$** is defined to be*

$$S^\dagger_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := \lim_{w\to\infty} S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

Remark 3.1.17. Notice that, in Definition 3.1.16 (iii), for every $\boldsymbol{x} \in \overline{\mathcal{X}}_{\Gamma(p^\infty),w}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ and any $\begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$, we have $\boldsymbol{\gamma}_a + \mathfrak{z}(\boldsymbol{x})\,\boldsymbol{\gamma}_c \in \mathrm{Iw}^{+,(w)}_{\mathrm{GL}_g}$. Hence, $\rho_{\kappa_\mathcal{U}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)$ is well-defined.

Remark 3.1.18. The definition above yields an analogue to the complex theory, which we describe in the following.

Suppose $k = (k_1, \ldots, k_g) \in \mathbf{Z}^g_{\geq 0}$ is a dominant weight for GL_g and let $\rho_k : \mathrm{GL}_g(\mathbf{C}) \to \mathrm{GL}(\mathbf{V}_k)$ be the corresponding irreducible representation of GL_g of highest weight k. Recall the Siegel upper-half space $\mathbb{H}^+_g$. Then a classical (complex) Siegel modular form of weight k and level Γ is a holomorphic function $f : \mathbb{H}^+_g \to \mathbf{V}_k$ such that

$$f(\boldsymbol{\gamma} \cdot \boldsymbol{x}) = \rho_k(\boldsymbol{\gamma}_c\, \boldsymbol{x} + \boldsymbol{\gamma}_d) f(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{H}^+_g$ and $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \Gamma \subset \mathrm{GSp}_{2g}(\mathbf{Z})$.

In our case, a w-overconvergent Siegel modular form $f \in M^{\kappa_\mathcal{U}}_{\mathrm{Iw}^+,w}$ can be viewed as a function

$$f : \overline{\mathcal{X}}_{\Gamma(p^\infty),w} \to C^{w-\mathrm{an}}_{\kappa_\mathcal{U}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathbf{C}_p \widehat{\otimes} R_\mathcal{U})$$

satisfying

$$f(\boldsymbol{x} \cdot \boldsymbol{\gamma}) = \rho_{\kappa_\mathcal{U}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ and $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \subset \mathrm{GSp}_{2g}(\mathbf{Z}_p)$. Notice that $C^{w-\mathrm{an}}_{\kappa_\mathcal{U}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathbf{C}_p \widehat{\otimes} R_\mathcal{U})$ is an analytic analogue of the algebraic representation $\mathbf{V}_k$.

Remark 3.1.19. The sheaf $\underline{\omega}^{\kappa_\mathcal{U}}_w$ is functorial in the weight $(R_\mathcal{U}, \kappa_\mathcal{U})$. Given a map of weights $R_\mathcal{U} \to R_{\mathcal{U}'}$ and $w > \max\{1 + r_\mathcal{U}, 1 + r_{\mathcal{U}'}\}$, we obtain a natural map $\underline{\omega}^{\kappa_\mathcal{U}}_w \to \underline{\omega}^{\kappa_{\mathcal{U}'}}_w$ induced from

$$C^{w-\mathrm{an}}_{\kappa_\mathcal{U}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_\mathcal{U}) \to C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}'}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}'}).$$

3.1.20. Finally, to simplify the notation, we introduce a 'twisted' left action of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ on the sheaf $\mathscr{C}^{w-\mathrm{an}}_{\kappa_\mathcal{U}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_\mathcal{U})$ by the formula

$$\boldsymbol{\gamma} \,.\, f := \rho_{\kappa_\mathcal{U}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)\,\boldsymbol{\gamma}^* f.$$

Then, sections of $\underline{\omega}^{\kappa_\mathcal{U}}_w$ are precisely the $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-invariant sections of $h_{\mathrm{Iw}^+,*}\,\mathscr{C}^{w-\mathrm{an}}_{\kappa_\mathcal{U}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_\mathcal{U})$ under this twisted action.

3.2 Local description

3.2.1. Throughout this subsection, let $(R_\mathcal{U}, \kappa_\mathcal{U})$ be a small weight and $w > 1 + r_\mathcal{U}$. We fix an ideal $\mathfrak{a}_\mathcal{U} \subset R_\mathcal{U}$ defining the profinite adic topology on $R_\mathcal{U}$. In addition, we assume $p \in \mathfrak{a}_\mathcal{U}$.

The purpose of this subsection is to give a local description of the overconvergent Siegel modular sheaf $\underline{\omega}^{\kappa_\mathcal{U}}_w$. More precisely, we show that $\underline{\omega}^{\kappa_\mathcal{U}}_w$ can be identified with the G-invariants of an *admissible Kummer étale Banach sheaf* in the sense of Definition A.2.11, where G is a finite group. Such a description allows us to apply Corollary A.2.18 to the sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$, which is crucial in our construction of the *overconvergent Eichler–Shimura morphism.*

Definition 3.2.2. *Let R be a flat $\mathcal{O}_{\mathbf{C}_p}$-algebra and suppose M is a free R-module of rank g. We write $R_w := R \otimes_{\mathcal{O}_{\mathbf{C}_p}} \mathcal{O}_{\mathbf{C}_p}/p^w$ and $M_w := M \otimes_R R_w$. Let $\underline{\boldsymbol{m}} := (\boldsymbol{m}_1, \ldots, \boldsymbol{m}_g)$ be an R_w-basis for M_w. We denote by $\mathrm{Fil}_\bullet^{\underline{\boldsymbol{m}}}$ the full flag*

$$0 \subset \langle \boldsymbol{m}_1 \rangle \subset \langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle \subset \cdots \subset \langle \boldsymbol{m}_1, \ldots, \boldsymbol{m}_g \rangle$$

of the free R_w-module M_w. Namely, $\mathrm{Fil}_i^{\underline{\boldsymbol{m}}} = \langle \boldsymbol{m}_1, \ldots, \boldsymbol{m}_i \rangle$ for all $i = 1, \ldots, g$.

*(i) A full flag $\mathrm{Fil}_\bullet$ of the free R-module M is called w-**compatible with** $\underline{\boldsymbol{m}}$ if*

$$\mathrm{Fil}_i \otimes_R R_w = \mathrm{Fil}_i^{\underline{\boldsymbol{m}}}$$

for all $i = 1, \ldots, g$.

*(ii) Suppose $\mathrm{Fil}_\bullet$ is a w-compatible full flag as in (i). Consider a collection $\{w_i : i = 1, \ldots, g\}$ where each w_i is an R-basis for $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$. Then $\{w_i : i = 1, \ldots, g\}$ is called w-**compatible with** $\underline{\boldsymbol{m}}$ if*

$$w_i \mod (p^w M + \mathrm{Fil}_{i-1}) = \boldsymbol{m}_i \mod \mathrm{Fil}_{i-1}^{\underline{\boldsymbol{m}}}$$

for all $i = 1, \ldots, g$.

3.2.3. Pick a positive integer $n > \sup\{w, \frac{g}{p-1}\}$. Recall from §2.3 the locally free $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n)}}$-module $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}$ over $\overline{\mathcal{X}}_{\Gamma(p^n)}$. Also recall the Hodge–Tate map

$$\mathrm{HT}_{\Gamma(p^n)} : \mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^n\mathbf{Z}) \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}/p^n \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}$$

over $\overline{\mathcal{X}}_{\Gamma(p^n)}$. Restricting to the w-ordinary locus $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ and composing with a natural projection, we obtain

$$\mathrm{HT}_{\Gamma(p^n),w} : \mathbf{V} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^n\mathbf{Z}) \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}/p^n \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w} \twoheadrightarrow \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}/p^w \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}$$

where $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}$ is the restriction of $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)}$ on $\overline{\mathcal{X}}_{\Gamma(p^n),w}$.

Lemma 3.2.4. *The sheaf $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}/p^w \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}$ is a free $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n),w}}/p^w$-module of rank g generated by the basis $\mathrm{HT}_{\Gamma(p^n),w}(e_{g+1})$, ..., $\mathrm{HT}_{\Gamma(p^n),w}(e_{2g})$.*

Proof. Notice that $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}/p^w \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}$ is locally free of rank g. It follows from the definition of w-ordinary locus that $\mathrm{HT}_{\Gamma(p^n),w}(e_{g+1})$, ..., $\mathrm{HT}_{\Gamma(p^n),w}(e_{2g})$ span $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}/p^w \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n),w}$. Hence they must form a set of free generators. □

3.2.5. We consider an adic space $\mathcal{IW}_w^+$ over $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ parameterising certain w-compatible objects. More precisely, for every affinoid open subset $\mathrm{Spa}(R, R^+) \subset \overline{\mathcal{X}}_{\Gamma(p^n),w}$, the set $\mathcal{IW}_w^+(R, R^+)$ consists of triples

$$(\boldsymbol{\sigma}, \mathrm{Fil}_\bullet, \{w_i : i = 1, \ldots, g\})$$

where

(i) $\boldsymbol{\sigma}$ is a matrix in $\mathrm{Iw}^+_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$ where $\mathrm{Iw}^+_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$ is the preimage of $T_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z})$ under the surjection $\mathrm{GL}_g(\mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\mathrm{mod}\ p} \mathrm{GL}_g(\mathbf{Z}/p\mathbf{Z})$;

(ii) both $\mathrm{Fil}_\bullet$ and $\{w_i : i = 1, \ldots, g\}$ are w-compatible with $\big(\mathrm{HT}_{\Gamma(p^n),w}(e_{g+1}), \ldots, \mathrm{HT}_{\Gamma(p^n),w}(e_{2g})\big) \cdot \boldsymbol{\sigma}$.

We write $\pi : \mathcal{IW}^+_w \to \overline{\mathcal{X}}_{\Gamma(p^n),w}$ for the natural projection.

3.2.6. In order to further understand $\mathcal{IW}^+_w$, we consider the following group objects in adic spaces:

(i) Define

$$\mathcal{B}^{\mathrm{opp}}_w = \begin{pmatrix} 1+p^w\mathbf{B}(0,1) & & & \\ p^w\mathbf{B}(0,1) & 1+p^w\mathbf{B}(0,1) & & \\ \vdots & \vdots & \ddots & \\ p^w\mathbf{B}(0,1) & p^w\mathbf{B}(0,1) & \cdots & 1+p^w\mathbf{B}(0,1) \end{pmatrix},$$

where $\mathbf{B}(0,1) = \mathrm{Spa}(\mathbf{C}_p\langle X\rangle, \mathcal{O}_{\mathbf{C}_p}\langle X\rangle)$ stands for the closed unit ball. In particular, the underlying adic space is isomorphic to a $\frac{1}{2}g(g+1)$-dimension ball of radius p^{-w}.

(ii) Define

$$\begin{aligned}\mathcal{T}^{(w)}_{\mathrm{GL}_g,0} &= \begin{pmatrix} \mathbf{Z}_p^\times + p^w\mathbf{B}(0,1) & & & \\ & \mathbf{Z}_p^\times + p^w\mathbf{B}(0,1) & & \\ & & \ddots & \\ & & & \mathbf{Z}_p^\times + p^w\mathbf{B}(0,1) \end{pmatrix} \\ &= \bigsqcup_{h_1,\ldots,h_g \in S} \begin{pmatrix} h_1 + p^w\mathbf{B}(0,1) & & & \\ & h_2 + p^w\mathbf{B}(0,1) & & \\ & & \ddots & \\ & & & h_g + p^w\mathbf{B}(0,1) \end{pmatrix}\end{aligned}$$

where $S \subset \mathbf{Z}_p^\times$ is a set of representatives of $\mathbf{Z}_p^\times/(1+p^n\mathbf{Z}_p)$. In particular, the underlying adic space is isomorphic to the disjoint union of $(p-1)^g p^{g(n-1)}$ copies of g-dimension ball of radius p^{-w}.

(iii) Define

$$\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} = \begin{pmatrix} 1 & & & \\ p\,\mathbf{Z}_p + p^w\mathbf{B}(0,1) & 1 & & \\ \vdots & \vdots & \ddots & \\ p\,\mathbf{Z}_p + p^w\mathbf{B}(0,1) & p\,\mathbf{Z}_p + p^w\mathbf{B}(0,1) & \cdots & 1 \end{pmatrix}$$
$$= \bigsqcup_{\substack{h_{i,j}\in S\\ 1\le j<i\le g}} \begin{pmatrix} 1 & & & \\ h_{2,1} + p^w\mathbf{B}(0,1) & 1 & & \\ \vdots & \vdots & \ddots & \\ h_{g,1} + p^w\mathbf{B}(0,1) & h_{g,2} + p^w\mathbf{B}(0,1) & \cdots & 1 \end{pmatrix}$$

where $S \subset \mathbf{Z}_p^\times$ is a set of representatives of $p\,\mathbf{Z}_p/p^n\,\mathbf{Z}_p$. In particular, the underlying adic space is isomorphic to the disjoint union of $p^{\frac{1}{2}g(g-1)(n-1)}$ copies of $\frac{1}{2}g(g-1)$-dimension ball of radius p^{-w}.

The adic spaces $\mathcal{B}^{\mathrm{opp}}_w$, $\mathcal{T}^{(w)}_{\mathrm{GL}_g,0}$ and $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}$ are equipped with group structures given by the matrix multiplications. Note that the $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$-points of $\mathcal{B}^{\mathrm{opp}}_w$, $\mathcal{T}^{(w)}_{\mathrm{GL}_g,0}$, and $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}$ coincide with the groups B^{opp}_w, $T^{(w)}_{\mathrm{GL}_g,0}$, and $U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}$, respectively. This justifies the notations.

Lemma 3.2.7. *$\mathcal{IW}^+_w$ is a $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ where $U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z})$ is the kernel of the natural surjection $U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\,\mathbf{Z}) \xrightarrow{\mathrm{mod}\ p} U_{\mathrm{GL}_g}(\mathbf{Z}/p\,\mathbf{Z})$. Namely, locally on $\overline{\mathcal{X}}_{\Gamma(p^n),w}$, we have identification*

$$\mathcal{IW}^+_w \simeq \overline{\mathcal{X}}_{\Gamma(p^n),w} \times_{\mathrm{Spa}(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})} \left(\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z})\right)$$

where $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z})$ acts from the right by matrix multiplication.

Proof. This is clear from the construction. □

3.2.8. Using the adic space $\mathcal{IW}^+_w$, we construct two auxiliary sheaves $\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}$ and $\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}}}_{n,w}$:

(i) The sheaf $\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}$ over $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ is defined to be

$$\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w} := \left(\pi_*\,\mathscr{O}^+_{\mathcal{IW}^+_w}\widehat{\otimes} R_{\mathcal{U}}\right)[\kappa^\vee_{\mathcal{U}}];$$

i.e., the subsheaf of $\pi_*\,\mathscr{O}^+_{\mathcal{IW}^+_w}\widehat{\otimes} R_{\mathcal{U}}$ consisting of those sections on which $T_{\mathrm{GL}_g,0}$-acts through the character $\kappa^\vee_{\mathcal{U}}$ and $U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z})$ acts trivially.

(ii) The sheaf

$$\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}}}_{n,w} := \left(\pi_*\,\mathscr{O}_{\mathcal{IW}^+_w}\widehat{\otimes} R_{\mathcal{U}}\right)[\kappa^\vee_{\mathcal{U}}]$$

is defined similarly.

Remark that since $\kappa_{\mathcal{U}}$ is w-analytic, the character $\kappa_{\mathcal{U}}^{\vee} : T_{\mathrm{GL}_g,0} \to R_{\mathcal{U}}^{\times}$ extends to a character on $\mathcal{T}_{\mathrm{GL}_g,0}^{(w)}$; namely, a character

$$\kappa_{\mathcal{U}}^{\vee} : \mathcal{T}_{\mathrm{GL}_g,0}^{(w)}(R, R^+) \to (R^+ \widehat{\otimes} R_{\mathcal{U}})^{\times}$$

for every affinoid $(\mathbf{C}_p, \mathscr{O}_{\mathbf{C}_p})$-algebra (R, R^+). It turns out, in the definitions of $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}$ and $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$, there is no difference between taking $\kappa_{\mathcal{U}}^{\vee}$-eigenspaces with respect to $T_{\mathrm{GL}_g,0}$- or $\mathcal{T}_{\mathrm{GL}_g,0}^{(w)}$-actions.

Lemma 3.2.9. *The sheaf $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$ is a projective Banach sheaf of $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^n),w}} \widehat{\otimes} R_{\mathcal{U}}$-modules in the sense of Definition A.2.4 (ii). Moreover, $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}$ is an integral model of $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$ in the sense of Definition A.2.4 (iv).*

Proof. Let $\{\mathcal{V}_{n,i} : i \in I\}$ be an affinoid open covering of $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ such that $\underline{\omega}_{\Gamma(p^n)}^{\mathrm{mod},+}|_{\mathcal{V}_{n,i}}$ is free, for every $i \in I$. By choosing a basis for $\underline{\omega}_{\Gamma(p^n)}^{\mathrm{mod},+}|_{\mathcal{V}_{n,i}}$, we can identify

$$\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}|_{\mathcal{V}_{n,i}} \simeq \mathscr{O}_{\mathcal{V}_{n,i}}^+\langle T_{st} : 1 \le s < t \le g\rangle \widehat{\otimes} R_{\mathcal{U}}$$

which is the p-adic completion of a free $\mathscr{O}_{\mathcal{V}_{n,i}}^+ \widehat{\otimes} R_{\mathcal{U}}$-module, as desired. $\square$

3.2.10. The p-adically completed sheaves on the Kummer étale site associated with $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}$ and $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$ are

$$\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}},+} := \varprojlim_m \left(\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+} \otimes_{\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n),w}}} \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n),w,\mathrm{k\acute{e}t}}} / p^m \right) \quad \text{and} \quad \underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}} := \underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}},+}[\frac{1}{p}]$$

respectively.

By Lemma 3.2.9 and Corollary A.2.9, $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}}$ is a projective Kummer étale Banach sheaf of $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^n),w,\mathrm{k\acute{e}t}}} \widehat{\otimes} R_{\mathcal{U}}$-modules in the sense of Definition A.2.6 (ii). Moreover, $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}},+}$ is an integral model of $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}}$ in the sense of Definition A.2.6 (iv). In fact, the Kummer étale Banach sheaf $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}}$ is *admissible* by the following lemma.

Lemma 3.2.11. *The sheaf $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}}$ is an admissible Kummer étale Banach sheaf of $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^n),w,\mathrm{k\acute{e}t}}} \widehat{\otimes} R_{\mathcal{U}}$-modules (in the sense of Definition A.2.11) with integral model $\underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}},+}$.*

Proof. The proof is inspired by the discussion in [AIP15, §8.1]. We provide a sketch of proof.

To simplify the notation, we write $\mathscr{F}^+ = \underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}},+}$ and $\mathscr{F} = \underline{\widetilde{\omega}}_{n,w,\mathrm{k\acute{e}t}}^{\kappa_{\mathcal{U}}}$. We also write $\mathscr{F}_m^+ := \mathscr{F}^+ / \mathfrak{a}_{\mathcal{U}}^m$, for every $m \in \mathbf{Z}_{\ge 1}$.

Let $\mathfrak{U} = \{\mathcal{V}_{n,i} : i \in I\}$ be an open affinoid covering for $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ such that $\underline{\omega}_{\Gamma(p^n)}^{\mathrm{mod},+}|_{\mathcal{V}_{n,i}}$ is free, for every $i \in I$. We equip each $\mathcal{V}_{n,i}$ the induced log structure from $\overline{\mathcal{X}}_{\Gamma(p^n),w}$. By choosing a basis for $\underline{\omega}_{\Gamma(p^n)}^{\mathrm{mod},+}|_{\mathcal{V}_{n,i}}$, we can identify

$$\mathscr{F}^+|_{\mathcal{V}_{n,i}} \simeq \mathscr{O}_{\mathcal{V}_{n,i}}^+\langle T_{st} : 1 \le s < t \le g\rangle \widehat{\otimes} R_{\mathcal{U}}$$

which is the p-adic completion of a free $\mathscr{O}^+_{\mathcal{V}_{n,i}}\widehat{\otimes} R_{\mathcal{U}}$-module. Modulo $\mathfrak{a}_{\mathcal{U}}^m$, we obtain

$$\mathscr{F}_m^+|_{\mathcal{V}_{n,i}} \simeq \left(\mathscr{O}^+_{\mathcal{V}_{n,i}}\otimes_{\mathbf{Z}_p}(R_{\mathcal{U}}/\,\mathfrak{a}_{\mathcal{U}}^m)\right)[T_{st}:1\leq s<t\leq g]$$

For any $d\in\mathbf{Z}_{\geq 0}$, consider the subsheaf $(\mathscr{F}_m^+|_{\mathcal{V}_{n,i}})^{\leq d}\subset \mathscr{F}_m^+|_{\mathcal{V}_{n,i}}$ consisting of those polynomials of degree $\leq d$, and consider

$$\mathscr{F}^+_{m,d}:=\ker\left(\prod_{i\in I}(\mathscr{F}_m^+|_{\mathcal{V}_{n,i}})^{\leq d}\to\prod_{i,j\in I}\mathscr{F}_m^+|_{\mathcal{V}_{n,i}\cap\mathcal{V}_{n,j}}\right).$$

Then each $\mathscr{F}^+_{m,d}$ is a coherent $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n),w,\text{két}}}\otimes_{\mathbf{Z}_p}(R_{\mathcal{U}}/\,\mathfrak{a}_{\mathcal{U}}^m)$-module and we have $\mathscr{F}^+_m=\varinjlim_d\mathscr{F}^+_{m,d}$, as desired. □

3.2.12. Next, we are going to relate the overconvergent Siegel modular sheaves $\underline{\omega}_w^{\kappa_{\mathcal{U}},+}$ and $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ with the auxiliary sheaves $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}$ and $\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$. To this end, we need two intermediate sheaves $\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}},+}$ and $\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}$ over $\overline{\mathcal{X}}_{\Gamma(p^n),w}$ defined as follows:

(i) The subsheaf $\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}$ of $h_{\Gamma(p^n),*}\mathscr{C}^{w-\text{an}}_{\kappa_{\mathcal{U}}}(\text{Iw}_{\text{GL}_g},\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}\widehat{\otimes}R_{\mathcal{U}})$ is defined as follows. For every affinoid open subset $\mathcal{V}\subset\overline{\mathcal{X}}_{\Gamma(p^n),w}$ with $\mathcal{V}_\infty=h^{-1}_{\Gamma(p^n)}(\mathcal{V})$, we put

$$\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}(\mathcal{V}):=\left\{f\in C^{w-\text{an}}_{\kappa_{\mathcal{U}}}(\text{Iw}_{\text{GL}_g},\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}}):\begin{array}{l}\boldsymbol{\gamma}^*f=\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a+\mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1}f,\\ \forall\boldsymbol{\gamma}=\begin{pmatrix}\boldsymbol{\gamma}_a&\boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c&\boldsymbol{\gamma}_d\end{pmatrix}\in\Gamma(p^n)\end{array}\right\}.$$

(ii) The subsheaf $\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}},+}$ of $h_{\Gamma(p^n),*}\mathscr{C}^{w-\text{an}}_{\kappa_{\mathcal{U}}}(\text{Iw}_{\text{GL}_g},\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}\widehat{\otimes}R_{\mathcal{U}})$ is defined as follows. For every affinoid open subset $\mathcal{V}\subset\overline{\mathcal{X}}_{\Gamma(p^n),w}$ with $\mathcal{V}_\infty=h^{-1}_{\Gamma(p^n)}(\mathcal{V})$, we put

$$\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}},+}(\mathcal{V}):=\left\{f\in C^{w-\text{an}}_{\kappa_{\mathcal{U}}}(\text{Iw}_{\text{GL}_g},\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}}):\begin{array}{l}\boldsymbol{\gamma}^*f=\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a+\mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1}f,\\ \forall\boldsymbol{\gamma}=\begin{pmatrix}\boldsymbol{\gamma}_a&\boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c&\boldsymbol{\gamma}_d\end{pmatrix}\in\Gamma(p^n)\end{array}\right\}.$$

Here, recall that $h_{\Gamma(p^n)}:\overline{\mathcal{X}}_{\Gamma(p^\infty),w}\to\overline{\mathcal{X}}_{\Gamma(p^n),w}$ is the natural projection.

One observes immediately that if $h_n:\overline{\mathcal{X}}_{\Gamma(p^n),w}\to\overline{\mathcal{X}}_{\text{Iw}^+,w}$ denotes the natural projection, then the overconvergent Siegel modular sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ can be identified as the $\text{Iw}^+_{\text{GSp}_{2g}}/\Gamma(p^n)$-invariants of the sheaf $h_{n,*}\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}$ with respect to the 'twisted' action $\boldsymbol{\gamma}\,.f:=\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a+\mathfrak{z}\,\boldsymbol{\gamma}_c)\,\boldsymbol{\gamma}^*f$ for every $\boldsymbol{\gamma}=\begin{pmatrix}\boldsymbol{\gamma}_a&\boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c&\boldsymbol{\gamma}_d\end{pmatrix}\in\Gamma(p^n)$ and $f\in\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}$. Similar result holds for the integral sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}},+}$.

Proposition 3.2.13. *There is a natural isomorphism of $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n),w}}\widehat{\otimes}R_{\mathcal{U}}$-modules $\Psi^+:\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}},+}\simeq\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}},+}$. Inverting p, we obtain a natural isomorphism of $\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^n),w}}\widehat{\otimes}R_{\mathcal{U}}$-modules $\Psi:\underline{\omega}_{n,w}^{\kappa_{\mathcal{U}}}\simeq\underline{\widetilde{\omega}}_{n,w}^{\kappa_{\mathcal{U}}}$.*

Proof. As a preparation, consider the pullback

$$\begin{tikzcd}
\mathcal{IW}^+_{w,\infty} \arrow[r] \arrow[d, "\pi_\infty"'] & \mathcal{IW}^+_w \arrow[d, "\pi"] \\
\overline{\mathcal{X}}_{\Gamma(p^\infty),w} \arrow[r, "h_{\Gamma(p^n)}"] & \overline{\mathcal{X}}_{\Gamma(p^n),w}
\end{tikzcd}$$

in the category of adic spaces. To show the existence of such a pullback, it suffices to check this locally on $\overline{\mathcal{X}}_{\Gamma(p^n),w}$. Recall that $\mathcal{IW}^+_w$ is a $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}\times\mathcal{T}^{(w)}_{\mathrm{GL}_g,0}\times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\Gamma(p^n),w}$, and $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}\times\mathcal{T}^{(w)}_{\mathrm{GL}_g,0}$ is isomorphic to finitely many copies of $\mathbf{B}(0,1)^{\frac{g(g+1)}{2}}$. It remains to show that the fibred product $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}\times_{\mathrm{Spa}(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})}\mathbf{B}(0,1)^{\frac{g(g+1)}{2}}$ exists. Indeed, by [SW20, Proposition 6.3.3 (3)], such a fibred product exists and is a sousperfectoid space. In addition, we know that the pullback $\mathcal{IW}^+_{w,\infty}$ is likewise a $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1}\times\mathcal{T}^{(w)}_{\mathrm{GL}_g,0}\times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$.

For every affinoid open $\mathcal{V}\subset\overline{\mathcal{X}}_{\Gamma(p^n),w}$ and $\mathcal{V}_\infty := h^{-1}_{\Gamma(p^n)}\mathcal{V}$, the desired isomorphism Ψ^+ will be established via a sequence of isomorphisms

$$\Psi^+ : \underline{\omega}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V}) \xrightarrow[\Psi_1]{\simeq} \omega^{(1)} \xrightarrow[\Psi_2]{\simeq} \omega^{(2)} \xrightarrow[\Psi_3]{\simeq} \underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V}),$$

where

$$\omega^{(1)} := \left\{ f\in C^{w-\mathrm{an}}_{\kappa^\vee_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g},\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}}) : \boldsymbol{\gamma}^* f = \rho_{\kappa^\vee_{\mathcal{U}}}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger) f, \quad \forall\, \boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix}\in\Gamma(p^n)\right\}$$

and

$$\omega^{(2)} := \left\{ f\in \pi_{\infty,*}\,\mathscr{O}^+_{\mathcal{IW}^+_{w,\infty}}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}} : \begin{array}{l} \boldsymbol{\gamma}^* f = f, \quad \boldsymbol{\tau}^* f = \kappa^\vee_{\mathcal{U}}(\boldsymbol{\tau}) f, \quad \boldsymbol{\nu}^* f = f \\ \forall (\boldsymbol{\gamma},\boldsymbol{\tau},\boldsymbol{\nu})\in\Gamma(p^n)\times T_{\mathrm{GL}_g,0}\times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\mathbf{Z}) \end{array}\right\}.$$

Here, for any $\boldsymbol{\delta}\in M_g$, we write $\boldsymbol{\delta}^\ddagger := \breve{\mathbb{1}}_g\,{}^{\mathsf{t}}\boldsymbol{\delta}\,\breve{\mathbb{1}}_g$, which can be viewed as the "transpose with respect to the anti-diagonal". Notice that $\mathfrak{z}^\ddagger = \mathfrak{z}$.

Construction of Ψ_1. Observe that there is an isomorphism

$$\Psi_1 : C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g},\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}}) \to C^{w-\mathrm{an}}_{\kappa^\vee_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g},\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}})$$

defined by

$$\Psi_1(f)(\boldsymbol{\gamma}') := f(\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\,\breve{\mathbb{1}}_g)$$

for all $f\in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g},\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}})$ and $\boldsymbol{\gamma}'\in\mathrm{Iw}_{\mathrm{GL}_g}$.

We claim that Ψ_1 induces an isomorphism $\underline{\omega}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V})\simeq\omega^{(1)}$. It suffices to check that if $\boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f$ for every $\boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix}\in\Gamma(p^n)$, then $\boldsymbol{\gamma}^*(\Psi_1(f)) = \rho_{\kappa^\vee_{\mathcal{U}}}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)\Psi_1(f)$.

Indeed, for any $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, we have

$$\begin{aligned}
\boldsymbol{\gamma}^*(\Psi_1(f))(\boldsymbol{\gamma}') &= \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f(\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\,\breve{\mathbb{1}}_g) \\
&= f\left({}^{\mathsf{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1}\,\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\,\breve{\mathbb{1}}_g\right) \\
&= f\left(\breve{\mathbb{1}}_g\,\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1}\,\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\,\breve{\mathbb{1}}_g\right) \\
&= f\left(\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}(\breve{\mathbb{1}}_g\,\boldsymbol{\gamma}_a\,\breve{\mathbb{1}}_g + \breve{\mathbb{1}}_g\,\mathfrak{z}\,\breve{\mathbb{1}}_g\,\breve{\mathbb{1}}_g\,\boldsymbol{\gamma}_c\,\breve{\mathbb{1}}_g)^{-1}\,{}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\,\breve{\mathbb{1}}_g\right) \\
&= f\left(\breve{\mathbb{1}}_g\,{}^{\mathsf{t}}({}^{\mathsf{t}}(\boldsymbol{\gamma}_a^{\ddagger} + \mathfrak{z}\,\boldsymbol{\gamma}_c^{\ddagger})\,\boldsymbol{\gamma}')^{-1}\,\breve{\mathbb{1}}_g\right) \\
&= \rho_{\kappa_{\mathcal{U}}^{\vee}}(\boldsymbol{\gamma}_a^{\ddagger} + \mathfrak{z}\,\boldsymbol{\gamma}_c^{\ddagger})\Psi_1(f)(\boldsymbol{\gamma}').
\end{aligned}$$

Construction of Ψ_2. To construct Ψ_2, consider $\mathfrak{s}^{\ddagger} = \begin{pmatrix}\mathfrak{s}_g & \cdots & \mathfrak{s}_1\end{pmatrix} \in \underline{\omega}_{\Gamma(p^\infty)}(\mathcal{V}_\infty)^g$. Recall that $\mathfrak{s} = \begin{pmatrix}\mathfrak{s}_1 \\ \vdots \\ \mathfrak{s}_g\end{pmatrix}$ and thus

$$\mathfrak{s}^{\ddagger} = {}^{\mathsf{t}}\mathfrak{s}\,\breve{\mathbb{1}}_g\,.$$

Moreover, for any $\boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix} \in \Gamma(p^n)$, we have $\boldsymbol{\gamma}^*\,\mathfrak{s} = {}^{\mathsf{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)\,\mathfrak{s}$ by (2.5). Hence

$$\boldsymbol{\gamma}^*\,\mathfrak{s}^{\ddagger} = {}^{\mathsf{t}}(\boldsymbol{\gamma}^*\,\mathfrak{s})\,\breve{\mathbb{1}}_g = {}^{\mathsf{t}}({}^{\mathsf{t}}(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)\,\mathfrak{s})\,\breve{\mathbb{1}}_g = {}^{\mathsf{t}}\mathfrak{s}\,\breve{\mathbb{1}}_g\,\breve{\mathbb{1}}_g(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)\,\breve{\mathbb{1}}_g = ({}^{\mathsf{t}}\mathfrak{s}\,\breve{\mathbb{1}}_g)\,{}^{\mathsf{t}}(\boldsymbol{\gamma}_a^{\ddagger} + \mathfrak{z}\,\boldsymbol{\gamma}_c^{\ddagger}) = \mathfrak{s}^{\ddagger}\,{}^{\mathsf{t}}(\boldsymbol{\gamma}_a^{\ddagger} + \mathfrak{z}\,\boldsymbol{\gamma}_c^{\ddagger}).$$

Let $\mathrm{Fil}^{\ddagger}_{\bullet}$ be the full flag of the free $\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)$-module $\underline{\omega}_{\Gamma(p^\infty)}(\mathcal{V}_\infty)$ given by

$$\mathrm{Fil}^{\ddagger}_{\bullet} = 0 \subset \langle \mathfrak{s}_g \rangle \subset \langle \mathfrak{s}_g, \mathfrak{s}_{g-1} \rangle \subset \cdots \langle \mathfrak{s}_g, \ldots, \mathfrak{s}_1 \rangle$$

and let $w_i^{\ddagger}$ be the image of $\mathfrak{s}_{g+1-i}$ in $\mathrm{Fil}^{\ddagger}_i / \mathrm{Fil}^{\ddagger}_{i-1}$, for all $i = 1, \ldots, g$. Then the triple $(\mathbb{1}_g, \mathrm{Fil}^{\ddagger}_{\bullet}, \{w_i^{\ddagger}\})$ defines a section of the $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\,\mathbf{Z})$-torsor $\pi_\infty^{-1}(\mathcal{V}_\infty) \to \mathcal{V}_\infty$. Consequently, one obtains an isomorphism

$$\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z}) \xrightarrow{\sim} \pi_\infty^{-1}(\mathcal{V}_\infty), \quad \boldsymbol{\gamma}' \mapsto (\mathbb{1}_g, \mathrm{Fil}^{\ddagger}_{\bullet}, \{w_i^{\ddagger}\}) \cdot \boldsymbol{\gamma}'$$

and thus an isomorphism

$$\begin{aligned}
\Phi : \pi_{\infty,*}\,\mathscr{O}^+_{\mathcal{IW}^+_{w,\infty}}(\mathcal{V}_\infty)\widehat{\otimes} R^+_{\mathcal{U}} &\xrightarrow{\simeq} \left\{ \begin{array}{c} \text{analytic functions} \\ U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times T^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\,\mathbf{Z}) \to \mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes} R_{\mathcal{U}} \end{array} \right\} \\
f &\mapsto \left(\boldsymbol{\gamma}' \mapsto f((\mathbb{1}_g, \mathrm{Fil}^{\ddagger}_{\bullet}, \{w_i^{\ddagger}\}) \cdot \boldsymbol{\gamma}')\right).
\end{aligned}$$

We claim that if $\boldsymbol{\gamma}^*\,f = f$ for any $\boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix} \in \Gamma(p^n)$, then $\boldsymbol{\gamma}^*\,\Phi(f) = \rho_{\kappa_{\mathcal{U}}^{\vee}}(\boldsymbol{\gamma}_a^{\ddagger} + \mathfrak{z}\,\boldsymbol{\gamma}_c^{\ddagger})\Phi(f)$.

Indeed, for any $\boldsymbol{\gamma}' \in U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times T^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\mathbf{Z})$, we have

$$\begin{aligned}(\boldsymbol{\gamma}^*\Phi(f))(\boldsymbol{\gamma}') &= (\boldsymbol{\gamma}^* f)(\boldsymbol{\gamma}^*(\psi^\ddagger, \mathrm{Fil}^\ddagger_\bullet, \{w_i^\ddagger\})\cdot\boldsymbol{\gamma}')\\ &= f\left((\psi^\ddagger, \mathrm{Fil}^\ddagger_\bullet, \{w_i^\ddagger\})\cdot {}^{\mathsf{t}}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)\cdot\boldsymbol{\gamma}'\right)\\ &= \rho_{\kappa_{\mathcal{U}}^\vee}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)\Phi(f)(\boldsymbol{\gamma}'),\end{aligned}$$

where the second equation follows from the identity $\boldsymbol{\gamma}^*\mathfrak{s}^\ddagger = \mathfrak{s}^\ddagger\,{}^{\mathsf{t}}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)$.

On the other hand, we can identify $\omega^{(1)}$ with the set of analytic functions

$$f : U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times T^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g,1}(\mathbf{Z}/p^n\mathbf{Z}) \to \mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes}R_{\mathcal{U}}$$

satisfying

- $f(\boldsymbol{\upsilon}\boldsymbol{\tau}\boldsymbol{\nu}) = \kappa_{\mathcal{U}}^\vee(\boldsymbol{\tau})f(\boldsymbol{\upsilon})$ for all $(\boldsymbol{\upsilon}, \boldsymbol{\tau}, \boldsymbol{\nu}) \in U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times T^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$;
- $\boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}^\vee}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)f$ for all $\boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b\\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d\end{pmatrix} \in \Gamma(p^n)$.

Therefore, putting $\Psi_2 := \Phi^{-1}$, one obtains the desired isomorphism

$$\Psi_2 : \omega^{(1)} \xrightarrow{\simeq} \omega^{(2)}.$$

Construction of Ψ_3. By the construction of $\underline{\omega}^{\kappa_{\mathcal{U}},+}_{n,w}$ and Lemma 2.3.6, one immediately obtains an identification of $\omega^{(2)}$ with $\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V})$. We simply take Ψ_3 to be this identification.

Putting everything together, the composition $\Psi^+ = \Psi_3\circ\Psi_2\circ\Psi_1$ yields an isomorphism

$$\Psi^+ : \underline{\omega}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V}) \xrightarrow{\simeq} \underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}(\mathcal{V}).$$

It is also straightforward to check that the construction is functorial in $\mathcal{V}$. By gluing, we arrive at an isomorphism

$$\Psi^+ : \underline{\omega}^{\kappa_{\mathcal{U}},+}_{n,w} \xrightarrow{\simeq} \underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}},+}_{n,w}.$$

□

3.2.14. By the observation in 3.2.12 and Proposition 3.2.13, $\underline{\omega}^{\kappa_{\mathcal{U}}}_w$ can be identified with the sheaf of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}/\Gamma(p^n)$-invariants of $h_{n,*}\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}}}_{n,w}$. Hence, $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\text{két}}$ can be identified with the sheaf of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}/\Gamma(p^n)$-invariants of $h_{n,*}\underline{\widetilde{\omega}}^{\kappa_{\mathcal{U}}}_{n,w,\text{két}}$, later of which is an admissible Kummer étale Banach sheaf of $\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}\widehat{\otimes}R_{\mathcal{U}}$-modules by Lemma 3.2.11 and Lemma A.2.12. Consequently, such a description allows us to apply Corollary A.2.18 to the sheaf $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\text{két}}$. This will be used in the construction of the overconvergent Eichler–Shimura morphism in Chapter 6.

3.3 Hecke operators

3.3.1. In this section, we spell out how the Hecke operators act on the overconvergent Siegel modular forms. Those Hecke operators at the primes dividing the tame level N are not considered in this thesis. Through out this section, let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a weight and $w > 1 + r_{\mathcal{U}}$.

3.3.2 (Hecke operators outside Np). We define the Hecke operators outside Np using correspondences. Let ℓ be a rational prime that does not divide Np. For every $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell)$, consider the moduli space $X_{\boldsymbol{\gamma},\mathrm{Iw}^+}$ over X_{Iw^+} parameterising *isogenies of type* $\boldsymbol{\gamma}$. More precisely, $X_{\boldsymbol{\gamma},\mathrm{Iw}^+}$ is the moduli space of sextuples

$$(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], \{C_i : i = 1, \dots, g\}, L)$$

where $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], \{C_i : i = 1, \dots, g\}) \in X_{\mathrm{Iw}^+}$ and $L \subset A$ is a subgroup of finite order such that the isogeny $(A, \lambda) \to (A/L, \lambda')$ is of type $\boldsymbol{\gamma}$ in the sense of [FC90, Chapter VII, §3], where λ' stands for the induced principal polarisation. According to *loc. cit.*, for every isogeny of type $\boldsymbol{\gamma}$, its dual isogeny is also of type $\boldsymbol{\gamma}$. In particular, the assignment

$$(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], \{C_i : i = 1, \dots, g\}, L) \mapsto (A' = A/L, \lambda', \psi'_N, \mathrm{Fil}_\bullet A'[p], \{C'_i : i = 1, \dots, g\}, L')$$

defines an isomorphism $\Phi_{\boldsymbol{\gamma}} : X_{\boldsymbol{\gamma},\mathrm{Iw}^+} \xrightarrow{\sim} X_{\boldsymbol{\gamma},\mathrm{Iw}^+}$, where

- λ' is the induced polarisation on A';
- ψ'_N, $\mathrm{Fil}_\bullet A'[p]$, and C'_i's are induced from ψ_N, $\mathrm{Fil}_\bullet A[p]$, and C_i's, respectively, via the isomorphisms $A[N] \simeq A'[N]$ and $A[p] \simeq A'[p]$;
- L' is defined by the dual isogeny of $(A, \lambda) \to (A', \lambda')$.

There are two finite étale projections

$$\begin{tikzcd} & X_{\boldsymbol{\gamma},\mathrm{Iw}^+} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ X_{\mathrm{Iw}^+} & & X_{\mathrm{Iw}^+} \end{tikzcd}$$

where pr_1 is the forgetful map and pr_2 sends the sextuple $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet A[p], \{C_i : i = 1, \dots, g\}, L)$ to the quintuple $(A' = A/L, \lambda', \psi'_N, \mathrm{Fil}_\bullet A'[p], \{C'_i : i = 1, \dots, g\})$ described as above. Clearly, we have $\mathrm{pr}_1 = \mathrm{pr}_2 \circ \Phi_{\boldsymbol{\gamma}}$.

Let $\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+}$ be the adic space associated with $X_{\boldsymbol{\gamma},\mathrm{Iw}^+}$ by taking analytification. We obtain finite étale morphisms $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+} \rightrightarrows \mathcal{X}_{\mathrm{Iw}^+}$ as well as an isomorphism $\Phi_{\boldsymbol{\gamma}} : \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+} \to \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+}$. We further pass to the w-ordinary locus. More precisely, let $\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w}$ denote the preimage of $\mathcal{X}_{\mathrm{Iw}^+,w}$ under the projection pr_1. Notice that $\Phi_{\boldsymbol{\gamma}}$ preserves $\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w}$ as the isogeny $(A, \lambda) \to (A', \lambda')$ induces a symplectic isomorphism $T_pA \simeq T_pA'$. Hence, we obtain finite étale morphisms

$$\begin{tikzcd} & \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{X}_{\mathrm{Iw}^+,w} & & \mathcal{X}_{\mathrm{Iw}^+,w} \end{tikzcd} \tag{3.1}$$

and an isomorphism $\Phi_{\boldsymbol{\gamma}} : \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w} \xrightarrow{\simeq} \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w}$. We still have $\mathrm{pr}_1 = \mathrm{pr}_2 \circ \Phi_{\boldsymbol{\gamma}}$.

In order to define the Hecke operator, we shall first construct a natural isomorphism

$$\varphi_{\boldsymbol{\gamma}} : \mathrm{pr}_2^* \underline{\omega}_w^{\kappa_{\mathcal{U}}} \xrightarrow{\simeq} \mathrm{pr}_1^* \underline{\omega}_w^{\kappa_{\mathcal{U}}}.$$

Here we have abused the notation and still write $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ for its restriction to $\mathcal{X}_{\mathrm{Iw}^+,w}$. Indeed, pulling back the diagram (3.1) along the projection $h_{\mathrm{Iw}^+} : \mathcal{X}_{\Gamma(p^\infty),w} \to \mathcal{X}_{\mathrm{Iw}^+,w}$, we obtain finite étale morphisms

$$\begin{tikzcd} & \mathcal{X}_{\boldsymbol{\gamma},\Gamma(p^\infty),w} \arrow[dl, "\mathrm{pr}_{1,\infty}"'] \arrow[dr, "\mathrm{pr}_{2,\infty}"] & \\ \mathcal{X}_{\Gamma(p^\infty),w} & & \mathcal{X}_{\Gamma(p^\infty),w} \end{tikzcd}$$

between perfectoid spaces and an $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant isomorphism $\Phi_{\boldsymbol{\gamma},\infty} : \mathcal{X}_{\boldsymbol{\gamma},\Gamma(p^\infty),w} \xrightarrow{\simeq} \mathcal{X}_{\boldsymbol{\gamma},\Gamma(p^\infty),w}$. The isomorphism $\Phi_{\boldsymbol{\gamma},\infty}$ induces an isomorphism

$$\Phi^*_{\boldsymbol{\gamma},\infty} : \mathrm{pr}^*_{2,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \xrightarrow{\simeq} \mathrm{pr}^*_{1,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}}.$$

It then induces an isomorphism

$$\Phi^*_{\boldsymbol{\gamma},\infty} : \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}^*_{2,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}) \xrightarrow{\simeq} \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}^*_{1,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$$

by taking the identity on $R_{\mathcal{U}}$.

Recall that $\mathfrak{z}$ is the pullback of the coordinate $\boldsymbol{z}$ via the Hodge–Tate period map $\pi_{\mathrm{HT}} : \mathcal{X}_{\Gamma(p^\infty),w} \to \mathcal{F}\ell^\times_w$. Let $\mathfrak{z}' := \mathrm{pr}^*_{1,\infty} \mathfrak{z}$ and $\mathfrak{z}'' := \mathrm{pr}^*_{2,\infty} \mathfrak{z}$. Since $\Phi_{\boldsymbol{\gamma},\infty}$ induces an isomorphism on the p-adic Tate module, we have $\mathfrak{z}' = \mathfrak{z}''$. Consequently, a section f of the sheaf $\mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}^*_{2,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ satisfies

$$\boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}'' \boldsymbol{\gamma}_c)^{-1} f \quad \text{for all } \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}},$$

if and only if the section $\Phi^*_{\boldsymbol{\gamma},\infty}(f)$ of the sheaf $\mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}^*_{1,\infty} \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ satisfies

$$\boldsymbol{\gamma}^*(\Phi^*_{\boldsymbol{\gamma},\infty}(f)) = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}' \boldsymbol{\gamma}_c)^{-1}(\Phi^*_{\boldsymbol{\gamma},\infty}(f)) \quad \text{for all } \boldsymbol{\gamma} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}.$$

This yields the desired isomorphism

$$\varphi_{\boldsymbol{\gamma}} : \mathrm{pr}_2^* \underline{\omega}_w^{\kappa_{\mathcal{U}}} \xrightarrow{\simeq} \mathrm{pr}_1^* \underline{\omega}_w^{\kappa_{\mathcal{U}}}.$$

Given this, we consider the composition

$$\begin{tikzcd} T_{\boldsymbol{\gamma}} : & H^0(\mathcal{X}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}}) \arrow[r, "\mathrm{pr}_2^*"] & H^0(\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w}, \mathrm{pr}_2^* \underline{\omega}_w^{\kappa_{\mathcal{U}}}) \arrow[dll, "\varphi_{\boldsymbol{\gamma}}"', rounded corners] & \\ & H^0(\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+,w}, \mathrm{pr}_1^* \underline{\omega}_w^{\kappa_{\mathcal{U}}}) \arrow[r, "\mathrm{Tr}\,\mathrm{pr}_1"] & H^0(\mathcal{X}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}}). \end{tikzcd}$$

Finally, we have to extend the construction to the boundary. In fact, we shall prove that the sections of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ on $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ are precisely the *bounded* sections of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ over the open part $\mathcal{X}_{\mathrm{Iw}^+,w}$.

Lemma 3.3.3. *Every bounded section of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ on $\mathcal{X}_{\mathrm{Iw}^+,w}$ uniquely extends to a section of $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$.*

Proof. The argument is similar to the proof of [AIP15, Proposition 5.5.2]. Recall that we can view sections of $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ as a sections on $\mathcal{IW}_w^+$. By applying [Lüt74, Theorem 1.6] to $\mathcal{IW}_w^+$, the result follows. □

3.3.4. Thanks to Lemma 3.3.3, and observe that

$$\Phi_{\gamma,\infty}^* : \mathscr{C}_{\kappa_{\mathcal{U}}}^{w-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}_{2,\infty}^* \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}) \xrightarrow{\sim} \mathscr{C}_{\kappa_{\mathcal{U}}}^{w-\mathrm{an}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathrm{pr}_{1,\infty}^* \mathscr{O}_{\mathcal{X}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$$

sends bounded sections to bounded sections, we know that T_γ extends to the boundary. We arrive at the Hecke operator

$$T_\gamma : M_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}^w = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}}) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}}) = M_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}^w.$$

3.3.5 (Hecke operators at p)**.** For $1 \leq i \leq g$, we consider matrices $\mathbf{u}_{p,i} \in \mathrm{GSp}_{2g}(\mathbf{Q}_p) \cap M_{2g}(\mathbf{Z}_p)$ defined by

$$\mathbf{u}_{p,i} := \begin{pmatrix} \mathbb{1}_i & & & \\ & p\,\mathbb{1}_{g-i} & & \\ & & p\,\mathbb{1}_{g-i} & \\ & & & p^2\,\mathbb{1}_i \end{pmatrix}$$

for $1 \leq i \leq g-1$, and

$$\mathbf{u}_{p,g} := \begin{pmatrix} \mathbb{1}_g & \\ & p\,\mathbb{1}_g \end{pmatrix}.$$

For later use, we write

$$\mathbf{u}_{p,i} = \begin{pmatrix} \mathbf{u}_{p,i}^{\square} & \\ & \mathbf{u}_{p,i}^{\blacksquare} \end{pmatrix}$$

where $\mathbf{u}_{p,i}^{\square}$ and $\mathbf{u}_{p,i}^{\blacksquare}$ are the corresponding $g \times g$ diagonal matrices.

Notice that the $\mathbf{u}_{p,i}$-action on $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ preserves $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$. This can be checked at the infinite level via local coordinates; *i.e.*, the action of $\mathbf{u}_{p,i}$ on $\boldsymbol{z}$ is given by

$$\boldsymbol{z} \,.\, \mathbf{u}_{p,i} = \mathbf{u}_{p,i}^{\square,-1}\, \boldsymbol{z}\, \mathbf{u}_{p,i}^{\blacksquare} = \begin{cases} \begin{pmatrix} p\,\boldsymbol{z}_{1,1} & \cdots & p\,\boldsymbol{z}_{1,g-i} & p^2\,\boldsymbol{z}_{1,g+1-i} & \cdots & p^2\,\boldsymbol{z}_{1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ p\,\boldsymbol{z}_{i,1} & \cdots & p\,\boldsymbol{z}_{i,g-i} & p^2\,\boldsymbol{z}_{i,g+1-i} & \cdots & p^2\,\boldsymbol{z}_{i,g} \\ \boldsymbol{z}_{i+1,1} & \cdots & \boldsymbol{z}_{i+1,g-i} & p\,\boldsymbol{z}_{i+1,g+1-i} & \cdots & p\,\boldsymbol{z}_{i+1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ \boldsymbol{z}_{g,1} & \cdots & \boldsymbol{z}_{g,g-i} & p\,\boldsymbol{z}_{g,g+1-i} & \cdots & p\,\boldsymbol{z}_{g,g} \end{pmatrix}, & \text{if } i = 1, ..., g-1 \\ p\,\boldsymbol{z}, & \text{if } i = g \end{cases}.$$

In particular, when $i = g$, the $\mathbf{u}_{p,g}$-action actually sends $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$ into $\overline{\mathcal{X}}_{\Gamma(p^\infty),w+1}$.

The operator $U_{p,i}$ is defined in two steps:

(i) For $f \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$, we define $\mathbf{u}_{p,i} .f \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ by
$$\mathbf{u}_{p,i} .f(\boldsymbol{\gamma}') := \mathbf{u}^*_{p,i}\, f(\mathbf{u}^{\square}_{p,i}\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0)$$
where $\boldsymbol{\gamma}' = \boldsymbol{\gamma}'_0 \boldsymbol{\beta}'_0 \in \mathrm{Iw}_{\mathrm{GL}_g}$ with $\boldsymbol{\gamma}'_0 \in U^{\mathrm{opp}}_{\mathrm{GL}_g,1}$ and $\boldsymbol{\beta}'_0 \in B_{\mathrm{GL}_g,0}$.

(ii) Suppose $f \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ satisfies
$$\boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_a + \mathfrak{z}\, \boldsymbol{\gamma}_c)^{-1} f$$
for all $\boldsymbol{\gamma} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$; *i.e.*, $\boldsymbol{\gamma} .f = f$. Pick a decomposition of the double coset
$$\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} = \bigsqcup_{j=1}^{m} \boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$$
with $\boldsymbol{\delta}_{i,j} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. Define
$$U_{p,i}(f) := p^{\nu_i} \sum_{j=1}^{m} \boldsymbol{\delta}_{i,j} .(\mathbf{u}_{p,i} .f) \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}),$$
where $\nu_i = -(g-i)(g+1)$ for $i = 1, ..., g-1$ and $\nu_g = \frac{-g(g+1)}{2}$. Here, we follow the normalisation as in [AIP15, §6.2].

Of course, we have to verify that $U_{p,i}(f)$ is independent of the choice to the representatives $\boldsymbol{\delta}_{ij}$'s. Suppose $\{\boldsymbol{\delta}'_{i,j}\}_{j=1}^m$ is another set of representatives. Up to re-labelling, we may assume that
$$\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} = \boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$$
for every $j = 1, \dots, m$. Write $\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i} = \boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i}\, \boldsymbol{\gamma}_j$ for some $\boldsymbol{\gamma}_j \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. We have to check that $\boldsymbol{\delta}_{ij} .(\mathbf{u}_{p,i} .f) = \boldsymbol{\delta}_{ij} .(\mathbf{u}_{p,i} .f)$. Indeed, if we write $\boldsymbol{\delta}_{ij} = \begin{pmatrix} \boldsymbol{\delta}_{ija} & \boldsymbol{\delta}_{ijb} \\ \boldsymbol{\delta}_{ijc} & \boldsymbol{\delta}_{ijd} \end{pmatrix}$, $\boldsymbol{\delta}'_{ij} = \begin{pmatrix} \boldsymbol{\delta}'_{ija} & \boldsymbol{\delta}'_{ijb} \\ \boldsymbol{\delta}'_{ijc} & \boldsymbol{\delta}'_{ijd} \end{pmatrix}$,

and $\boldsymbol{\gamma}_j = \begin{pmatrix} \gamma_{ja} & \gamma_{jb} \\ \gamma_{jc} & \gamma_{jd} \end{pmatrix}$, then for every $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, we have

$$\begin{aligned}
\boldsymbol{\delta}'_{ij} .(\mathbf{u}_{p,i} .f)(\boldsymbol{\gamma}') =& \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\delta}'_{ija} + \mathfrak{z}\, \boldsymbol{\delta}'_{ijc})\, \boldsymbol{\delta}^*_{ij}(\mathbf{u}^*_{p,i}\, f)(\mathbf{u}^{\square}_{p,i}\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& (\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i})^* f(\mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}'_{ija} + \mathfrak{z}\, \boldsymbol{\delta}'_{ijc})\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& (\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i})^* f({}^{\mathfrak{t}}(\boldsymbol{\gamma}_{ja} + \mathfrak{z}\, \boldsymbol{\gamma}_{jc})\, \mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc})\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& (\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i})^* (\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}_{ja} + \mathfrak{z}\, \boldsymbol{\gamma}_{jc}) f)(\mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc})\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& (\boldsymbol{\delta}'_{ij}\, \mathbf{u}_{p,i})^* (\boldsymbol{\gamma}_j^{-1,*}\, f)(\mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc})\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& (\boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i})^* f(\mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc})\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc})(\boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i})^* f(\mathbf{u}^{\square}_{p,i}\, \boldsymbol{\gamma}'_0\, \mathbf{u}^{\square,-1}_{p,i}\, \boldsymbol{\beta}'_0) \\
=& \boldsymbol{\delta}_{ij} .(\mathbf{u}_{p,i} .f)(\boldsymbol{\gamma}')
\end{aligned}$$

as desired. Here, the third equality follows from the identity

$$\mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}'_{ija} + \mathfrak{z}\, \boldsymbol{\delta}'_{ijc}) = {}^{\mathfrak{t}}(\boldsymbol{\gamma}_{ja} + \mathfrak{z}\, \boldsymbol{\gamma}_{jc})\, \mathbf{u}^{\square}_{p,i}\, {}^{\mathfrak{t}}(\boldsymbol{\delta}_{ija} + \mathfrak{z}\, \boldsymbol{\delta}_{ijc}).$$

Lemma 3.3.6. *Suppose* $f \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ *such that* $\boldsymbol{\gamma} .f = f$ *for all* $\boldsymbol{\gamma} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. *Then, the section* $U_{p,i}(f) \in \mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ *satisfies* $\boldsymbol{\gamma} .(U_{p,i}(f)) = U_{p,i}(f)$ *for all* $\boldsymbol{\gamma} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$.

Proof. We have

$$\boldsymbol{\gamma} .(U_{p,i}(f)) = p^{\nu_i} \sum_{j=1}^m \boldsymbol{\gamma} .(\boldsymbol{\delta}_{ij} .(\mathbf{u}_{p,i}(f))) = p^{\nu_i} \sum_{j=1}^m (\boldsymbol{\gamma}\, \boldsymbol{\delta}_{ij}).(\mathbf{u}_{p,i}(f)).$$

The last term indeed computes $U_{p,i}(f)$ because $\{\boldsymbol{\gamma}\, \boldsymbol{\delta}_{ij} : 1 \leq j \leq m\}$ is also a valid set of representatives. □

3.3.7. Consequently, we arrive at the Hecke operator

$$U_{p,i} : M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_w) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_w) = M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

Moreover, for any $x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$, we denote by $U^x_{p,i}$ the Hecke operator defined by the double coset

$$\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}(x \cdot \mathbf{u}_{p,i})\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}},$$

whose action is defined analogously as above.

Definition 3.3.8. *The* ***Hecke algebra outside*** Np *is defined to be*

$$\mathbb{T}^{Np} := \mathbf{Z}_p \left[T_{\boldsymbol{\gamma}}; \boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell),\ \ell \nmid Np\right]$$

and the ***total Hecke algebra*** *is defined to be*

$$\mathbb{T} := \mathbb{T}^{Np} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[U^x_{p,i} : i = 0, 1, ..., g-1, x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}].$$

Proposition 3.3.9. *The operator $U_p := \prod_{i=1}^g U_{p,i}$ is a compact operator on $M^{\kappa_{\mathcal{U}}}_{\mathrm{Iw}^+,w}$.*

Proof. Note that the action of $\mathbf{u}_{p,g}$ on $\boldsymbol{z}$ is given by $p\,\boldsymbol{z}$ and that, by definition, the action of $\prod_{i=1}^g \mathbf{u}_{p,i}$ on $\mathscr{C}^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}})$ factors through the inclusion

$$\mathscr{C}^{(w-1)-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}) \hookrightarrow \mathscr{C}^{w-an}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}} \widehat{\otimes} R_{\mathcal{U}}).$$

This means that U_p factors as

$$U_p : H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_w) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w+1}, \underline{\omega}^{\kappa_{\mathcal{U}}}_w) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w+1}, \underline{\omega}^{\kappa_{\mathcal{U}}}_{w-1}) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_w),$$

where the first arrow is the natural restriction map.

To show the desired result, note that it is known that restrictions of the structure sheaf of $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ are compact operators. Moreover, by the discussion in [Han17, §2.2], the injection of $(w-1)$-analytic functions into w-analytic functions is compact. The assertion then follows by combining these two facts. □

Remark 3.3.10. Note that the subspace $S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} \subset M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$ of w-overconvergent Siegel cuspforms of weight $\kappa_{\mathcal{U}}$ is stable under the action of $\mathbb{T}$. Moreover, as U_p is a compact operator on $M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$, it is also a compact operator on $S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$.

3.4 Classical Siegel modular forms

3.4.1. The goal of this section is to show that our space of overconvergent Siegel modular forms does contain the space of classical (algebraic) Siegel modular forms. This justifies the name

3.4.2. Let $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{\geq 0}$ be a dominant weight and consider $k^\vee = (-k_g, \ldots, -k_1)$. Let $\mathcal{M} := \mathrm{Isom}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}(\mathscr{O}^g_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}, \underline{\omega}_{\mathrm{Iw}^+})$ be the GL_g-torsor over $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ together with the structure morphism $\vartheta : \mathcal{M} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+}$. Then the sheaf $\underline{\omega}^k_{\mathrm{Iw}^+}$ of ***classical Siegel modular forms of weight** k **(of strict Iwahori level)*** is defined to be

$$\underline{\omega}^k_{\mathrm{Iw}^+} := \vartheta_* \mathscr{O}_{\mathcal{M}}[k^\vee];$$

namely, the subsheaf of $\vartheta_* \mathscr{O}_{\mathcal{M}}$ on which T_{GL_g} acts through the character $k^\vee$. The ***space of classical Siegel modular forms of weight** k **(of strict Iwahori level)*** is defined to be

$$M^{\mathrm{cl}}_{\mathrm{Iw}^+,k} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^k_{\mathrm{Iw}^+})$$

equipped with naturally defined Hecke operators.

Remark 3.4.3. One can also define the sheaf of integral classical Siegel modular forms by

$$\underline{\omega}^{k,+}_{\mathrm{Iw}^+} := \vartheta_* \mathscr{O}^+_{\mathcal{M}}[k^\vee].$$

But we do not need this in this thesis.

3.4.4. We would like to have a *perfectoid description* of the sheaf $\underline{\omega}^k_{\mathrm{Iw}^+}$. To this end, we first introduce the following notations:

(i) Let $P(\mathrm{GL}_g, \mathbb{A}^1)$ denote the $\mathbf{Q}_p$-vector space of maps $\mathrm{GL}_g \to \mathbb{A}^1$ between algebraic varieties over $\mathbf{Q}_p$.

(ii) For every uniform $\mathbf{C}_p$-Banach algebra B, define

$$P(\mathrm{GL}_g, B) := P(\mathrm{GL}_g, \mathbb{A}^1)\widehat{\otimes}_{\mathbf{Q}_p} B$$

and let $P_k(\mathrm{GL}_g, B)$ denote the subspace of $P(\mathrm{GL}_g, B)$ consisting of those $f : \mathrm{GL}_g \to B$ such that $f(\boldsymbol{\gamma}\,\boldsymbol{\beta}) = k(\boldsymbol{\beta})f(\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} \in \mathrm{GL}_g$ and $\boldsymbol{\beta} \in B_{\mathrm{GL}_g}$.

(iii) There is a natural left action of GL_g on $P_k(\mathrm{GL}_g, B)$ given by

$$(\boldsymbol{\gamma} . f)(\boldsymbol{\gamma}') = f({}^{\mathrm{t}}\boldsymbol{\gamma}\,\boldsymbol{\gamma}')$$

for all $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathrm{GL}_g$ and $f \in P_k(\mathrm{GL}_g, B)$. This left action is denoted by

$$\rho_k : \mathrm{GL}_g \to \mathrm{Aut}(P_k(\mathrm{GL}_g, B)).$$

Proposition 3.4.5. *For any affinoid open $\mathcal{V} \subset \overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ with preimage $\mathcal{V}_\infty$ in $\overline{\mathcal{X}}_{\Gamma(p^\infty),w}$, we have a natural identification*

$$\underline{\omega}^k_{\mathrm{Iw}^+}(\mathcal{V}) = \left\{ f \in P_k(\mathrm{GL}_g, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty)) : \boldsymbol{\gamma}^* f = \rho_k(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f,\ \ \forall\, \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \right\}.$$

In particular, there is a natural injection

$$\underline{\omega}^k_{\mathrm{Iw}^+}|_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}} \hookrightarrow \underline{\omega}^k_w. \tag{3.2}$$

Proof. For the first statement, the strategy in the proof of Proposition 3.2.13 applies verbatim, except that we consider the torsor $\mathcal{M}$ in place of $\mathcal{IW}^+_w$. The details are left to the reader. The inclusion $\underline{\omega}^k_{\mathrm{Iw}^+}|_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}} \hookrightarrow \underline{\omega}^k_w$ follows from the natural inclusion from $P_k(\mathrm{GL}_g, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty))$ into $C^{w-\mathrm{an}}_k(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty),w}}(\mathcal{V}_\infty))$. □

Lemma 3.4.6. *The Hecke-equivariant composition of maps*

$$M^{\mathrm{cl}}_{\mathrm{Iw}^+,k} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^k_{\mathrm{Iw}^+}) \xrightarrow{\mathrm{Res}} H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^k_{\mathrm{Iw}^+}) \hookrightarrow M^w_{\mathrm{Iw}^+,k}$$

is injective.

Proof. It suffices to show that

$$\mathrm{Res} : H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^k_{\mathrm{Iw}^+}) \to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^k_{\mathrm{Iw}^+})$$

is injective; namely, given any global section f of $\underline{\omega}^k_{\mathrm{Iw}^+}$ that vanishes on $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$, we have to show that $f = 0$ on every irreducible component of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$.

For every algebraic variety Y over $\mathbf{C}_p$, we know that Y is irreducible if and only if the associated adic space $\mathcal{Y}$ over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ is irreducible (see [Con99, Theorem 2.3.1] and [Hub13, §1.1.11.(c)]). In particular, the irreducible components of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ coincide with the irreducible components of $\overline{X}^{\mathrm{tor}}_{\mathrm{Iw}^+}$. As $\overline{X}^{\mathrm{tor}}_{\mathrm{Iw}^+}$ is a compactification of X_{Iw^+}, its irreducible components correspond to the irreducible components of X_{Iw^+}. Under the identification

$$X_{\mathrm{Iw}^+}(\mathbf{C}) = \mathrm{GSp}_{2g}(\mathbf{Q})\backslash \mathrm{GSp}_{2g}(\mathbf{A}_f) \times \mathbb{H}_g \,/\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \Gamma(N),$$

[Del71, §2] provides the following description of the irreducible components of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$:

$$\pi_0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}) = \mathbf{Q}_{>0}\backslash \mathbb{G}_m(\mathbf{A}_f)/\varsigma\left(\Gamma(N)\,\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}\right).$$

where ς is the character of similitude involved in the definition of GSp_{2g}. There is a similar description for $\pi_0(\overline{\mathcal{X}})$. Note that $\pi_0(\overline{\mathcal{X}}_{\mathrm{Iw}^+})$ is the same as $\pi_0(\overline{\mathcal{X}})$ because $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ and $\mathrm{GSp}_{2g}(\mathbf{Z}_p)$ have the same image via ς. In particular, since every irreducible component in $\pi_0(\overline{\mathcal{X}})$ contains an ordinary point, every irreducible component of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ intersects $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$.

By definition, f can be viewed as a global section of the structure sheaf of $\mathcal{M}$. Let $\mathcal{C}$ be any irreducible component of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$, it remains to show that f vanishes on $\mathcal{M} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \mathcal{C}$. Indeed, observe that $\mathcal{M} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \mathcal{C}$ is irreducible and f vanishes on $\mathcal{M} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}}} (\mathcal{C} \cap \overline{\mathcal{X}}_{\mathrm{Iw}^+,w})$. Hence, the desired vanishing follows from [Ber96, Proposition 0.1.13] which states that a rigid analytic function vanishing on an open subset of an irreducible rigid analytic variety is identically zero. $\square$

3.5 The construction à la Andreatta–Iovita–Pilloni

3.5.1. We dedicate in this section to briefly recall the construction of sheaves of overconvergent Siegel modular forms introduced in [AIP15]. In fact, the readers may find that §3.2 is highly inspired by [AIP15].

3.5.2. Choose $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2})$ and let n be a positive integer such that $v < \frac{1}{2p^{n-1}}$. Consider the open subset

$$\overline{\mathcal{X}}(v) := \{\boldsymbol{x} \in \overline{\mathcal{X}} : |\widetilde{\mathrm{Ha}}(\boldsymbol{x})| \geq p^{-v}\} \subset \overline{\mathcal{X}},$$

where $\widetilde{\mathrm{Ha}}$ is a fixed lift of the Hasse invariant.[2] Thanks to [AIP15, Proposition 4.1.3], for every $1 \leq m \leq n$, there is a universal canonical subgroup $\mathcal{H}_m$ of level m of the tautological semiabelian variety over $\overline{\mathcal{X}}(v)$. Let $\underline{\omega}_v$ denote the restriction of $\underline{\omega}$ on $\overline{\mathcal{X}}(v)$.

3.5.3. Andreatta–Iovita–Pilloni's construction of the sheaves concerns the following finite covers of $\overline{\mathcal{X}}(v)$:

- Let

$$\overline{\mathcal{X}}_1(p^n)(v) := \mathrm{Isom}_{\overline{\mathcal{X}}(v)}((\mathbf{Z}/p^n\mathbf{Z})^g, \mathcal{H}_n^\vee)$$

[2] We point out that, for those $\boldsymbol{x}$ at the boundary, the Hasse invariant of $\boldsymbol{x}$ is defined to be the Hasse invariant of the abelian part of the semiabelian scheme associated with $\boldsymbol{x}$.

be the adic space over $\overline{\mathcal{X}}(v)$ which parameterises trivialisations of $\mathcal{H}_n^\vee$. Notice that the group $\mathrm{GL}_g(\mathbf{Z}/p^n\mathbf{Z})$ naturally acts on $\overline{\mathcal{X}}_1(p^n)(v)$ from the right by permuting the trivialisations.

- Let
$$\overline{\mathcal{X}}_1(v) := \mathrm{Isom}_{\overline{\mathcal{X}}(v)}((\mathbf{Z}/p\mathbf{Z})^g, \mathcal{H}_1^\vee)$$
be the adic space over $\overline{\mathcal{X}}(v)$ which parameterises trivialisations of $\mathcal{H}_1^\vee$.

- The group $\mathrm{GL}_g(\mathbf{Z}/p\mathbf{Z})$ naturally acts on $\overline{\mathcal{X}}_1(v)$ from the right by permuting the trivialisations. By taking the quotient
$$\overline{\mathcal{X}}_{\mathrm{Iw}}(v) := \overline{\mathcal{X}}_1(v)/B_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z}),$$
we obtain an adic space $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ over $\overline{\mathcal{X}}(v)$ which parameterises full flags $\mathrm{Fil}_\bullet \mathcal{H}_1^\vee$ of $\mathcal{H}_1^\vee$.

We let $\underline{\omega}_{n,v}$ (resp., $\underline{\omega}_{\mathrm{Iw},v}$) be the pullback of $\underline{\omega}_v$ along $\overline{\mathcal{X}}_1(p^n)(v) \to \overline{\mathcal{X}}(v)$ (resp., $\overline{\mathcal{X}}_{\mathrm{Iw}}(v) \to \overline{\mathcal{X}}(v)$).

3.5.4. We will also need the following integral models of the aforementioned geometric objects:

- Recall that $\overline{\mathfrak{X}}$ is the formal completion of $\overline{X}_0^{\mathrm{tor}}$ along the special fibre. Let $\widetilde{\mathfrak{X}}(v)$ be the blowup of $\overline{\mathfrak{X}}$ along the ideal $(\widetilde{\mathrm{Ha}}, p^v)$. Let $\overline{\mathfrak{X}}(v)$ be the p-adic completion of the normalisation of the largest open formal subscheme of $\widetilde{\mathfrak{X}}(v)$ where the ideal $(\widetilde{\mathrm{Ha}}, p^v)$ is generated by $\widetilde{\mathrm{Ha}}$. Then $\overline{\mathfrak{X}}(v)$ is a formal model of $\overline{\mathcal{X}}(v)$.

- Let $\overline{\mathfrak{X}}_1(p^n)(v)$ be the normalisation of $\overline{\mathfrak{X}}(v)$ in $\overline{\mathcal{X}}_1(p^n)(v)$. The group $\mathrm{GL}_g(\mathbf{Z}/p^n\mathbf{Z})$ naturally acts on $\overline{\mathfrak{X}}_1(p^n)(v)$.

- Let $\overline{\mathfrak{X}}_1(v)$ be the normalisation of $\overline{\mathfrak{X}}(v)$ in $\overline{\mathcal{X}}_1(v)$. The group $\mathrm{GL}_g(\mathbf{Z}/p\mathbf{Z})$ naturally acts on $\overline{\mathfrak{X}}_1(v)$.

- Let $\overline{\mathfrak{X}}_{\mathrm{Iw}}(v)$ be the normalisation of $\overline{\mathfrak{X}}(v)$ in $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$. We can identify $\overline{\mathfrak{X}}_{\mathrm{Iw}}(v)$ with the quotient $\overline{\mathfrak{X}}_1(v)/B_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z})$.

Moreover, the integral models of $\underline{\omega}_{n,v}$ and $\underline{\omega}_{\mathrm{Iw},v}$ are given as follows. Let $\mathfrak{G}_v^{\mathrm{univ}}$ be the tautological semiabelian scheme over $\overline{\mathfrak{X}}(v)$ with the structure morphism
$$\pi : \mathfrak{G}_v^{\mathrm{univ}} \to \overline{\mathfrak{X}}(v).$$
Define
$$\underline{\Omega}_v := \pi_* \Omega^1_{\mathfrak{G}_v^{\mathrm{univ}}/\overline{\mathfrak{X}}(v)}.$$
Then, let $\underline{\Omega}_{n,v}$ (resp., $\underline{\Omega}_{\mathrm{Iw},v}$) be the pullback of $\underline{\Omega}_v$ along $\overline{\mathfrak{X}}_1(p^n)(v) \to \overline{\mathfrak{X}}(v)$ ($\overline{\mathfrak{X}}_{\mathrm{Iw}}(v) \to \overline{\mathfrak{X}}(v)$).

3.5.5. Now suppose $w \in \mathbf{Q}_{>0}$ lies in the interval $\left(n-1+\frac{v}{p-1}, n-\frac{vp^n}{p-1}\right]$. Let
$$\psi_n^{\mathrm{univ}} : (\mathbf{Z}/p^n\mathbf{Z})^g \simeq \mathcal{H}_n^\vee$$

denote the universal trivialisation of $\mathcal{H}_n^\vee$ over $\overline{\mathfrak{X}}_1(p^n)(v)$. Then [AIP15, Proposition 4.3.1] yields a locally free $\mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}$-submodule $\mathscr{F} \subset \underline{\Omega}_{n,v}$ of rank g, equipped with a map

$$\mathrm{HT}_{n,v,w} : (\mathbf{Z}/p^n\mathbf{Z})^g \overset{\psi_n^{\mathrm{univ}}}{\simeq} \mathcal{H}_n^\vee \to \mathscr{F} \otimes_{\mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}} \mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}/p^w$$

which induces an isomorphism

$$\mathrm{HT}_{n,v,w} \otimes \mathrm{id} : (\mathbf{Z}/p^n\mathbf{Z})^g \otimes_{\mathbf{Z}} \mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}/p^w \simeq \mathscr{F} \otimes_{\mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}} \mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}/p^w.$$

More precisely, locally on $\overline{\mathfrak{X}}_1(p^n)(v)$, consider the family version of the Hodge–Tate map

$$\mathrm{HT}_n : (\mathbf{Z}/p^n\mathbf{Z})^g \overset{\psi_n^{\mathrm{univ}}}{\simeq} \mathcal{H}_n^\vee \to \omega_{\mathcal{H}_n}$$

studied in [AIP15, §4]. Let $\epsilon_1, \dots, \epsilon_g$ be the standard $(\mathbf{Z}/p^n\mathbf{Z})$-basis for $(\mathbf{Z}/p^n\mathbf{Z})^g$ and let $\widetilde{\mathrm{HT}}_n(\epsilon_i)$ be lifts of $\mathrm{HT}_n(\epsilon_i)$ from $\omega_{\mathcal{H}_n}$ to $\underline{\Omega}_{n,v}$. Then $\mathscr{F}$ is generated by $\widetilde{\mathrm{HT}}_n(\epsilon_1), \dots, \widetilde{\mathrm{HT}}_n(\epsilon_g)$. It turns out this local construction glues to a locally free $\mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}$-module of rank g.

3.5.6. In [AIP15, §4.5], Andreatta–Iovita–Pilloni constructs a formal scheme $\mathfrak{IW}_{w,v}^+$ over $\overline{\mathfrak{X}}_1(p^n)(v)$ which parameterises such w-compatible objects. More precisely, $\mathfrak{IW}_{w,v}^+$ is the formal schemes over $\overline{\mathfrak{X}}_1(p^n)(v)$ such that for every affine open subset $\mathrm{Spf}\, R \subset \overline{\mathfrak{X}}_1(p^n)(v)$ on which $\mathscr{F}$ is free, $\mathfrak{IW}_{w,v}^+(R)$ consists of pairs $(\mathrm{Fil}_\bullet, \{w_i : i = 1, \dots, g\})$ where both $\mathrm{Fil}_\bullet$ and $\{w_i : i = 1, \dots, g\}$ are w-compatible with $\mathrm{HT}_n(\epsilon_1), \dots, \mathrm{HT}_n(\epsilon_g)$ in the sense of Definition 3.2.2.

Let $\mathcal{IW}_{w,v}^+$ be the adic space associated with the formal scheme $\mathfrak{IW}_{w,v}^+$ over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Then we have a chain of morphisms of adic spaces

$$\pi^{\mathrm{AIP}} : \mathcal{IW}_{w,v}^+ \to \overline{\mathcal{X}}_1(p^n)(v) \to \overline{\mathcal{X}}_1(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v).$$

Lemma 3.5.7. *Recall the group adic spaces $\mathcal{B}_w^{\mathrm{opp}}$, $\mathcal{T}_{\mathrm{GL}_g,0}^{(w)}$, and $\mathcal{U}_{\mathrm{GL}_g,1}^{\mathrm{opp},(w)}$ defined in Definition 3.2.6.*

(i) $\mathcal{IW}_{w,v}^+$ is a $\mathcal{B}_w^{\mathrm{opp}}$-torsor over $\overline{\mathcal{X}}_1(p^n)(v)$. Namely, locally on $\overline{\mathcal{X}}_1(p^n)(v)$, we have identification

$$\mathcal{IW}_{w,v}^+ \simeq \overline{\mathcal{X}}_1(p^n)(v) \times_{\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})} \mathcal{B}_w^{\mathrm{opp}}$$

where $\mathcal{B}_w^{\mathrm{opp}}$ permutes the points $(\mathrm{Fil}_\bullet, \{w_i\})$ from the right.

(ii) Similarly, $\mathcal{IW}_{w,v}^+$ is a $\mathcal{U}_{\mathrm{GL}_g,1}^{\mathrm{opp},(w)} \times \mathcal{T}_{\mathrm{GL}_g,0}^{(w)} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$.

Proof. These are clear from the construction. □

Definition 3.5.8. *Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a w-analytic weight.*

*(i) Andreatta–Iovita–Pilloni's **sheaf of w-analytic v-overconvergent Siegel modular forms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)** is defined to be*

$$\underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}} := \pi_*^{\mathrm{AIP}} \mathscr{O}_{\mathcal{IW}_{w,v}^+}[\kappa_{\mathcal{U}}^\vee],$$

where $\pi_*^{\mathrm{AIP}}\mathscr{O}_{\mathcal{IW}^+_{w,v}}[\kappa_{\mathcal{U}}^\vee]$ stands for the subsheaf of $\pi_*^{\mathrm{AIP}}(\mathscr{O}_{\mathcal{IW}^+_{w,v}}\widehat{\otimes}R_{\mathcal{U}})$ consisting of sections on which $T_{\mathrm{GL}_g,0}$ acts via the character $\kappa_{\mathcal{U}}^\vee$ and $U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$ acts trivially.

(ii) *Andreatta–Iovita–Pilloni's* ***space of w-analytic v-overconvergent Siegel modular forms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)*** *is*

$$M_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{w,v,\mathrm{AIP}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}}(v), \underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}).$$

(iii) *The* ***space of locally analytic overconvergent Siegel modular forms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)*** *is*

$$M_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{\dagger,\mathrm{AIP}} := \lim_{\substack{v\to 0\\ w\to\infty}} M_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{w,v,\mathrm{AIP}}.$$

(iv) *Recall that $\mathcal{Z}_{\mathrm{Iw}} = \overline{\mathcal{X}}_{\mathrm{Iw}}\smallsetminus\mathcal{X}_{\mathrm{Iw}}$ is the boundary divisor. Andreatta–Iovita–Pilloni's* ***sheaf of w-analytic v-overconvergent Siegel cuspforms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)*** *is defined to be the subseaf $\underline{\omega}_{w,v,\mathrm{cusp}}^{\kappa_{\mathcal{U}},\mathrm{AIP}} = \underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}(-\mathcal{Z}_{\mathrm{Iw}})$ of $\underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}$ consisting of sections that vanish along $\mathcal{Z}_{\mathrm{Iw}}$.*

Andreatta–Iovita–Pilloni's ***space of w-analytic v-overconvergent Siegel cuspforms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)*** *is defined to be*

$$S_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{w,v,\mathrm{AIP}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}}(v), \underline{\omega}_{w,v,\mathrm{cusp}}^{\kappa_{\mathcal{U}},\mathrm{AIP}}),$$

and the ***space of locally analytic overconvergent Siegel cuspforms of weight $\kappa_{\mathcal{U}}$ (of Iwahori level)*** *is*

$$S_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{\dagger,\mathrm{AIP}} := \lim_{\substack{v\to 0\\ w\to\infty}} S_{\mathrm{Iw},\kappa_{\mathcal{U}}}^{\dagger,\mathrm{AIP}}.$$

Remark 3.5.9. Similar to 3.2.8, in Definition 3.5.8 (i), there is no difference between taking $\kappa_{\mathcal{U}}^\vee$-eigenspaces with respect to $T_{\mathrm{GL}_g,0}$- or $\mathcal{T}_{\mathrm{GL}_g,0}^{(w)}$-actions.

3.6 Pseudocanonical subgroups

3.6.1. In §3.7, we will prove the comparison between our perfectoid construction of the overconvergent Siegel modular forms and the construction of Andreatta–Iovita–Piloni. Immediate from the definitions, one observes the incompatibility of the underlying adic spaces used in the two constructions. That is, we employ the w-ordinary locus in the perfectoid construction while the authors of [AIP15] make use of the 'v-locus' $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$. Therefore, as a preparation for the comparison result, we have to first compare these two different loci. The main result of this section is Theorem 3.6.4. Due to technical reasons, we assume $p > 2g$ in this section.

3.6.2. We begin with recalling the homogeneous coordinates $\begin{pmatrix}\mathbb{1}_g & \boldsymbol{z}\end{pmatrix}$ on $\mathscr{F}\ell^\times \subset \mathscr{F}\ell$. We define the locus $\mathscr{F}\ell^{\times}_{\mathrm{can}} \subset \mathscr{F}\ell$, whose homogeneous coordinate is given by

$$\begin{pmatrix} \boldsymbol{z}_{1,1} & \cdots & \boldsymbol{z}_{1,g} & 1 & & \\ \vdots & & \vdots & & \ddots & \\ \boldsymbol{z}_{g,1} & \cdots & \boldsymbol{z}_{1,g} & & & 1 \end{pmatrix},$$

i.e., the translate of $\mathscr{F}\ell^{\times}$ by the longest Weyl element of the Weyl group of GSp_{2g}. For any $w \in \mathbf{Q}_{>0}$, we then define $\mathscr{F}\ell_{\mathrm{can},w} \subset \mathscr{F}\ell^{\times}_{\mathrm{can}}$ to be

$$\mathscr{F}\ell_{\mathrm{can},w} := \left\{ \boldsymbol{x} \in \mathscr{F}\ell^{\times}_{\mathrm{can}} : \max_{i,j} \inf_{t \in p\,\mathbf{Z}_p} \{ |\, \boldsymbol{z}_{i,j}(\boldsymbol{x}) - t| \leq p^{-w} \} \right\}.$$

Similar as before, we define

$$\begin{aligned} \overline{\mathcal{X}}_{\Gamma(p^\infty),\mathrm{can},w} &:= \pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell_{\mathrm{can},w}), \\ \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{can},w} &:= h_{\mathrm{Iw}^+}(\overline{\mathcal{X}}_{\Gamma(p^\infty),\mathrm{can},w}), \\ \overline{\mathcal{X}}_{\mathrm{Iw},\mathrm{can},w} &:= h_{\mathrm{Iw}}(\overline{\mathcal{X}}_{\Gamma(p^\infty),\mathrm{can},w}), \\ \overline{\mathcal{X}}_{\mathrm{can},w} &:= h(\overline{\mathcal{X}}_{\Gamma(p^\infty),\mathrm{can},w}). \end{aligned}$$

We call them the ***canonical w-ordinary loci***.

3.6.3. We also need the following definition of the v-locus at the strict Iwahori level. Recall from §3.5 that, for any $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2})$, $\overline{\mathcal{X}}_1(p^n)(v)$ (resp., $\overline{\mathcal{X}}_1(v)$; resp., $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$) is the adic space over $\overline{\mathcal{X}}(v)$ which parameterises trivialisations of $\mathcal{H}_n^\vee$ (resp., trivialisations of $\mathcal{H}_1^\vee$; resp., full flags of $\mathcal{H}_1^\vee$). In particular, $\overline{\mathcal{X}}_1(v)$ is equipped with a natural right action of $\mathrm{GL}_g(\mathbf{Z}/p\,\mathbf{Z})$ permuting the trivialisations. Consider the quotient

$$\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v) := \overline{\mathcal{X}}_1(v)/T_{\mathrm{GL}_g}(\mathbf{Z}/p\,\mathbf{Z})$$

which is an adic space over $\overline{\mathcal{X}}(v)$ parametersing the 'strict Iwahori structures' of $\mathcal{H}_1^\vee$; namely, it parameterises full flags $\mathrm{Fil}_\bullet\,\mathcal{H}_1^\vee$ of $\mathcal{H}_1^\vee$ together with a collection of subgroups $\{D_i : i = 1, \ldots, g\}$ of $\mathcal{H}_1^\vee$ of order p such that

$$\mathrm{Fil}_i\,\mathcal{H}_1^\vee = \langle D_1, \ldots, D_i \rangle$$

for all $i = 1, \ldots, g$. There is a chain of natural projections among these v-loci

$$\overline{\mathcal{X}}_1(p^n)(v) \to \overline{\mathcal{X}}_1(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}^+}(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v) \to \overline{\mathcal{X}}(v).$$

One can identify $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ as the quotient of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v)$ by the finite group $U_{\mathrm{GL}_g}(\mathbf{Z}/p\,\mathbf{Z})$.

Theorem 3.6.4. *Given $\Gamma \in \{\mathrm{Iw}^+, \mathrm{Iw}\}$, the system of canonical w-ordinary loci $\{\overline{\mathcal{X}}_{\Gamma,\mathrm{can},w} : w \in \mathbf{Q}_{>0}\}$ and the system of v-loci $\{\overline{\mathcal{X}}_\Gamma(v) : v \in \mathbf{Q}_{>0} \cap [0, 1/2)\}$ are mutually cofinal. More precisely,*

(i) For any given $v \in \mathbf{Q}_{>0} \cap [0, 1/2)$, there exists sufficiently large $w \in \mathbf{Q}_{>0}$ such that $\overline{\mathcal{X}}_{\Gamma,\mathrm{can},w} \subset \overline{\mathcal{X}}_\Gamma(v)$.

(ii) *For any given $w \in \mathbf{Q}_{>0}$, there exists sufficiently small $v \in \mathbf{Q}_{>0} \cap [0, 1/2)$ such that $\overline{\mathcal{X}}_{\Gamma}(v) \subset \overline{\mathcal{X}}_{\Gamma,\mathrm{can},w}$.*

Remark 3.6.5. To go back to the w-ordinary loci, note that we have the *Atkin–Lehner operator*

$$\mathrm{AL} : \mathcal{F}\ell_w^{\times} \to \mathcal{F}\ell_{\mathrm{can},w}, \quad \begin{pmatrix} \mathbb{1}_g & \boldsymbol{z} \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{1}_g & \boldsymbol{z} \end{pmatrix} \begin{pmatrix} & \mathbb{1}_g \\ -p\,\mathbb{1}_g & \end{pmatrix} = \begin{pmatrix} -p\,\boldsymbol{z} & \mathbb{1}_g \end{pmatrix},$$

moving from $\mathcal{F}\ell_w^{\times}$ to $\mathcal{F}\ell_{\mathrm{can},w}$. Obviously, we also have $\mathrm{AL}^{-1} : \mathcal{F}\ell_{\mathrm{can},w} \to \mathcal{F}\ell_w^{\times}$, given by the matrix $\begin{pmatrix} & \frac{-1}{p}\mathbb{1}_g \\ \mathbb{1}_g & \end{pmatrix}$. Therefore, as an immediate corollary of Theorem 3.6.4 and the Atkin–Lehner operator, $\{\overline{\mathcal{X}}_{\Gamma,w} : w \in \mathbf{Q}_{>0}\}$ and $\{\mathrm{AL}^{-1}\,\overline{\mathcal{X}}_{\Gamma}(v) : v \in \mathbf{Q}_{>0} \cap [0, 1/2)\}$ are mutually cofinal.

3.6.6. To prove Theorem 3.6.4, we follow the strategy in [CHJ17, §2.3]. However, we have to generalise their study of pseudocanonical subgroups to the case of semiabelian schemes with constant toric rank.

Let C be an algebraically closed complete nonarchimedean field containing $\mathbf{Q}_p$ and let $\mathcal{O}_C$ be its ring of integers. Suppose the valuation v_p on C is normalised so that $v_p(p) = 1$. Let G be a semiabelian scheme over $\mathcal{O}_C$ of dimension g with constant toric rank $r \leq g$. That is, G sits inside an extension

$$0 \to T \to G \to A \to 0,$$

where T is a torus of rank r over $\mathcal{O}_C$ and A is an abelian scheme of dimension $g - r$ over $\mathcal{O}_C$. (We say that G is ***principally polarised*** if A is principally polarised.) One sees that the p-adic Tate module $T_pG := \varprojlim_n G[p^n](C)$ is isomorphic to $\mathbf{Z}_p^{2g-r}$.

Recall the Hodge–Tate complex over $\mathcal{O}_C$

$$0 \to \mathrm{Lie}\, G \to T_pG \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \omega_{G^\vee} \to 0,$$

where $\omega_{G^\vee}$ is the dual of the Lie algebra $\mathrm{Lie}\, G^\vee$ of the dual semiabelian scheme $G^\vee$, and the second last map is induced from the Hodge–Tate map $\mathrm{HT}_G : T_pG \to \omega_{G^\vee}$. By [FGL08, Théorème II. 1.1], the cohomology of this complex is killed by $p^{1/(p-1)}$.

3.6.7. We set up the following notation. Recall that $\mathbf{V}_p = \mathbf{V} \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq \mathbf{Z}_p^{2g}$ is equipped with the standard basis $e_1, \ldots, e_{2g}$ together with a symplectic pairing. For every $0 \leq r \leq g$, let $\mathbf{V}_{p,r}$ denote the $\mathbf{Z}_p$-submodule of $\mathbf{V}_p$ spanned by $e_{r+1}, e_{r+2}, \ldots, e_{2g-r}$, equipped with the induced symplectic pairing. We also write $\mathbf{V}'_{p,r}$ to be $\mathbf{Z}_p$-submodule of $\mathbf{V}_p$ spanned by $e_1, \ldots, e_{2g-r}$ and write $\mathbf{W}_{p,r}$ to be the one spanned by $e_1, \ldots, e_r$. There is an obvious split exact sequence

$$0 \to \mathbf{W}_{p,r} \to \mathbf{V}'_{p,r} \to \mathbf{V}_{p,r} \to 0.$$

Definition 3.6.8. *Let G be a principally polarised semiabelian scheme over $\mathcal{O}_C$ of dimension g with constant toric rank $r \leq g$.*

(i) *An isomorphism $\alpha : \mathbf{V}'_{p,r} \xrightarrow{\sim} T_pG$ is called a **trivialisation** of T_pG if it is part of a* *commutative diagram*

$$\begin{tikzcd}
\mathbf{V}_{p,r} \arrow[r, "\simeq"] & T_pA \\
\mathbf{V}'_{p,r} \arrow[r, "\simeq"] \arrow[u, two heads] & T_pG \arrow[u, two heads] \\
\mathbf{W}_{p,r} \arrow[r, "\simeq"] \arrow[u, hook] & T_pT \arrow[u, hook]
\end{tikzcd}$$

where

- *the vertical arrows on the left are the ones as in Definition 3.6.7;*
- *the vertical arrows on the right are induced from the exact sequence* $0 \to T \to G \to A \to 0$*;*
- *the top arrow preserves the symplectic pairings.*

(ii) A trivialisation $\alpha : \mathbf{V}'_{p,r} \to T_pG$ *is* w***-ordinary*** *if* $\mathrm{HT}_G(\alpha(e_i)) \in p^w\omega_{G^\vee}$ *for all* $i = 1, ..., g$.

(iii) We say that G *is* w***-ordinary*** *if it admits a* w*-ordinary trivialisation.*

Remark 3.6.9. From the definition, if G is w-ordinary, it is w'-ordinary for any $w' > w$. It is also clear that G is ordinary if and only if it is w-ordinary for all $w \in \mathbf{Q}_{>0}$.

Lemma 3.6.10. *Let* G *be a* w*-ordinary semiabelian scheme (of dimension* g *with constant toric rank* r*) over* $\mathcal{O}_C$ *and let* $n \in \mathbf{Z}_{\geq 1}$ *such that* $n < w + 1$*. The Hodge-Tate map* HT_G *induces a map*

$$G[p^n](C) \to (\operatorname{image} \mathrm{HT}_G)/p^{\min\{n,w\}}(\operatorname{image} \mathrm{HT}_G).$$

Then the schematic closure of the kernel of this map defines a flat subgroup scheme $H_n \subset G[p^n]$ *whose generic fibre is isomorphic to* $(\mathbf{Z}/p^n\mathbf{Z})^g$*. Moreover, if* α *is a* w*-ordinary trivialisation of* T_pG*, then* $H_n(C)$ *is generated by* $\alpha(e_1)$*, ...,* $\alpha(e_g)$*. Here we have abused the notations and still use* $\alpha(e_i)$*'s to denote their images in* $G[p^n](C)$*.*

Proof. Since the Hodge–Tate complex is exact after inverting p, the image of $\operatorname{Lie} G$ in $T_pG \otimes_{\mathbf{Z}_p} \mathcal{O}_C$ is a rank g sub-lattice in the kernel of $T_pG \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \omega_{G^\vee}$. Hence, the kernel of $\mathrm{HT}_G : T_pG \to \omega_{G^\vee}$ has rank at most g.

On the other hand, there is a commutative diagram

$$\begin{tikzcd}
T_pG \arrow[r, "\mathrm{HT}_G"] \arrow[d] & \omega_{G^\vee} \arrow[d] \\
G[p^n](C) \arrow[r, "\mathrm{HT}_{G[p^n]}"] & \omega_{G[p^n]^\vee}
\end{tikzcd},$$

where the right vertical arrow is induced from the natural identification $\omega_{G[p^n]^\vee} = \omega_{G^\vee}/p^n\omega_{G^\vee}$. Consequently, $\ker \mathrm{HT}_{G[p^n]}$ also has rank at most g.

Let α be a w-ordinary trivialisation of T_pG. Since $n < w+1$, the kernel of the composition

$$T_pG \xrightarrow{\mathrm{HT}_G} \omega_{G^\vee} \to \omega_{G^\vee}/p^n\omega_{G^\vee}$$

necessarily contains $\alpha(e_i)$, for all $i = 1, ..., g$. Since $\alpha(e_i)$'s are $\mathbf{Z}_p$-linearly independent, their images in $G[p^n](C)$ are $(\mathbf{Z}/p^n\mathbf{Z})$-linearly independent and hence generate $\ker \mathrm{HT}_{G[p^n]}$. Consequently, H_n is precisely the schematic closure in $G[p^n]$ of the subgroup of $G[p^n](C)$ generated by $\{\alpha(e_i) : i = 1, ..., g\}$. Flatness of H_n follows from the flatness of G. □

Definition 3.6.11. *The subgroup scheme H_n defined in Lemma 3.6.10 is called the **pseudocanonical subgroup of level** n. When $n = 1$, we simply call H_1 the **pseudocanonical subgroup** of G.*

Lemma 3.6.12. *Let $m \leq n$ be positive integers and let $w \in \mathbf{Q}_{>0}$ such that $w > n$. Let G be a semiabelian scheme (of dimension g with constant toric rank r) over $\mathcal{O}_C$. Suppose G is w-ordinary. Then, G/H_m is $(w-m)$-ordinary, and for any $m' \in \mathbf{Z}$ with $m < m' \leq n$, we have $H'_{m'-m} = H_{m'}/H_m$, where $H'_{m'-m}$ is the pseudocanonical subgroup of G/H_m of level $m'-m$.*

Proof. The proof is the same as in [CHJ17, Lemma 2.11] as long as we use the matrix $\mathrm{diag}(p^m\,\mathbb{1}_g, \mathbb{1}_{g-r})$ in place of $\mathrm{diag}(1, p^m)$. Notice that the "p^m" factor appears at the bottom right corner in *loc. cit.* because they work with a slightly different action of $\mathrm{GL}_2(\mathbf{Q}_p)$. □

3.6.13. Before stating the next lemma, let us recall the notion of the *degree* of a finite flat group scheme over $\mathcal{O}_C$ studied in [Far11]. If M is a p-power torsion $\mathcal{O}_C$-module of finite presentation, we can write

$$M \simeq \bigoplus_{i=1}^{l} \mathcal{O}_C / a_i \mathcal{O}_C$$

for some $a_i \in \mathcal{O}_C$, $i = 1, \ldots, l$. Then the degree of M is defined to be $\deg M := \sum_{i=1}^{l} v_p(a_i)$. Now, if H is a finite flat group scheme over $\mathcal{O}_C$ and let ω_H denote the $\mathcal{O}_C$-module of invariant differentials on H, then we define the ***degree*** of H to be $\deg H := \deg \omega_H$.

Lemma 3.6.14. *Let G be a w-ordinary semiabelian scheme (of dimension g with constant toric rank r) over $\mathcal{O}_C$ and let α be a w-ordinary trivialisation. Let ω_{H_1} be the dual of $\mathrm{Lie}\, H_1$ and let $\omega_{H_1^\vee}$ be the dual of $\mathrm{Lie}\, H_1^\vee$. For $i = 1, ..., g$, let $H_{1,i}$ be the schematic closure in H_1 of the subgroup generated by $\alpha(e_i)$. Then*

(i) Each $H_{1,i}$ is isomorphic to $\mathrm{Spec}(\mathcal{O}_C[X]/(X^p - a_iX))$ for some $a_i \in \mathcal{O}_C$. The dual $H_{1,i}^\vee$ is isomorphic to $\mathrm{Spec}(\mathcal{O}_C[X]/(X^p - b_iX))$ with $a_ib_i = p$.

(ii) We have isomorphisms $\omega_{H_1} \simeq \bigoplus_{i=1}^{g} \mathcal{O}_C / a_i \mathcal{O}_C$ and $\omega_{H_1^\vee} \simeq \bigoplus_{i=1}^{g} \mathcal{O}_C / b_i \mathcal{O}_C$. In particular, we have $\deg H_1 = \sum_{i=1}^{g} v_p(a_i)$ and $\deg H_1^\vee = \sum_{i=1}^{g} v_p(b_i) = g - \sum_{i=1}^{g} v_p(a_i)$.

(iii) Under the identification $\omega_{H_1^\vee} \simeq \bigoplus_{i=1}^{g} \mathcal{O}_C / b_i \mathcal{O}_C$, the image of the (linearised) Hodge–Tate map

$$H_1(C) \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \omega_{H_1^\vee}$$

is equal to $\bigoplus_{i=1}^{g} c_i \mathcal{O}_C / b_i \mathcal{O}_C$ for some $c_i \in \mathcal{O}_C$ such that $v_p(c_i) = v_p(a_i)/(p-1)$, $i = 1, \ldots, g$.

Proof. Since each $H_{1,i}$ is a finite flat group scheme over $\mathcal{O}_C$ of degree p, the assertion follows from classical Oort–Tate theory. See, for example, [Far11, §6.5, Lemme 9]. □

Lemma 3.6.15. *Let G be a w-ordinary semiabelian scheme (of dimension g with constant toric rank r) over $\mathcal{O}_C$. Suppose $\frac{(2g-1)p}{2g(p-1)} < w \leq 1$.* [3] *Then H_1 coincides with the canonical subgroup of G. Moreover, the Hodge height*[4] *of G is smaller than $1/2$.*

Proof. We follow the strategy of the proof of [CHJ17, Lemma 2.14]. Consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_1(C) & \longrightarrow & G[p](C) \\ & & \Big\downarrow{\scriptstyle \mathrm{HT}_{H_1}} & & \Big\downarrow{\scriptstyle \mathrm{HT}_{G[p]}} \\ 0 & \longrightarrow & \omega_{H_1^\vee} & \longrightarrow & \omega_{G[p]^\vee} \end{array}$$

with exact rows. Notice that we have an identification $\omega_{G[p]^\vee} = \omega_{G^\vee}/p\omega_{G^\vee}$. Let α be a w-ordinary trivialisation of T_pG. According to Lemma 3.6.10, $\alpha(e_1), \dots, \alpha(e_g)$ form a basis for $H_1(C)$. Also, by definition, we have $\mathrm{HT}_{G[p]}(\alpha(e_i)) \in p^w\omega_{G[p]^\vee}$.

Now, with respect to the generators $\alpha(e_1)\dots, \alpha(e_g)$ of $H_1(C)$, the map $\omega_{H_1^\vee} \to \omega_{G[p]^\vee}$ can be identified with the inclusion

$$\bigoplus_{i=1}^{g} \mathcal{O}_C/b_i\,\mathcal{O}_C \to (\mathcal{O}_C/p\,\mathcal{O}_C)^g, \quad (x_1, ..., x_g) \mapsto (a_1x_1, ..., a_gx_g).$$

Therefore, we see that

$$a_i\,\mathrm{HT}_{H_1}(\alpha(e_i)) = \mathrm{HT}_{G[p]}(\alpha(e_i)) \in p^w\omega_{G[p]^\vee}.$$

By Lemma 3.6.14 (iii), we know that $\mathrm{HT}_{H_1}(\alpha(e_i))$ has valuation $v_p(a_i)/(p-1)$. This implies

$$w \leq v_p(a_i) + \frac{v_p(a_i)}{p-1} = \frac{pv_p(a_i)}{p-1}.$$

Consequently, we have

$$\deg H_1 = \sum_{i=1}^{g} v_p(a_i) \geq \frac{gw(p-1)}{p} > \frac{g(p-1)}{p}\cdot\frac{(2g-1)p}{2g(p-1)} = \frac{2g-1}{2} = g - \frac{1}{2}.$$

It follows from [AIP15, Proposition 3.1.2] that H_1 is exactly the canonical subgroup of G and the Hodge height of G is less than $\frac{1}{2}$. □

Remark 3.6.16. The lemma might hold without the assumption $p > 2g$ as long as one can produce finer estimates on the degree and the Hodge height. However, we do not attempt to find these better estimates.

[3]The inequalities are valid because of the assumption $p > 2g$ at the beginning of this section.

[4]Recall from [AIP15, §3.1] that the *Hodge height* of G is defined to be the 'truncated' p-adic valuation of the Hasse invariant of G. See *loc. cit.* for details.

Proposition 3.6.17. *Let G be a w-ordinary semiabelian scheme (of dimension g with constant toric rank r) over $\mathcal{O}_C$. Suppose $\frac{(2g-1)p}{2g(p-1)} + n - 1 < w \leq n$, then H_n coincides with the canonical subgroup of G of level n. In this case, the Hodge height of G is less than $\frac{1}{2p^{n-1}}$.*

Proof. The proof follows from induction. The case for $n = 1$ is precisely Lemma 3.6.15.

Assume that the statement is affirmative for $n-1$. By Lemma 3.6.12, G/H_1 is $(w-1)$-ordinary and we have $\frac{(2g-1)p}{2g(p-1)} + n - 2 < w - 1 \leq n - 1$. The induction hypothesis implies that the pseudocanonical subgroup H_n/H_1 of of level $n-1$ of G/H_1 is the canonical subgroup of level $n-1$ and that the Hodge height of G/H_1 is less than $\frac{1}{2p^{n-2}}$.

However, H_1 coincides with the canonical subgroup of G by Lemma 3.6.15. Hence, by [Far11, Théorèm 6 (4)] (see also [AIP15, Theorem 3.1.1 (5)]), we see that the Hodge height of G is bounded by $\frac{1}{2p^{n-1}}$ and that H_n is the canonical subgroup of level n of G. □

Corollary 3.6.18. *Let $n \in \mathbf{Z}_{\geq 1}$ and suppose $w \in \mathbf{Q}_{>0}$ such that $\frac{(2g-1)p}{2g(p-1)} < w \leq n$. Then there exists $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2p^{n-1}})$ and a natural inclusion $\overline{\mathcal{X}}_{\mathrm{can},w} \hookrightarrow \overline{\mathcal{X}}(v)$.*

Proof. It suffices to work with $(C, \mathcal{O}_C)$-points for algebraically closed complete nonarchimedean field C containing $\mathbf{Q}_p$.[5] Let $\boldsymbol{x} \in \overline{\mathcal{X}}_{\mathrm{can},w}(C, \mathcal{O}_C)$. By the properness of $\overline{\mathcal{X}}$, the point $\boldsymbol{x}$ extends to an $\mathcal{O}_C$-point $\tilde{\boldsymbol{x}}$ of $\overline{\mathfrak{X}}^{\mathrm{tor}}$. One can associate with $\tilde{\boldsymbol{x}}$ a 1-motive $\widetilde{M}_{\tilde{\boldsymbol{x}}} = [Y \to \widetilde{G}_{\tilde{\boldsymbol{x}}}]$ where $\widetilde{G}_{\tilde{\boldsymbol{x}}}$ is a semiabelian scheme (of dimension g with constant toric rank) over $\mathcal{O}_C$ and Y is a free $\mathbf{Z}$-module of finite rank (see, for example, [Str10]).

From the definition of the Hodge–Tate period map, we see that $\widetilde{G}_{\tilde{\boldsymbol{x}}}$ is w-ordinary. By Proposition 3.6.17, the Hodge height of $\widetilde{G}_{\tilde{\boldsymbol{x}}}$ is smaller than $\frac{1}{2p^{n-1}}$. This means $\boldsymbol{x} \in \overline{\mathcal{X}}(v)(C, \mathcal{O}_C)$ for some $v < \frac{1}{2p^{n-1}}$. □

3.6.19. Recall that, for any $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2})$, $\mathcal{H}_1$ is the universal canonical subgroup of the tautological semiabelian variety over $\overline{\mathcal{X}}(v)$. Let $w > \frac{(2g-1)p}{2g(p-1)}$ and pick v so that $\overline{\mathcal{X}}_{\mathrm{can},w} \hookrightarrow \overline{\mathcal{X}}(v)$ as in Corollary 3.6.18. We still write $\mathcal{H}_1$ for its pullback to $\overline{\mathcal{X}}_w$. Therefore, we can consider

$$\overline{\mathcal{X}}_{1,\mathrm{can},w} := \mathrm{Isom}_{\overline{\mathcal{X}}_{\mathrm{can},w}}((\mathbf{Z}/p\mathbf{Z})^g, \mathcal{H}_1^\vee);$$

namely, the adic space over $\overline{\mathcal{X}}_{\mathrm{can},w}$ which parameterises trivialisations of $\mathcal{H}_1^\vee$. The group $\mathrm{GL}_g(\mathbf{Z}/p\mathbf{Z})$ naturally acts on $\overline{\mathcal{X}}_{1,\mathrm{can},w}$ by permuting the trivialisations.

Lemma 3.6.20. *For $w > \frac{(2g-1)p}{2g(p-1)}$, there are natural identifications*

$$\overline{\mathcal{X}}_{1,\mathrm{can},w}/B_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z}) = \overline{\mathcal{X}}_{\mathrm{Iw},\mathrm{can},w} \quad \text{and} \quad \overline{\mathcal{X}}_{1,\mathrm{can},w}/T_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z}) = \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{can},w}.$$

Proof. We only give the proof for the first identity. The second one is similar and left to the readers.

We first focus on the part away from the boundary. Let $\mathcal{X}_{\mathrm{can},w} = \overline{\mathcal{X}}_{\mathrm{can},w} \cap \mathcal{X}$ and let $\mathcal{A}_w^{\mathrm{univ}}$ be the universal abelian variety over $\mathcal{X}_{\mathrm{can},w}$.

[5] Notice that the classical points determine these adic spaces by [Hub13, (1.1.11)].

The key observation is that any trivialisation $\psi : (\mathbf{Z}/p\mathbf{Z})^g \to \mathcal{H}_1^\vee$ induces a full flag $\mathrm{Fil}^\psi_\bullet \mathcal{A}^{\mathrm{univ}}_w[p]$ on $\mathcal{A}^{\mathrm{univ}}_w[p]$. Indeed, let $\epsilon_1, \ldots, \epsilon_g$ denote the standard basis for $(\mathbf{Z}/p\mathbf{Z})^g$ and let $\mathrm{Fil}^\psi_\bullet \mathcal{H}_1^\vee$ be the full flag of $\mathcal{H}_1^\vee$ given by

$$0 \subset \langle \psi(\epsilon_1) \rangle \subset \langle \psi(\epsilon_1), \psi(\epsilon_2) \rangle \subset \cdots \subset \langle \psi(\epsilon_1), ..., \psi(\epsilon_g) \rangle.$$

Consider the natural projection

$$\mathrm{pr} : \mathcal{A}^{\mathrm{univ}}_w[p] \xrightarrow{\simeq} \mathcal{A}^{\mathrm{univ}}_w[p]^\vee \twoheadrightarrow \mathcal{H}_1^\vee$$

where the first isomorphism is induced from the principal polarisation. Then the desired full flag $\mathrm{Fil}^\psi_\bullet \mathcal{A}^{\mathrm{univ}}_w[p]$ is given by

$$\mathrm{Fil}^\psi_i \mathcal{A}^{\mathrm{univ}}_w[p] := \begin{cases} \mathrm{pr}^{-1} \mathrm{Fil}^\psi_{i-g} \mathcal{H}_1^\vee, & i > g \\ (\mathrm{pr}^{-1} \mathrm{Fil}^\psi_{g-i} \mathcal{H}_1^\vee)^\perp, & i \leq g \end{cases}.$$

Moreover, if two such ψ's induce the same $\mathrm{Fil}^\psi_\bullet \mathcal{H}_1^\vee$, then the associated $\mathrm{Fil}^\psi_\bullet \mathcal{A}^{\mathrm{univ}}_w[p]$ coincide. Hence, the assignment $\psi \mapsto \mathrm{Fil}^\psi_\bullet \mathcal{A}^{\mathrm{univ}}_w[p]$ induces a natural inclusion $\mathcal{X}_{1,w}/B_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z}) \subset \mathcal{X}_{\mathrm{Iw},w}$ away from the boundary.

Conversely, using the w-ordinarity, one sees that the universal full flag $\mathrm{Fil}_\bullet \mathcal{A}^{\mathrm{univ}}_w[p]$ on $\mathcal{X}_{\mathrm{Iw},w}$ induces a full flag $\mathrm{Fil}_\bullet \mathcal{H}_1^\vee$ of $\mathcal{H}_1^\vee$ given by

$$\mathrm{Fil}_i \mathcal{H}_1^\vee = \mathrm{pr}\left((\mathrm{Fil}_{g-i} \mathcal{A}^{\mathrm{univ}}_w[p])^\perp\right)$$

for $i = 1, ..., g$. This yields the opposite inclusion away from the boundary.

In order to extend to the boundary, one considers the 1-motives on the boundary strata and same argument as above applies verbatim. The details are left to the reader. □

Proof of Theorem 3.6.4. (i) We may assume $v = \frac{1}{2p^{n-1}}$ for some sufficiently large n. In this case, we can take any $\frac{(2g-1)p}{2g(p-1)} + n - 1 < w \leq n$. Indeed, by Corollary 3.6.18, we have a Cartesian diagram

$$\begin{tikzcd} \overline{\mathcal{X}}_{1,\mathrm{can},w} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}}_1(v) \arrow[d] \\ \overline{\mathcal{X}}_{\mathrm{can},w} \arrow[r, hook] & \overline{\mathcal{X}}(v) \end{tikzcd}$$

where the top arrow is equivariant under the action of $\mathrm{GL}_g(\mathbf{Z}/p\mathbf{Z})$. Taking the quotient by either $B_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z})$ or $T_{\mathrm{GL}_g}(\mathbf{Z}/p\mathbf{Z})$, and applying Lemma 3.6.20, we obtain the desired inclusions.

(ii) We may assume $n-1 < w < n$ for some sufficiently large n. Pick $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2p^{n-1}})$ such that $w \in \left(n - 1 + \frac{v}{p-1}, n - \frac{vp^n}{p-1}\right]$. Applying [AIP15, Proposition 3.2.1], on the level of classical points, we obtain a natural inclusion $\overline{\mathcal{X}}(v)(C, \mathcal{O}_C) \hookrightarrow \overline{\mathcal{X}}_{\mathrm{can},w}(C, \mathcal{O}_C)$

and hence an inclusion $\overline{\mathcal{X}}(v) \hookrightarrow \overline{\mathcal{X}}_{\mathrm{can},w}$. There is a Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{X}}_1(v) & \hookrightarrow & \overline{\mathcal{X}}_{1,\mathrm{can},w} \\ \downarrow & & \downarrow \\ \overline{\mathcal{X}}(v) & \hookrightarrow & \overline{\mathcal{X}}_{\mathrm{can},w} \end{array}$$

Once again, applying Lemma 3.6.20 and taking the corresponding quotients yield the desired inclusions.

□

3.7 The comparison of constructions

3.7.1. In this section, we still assume $p > 2g$. The aim of this section is to prove the following theorem which compares the overconvergent automorphic sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}}}$ constructed in §3.1 and the sheaf $\underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}$ of Andreatta–Iovita–Pilloni.

For any $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2})$, we identify $\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v)$ and $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ with their image under AL^{-1} in this subsection. Let $h_\diamond : \overline{\mathcal{X}}_{\mathrm{Iw}^+}(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ denote the natural projection.

Theorem 3.7.2. *Suppose $n > \frac{g}{p-1}$ and let $v \in \mathbf{Q}_{>0} \cap [0, \frac{1}{2p^{n-1}})$, $w \in \mathbf{Q}_{>0} \cap (n-1+\frac{v}{p-1}, n - \frac{vp^n}{p-1}]$. (In particular, Theorem 3.6.4 (ii) and Remark 3.6.5, there are natural inclusions $\overline{\mathcal{X}}_{\mathrm{Iw}}(v) \hookrightarrow \overline{\mathcal{X}}_{\mathrm{Iw},w}$ and $\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v) \hookrightarrow \overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$.) Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a weight such that $w > 1 + r_{\mathcal{U}}$. Then, over $\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v)$, there is a canonical isomorphism of sheaves*

$$\Psi : \underline{\omega}_w^{\kappa_{\mathcal{U}}}|_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v)} \xrightarrow{\simeq} h_\diamond^* \underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}},$$

where $h_\diamond : \overline{\mathcal{X}}_{\mathrm{Iw}^+}(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ denote the natural projection.

3.7.3. Recall that the space of overconvergent Siegel modular forms of weight $\kappa_{\mathcal{U}}$ of strict Iwahori level (see Definition 3.1.16 (v)) is defined to be

$$M^\dagger_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} = \varinjlim_{w\to\infty} M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$$

where

$$M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\ \underline{\omega}_w^{\kappa_{\mathcal{U}}}).$$

We can also extend the notion of overconvergent Siegel modular forms of Andreatta–Iovita–Pilloni to the case of strict Iwahori level.

(i) Let $v \in \mathbf{Q}_{>0} \cap [0, 1/2)$ and $w \in \mathbf{Q}_{>0}$. Suppose $\kappa_{\mathcal{U}}$ is w-analytic. The ***space of w-analytic v-overconvergent Siegel modular forms of weight $\kappa_{\mathcal{U}}$ (of strict Iwahori level)*** of Andreatta–Iovita–Pilloni is defined to be

$$M^{w,v,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v), h_\diamond^* \underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}).$$

(ii) The ***space of locally analytic overconvergent Siegel modular forms of weight*** $\kappa_{\mathcal{U}}$ ***(of strict Iwahori level)*** of Andreatta–Iovita–Pilloni is defnied to be

$$M^{\dagger,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := \lim_{\substack{v\to 0\\ w\to\infty}} M^{w,v,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

(iii) Similarly, the ***space of*** w***-analytic*** v***-overconvergent Siegel cuspforms of weight*** $\kappa_{\mathcal{U}}$ ***(of strict Iwahori level)*** of Andreatta–Iovita–Pilloni is defined to be

$$S^{w,v,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}(v), h_{\diamond}^*\underline{\omega}^{\kappa_{\mathcal{U}},\mathrm{AIP}}_{w,v,\mathrm{cusp}}),$$

and the ***space of locally analytic overconvergent Siegel cuspforms of weight*** $\kappa_{\mathcal{U}}$ ***(of strict Iwahori level)*** of Andreatta–Iovita–Pilloni is defined to be

$$S^{\dagger,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} := \lim_{\substack{v\to 0\\ w\to\infty}} S^{w,v,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

Then we have the following immediate corollary of Theorem 3.7.2 and Theorem 3.6.4.

Corollary 3.7.4. *There are canonical isomorphisms*

$$M^{\dagger}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} \simeq M^{\dagger,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} \quad \textit{and} \quad S^{\dagger}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} \simeq S^{\dagger,\mathrm{AIP}}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}.$$

Remark 3.7.5. In fact, it will follow from the construction of Ψ that the isomorphisms in Corollary 3.7.4 is also Hecke-equivariant.

3.7.6. The rest of the section is dedicated to the proof of Theorem 3.7.2. Our strategy is simple. Let n, v, w, and $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be as in Theorem 3.7.2. Recall that the $\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+(v)}}$-module (resp., $\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}(v)}}$-module) $\underline{\omega}_{\mathrm{Iw}^+,v}$ (resp., $\underline{\omega}_{\mathrm{Iw},v}$) is locally free of rank g. Let $\mathcal{V}' \subset \overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ be an affinoid open subset such that $\underline{\omega}_{\mathrm{Iw},v}|_{\mathcal{V}'}$ is free, and let $\mathcal{V} \subset \overline{\mathcal{X}}_{\mathrm{Iw}^+}(v)$ be the preimage of $\mathcal{V}'$. To construct Ψ, it suffices to establish a canonical isomorphism

$$\Psi : \underline{\omega}_w^{\kappa_{\mathcal{U}}}(\mathcal{V}) \xrightarrow{\sim} h_{\diamond}^*\underline{\omega}^{\kappa_{\mathcal{U}},\mathrm{AIP}}_{w,v}(\mathcal{V})$$

for every such $\mathcal{V}$, which is also functorial in $\mathcal{V}$.

3.7.7. As a preparation, consider the pullback

$$\begin{tikzcd} \mathcal{IW}^+_{w,v,\infty} \arrow[r] \arrow[d, "\pi^{\mathrm{AIP}}_{\infty}"'] & \mathcal{IW}^+_{w,v} \arrow[d, "\pi^{\mathrm{AIP}}"] \\ \overline{\mathcal{X}}_{\Gamma(p^\infty)}(v) \arrow[r, "h_{\mathrm{Iw}}"] & \overline{\mathcal{X}}_{\mathrm{Iw}}(v) \end{tikzcd}$$

where $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$ is the preimage of $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ under the natural morphism $h_{\mathrm{Iw}} : \overline{\mathcal{X}}_{\Gamma(p^\infty),w} \to \overline{\mathcal{X}}_{\mathrm{Iw},w}$. For later usage, we denote by $\mathcal{V}_\infty$ (resp., $\mathcal{V}^+_\infty$) the preimage of $\mathcal{V}'$ in $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$ (resp., in $\mathcal{IW}^+_{w,v,\infty}$) under the projection h_{Iw} (resp., $h_{\mathrm{Iw}} \circ \pi^{\mathrm{AIP}}_\infty$).

Since $\mathcal{IW}^+_{w,v}$ is a $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\mathrm{Iw}}$, we know that $\mathcal{IW}^+_{w,v,\infty}$ is likewise a $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$-torsor over $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$. In what follows, we provide an explicit moduli interpretation of this torsor, in three steps.

Step 1. Observe that the natural projection $h_{\mathrm{Iw}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)}(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v)$ factors as

$$h_{\mathrm{Iw}} : \overline{\mathcal{X}}_{\Gamma(p^\infty)}(v) \xrightarrow{h_1} \overline{\mathcal{X}}_1(p^n)(v) \to \overline{\mathcal{X}}_{\mathrm{Iw}}(v).$$

Indeed, away from the boundary, the map h_1 can be described as follows. Let $\mathcal{X}_{\Gamma(p^\infty)}(v)$ be the part of $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$ away from the boundary. For every point $(A, \lambda, \psi_N, \psi_{p^\infty}) \in \mathcal{X}_{\Gamma(p^\infty)}(v)$, consider the dual trivialisation

$$\psi^\vee_{p^\infty} : \mathbf{V}^\vee_p \xrightarrow{\sim} T_pA^\vee.$$

Modulo p^n, we obtain a symplectic isomorphism

$$\psi^\vee_{p^n} : \mathbf{V}^\vee_p \otimes_{\mathbf{Z}_p} (\mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\sim} A[p^n]^\vee.$$

Then h_1 sends $(A, \lambda, \psi_N, \psi_{p^\infty})$ to $(A, \lambda, \psi_N, \psi)$ where ψ is the composition

$$\psi : (\mathbf{Z}/p^n\mathbf{Z})^g \hookrightarrow \mathbf{V}^\vee_p \otimes_{\mathbf{Z}_p} (\mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\psi^\vee_{p^n}} A[p^n]^\vee \twoheadrightarrow H^\vee_n$$

with the first arrow sending ϵ_i to $e^\vee_{g+1-i} \otimes 1$, for all $i = 1, \ldots, g$, and the last arrow being the natural surjection. From the proof of Lemma 3.6.20, we see that ψ is indeed a trivialisation of $H^\vee_n$.

Using the language of 1-motives, this description of h_1 also extends to the boundary. The details are left to the readers.

Step 2. Recall that, in 3.5.5, we defined a locally free $\mathscr{O}_{\overline{\mathfrak{X}}_1(p^n)(v)}$-submodule $\mathscr{F} \subset \underline{\Omega}_{n,v}$ on $\overline{\mathfrak{X}}_1(p^n)(v)$. Passing to the adic generic fibre, let $\underline{\omega}^+_{n,v}$ denote the sheaf of $\mathscr{O}^+_{\overline{\mathcal{X}}_1(p^n)(v)}$-module on $\overline{\mathcal{X}}_1(p^n)(v)$ associated with $\underline{\Omega}_{n,v}$. Then $\mathscr{F}$ can be identified with a locally free $\mathscr{O}^+_{\overline{\mathcal{X}}_1(p^n)(v)}$-submodule of $\underline{\omega}^+_{n,v}$, which is still denoted by $\mathscr{F}$. Moreover, let $\mathscr{F}_\infty$ be the pullback of $\mathscr{F}$ to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$ along h_1.

Recall as well the $\mathscr{O}_{\overline{\mathfrak{X}}_{\Gamma(p^n)}}$-modules $\underline{\Omega}^{\mathrm{mod}}_{\Gamma(p^n)} \subset \underline{\Omega}_{\Gamma(p^n)}$ constructed in 2.3.1. Passing to the adic generic fibre, they induce $\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^n)}}$-modules $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^n)} \subset \underline{\omega}^+_{\Gamma(p^n)}$ on $\overline{\mathcal{X}}_{\Gamma(p^n)}$. Let $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty)} \subset \underline{\omega}^+_{\Gamma(p^\infty)}$ be their pullbacks to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ and let $\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v} \subset \underline{\omega}^+_{\Gamma(p^\infty),v}$ be their restrictions on $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$.

We claim that there is a natural inclusion

$$\mathscr{F}_\infty \subset \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v}.$$

Indeed, recall the map

$$\mathrm{HT}_n : (\mathbf{Z}/p^n\mathbf{Z})^g \to \omega_{\mathcal{H}_n}$$

on $\overline{\mathcal{X}}_1(p^n)(v)$ constructed in 3.5.5. Pulling back to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$, we obtain a map

$$\mathrm{HT}_{n,\infty} : (\mathbf{Z}/p^n\mathbf{Z})^g \to \omega_{\mathcal{H}_{n,\infty}}$$

where $\mathcal{H}_{n,\infty}$ is the pullback of $\mathcal{H}_n$ along the projection $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v) \to \overline{\mathcal{X}}_1(p^n)(v)$. On the other hand, recall the map $\mathrm{HT}_{\Gamma(p^\infty)}$ on $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ constructed in 2.5.1. Restricting to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$ and modulo p^n, we obtain a map

$$\mathrm{HT}_{\Gamma(p^\infty),n,v} : \mathbf{V}\otimes_{\mathbf{Z}}(\mathbf{Z}/p^n\mathbf{Z}) \to \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v}/p^n\underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v}.$$

These maps fit into a commutative diagram

$$\begin{tikzcd}
 & & \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v} \arrow[rr, hook] \arrow[d, dash] & & \underline{\omega}^{+}_{\Gamma(p^\infty),v} \arrow[d, two heads] \\
(\mathbf{Z}/p^n\mathbf{Z})^g \arrow[rrrr, "\mathrm{HT}_{n,\infty}"] \arrow[d, hook] & & {} \arrow[d, two heads] & & \omega_{\mathcal{H}_{n,\infty}} \arrow[d, two heads] \\
\mathbf{V}\otimes_{\mathbf{Z}}(\mathbf{Z}/p^n\mathbf{Z}) \arrow[rr, "\mathrm{HT}_{\Gamma(p^\infty),n,v}"] & & \underline{\omega}^{\mathrm{mod},+}_{\Gamma(p^\infty),v}/p^n \arrow[r, hook] & \underline{\omega}^{+}_{\Gamma(p^\infty),v}/p^n \arrow[r, equal] & \omega_{\mathcal{H}_{n,\infty}}/p^n
\end{tikzcd}.$$

where the left inclusion sends ϵ_i to $e_{2g+1-i}\otimes 1$, for all $i = 1,\dots,g$. The equality at the bottom right corner follows from [AIP15, Proposition 3.2.1]. By definition, $\mathscr{F}_\infty$ is generated by the lifts of $\mathrm{HT}_{n,\infty}(\epsilon_i)$'s from $\omega_{\mathcal{H}_{n,\infty}}$ to $\underline{\omega}^{+}_{\Gamma(p^\infty),v}$ and hence the desired inclusion follows.

Step 3. We are now able to describe the torsor. Recall that there is a universal full flag $\mathrm{Fil}^{\mathrm{univ}}_\bullet\mathcal{H}_1^\vee$ of $\mathcal{H}_1^\vee$ on $\overline{\mathcal{X}}_{\mathrm{Iw}}(v)$. Pulling back to $\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$, we obtain universal full flag $\mathrm{Fil}^{\mathrm{univ}}_\bullet\mathcal{H}_{1,\infty}^\vee$ of $\mathcal{H}_{1,\infty}^\vee$. There is a natural projection $\Theta : \mathcal{H}^\vee_{n,\infty} \to \mathcal{H}^\vee_{1,\infty}$. Moreover, the Hodge–Tate map on $\mathcal{H}^\vee_{n,\infty}$ induces a map

$$\mathrm{HT}_{\mathcal{H}^\vee_{n,\infty}} : \mathcal{H}^\vee_{n,\infty} \to \mathscr{F}_\infty \otimes_{\mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)}} \mathscr{O}^+_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)}/p^w.$$

Then, for every affinoid open $\mathcal{Y} = \mathrm{Spa}(R,R^+) \subset \overline{\mathcal{X}}_{\Gamma(p^\infty)}(v)$, the sections $\mathcal{IW}^+_{w,\infty}(\mathcal{Y})$ parametrise triples $(\psi, \mathrm{Fil}_\bullet, \{w_i : i = 1,\dots,g\})$ where

- $\psi : (\mathbf{Z}/p^n\mathbf{Z})^g \xrightarrow{\simeq} \mathcal{H}^\vee_{n,\infty}|_{\mathcal{Y}}$ is a trivialisation such that

  $$\psi\langle\epsilon_1,\dots,\epsilon_i\rangle = \Theta(\mathrm{Fil}^{\mathrm{univ}}_i\mathcal{H}^\vee_{1,\infty})$$

  for all $i = 1,\dots,g$.

- $\mathrm{Fil}_\bullet$ is a full flag of the free R^+-module $\mathscr{F}_\infty(\mathcal{Y})$, which is w-compatible with respect to the basis $\mathrm{HT}_{\mathcal{H}^\vee_{n,\infty}}(\psi(\epsilon_1)),\dots,\mathrm{HT}_{\mathcal{H}^\vee_{n,\infty}}(\psi(\epsilon_g))$ in the sense of Definition 3.2.2 (i).

- Each w_i is an R^+-basis for $\mathrm{Fil}_i/\mathrm{Fil}_{i-1}$, which is w-compatible with respect to the basis $\mathrm{HT}_{\mathcal{H}^\vee_{n,\infty}}(\psi(\epsilon_1)),\dots,\mathrm{HT}_{\mathcal{H}^\vee_{n,\infty}}(\psi(\epsilon_g))$ in the sense of Definition 3.2.2 (ii).

Moreover, $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$ permutes these triples by right multiplication.

Proof of Theorem 3.7.2. The construction of Ψ is similar to the proof of Proposition 3.2.13. We only give a sketch of the proof. Indeed, the isomorphism Ψ is established via a sequence of isomorphisms

$$\Psi : \underline{\omega}_w^{\kappa_{\mathcal{U}}}(\mathcal{V}) \xrightarrow[\Psi_1]{\simeq} \omega^{(1)} \xrightarrow[\Psi_2]{\simeq} \omega^{(2)} \xrightarrow[\Psi_3]{\simeq} h_\diamond^* \underline{\omega}_{w,v}^{\kappa_{\mathcal{U}},\mathrm{AIP}}(\mathcal{V}),$$

where

$$\omega^{(1)} := \left\{ f \in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}^\vee}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathscr{O}_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes} R_{\mathcal{U}}) : \boldsymbol{\gamma}^* f = \rho_{\kappa_{\mathcal{U}}^\vee}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger) f, \quad \forall\, \boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \right\}$$

and

$$\omega^{(2)} := \left\{ f \in \pi^{\mathrm{AIP}}_{\infty,*}\, \mathscr{O}_{\mathcal{IW}^+_{w,\infty}}(\mathcal{V}_\infty)\widehat{\otimes} R_{\mathcal{U}} : \begin{array}{l} \boldsymbol{\gamma}^* f = f, \quad \boldsymbol{\tau}^* f = \kappa_{\mathcal{U}}^\vee(\boldsymbol{\tau}) f, \quad \boldsymbol{\nu}^* f = f \\ \forall (\boldsymbol{\gamma}, \boldsymbol{\tau}, \boldsymbol{\nu}) \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \times T_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z}) \end{array} \right\}.$$

The construction of Ψ_1 and Ψ_3 follows verbatim as in Proposition 3.2.13. To construct Ψ_2, consider $\mathfrak{s}^\ddagger = \begin{pmatrix} \mathfrak{s}_g & \cdots & \mathfrak{s}_1 \end{pmatrix} \in \mathscr{F}_\infty(\mathcal{V}_\infty)^g$. Let $\mathrm{Fil}^\ddagger_\bullet$ be the full flag of the free $\mathscr{O}^+_{\mathcal{V}_\infty}(\mathcal{V}_\infty)$-module $\mathscr{F}_\infty(\mathcal{V}_\infty)$ given by

$$\mathrm{Fil}^\ddagger_\bullet = 0 \subset \langle \mathfrak{s}_g \rangle \subset \langle \mathfrak{s}_g, \mathfrak{s}_{g-1} \rangle \subset \cdots \langle \mathfrak{s}_g, \ldots, \mathfrak{s}_1 \rangle$$

and let $w_i^\ddagger$ be the image of $\mathfrak{s}_{g+1-i}$ in $\mathrm{Fil}^\ddagger_i / \mathrm{Fil}^\ddagger_{i-1}$, for all $i = 1, \ldots, g$. Moreover, consider the trivialisation

$$\psi^\ddagger : (\mathbf{Z}/p^n\mathbf{Z})^g \xrightarrow{\simeq} \mathcal{H}^\vee_{n,\infty}$$

obtained by pulling back the universal trivialisation of $\mathcal{H}_n^\vee$ on $\overline{\mathcal{X}}_1(p^n)(v)$ along $h_1 : \overline{\mathcal{X}}_{\Gamma(p^\infty)}(v) \to \overline{\mathcal{X}}_1(p^n)(v)$. Then the triple $(\psi^\ddagger, \mathrm{Fil}^\ddagger_\bullet, \{w_i^\ddagger\})$ defines a section of the $\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z})$-torsor $\pi^{\mathrm{AIP}}_\infty : \mathcal{V}^+_\infty \to \mathcal{V}_\infty$. Consequently, one obtains an isomorphism

$$\mathcal{U}^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times \mathcal{T}^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\simeq} \mathcal{V}^+_\infty, \quad \boldsymbol{\gamma}' \mapsto (\psi^\ddagger, \mathrm{Fil}^\ddagger_\bullet, \{w_i^\ddagger\}) \cdot \boldsymbol{\gamma}'$$

and thus an isomorphism

$$\Phi : \pi^{\mathrm{AIP}}_{\infty,*}\, \mathscr{O}_{\mathcal{IW}^+_{w,\infty}}(\mathcal{V}_\infty)\widehat{\otimes} R_{\mathcal{U}} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{analytic functions} \\ U^{\mathrm{opp},(w)}_{\mathrm{GL}_g,1} \times T^{(w)}_{\mathrm{GL}_g,0} \times U_{\mathrm{GL}_g}(\mathbf{Z}/p^n\mathbf{Z}) \to \mathscr{O}_{\mathcal{V}_\infty}(\mathcal{V}_\infty)\widehat{\otimes} R_{\mathcal{U}} \end{array} \right\}$$
$$f \mapsto \Big(\boldsymbol{\gamma}' \mapsto f\big((\psi^\ddagger, \mathrm{Fil}^\ddagger_\bullet, \{w_i^\ddagger\}) \cdot \boldsymbol{\gamma}'\big) \Big).$$

By the same calculation as in Proposition 3.2.13, one sees that, if $\boldsymbol{\gamma}^* f = f$ for any $\boldsymbol{\gamma} = \begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$, then $\boldsymbol{\gamma}^* \Phi(f) = \rho_{\kappa_{\mathcal{U}}^\vee}(\boldsymbol{\gamma}_a^\ddagger + \mathfrak{z}\,\boldsymbol{\gamma}_c^\ddagger)\Phi(f)$. This induces an isomorphsm $\Phi : \omega^{(2)} \xrightarrow{\simeq} \omega^{(1)}$. Taking $\Psi_2 = \Phi^{-1}$ does the job. $\square$

Remark 3.7.8. Notice that $\mathfrak{s}_i$'s are, in fact, integral. Hence, by pulling back the formal scheme $\mathfrak{IW}^+_w$ to the modified integral model, the method above provides a strategy to compare our integral sheaf $\underline{\omega}_w^{\kappa_{\mathcal{U}},+}$ with the integral overconvergent automorphic sheaf constructed in [AIP15].

Chapter 4

Overconvergent cohomology groups

In this chapter, we study the *overconvergent cohomology groups*. Such a notion was first introduced in [AS08] and taken to study eigenvarieties for reductive groups in [Urb11; Han17; JN19]. We shall discuss the construction in §4.1 by following [Han17] closely. In §4.2, we again follow *loc. cit.* to define the Hecke operators. We will close this chapter with the algebraic counterparts of the overconvergent cohomology groups.

4.1 Overconvergent cohomology groups

4.1.1. Consider

$$\mathbf{T}_0 := \left\{(\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \in \mathrm{Iw}_{\mathrm{GL}_g} \times M_g(p\,\mathbf{Z}_p) : {}^{\mathtt{t}}\boldsymbol{\gamma}\,\breve{\mathbb{1}}_g\,\boldsymbol{\upsilon} = {}^{\mathtt{t}}\boldsymbol{\upsilon}\,\breve{\mathbb{1}}_g\,\boldsymbol{\gamma}\right\}.$$

Notice that a pair $(\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \in \mathrm{Iw}_{\mathrm{GL}_g} \times M_g(p\,\mathbf{Z}_p)$ lies in $\mathbf{T}_0$ if and only if there exist $\boldsymbol{\alpha}_b, \boldsymbol{\alpha}_d \in M_g(\mathbf{Z}_p)$ such that

$$\begin{pmatrix} \boldsymbol{\gamma} & \boldsymbol{\alpha}_b \\ \boldsymbol{\upsilon} & \boldsymbol{\alpha}_d \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbf{Q}_p) \cap M_{2g}(\mathbf{Z}_p).$$

In fact, there is a natural embedding

$$\mathbf{T}_0 \hookrightarrow \mathrm{Iw}_{\mathrm{GSp}_{2g}}, \quad (\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \mapsto \begin{pmatrix} \boldsymbol{\gamma} & \\ \boldsymbol{\upsilon} & \breve{\mathbb{1}}_g\,{}^{\mathtt{t}}\boldsymbol{\gamma}^{-1}\,\breve{\mathbb{1}}_g \end{pmatrix}.$$

Also consider the subset $\mathbf{T}_{00}$ of $\mathbf{T}_0$ defined by

$$\mathbf{T}_{00} := \left\{(\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \in \mathbf{T}_0 : \boldsymbol{\gamma} \in U_{\mathrm{GL}_g,1}^{\mathrm{opp}}\right\}.$$

We can identify $\mathbf{T}_{00}$ with $U_{\mathrm{GSp}_{2g},1}^{\mathrm{opp}}$ through the bijection

$$\mathbf{T}_{00} \to U_{\mathrm{GSp}_{2g},1}^{\mathrm{opp}}, \quad (\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \mapsto \begin{pmatrix} \boldsymbol{\gamma} & \\ \boldsymbol{\upsilon} & \breve{\mathbb{1}}_g\,{}^{\mathtt{t}}\boldsymbol{\gamma}^{-1}\,\breve{\mathbb{1}}_g \end{pmatrix}.$$

Observe that $\mathbf{T}_0$ admits two natural actions:

(i) There is a right action of $\mathrm{Iw}_{\mathrm{GL}_g}$ given by

$$\mathbf{T}_0 \times \mathrm{Iw}_{\mathrm{GL}_g} \to \mathbf{T}_0, \quad ((\boldsymbol{\gamma}, \boldsymbol{\upsilon}), \boldsymbol{\gamma}') \mapsto (\boldsymbol{\gamma}\,\boldsymbol{\gamma}', \boldsymbol{\upsilon}\,\boldsymbol{\gamma}').$$

To see that this is indeed a right action, we embed $\mathrm{Iw}_{\mathrm{GL}_g}$ into $\mathrm{Iw}_{\mathrm{GSp}_{2g}}$ through $\boldsymbol{\gamma}' \mapsto \begin{pmatrix} \boldsymbol{\gamma}' & \\ & \breve{\mathbb{1}}_g\, {}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\, \breve{\mathbb{1}}_g \end{pmatrix}$ and verify that

$$\begin{pmatrix} \boldsymbol{\gamma} & * \\ \boldsymbol{\upsilon} & * \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}' & \\ & \breve{\mathbb{1}}_g\, {}^{\mathsf{t}}\boldsymbol{\gamma}'^{-1}\, \breve{\mathbb{1}}_g \end{pmatrix} = \begin{pmatrix} \boldsymbol{\gamma}\,\boldsymbol{\gamma}' & * \\ \boldsymbol{\upsilon}\,\boldsymbol{\gamma}' & * \end{pmatrix}$$

(ii) There is a left action of $\Xi := \begin{pmatrix} \mathrm{Iw}_{\mathrm{GL}_g} & M_g(\mathbf{Z}_p) \\ M_g(p\,\mathbf{Z}_p) & M_g(\mathbf{Z}_p) \end{pmatrix} \cap \mathrm{GSp}_{2g}(\mathbf{Q}_p)$ given by

$$\Xi \times \mathbf{T}_0 \to \mathbf{T}_0, \quad \left(\begin{pmatrix} \boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d \end{pmatrix}, (\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \right) \mapsto (\boldsymbol{\alpha}_a\,\boldsymbol{\gamma} + \boldsymbol{\alpha}_b\,\boldsymbol{\upsilon}, \boldsymbol{\alpha}_c\,\boldsymbol{\gamma} + \boldsymbol{\alpha}_d\,\boldsymbol{\upsilon}).$$

To see this is indeed a left action, it suffices to observe that

$$\begin{pmatrix} \boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} & * \\ \boldsymbol{\upsilon} & * \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_a\,\boldsymbol{\gamma} + \boldsymbol{\alpha}_b\,\boldsymbol{\upsilon} & * \\ \boldsymbol{\alpha}_c\,\boldsymbol{\gamma} + \boldsymbol{\alpha}_d\,\boldsymbol{\upsilon} & * \end{pmatrix}.$$

Since $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ is a subset of Ξ, we also obtain a natural left action of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ on $\mathbf{T}_0$.

4.1.2. Let $r \in \mathbf{Q}_{>0}$ and let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be an r-analytic weight. We employ the notion of *r-analytic functions* on $U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}$, $\mathbf{T}_{00}$, and $\mathbf{T}_0$ as follows.

Fix a (topological) isomorphism

$$\mathbf{Z}_p^{g^2} \simeq U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}.$$

(i) We say that a function $f : U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1} \to R^+_{\mathcal{U}}$ is ***r-analytic*** if the composition

$$\mathbf{Z}_p^{g^2} \simeq U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1} \xrightarrow{f} R^+_{\mathcal{U}} \hookrightarrow \mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}}$$

is r-analytic in the sense of Definition 3.1.8 (i).

(ii) We say that a function $f : \mathbf{T}_{00} \to R^+_{\mathcal{U}}$ is ***r-analytic*** if it is r-analytic viewed as a function on $U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}$, via the identification $\mathbf{T}_{00} \simeq U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}$.

Before proceeding, we need the following statement.

Lemma 4.1.3 (Amice)**.** *Let $r \in \mathbf{Q}_{\geq 0}$. For any $d \in \mathbf{Z}_{>0}$ and for any $i = (i_1, ..., i_d) \in \mathbf{Z}^d_{\geq 0}$, define the function*

$$e_i^{(r)} : \mathbf{Z}_p^d \to \mathbf{Z}_p, \quad (x_1, ..., x_d) \mapsto \prod_{t=1}^{d} \lfloor p^{-r} i_t \rfloor! \binom{x_t}{i_t}.$$

Then, $\{e_i^{(r)}\}_i$ *provides an othonormal basis for* $C^{r-\mathrm{an}}(\mathbf{Z}_p^d, \mathbf{Z}_p)$.

Proof. This is the multivariable version of a theorem of Y. Amice, which is presumebaly well-known. We deduce the statement from [Laz65, Chapter III, 1.3.8], which is based on the work of Amice [Ami64, §10].

By replacing r with $\lceil r \rceil$, we may assume $r \in \mathbf{Z}$. By Mahler expansion, we know a continuous function $f : \mathbf{Z}_p^d \to \mathbf{Z}_p$ can be written as

$$f(x_1, ..., x_d) = \sum_{i=(i_1,...,i_d)\in \mathbf{Z}_{\geq 0}^d} c_i \prod_{t=1}^{d} \binom{x_t}{i_t}$$

for some $c_i \in \mathbf{Z}_p$. By [Laz65, Chapter III, 1.3.8], f is r-analytic if and only if

$$v_p(c_i) - \sum_{t=1}^{d} \left(v_p(i_t!) - \frac{i_t(1-p^{-r})}{p-1} \right) \to \infty \quad \text{as } \sum_{t=1}^{d} i_t \to \infty.$$

On the other hand, for each i_t, we have

$$v_p(i_t!) - v_p(\lfloor p^{-r} i_t \rfloor !) = \sum_{m=1}^{\infty} \lfloor p^{-m} i_t \rfloor - \sum_{m=1}^{\infty} \lfloor p^{-(m+r)} i_t \rfloor$$

by Legendre's formula.

Observe that

$$\sum_{m=1}^{\infty} \frac{i_t}{p^m} - \sum_{m=1}^{\infty} \left\lfloor \frac{i_t}{p^m} \right\rfloor = \frac{\mathrm{Schiff}\, i_t}{p-1}$$[1]

Hence,

$$\begin{aligned} \left(\sum_{m=1}^{\infty} \frac{i_t}{p^m} - \sum_{m=1}^{\infty} \frac{i_t}{p^{m+r}} \right) - \left(\sum_{m=1}^{\infty} \left\lfloor \frac{i_t}{p^m} \right\rfloor - \sum_{m=1}^{\infty} \left\lfloor \frac{i_t}{p^{m+r}} \right\rfloor \right) \\ &= \frac{1}{p-1} \left(\mathrm{Schiff}\, i_t - \mathrm{Schiff}\, \lfloor i_t / p^r \rfloor \right) \\ &= \frac{1}{p-1} \left(\sum_{j=1}^{r-1} i_{t,j} \right) \\ &\leq \frac{(r-1)(p-1)}{p-1} = r - 1. \end{aligned}$$

[1]Here, the function Schiff is defined as in [Laz65], *i.e.*, for any integer $n = n_k p^k + n_{k-1} p^{k-1} + ... + n_1 p + n_0$ for $0 \leq n_j \leq p-1$, then $\mathrm{Schiff}\, n := \sum_j n_j$. The name of this function should be understood as 'la *somme* des *chiffres* du développement de n' (the sum of the digits of the expansion of n). It should not be confused with the German word 'das Schiff' (the ship).

Here, in the second equation above, we write $i_t = \sum_{j=1}^k i_{t,j}p^j$ with $0 \le i_{t,j} \le p-1$. Since

$$\sum_{m=1}^{\infty} p^{-m} i_t - \sum_{m=1}^{\infty} p^{-(m+r)} i_t = \frac{i_t(1-p^{-r})}{p-1},$$

we have

$$\left(\sum_{m=1}^{\infty}\left\lfloor\frac{i_t}{p^m}\right\rfloor - \sum_{m=1}^{\infty}\left\lfloor\frac{i_t}{p^{m+r}}\right\rfloor\right) + (r-1) \ge \frac{i_t(1-p^{-r})}{p-1} \ge \left(\sum_{m=1}^{\infty}\left\lfloor\frac{i_t}{p^m}\right\rfloor - \sum_{m=1}^{\infty}\left\lfloor\frac{i_t}{p^{m+r}}\right\rfloor\right)$$

and so

$$v_p(i_t!) - \frac{i_t(1-p^{-r})}{p-1} + (r-1) \ge v_p(\lfloor p^{-r} i_t\rfloor!) \ge v_p(i_t!) - \frac{i_t(1-p^{-r})}{p-1}.$$

Therefore, one concludes that f is r-analytic if and only if

$$v_p(c_i) - \sum_{t=1}^{d} v_p(\lfloor p^{-r} i_t\rfloor!) \to \infty \quad \text{as } \sum_{t=1}^{d} i_t \to \infty.$$

This then implies the desired result. $\square$

4.1.4. Given an r-analytic weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$, we define

$$A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}}) := C^{r-\mathrm{an}}(\mathbf{T}_{00}, \mathbf{Z}_p)\widehat{\otimes} R_{\mathcal{U}}^+ \quad \text{and} \quad A^r(\mathbf{T}_{00}, R_{\mathcal{U}}) := A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}})[\frac{1}{p}].$$

By identifying $\mathbf{T}_{00}$ with $\mathbf{Z}_p^{g^2}$, Lemma 4.1.3 implies that

$$A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}}) \simeq \widehat{\bigoplus}_{i\in\mathbf{Z}_{\ge 0}^{g^2}} R_{\mathcal{U}}^+ e_i^{(r)}$$

and so we view elements in $A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}})$ as functions from $\mathbf{T}_{00}$ to $R_{\mathcal{U}}^+$. In other words, we have

$$A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}}) = \left\{ \sum_{i\in\mathbf{Z}_{\ge 0}^{g^2}} c_i e_i^{(r)} : c_i \in R_{\mathcal{U}}^+ \text{ and } c_i \to 0\ \mathfrak{a}_{\mathcal{U}}\text{-adically} \right\},$$

where $\mathfrak{a}_{\mathcal{U}} = pR_{\mathcal{U}}^+$ if $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is an affinoid weight and $\mathfrak{a}_{\mathcal{U}}$ is an ideal of definition of the profinite topology on $R_{\mathcal{U}}^+$ if $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is a small weight. By definition, these functions are r-analytic. In fact, if $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is an affinoid weight, we have the identification

$$A^r(\mathbf{T}_{00}, R_{\mathcal{U}}) = \{r\text{-analytic functions } f : \mathbf{T}_{00} \to R_{\mathcal{U}}\}.$$

On the other hand, define

$$A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := \left\{ f : \mathbf{T}_0 \to R_{\mathcal{U}}^+ : \begin{array}{l} f(\boldsymbol{\gamma}\boldsymbol{\beta}, \boldsymbol{v}\boldsymbol{\beta}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta}) f(\boldsymbol{\gamma}, \boldsymbol{v}), \forall (\boldsymbol{\gamma}, \boldsymbol{v}) \in \mathbf{T}_0, \ \boldsymbol{\beta} \in B_{\mathrm{GL}_g,0} \\ f|_{\mathbf{T}_{00}} \in A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}}) \end{array} \right\} \text{ and}$$

$$A^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})[\frac{1}{p}].$$

We have an identification

$$A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \xrightarrow{\sim} A^{r,\circ}(\mathbf{T}_{00}, R_{\mathcal{U}}), \quad f \mapsto f|_{\mathbf{T}_{00}}.$$

Taking continuous duals, we obtain the corresponding spaces of r-analytic distributions

$$D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := \mathrm{Hom}^{\mathrm{cts}}_{R_{\mathcal{U}}^+}(A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}), R_{\mathcal{U}}^+) \quad \text{and} \quad D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})[\frac{1}{p}].$$

Here, we remark that if $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is an affinoid (resp., a small) weight, the continuous dual is taken with respect to the p-adic (resp., profinite) topology on $R_{\mathcal{U}}^+$.

From the construction, we see that the left action of Ξ on $\mathbf{T}_0$ then induce a left action of Ξ on both $D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ and $D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$. Furthermore, if $r' \geq r$, there is a natural injection $A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \hookrightarrow A^{r',\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ which induces injections (see [Han17, §2.2])

$$D^{r',\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \hookrightarrow D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \quad \text{and} \quad D^{r'}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \hookrightarrow D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}).$$

We then write

$$A^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := \varinjlim_r A^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \quad \text{and} \quad D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := \varprojlim_r D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}).$$

Example 4.1.5. An example of elements in $A^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is the ***(analytic) highest weight vector*** $e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}$ defined as follows.

Given any $X = (X_{ij})_{1 \leq i,j \leq g} \in \mathrm{Iw}_{\mathrm{GL}_g}$, define

$$e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(X) = \frac{\kappa_{\mathcal{U},1}(X_{11})}{\kappa_{\mathcal{U},2}(X_{11})} \times \frac{\kappa_{\mathcal{U},2}(\det((X_{ij})_{1 \leq i,j \leq 2}))}{\kappa_{\mathcal{U},3}(\det((X_{ij})_{1 \leq i,j \leq 2}))} \times \cdots \times \kappa_{\mathcal{U},g}(\det(X)).$$

Then, we view $e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}$ as a function on $\mathbf{T}_0$ via

$$(\boldsymbol{\gamma}, \boldsymbol{v}) \mapsto e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}).$$

By direct computation, one easily checks that

$$e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}\boldsymbol{\beta}, \boldsymbol{v}\boldsymbol{\beta}) = e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}\boldsymbol{\beta}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}(\boldsymbol{\gamma}, \boldsymbol{v}).$$

Moreover, by calculation in the proof of [CHJ17, Proposition 2.6], one concludes that $e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}} \in A^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$.

4.1.6. Suppose now that $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is a small weight and take $r > 1 + r_{\mathcal{U}}$ (see Definition 3.1.10). Fix an ideal $\mathfrak{a}_{\mathcal{U}}$ of $R_{\mathcal{U}}$ defining the profinite topology on $R_{\mathcal{U}}$ and such that $p \in \mathfrak{a}_{\mathcal{U}}$.

Similar to [CHJ17, Proposition 3.1] (see also [Han15, §2.1]), $D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ admits a decreasing filtration $\mathrm{Fil}^\bullet D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ defined by

$$\mathrm{Fil}^j D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) := \ker\left(D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \to D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})/\, \mathfrak{a}_{\mathcal{U}}^j D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})\right).$$

Write

$$D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}) := D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})/\, \mathrm{Fil}^j D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$$

for every $j \in \mathbf{Z}_{\geq 1}$.

Lemma 4.1.7. *Given a small weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and $r > 1 + r_{\mathcal{U}}$.*

(i) For any $j \in \mathbf{Z}_{\geq 0}$, $\mathrm{Fil}^j D^{r,\circ}_{\kappa_{\mathcal{U}}}$ is Ξ-stable.

(ii) For any $j \in \mathbf{Z}_{\geq 0}$, $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ is a finitely abelian group. Therefore,

$$D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) = \varprojlim_j D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}),$$

is a profinite flat $\mathbf{Z}_p$-module in the sense of [CHJ17, Definition 6.1].

Proof. To show (i), one observes that

$$\mathfrak{a}_{\mathcal{U}}^j D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) = \left\{\mu \in D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) : \mu(f) \in \mathfrak{a}_{\mathcal{U}}^j,\ \forall f \in A^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})\right\}$$

Since $A^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is stable under the action of Ξ, $\mathfrak{a}_{\mathcal{U}}^j D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is stable under the action of Ξ. This then implies the desired result.

The proof for (ii) is inspired by the discussion in [Han15, §2.1]. We first fix identifications $\mathbf{T}_{00} \simeq U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1} \simeq \mathbf{Z}_p^{g^2}$ and simplify the notation by writing $d = g^2$. From the construction and by Lemma 4.1.3, the collection $\{e_i^{(r)}\}_i$ provides an orthonormal basis for $A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$, *i.e.*, we have an isomorphism

$$A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \simeq \widehat{\bigoplus}_{i\in\mathbf{Z}^d_{\geq 0}} R_{\mathcal{U}} e_i^{(r)}.$$

Consequently, we have an isomorphism

$$D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \simeq \prod_{i\in\mathbf{Z}^d_{\geq 0}} R_{\mathcal{U}}, \quad \mu \mapsto (\mu(e_i^{(r)}))_i.$$

For any $i \in \mathbf{Z}^d_{\geq 0}$, write $c_{r,i} := \prod_t \frac{\lfloor p^{-(r-1)} i_t\rfloor!}{\lfloor p^{-r} i_t\rfloor!}$. Then, the natural injection $A^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \hookrightarrow A^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is given by

$$\widehat{\bigoplus}_i R_{\mathcal{U}} e_i^{(r)} \to \widehat{\bigoplus}_i R_{\mathcal{U}} e_i^{(r-1)}, \quad e_i^{(r)} \mapsto e_i^{(r)} = c_{r,i} e_i^{(r-1)}.$$

Hence, the natural inclusion $D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \hookrightarrow D^{r-1,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is given by

$$\prod_{i\in\mathbf{Z}^d_{\geq 0}} R_{\mathcal{U}} \to \prod_{i\in\mathbf{Z}^d_{\geq 0}} R_{\mathcal{U}}, \quad (\mu(e_i^{(r)}))_i \mapsto (c_{r,i}\mu(e_i^{(r-1)})).$$

Moreover, by Legendre's formula, we have $v_p(c_{r,i}) = \sum_{t=1}^{d} \lfloor p^{-r} i_t \rfloor$. Therefore, we see that

$$D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}) \simeq \bigoplus_{\substack{i \in \mathbf{Z}^d_{\geq 0} \\ v_p(c_{r,i}) < j}} R_{\mathcal{U}}/(\mathfrak{a}^j_{\mathcal{U}}, p^{j - v_p(c_{r,i})}).$$

Since this is a finite direct sum and each direct summand is a finite abelian group, we conclude that each $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ is a finite abelian group.

Finally, from the construction, we see that the natural map

$$D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \to \varprojlim_j D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}),$$

has dense image. Since both sides are compact, this natural map is an isomorphism. $\square$

4.1.8. Consider the étale site $\mathcal{X}_{\mathrm{Iw}^+,\text{ét}}$. Recall that, for every $n \in \mathbf{Z}_{\geq 1}$, $\mathcal{X}_{\Gamma(p^n)}$ is a finite étale Galois cover over $\mathcal{X}_{\mathrm{Iw}^+}$ with Galois group $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}/\Gamma(p^n)$, and hence $\varprojlim_n \mathcal{X}_{\Gamma(p^n)}$ is a pro-étale Galois cover of $\mathcal{X}_{\mathrm{Iw}^+}$ with Galois group $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. For each $j \in \mathbf{Z}_{\geq 1}$, let $\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}$ be the locally constant sheaf on $\mathcal{X}_{\mathrm{Iw}^+,\text{ét}}$ associated with $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ via

$$\pi_1^{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}) \to \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \to \mathrm{Aut}\left(D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})\right).$$

We obtain an inverse system of étale locally constant sheaves $(\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j})_{j \in \mathbf{Z}_{\geq 1}}$ on $\mathcal{X}_{\mathrm{Iw}^+,\text{ét}}$. This allows us to consider the étale cohomology groups

$$H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}}}) := \varprojlim_j H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}),$$

$$H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r}_{\kappa_{\mathcal{U}}}) := H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}}})[\frac{1}{p}]$$

for every $t \in \mathbf{Z}_{\geq 0}$.

4.1.9. Recall the locally symmetric space

$$X_{\mathrm{Iw}^+}(\mathbf{C}) = \mathrm{GSp}_{2g}(\mathbf{Q}) \backslash \mathrm{GSp}_{2g}(\mathbf{A}_f) \times \mathbb{H}_g / \Gamma^{(p)} \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}.$$

By taking the trivial $\mathrm{GSp}_{2g}(\mathbf{Z}_\ell)$-action on $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ for every prime number $\ell \neq p$ and letting $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ act on $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ via the left action of Ξ, we see that $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ defines a local system on the locally symmetric space $X_{\mathrm{Iw}^+}(\mathbf{C})$. In particular, for every $t \in \mathbf{Z}_{\geq 0}$, we can consider the Betti cohomology group

$$H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

The following proposition compares these two cohomology groups.

Proposition 4.1.10. *For every $t \in \mathbf{Z}_{\geq 0}$, there is a natural isomorphism*

$$H^t_{\acute{e}t}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \simeq H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

Proof. For any $j \in \mathbf{Z}_{>0}$, we have isomorphisms

$$H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \simeq H^t_{\text{ét}}(X_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \simeq H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})),$$

where

- the first isomorphism follows from the comparison isomorphism between the étale cohomology groups of an algebraic variety and the ones on the corresponding adic spaces (see [Hub13, Theorem 3.8.1]);[2] and

- the second isomorphism follows from the fact that $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ acts continuously on the module $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ and the well-known Artin comparison between the étale cohomology of a complex algebraic variety and the Betti cohomology of the associated complex manifold.

Note that we have used the algebraic isomorphism $\mathbf{C}_p \simeq \mathbf{C}$ fixed at the beginning of the thesis.

Taking limit and inverting p, we then arrive at the isomorphisms

$$H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r}_{\kappa_{\mathcal{U}}}) \simeq H^t_{\text{ét}}(X_{\mathrm{Iw}^+}, \mathscr{D}^{r}_{\kappa_{\mathcal{U}}}) \simeq \left(\varprojlim_j H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}))\right)[1/p].$$

To finish the proof, we claim

$$\left(\varprojlim_j H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}))\right)[1/p] = H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

It suffices to show that

$$\varprojlim_j H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})) = H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})),$$

i.e., the inverse limit commute with cohomology. Note that Betti cohomology can be computed via sheaf cohomology. Hence, by viewing each $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ as a locally constant sheaf on $X_{\mathrm{Iw}^+}(\mathbf{C})$, we need to show

$$R^i \varprojlim_j D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}) = 0 \tag{4.1}$$

for all $i > 0$.

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover for $X_{\mathrm{Iw}^+}(\mathbf{C})$ given by contractible open subsets. Then,

$$H^i(U_\lambda, D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})) = 0$$

[2] On the algebraic variety $X_{\mathrm{Iw}^+} = X_{\mathrm{Iw}^+,\mathbf{C}_p}$, the locally constant sheaves $\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}$ and étale cohomology groups $H^t_{\text{ét}}(X_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}}})$ and $H^t_{\text{ét}}(X_{\mathrm{Iw}^+}, \mathscr{D}^{r}_{\kappa_{\mathcal{U}}})$ are defined analogously as those on $\mathcal{X}_{\mathrm{Iw}^+}$.

for all $i, j > 0$ and for all $\lambda \in \Lambda$. Moreover, for each $\lambda \in \Lambda$, the natural map

$$D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}) = H^0(U_\lambda, D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^0(U_\lambda, D^{r,\circ}_{\kappa_{\mathcal{U}},j-1}(\mathbf{T}_0, R_{\mathcal{U}})) = D^{r,\circ}_{\kappa_{\mathcal{U}},j-1}(\mathbf{T}_0, R_{\mathcal{U}})$$

is surjective. Thus, the inverse system $\{H^0(U_\lambda, D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}))\}_{j\in\mathbf{Z}_{>0}}$ satisfies the Mittag–Leffler condition and so

$$R^1 \varprojlim_j H^0(U_\lambda, D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})) = 0.$$

We then can conclude (4.1) by applying [Sch13, Lemma 3.18]. □

4.2 Hecke operators

4.2.1. Let us discuss the Hecke operators acting on $H^t_{\text{ét}}(\mathcal{X}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$. Similar as before, the definition of Hecke operators splits into two cases: Hecke operators outside Np and Hecke operators at p. Our strategy is to describe Hecke operators on the Betti cohomology groups $H^t(X_{\text{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ and then use Proposition 4.1.10 to make these operators acting on $H^t_{\text{ét}}(\mathcal{X}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$. Therefore, we begin with a brief recollection of the Hecke operators on $H^t(X_{\text{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ studied in [Han17]. We refer the readers to *loc. cit.* for a more detailed discussion.

4.2.2 (Hecke operators outside pN)**.** Let ℓ be a prime number not dividing pN. For any $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell)$, consider a double coset decomposition

$$\mathrm{GSp}_{2g}(\mathbf{Z}_\ell)\,\boldsymbol{\gamma}\,\mathrm{GSp}_{2g}(\mathbf{Z}_\ell) = \bigsqcup_j \boldsymbol{\delta}_j\,\boldsymbol{\gamma}\,\mathrm{GSp}_{2g}(\mathbf{Z}_\ell)$$

for some $\boldsymbol{\delta}_j \in \mathrm{GSp}_{2g}(\mathbf{Z}_\ell)$. If we take the trivial $\mathrm{GSp}_{2g}(\mathbf{Q}_\ell)$-action on $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$, then the natural left action of $\mathrm{GSp}_{2g}(\mathbf{Q}_\ell)$ on $X_{\text{Iw}^+}(\mathbf{C})$ induces the Hecke operator

$$T_{\boldsymbol{\gamma}} : H^t(X_{\text{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\text{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})), \quad [\mu] \mapsto \sum_j (\boldsymbol{\delta}_j\,\boldsymbol{\gamma}).[\mu]. \tag{4.2}$$

4.2.3. We specify out a special element $\mathbf{t}_{\ell,0} = \mathrm{diag}(\mathbb{1}_g, \ell\,\mathbb{1}_g) \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell)$. For any $x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$, denote by $T^w_{\ell,0}$ the Hecke operator defined by the double coset

$$\mathrm{GSp}_{2g}(\mathbf{Z}_\ell)(x \cdot \mathbf{t}_{\ell,0})\,\mathrm{GSp}_{2g}(\mathbf{Z}_\ell).$$

Following [GT05, §3], we define the ***Hecke polynomial at*** ℓ to be

$$P_{\mathrm{Hecke},\ell}(Y) := \prod_{x\in \mathrm{Weyl}^H} (Y - T^x_{\ell,0}) \in \mathbb{T}_\ell[Y]. \tag{4.3}$$

One sees immediately that this is a polynomial of degree 2^g.

4.2.4 (Hecke operators at p)**.** For the Hecke operators at p, recall the matrices

$$\mathbf{u}_{p,i} = \begin{cases} \begin{pmatrix} \mathbb{1}_i & & & \\ & p\,\mathbb{1}_{g-i} & & \\ & & p\,\mathbb{1}_{g-i} & \\ & & & p^2\,\mathbb{1}_i \end{pmatrix}, & 1 \le i \le g-1 \\ \\ \begin{pmatrix} \mathbb{1}_g & \\ & p\,\mathbb{1}_g \end{pmatrix}, & i = g \end{cases}$$

and we write

$$\mathbf{u}_{p,i} = \begin{pmatrix} \mathbf{u}_{p,i}^{\square} & \\ & \mathbf{u}_{p,i}^{\blacksquare} \end{pmatrix}.$$

For every $i = 1, \dots, g$, consider a $\mathbf{u}_{p,i}$-action on $\mathbf{T}_0$ defined as follows: for every $(\boldsymbol{\gamma}, \boldsymbol{v}) \in \mathbf{T}_0$, we put

$$\mathbf{u}_{p,i} \,.(\boldsymbol{\gamma}, \boldsymbol{v}) = (\mathbf{u}_{p,i}^{\square}\, \boldsymbol{\gamma}_0\, \mathbf{u}_{p,i}^{\square,-1}, \mathbf{u}_{p,i}^{\blacksquare}\, \boldsymbol{v}_0\, \mathbf{u}_{p,i}^{\square,-1})\, \boldsymbol{\beta}$$

where we write $(\boldsymbol{\gamma}, \boldsymbol{v}) = (\boldsymbol{\gamma}_0, \boldsymbol{v}_0)\, \boldsymbol{\beta}$ with $\boldsymbol{\gamma}_0 \in U^{\mathrm{opp}}_{\mathrm{GL}_g,1}$ and $\boldsymbol{\beta} \in B_{\mathrm{GL}_g,0}$. This then induces a $\mathbf{u}_{p,i}$-action on $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$.

Similar to §3.3, for every $i = 1, \dots, g$, choose a double coset decomposition

$$\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} = \bigsqcup_j \boldsymbol{\delta}_{ij}\, \mathbf{u}_{p,i}\, \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}\,.$$

with $\boldsymbol{\delta}_{ij} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. The natural left action of $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$ on $X_{\mathrm{Iw}^+}(\mathbf{C})$ together with the actions of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ and $\mathbf{u}_{p,i}$ on $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ induce the Hecke operator

$$\begin{aligned} U_{p,i} : H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) &\to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})), \\ [\mu] &\mapsto p^{\nu_i} \textstyle\sum_j \boldsymbol{\delta}_{ij} \,.(\mathbf{u}_{p,i} \,.[\mu]) \end{aligned} \quad . \tag{4.4}$$

Here, again, $\nu_i = -(g-i)(g+1)$ for $i = 1, ..., g-1$ and $\nu_g = \frac{-g(g+1)}{2}$. Similarly, we have Hecke operators $U^x_{p,i}$ for any $x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$.

4.2.5. Finally, as mentioned, the Hecke operators acting on $H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$ are defined as follows:

(i) The Hecke operators $T_{\boldsymbol{\gamma}}$ (for $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell)$ with $\ell \nmid Np$) and $U^x_{p,i}$ (for $i = 1, ..., g$ and $x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$) acting on the overconvergent cohomology groups $H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$ are defined to be the operators $T_{\boldsymbol{\gamma}}$ and $U^x_{p,i}$ acting on the overconvergent cohomology group $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ via the isomorphism in Proposition 4.1.10.

(ii) We define the operator U_p as the composition $U_p = \prod_{i=1}^g U_{p,i}$.

4.3 Overconvergent parabolic cohomology groups

4.3.1. We spell out the *overconvergent parabolic cohomology groups* in this section, which is essential in the construction of the cuspidal eigenvariety. These groups are nothing but the image of the natural map from the compactly supported cohomology groups into the cohomology groups. Let us discuss about this in more details.

4.3.2. Consider the *Borel–Serre compactification* $\overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C})$ of the locally symmetric space $X_{\mathrm{Iw}^+}(\mathbf{C})$ (see [BS73]). By choosing a triangulation on $\overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C})$, one can form the so-called *Borel–Serre cochain complex* $C^{\bullet}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ which computes the Betti cohomology groups $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ (see [Han17, §2.1]).

The fixed triangulation on $\overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C})$ provides also a triangulation on the boundary $\partial\overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C}) := \overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C})\smallsetminus X_{\mathrm{Iw}^+}(\mathbf{C})$ and hence defines a cochain complex $C^{\bullet}_{\partial}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ that computes the cohomology groups at the boundary. The natural closed embedding $\partial\overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C}) \hookrightarrow \overline{X}^{\mathrm{BS}}_{\mathrm{Iw}^+}(\mathbf{C})$ then induces a morphism of cochain complexes

$$\pi : C^{\bullet}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to C^{\bullet}_{\partial}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

Following [Bar18, §3.1.3], we define $C^{\bullet}_c(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) := \mathrm{Cone}(\pi)$ the mapping cone of π, *i.e.*,

$$\begin{aligned}
&\mathrm{Cone}(\pi)^t = C^t(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \oplus C^{t-1}_{\partial}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \text{ with}\\
&d^t_c : \mathrm{Cone}(\pi)^t \to \mathrm{Cone}(\pi)^{t+1}, \quad (\sigma, \sigma_{\partial}) \mapsto (-d^t\sigma, -\pi^i\sigma + d^{t-1}_{\partial}\sigma_{\partial}),
\end{aligned}$$

where d and d_{∂} are differentials on $C^{\bullet}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ and $C^{\bullet}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ respectively. The strategy of the proof of [Bar18, Proposition 3.5] applies here and one sees that $C^{\bullet}_c(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ computes the compactly supported cohomology groups $H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$. Moreover, the natural morphism

$$C^{\bullet}_c(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to C^{\bullet}(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$$

induces a morphism on the cohomology groups

$$H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

For each t, we let

$$H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) := \mathrm{image}\left(H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))\right),$$

and call them the parabolic cohomology groups.

Lemma 4.3.3. *The parabolic cohomology groups $H^r_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ are Hecke-stable.*

Proof. Due to the nature of the Borel–Serre compactification, $C^{\bullet}_{\partial}(\mathrm{Iw}_{\mathrm{GSp}^+_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ admits Hecke actions as the ones defined above. Hence

$$\pi : C^\bullet(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to C^\bullet_\partial(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$$

is a Hecke equivariant morphism of cochain complexes and hence

$$C^\bullet_c(\mathrm{Iw}_{\mathrm{GSp}^+_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to C^\bullet(\mathrm{Iw}_{\mathrm{GSp}^+_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$$

is also Hecke-equivariant and induces a Hecke-equivariant map on cohomology groups

$$H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \to H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

This then shows the desired result. $\square$

4.3.4. Finally, we discuss the relation between $H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ with étale cohomology. Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a small weight and let $r > 1 + r_{\mathcal{U}}$. Then, Proposition 4.1.10 and the Poincaré duality (for both Betti cohomology and étale cohomology) allows us to deduce an isomorphism

$$H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \simeq H^t_{\text{ét},c}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$$

for any $t \in \mathbf{Z}$. Again, we define Hecke operators acting on $H^t_{\text{ét},c}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$ via this isomorphism. In particular, we have a commutative diagram

$$\begin{tikzcd}
H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \arrow[r, "\simeq"] & H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \\
H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \arrow[r, "\simeq"] \arrow[u] & H^t_{\text{ét},c}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \arrow[u]
\end{tikzcd},$$

where the vertical arrows are Hecke-equivariant.

4.4 Algebraic counterparts

4.4.1. The modules $A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ and $D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ introduced in 4.1.4 have algebraic counterparts, which we now explain.

Let $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{>0}$ with $k_1 \geq \cdots \geq k_g$. One can view k as a character on $T_{\mathrm{GSp}_{2g}}$ via

$$k : T_{\mathrm{GSp}_{2g}} \to \mathbb{G}_m, \quad \mathrm{diag}(\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_g, \boldsymbol{\tau}_0 \boldsymbol{\tau}_g^{-1}, ..., \boldsymbol{\tau}_0 \boldsymbol{\tau}_1^{-1}) \mapsto \prod_{i=1}^g \boldsymbol{\tau}_i^{k_i}.$$

One extends k to $B_{\mathrm{GSp}_{2g}}$ by setting $k(U_{\mathrm{GSp}_{2g}}) = \{1\}$. Consider the irreducible representation for GSp_{2g}

$$\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} := \left\{ \phi : \mathrm{GSp}_{2g} \to \mathbb{A}^1 : \begin{array}{l} \phi \text{ is a morphism of schemes} \\ \phi(\boldsymbol{\gamma}\boldsymbol{\beta}) = k(\boldsymbol{\beta})\phi(\boldsymbol{\gamma}) \text{ for any } (\boldsymbol{\gamma}, \boldsymbol{\beta}) \in \mathrm{GSp}_{2g} \times B_{\mathrm{GSp}_{2g}} \end{array} \right\}.$$

One can consider the following actions of GSp_{2g} on $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$:

(i) The right action given by
$$(\phi\cdot\boldsymbol{\gamma})(\boldsymbol{\gamma}') = \phi(\boldsymbol{\gamma}\,\boldsymbol{\gamma}').$$

(ii) The left action given by
$$(\boldsymbol{\gamma}\cdot\phi)(\boldsymbol{\gamma}') = \phi({}^{\mathtt{t}}\boldsymbol{\gamma}\,\boldsymbol{\gamma}').$$

(iii) The left action given by
$$(\boldsymbol{\gamma}\cdot\phi)(\boldsymbol{\gamma}') = \phi(\boldsymbol{\gamma}^{-1}\,\boldsymbol{\gamma}').$$

Notice that the second action is valid since GSp_{2g} is stable under transpose. In fact, one deduces easily from the definition that

$$ {}^{\mathtt{t}}\boldsymbol{\gamma} = \varsigma(\boldsymbol{\gamma})\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}^{-1}\begin{pmatrix} & \breve{\mathbb{1}}_g\\ -\breve{\mathbb{1}}_g & \end{pmatrix}$$

for any $\boldsymbol{\gamma}\in\mathrm{GSp}_{2g}$. Therefore, the second action is nothing but a twisted action of the third one. In what follows, we equip $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ with the left GSp_{2g}-action given by (ii).

Denote by $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ the linear dual of $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$. We equip with it a left GSp_{2g}-action given by the right action (i) on $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$.

4.4.2. From now on, we abuse the notation, writing $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ and $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ for their $\mathbf{Q}_p$-realisation. That is,

$$\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} = \left\{\phi:\mathrm{GSp}_{2g}(\mathbf{Q}_p)\to\mathbf{Q}_p : \begin{array}{l}\phi \text{ is a polynomial function}\\ \phi(\boldsymbol{\gamma}\,\boldsymbol{\beta}) = k(\boldsymbol{\beta})\phi(\boldsymbol{\gamma})\ \forall(\boldsymbol{\gamma},\boldsymbol{\beta})\in\mathrm{GSp}_{2g}(\mathbf{Q}_p)\times B_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)\end{array}\right\}$$
$$\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} = \mathrm{Hom}_{\mathbf{Q}_p}(\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k},\mathbf{Q}_p).$$

There is an obvious injective morphism

$$\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}\to A^r_k(\mathbf{T}_0,\mathbf{Q}_p),\quad \phi\mapsto\left((\boldsymbol{\gamma},\boldsymbol{v})\mapsto k(\boldsymbol{\beta})\phi\left(\begin{pmatrix}\boldsymbol{\gamma}_0 & \\ \boldsymbol{v}_0 & \breve{\mathbb{1}}_g\,{}^{\mathtt{t}}\boldsymbol{\gamma}_0^{-1}\,\breve{\mathbb{1}}_g\end{pmatrix}\right)\right)$$

for any r, where $(\boldsymbol{\gamma},\boldsymbol{v}) = (\boldsymbol{\gamma}_0,\boldsymbol{v}_0)\,\boldsymbol{\beta}$ with $\boldsymbol{\gamma}_0\in U^{\mathrm{opp}}_{\mathrm{GL}_g,1}$ and $\boldsymbol{\beta}\in B_{\mathrm{GL}_g,0}$. Therefore, there is a natural surjection $D^r_k(\mathbf{T}_0,\mathbf{Q}_p)\to\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ for any r, which is $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant.

Example 4.4.3. Similarly, the analytic highest weight vector also has an algebraic counterpart: Given $X = (X_{ij})_{1\leq i,j\leq 2g}\in\mathrm{GSp}_{2g}$, we consider

$$e^{\mathrm{hst}}_k(X) = X_{11}^{k_1-k_2}\times\det((X_{ij})_{1\leq i,j\leq 2})^{k_2-k_3}\times\cdots\times\det((X_{ij})_{1\leq i,j\leq g})^{k_g}.$$

Similar as in Example 4.1.5, one sees that $e^{\mathrm{hst}}_k\in\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$.

4.4.4. Notice that the left GSp_{2g}-actions on $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ and $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ induce étale $\mathbf{Q}_p$-local systems on $\mathcal{X}_{\mathrm{Iw}^+}$ which we still denote by the same symbols. In particular, we can consider the cohomology groups $H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+},\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k})$ for $t\in\mathbf{Z}_{\geq 0}$.

On the other hand, we can also consider the Betti cohomology groups $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$. By the proof of Proposition 4.1.10, we know that there is an isomorphism

$$H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}) \simeq H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$$

for any $t \in \mathbf{Z}_{\geq 0}$. Therefore, the $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant morphism $D^r_k(\mathbf{T}_0, \mathbf{Q}_p) \to \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$ then induces a commutative diagram

$$\begin{tikzcd} H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \arrow[r, "\simeq"] \arrow[d] & H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \arrow[d] \\ H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}) \arrow[r, "\simeq"] & H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}) \end{tikzcd} . \tag{4.5}$$

4.4.5. We wrap up this section by discussing the Hecke operators acting on $H^t_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$. As before, we only need to define them on the Betti cohomology groups $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$:

- For any Hecke operator $T_{\boldsymbol{\gamma}}$ away from Np, its action on $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ is defined by the same formula as (4.2).
- For the $U_{p,i}$-action, let $\mathbf{u}_{p,i}$ act on $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$ via conjugation

  $$\mathbf{u}_{p,i} \,.\, \boldsymbol{\gamma} = \mathbf{u}_{p,i}\, \boldsymbol{\gamma}\, \mathbf{u}_{p,i}^{-1} \,.$$

  Observe that if $\boldsymbol{\gamma} \in B_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)$, then $\mathbf{u}_{p,i} \,.\, \boldsymbol{\gamma} \in B_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)$ and the diagonal entries of $\boldsymbol{\gamma}$ coincide with the diagonal entries of $\mathbf{u}_{p,i} \,.\, \boldsymbol{\gamma}$. This action then induces a left $\mathbf{u}_{p,i}$-action on $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$. The operator $U_{p,i}$ acting on $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ is defined by the same formula as (4.4).

Using these operators, together with the $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant surjection $D^r_k(\mathbf{T}_0, \mathbf{Q}_p) \to \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$, one sees that the commutative diagram (4.5) is moreover Hecke-equivariant.

Remark 4.4.6. There are obvious versions of compactly supported cohomology groups and parabolic cohomology groups with coefficients in $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$. We shall use similar notations $H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ and $H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ to denote these groups.

Chapter 5

The cuspidal eigenvarieties

In this chapter, we construct the cuspidal eigenvarieties arising from overconvergent cohomology groups and from overconvergent Siegel modular forms and then study their interaction with each other. For this purpose, we will recall some preliminaries about slope decompositions from [AS08, §4] in §5.1. The cuspidal eigenvariety associated with overconvergent cohomology groups are discussed in §5.2 whereas the one associated with overconvergent Siegel modular forms are elaborated in §5.3.

Convention. We will use the following convention from now on:

(i) Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a small weight. We say it is ***open*** if the natural map

$$\mathcal{U}^{\mathrm{rig}} = \mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}})^{\mathrm{rig}} \to \mathcal{W}$$

is an open immersion.

(ii) Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be an affinoid weight. We say it is ***open*** if the natural map

$$\mathcal{U}^{\mathrm{rig}} = \mathcal{U} = \mathrm{Spa}(R_{\mathcal{U}}, R_{\mathcal{U}}^{\circ}) \to \mathcal{W}$$

is an open immersion.

(iii) A weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is called an ***open weight*** if it is either an small open weight or an affinoid open weight.

5.1 Preliminaries on slope decompositions

5.1.1. Let R be a $\mathbf{Q}_p$-Banach algebra whose norm is denoted by $|\cdot|_R$. We define the valuation v_R on R by

$$v_R : R \to \mathbf{R} \cup \{\infty\}, \quad a \mapsto \begin{cases} \infty, & \text{if } a = 0 \\ v_R(a) \text{ s.t. } |a|_R = p^{-v_R(a)}, & \text{else} \end{cases}$$

We shall always normalise so that $v_R(p) = 1$.

Given a power series $F = \sum_{n\geq 0} a_n T^n \in R[\![T]\!]$, we define the ***Newton polygon*** Newt_F of F by

$$\mathrm{Newt}_F := \text{the lower convex hull of } \{(n, v_R(a_n)) \in \mathbf{R}^2 : a_n \neq 0\} \text{ in } \mathbf{R}^2.$$

We call the line segments in the boundary of Newt_F the ***edges*** of Newt_F; and we call the slope of any edge of Newt_F a ***slope*** of Newt_F.

Definition 5.1.2. *Let $F \in R[\![T]\!]$ and $h \in \mathbf{R}_{>0}$. A **slope-$\leq h$ factorisation** of F is a factorisation of power series*

$$F = Q \cdot S$$

such that

- *$Q \in R[T]$ with $Q^*(0) \in A^\times$, where $Q^*(T) = T^{\deg Q} Q(1/T)$;*
- *$S \in R[\![T]\!]$ such that $S(0) = 1$ (i.e., S is a **Fredholm series**);*
- *every slope of Newt_Q is $\leq h$ (i.e., Q has slope $\leq h$);*
- *every slope of Newt_S is $> h$ (i.e., S has slope $> h$);*
- *$S(p^h)$ converges.*

5.1.3. There is a similar notion related to modules, which we now discuss.

Let R again be a $\mathbf{Q}_p$-Banach algebra and M be an R-module, equipped with an endomorphism $u : M \to M$. An element $x \in M$ is said to have ***slope*** $\leq h$ if there exists a polynomial $Q \in R[T]$ with $Q^*(0) \in R^\times$ and having slope $\leq h$ such that $Q^*(u)m = 0$. We denote by

$$M^{\leq h} := \{x \in M : x \text{ has slope } \leq h\},$$

which turns out to be a R-submodule of M by [AS08, Proposition 4.6.2].

Definition 5.1.4. *Assume we are in the situation above. A **slope-$\leq h$ decomposition** of M is an isomorphism*

$$M \simeq M^{\leq h} \oplus M^{>h}$$

for some R-module $M^{>h}$ such that

- *$M^{\leq h}$ is finitely generated over R and*
- *for any polynomial $Q \in R[T]$ with $Q^*(0) \in R^\times$ and having slope $\leq h$, the map*

  $$Q^*(u) : M^{>h} \to M^{>h}$$

  is an isomorphism of A-modules.

Theorem 5.1.5 ([Buz07, Theorem 3.3]). *Suppose R is a reduced affinoid algebra. Let M be an R-module having (Pr) (in the sense of [Buz07]) and u be a compact operator acting on M. Let $F_u(T)$ be the Fredholm determinant of u acting on M. Then, for any $h \in \mathbf{R}_{>0}$, F_u has a slope-$\leq h$ factorisation if and only if M has a slope-$\leq h$ decomposition.*

Sketch of proof. Suppose $M \simeq M^{\leq h} \oplus M^{>h}$ is a slope-$\leq h$ decomposition. One sees that

$$F_u(T) = \det(1 - uT|M) = \det(1 - uT|M^{\leq h}) \det(1 - uT|M^{>h}).$$

One checks that this is a slope-$\leq h$ factorisation of F_u.

On the other hand, if $F_u = QS$ is a slope-$\leq h$ factorisation, then [Buz07, Proposition 3.2] provides a decomposition

$$M \simeq \ker Q^*(u) \oplus \operatorname{image} Q^*(u).$$

One checks that this is a slope-$\leq h$ decomposition of M. □

5.2 The cuspidal eigenvariety for overconvergent cohomology

5.2.1. Given an open weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and an integer $r > 1 + r_{\mathcal{U}}$, we have discussed about the Borel–Serre cochain complex $C^\bullet(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ in 4.3.2. One can similarly consider the *Borel–Serre chain complex* $C_\bullet(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ which computes the Betti homolgy groups $H_t(X_{\mathrm{Iw}^+}(\mathbf{C}), A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ (see [Han17]). The Borel–Serre chain complex is a finite complex as it is constructed by a fixed triangulation on the Borel–Serre compactification of the locally symmetric space $X_{\mathrm{Iw}^+}(\mathbf{C})$. We write

$$\begin{aligned} C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}} &:= \oplus_t C_t(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \\ C^{\mathrm{tol}}_{\kappa_{\mathcal{U}}, r} &:= \oplus_t C^t(\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}, D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})). \end{aligned}$$

Then $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}}$ is an ON-able $R_{\mathcal{U}}[1/p]$-module as $A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is ON-able (see [Han17, §2.2, Remarks]). Moreover, there are naturally defined Hecke operators on $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}}$ and the action of U_p is compact (see [*op. cit.*, §2.2]). We define $F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}, r} \in R_{\mathcal{U}}[1/p][\![T]\!]$ to be the Fredholm determinant of U_p acting on $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}}$. One observes that [Han17, Proposition 3.1.1] goes through for small weights, showing that $F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}, r}$ is independent to r. Thus, for any $h \in \mathbf{Q}_{\geq 0}$, the existence of a slope-$\leq h$ decomposition of $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}}$ is equivalent to the existence of a slope-$\leq h$ factorisation of $F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}} = F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}, r}$[1] (see Theorem 5.1.5). We call the pair $(\mathcal{U}, h)$ a ***slope datum*** if $F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}, r}$ admits a slope-$\leq h$ decomposition. Moreover, if $\mathcal{U}' = (R_{\mathcal{U}'}, \kappa_{\mathcal{U}'})$ is another open weight with $\mathcal{U}'^{\mathrm{rig}} \subset \mathcal{U}^{\mathrm{rig}}$, the relation $A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \widehat{\otimes} R_{\mathcal{U}'}[1/p] \cong A^r_{\kappa_{\mathcal{U}'}}(\mathbf{T}_0, R_{\mathcal{U}'})$ implies that $F^{\mathrm{oc}}_{\kappa_{\mathcal{U}}}$ glue to a function $F^{\mathrm{oc}}_{\mathcal{W}}$ over $\mathcal{W}$ (see also [Han17, §4.3]).

Observe also that [Han17, Proposition 3.1.2] also goes through for small weights. Hence, if $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}, \leq h}$ is the slope-$\leq h$ submodule of $C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}}$ and suppose $\mathcal{U}' = (R_{\mathcal{U}'}, \kappa_{\mathcal{U}'})$ is another open weight such that $\mathcal{U}'^{\mathrm{rig}} \subset \mathcal{U}^{\mathrm{rig}}$, there is a canonical isomorphism

$$C^{\kappa_{\mathcal{U}}, r}_{\mathrm{tol}, \leq h} \otimes_{R_{\mathcal{U}}[\frac{1}{p}]} R_{\mathcal{U}'}[\frac{1}{p}] \simeq C^{\kappa_{\mathcal{U}'}, r}_{\mathrm{tol}, \leq h}.$$

[1] We drop the 'r' in the notation as the Fredholm determinant is independent to r.

Proposition 5.2.2. *Let $(\mathcal{U}, h)$ be a slope datum and let $(R_{\mathcal{U}'}, \kappa_{\mathcal{U}'})$ be an affinoid open weight such that $\mathcal{U}' \subset \mathcal{U}^{\mathrm{rig}}$.*

(i) There is a canonical isomorphism

$$H_t(X_{\mathrm{Iw}^+}(\mathbf{C}), A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))_{\leq h} \otimes_{R_{\mathcal{U}}[\frac{1}{p}]} R_{\mathcal{U}'} \simeq H_t(X_{\mathrm{Iw}^+}(\mathbf{C}), A^r_{\kappa_{\mathcal{U}'}}(\mathbf{T}_0, R_{\mathcal{U}'}))_{\leq h}$$

for all $t \in \mathbf{Z}$, where the subscript '$\leq h$' stands for the slope-$\leq h$ submodule.

(ii) The cochain complex $C^{\mathrm{tol}}_{\kappa_{\mathcal{U}},r}$ and the cohomology groups $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ admit slope-$\leq h$ decompositions. The corresponding slope-$\leq h$ submodules are denoted by $C^{\mathrm{tol},\leq h}_{\kappa_{\mathcal{U}},r}$ and $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))^{\leq h}$, respectively.

(iii) There are canonical isomorphisms

$$C^{\mathrm{tol},\leq h}_{\kappa_{\mathcal{U}},r} \otimes_{R_{\mathcal{U}}[\frac{1}{p}]} R_{\mathcal{U}'} \simeq C^{\mathrm{tol},\leq h}_{\kappa_{\mathcal{U}'},r}$$

and

$$H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))^{\leq h} \otimes_{R_{\mathcal{U}}[\frac{1}{p}]} R_{\mathcal{U}'} \simeq H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}'}}(\mathbf{T}_0, R_{\mathcal{U}'}))^{\leq h}.$$

Proof. The proof follows verbatim as in the proofs of [CHJ17, Proposition 3.3 & Proposition 3.4]. □

5.2.3. Let $\mathbb{A}^1_{\mathbf{Q}_p}$ be the affine line over $(\mathbf{Q}_p, \mathbf{Z}_p)$ and let

$$\mathbb{A}^1_{\mathcal{W}} := \mathcal{W} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathbb{A}^1_{\mathbf{Q}_p}.$$

Then, the ***spectral variety*** (or the ***Fredholm hypersurface***) $\mathcal{S}^{\mathrm{oc}}$ (associated with $F^{\mathrm{oc}}_{\mathcal{W}}$) is defined to be

$$\mathcal{S}^{\mathrm{oc}} := \text{ the zero locus of } F^{\mathrm{oc}}_{\mathcal{W}} \text{ in } \mathbb{A}^1_{\mathcal{W}}.$$

5.2.4. For any open weight $\mathcal{U}$ so that $\mathcal{U}^{\mathrm{rig}} \subset \mathcal{W}$, consider $\mathbb{A}^1_{\mathcal{U}} := \mathcal{U}^{\mathrm{rig}} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathbb{A}^1_{\mathbf{Q}_p}$. In particular, we have an open embedding

$$\mathbb{A}^1_{\mathcal{U}} \hookrightarrow \mathbb{A}^1_{\mathcal{W}}.$$

For any $h \in \mathbf{Q}_{>0}$, let $\mathbf{B}(0, p^h) \subset \mathbb{A}^1_{\mathbf{Q}_p}$ be the closed ball of radius p^h. We also consider

$$\mathbf{B}_{\mathcal{U},h} := \mathcal{U}^{\mathrm{rig}} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathbf{B}(0, p^h) \subset \mathbb{A}^1_{\mathcal{U}}.$$

Then, we say the pair $(\mathcal{U}, h)$ is ***slope-adapted*** if the natural map

$$\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h} := \mathcal{S}^{\mathrm{oc}} \cap \mathbf{B}_{\mathcal{U},h} \to \mathcal{U}^{\mathrm{rig}}$$

is finite flat.

Consider the collections

$$\begin{aligned}\mathrm{Cov}(\mathcal{S}^{\mathrm{oc}}) &:= \left\{\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h} : (\mathcal{U},h) \text{ is slope-adapted}\right\}\\ \mathrm{Cov}_{\mathrm{aff}}(\mathcal{S}^{\mathrm{oc}}) &:= \left\{\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h} \in \mathrm{Cov}(\mathcal{S}^{\mathrm{oc}}) : \mathcal{U} \text{ is an affinoid weight}\right\}.\end{aligned}$$

Therefore, by [Buz07, Theorem 4.6] (see also [Han17, Proposition 4.1.4]), we know that $\mathrm{Cov}_{\mathrm{aff}}(\mathcal{S}^{\mathrm{oc}})$ is an open cover for $\mathcal{S}^{\mathrm{oc}}$ (hence so is $\mathrm{Cov}(\mathcal{S}^{\mathrm{oc}})$).

5.2.5. Recall that $D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ is defined to be the inverse limit of $D^{r}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$ with respect to r. We define the coherent sheaf $\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}$ on $\mathcal{S}^{\mathrm{oc}}$ by assigning each $\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h} \in \mathrm{Cov}(\mathcal{S}^{\mathrm{oc}})$ to the module

$$H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} := \oplus_t \left(H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \cap H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))^{\leq h}\right).$$

Notice that the intersection $H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})) \cap H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))^{\leq h}$ is a direct summand of the parabolic cohomology group $H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ and such a decomposition gives a slope-$\leq h$ decomposition for $H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ since it is Hecke-stable in $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))$ by Lemma 4.3.3. Note also that $\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}$ is, indeed, a well-defined coherent sheaf on $\mathcal{S}^{\mathrm{oc}}$ by the discussion in [Han17, §4.3].

Furthermore, the Hecke algebra $\mathbb{T}$ acts on the coherent sheaf $\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}$. Thus, for each slope-adapted pair $(\mathcal{U}, h)$, we can define

$$\begin{aligned}\mathbb{T}^{\mathrm{oc}}_{\mathcal{U},h} &:= \text{ the reduced } \mathscr{O}_{\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h}}(\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h})\text{-algebra generated by the image of } \mathbb{T} \to \mathrm{End}\left(\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}(\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h})\right)\\ \mathbb{T}^{\mathrm{oc},\circ}_{\mathcal{U},h} &:= \text{ the integral closure of } \mathscr{O}_{\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h}}(\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h})^{\circ} \text{ inside } \mathbb{T}^{\mathrm{oc}}_{\mathcal{U},h}.\end{aligned}$$

Since $\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}$ is a coherent sheaf on $\mathcal{S}^{\mathrm{oc}}$, these algebras glue to coherent sheaves of algebras $\mathscr{T}_{\mathrm{oc}}$ and $\mathscr{T}^{\circ}_{\mathrm{oc}}$ on $\mathcal{S}^{\mathrm{oc}}$ respectively.

Definition 5.2.6. *The* ***equidimensional reduced cuspidal eigenvariety*** *for* GSp_{2g} *is defined to be*

$$\mathcal{E}^{\mathrm{oc}}_0 := \textit{ the equidimensional locus of } \mathrm{Spa}_{\mathcal{S}^{\mathrm{oc}}}(\mathscr{T}_{\mathrm{oc}}, \mathscr{T}^{\circ}_{\mathrm{oc}}),$$

where $\mathrm{Spa}_{\mathcal{S}^{\mathrm{oc}}}$ *is the relative adic spectrum over* $\mathcal{S}^{\mathrm{oc}}$.

5.2.7. We close our discussion about the cuspidal eigenvariety for GSp_{2g} with the following control theorem.

Theorem 5.2.8 (Control theorem)**.** *For* $g \in \mathbf{Z}_{>0}$, *let* $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{>0}$ *be a dominant algebraic weight. Let* $\mathbf{u}_p = \prod_{i=1}^g \mathbf{u}_{p,i}$ *and let*

$$h_k := \min_{\alpha \in \Phi^+_{\mathrm{GSp}_{2g}}} \left\{-v_p(\alpha(\mathbf{u}_p))(1 + \langle k, \alpha^{\vee}\rangle)\right\},$$

where $\alpha^{\vee}$ *denotes the coroot corresponds to* α. *Then, for any* $\mathbf{Q}_{>0} \ni h < h_k$, *we have a canonical isomorphism*

$$H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_k(\mathbf{T}_0, \mathbf{Q}_p))^{\leq h} \simeq H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k})^{\leq h}.$$

(see also [AS08, Theorem 6.4.1])

Proof. Let $\mathbf{K} := \ker(D_k^\dagger(\mathbf{T}_0, \mathbf{Q}_p) \to \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ and so we have an exact sequence

$$0 \to C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{K}) \to C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, D_k^\dagger(\mathbf{T}_0, \mathbf{Q}_p)) \to C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}) \to 0.$$

Observe that the map

$$C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, D_k^\dagger) \to C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$$

is Hecke equivariant and so $C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{K})$ is Hecke stable. Denote by $C_{\mathbf{K}}^{\mathrm{tol}}$ and $C_{k,\mathrm{alg}}^{\mathrm{tol}}$ the total cochain complexes of $C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{K})$ and $C^\bullet(\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+, \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})$ respectively. Then, by [AS08, Theorem 3.11.1], we know that the norm of U_p on $C_{\mathbf{K}}^{\mathrm{tol}}$ satisfies

$$||U_p||_{\mathbf{K}} \leq p^{-h_k}.$$

Now, we claim the following: Fix $\mathbf{Q}_{>0} \ni h < h_k$, if $Q \in \mathbf{Q}_p[X]$ with $Q^*(0) \in \mathbf{Q}_p^\times$ and the slope of Q is $\leq h$, then $Q^*(U_p)$ acts on $C_{\mathbf{K}}^{\mathrm{tol}}$ invertibly. Write $Q = a_0 + a_1X + \cdots + a_nX^n$. The two conditions on Q means

- $a_n \in \mathbf{Q}_p^\times$
- $v_p(a_n) - v_p(a_i) \leq (n-i)h$ for all $i = 0, ..., n-1$.

Therefore, we have

$$|a_i/a_n| < p^{(n-i)h} \text{ and } \left|\left|\frac{a_i}{a_n}U_p^{n-i}\right|\right|_{\mathbf{K}} < 1.$$

Let $P(X) = -\frac{a_0}{a_n}X^n - \frac{a_1}{a_n}X^{n-1} - \cdots - \frac{a_{n-1}}{a_n}X$, then $\frac{1}{a_n}Q^*(X) = 1 - P(X)$. We can deduce that $||P(U_p)||_{\mathbf{K}} < 1$ and so $Q^*(U_p)$ acts on $C_{\mathbf{K}}^{\mathrm{tol}}$ invertibly with inverse given explicitly by

$$Q^*(U_p)^{-1} = \frac{1}{a_n}\sum_{j\geq 0} P(U_p)^j.$$

Now fix $h < h_k$, then by [Han17, Proposition 2.3.3], we know that C_k^{tol} and $C_{k,\mathrm{alg}}^{\mathrm{tol}}$ have slope-$\leq h$ decomposition. Hence, if $F_k^\dagger$ and F_k^{alg} denote the corresponding Fredholm determinant of U_p on C_k^{tol} and $C_{k,\mathrm{alg}}^{\mathrm{tol}}$ respectively, we have the corresponding slope-$\leq h$ factorisation $F_k^\dagger = Q_h^\dagger S_h^\dagger$ and $F_k^{\mathrm{alg}} = Q_h^{\mathrm{alg}} S_h^{\mathrm{alg}}$ and

$$C_k^{\mathrm{tol},\leq h} \twoheadrightarrow C_{k,\mathrm{alg}}^{\mathrm{tol},\leq h}$$

with $C_k^{\mathrm{tol},\leq h} = \ker Q_h^{\dagger,*}(U_p|_{C_k^{\mathrm{tol}}})$ and $C_{k,\mathrm{alg}}^{\mathrm{tol},\leq h} = \ker Q_h^{\mathrm{alg},*}(U_p|_{C_{k,\mathrm{alg}}^{\mathrm{tol}}})$. Let $C_{\mathbf{K}}^{\mathrm{tol},\leq h}$ be the kernel of the surjection, then, by taking cohomology, we have the corresponding long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{K})^{\leq h} \arrow[r] & H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D_k^{\dagger}(\mathbf{T}_0, \mathbf{Q}_p))^{\leq h} \arrow[dll] \\ H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee})^{\leq h} \arrow[r] & H^{t+1}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{K})^{\leq h} \arrow[r] & \cdots \end{tikzcd}.$$

The above claim shows that both $Q_h^{\dagger,*}(U_p)$ and $Q_h^{\mathrm{alg},*}(U_p)$ act on $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{K})^{\leq h}$ invertibly. Take any $\sigma \in H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{K})^{\leq h}$, the image of $Q_h^{\dagger}(U_p)\sigma$ in $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D_k^{\dagger}(\mathbf{T}_0, \mathbf{Q}_p))^{\leq h}$ is zero, thus there exists $\sigma' \in H^{t-1}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}_{\mathrm{GSp}_4,k}^{\mathrm{alg},\vee})^{\leq h}$ whose image in $H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{K})^{\leq h}$ is $Q_h^{\dagger,*}(U_p)\sigma$. Since $Q_h^{\mathrm{alg},*}(U_p)\sigma' = 0$, thus $Q_h^{\mathrm{alg},*}(U_p)Q_h^{\dagger,*}(U_p)\sigma = 0$. This implies $\sigma = 0$ so the desired isomorphism follows. $\square$

Remark 5.2.9. The above control theorem is basically [AS08, Theorem 6.4.1] with only a slight modification. There is another version of the control theorem by [Urb11, Proposition 4.3.10] (see also [Han17, Theorem 3.2.5]). However, the control theorem in [Urb11] requires a modification on the Shimura varieties while this is not the case in our version.

5.3 The cuspidal eigenvariety for overconvergent Siegel modular forms

5.3.1. Throughout this section, we assume $p > 2g$ so that we can apply results in [AIP15] via the comparison in §3.7. On the other hand, we believe that the results in this section hold for smaller primes as well. In order to deal with these smaller primes, one would have to reprove several results in [AIP15] in our context; *e.g.*, the classicality result and the fact that $S^{\dagger}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$ has (Pr). We leave these generalities to the readers in order to keep this thesis within a reasonable length.

5.3.2. Given an affinoid weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and $w > 1 + r_{\mathcal{U}}$, by [AIP15, Proposition 8.1.3.1] and Theorem 3.7.2 (see also [*op. cit.*, Proposition 8.2.3.3]), the space of cuspforms $S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{cusp}})$ has property (Pr) in the sense of [Buz07]; namely, it is a direct summand of a potentially ON-able $\mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}}$-Banach space. Also recall that U_p as compactly on the space of overconvergent Siegel modular forms. Therefore, we can define the Fredholm determinant $F^{\mathrm{mf}}_{\kappa_{\mathcal{U}},w}$ of U_p acting on $S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}}$. Note that the Fredholm determinant is independent to w. When we vary the affinoid weights, the Fredholm determinants glue together and so we arrive at a power series $F^{\mathrm{mf}}_{\mathcal{W}} \in \mathscr{O}_{\mathcal{W}}(\mathcal{W})\{\{T\}\}\widehat{\otimes}_{\mathbf{Q}_p} \mathbf{C}_p$.

Consider $\mathbb{A}^1_{\mathbf{C}_p} = \mathbb{A}^1_{\mathbf{Q}_p} \times_{\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ and let $\mathbb{A}^1_{\mathcal{W},\mathbf{C}_p} := \mathcal{W} \times_{\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)} \mathbb{A}^1_{\mathbf{C}_p}$. The ***spectral variety*** $\mathcal{S}_{\mathbf{C}_p}$ (associated with $F^{\mathrm{mf}}_{\mathcal{W}}$) is defined to be

$$\mathcal{S}_{\mathbf{C}_p} := \text{ the zero locus of } F^{\mathrm{mf}}_{\mathcal{W}} F^{\mathrm{oc}}_{\mathcal{W}} \text{ in } \mathbb{A}^1_{\mathcal{W},\mathbf{C}_p}.$$

By construction, we see that there is a closed immersion

$$\mathcal{S}^{\mathrm{oc}}_{\mathbf{C}_p} := \mathcal{S}^{\mathrm{oc}} \times_{\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p}) \hookrightarrow \mathcal{S}_{\mathbf{C}_p}\,.$$

Consequently, denote by $\mathcal{E}^{\mathrm{oc}}_{0,\mathbf{C}_p}$ the base change of $\mathcal{E}^{\mathrm{oc}}_0$ to $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, we can view $\mathcal{E}^{\mathrm{oc}}_{0,\mathbf{C}_p}$ as an adic space over $\mathcal{S}_{\mathbf{C}_p}$.

5.3.3. One can now run through the strategy in §5.2 again so that we have the following.

- A pair $(\mathcal{U}, h)$ with $\mathcal{U}$ an open weight and $h \in \mathbf{Q}_{>0}$ is ***slope-adapted*** if the natural map

$$\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h} := \mathcal{S}_{\mathbf{C}_p} \cap \Big(\mathbf{B}(0,p^h) \times_{\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})\Big) \to \mathcal{U}^{\mathrm{rig}} \times_{\mathrm{Spa}(\mathbf{Q}_p,\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$$

  is finite flat.

- The collection $\mathrm{Cov}(\mathcal{S}_{\mathbf{C}_p}) = \{\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h} : (\mathcal{U}, h) \text{ is slope-adapted}\}$ is again an open cover for $\mathcal{S}_{\mathbf{C}_p}$.

- We define the coherent sheaf $\mathscr{S}^{\dagger}_{\mathrm{Iw}^+}$ by assigning each $\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h} \in \mathrm{Cov}(\mathcal{S}_{\mathbf{C}_p})$ to the module $S^{\dagger,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}$.

- The reduced $\mathscr{O}_{\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h}}(\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h})$-algebra $\mathbb{T}^{\mathrm{mf}}_{\mathcal{U},h}$ generated by the image of $\mathbb{T}$ in $\mathrm{End}(\mathscr{S}^{\dagger}_{\mathrm{Iw}^+}(\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h}))$ then gives rise to coherent sheaves of algebras $\mathscr{T}_{\mathrm{mf}}$ and $\mathscr{T}^{\circ}_{\mathrm{mf}}$.

Definition 5.3.4. *The **equidimensional reduced cuspidal eigenvariety** for overconvergent Siegel cuspforms is defined to be*

$$\mathcal{E}^{\mathrm{mf}}_0 := \text{ the equidimensional locus of } \mathrm{Spa}_{\mathcal{S}_{\mathbf{C}_p}}(\mathscr{T}_{\mathrm{mf}}, \mathscr{T}^{\circ}_{\mathrm{mf}}),$$

where $\mathrm{Spa}_{\mathcal{S}_{\mathbf{C}_p}}$ *is the relative adic spectrum over* $\mathcal{S}_{\mathbf{C}_p}$.

Remark 5.3.5. Notice that $\mathcal{E}^{\mathrm{mf}}_0$ is (the stricit Iwahori version of) the equidimensional cuspidal eigenvariety constructed in [AIP15] after base change to $\mathbf{C}_p$.

Proposition 5.3.6. *There is a natural closed immersion* $\mathcal{E}^{\mathrm{mf}}_0 \hookrightarrow \mathcal{E}^{\mathrm{oc}}_{0,\mathbf{C}_p}$.

Proof. The strategy is to apply [Han17, Theorem 5.1.2]. To this end, we need to find a *very Zariski-dense* subset $\mathcal{S}^{\mathrm{cl}}$ of $\mathcal{S}_{\mathbf{C}_p}$ such that for every $\boldsymbol{x} \in \mathcal{S}^{\mathrm{cl}}$ with dominate algebraic weight $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{\geq 0}$ and any $Y \in \mathbb{T}$, we have

$$\det\Big(1 - TY|\,\mathscr{S}^{\dagger}_{\mathrm{Iw}^+,\boldsymbol{x}}\Big) \;\Big|\; \det\Big(1 - TY|\,\mathscr{H}^{\mathrm{tol}}_{\mathrm{par},\boldsymbol{x}} \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{C}_p\Big).$$

By Theorem 5.2.8, there exists an $h_k \in \mathbf{R}_{>0}$ such that for all $h \in \mathbf{Q} \cap (0, h_k]$, the canonical map

$$H^{n_0}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^{\dagger}_k(\mathbf{T}_0, \mathbf{Q}_p))^{\leq h} \to H^{n_0}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k})^{\leq h}$$

is an isomorphism. On the other hand, let

$$\underline{\omega}^k_{\mathrm{Iw}^+,\mathrm{cusp}} := \underline{\omega}^k_{\mathrm{Iw}^+} \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}} \mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}(-\mathcal{Z}_{\mathrm{Iw}^+})$$

be the sheaf of classical cuspidal Siegel modular forms of weight k on $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$. The classicality theorem [AIP15, Theorem 7.1.1] provides an $a_k \in \mathbf{Q}_{>0}$ such that for all $h \in \mathbf{Q} \cap (0, a_k]$, the slope-$\leq h$ overconvergent Siegel cuspforms of weight k are classical; namely,

$$H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^k_{w,\mathrm{cusp}})^{\leq h} \subset H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^k_{\mathrm{Iw}^+,\mathrm{cusp}}).$$

Now, let $\ell_k = \min\{h_k, a_k\}$ and take $h \leq \ell_k$. Applying the *generalised Eichler–Shimura morphism* in [Hid02, Theorem 3.8], we obtain an injection from the space of slope-$\leq h$ overconvergent Siegel cuspforms of classical weight into the slope-$\leq h$ cohomology group with coefficient in the algebraic representation. Consequently, the desired very Zariski-dense subset of $\mathcal{S}$ can be taken to be

$$\mathcal{S}^{\mathrm{cl}} = \cup_{\mathcal{S}_{\mathcal{U},h} \in \mathrm{Cov}_{\mathrm{aff}}(\mathcal{S})} \{\boldsymbol{x} \in \mathcal{S}_{\mathbf{C}_p,\mathcal{U},h} : \boldsymbol{x} \text{ has classical weight } k \in \mathbf{Z}^g_{\geq 0} \text{ and } h \leq \ell_k\}$$

Finally, [Han17, Theorem 5.1.2] yields the result. □

5.3.7. Given Proposition 5.3.6, we may identify $\mathcal{E}_0^{\mathrm{mf}}$ with its image in $\mathcal{E}^{\mathrm{oc}}_{0,\mathbf{C}_p}$ and denote it by $\mathcal{E}_0$ for simplicity. We have a diagram

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r] \arrow[rr, bend right] & \mathcal{S}_{\mathbf{C}_p} \arrow[r] & \mathcal{W} \end{tikzcd}$$

Chapter 6

Overconvergent Eichler–Shimura morphisms

The goal of this chapter is to answer the second part of Question 1.3.2 (i). That is, we explicitly construct an overocnvergent Eichler–Shimura morphism for overconvergent Siegel modular forms by using perfectoid methods. Our approach is similar to [CHJ17], but we have to overcome several technicalities.

We organise this chapter as follows. The purpose of §6.1 is to show that the overconvergent cohomology groups can be computed using the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$. Working with the pro-Kummer étale sites, we construct explicitly the overconvergent Eichler–Shimura morphism in §6.2. Then, in §6.3, we study the image of the overconvergent Eichler–Shimura morphism at classical weights. Finally, we show in §6.4 that such morphisms can be promoted as a morphism between coherent sheaves on the cuspidal eigenvariety $\mathcal{E}_0$.

6.1 The (pro-)Kummer étale cohomology groups

6.1.1. Consider the natural morphism of sites

$$\jmath_{\mathrm{k\acute{e}t}} : \mathcal{X}_{\mathrm{Iw}^+,\mathrm{\acute{e}t}} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{k\acute{e}t}}.$$

Recall that, for every small weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and any integer $r \geq 1 + r_{\mathcal{U}}$, there is an inverse system of étale locally constant sheaves $(\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j})_{j\in\mathbf{Z}_{\geq 1}}$ on $\mathcal{X}_{\mathrm{Iw}^+,\mathrm{\acute{e}t}}$. Applying [DLLZ19, Corollary 4.6.7], we obtain an isomorphism

$$\varprojlim_j H^t_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \simeq \varprojlim_j H^t_{\mathrm{k\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j})$$

for every $t \in \mathbf{Z}_{\geq 0}$. Write

$$H^t_{\mathrm{k\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{D}^{r}_{\kappa_{\mathcal{U}}}) := \varprojlim_j H^t_{\mathrm{k\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j})[1/p].$$

By Proposition 4.1.10, we arrive at isomorphisms

$$H^t_{\text{két}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \simeq H^t_{\text{ét}}(\mathcal{X}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \simeq H^t(X_{\text{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})).$$

To simplify the notation, we introduce the following abbreviations:

$$\begin{array}{ll} \mathbf{O\!C}^{r,\circ}_{\kappa_{\mathcal{U}}} & := \varprojlim_j H^{n_0}_{\text{két}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \\ \mathbf{O\!C}^{r}_{\kappa_{\mathcal{U}}} & := \mathbf{O\!C}^{r,\circ}_{\kappa_{\mathcal{U}}}[\frac{1}{p}] = H^{n_0}_{\text{két}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}}) \\ \mathbf{O\!C}^{r,\circ}_{\kappa_{\mathcal{U}},\mathcal{O}_{\mathbf{C}_p}} & := \varprojlim_j \left(H^{n_0}_{\text{két}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathbf{C}_p}\right) \\ \mathbf{O\!C}^{r}_{\kappa_{\mathcal{U}},\mathbf{C}_p} & := \mathbf{O\!C}^{r,\circ}_{\kappa_{\mathcal{U}},\mathcal{O}_{\mathbf{C}_p}}[\frac{1}{p}] \end{array}.$$

where $n_0 = \dim_{\mathbf{C}_p} \mathcal{X}_{\text{Iw}^+}$.

6.1.2. Let

$$\nu : \overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}} \to \overline{\mathcal{X}}_{\text{Iw}^+,\text{két}}$$

be the natural projection of sites. Consider the sheaf $\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}$ on the pro-Kummer étale site $\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}$ defined by

$$\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} := \left(\varprojlim_j \left(\nu^{-1}\jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j} \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}}\right)\right)[\frac{1}{p}].$$

Proposition 6.1.3. *There is a* $\text{Gal}_{\mathbf{Q}_p}$*-equivariant isomorphism*

$$\mathbf{O\!C}^r_{\kappa_{\mathcal{U}},\mathbf{C}_p} \simeq H^{n_0}_{\textit{prokét}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}).$$

Proof. By [DLLZ19, Theorem 6.2.1 & Corollary 6.3.4], there is an almost isomorphism

$$\left(H^{n_0}_{\text{két}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathbf{C}_p}\right)^a \simeq H^{n_0}_{\text{prokét}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \nu^{-1}\jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j} \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}})^a.$$

It remains to establish an almost isomorphism

$$\begin{aligned} &\varprojlim_j H^{n_0}_{\text{prokét}}\left(\overline{\mathcal{X}}_{\text{Iw}^+}, \nu^{-1}\jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j} \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}}\right)^a \\ &\qquad \simeq H^{n_0}_{\text{prokét}}\left(\overline{\mathcal{X}}_{\text{Iw}^+}, \varprojlim_j \left(\nu^{-1}\jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j} \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}}\right)\right)^a. \end{aligned}$$

Indeed, observe that the higher inverse limit $R^i\varprojlim_j \left(\nu^{-1}\jmath_{\text{két},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j} \otimes_{\mathbf{Z}_p} \mathscr{O}^+_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}}\right)$ almost vanishes for $i \geq 1$ by an almost version of [Sch13, Lemma 3.18] and [DLLZ19, Proposition 6.1.11]. This then allows us to commute the inverse limit with taking cohomology, hence the result. $\square$

6.1.4. Thanks to Proposition 6.1.3, $H^{n_0}_{\text{prokét}}(\overline{\mathcal{X}}_{\text{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}})$ inherits actions of the Hecke operators $T_{\boldsymbol{\gamma}}$ and $U_{p,i}$ from $H^{n_0}_{\text{ét}}(\mathcal{X}_{\text{Iw}^+}, \mathscr{D}^r_{\kappa_{\mathcal{U}}})$. On the other hand, thanks to the $\mathscr{O}_{\overline{\mathcal{X}}_{\text{Iw}^+,\text{prokét}}}$-module structure on $\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}$, there is an alternative way to define the Hecke operators $T_{\boldsymbol{\gamma}}$'s using correspondences. More precisely, for any prime number $\ell \nmid Np$ and any $\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}(\mathbf{Q}_\ell) \cap M_{2g}(\mathbf{Z}_\ell)$, consider the correspondence

$$\begin{tikzcd} & \mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathcal{X}_{\mathrm{Iw}^+} & & \mathcal{X}_{\mathrm{Iw}^+} \end{tikzcd},$$

studied in 3.3.2. One then similarly obtains an isomorphism

$$\varphi_{\boldsymbol{\gamma}} : \mathrm{pr}_2^* \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}} \xrightarrow{\simeq} \mathrm{pr}_1^* \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}.$$

Consider the composition

$$\begin{tikzcd} T'_{\boldsymbol{\gamma}} : & H^{n_0}_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}) \arrow[r, "\mathrm{pr}_2^*"] & H^{n_0}_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\boldsymbol{\gamma},\mathrm{Iw}^+}, \mathrm{pr}_2^* \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}) \arrow[dl, "\varphi_{\boldsymbol{\gamma}}"] \\ & H^{n_0}_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathrm{pr}_1^* \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}) \arrow[r, "\mathrm{Tr}_{\mathrm{pr}_1}"] & H^{n_0}_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}) \end{tikzcd}$$

However, since $H^{n_0}_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}) \simeq H^{n_0}_{\mathrm{k\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \jmath_{\mathrm{k\acute{e}t},*}\, \mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j})$ for every j, we have an identification

$$H^{n_0}_{\mathrm{pro\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} |_{\mathcal{X}_{\mathrm{Iw}^+}}) \simeq H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}})$$

and hence an operator $T'_{\boldsymbol{\gamma}}$ on $H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}})$. One checks that $T_{\boldsymbol{\gamma}}$ coincides with $T'_{\boldsymbol{\gamma}}$.

6.2 Overconvergent Eichler–Shimura morphisms

6.2.1. The strategy of the construction of our overconvergent Eichler–Shimura morphism is similar to [CHJ17], *i.e.*, we first construct a morphism between sheaves on the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}$, which then induces the desired map on the spaces.

Let $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ be a small weight and let $w \geq r \geq 1 + r_{\mathcal{U}}$. Recall that we have defined a sheaf $\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}$ on the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$ in the previous section. The following lemma is an analogue of [CHJ17, Lemma 4.5].

Lemma 6.2.2. *Let $\mathcal{V} = \varprojlim_n \mathcal{V}_n \to \overline{\mathcal{X}}_{\mathrm{Iw}^+}$ be a pro-Kummer étale presentation of a log affinoid perfectoid object in $\overline{\mathcal{X}}_{\mathrm{Iw}^+,prok\acute{e}t}$. Let $\mathcal{V}_\infty := \mathcal{V} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \overline{\mathcal{X}}_{\Gamma(p^\infty)}$. (Here we have abused the notation and identify $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ with the object $\varprojlim_n \overline{\mathcal{X}}_{\Gamma(p^n)}$ in $\overline{\mathcal{X}}_{\mathrm{Iw}^+,prok\acute{e}t}$.) Then there is a natural isomorphism*

$$\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}(\mathcal{V}) \simeq \left(D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,prok\acute{e}t}}(\mathcal{V}_\infty) \right)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}.$$

Proof. Recall that $\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}$ is the locally constant sheaf on $\mathcal{X}_{\mathrm{Iw}^+,\mathrm{\acute{e}t}}$ induced by

$$\pi_1^{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}) \to \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \to \mathrm{Aut}\left(D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}}) \right).$$

Since $\overline{\mathcal{X}}_{\Gamma(p^\infty)}$ is a profinite Galois cover of $\overline{\mathcal{X}}_{\mathrm{Iw}^+}$ with Galois group $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$, one sees that $\nu^{-1}\jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}$ becomes the constant local system associated with $D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})$ after restricting to the localised site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}/\overline{\mathcal{X}}_{\Gamma(p^\infty)}$.

Applying [DLLZ19, Theorem 5.4.3], we obtain an almost isomorphism

$$\Big(D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^a \simeq \Big(\Big(\nu^{-1}\jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}\otimes_{\mathbf{Z}_p}\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}\Big)(\mathcal{V}_\infty)\Big)^a.$$

By taking $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-invariants, we obtain almost isomorphisms

$$\begin{aligned}
&\left(\Big(D^{r,\circ}_{\kappa_{\mathcal{U}},j}(\mathbf{T}_0, R_{\mathcal{U}})\otimes_{\mathbf{Z}_p}\widehat{\mathscr{O}}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}\right)^a\\
&\simeq \left(\Big(\Big(\nu^{-1}\jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}\otimes_{\mathbf{Z}_p}\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}\Big)(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}\right)^a\\
&= \Big(\Big(\nu^{-1}\jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}\otimes_{\mathbf{Z}_p}\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}\Big)(\mathcal{V})\Big)^a.
\end{aligned}$$

Finally, taking inverse limit over j and inverting p, we conclude that

$$\begin{aligned}
\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}(\mathcal{V}) &= \left(\varinjlim_j \Big(\nu^{-1}\jmath_{\mathrm{k\acute{e}t},*}\mathscr{D}^{r,\circ}_{\kappa_{\mathcal{U}},j}\otimes_{\mathbf{Z}_p}\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}\Big)(\mathcal{V})\right)[\frac{1}{p}]\\
&\simeq \Big(D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}.
\end{aligned}$$

□

6.2.3. To deal with the overconvergent automorphic sheaves, we recall the Kummer étale sheaves $\underline{\omega}^{\kappa_{\mathcal{U}},+}_{w,\mathrm{k\acute{e}t}}$ and $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{k\acute{e}t}}$ associated with $\underline{\omega}^{\kappa_{\mathcal{U}},+}_{w}$ and $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w}$ considered in 3.2.14. Then we consider the p-adically completed pullback of them to the pro-Kummer étale site; namely,

$$\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}},+}_w := \varprojlim_m\Big(\underline{\omega}^{\kappa_{\mathcal{U}},+}_{w,\mathrm{k\acute{e}t}}\otimes_{\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{k\acute{e}t}}}}\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}/p^m\Big)$$

and

$$\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w := \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}},+}_w[\frac{1}{p}].$$

Lemma 6.2.4. *There is a canonical Hecke- and $\mathrm{Gal}_{\mathbf{Q}_p}$-equivariant morphism*

$$H^{n_0}_{\mathit{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w)\to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\underline{\omega}^{\kappa_{\mathcal{U}}+g+1}_w)(-n_0).$$

Proof. By the discussion in 3.2.14, we have seen that $\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{k\acute{e}t}}$ can be identified with the sheaf of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}/\Gamma(p^n)$-invariants of an admissible Kummer étale Banach sheaf of $\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{k\acute{e}t}}}\widehat{\otimes}R_{\mathcal{U}}$-modules. Corollary A.2.18 then yields a canonical isomorphism

$$\underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\mathrm{k\acute{e}t}}\otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{k\acute{e}t}}}}R^i\nu_*\widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}\xrightarrow{\sim}R^i\nu_*\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w$$

for every $i \in \mathbf{Z}_{\geq 0}$. On the other hand, by [DRW22, Proposition A.2.3], we have a canonical isomorphism

$$R^i \nu_* \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{prokét}}} \simeq \Omega^{\log,i}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}(-i).$$

Combining the two isomorphisms, we obtain

$$R^i \nu_* \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w \simeq \underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\text{két}} \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}} \Omega^{\log,i}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}(-i).$$

Moreover, there is a Leray spectral sequence

$$E_2^{j,i} = H^j_{\text{két}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, R^i\nu_* \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w) \Rightarrow H^{j+i}_{\text{prokét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w).$$

The edge map yields a Galois-equivariant morphism

$$H^{n_0}_{\text{prokét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w) \to H^0_{\text{két}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, R^{n_0}\nu_* \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w) \simeq H^0_{\text{két}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\text{két}} \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}} \Omega^{\log,n_0}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}})(-n_0).$$

Finally, let $\pi_{\mathrm{Iw}^+} : \mathcal{G}^{\text{univ}}_{\mathrm{Iw}^+,w} \to \overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ denote the universal semiabelian variety over $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$ and let

$$\underline{\omega}_{\mathrm{Iw}^+,w} := \pi_{\mathrm{Iw}^+,*} \Omega^1_{\mathcal{G}^{\text{univ}}_{\mathrm{Iw}^+,w}/\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}}.$$

Note that $\underline{\omega}_{\mathrm{Iw}^+,w}$ agrees with $\underline{\omega}^k_{\mathrm{Iw}^+}|_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}}$ studied in §3.4 for $k = (1, 0, \ldots, 0)$. The Kodaira–Spencer isomorphism [Lan12, Theorem 1.41 (4)] yields an isomorphism

$$\mathrm{Sym}^2 \underline{\omega}_{\mathrm{Iw}^+,w} \simeq \Omega^{\log,1}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}}.$$

Hence,

$$\Omega^{\log,n_0}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}} \simeq \wedge^{n_0}\left(\mathrm{Sym}^2 \underline{\omega}_{\mathrm{Iw}^+,w}\right) = \underline{\omega}^{g+1}_{\mathrm{Iw}^+,w} \subset \underline{\omega}^{g+1}_w$$

where the last inclusion follows from Lemma 3.4.6. We obtain an injection

$$\begin{array}{ccc} H^0_{\text{két}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}}_{w,\text{két}} \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}}} \Omega^{\log,n_0}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{két}}})(-n_0) & \longhookrightarrow & H^0_{\text{két}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}+g+1}_{w,\text{két}})(-n_0) \\ & & \parallel \\ & & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}+g+1}_w)(-n_0) \end{array}.$$

Note that, due to the normalisation of the Hecke operators, the Kodaira–Spencer isomorphism is Hecke-equivariant (see [FC90, pp. 258]). □

6.2.5. For any matrix $\boldsymbol{\sigma} \in M_g(\mathcal{O}_{\mathbf{C}_p})$ and $\mu \in D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$, we define a function $f_{\mu,\boldsymbol{\sigma}} \in C^{w-\text{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \mathbf{C}_p \widehat{\otimes} R_{\mathcal{U}})$ as follows. For any $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, we define

$$f_{\mu,\boldsymbol{\sigma}}(\boldsymbol{\gamma}') := \int_{(\boldsymbol{\gamma},\boldsymbol{\upsilon}) \in \mathbf{T}_0} e^{\text{hst}}_{\kappa_{\mathcal{U}}}({}^{\mathrm{t}}\boldsymbol{\gamma}'(\boldsymbol{\gamma} + \boldsymbol{\sigma}\,\boldsymbol{\upsilon})) \quad d\mu,$$

where $e^{\text{hst}}_{\kappa_{\mathcal{U}}}$ is as defined in Example 4.1.5. The following lemma justifies this definition.

Lemma 6.2.6. *(i) For every $\boldsymbol{\sigma} \in M_g(\mathcal{O}_{\mathbf{C}_p})$ and $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, the assignment*

$$(\boldsymbol{\gamma}, \boldsymbol{\upsilon}) \mapsto e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}({}^{\mathrm{t}}\boldsymbol{\gamma}'(\boldsymbol{\gamma} + \boldsymbol{\sigma}\,\boldsymbol{\upsilon}))$$

defines an element in $A^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}})$.

(ii) For every $\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$ and $\boldsymbol{\beta} \in B_{\mathrm{GL}_g,0}$, we have

$$f_{\mu,\boldsymbol{\sigma}}(\boldsymbol{\gamma}'\,\boldsymbol{\beta}) = \kappa_{\mathcal{U}}(\boldsymbol{\beta}) f_{\mu,\boldsymbol{\sigma}}(\boldsymbol{\gamma}').$$

Proof. This is straightforward. □

6.2.7. We are ready to construct the desired morphism $\eta_{\kappa_{\mathcal{U}}} : \mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}} \to \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w$ between sheaves on the pro-Kummer étale site $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}$. Indeed, it suffices to construct a map $\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}(\mathcal{V}) \to \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w(\mathcal{V})$ for every log affinoid perfectoid object $\mathcal{V}$ in $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$. By Lemma 6.2.2, we have

$$\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}(\mathcal{V}) \simeq \Big(D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}$$

where $\mathcal{V}_\infty := \mathcal{V} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \overline{\mathcal{X}}_{\Gamma(p^\infty)}$.

On the other hand, by definition, $\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w(\mathcal{V})$ consists of $f \in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty) \widehat{\otimes} R_{\mathcal{U}})$ satisfying $\boldsymbol{\alpha}^* f = \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\alpha}_a + \mathfrak{z}\,\boldsymbol{\alpha}_c)^{-1} f$, for all $\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$. This is equivalent to saying that $\widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w(\mathcal{V})$ consists of $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-invariant elements $f \in C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty) \widehat{\otimes} R_{\mathcal{U}})$ with respect to the twisted $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-action

$$\boldsymbol{\alpha}\,.f := \rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\alpha}_a + \mathfrak{z}\,\boldsymbol{\alpha}_c)(\boldsymbol{\alpha}^* f).$$

Consider the map

$$D^{r,\circ}_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}) \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty) \to C^{w-\mathrm{an}}_{\kappa_{\mathcal{U}}}(\mathrm{Iw}_{\mathrm{GL}_g}, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty) \widehat{\otimes} R_{\mathcal{U}}), \quad \mu \otimes \delta \mapsto \delta f_{\mu,\mathfrak{z}}.$$

We claim that this map is $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant, and hence taking the $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-invariants yields the desired map $\mathscr{O}\mathscr{D}^r_{\kappa_{\mathcal{U}}}(\mathcal{V}) \to \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w(\mathcal{V})$. Indeed, for any $\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$ and any

$\boldsymbol{\gamma}' \in \mathrm{Iw}_{\mathrm{GL}_g}$, we have

$$\begin{aligned}
(\boldsymbol{\alpha}^* \delta) f_{\boldsymbol{\alpha}\cdot\mu,\mathfrak{z}}(\boldsymbol{\gamma}') &= (\boldsymbol{\alpha}^* \delta)\left(\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}({}^{\mathtt{t}}\boldsymbol{\gamma}'(\boldsymbol{\gamma}+\mathfrak{z}\,\boldsymbol{v})) \quad d\,\boldsymbol{\alpha}\cdot\mu\right)\\
&= (\boldsymbol{\alpha}^* \delta)\left(\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}'\left((\boldsymbol{\alpha}_a\boldsymbol{\gamma}+\boldsymbol{\alpha}_b\boldsymbol{v})+\mathfrak{z}(\boldsymbol{\alpha}_c\boldsymbol{\gamma}+\boldsymbol{\alpha}_d\boldsymbol{v})\right)\right) \quad d\mu\right)\\
&= (\boldsymbol{\alpha}^* \delta)\left(\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}'((\boldsymbol{\alpha}_a+\mathfrak{z}\,\boldsymbol{\alpha}_c)\,\boldsymbol{\gamma}+(\boldsymbol{\alpha}_b+\mathfrak{z}\,\boldsymbol{\alpha}_d)\,\boldsymbol{v})\right) \quad d\mu\right)\\
&= (\boldsymbol{\alpha}^* \delta)\left(\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}'(\boldsymbol{\alpha}_a+\mathfrak{z}\,\boldsymbol{\alpha}_c)(\boldsymbol{\gamma}+(\boldsymbol{\alpha}_a+\mathfrak{z}\,\boldsymbol{\alpha}_c)^{-1}(\boldsymbol{\alpha}_b+\mathfrak{z}\,\boldsymbol{\alpha}_d)\,\boldsymbol{v})\right) \quad d\mu\right)\\
&= (\boldsymbol{\alpha}^* \delta)\left(\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}({}^{\mathtt{t}}(\boldsymbol{\alpha}_a+\mathfrak{z}\,\boldsymbol{\alpha}_c)\,\boldsymbol{\gamma}')(\boldsymbol{\gamma}+(\mathfrak{z}\cdot\boldsymbol{\alpha})\,\boldsymbol{v})\right) \quad d\mu\right)\\
&= (\boldsymbol{\alpha}^* \delta)\left(\rho_{\kappa_{\mathcal{U}}}(\boldsymbol{\alpha}_a+\mathfrak{z}\,\boldsymbol{\alpha}_c)\int_{\mathbf{T}_0} e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}'(\boldsymbol{\gamma}+(\mathfrak{z}\cdot\boldsymbol{\alpha})\,\boldsymbol{v})\right) \quad d\mu\right)\\
&= \boldsymbol{\alpha}\,.(\delta f_{\mu,\mathfrak{z}})(\boldsymbol{\gamma}')
\end{aligned}$$

as desired.

Putting everything together, the composition

$$\begin{tikzcd}
\mathbf{OC}^r_{\kappa_{\mathcal{U}},\mathbf{C}_p} \arrow[r, "\simeq"] \arrow[drrr, bend right, "\mathrm{ES}_{\kappa_{\mathcal{U}}}"'] & H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{OD}^r_{\kappa_{\mathcal{U}}}) \arrow[r, "\mathrm{Res}"] & H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \mathscr{OD}^r_{\kappa_{\mathcal{U}}}) \arrow[d, "\eta_{\kappa_{\mathcal{U}}}"] \\
& & H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \widehat{\underline{\omega}}^{\kappa_{\mathcal{U}}}_w) \arrow[d, "\text{Lemma 6.2.4}"] \\
& & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}^{\kappa_{\mathcal{U}}+g+1}_w)(-n_0) \arrow[d, equal] \\
& & M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0)
\end{tikzcd}$$

is called the ***overconvergent Eichler–Shimura morphism*** (of weight $\kappa_{\mathcal{U}}$).

Proposition 6.2.8. *The overconvergent Eichler–Shimura morphism*

$$\mathrm{ES}_{\kappa_{\mathcal{U}}} : \mathbf{OC}^r_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to M^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0)$$

is Hecke- and $\mathrm{Gal}_{\mathbf{Q}_p}$*-equivariant.*

Proof. The Galois-equivariance follows immediately from Lemma 6.2.4. For Hecke operators away from Np, notice that the operators $T_{\boldsymbol{\gamma}}$'s on both sides are defined in the same way using correspondences. Hence, it is straightforward to verify the $T_{\boldsymbol{\gamma}}$-equivariances. It remains to check the $U_{p,i}$-equivariance for all $i = 1, ..., g$.

To this end, due to the $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariance of $\eta_{\kappa_{\mathcal{U}}}$, we only have to check the $\mathbf{u}_{p,i}$-equivariance. Indeed, for every $\boldsymbol{\gamma}' = \boldsymbol{\gamma}'_0\,\boldsymbol{\beta}'_0 \in \mathrm{Iw}_{\mathrm{GL}_g}$ with $\boldsymbol{\gamma}'_0 \in U^{\mathrm{opp}}_{\mathrm{GL}_g,1}$ and $\boldsymbol{\beta}'_0 \in B_{\mathrm{GL}_g,0}$, we have

$$
\begin{aligned}
(\mathbf{u}_{p,i}^{*}\,\delta) f_{\mathbf{u}_{p,i}\cdot\mu,\mathfrak{z}}(\boldsymbol{\gamma}') &= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}({}^{\mathsf{t}}\boldsymbol{\gamma}_0'(\boldsymbol{\gamma}+\mathfrak{z}\,\boldsymbol{v})) \quad d\,\mathbf{u}_{p,i}\cdot\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}({}^{\mathsf{t}}\boldsymbol{\gamma}_0'(\boldsymbol{\gamma}_0+\mathfrak{z}\,\boldsymbol{v}_0)) \quad d\,\mathbf{u}_{p,i}\cdot\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left({}^{\mathsf{t}}\boldsymbol{\gamma}_0'(\mathbf{u}_{p,i}^{\square}\,\boldsymbol{\gamma}_0\,\mathbf{u}_{p,i}^{\square,-1}+\mathfrak{z}\,\mathbf{u}_{p,i}^{\blacksquare}\,\boldsymbol{v}_0\,\mathbf{u}_{p,i}^{\square,-1})\right) \quad d\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left({}^{\mathsf{t}}\boldsymbol{\gamma}_0'\,\mathbf{u}_{p,i}^{\square}(\boldsymbol{\gamma}_0+\mathbf{u}_{p,i}^{\square,-1}\,\mathfrak{z}\,\mathbf{u}_{p,i}^{\blacksquare}\,\boldsymbol{v}_0)\,\mathbf{u}_{p,i}^{\square,-1}\right) \quad d\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left({}^{\mathsf{t}}\boldsymbol{\gamma}_0'\,\mathbf{u}_{p,i}^{\square}(\boldsymbol{\gamma}_0+(\mathfrak{z}\cdot\mathbf{u}_{p,i})\,\boldsymbol{v}_0)\,\mathbf{u}_{p,i}^{\square,-1}\right) \quad d\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left(\mathbf{u}_{p,i}^{\square,-1}\,{}^{\mathsf{t}}\boldsymbol{\gamma}_0'\,\mathbf{u}_{p,i}^{\square}(\boldsymbol{\gamma}_0+(\mathfrak{z}\cdot\mathbf{u}_{p,i})\,\boldsymbol{v}_0)\right) \quad d\mu\right)\\
&= (\mathbf{u}_{p,i}^{*}\,\delta)\left(\kappa_{\mathcal{U}}(\boldsymbol{\beta}_0')\int_{\mathbf{T}_0} \kappa_{\mathcal{U}}(\boldsymbol{\beta}) e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left({}^{\mathsf{t}}(\mathbf{u}_{p,i}^{\square}\,\boldsymbol{\gamma}_0'\,\mathbf{u}_{p,i}^{\square,-1})(\boldsymbol{\gamma}_0+(\mathfrak{z}\cdot\mathbf{u}_{p,i})\,\boldsymbol{v}_0)\right) \quad d\mu\right)\\
&= \mathbf{u}_{p,i}\,.(\delta f_{\mu,\mathfrak{z}}),
\end{aligned}
$$

where we have written $(\boldsymbol{\gamma},\boldsymbol{v})=(\boldsymbol{\gamma}_0,\boldsymbol{v}_0)\,\boldsymbol{\beta}$ for $(\boldsymbol{\gamma}_0,\boldsymbol{v}_0)\in\mathbf{T}_{00}$ and $\boldsymbol{\beta}\in B_{\mathrm{GL}_g,0}$. The antepenultimate equation follows from the property of matrix determinants. $\square$

6.2.9. There is an analogue for compactly supported cohomology groups and overconvergent cuspforms. Let r, w, and $(R_{\mathcal{U}},\kappa_{\mathcal{U}})$ be the same as before. On one hand, consider

$$
\mathscr{O}\mathscr{D}_{\kappa_{\mathcal{U}}}^{r,\mathrm{cusp}} := \left(\varprojlim_{j}\left(\nu^{-1}\jmath_{\mathrm{k\acute{e}t},!}\,\mathscr{D}_{\kappa_{\mathcal{U}},j}^{r,\circ}\otimes_{\mathbf{Z}_p}\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}^{+}\right)\right)[\frac{1}{p}].
$$

Since

$$
H_{\mathrm{k\acute{e}t}}^{n_0}(\overline{\mathcal{X}}_{\mathrm{Iw}^+},\jmath_{\mathrm{k\acute{e}t},!}\,\mathscr{D}_{\kappa_{\mathcal{U}},j}^{r,\circ}) = H_{\mathrm{\acute{e}t},c}^{n_0}(\mathcal{X}_{\mathrm{Iw}^+},\mathscr{D}_{\kappa_{\mathcal{U}},j}^{r,\circ}),
$$

an analogue of Proposition 6.1.3 implies that $\mathscr{O}\mathscr{D}_{\kappa_{\mathcal{U}}}^{r,\mathrm{cusp}}$ computes

$$
\mathbb{O}_{\kappa_{\mathcal{U}},\mathbf{C}_p}^{r,c} := \left(\varprojlim_{j} H_{\mathrm{\acute{e}t},c}^{n_0}(\mathcal{X}_{\mathrm{Iw}^+},\mathscr{D}_{\kappa_{\mathcal{U}},j}^{r,\circ})\otimes_{\mathbf{Z}_p}\mathcal{O}_{\mathbf{C}_p}\right)[\frac{1}{p}].
$$

On the other hand, recall the sheaf $\underline{\omega}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}}$ of w-overconvergent Siegel cuspforms of weight $\kappa_{\mathcal{U}}$ and consider the p-adically completed pullback $\widehat{\underline{\omega}}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}}$ to the pro-Kummer étale site. Repeating the construction above, we obtain a morphism $\eta_{\kappa_{\mathcal{U}}}^{\mathrm{cusp}}:\mathscr{O}\mathscr{D}_{\kappa_{\mathcal{U}}}^{r,\mathrm{cusp}}\to\widehat{\underline{\omega}}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}}$ which induces a morphism

$$
\mathrm{ES}_{\kappa_{\mathcal{U}}}^{\mathrm{cusp}}:\mathbb{O}_{\kappa_{\mathcal{U}},\mathbf{C}_p}^{r,c}\to H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w},\underline{\omega}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}+g+1})(-n_0)
$$

rendering the following Galois- and Hecke-equivariant diagram commutative:

$$\begin{tikzcd} \mathbb{O}^{r}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \arrow[r, "\mathrm{ES}_{\kappa_{\mathcal{U}}}"] & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{\kappa_{\mathcal{U}}+g+1})(-n_0) \\ \mathbb{O}^{r,c}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \arrow[u] \arrow[r, "\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}}"] & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}+g+1})(-n_0) \arrow[u, hook] \end{tikzcd},$$

where the vertical arrow on the left is the natural map from the compactly supported cohomology group to the usual cohomology group. Let

$$\mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p} := \mathrm{image}\left(\mathbb{O}^{r,c}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to \mathbb{O}^{r}_{\kappa_{\mathcal{U}},\mathbf{C}_p}\right).$$

We arrive at the ***overconvergent Eichler–Shimura morphism for overconvergent Siegel cuspforms*** (of weight $\kappa_{\mathcal{U}}$)

$$\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}} : \mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0),$$

where

$$S^w_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_{w,\mathrm{cusp}}^{\kappa_{\mathcal{U}}+g+1})$$

is the space of w-overconvergent Siegel cuspforms of strict Iwahori level and weight $\kappa_{\mathcal{U}}+g+1$.

Remark 6.2.10. We finally remark that, by construction, both $\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}}$ and $\mathrm{ES}_{\kappa_{\mathcal{U}}}$ are functorial in the small weights $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

6.3 The image of overconvergent Eichler–Shimura morphisms at classical weights

6.3.1. The aim of this last part of the section is to describe the image of the overconvergent Eichler–Shimura morphism at classical algebraic weights. Let $k = (k_1, \ldots, k_g) \in \mathbf{Z}^g_{\geq 0}$ be a dominant weight and recall the representations $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ and $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ of GSp_{2g} studied in §4.4.

Similar to 6.1.2, we introduce the sheaves $\mathscr{O}\mathscr{V}_k$ and $\mathscr{O}\mathscr{V}_k^\vee$ on $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$ defined by

$$\mathscr{O}\mathscr{V}_k := \nu^{-1} j_{\mathrm{k\acute{e}t},*}\, \mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}},$$
$$\mathscr{O}\mathscr{V}_k^\vee := \nu^{-1} j_{\mathrm{k\acute{e}t},*}\, \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}.$$

By the same argument as in Proposition 6.1.3, we obtain a natural identification

$$H^{n_0}_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \simeq H^{n_0}_{\mathrm{prok\acute{e}t}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{V}_k^\vee).$$

Moreover, if $\mathcal{V} = \varprojlim_n \mathcal{V}_n \to \overline{\mathcal{X}}_{\mathrm{Iw}^+}$ is a pro-Kummer étale presentation of a log affinoid perfectoid object in $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$ and let $\mathcal{V}_\infty := \mathcal{V} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \overline{\mathcal{X}}_{\Gamma(p^\infty)}$, then, following the same argument as in the proof of Lemma 6.2.2, we obtain identifications

$$\mathscr{O}\mathscr{V}_k(\mathcal{V}) = \Big(\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}},$$

$$\mathscr{O}\mathscr{V}^\vee_k(\mathcal{V}) = \Big(\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)\Big)^{\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}}.$$

6.3.2. We also consider the p-adically completed automorphic sheaf $\widehat{\underline{\omega}}^k_{\mathrm{Iw}^+}$ on $\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$ defined by

$$\widehat{\underline{\omega}}^k_{\mathrm{Iw}^+} := \varprojlim_m \Big(\underline{\omega}^{k,+}_{\mathrm{Iw}^+} \otimes_{\mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}} \mathscr{O}^+_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}}/p^m\Big)[\frac{1}{p}]$$

where $\underline{\omega}^{k,+}_{\mathrm{Iw}^+}$ is defined in Remark 3.4.3. It follows from Proposition 3.4.5 that

$$\widehat{\underline{\omega}}^k_{\mathrm{Iw}^+}(\mathcal{V}) = \left\{ f \in P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty)) : \boldsymbol{\gamma}^* f = \rho_k(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)^{-1} f,\ \ \forall\, \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_a & \boldsymbol{\gamma}_b \\ \boldsymbol{\gamma}_c & \boldsymbol{\gamma}_d \end{pmatrix} \in \mathrm{Iw}^+_{\mathrm{GSp}_{2g}} \right\}.$$

for any log affinoid perfectoid object $\mathcal{V} \in \overline{\mathcal{X}}_{\mathrm{Iw}^+,\mathrm{prok\acute{e}t}}$ and $\mathcal{V}_\infty = \mathcal{V} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}} \overline{\mathcal{X}}_{\Gamma(p^\infty)}$.

6.3.3. Recall the Hodge–Tate morphism

$$\mathrm{HT}_{\Gamma(p^\infty)} : \mathbf{V}_p \to \underline{\omega}_{\Gamma(p^\infty)}.$$

It follows from the definition that

$$\underline{\omega}^k_{\mathrm{Iw}^+} = (\mathrm{Sym}^{k_1-k_2}\, \underline{\omega}_{\mathrm{Iw}^+}) \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}} (\mathrm{Sym}^{k_2-k_3} \wedge^2 \underline{\omega}_{\mathrm{Iw}^+}) \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}} \cdots \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+}}} (\mathrm{Sym}^{k_g} \det \underline{\omega}_{\mathrm{Iw}^+})$$

and hence

$$\underline{\omega}^k_{\Gamma(p^\infty)} = (\mathrm{Sym}^{k_1-k_2}\, \underline{\omega}_{\Gamma(p^\infty)}) \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}} (\mathrm{Sym}^{k_2-k_3} \wedge^2 \underline{\omega}_{\Gamma(p^\infty)}) \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}} \cdots \otimes_{\mathscr{O}_{\overline{\mathcal{X}}_{\Gamma(p^\infty)}}} (\mathrm{Sym}^{k_g} \det \underline{\omega}_{\Gamma(p^\infty)}).$$

Let $\mathbf{V}_{\mathrm{std}}$ denote the standard representation of GSp_{2g} over $\mathbf{Q}_p$, with standard basis $x_1, \ldots, x_{2g}$. There is an isomorphism of $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-representations $\mathbf{V}_{\mathrm{std}} \simeq \mathbf{V}_{\mathbf{Q}_p} := \mathbf{V}_p \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ sending x_i to e_{2g+1-i}, for $i = 1, \ldots, g$, and sending x_i to $-e_{2g+1-i}$, for $i = g+1, \ldots, 2g$. If we write

$$\mathbf{V}^k_{\mathrm{std}} := (\mathrm{Sym}^{k_1-k_2}\, \mathbf{V}_{\mathrm{std}}) \otimes_{\mathbf{Q}_p} (\mathrm{Sym}^{k_2-k_3}(\wedge^2 \mathbf{V}_{\mathrm{std}})) \otimes_{\mathbf{Q}_p} \cdots \otimes_{\mathbf{Q}_p} (\mathrm{Sym}^{k_g}(\wedge^g \mathbf{V}_{\mathrm{std}}))$$

and

$$\mathbf{V}^k_{\mathbf{Q}_p} := (\mathrm{Sym}^{k_1-k_2}\, \mathbf{V}_{\mathbf{Q}_p}) \otimes_{\mathbf{Q}_p} (\mathrm{Sym}^{k_2-k_3}(\wedge^2 \mathbf{V}_{\mathbf{Q}_p})) \otimes_{\mathbf{Q}_p} \cdots \otimes_{\mathbf{Q}_p} (\mathrm{Sym}^{k_g}(\wedge^g \mathbf{V}_{\mathbf{Q}_p})),$$

the Hodge-Tate map induces a map $\mathbf{V}^k_{\mathrm{std}} \simeq \mathbf{V}^k_{\mathbf{Q}_p} \to \underline{\omega}^k_{\mathrm{Iw}^+}$. Moreover, it is well-known that $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ is an irreducible GSp_{2g}-subrepresentation of $\mathbf{V}^k_{\mathrm{std}}$ (see for example [FH91, Lecture 17]). In particular, the highest weight vector e^{hst}_k in $\mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k}$ corresponds to the element

$$x_1^{k_1-k_2} \otimes (x_1 \wedge x_2)^{k_2-k_3} \otimes \cdots \otimes (x_1 \wedge \cdots \wedge x_g)^{k_g}$$

in $\mathbf{V}^k_{\mathrm{std}}$.

The composition

$$\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} \xrightarrow{\beta} \mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} \hookrightarrow \mathbf{V}^k_{\mathrm{std}} \simeq \mathbf{V}^k_{\mathbf{Q}_p} \to \underline{\omega}^k_{\mathrm{Iw}^+}$$

then induces a map

$$\eta^{\mathrm{alg}}_k : \mathscr{O}\mathscr{V}^\vee_k \to \widehat{\underline{\omega}}^k_{\mathrm{Iw}^+}.$$

Eventually, we arrive at the ***algebraic Eichler-Shimura morphism*** (of weight k)

$$\begin{tikzcd}
H^{n_0}_{\text{ét}}(\mathcal{X}_{\mathrm{Iw}^+}, \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \arrow[r, "\simeq"] \arrow[drr, bend right, "\mathrm{ES}^{\mathrm{alg}}_k"'] & H^{n_0}_{\text{prokét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{V}^\vee_k) \arrow[r, "\eta^{\mathrm{alg}}_k"] & H^{n_0}_{\text{prokét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \widehat{\underline{\omega}}^k_{\mathrm{Iw}^+}) \arrow[d] \\
 & & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^{k+g+1}_{\mathrm{Iw}^+})(-n_0)
\end{tikzcd},$$

where the last map follows from the same argument as in the proof of Lemma 6.2.4. We remark that $\mathrm{ES}^{\mathrm{alg}}_k$ coincides with the one induced from [FC90, Theorem VI. 6.2]. It is Hecke- and $\mathrm{Gal}_{\mathbf{Q}_p}$-equivariant, and also surjective.

Lemma 6.3.4. *Over the w-ordinary locus $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}$, the map η^{alg}_k has the following explicit description.*

(i) Let $\mathcal{V} = \varprojlim_n \mathcal{V}_n$ be a pro-Kummer étale presentation of a log affinoid perfectoid object in $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,prok\acute{e}t}$ and let $\mathcal{V}_\infty = \mathcal{V} \times_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}} \overline{\mathcal{X}}_{\Gamma(p^\infty),w}$. There is a well-defined $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-equivariant map

$$\widetilde{\eta}^{\mathrm{alg}}_k : \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,prok\acute{e}t}}(\mathcal{V}_\infty) \to P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,prok\acute{e}t}}(\mathcal{V}_\infty))$$

defined by $\mu \otimes \delta \mapsto \delta f^{\mathrm{alg}}_{\mu,\mathfrak{z}}$ where

$$f^{\mathrm{alg}}_{\mu,\mathfrak{z}}(\boldsymbol{\gamma}') = \int_{\boldsymbol{\alpha}\in\mathrm{GSp}_{2g}} e^{\mathrm{hst}}_k\left(\begin{pmatrix} {}^{\mathsf{t}}\boldsymbol{\gamma}' & \\ & \breve{\mathbb{1}}_g\, \boldsymbol{\gamma}'^{-1}\, \breve{\mathbb{1}}_g\end{pmatrix}\begin{pmatrix}\mathbb{1}_g & \mathfrak{z} \\ & \mathbb{1}_g\end{pmatrix}\boldsymbol{\alpha}\right) \quad d\mu.$$

Here, the $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-action on the right hand side is given by

$$\boldsymbol{\gamma} \cdot f := \rho_k(\boldsymbol{\gamma}_a + \mathfrak{z}\,\boldsymbol{\gamma}_c)(\boldsymbol{\gamma}^* f)$$

for every $\boldsymbol{\gamma} = \begin{pmatrix}\gamma_a & \gamma_b \\ \gamma_c & \gamma_d\end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbf{Q}_p)$ and $f \in P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,prok\acute{e}t}}(\mathcal{V}_\infty))$.

(ii) The map η^{alg}_k is obtained from $\widetilde{\eta}^{\mathrm{alg}}_k$ by taking $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-invariants on both sides.

Proof. (i) Notice that $\widetilde{\eta}^{\mathrm{alg}}_k$ is the composition of β with the map

$$\xi^{\mathrm{alg}}_k : \mathbf{V}^{\mathrm{alg}}_{\mathrm{GSp}_{2g},k} \otimes_{\mathbf{Q}_p} \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty) \to P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty))$$

defined by $\phi \otimes \delta \mapsto \delta g_{\phi,\mathfrak{z}}$ where

$$g_{\phi,\mathfrak{z}}(\boldsymbol{\gamma}') = \phi\left(\begin{pmatrix} \mathbb{1}_g & \\ {}^{\mathtt{t}}\mathfrak{z} & \mathbb{1}_g \end{pmatrix}\begin{pmatrix} \boldsymbol{\gamma}' & \\ & \breve{\mathbb{1}}_g\, {}^{\mathtt{t}}(\boldsymbol{\gamma}')^{-1}\, \breve{\mathbb{1}}_g \end{pmatrix}\right)$$

for all $\boldsymbol{\gamma}' \in \mathrm{GL}_g(\mathbf{C}_p)$.

Recall that β is $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-equivariant. It remains to check that ξ_k^{alg} is $\mathrm{GSp}_{2g}(\mathbf{Q}_p)$-equivariant, which follows from a straightforward calculation.

(ii) It suffices to check that the $\mathrm{Iw}_{\mathrm{GSp}_{2g}}^+$-invariance of ξ_k^{alg} coincides with the map induced from the composition $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}} \hookrightarrow V_{\mathrm{std}}^k \simeq V_{\mathbf{Q}_p}^k \to \underline{\omega}_{\mathrm{Iw}^+}^k$. Notice that $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}}$ is spanned by GSp_{2g}-translations of the highest weight vector e_k^{hst}. Therefore, we only need to check that $\xi_k^{\mathrm{alg}}(e_k^{\mathrm{hst}} \otimes 1)$ gives the correct element in $\widehat{\underline{\omega}}_{\mathrm{Iw}^+}^k$.

Indeed, since the Hodge–Tate map $V_p \to \underline{\omega}_{\mathrm{Iw}^+}$ sends e_{2g+1-i} to $\mathfrak{s}_i$, for $i = 1, \ldots, g$, we see that the composition $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}} \hookrightarrow V_{\mathrm{std}}^k \simeq V_{\mathbf{Q}_p}^k \to \underline{\omega}_{\mathrm{Iw}^+}^k$ sends the highest weight vector e_k^{hst} to

$$\mathfrak{s}_1^{k_1-k_2} \otimes (\mathfrak{s}_1 \wedge \mathfrak{s}_2)^{k_2-k_3} \otimes \cdots \otimes (\mathfrak{s}_1 \wedge \cdots \wedge \mathfrak{s}_g)^{k_g}.$$

On the other hand, notice that the element $\mathfrak{s}_1 \wedge \cdots \wedge \mathfrak{s}_t$ corresponds to the function $X = (X_{ij})_{1\leq i,j\leq g} \mapsto \det((X_{ij})_{1\leq i,j\leq t})$ in $P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty))$. Therefore, e_k^{hst} is sent to the function

$$X \mapsto X_{11}^{k_1-k_2} \times \det((X_{ij})_{1\leq i,j\leq 2})^{k_2-k_3} \times \cdots \times \det((X_{ij})_{1\leq i,j\leq g})^{k_g}$$

in $P_k(\mathrm{GL}_g, \widehat{\mathscr{O}}_{\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}}(\mathcal{V}_\infty))$. This element coincides with $\xi_k^{\mathrm{alg}}(e_k^{\mathrm{hst}} \otimes 1)$, as desired.

□

6.3.5. Recall the natural inclusion $M_{\mathrm{Iw}^+}^{k,\mathrm{cl}} = H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}_{\mathrm{Iw}^+}^k) \hookrightarrow H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^k) = M_{\mathrm{Iw}^+,w}^k$ from Lemma 3.4.6. The main result of this section is the following.

Theorem 6.3.6. *Let $k = (k_1, ..., k_g) \in \mathbf{Z}_{\geq 0}^g$ be a dominant weight. Then the image of*

$$\mathrm{ES}_k : \mathbf{\Omega}_{k,\mathbf{C}_p}^r \longrightarrow M_{\mathrm{Iw}^+,k+g+1}^w(-n_0)$$

is contained in the space of the classical forms $M_{\mathrm{Iw}^+,k+g+1}^{\mathrm{cl}}(-n_0)$.

Proof. Recall the map

$$D_k^r(\mathbf{T}_0, \mathbf{Q}_p) \to \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$$

from 4.4.2. This map then induces a map of sheaves

$$\mathscr{O}\mathscr{D}_k^r \to \mathscr{O}\mathscr{V}_k^\vee$$

over $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\mathrm{prok\acute{e}t}}$. Hence, the theorem follows once we show that the following diagram commutes

$$\begin{tikzcd}
H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{D}^r_k) \arrow[r, "\mathrm{ES}_k"] \arrow[d] & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{k+g+1})(-n_0) \\
H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{V}^\vee_k) \arrow[r, "\mathrm{ES}_k^{\mathrm{alg}}"] & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}_{\mathrm{Iw}^+}^{k+g+1})(-n_0) \arrow[u]
\end{tikzcd}.$$

Over $\overline{\mathcal{X}}_{\mathrm{Iw}^+,w,\text{proét}}$, it follows from the construction that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}\mathscr{D}^r_k \arrow[r, "\eta_k"] \arrow[d] & \widehat{\underline{\omega}}^k_w \\
\mathscr{O}\mathscr{V}^\vee_k \arrow[r, "\eta_k^{\mathrm{alg}}"] & \widehat{\underline{\omega}}^k_{\mathrm{Iw}^+} \arrow[u, hook]
\end{tikzcd},$$

where the inclusion on the right-hand side is given by the inclusion (3.2). Consequently, there is a commutative diagram on the cohomology groups

$$\begin{tikzcd}
H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+} \mathscr{O}\mathscr{D}^r_k) \arrow[r] \arrow[d, "\mathrm{Res}"] \arrow[dddd, bend right=60, "\mathrm{ES}_k"'] & H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \mathscr{O}\mathscr{V}^\vee_k) \arrow[d, "\mathrm{Res}"] \arrow[ddddr, bend left, "\mathrm{ES}_k^{\mathrm{alg}}"] & \\
H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \mathscr{O}\mathscr{D}^r_k) \arrow[r] \arrow[d, "\eta_k"] & H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \mathscr{O}\mathscr{V}^\vee_k) \arrow[d, "\eta_k^{\mathrm{alg}}"] & \\
H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \widehat{\underline{\omega}}^k_w) \arrow[d] & H^{n_0}_{\text{proét}}(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \widehat{\underline{\omega}}^k_{\mathrm{Iw}^+}) \arrow[l, hook'] \arrow[d] & \\
 & & \\
H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_w^{k+g+1})(-n_0) & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+,w}, \underline{\omega}_{\mathrm{Iw}^+}^{k+g+1})(-n_0) \arrow[l, hook'] & H^0(\overline{\mathcal{X}}_{\mathrm{Iw}^+}, \underline{\omega}^k_{\mathrm{Iw}^+})(-n_0) \arrow[l, hook', "\mathrm{Res}"]
\end{tikzcd}.$$

This finishes the proof. $\square$

6.4 Sheaves on the cuspidal eigenvariety

6.4.1. In this section, we glue the overconvergent Eichler–Shimura morphism over the cuspidal eigenvariety $\mathcal{E}_0$. Due to our construction of $\mathcal{E}_0$, we shall again assume $p > 2g$ in this section. We begin with some setup of notations:

Given a weight $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ and an integer $r > 1 + r_{\mathcal{U}}$, we write

$$\mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p} = \begin{cases} \mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p}, & \text{if } \mathcal{U} \text{ is a small weight} \\ H^{n_0}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0, R_{\mathcal{U}}))\widehat{\otimes}_{\mathbf{Q}_p}\mathbf{C}_p, & \text{if } \mathcal{U} \text{ is an affinoid weight} \end{cases}.$$

We also write

$$\mathbb{O}^{\dagger,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p} := \varprojlim_r \mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p}.$$

6.4.2. Suppose that $(R_{\mathcal{U}}, \kappa_{\mathcal{U}})$ is an small open weight and recall the overconvergent Eichler–

Shimura morphism for overconvergent Siegel cuspforms

$$\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}} : \mathbb{O}^{r,\mathrm{cusp}}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to S^{w}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0).$$

If $(\mathcal{U},h)$ slope-adapted, then the Hecke-equivariance of $\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}}$ induces a $\mathbf{C}_p\widehat{\otimes}R_{\mathcal{U}}$-linear map

$$\mathrm{ES}^{\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}}} : \mathbb{O}^{r\ \mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to S^{w,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0).$$

of finite projective $\mathbf{C}_p\widehat{\otimes}R_{\mathcal{U}}$-modules.

Now, if $\mathcal{U}'\subset\mathcal{U}^{\mathrm{rig}}$ is an affinoid weight, the $\mathbf{C}_p\widehat{\otimes}R_{\mathcal{U}'}$-linear map $\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}'}}$ is defined to be the composition

$$\mathrm{ES}^{\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}'}} : \mathbb{O}^{r,\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}'},\mathbf{C}_p} \simeq \mathbb{O}^{r,\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}},\mathbf{C}_p}\otimes_{R_{\mathcal{U}}[\frac{1}{p}]}R_{\mathcal{U}'} \to S^{w,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0) \to S^{w,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}'}+g+1}(-n_0), \tag{6.1}$$

where the first isomorphism follows from Proposition 5.2.2.

6.4.3. Recall the natural map $\mathcal{E}_0\to\mathcal{S}_{\mathbf{C}_p}$ and let $\mathcal{E}_{\mathcal{U},h}$ be the preimage of $\mathcal{S}_{\mathbf{C}_p,\mathcal{U},h}$. On the cuspidal eigenvariety $\mathcal{E}_0$, we consider two coherent sheaves $\mathscr{OC}^{\dagger}_{\mathrm{cusp}}$ and $\mathscr{S}^{\dagger}_{\mathrm{Iw}^+}(-n_0)$ given by

$$\mathscr{OC}^{\dagger}_{\mathrm{cusp}}(\mathcal{E}_{\mathcal{U},h}) := \mathbb{O}^{\dagger,\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}},\mathbf{C}_p}$$

and

$$\mathscr{S}^{\dagger}_{\mathrm{Iw}^+}(-n_0)(\mathcal{E}_{\mathcal{U},h}) := S^{\dagger,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0),$$

for all $\mathcal{S}_{\mathcal{U},h}\in\mathrm{Cov}_{\mathrm{aff}}(\mathcal{S})$.

Theorem 6.4.4. *There exists a morphism*

$$\mathscr{ES} : \mathscr{OC}^{\dagger}_{\mathrm{cusp}} \to \mathscr{S}^{\dagger}_{\mathrm{Iw}^+}(-n_0)$$

of coherent sheaves over $\mathcal{E}_0$ such that if $(\mathcal{U},h)$ is a slope-adapted pair, then $\mathscr{ES}(\mathcal{E}_{\mathcal{U},h})$ is exactly the overconvergent Eichler–Shimura morphism for overconvergent Siegel cuspforms

$$\mathrm{ES}^{\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}}} : \mathbb{O}^{\dagger,\mathrm{cusp},\leq h}_{\kappa_{\mathcal{U}},\mathbf{C}_p} \to S^{\dagger,\leq h}_{\mathrm{Iw}^+,\kappa_{\mathcal{U}}+g+1}(-n_0).$$

Proof. It follows from (6.1) and the functoriality of $\mathrm{ES}^{\mathrm{cusp}}_{\kappa_{\mathcal{U}}}$ in small open weights $\mathcal{U}$. □

6.4.5. Denote by $\mathscr{Im}$ and $\mathscr{Ker}$ the image and the kernel of $\mathscr{ES}$, respectively. We obtain a short exact sequence of sheaves on $\mathcal{E}_0$

$$0\to\mathscr{Ker}\to\mathscr{OC}^{\dagger}_{\mathrm{cusp}}\to\mathscr{Im}\to 0.$$

We remind the readers that this short exact sequence is Galois- and Hecke-equivariant. Let $\mathcal{V}=\mathrm{Spa}(R_{\mathcal{V}},R_{\mathcal{V}}^+)$ be an affinoid open subsapce of $\mathcal{E}_0$ such that $\mathscr{Ker}(\mathcal{V})$, $\mathscr{OC}^{\dagger}_{\mathrm{cusp}}(\mathcal{V})$, and $\mathscr{Im}(\mathcal{V})$ are free and such that the sequence

$$0\to\mathscr{Ker}(\mathcal{V})\to\mathscr{OC}^{\dagger}_{\mathrm{cusp}}(\mathcal{V})\to\mathscr{Im}(\mathcal{V})\to 0$$

is exact. Consider

$$\mathscr{H}(\mathcal{V}) := \mathrm{Hom}_{R_{\mathcal{V}}}(\mathscr{I}\!m(\mathcal{V}), \mathscr{K}\!er(\mathcal{V})).$$

Recall that we have the Sen operator $\varphi_{\mathrm{Sen}} = \varphi_{\mathrm{Sen},\mathcal{V}}$ associated with $\mathscr{H}(\mathcal{V})$, which was introduced in [Sen88] (see also [Kis03]).

The following result is an analogue to [AIS15, Theorem 6.1(c)].

Theorem 6.4.6. *Let $\mathcal{V} = \mathrm{Spa}(R_{\mathcal{V}}, R_{\mathcal{V}}^+) \subset \mathcal{E}_0$ be an affinoid open subspace such that $\mathscr{K}\!er(\mathcal{V})$, $\mathscr{O}\mathscr{C}^{\dagger}_{\mathrm{cusp}}(\mathcal{V})$, and $\mathscr{I}\!m(\mathcal{V})$ are free and such that the sequence*

$$0 \to \mathscr{K}\!er(\mathcal{V}) \to \mathscr{O}\mathscr{C}^{\dagger}_{\mathrm{cusp}}(\mathcal{V}) \to \mathscr{I}\!m(\mathcal{V}) \to 0$$

is exact. Suppose φ_{Sen} is non-vanishing. Then the short exact sequence

$$0 \to \mathscr{K}\!er \to \mathscr{O}\mathscr{C}^{\dagger}_{\mathrm{cusp}} \to \mathscr{I}\!m \to 0$$

splits locally over $\mathcal{V}$.

Proof. We follow the same strategy as in [AIS15, Theorem 6.1(c)]. Observe that we have an isomorphism $H^1(\mathrm{Gal}_{\mathbf{Q}_p}, \mathscr{H}(\mathcal{V})) \simeq \mathrm{Ext}^1_{R_{\mathcal{V}}[\mathrm{Gal}_{\mathbf{Q}_p}]}(\mathscr{I}\!m(\mathcal{V}), \mathscr{K}\!er(\mathcal{V}))$. Thus, the $\mathrm{Gal}_{\mathbf{Q}_p}$-equivariance of the short exact sequence defines a class in $H^1(\mathrm{Gal}_{\mathbf{Q}_p}, \mathscr{H}(\mathcal{V}))$. Then by [Kis03, Proposition 2.3], $\det(\varphi_{\mathrm{Sen}}) \in R_{\mathcal{V}}$ kills the cohomology group $H^1(\mathrm{Gal}_{\mathbf{Q}_p}, \mathscr{H}(\mathcal{V}))$. On the other hand, $\det(\varphi_{\mathrm{Sen}})$ is non-zero. Therefore, after localising at this element, the short exact sequence splits as a sequence of semilinear $\mathrm{Gal}_{\mathbf{Q}_p}$-representations. Since the Galois-action commutes with the Hecke-actions, the splitting can be chosen to be Hecke-equivariant. $\square$

Chapter 7

A pairing on the cuspidal eigenvariety

The aim of this chapter is to construct a pairing on the cuspidal eigenvariety $\mathcal{E}_0^{\mathrm{oc}}$ so that we provide an answer to the first half of Question 1.3.2 (ii). In particular, after base change to $\mathbf{C}_p$, this yields a method to study the ramification locus of $\mathcal{E}_0$ (over $\mathcal{W} \times_{\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$).

We shall begin with the construction of a pairing on the overocnvergent cohomology groups in §7.1. Such a pairing is just defined on the space of distributions, inspired by an algebraic model on the irreducible representations $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$. In §7.2, we will recall some results in commutative algebra by following [Bel21]. The preparation on the commutative algebras then allows us to study the ramification locus of $\mathcal{E}_0^{\mathrm{oc}}$ in §7.3.

7.1 A pairing on the overconvergent cohomology groups

7.1.1. The pairing we shall construct has an algebraic model, which we now explain.

Given a dominant weight $k = (k_1, ..., k_g) \in \mathbf{Z}_{\geq 0}^g$, recall $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}}$ and $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$ with their left GSp_{2g}-actions from §4.4. Then, we have a morphism

$$\Phi_k^{\mathrm{alg}} : \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee} \to \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}}, \quad \mu \mapsto \left(\boldsymbol{\gamma}' \mapsto \int_{\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}({}^{\mathtt{t}}\boldsymbol{\gamma}' \boldsymbol{\gamma}) \quad d\mu \right),$$

where $e_k^{\mathrm{hst}} \in \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg}}$ is defined in Example 4.4.3. One sees that Φ_k^{alg} is GSp_{2g}-equivariant with respect to the left GSp_{2g}-actions on both spaces. Indeed, for any $\boldsymbol{\alpha}, \boldsymbol{\gamma}' \in \mathrm{GSp}_{2g}$ and $\mu \in \mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$, we have

$$\begin{aligned} \Phi_k^{\mathrm{alg}}(\boldsymbol{\alpha} \cdot \mu)(\boldsymbol{\gamma}') &= \int_{\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}({}^{\mathtt{t}}\boldsymbol{\gamma}' \boldsymbol{\alpha} \boldsymbol{\gamma}) \quad d\mu \\ &= \int_{\boldsymbol{\gamma} \in \mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}({}^{\mathtt{t}}({}^{\mathtt{t}}\boldsymbol{\alpha} \boldsymbol{\gamma}') \boldsymbol{\gamma}) \quad d\mu \\ &= \left(\boldsymbol{\alpha} \cdot \Phi_k^{\mathrm{alg}}(\mu) \right)(\boldsymbol{\gamma}'). \end{aligned}$$

Consequently, Φ_k^{alg} defines a pairing on $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$ by

$$(\mu_1,\mu_2)\mapsto \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\,\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2).$$

Remark 7.1.2. Notice that $\mathbf{V}_{\mathrm{GSp}_{2g},k}^{\mathrm{alg},\vee}$ is an irreducible representation of GSp_{2g}, thus it admits a pairing induced by the symplectic pairing $\langle\cdot,\cdot\rangle$ on $\mathbf{V}$. This pairing can be viewed by the following formula

$$\langle\cdot,\cdot\rangle_k:(\mu_1,\mu_2)\mapsto \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2).$$

Indeed, for any $\boldsymbol{\alpha}\in\mathrm{GSp}_{2g}$, we have

$$\begin{aligned}
\langle\,\boldsymbol{\alpha}\cdot\mu_1,\mu_2\,\rangle_k &= \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\alpha}\,\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2)\\
&= \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\,\varsigma(\boldsymbol{\alpha})\,{}^{\mathtt{t}}\boldsymbol{\alpha}^{-1}\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2)\\
&= \varsigma(\boldsymbol{\alpha})^{\sum k_i}\int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}(\boldsymbol{\alpha}^{-1}\boldsymbol{\gamma}_2)\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2)\\
&= \varsigma(\boldsymbol{\alpha})^{\sum k_i}\,\langle\,\mu_1,\boldsymbol{\alpha}^{-1}\cdot\mu_2\,\rangle_k,
\end{aligned}$$

where the second equality follows from the definition of GSp_{2g}.

7.1.3. Now, for any weight $(R_{\mathcal{U}},\kappa_{\mathcal{U}})$ and any $r>1+r_{\mathcal{U}}$, we consider

$$\begin{aligned}
\Phi_\kappa: D_{\kappa_{\mathcal{U}}}^r(\mathbf{T}_0,R_{\mathcal{U}})&\to A_{\kappa_{\mathcal{U}}}^r(\mathbf{T}_0,R_{\mathcal{U}}),\\
\mu&\mapsto\left((\boldsymbol{\gamma}',\boldsymbol{v}')\mapsto\int_{(\boldsymbol{\gamma},\boldsymbol{v})\in\mathbf{T}_{00}} e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left(\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\gamma}' & {}^{\mathtt{t}}\boldsymbol{v}'\end{pmatrix}\begin{pmatrix}\mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{v}\end{pmatrix}\right)\quad d\mu\right),
\end{aligned}$$

where $e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}$ is the function in $A_{\kappa_{\mathcal{U}}}^r(\mathbf{T}_0,R_{\mathcal{U}})$ defined in Example 4.1.5. Consequently, we have the pairing

$$[\cdot,\cdot]_{\kappa_{\mathcal{U}}}^{\circ}: D_{\kappa_{\mathcal{U}}}^r(\mathbf{T}_0,R_{\mathcal{U}})\times D_{\kappa_{\mathcal{U}}}^r(\mathbf{T}_0,R_{\mathcal{U}})\to R_{\mathcal{U}}$$

given by the formula

$$[\mu_1,\mu_2]_{\kappa_{\mathcal{U}}}^{\circ}=\int_{\mathbf{T}_{00}^2} e_{\kappa_{\mathcal{U}}}^{\mathrm{hst}}\left(\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{v}_2\end{pmatrix}\begin{pmatrix}\mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{v}_1\end{pmatrix}\right)\quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2).$$

Proposition 7.1.4. *For any* $\boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b\\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d\end{pmatrix}\in\Xi$ *with* $\boldsymbol{\alpha}_a\in\mathrm{Iw}_{\mathrm{GL}_g}^+$, *write*

$$\boldsymbol{\alpha}^{\llcorner}=\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\alpha}_a & {}^{\mathtt{t}}\boldsymbol{\alpha}_c/p\\ p\,{}^{\mathtt{t}}\boldsymbol{\alpha}_b & {}^{\mathtt{t}}\boldsymbol{\alpha}_d\end{pmatrix}\in\Xi.$$

Then, for any $\mu_1, \mu_2 \in D^r_\kappa(\mathbf{T}_0, R)$, we have

$$[\,\boldsymbol{\alpha} \cdot \mu_1, \mu_2\,]^\circ_{\kappa_{\mathcal{U}}} = [\,\mu_1, \boldsymbol{\alpha}^{\llcorner} \cdot \mu_2\,]^\circ_{\kappa_{\mathcal{U}}}\,.$$

Proof. The assertion follows from the computation

$$\begin{aligned}
&\begin{pmatrix} {}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{\upsilon}_2 \end{pmatrix} \begin{pmatrix} \mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_a & \boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c & \boldsymbol{\alpha}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\upsilon}_1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} {}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{\upsilon}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_a & p\,\boldsymbol{\alpha}_b \\ \boldsymbol{\alpha}_c\,/p & \boldsymbol{\alpha}_d \end{pmatrix} \begin{pmatrix} \mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\upsilon}_1 \end{pmatrix} \\
&\qquad = {}^{\mathtt{t}}\!\left(\begin{pmatrix} {}^{\mathtt{t}}\boldsymbol{\alpha}_a & {}^{\mathtt{t}}\boldsymbol{\alpha}_c\,/p \\ p\,{}^{\mathtt{t}}\boldsymbol{\alpha}_b & {}^{\mathtt{t}}\boldsymbol{\alpha}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_2 \\ \boldsymbol{\upsilon}_2 \end{pmatrix} \right) \begin{pmatrix} \mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\upsilon}_1 \end{pmatrix}.
\end{aligned}$$

□

Remark 7.1.5. When comparing with our algebraic model, one notices that the definition of the pairing $[\,\cdot,\cdot\,]^\circ_{\kappa_{\mathcal{U}}}$ involves a 'normalisation' by p^{-1}. Such a normalisation is due to our model for $g = 1$. More precisely, when $g = 1$, elements in $\mathbf{T}_0$ can be written as $(1, pc)a$ for some $a \in \mathbf{Z}_p^\times$ and $c \in \mathbf{Z}_p$. Then, for any $\mu_1, \mu_2 \in D^r_\kappa(\mathbf{T}_0, R)$, we have

$$[\,\mu_1, \mu_2\,]^\circ_{\kappa_{\mathcal{U}}} = \int_{\mathbf{T}_{00}^2} \kappa(1 + pc_1c_2) \quad d\mu_1(1, c_1)d\mu_2(1, c_2),$$

which then coincides with the interpretation in Hansen's unpublished notes [Han12]. In particular, by applying [Bel21, Definition VIII.2.4], we have the formula

$$[\,\mu_1, \mu_2\,]^\circ_{\kappa_{\mathcal{U}}} = \sum_{i=0}^{\infty} p^i \binom{\kappa_{\mathcal{U}}}{i} \mu_1(c_1^i)\mu_2(c_2^i),$$

which is (almost) the same formula given by [*op. cit.*, (VIII.2.4)]. Here, for $j = 1, 2$, we view c_j^i as a function on $\mathbf{T}_0$ via

$$c_j^i : \mathbf{T}_0 \ni (a, pc) \mapsto \kappa_{\mathcal{U}}(a)(c/a)^i.$$

Remark 7.1.6. Following Remark 7.1.2, for any dominant $k \in \mathbf{Z}^g_{\geq 0}$, we may consider the pairing $[\,\cdot,\cdot\,]^\circ_k$ to be the twist of $\langle\,\cdot,\cdot\,\rangle_k$ by an *Atkin–Lehner operator*. More precisely, let

$$\mathbf{w}_p := \begin{pmatrix} & -p^{-1}\,\breve{\mathbb{1}}_g \\ \breve{\mathbb{1}}_g & \end{pmatrix},$$

then

$$\begin{aligned}
[\,\mu_1,\mu_2\,]_k^\circ &= \int_{\mathbf{T}_{00}^2} e_k^{\mathrm{hst}}\left(\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{\upsilon}_2\end{pmatrix}\begin{pmatrix}\mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\upsilon}_1\end{pmatrix}\right) \quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{\upsilon}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{\upsilon}_2)\\
&= \int_{\mathbf{T}_{00}^2} e_k^{\mathrm{hst}}\left(\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{\upsilon}_2\end{pmatrix}\begin{pmatrix} & \breve{\mathbb{1}}_g\\ -p^{-1}\,\breve{\mathbb{1}}_g & \end{pmatrix}\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\upsilon}_1\end{pmatrix}\right) \quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{\upsilon}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{\upsilon}_2)\\
&= \int_{\mathbf{T}_{00}^2} e_k^{\mathrm{hst}}\left({}^{\mathtt{t}}\!\left(\mathbf{w}_p\begin{pmatrix}\boldsymbol{\gamma}_2\\ \boldsymbol{\upsilon}_2\end{pmatrix}\right)\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\upsilon}_1\end{pmatrix}\right) \quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{\upsilon}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{\upsilon}_2).
\end{aligned}$$

In particular, this viewpoint coincides with the perspectives in [Kim06; Bel21; Han12] when $g=1$.

Proposition 7.1.7. *We have a well-defined pairing*

$$[\,\cdot,\cdot\,]_\kappa : H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\times H^{2n_0-t}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\to R_{\mathcal{U}}$$

for any $0\le t\le 2n_0$.

Proof. Together with the cup product on cohomology groups, the pairing defined in 7.1.3 induces a pairing $[\,\cdot,\cdot\,]^*_\kappa$ defined as the composition

$$\begin{tikzcd}
H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\times H^{2n_0-t}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[r, "\smile"] \arrow[ddr, bend right, "{[\,\cdot,\cdot\,]^*_\kappa}"'] & H^{2n_0}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})\widehat{\otimes}_R D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[d, "{[\,\cdot,\cdot\,]^\circ_\kappa}"] \\
 & H^{2n_0}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), R_{\mathcal{U}}) \arrow[d, "\simeq"] \\
 & R_{\mathcal{U}},
\end{tikzcd}$$

where '$\smile$' denotes the cup product.

The compatibility of cup products (see, for example, [Mun84, Chapter 5, §48, Exercise 2]) yields the commutative diagram

$$\begin{tikzcd}
H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\times H^{2n_0-t}(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[r, "\smile"] & H^{2n_0}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D^r_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})\widehat{\otimes}_{R_{\mathcal{U}}} D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[d, equal] \\
H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\times H^{2n_0-t}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[u] \arrow[r, "\smile"] \arrow[d] & H^{2n_0}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})\widehat{\otimes}_{R_{\mathcal{U}}} D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[d, equal] \\
H^t(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))\times H^{2n_0-t}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})) \arrow[r, "\smile"] & H^{2n_0}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})\widehat{\otimes}_{R_{\mathcal{U}}} D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))
\end{tikzcd}.$$

In particular, if $[\mu_1]\in H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))$ and $[\mu_2]\in H^{2n_0-t}_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))$ with $[\mu_1']\in H^t_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}})$ and $[\mu_2']\in H^{2n_0-t}_c(X_{\mathrm{Iw}^+}(\mathbf{C}), D_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}}))$ such that $[\mu_i']\mapsto[\mu_i]$ for $i=1,2$, then

$$[\mu_1]\smile[\mu_2']=[\mu_1']\smile[\mu_2']=[\mu_1']\smile[\mu_2].$$

Hence we define

$$[[\mu_1],[\mu_2]]_{\kappa_{\mathcal{U}}}=[[\mu_1'],[\mu_2]]^*_{\kappa_{\mathcal{U}}}=[[\mu_1],[\mu_2']]^*_{\kappa_{\mathcal{U}}}\,.$$

We see that $[\,\cdot,\cdot\,]_{\kappa_{\mathcal{U}}}$ is well-defined, *i.e.*, independent of the choice of the lifting, due to the commutativity of the above diagram. $\square$

7.1.8. Recall the equidimensional reduced cuspidal eigenvariety $\mathcal{E}_0^{\mathrm{oc}}$ from §5.2. We name the natural morphisms

$$\begin{tikzcd} \mathcal{E}_0^{\mathrm{oc}} \arrow[r, "\pi"] \arrow[rr, bend right, "\mathrm{wt}"'] & \mathcal{S}^{\mathrm{oc}} \arrow[r] & \mathcal{W} \end{tikzcd}\,.$$

Then, we have the following corollary.

Corollary 7.1.9. *The pairing in Proposition 7.1.7 induce pairings*

$$[\,\cdot,\cdot\,]:\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}\times\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}\to\mathscr{O}_{\mathcal{S}^{\mathrm{oc}}}\ \ \textit{and}\ \ [\,\cdot,\cdot\,]:\pi^*\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}\times\pi^*\mathscr{H}^{\mathrm{tol}}_{\mathrm{par}}\to\mathscr{O}_{\mathcal{E}_0^{\mathrm{oc}}}$$

of coherent sheaves on $\mathcal{S}^{\mathrm{oc}}$ and $\mathcal{E}_0^{\mathrm{oc}}$ respectively. Moreover, the first pairing is $\mathbb{T}$-equivariant.

Proof. First of all, we claim that the pairing $[\,\cdot,\cdot\,]^\circ_{\kappa_{\mathcal{U}}}$ is $\mathbf{u}_{p,i}$-equivariant for any $i=0,1,...,g-1$ and for any weight $(R_{\mathcal{U}},\kappa_{\mathcal{U}})$. Take any $\mu_1,\mu_2\in D^\dagger_{\kappa_{\mathcal{U}}}(\mathbf{T}_0,R_{\mathcal{U}})$, we have

$$\begin{aligned}
&[\,\mathbf{u}_{p,i}\cdot\mu_1,\mu_2\,]^\circ_{\kappa_{\mathcal{U}}}\\
&\quad=\int_{\mathbf{T}_{00}^2}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left(\begin{pmatrix}{}^{\mathtt{t}}\boldsymbol{\gamma}_2 & {}^{\mathtt{t}}\boldsymbol{v}_2\end{pmatrix}\begin{pmatrix}\mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{v}_1\end{pmatrix}\right)\quad d\,\mathbf{u}_{p,i}\cdot\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=\int_{\mathbf{T}_{00}}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}({}^{\mathtt{t}}\boldsymbol{\gamma}_2\,\boldsymbol{\gamma}_1+{}^{\mathtt{t}}\boldsymbol{v}_2\,\boldsymbol{v}_1\,/p)\quad d\,\mathbf{u}_{p,i}\cdot\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=\int_{\mathbf{T}_{00}^2}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2(\mathbf{u}^{\square}_{p,i}\,\boldsymbol{\gamma}_1\,\mathbf{u}^{\square,-1}_{p,i})+{}^{\mathtt{t}}\boldsymbol{v}_2(\mathbf{u}^{\blacksquare}_{p,i}\,\boldsymbol{v}_1\,\mathbf{u}^{\square,-1}_{p,i})/p\right)\quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=\int_{\mathbf{T}_{00}^2}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left(({}^{\mathtt{t}}\boldsymbol{\gamma}_2\,\mathbf{u}^{\square}_{p,i}\,\boldsymbol{\gamma}_1+{}^{\mathtt{t}}\boldsymbol{v}_2\,\mathbf{u}^{\blacksquare}_{p,i}\,\boldsymbol{v}_1\,/p)\,\mathbf{u}^{\square,-1}_{p,i}\right)\quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_2)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=\int_{\mathbf{T}_{00}}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left(\mathbf{u}^{\square,-1}_{p,i}\,{}^{\mathtt{t}}\boldsymbol{\gamma}_2\,\mathbf{u}^{\square}_{p,i}\,\boldsymbol{\gamma}_1+\mathbf{u}^{\square,-1}_{p,i}\,{}^{\mathtt{t}}\boldsymbol{v}_2\,\mathbf{u}^{\blacksquare}_{p,i}\,\boldsymbol{v}_1\,/p\right)\quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=\int_{\mathbf{T}_{00}}e^{\mathrm{hst}}_{\kappa_{\mathcal{U}}}\left({}^{\mathtt{t}}(\mathbf{u}^{\square}_{p,i}\,\boldsymbol{\gamma}_2\,\mathbf{u}^{\square,-1}_{p,i})\,\boldsymbol{\gamma}_1+{}^{\mathtt{t}}(\mathbf{u}^{\blacksquare}_{p,i}\,\boldsymbol{v}_2\,\mathbf{u}^{\square,-1}_{p,i})\,\boldsymbol{v}_1\,/p\right)\quad d\mu_1(\boldsymbol{\gamma}_1,\boldsymbol{v}_1)d\mu_2(\boldsymbol{\gamma}_2,\boldsymbol{v}_2)\\
&\quad=[\,\mu_1,\mathbf{u}_{p,i}\cdot\mu_2\,]^\circ_{\kappa_{\mathcal{U}}},
\end{aligned}$$

where the antepenultimate equation follows from the nature of determinants (again).

This claim then implies that we have a $U_{p,i}$-equivariant (and hence $U^x_{p,i}$-equivariant for any $x\in\mathrm{Weyl}_{\mathrm{GSp}_{2g}}$) pairing

$$[\,\cdot,\cdot\,]_{\kappa_{\mathcal{U}}}:H^{\mathrm{tol},\le h}_{\mathrm{par},\kappa_{\mathcal{U}}}\times H^{\mathrm{tol},\le h}_{\mathrm{par},\kappa_{\mathcal{U}}}\to R_{\mathcal{U}}.$$

Thus, by gluing, one obtains the first desired pairing. It is furthermore $\mathbb{T}^p$-equivariant since the Hecke operators outside p acts on the ananlytic distributions trivially. The second one follows immediately. $\square$

7.2 Some commutative algebras

7.2.1. Let A be a noetherian domain and B be a finite flat A-algebra. Consider

$$\mathrm{mult} : B \otimes_A B \to B, \quad b \otimes b' \mapsto bb'$$

and write $\mathfrak{mult} = \ker(\mathrm{mult})$. Let

$$(B \otimes_A B)[\mathfrak{mult}] := \{x \in B \otimes_A B : y \cdot x = 0 \ \forall y \in \mathfrak{mult}\},$$

then the ***Noether's different*** of B over A is defined to be the ideal

$$\mathfrak{d}(B/A) := \mathrm{image}\left((B \otimes_A B)[\mathfrak{mult}] \xrightarrow{\mathrm{mult}} B\right)$$

in B.

Theorem 7.2.2 (Auslander–Buchsbaum)**.** *A prime ideal $\mathfrak{P}$ of B is ramified over A if and only if $\mathfrak{d}(B/A) \subset \mathfrak{P}$. Equivalently,* $\operatorname{Spec} B/\mathfrak{d}(B/A)$ *is the ramification locus of* $\operatorname{Spec} B$ *over* $\operatorname{Spec} A$*.*

Proof. See [AB59, Theorem 2.7]. $\square$

7.2.3. Suppose M, N are two B-modules which are finite flat over A and assume we are in the following situation:

- There exists an A-linear pairing

$$\beta : M \times N \to A$$

  such that β is B-equivariant.

- We have isomorphisms $M \simeq N \simeq B^\vee := \mathrm{Hom}_A(B, A)$ of B-modules.

Lemma 7.2.4 ([Bel21, Proposition VIII.1.11])**.** *Denote by β_B the base change of β to B on $M \otimes_A B \times N \otimes_A B$. Let*

$$(M \otimes_A B)[\mathfrak{mult}] = \{x \in M \otimes_A B : y \cdot x = 0 \ \forall y \in \mathfrak{mult}\}$$
$$(N \otimes_A B)[\mathfrak{mult}] = \{x \in N \otimes_A B : y \cdot x = 0 \ \forall y \in \mathfrak{mult}\}.$$

Then the ideal

$$\mathfrak{L}_\beta := \mathrm{image}\,(\beta_B : (M \otimes_A B)[\mathfrak{mult}] \times (N \otimes_A B)[\mathfrak{mult}] \to B)$$

is a principal ideal in B.

Proof. We claim first that for any B-module M which is finite flat over A, we have an isomorphism $M^\vee \otimes_A B[\mathfrak{mult}] \simeq \mathrm{Hom}_B(M, B)$, where $M^\vee = \mathrm{Hom}_A(M, A)$. Notice that $M^\vee$ also admits a B-module structure by $b\psi : m \mapsto \psi(bm)$ for all $b \in B$, $\psi \in M^\vee$ and $m \in M$. We have a natural isomorphism

$$M^\vee \otimes_A B = \mathrm{Hom}_A(M, A) \otimes_A B \to \mathrm{Hom}_A(M, B), \quad \psi \otimes b \mapsto (m \mapsto \psi(m)b).$$

Since $\mathfrak{mult} = \sum_{b\in B}(b\otimes 1 - 1\otimes b)B\otimes_A B$, thus

$$\begin{aligned}\psi\otimes b \in M^\vee \otimes_A B[\mathfrak{mult}] &\Leftrightarrow (b'\otimes 1 - 1\otimes b')\psi\otimes b = 0\ \forall b'\in B\\ &\Leftrightarrow b'\psi\otimes b = \psi\otimes bb'\ \forall b'\in B\\ &\Leftrightarrow \psi(b'm)b = \psi(m)bb'\ \forall b'\in B, m\in M\\ &\Leftrightarrow (m\mapsto \psi(m)b)\in \mathrm{Hom}_B(M,B).\end{aligned}$$

Apply the claim in our situation, we have isomorphisms of B-modules

$$(M\otimes_A B)[\mathfrak{mult}] \simeq (B^\vee\otimes_A B)[\mathfrak{mult}] \simeq \mathrm{Hom}_B(B,B)\simeq B$$

and same for $(N\otimes_A B)[\mathfrak{mult}]$. Hence, let $\widetilde{m}$ and $\widetilde{n}$ be generators of $(M\otimes_A B)[\mathfrak{mult}]$ and $(N\otimes_A B)[\mathfrak{mult}]$ respectively as B-modules. Then $\mathfrak{L}_\beta = \beta_B(\widetilde{m},\widetilde{n})B$. $\square$

Proposition 7.2.5 ([Bel21, Corollary VIII.1.13])**.** *Suppose B is Gorenstein over A, i.e., $B^\vee$ is flat of constant rank 1 over B, and M,N are B-modules which are finite flat over A and flat of rank 1 over B. Assume there is an A-linear pairing $\beta : M\times N\to A$ which is B-equivariant. We retain the notation β_B and $\mathfrak{L}_\beta$ as in Lemma 7.2.4. Then*

(i) Both ideals $\mathfrak{d}(B/A)$ and $\mathfrak{L}_\beta$ are locally principal. Moreover, there exists $b_0\in B$ such that $\mathfrak{L}_\beta = b_0\,\mathfrak{d}(B/A)$.

(ii) We have $\mathfrak{L}_\beta = \mathfrak{d}(B/A)$ if and only if β is non-degenerate.

Proof. We are in a special case of Lemma 7.2.4 that we can identify (locally) $M\simeq N\simeq B^\vee\simeq B$ and hence we know $\mathfrak{L}_\beta$ is principal. Moreover, the identification $B\otimes_A B[\mathfrak{mult}]\simeq \mathrm{Hom}_B(B,B)\simeq B$ implies that $\mathfrak{d}(B/A)$ is also principal.

Observe that we can identify $\beta : M\times N\to A$ as a linear morphism $B^\vee\otimes_A B^\vee\to A$. Hence by duality, we identify β with an element $b\in B\otimes_A B$. We claim that $\mathfrak{L}_\beta = \mathrm{mult}(b)B$. As we are working locally, we assume $b_1,...,b_n$ is a basis of B over A, then $b_1^\vee,...,b_n^\vee$ is a basis of $B^\vee$ over A. Observe that $b^\square := \sum_i b_i^\vee\otimes b_i$ is a generator of $B^\vee\otimes_A B[\mathfrak{mult}]\simeq \mathrm{Hom}_B(B,B)$ as it maps to the identity in $\mathrm{Hom}_B(B,B)$. Hence by definition

$$\mathfrak{L}_\beta = \beta_B(b^\square,b^\square)B = \left(\sum_{i,j}\beta(b_i^\vee,b_j^\vee)b_ib_j\right)B.$$

On the other hand, by the above construction, we see that $b=\sum_{i,j}\beta(b_i^\vee,b_j^\vee)b_i\otimes b_j$ with $\mathrm{mult}(b)=\sum_{i,j}\beta(b_i^\vee,b_j^\vee)b_ib_j$.

Let $\widetilde{b^\square} = \sum_i b_i\otimes b_i$, then it is a generator of $B\otimes_A B[\mathfrak{mult}]\simeq B$. Thus, there exists $b_0\in B$ such that $b_0\widetilde{b^\square} = b$. We conclude that

$$\mathfrak{L}_\beta = \mathrm{mult}(b)B = \mathrm{mult}(b_0\widetilde{b^\square})B = b_0\,\mathrm{mult}(\widetilde{b^\square})B = b_0\,\mathfrak{d}(B/A).$$

Finally, we have

$$\begin{aligned}\mathfrak{L}_\beta = \mathfrak{d}(B/A) &\Leftrightarrow b_0 \in B^\times \\ &\Leftrightarrow \beta(b_i^\vee, b_j^\vee) = \begin{cases} b_0 \in B^\times & i = j \\ 0 & i \neq j \end{cases} \\ &\Leftrightarrow \beta \text{ is non-degenerate.}\end{aligned}$$

□

7.3 The ramification locus of the cuspidal eigenvariety

7.3.1. Recall the open cover $\mathrm{Cov}(\mathcal{S}^{\mathrm{oc}})$ for $\mathcal{S}^{\mathrm{oc}}$, consisting of open subsets of the form $\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h}$ with $(\mathcal{U}, h)$ being slope-adapted. We denote by $\mathcal{E}^{\mathrm{oc}}_{0,\mathcal{U},h}$ the inverse image of $\mathcal{S}^{\mathrm{oc}}_{\mathcal{U},h}$ in $\mathcal{E}^{\mathrm{oc}}_0$. We adapt the definitions of 'clean neighbourhoods' and 'good points' in [Bel21] in our situation.

Definition 7.3.2. *(i) Let $\boldsymbol{x} \in \mathcal{E}^{\mathrm{oc}}_0$ and $\mathcal{V} = \mathrm{Spa}(R_{\mathcal{V}}, R^+_{\mathcal{V}})$ be an open affinoid neighbourhood of $\boldsymbol{x}$. We say $\mathcal{V}$ is a* ***clean neighbourhood*** *of $\boldsymbol{x}$ if it satisfies the following properties:*

- $\mathrm{wt}(\mathcal{V}) = \mathcal{Y} = \mathrm{Spa}(R_{\mathcal{Y}}, R^+_{\mathcal{Y}}) \subset \mathcal{W}$ *is an open affinoid subset of $\mathcal{W}$ and there exists a slope-adapted pair $(\mathcal{U}, h)$ such that $\mathcal{V}$ is the connected component of $\boldsymbol{x}$ in $\mathcal{E}^{\mathrm{oc}}_{0,\mathcal{U},h}$;*
- *$\boldsymbol{x}$ is the only point of $\mathcal{V}$ sitting above* $\mathrm{wt}(\boldsymbol{x})$*;*
- *the map* $\mathrm{wt} : \mathcal{V} \to \mathcal{Y}$ *is flat and is moreover étale except perhaps at $\boldsymbol{x}$.*

In this case, there exists an idempotent $\eta = \eta_{\mathcal{V}} \in \mathbb{T}^{\mathrm{oc}}_{\mathcal{U},h}$ such that $\mathcal{V}$ is defined by the equation $\eta = 1$ and the module $\eta H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}}$ is a direct summand of $H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}}$.

(ii) A point $\boldsymbol{x} \in \mathcal{E}_0$ is said to be a ***good point*** *if it admits a sufficiently small clean neighbourhood $\mathcal{V}$ with $\mathrm{wt}(\mathcal{V}) = \mathcal{Y}$ such that the modules $\eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}}$ and $(\eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}})^\vee$ are free of rank one over $R_{\mathcal{V}}$, where the dual is taken to be an $R_{\mathcal{Y}}$-dual.*

Remark 7.3.3. We remark the following:

- In the GL_2 case, the eigencurve is locally finite flat over the weight space ([Bel21, §VI.1.4]) and so the author of *op. cit.* can consequently deduce that the collection of clean neighbourhoods of points on the eigencurve gives a open cover of the eigencurve. In our case, the Fredholm hypersurface $\mathcal{S}^{\mathrm{oc}}$ is finite flat over $\mathcal{W}$ by [AIP18, Theorem B.1]. However, we don't know if $\mathcal{E}^{\mathrm{oc}}_0$ is flat over $\mathcal{S}^{\mathrm{oc}}$. Therefore, instead of considering $\mathcal{E}^{\mathrm{oc}}_0$, we consider $\mathcal{E}^{\mathrm{oc,fl}}_0 \subset \mathcal{E}^{\mathrm{oc}}_0$ the flat locus over $\mathcal{W}$, which is open over $\mathcal{W}$, and let $\mathrm{Cov}_{\mathrm{cl}}(\mathcal{E}^{\mathrm{oc,fl}}_0)$ be the open cover of clean neighbourhoods.
- In the definition of good points, we see immediately that $R_{\mathcal{V}}$ is Gorenstein over $R_{\mathcal{Y}}$.

7.3.4. Following [Bel21, §VIII. 4], we study the *adjoint L-ideal* and define the *p-adic adjoint L-function* here. Let $\boldsymbol{x} \in \mathcal{E}^{\mathrm{oc,fl}}_0$ and $\mathcal{V}$ be a clean neighbourhood of $\boldsymbol{x}$ with weight $\mathrm{wt}(\mathcal{V}) = \mathcal{Y}$. There is a natural multiplication map

$$\mathrm{mult} : R_{\mathcal{V}} \widehat{\otimes}_{R_{\mathcal{Y}}} R_{\mathcal{V}} \to R_{\mathcal{V}}, \quad b \otimes b' \mapsto bb'.$$

Let $\mathfrak{mult} := \ker \mathrm{mult}$ and define

$$M\widehat{\otimes}_{R_{\mathcal{Y}}} R_{\mathcal{V}}[\mathfrak{mult}] := \{m \in M\widehat{\otimes}_{R_{\mathcal{Y}}} R_{\mathcal{V}} : \mathfrak{mult} \cdot m = 0\}$$

for any Banach $R_{\mathcal{V}}$-module M.

Definition 7.3.5. *Keep the notations above. The* ***adjoint*** L***-ideal*** *of* $\mathcal{V}$ *is defined to be*

$$\mathscr{L}^{\mathrm{adj}}(\mathcal{V}) := \mathrm{image}\left([\cdot,\cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}}\widehat{\otimes}_{R_{\mathcal{Y}}} R_{\mathcal{V}}[\mathfrak{mult}] \times \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}}\widehat{\otimes}_{R_{\mathcal{Y}}} R_{\mathcal{V}}[\mathfrak{mult}] \to R_{\mathcal{V}}\right).$$

Remark 7.3.6. Since the clean neighbourhoods cover $\mathcal{E}_0^{\mathrm{oc,fl}}$, the collection $\{\mathscr{L}^{\mathrm{adj}}(\mathcal{V}) : \mathcal{V} \in \mathrm{Cov}_{\mathrm{cl}}(\mathcal{E}_0^{\mathrm{oc,fl}})\}$ glues to a coherent sheaf $\mathscr{L}^{\mathrm{adj}}$ on $\mathcal{E}_0^{\mathrm{oc,fl}}$.

Proposition 7.3.7. *Let $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ be a good point. Then there exists a sufficiently small clean neighbourhood $\mathcal{V}$ of $\boldsymbol{x}$ with $\mathrm{wt}(\mathcal{V}) = \mathcal{Y}$ such that $\mathscr{L}^{\mathrm{adj}}(\mathcal{V})$ is a principal ideal in $R_{\mathcal{V}}$.*

Proof. The assertion follows from Lemma 7.2.4. □

Definition 7.3.8. *Let $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ be a good point and $\mathcal{V}$ be a sufficiently small clean neighbourhood such that $\mathscr{L}^{\mathrm{adj}}(\mathcal{V})$ is principal. We define the* ***adjoint*** p***-adic*** L***-function*** *on $\mathcal{V}$ to be $L^{\mathrm{adj}}_{\mathcal{V}} \in R_{\mathcal{V}}$ such that $L^{\mathrm{adj}}_{\mathcal{V}}$ generates $\mathscr{L}^{\mathrm{adj}}(\mathcal{V})$. The value of $L^{\mathrm{adj}}_{\mathcal{V}}$ at $\boldsymbol{x}$ is denoted by $L^{\mathrm{adj}}(\boldsymbol{x})$ as it doesn't depend on the clean neighbourhood.*

7.3.9. Let $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ be a good point and let $\mathcal{V}$ be a sufficiently small clean neighbourhood of $\boldsymbol{x}$ such that $L^{\mathrm{adj}}_{\mathcal{V}}$ is defined. Let $(\mathcal{U}, h)$ be the slope datum that defines $\mathcal{V}$ and let $\mathrm{wt}(\mathcal{V}) = \mathcal{Y}$. Corollary 7.1.9 yields an $R_{\mathcal{V}}$-equivariant pairing

$$[\cdot,\cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \times \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \to R_{\mathcal{Y}}.$$

Together with the definition of good points, we are in the situation of Proposition 7.2.5.

Theorem 7.3.10. *Let $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ be a good point and let $\kappa = \mathrm{wt}(\boldsymbol{x})$. Suppose the pairing*

$$[\cdot,\cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \times \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \to R_{\mathcal{Y}}$$

is non-degenerate at $\mathrm{wt}(\boldsymbol{x})$, then

$$L^{\mathrm{adj}}(\boldsymbol{x}) = 0 \text{ if and only if } \mathrm{wt} \text{ is ramified at } \boldsymbol{x}.$$

Proof. Let $\mathcal{V}$ be a sufficiently small clean neighbourhood of $\boldsymbol{x}$ which is defined by the slope datum $(\mathcal{U}, h)$ and $\mathrm{wt}(\mathcal{V}) = \mathcal{Y}$. Since the pairing

$$[\cdot,\cdot]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \times \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \to R_{\mathcal{Y}}$$

is assumed to be non-degenerate, then by Proposition 7.2.5, $\mathscr{L}^{\mathrm{adj}}(\mathcal{V}) = \mathfrak{d}(R_{\mathcal{V}}/R_{\mathcal{Y}})$. Thus,

$$L^{\mathrm{adj}}(\boldsymbol{x}) = 0 \Leftrightarrow L^{\mathrm{adj}} \in \mathrm{supp}\,\boldsymbol{x} \Leftrightarrow \mathfrak{d}(R_{\mathcal{V}}/R_{\mathcal{Y}}) \subset \mathrm{supp}\,\boldsymbol{x} \Leftrightarrow \mathrm{wt} \text{ is ramified at } \boldsymbol{x},$$

where the last equivalence is due to Auslander–Buchsbaum's theorem. □

Theorem 7.3.11. *Let $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{fl}}$ be a good and smooth point and let $\kappa = \mathrm{wt}(\boldsymbol{x})$. Assume again that the pairing*

$$[\,\cdot\,,\cdot\,]_{\kappa_{\mathcal{U}}} : \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \times \eta_{\mathcal{V}} H^{\mathrm{tol},\leq h}_{\mathrm{par},\kappa_{\mathcal{U}}} \to R_{\mathcal{Y}}$$

is non-degenerate at $\mathrm{wt}(\boldsymbol{x})$. *Let $R_{\mathrm{wt}(\boldsymbol{x})}$ and $R_{\boldsymbol{x}}$ be the local rings at $\mathrm{wt}(\boldsymbol{x})$ and $\boldsymbol{x}$ respectively and denote by $\mathfrak{m}_{\mathrm{wt}(\boldsymbol{x})}$, $\mathfrak{m}_{\boldsymbol{x}}$ their maximal ideals respectively. Define*

$$e(\boldsymbol{x}) := \max\{e \in \mathbf{Z}_{\geq 0} : \mathfrak{d}(R_{\boldsymbol{x}}/R_\kappa) \subset \mathfrak{m}_{\boldsymbol{x}}^e\}.$$

Then, we have

$$\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} = e(\boldsymbol{x}).$$

Proof. Note that

$$\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} := \max\{e \in \mathbf{Z}_{\geq 0} : L^{\mathrm{adj}}(\boldsymbol{x}) \in \mathfrak{m}_{\boldsymbol{x}}^e\}.$$

In our situation, we see that

$$\mathfrak{m}_{\boldsymbol{x}}^{e(\boldsymbol{x})} \supset \mathfrak{d}(R_{\boldsymbol{x}}/R_\kappa) = L^{\mathrm{adj}}(\boldsymbol{x}) R_{\boldsymbol{x}} \subset \mathfrak{m}_{\boldsymbol{x}}^{\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}}}.$$

As the inclusions on both sides satisfy the same condition, the exponents coincide. $\square$

Remark 7.3.12. We remark that the above two theorems have their roots in the GL_2 case. Theorem 7.3.10 is an analogue of [Bel21, Theorem VIII.4.7] while Theorem 7.3.11 is inspired by [*op. cit.*, Theorem VIII.4.8(i)].

7.3.13. Our next task is to justify that there exists some $\boldsymbol{x} \in \mathcal{E}_0^{\mathrm{oc,fl}}$ such that the pairing is non-degenerate at $\mathrm{wt}(\boldsymbol{x})$. To this end, recall that we have a $\mathrm{Iw}^+_{\mathrm{GSp}_{2g}}$-equivariant surjection

$$D_k^{\dagger}(\mathbf{T}_0, \mathbf{Q}_p) \to \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$$

for any dominant weight $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{\geq 0}$. We can then descend the pairing $[\,\cdot\,,\cdot\,]_k^{\circ}$ to $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ by the same formula

$$\begin{aligned}[\,\cdot\,,\cdot\,]_k^{\circ} : \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} \times \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} &\to \mathbf{Q}_p, \\ (\mu_1, \mu_2) &\mapsto \int_{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}} e_k^{\mathrm{hst}} \left({}^{\mathtt{t}}\boldsymbol{\gamma}_2 \begin{pmatrix} \mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g \end{pmatrix} \boldsymbol{\gamma}_1 \right) \quad d\mu_1(\boldsymbol{\gamma}_1) d\mu_2(\boldsymbol{\gamma}_2).\end{aligned}$$

Proposition 7.3.14. *Let $k \in \mathbf{Z}^g_{>0}$ be a dominant weight. Then the pairing $[\,\cdot\,,\cdot\,]_k^{\circ}$ on $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ is non-degenerate.*

Proof. Recall the symplectic pairing $\langle\,\cdot\,,\cdot\,\rangle_k$ on $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ from Remark 7.1.2

$$\langle\,\mu_1, \mu_2\,\rangle_k = \int_{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathrm{GSp}_{2g}(\mathbf{Q}_p)} e_k^{\mathrm{hst}} \left({}^{\mathtt{t}}\boldsymbol{\gamma}_2 \begin{pmatrix} & -\breve{\mathbb{1}}_g \\ \breve{\mathbb{1}}_g & \end{pmatrix} \boldsymbol{\gamma}_1 \right) \quad d\mu_1(\boldsymbol{\gamma}_1) d\mu_2(\boldsymbol{\gamma}_2).$$

Since the symplectic pairing $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathbf{V}$ is non-degenerate, we know that $\langle\,\cdot\,,\cdot\,\rangle_k$ is non-degenerate.

Define

$$\begin{aligned}\langle\,\cdot,\cdot\,\rangle'_k : \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}\times\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k} &\to \mathbf{Q}_p,\\ (\mu_1,\mu_2)&\mapsto \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in U^{\mathrm{opp}}_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)} e^{\mathrm{hst}}_k\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\gamma_2).\end{aligned}$$

Then $\langle\,\cdot,\cdot\,\rangle'_k$ is a non-degenerate pairing. Indeed, we have

$$\begin{aligned}\langle\,\mu_1,\mu_2\,\rangle_k &= \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}(\mathbf{Q}_p)} e^{\mathrm{hst}}_k\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2)\\ &= \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in\mathrm{GSp}_{2g}(\mathbf{Q}_p)} k(\beta_1)k(\beta_2)e^{\mathrm{hst}}_k\left({}^{\mathtt{t}}\boldsymbol{\gamma}'_2\begin{pmatrix} & -\breve{\mathbb{1}}_g\\ \breve{\mathbb{1}}_g & \end{pmatrix}\boldsymbol{\gamma}'_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2),\end{aligned}$$

where $\boldsymbol{\gamma}_i=\boldsymbol{\gamma}'_i\boldsymbol{\beta}$ with $\boldsymbol{\gamma}'_i\in U^{\mathrm{opp}}_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)$ and $\boldsymbol{\beta}_i\in B_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)$ for $i=1,2$. As k is non-zero on $B_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)$, we see that $\langle\,\mu_1,\mu_2\,\rangle_k=0$ if and only if $\langle\,\mu_1,\mu_2\,\rangle'_k=0$.

Now, let $[\,\cdot,\cdot\,]'_k$ be the pairing on $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ defined by

$$\begin{aligned}[\,\mu_1,\mu_2\,]'_k &:= \langle\,\mu_1,\mathbf{w}_p\cdot\mu_2\,\rangle'_k\\ &= \int_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2\in U^{\mathrm{opp}}_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)} e^{\mathrm{hst}}_k\left({}^{\mathtt{t}}\boldsymbol{\gamma}_2\begin{pmatrix}\mathbb{1}_g & \\ & p^{-1}\,\mathbb{1}_g\end{pmatrix}\boldsymbol{\gamma}_1\right)\quad d\mu_1(\boldsymbol{\gamma}_1)d\mu_2(\boldsymbol{\gamma}_2).\end{aligned}$$

Then, $[\,\cdot,\cdot\,]'_k$ is again a non-degenerate pairing since $\mathbf{w}_p\in\mathrm{GSp}_{2g}(\mathbf{Q}_p)$. Recall that $U^{\mathrm{opp}}_{\mathrm{GSp}_{2g},1}\simeq\mathbf{Z}_p^{d_0}$, for some $d_0\in\mathbf{Z}_{>0}$, as p-adic manifolds, thus $U^{\mathrm{opp}}_{\mathrm{GSp}_{2g}}(\mathbf{Q}_p)\simeq\mathbf{Q}_p^{d_0}$. However, $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ is defined algebraically and $\mathbf{Z}_p^{d_0}\subset\mathbf{Q}_p^{d_0}$ is Zariski dense, thus the non-degeneracy of $[\,\cdot,\cdot\,]'_k$ implies the non-degeneracy of $[\,\cdot,\cdot\,]^{\circ}_k$. ☐

Corollary 7.3.15. *Let $\kappa=k\in\mathbf{Z}^g_{>0}$ be a dominant algebraic weight. Then the pairing*

$$[\,\cdot,\cdot\,]_k : H^{\mathrm{tol},\leq h}_{\mathrm{par},k}\times H^{\mathrm{tol},\leq h}_{\mathrm{par},k}\to\mathbf{Q}_p$$

is non-degenerate when $h<h_k$, where h_k is as defined in Theorem 5.2.8.

Proof. This is an easy consequence of Proposition 7.3.14 and Theorem 5.2.8. ☐

Corollary 7.3.16. *Suppose $\boldsymbol{x}\in\mathcal{E}_0^{\mathrm{oc,fl}}$ is a good classical point, i.e., $\boldsymbol{x}$ satisfies the following conditions*

- *$\boldsymbol{x}$ is a good point;*
- *$\mathrm{wt}(\boldsymbol{x})=k\in\mathbf{Z}^g_{\geq 0}$ is a dominant algebraic weight; and*
- *there is a slope-adapted pair $(\mathcal{U},h)$ such that $\boldsymbol{x}\in\mathcal{U}$ and $h<h_k$.*

Then

(i) The adjoint p-adic L-function L^{adj} *vanishes at* $\boldsymbol{x}$ *if and only if the weight map* $\mathrm{wt} : \mathcal{E}_0^{\mathrm{oc}} \to \mathcal{W}$ *is ramified at* $\boldsymbol{x}$*.*

(ii) If $\boldsymbol{x}$ *is furthermore a smooth point of* $\mathcal{E}_0^{\mathrm{oc,fl}}$*, let* $e(\boldsymbol{x})$ *be as defined in Theorem 7.3.11, then we have* $\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} = e(\boldsymbol{x})$*.*

Proof. This is an immediate consequence of Theorem 7.3.10, Theorem 7.3.11 and Corollary 7.3.15. $\square$

Chapter 8

Families of Galois representations and adjoint Bloch–Kato Selmer groups

The final chapter of this thesis is dedicated to the answer to the second half of Question 1.3.2 (ii). To this end, we will first review the study of families of Galois representations in §8.1 by following [BC09] and discuss the Galois representations for GSp_{2g} in §8.2. Inspired by the strategy presented in [BC09], we discuss in §8.4 some local and global Galois deformations that we are interested in. Our main results in the study of the adjoint Bloch–Kato Selmer groups are presented in §8.5.

8.1 Recapitulations of families of Galois representations

8.1.1. The purpose of this section is to recall several terminologies for studying families of Galois representations. Most of the materials presenting in this subsection are taken from [BC09].

8.1.2 (Determinants)**.** We briefly recall the notion of 'determinants' from [Che14] and refer the readers to *loc. cit.* for more detailed discussions. We remark in the beginning that the notion of determinants are used to strengthen the notion of *pseudocharacters* first introduced by R. Taylor in [Tay91] and studied by other mathematicians. We also remark that determinants are equivalent to pseudocharacters in characteristic 0.

Definition 8.1.3. *Let A be a commutative ring and R be an A-algebra (not necessarily commutative).*

*(i) For any A-module M, one can view M as a functor from the category of commutative A-algebras to the category of sets, sending B to $M \otimes_A B$. Let M, N be two A-modules. Then an A-**polynomial law** between M and N is a natural transformation*

$$M \otimes_A B \to N \otimes_A B$$

on the category of commutative A-algebras.

(ii) *Let $P : M \to N$ be an A-polynomial law and $d \in \mathbf{Z}_{>0}$. We say P is **homogeneous of dimension** d if for any commutative A-algebra B, any $b \in B$ and any $x \in M \otimes_A B$, we have $P(bx) = b^d P(x)$.*

(iii) *Let $P : R \to A$ be an A-polynomial law. We say P is **multiplicative** if, for any commutative A-algebra B, $P(1) = 1$ and $P(xy) = P(x)P(y)$ for any $x, y \in R \otimes_A B$.*

(iv) *For $d \in \mathbf{Z}_{>0}$, a d-**dimensional** A-**valued determinant on** R is a multiplicative A-polynomial law $D : R \to A$ which is homogeneous of dimension d.*

Example 8.1.4. Let G be a group and A be any ring. Let $\rho : G \to \mathrm{GL}_d(A)$ be a representation of dimension d. Then

$$D : A[G] \to A, \quad G \ni \sigma \mapsto \det \rho(\sigma)$$

is an A-valued determinant of dimension d on $A[G]$. We also say that D is an A-valued determinant of dimension d on G.

Theorem 8.1.5 ([Che14, Theorem A & Theorem B])**.** *Let G be a group.*

(i) *Let k be an algebraically closed field and let $D : k[G] \to k$ be a determinant of dimension d. Then, there exists a unique (up to isomorphism) semisimple representation $\rho : G \to \mathrm{GL}_d(k)$ such that for any $\sigma \in G$, we have*

$$\det(1 + Y\rho(\sigma)) = D(1 + Y\sigma) \in k[Y].$$

In particular, $\det \rho = D$.

(ii) *Let A be an henselian local ring with algebraically closed residue field k, $D : A[G] \to A$ be a d-dimensional determinant and let ρ be the semisimple representation attached to $D \otimes_A k$ in (i). Suppose ρ is irreducible, then there exists a unique (up to isomorphism) representation $\widetilde{\rho} : G \to \mathrm{GL}_d(A)$ such that*

$$\det(1 + Y\widetilde{\rho}(\sigma)) = D(1 + Y\sigma) \in A[Y]$$

for any $\sigma \in G$.

8.1.6 (Refinements of crystalline representations)**.** We recall the notion of 'refinements' of crystalline representations from [BC09, §2.4]. Let L be a finite extension of $\mathbf{Q}_p$ and let V be an n-dimensional L-representation of $\mathrm{Gal}_{\mathbf{Q}_p}$. Assume that V is crystalline. Also assume that the crystalline Frobenius $\varphi = \varphi_{\mathrm{cris}}$ acting on $\mathbf{D}_{\mathrm{cris}}(V)$ has all eigenvalues living in $L^\times$.

Definition 8.1.7 ([BC09, §2.4.1])**.** *A **refinement** of V is the data of a full φ-stable L-filtration*

$$\mathbb{F}_\bullet : 0 = \mathbb{F}_0 \subsetneq \mathbb{F}_1 \subsetneq \cdots \subsetneq \mathbb{F}_{n-1} \subsetneq \mathbb{F}_n = \mathbf{D}_{\mathrm{cris}}(V).$$

8.1.8. Suppose $\mathbb{F}_\bullet$ is a refinement of V, one sees immediately that it determines two orderings:

(Ref 1) An ordering $(\varphi_1, ..., \varphi_n)$ of the eigenvalues of φ by the formula

$$\det(X - \varphi|_{\mathbb{F}_i}) = \prod_{j=1}^{i}(X - \varphi_j).$$

Notice that if the φ_j's are all distinct, then such an ordering of eigenvalues of φ conversely determines the refinement.

(Ref 2) An ordering $(a_1, ..., a_n)$ of Hodge–Tate weights of V. More precisely, the jumps of the Hodge filtration of $\mathbf{D}_{\mathrm{cris}}(V)$ induced on $\mathbb{F}_i$ are $(a_1, ..., a_i)$.

Definition 8.1.9 ([BC09, Definition 2.4.5]). *Suppose the Hodge–Tate weights of V are all distinct $a_1 < \cdots < a_n$. Let $\mathbb{F}_\bullet$ be a refinement of V and let $\mathrm{Fil}^\bullet \mathbf{D}_{\mathrm{cris}}(V)$ be the Hodge filtration of $\mathbf{D}_{\mathrm{cris}}(V)$. We say $\mathbb{F}_\bullet$ is **non-critical** if, for all $1 \le i \le n$, we have*

$$\mathbf{D}_{\mathrm{cris}}(V) = \mathbb{F}_i \oplus \mathrm{Fil}^{a_i+1} \mathbf{D}_{\mathrm{cris}}(V).$$

8.1.10. Recall the Robba ring

$$\mathcal{R}_L := \left\{ f(Y) = \sum_{i \in \mathbf{Z}} t_n(Y-1)^n \in L[\![Y]\!] : \begin{array}{l} f(X) \text{ converges on some annulus of } \mathbf{C}_p \\ \text{of the form } r(f) \le |Y-1| \le 1 \end{array} \right\}.$$

Here the norm $|\cdot|$ is the p-adic norm on $\mathbf{C}_p$ with the normalisation $|p| = 1/p$. Let $\Gamma = \mathbf{Z}_p^\times$. The theory of (φ, Γ)-modules yields an equivalence of categories between the category finite-dimensional L-representations of $\mathrm{Gal}_{\mathbf{Q}_p}$ and the category of étale (φ, Γ)-modules over $\mathcal{R}_L$ (see, for example, [BC09, §2.2]). In particular, we have a (φ, Γ)-module $\mathbf{D}_{\mathrm{rig}}(V)$ over $\mathcal{R}_L$ associated with V.

Proposition 8.1.11 ([BC09, Proposition 2.4.1 & Proposition 2.4.7]). *Let $\mathbb{F}_\bullet$ be a refinement of V.*

(i) Then $\mathbb{F}_\bullet$ determines a unique filtration $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(V)$ of length n, i.e., a triangulation of $\mathbf{D}_{\mathrm{rig}}(V)$. Consequently, $\mathbb{F}_\bullet$ determines a unique collection of continuous characters $\delta_i : \mathbf{Q}_p^\times \to L^\times$ via the isomorphism

$$\mathrm{Fil}_i \mathbf{D}_{\mathrm{rig}}(V) / \mathrm{Fil}_{i-1} \mathbf{D}_{\mathrm{rig}}(V) \simeq \mathcal{R}_L(\delta_i)$$

*given by [BC09, Proposition 2.3.1]. Here, the tuple $\delta = (\delta_1, ..., \delta_n)$ is called the **parameter** of V.*

(ii) Moreover, suppose the Hodge–Tate weight of V are all distinct $h_1 < \cdots < h_n$. Then, $\mathbb{F}_\bullet$ is non-critical if and only if the sequence Hodge–Tate weights $(a_1, ..., a_n)$ associated with $\mathbb{F}_\bullet$ in (Ref 2) is increasing, i.e., $a_i = h_i$ for all $i = 1, ..., n$.

Remark 8.1.12. The theory of (φ, Γ)-modules can be worked out for local artinian $\mathbf{Q}_p$-algebras (see, for example, [BC09, §2]). Thus, it makes sense to consider the following deformation functor. Let $\mathbf{A}\mathbf{R}$ be the category of local artinian $\mathbf{Q}_p$-algebras whose residue field is isomorphic to L. Then, we define the ***(local) trianguline deformation*** functor

$$\mathscr{D}_{V,\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(V)} : \mathbf{A}\mathbf{R} \to \mathbf{S}\mathbf{ETS},$$

$$A \mapsto \left\{ (V_A, \rho_A, \mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(V_A)) : \begin{array}{l} V_A \simeq A^n \\ \rho_A : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}(V_A) \simeq \mathrm{GL}_n(A) \\ \qquad \text{s.t. } \rho_A \otimes_A L \simeq V \\ \mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(V_A) \otimes_{\mathcal{R}_A} \mathcal{R}_L \simeq \mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(V) \end{array} \right\} / \simeq$$

We will also denote the above deformation functor by $\mathscr{D}_{V,\mathbb{F}_\bullet}$ as the triangulation $\mathrm{Fil}_\bullet D_{\mathrm{rig}}(V)$ is uniquely determined by $\mathbb{F}_\bullet$. In fact, we will confuse the refinement $\mathbb{F}_\bullet$ with the triangulation $\mathrm{Fil}_\bullet D_{\mathrm{rig}}(V)$ in what follows.

8.1.13 (Families of representations)**.** Here, we collect some terminologies introduced in [BC09, §5] that will be needed in the later subsections. Note that the terminology of *psuedocharacters* is used in *op. cit.* since the notion of *determinants* was not yet discovered. In what follows, we shall adapt everything with the notion of determinants.

Let G be a topological group with a continuous group homomorphism $\mathrm{Gal}_{\mathbf{Q}_p} \to G$, *e.g.*, $G = \mathrm{Gal}_{\mathbf{Q}}$ with the natural inclusion $\mathrm{Gal}_{\mathbf{Q}_p} \hookrightarrow \mathrm{Gal}_{\mathbf{Q}}$. Therefore, any (continuous) representation ρ of G induces a (continuous) representation of $\mathrm{Gal}_{\mathbf{Q}_p}$, denoted by $\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}}$.

By a ***family of representations***, we mean a datum $(\mathcal{E}, D)$, where $\mathcal{E}$ is a reduced separated rigid analytic variety (viewed as an adic space) over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and a continuous determinant $D : \mathscr{O}_{\mathcal{E}}(\mathcal{E})[G] \to \mathscr{O}_{\mathcal{E}}(\mathcal{E})$. The dimension of this family is understood to be the dimension of the determinant D, denoted by n. For any $\boldsymbol{x} \in \mathcal{E}$, let $k_{\boldsymbol{x}}$ be the residue field of $\boldsymbol{x}$, then we have the specialisation

$$D|_{\boldsymbol{x}} : G \xrightarrow{D} \mathscr{O}_{\mathcal{E}}(\mathcal{E}) \to k_{\boldsymbol{x}}. \tag{8.1}$$

Applying Theorem 8.1.5 (i), we see that $D|_{\boldsymbol{x}}$ is nothing but the determinant of a (unique up to isomorphism) continuous semisimple representation $\rho_{\boldsymbol{x}} : G \to \mathrm{GL}_n(\overline{k_{\boldsymbol{x}}})$.

Definition 8.1.14 ([BC09, Definition 4.2.3])**.** *A **refined family of representations** of dimension n is a datum $(\mathcal{E}, D, \mathcal{Q}, \{\alpha_i : i = 1, ..., n\}, \{F_i : i = 1, ..., n\})$, where*

(a) $(\mathcal{E}, D)$ is a family of representations of dimension n,

(b) $\mathcal{Q} \subset \mathcal{E}$ is a Zariski dense subset,

(c) $\alpha_i \in \mathscr{O}_{\mathcal{E}}(\mathcal{E})$ is an analytic function for $i = 1, ..., n$,

(d) $F_i \in \mathscr{O}_{\mathcal{E}}(\mathcal{E})$ is an analytic function for $i = 1, ..., n$,

such that

(i) For every $\boldsymbol{x} \in \mathcal{E}$, the Hodge–Tate–Sen weights[1] for $\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ are $\alpha_1(\boldsymbol{x})$, ..., $\alpha_n(\boldsymbol{x})$.

[1] Here, the *Hodge–Tate–Sen weight* is defined to be the roots of the Sen polynomial (see, for example, [Liu15, Definition 2.24]).

(ii) For each $\boldsymbol{y} \in \mathcal{Q}$, the representation $\rho_{\boldsymbol{y}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is crystalline (so that $\alpha_i(\boldsymbol{y})$'s are integers) and $\alpha_1(\boldsymbol{y}) < \cdots < \alpha_n(\boldsymbol{y})$.

(iii) For each $\boldsymbol{y} \in \mathcal{Q}$, the eigenvalues of the crystalline Frobenius φ on $\mathbf{D}_{\mathrm{cris}}(\rho_{\boldsymbol{y}}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ are distinct and are $(p^{\alpha_1(\boldsymbol{y})}F_1(\boldsymbol{y}), ..., p^{\alpha_n(\boldsymbol{y})}F_n(\boldsymbol{y}))$.

(iv) For any $C \in \mathbf{Z}_{>0}$, define

$$\mathcal{Q}_C := \left\{ y \in \mathcal{Q} : \begin{array}{l} \alpha_{i+1}(\boldsymbol{y}) - \alpha_i(\boldsymbol{y}) > C(\alpha_i(\boldsymbol{y}) - \alpha_{i-1}(\boldsymbol{y})) \text{ for } i = 2, ..., n-1 \\ \alpha_2(\boldsymbol{y}) - \alpha_1(\boldsymbol{y}) > C \end{array} \right\}.$$

We request that $\mathcal{Q}_C$ accumulates at any point of $\mathcal{Q}$ for any C. In other words, for any $\boldsymbol{y} \in \mathcal{Q}$ and any $C \in \mathbf{Z}_{>0}$, there is a basis of affinoid neighbourhoods $\mathcal{U}$ of $\boldsymbol{x}$ such that $\mathcal{U} \cap \mathcal{Q}_C$ is Zariski dense in $\mathcal{U}$.

(*) For each $i = 1, ..., n$, there is a continuous character $\mathbf{Z}_p^\times \to \mathscr{O}_{\mathcal{E}}(\mathcal{E})^\times$ whose derivative at 1 is the map α_i and whose evaluation at any point $\boldsymbol{y} \in \mathcal{Q}$ is the elevation to the $\alpha_i(\boldsymbol{y})$-th power.

8.1.15. Let $(\mathcal{E}, D, \mathcal{Q}, \{\alpha_i : i = 1, ..., n\}, \{F_i : i = 1, ..., n\})$ be a refined family of dimension n. We fix a point $\boldsymbol{y} \in \mathcal{Q}$. Then $\rho_{\boldsymbol{y}}$ admits a natural refinement $\mathbb{F}_\bullet^{\boldsymbol{y}}$ given by the ordering of distinct eigenvalues

$$(p^{\alpha_1(y)}F_1(\boldsymbol{y}), ..., p^{\alpha_n(y)}F_n(\boldsymbol{y}))$$

of the crystalline Frobenius acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{\boldsymbol{y}}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ ([BC09, Definition 4.2.4]). We assume that $\rho_{\boldsymbol{y}}$ is irreducible and it satisfies the following two conditions:

(REG) The refinement $\mathbb{F}_\bullet^{\boldsymbol{y}}$ is ***regular***, *i.e.*, for any $i = 1, ..., n$, $p^{\alpha_1(\boldsymbol{y})+\cdots+\alpha_i(y)}F_1(\boldsymbol{y})\cdots F_i(\boldsymbol{y})$ is an eigenvalue of the crystalline Frobenius φ acting on $\mathbf{D}_{\mathrm{cris}}(\wedge^i \rho_{\boldsymbol{y}}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ of multiplicity one.

(NCR) The refinement $\mathbb{F}_\bullet^{\boldsymbol{y}}$ is non-critical.

Since $\rho_{\boldsymbol{y}}$ is assumed to be irreducible, Theorem 8.1.5 (ii) implies that there is a unique continuous representation

$$\rho_{\mathcal{E},\boldsymbol{y}} : G \to \mathrm{GL}_n(\mathscr{O}_{\mathcal{E},\boldsymbol{y}})$$

such that $\rho_{\mathcal{E},\boldsymbol{y}} \otimes_{\mathscr{O}_{\mathcal{E},\boldsymbol{y}}} k_{\boldsymbol{y}} = \rho_{\boldsymbol{y}}$ and so $\det \rho_{\boldsymbol{y}}$ coincides with the composition $G \xrightarrow{D} \mathscr{O}_{\mathcal{E}}(\mathcal{E}) \to \mathscr{O}_{\mathcal{E},\boldsymbol{y}}$. Following [BC09, §4.4], we define a continuous character $\delta_{\boldsymbol{y}} : \mathbf{Q}_p^\times \to (\mathscr{O}_{\mathcal{E},\boldsymbol{y}}^\times)^n$ by setting

$$\delta_{\boldsymbol{y}}(p) = (F_{1,\boldsymbol{y}}, ..., F_{n,\boldsymbol{y}}) \quad \text{and} \quad \delta_{\boldsymbol{y}}|_{\mathbf{Z}_p^\times} = (\alpha_{1,\boldsymbol{y}}^{-1}, ..., \alpha_{n,\boldsymbol{y}}^{-1}), \tag{8.2}$$

where $F_{i,\boldsymbol{y}}$ and $\alpha_{i,\boldsymbol{y}}$ are the images of F_i and α_i in $\mathscr{O}_{\mathcal{E},\boldsymbol{y}}$ respectively.

Theorem 8.1.16 ([BC09, Theorem 4.4.1])**.** *For any ideal $\mathfrak{I} \subsetneq \mathscr{O}_{\mathcal{E},\boldsymbol{y}}$ of cofinite length, $\rho_{\mathcal{E},\boldsymbol{y}} \otimes_{\mathscr{O}_{\mathcal{X},\boldsymbol{y}}} \mathscr{O}_{\mathcal{E},\boldsymbol{y}} / \mathfrak{I}$ is a trianguline deformation of $(\rho_{\boldsymbol{y}}, \mathbb{F}_\bullet^{\boldsymbol{y}})$, i.e., it belongs to $\mathscr{D}_{\rho_{\boldsymbol{y}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}, \mathbb{F}_\bullet^{\boldsymbol{y}}}(\mathscr{O}_{\mathcal{E},\boldsymbol{y}} / \mathfrak{I})$ (defined in Remark 8.1.12), whose parameter is $\delta_{\boldsymbol{y}} \otimes \mathscr{O}_{\mathcal{E},\boldsymbol{y}} / \mathfrak{I}$.*

8.2 Galois representations for GSp_{2g}

8.2.1. Before discussing about the Galois representations for GSp_{2g}, we shall briefly review the *spin representation* by following [FH91, Lecture 20].

Let $V \simeq \mathbf{Z}^{2g+1}$ be a free $\mathbf{Z}$-module, equipped with a quadratic form Q. The ***Clifford algebra*** Cliff associated to the pair (V, Q) is described by

$$\mathrm{Cliff} = \mathrm{Cliff}(V, Q) = \left(\oplus_{n\geq 0} V^{\otimes n}\right) / \langle v \otimes v - Q(v) \cdot 1 : \forall v \in V\rangle,$$

where 1 is the identity of the tensor algebra $\oplus_{n\geq 0} V^{\otimes n}$. The Clifford algebra admits a decomposition

$$\mathrm{Cliff} = \mathrm{Cliff}^+ \oplus \mathrm{Cliff}^-,$$

where Cliff^+ (resp. Cliff^-) consists of elements of even (resp. odd) degrees. On Cliff, there is an anti-involution $\bullet^*$, determined by the formula

$$(v_1 \cdot v_2 \cdots v_r)^* = (-1)^r v_r \cdot v_{r-1} \cdots v_1$$

for any $v_1, ..., v_r \in V$. Then for any ring R, we define

$$\mathrm{GSpin}_{2g+1}(R) := \left\{v \in (\mathrm{Cliff}^+ \otimes_{\mathbf{Z}} R)^\times : v \cdot R^g \cdot v^* := \{v \cdot w \cdot v^* : w \in R^g\} = R^g\right\}.$$

Here, the multiplication '$\cdot$' is the multiplication on the Clifford algebra. We remark that GSpin_{2g+1} is the dual group of GSp_{2g} in the sense of Langlands, *i.e.*, the algebraic characters of the maximal torus of GSp_{2g} define the algebraic cocharcters of the maximal torus of GSpin_{2g+1}.

Fix a maximal totally isotropic direct summand W of V with respect to Q. In particular, W is of rank g. Then [FH91, Lemma 20.16] yields an isomorphism

$$\mathrm{Cliff}^+ \simeq \mathrm{End}_{\mathbf{Z}}(\wedge^\bullet W),$$

where $\wedge^\bullet W = \oplus_{n\geq 0} \wedge^n W$ is the exterior algebra associated to W. Notice that $\wedge^\bullet W$ is of rank 2^g over $\mathbf{Z}$ and hence, via the isomorphism, we obtain the spin representation

$$\mathrm{spin} : \mathrm{GSpin}_{2g+1} \to \mathrm{GL}_{2^g}.$$

Lemma 8.2.2. *(i) The map* $\mathrm{spin} : \mathrm{GSpin}_{2g+1} \to \mathrm{GL}_{2^g}$ *is a closed immersion.*

(ii) If $g(g+1)/2$ is even (resp., odd), then there exists a symmetric (resp., symplectic) bilinear form on the 2^g-dimensional spin representation such that the bilinear form is preserved under GSpin_{2g+1} *up to scalar. In particular, the spin representation factors as*

$$\mathrm{spin} : \mathrm{GSpin}_{2g+1} \to \mathrm{GO}_{2^g} \hookrightarrow \mathrm{GL}_{2^g} \quad (\textit{resp.}, \ \mathrm{spin} : \mathrm{GSpin}_{2g+1} \to \mathrm{GSp}_{2^g} \hookrightarrow \mathrm{GL}_{2^g})$$

when $g(g+1)/2$ is even (resp. odd). Here, GO_{2^g} *is the algebraic group defined as*

$$\mathrm{GO}_{2^g} = \left\{\boldsymbol{\gamma} \in \mathrm{GL}_{2^g} : {}^{\mathtt{t}}\boldsymbol{\gamma}\, \breve{\mathbb{1}}_{2^g}\, \boldsymbol{\gamma} = \varsigma(\boldsymbol{\gamma})\, \breve{\mathbb{1}}_{2^g} \ \textit{for some } \varsigma(\boldsymbol{\gamma}) \in \mathbb{G}_m\right\}.$$

Proof. The first assertion is clear from the construction above while the second assertion is [KS20, Lemma 0.1]. $\square$

8.2.3. Given a dominant weight $k = (k_1, ..., k_g) \in \mathbf{Z}^g_{\geq 0}$, recall the GSp_{2g}-representation $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$. Although we considered the induced local system of $\mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}$ on $X_{\mathrm{Iw}^+}(\mathbf{C})$ in previous chapters, the left GSp_{2g}-action, in fact, induces a local system on $X(\mathbf{C})$, which we still denote by the same symbol. In particular, we can consider the parabolic cohomology group

$$H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k} := \oplus_{t=0}^{2n_0} H^t_{\mathrm{par}}(X(\mathbf{C}), \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}).$$

Note that the double cosets $[\mathrm{GSp}_{2g}(\mathbf{Z}_p)(x \cdot \mathbf{u}_{p,i}) \mathrm{GSp}_{2g}(\mathbf{Z}_p)]$ acts on $H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$ (as defined in 4.4.5) for any $x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$. We denote by

$$\mathbb{T}^{\mathrm{tame}} := \mathbb{T}^p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p \left[[\mathrm{GSp}_{2g}(\mathbf{Z}_p)(x \cdot \mathbf{u}_{p,i}) \mathrm{GSp}_{2g}(\mathbf{Z}_p)] : i = 0, 1, ..., g-1, x \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}} \right].$$

In particular, it makes sense to consider the Hecke polynomial $P_{\mathrm{Hecke},p}(Y)$ at p in this case and is defined as in (4.3).

Hypothesis 1. *For any $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu] \in H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$ with eigensystem $\lambda_{[\mu]} : \mathbb{T}^{\mathrm{tame}} \to \overline{\mathbf{Q}}_p$, there exists a (continuous) Galois representation*

$$\rho_{[\mu]} : \mathrm{Gal}_{\mathbf{Q}} \xrightarrow{\rho^{\mathrm{spin}}_{[\mu]}} \mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p) \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p)$$

such that

(i) The representation $\rho_{[\mu]}$ is unramified outside pN and

$$\mathrm{char.\,poly}(\mathrm{Frob}_\ell)(Y) = \lambda_{[\mu]}(P_{\mathrm{Hecke},\ell}(Y)) := \prod_{x \in \mathrm{Weyl}^H} (Y - \lambda_{[\mu]}(T^x_{\ell,0}))$$

for any $\ell \nmid pN$, where $\mathrm{char.\,poly}(\mathrm{Frob}_\ell)(Y)$ *stands for the characteristic polynomial of the Frobenius at ℓ and $P_{\mathrm{Hecke},\ell}(Y)$ is the Hecke polynomial defined in (4.3). Moreover, the coefficients of these two polynomials are algebraic integers over $\mathbf{Q}$.*

(ii) The representation $\rho_{[\mu]}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is crystalline with Hodge–Tate weights

$$(a_1, ..., a_{2^g}) = (0, a'_g, \cdots, a'_1, a'_g + a'_{g-1}, ..., a'_2 + a'_1, \cdots, a'_g + \cdots + a'_1),^2$$

where $a'_i = (g+1-i) + k_i$. Let $\varphi = \varphi_{\mathrm{cris}}$ be the crystalline Frobenius acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{[\mu]}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$, we moreover have

$$\mathrm{char.\,poly}(\varphi)(Y) = \lambda_{[\mu]}(P_{\mathrm{Hecke},p}(Y)),$$

[2] These numbers are all possibilities of sums of a_i's. The order is chosen so that if $k = (k_1, ..., k_g) = (k_g + g - 1, k_g + g - 2, ..., k_g)$, we have $a_1 < a_2 < \cdots < a_{2^g}$.

where $\mathrm{char.\,poly}(\varphi)(X)$ *is the characteristic polynomial of* φ *acting on* $\mathbf{D}_{\mathrm{cris}}(\rho_{[\mu]}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$*, and the coefficients of these two polynomials are algebraic integers over* $\mathbf{Q}$*. We order the eigenvalues of* φ *so they satisfy*

$$(\varphi_1, ..., \varphi_{2^g}) = \varphi_1(1, \varphi_2', ..., \varphi_{g+1}', \varphi_2'\varphi_3', ..., \varphi_g'\varphi_{g+1}', ..., \varphi_2' \cdots \varphi_{g+1}').$$

The order of the later tuple is chosen similarly as the Hodge–Tate weights. In particular, $\varphi_2, ..., \varphi_{g+1}$ *are divisible by* φ_1 *and the* 2^g *eigenvalues of* φ *depend only on* $\varphi_1, ..., \varphi_{g+1}$*.*

Remark 8.2.4. Recall that $\mathtt{S}_{\mathrm{bad}}$ is the finite set of prime numbers which divides N. From now on, we shall redefine $\mathtt{S}_{\mathrm{bad}}$ to be $\mathtt{S}_{\mathrm{bad}} \cup \{p\}$. Let $\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}$ be the Galois group of the maximal extension of $\mathbf{Q}$ which is unramified outside $\mathtt{S}_{\mathrm{bad}}$. Therefore, the representation $\rho_{[\mu]}$ in Hypothesis 1 can be regarded as a Galois representation of $\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}$.

Remark 8.2.5. Evidently, Hypothesis 1 comes from Global Langlands Correspondence. We comment briefly to this hypothesis.

(i) When $g \leq 2$, Hypothesis 1 (i) is well-known (see, for example, [Wei05]). The work of A. Kret and S. W. Shin ([KS20]) gave a positive answer to Hypothesis 1 (i) under some conditions on the automorphic representations for general g. Although their result is not complete unconditional, it suggests that Hypothesis 1 is reasonable to assume (but could be difficult to prove in general).

(ii) Hypothesis 1 (ii) is also well-studied when $g \leq 2$. In particular, E. Urban proved the case for $g = 2$ in [Urb05], result deduced from A. Scholl's motive for modular forms ([Sch90]). For general g, the property is expected if Hypothesis 2 below holds (see, for example, [PT15, Theorem 2.1 & Corollary 2.2]).

8.2.6. By Lemma 8.2.2 and under the assumption of Hypothesis 1, we know that given a $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu]$ as above, $\rho_{[\mu]}$ factors as

$$\rho_{[\mu]} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \xrightarrow{\rho_{[\mu]}^{\mathrm{spin}}} \mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p) \xrightarrow{\mathrm{spin}} \mathrm{GS}(\overline{\mathbf{Q}}_p) \to \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p),$$

where

$$\mathrm{GS} = \begin{cases} \mathrm{GO}_{2^g}, & \text{if } g(g+1)/2 \text{ is even} \\ \mathrm{GSp}_{2^g}, & \text{if } g(g+1)/2 \text{ is odd} \end{cases}$$

and the last arrow is nothing but the natural inclusion. Define

$$\begin{aligned} \mathfrak{gl}_{2^g} &:= \text{the Lie algebra of } \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p), \\ &\quad\ \text{equipped with the induced adjoint } \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}\text{-action by } \rho_{[\mu]} \\ \mathrm{ad}\,\rho_{[\mu]} &:= \text{the Lie algebra of } \mathrm{GS}(\overline{\mathbf{Q}}_p), \\ &\quad\ \text{equipped with the induced adjoint } \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}\text{-action by } \mathrm{spin} \circ \rho_{[\mu]}^{\mathrm{spin}} \\ \mathrm{ad}\,\rho_{[\mu]}^{\mathrm{spin}} &:= \text{the Lie algebra of } \mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p), \\ &\quad\ \text{equipped with the induced adjoint } \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}\text{-action by } \rho_{[\mu]}^{\mathrm{spin}}. \end{aligned}$$

Then, the inclusions

$$\mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p) \hookrightarrow \mathrm{GS}(\overline{\mathbf{Q}}_p) \hookrightarrow \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p)$$

induces $\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}$-equivariant inclusions

$$\mathrm{ad}\,\rho_{[\mu]}^{\mathrm{spin}} \hookrightarrow \mathrm{ad}\,\rho_{[\mu]} \hookrightarrow \mathfrak{gl}_{2^g},$$

which then further induces inclusions of the Galois cohomology groups

$$H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}\,\rho_{[\mu]}^{\mathrm{spin}}) \hookrightarrow H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}\,\rho_{[\mu]}) \hookrightarrow H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathfrak{gl}_{2^g}).$$

On the other hand, let $\mathfrak{sl}_{2^g}$ be the trace-zero part of $\mathfrak{gl}_{2^g}$ and let

$$\mathrm{ad}^0\,\rho_{[\mu]} := \mathrm{ad}\,\rho_{[\mu]} \cap \mathfrak{sl}_{2^g} \quad \text{and} \quad \mathrm{ad}^0\,\rho_{[\mu]}^{\mathrm{spin}} := \mathrm{ad}\,\rho_{[\mu]}^{\mathrm{spin}} \cap \mathfrak{sl}_{2^g}.$$

Note that the decomposition $\mathfrak{gl}_{2^g} = \mathfrak{sl}_{2^g} \oplus \mathfrak{gl}_1$ is $\mathrm{Gal}_{\mathbf{Q}}$-equivariant, we thus have a commutative diagram

$$\begin{tikzcd}
H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}\,\rho_{[\mu]}^{\mathrm{spin}}) \arrow[r, hook] & H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}\,\rho_{[\mu]}) \arrow[r, hook] & H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathfrak{gl}_{2^g}) \\
H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}^0\,\rho_{[\mu]}^{\mathrm{spin}}) \arrow[r, hook] \arrow[u, hook] & H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}^0\,\rho_{[\mu]}) \arrow[r, hook] \arrow[u, hook] & H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathfrak{sl}_{2^g}) \arrow[u, hook]
\end{tikzcd},$$

where the arrows are all inclusions.

8.2.7. Under the assumption of Hypothesis 1, one obtains a 2^g-dimensional Galois representation for each eigenclass $[\mu]$. It is then a natural question to ask whether the attached Galois representation admits an associated cuspidal automorphic representation of GL_{2^g}. The answer of this question is expected to be affirmative, which we state as the next hypothesis.

Hypothesis 2 (The potential spin functoriality). *Given a $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu] \in H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$, there exists a finite real extension $L \subset \overline{\mathbf{Q}}$ of $\mathbf{Q}$ with $\rho_{[\mu]}|_{\mathrm{Gal}_L}$ being irreducible and a generic cuspidal automorphic representation $\pi_{[\mu]}$ of $\mathrm{GL}_{2^g}(\mathbf{A}_L)$, where $\mathbf{A}_L$ is the ring of adèles of L, such that*

- *$\pi_{[\mu]}$ is unramified outside the places above $\mathtt{S}_{\mathrm{bad}}$ and*
- *the Galois representation associated with $\pi_{[\mu]}$ is isomorphic to $\rho_{[\mu]}|_{\mathrm{Gal}_L}$.*

Remark 8.2.8. We should remark that Kret and Shin verify the above hypothesis in [KS20, Theorem C] under some stronger conditions than the ones they verify Hypothesis 1.

8.2.9. On the other hand, we also write

$$H^{\mathrm{alg,tol}}_{\mathrm{par},k} := \oplus_{t=0}^{2n_0} H^t_{\mathrm{par}}(X_{\mathrm{Iw}^+}(\mathbf{C}), \mathbf{V}^{\mathrm{alg},\vee}_{\mathrm{GSp}_{2g},k}).$$

The forgetful map $X_{\mathrm{Iw}^+}(\mathbf{C}) \to X(\mathbf{C})$ then induces a morphism

$$\Lambda_p : H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k} \to H^{\mathrm{alg,tol}}_{\mathrm{par},k}. \tag{8.3}$$

Observe that this morphism is $\mathbb{T}^p$-equivariant. Moreover, we have slope decomposition on the later space with respect to the action of U_p since it is a finite dimensional $\mathbf{Q}_p$-vector space. Thus, for each $h \in \mathbf{Q}_{>0}$, we write

$$H^{\mathrm{alg,tol},\leq h}_{\mathrm{tame,par},k} := \mathrm{image}\left(H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k} \xrightarrow{\Lambda_p} H^{\mathrm{alg,tol}}_{\mathrm{par},k} \twoheadrightarrow H^{\mathrm{alg,tol},\leq h}_{\mathrm{par},k}\right),$$

where $H^{\mathrm{alg,tol},\leq h}_{\mathrm{par},k}$ is the '$\leq h$' part of $H^{\mathrm{alg,tol}}_{\mathrm{par},k}$ under the action of U_p. Thus, for any $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu]$ in $H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$, its image in $H^{\mathrm{alg,tol},\leq h}_{\mathrm{tame,par},k}$ can be decomposed as a sum of $\mathbb{T}$-eigenclasses. We call any of these factors a ***p*-stabilisation** of $[\mu]$.

It is a natural question asking how the eigenvalues of a $\mathbb{T}^{\mathrm{tame}}$-eigenclass interact with the eigenvalues of its p-stabilisations. Due to our lack knowledge on the Hecke algebra of the strict Iwahori level, we state such a conjectural interaction in the next hypothesis.

Hypothesis 3. *(i) Let $[\mu]$ be a $\mathbb{T}^{\mathrm{tame}}$-eigenclass with eigensystem $\lambda_{[\mu]}$ in $H^{\mathrm{alg,tol},\leq h}_{\mathrm{par},k}$. Then, there exist $2^g g!$ p-stabilisations $[\mu]^{(p)}$, indexed by $\mathrm{Weyl}_{\mathrm{GSp}_{2g}}$.*

(ii) Chose a bijection of sets $\iota : \{1, 2, ..., 2^g\} \xrightarrow{\sim} \mathrm{Weyl}^H$ so that $\lambda_{[\mu]}\left(T^{\iota(i)}_{p,0}\right) = \varphi_i$, where $T^{\iota(i)}_{p,0}$ is the Hecke operator defined by $[\mathrm{GSp}_{2g}(\mathbf{Z}_p)(\iota(i)\cdot\mathbf{u}_{p,0})\,\mathrm{GSp}_{2g}(\mathbf{Z}_p)]$ acting on $H^{\mathrm{alg,tol}}_{\mathrm{tame,par,wt}(\boldsymbol{x})}$ and φ_i is the i-th eigenvalue of the crystalline Frobenius associated with $\rho_{[\mu]}$.[3] Denote by $\lambda_i = \lambda_{[\mu]}(T^{\iota(i)}_{p,0})$ and let $[\mu]^{(p)}$ be any of the p-stablisation of $[\mu]$ with Hecke eigensystem $\lambda^{(p)}_{[\mu]}$. Then, there exists a constant $\theta \in \mathbf{Q}$ (depending only on g) such that, for $i = 1, ..., g+1$,

$$\lambda^{(p)}_{[\mu]}(U^{\iota(i)}_{p,0}) = p^{\theta} \cdot p^{-(g+1-i)} \lambda_1 \prod_{j=1}^{g} (\lambda_{j+1}/\lambda_1)^{a_{\nu(j)} \text{ or } 1-a_{\nu(j)}},$$

where

- *the index of $[\mu]^{(p)}$ is $(\varepsilon, \nu) \in \mathrm{Weyl}_{\mathrm{GSp}_{2g}} = \mathbf{\Sigma}_g \ltimes (\mathbf{Z}/2\mathbf{Z})^g$ and*

$$a_{\nu(j)} = \begin{cases} 1, & \nu(j) = i \\ 0, & \text{otherwise} \end{cases};$$

- *the exponent depends on whether $\varepsilon(\nu(j)) = 0$ or $1 \in \mathbf{Z}/2\mathbf{Z}$.*

Remark 8.2.10. (i) The above equation in the above hypothesis totally defines the values $\lambda^{(p)}_{[\mu]}(U^{\iota(i)}_{p,0})$ for $g+1 < i \leq 2^g$ due to the relations of eigenvalues φ_i of the crystalline Frobenius in Hypothesis 1 (ii).

(ii) We remark that Hypothesis 3 is inspired by [HJ17, Lemma 17]. However, one is not allowed to apply *loc. cit.* directly since the authors of *loc. cit.* considered the Iwahori subgroup while we are working with the *strict* Iwahori subgroup, which is a deeper level at p. A priori, given a class $[\mu]$ in $H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$, it might admit more p-stablisations

[3] This can be done due to Hypothesis 1 (ii).

at the strict Iwahori level than at the Iwahori level. The meaning of the hypothesis above means that these extra p-stablisations shall be killed after taking the finite-slope part. Such a phenomenon already happens in the classical theory of modular forms.

Assumption 8.2.11. In the rest of this chapter, we assume that Hypothesis 1, Hypothesis 2 and Hypothesis 3 hold.

8.3 Families of Galois representations on the cuspidal eigenvariety

8.3.1. The goal of this section is to construct families of Galois representations on a sublocus of the cuspidal eigenvariety $\mathcal{E}_0^{\mathrm{oc}}$ under the assumption of Hypothesis 1 and Hypothesis 3.

8.3.2. For any dominant algebraic weight $k \in \mathbf{Z}_{>0}^g$, recall from Theorem 5.2.8 that there is $h_k \in \mathbf{R}_{>0}$ such that for any $h \in \mathbf{Q}_{>0}$ with $h < h_k$, we have a canonical isomorphism

$$H_{\mathrm{par},k}^{\mathrm{tol},\leq h} \xrightarrow{\sim} H_{\mathrm{par},k}^{\mathrm{alg},\mathrm{tol},\leq h}.$$

We then define the ***p*-stabilised classical locus** of $\mathcal{E}_0^{\mathrm{oc}}$ to be the locus $\mathcal{G}^{\mathrm{cl}} \subset \mathcal{E}_0$, containing those $\boldsymbol{x}$ with the following conditions:

- $\mathrm{wt}(\boldsymbol{x}) = k \in \mathbf{Z}_{>0}^g$ is a dominant algebraic weight;
- there exists $h < h_k$ such that $\boldsymbol{x}$ corresponds to a p-stabilisation of slope $\leq h$ of a $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu]$ in $H_{\mathrm{tame},\mathrm{par},k}^{\mathrm{alg},\mathrm{tol}}$;
- the Galois representation $\rho_{[\mu]}^{\mathrm{spin}}$ attached to $[\mu]$ (by Hypothesis 1) is irreducible.

Consequently, we define

$$\mathcal{E}_0^{\mathrm{irr}} := \text{ the Zariski closure of } \mathcal{G}^{\mathrm{cl}} \text{ in } \mathcal{E}_0^{\mathrm{oc}}.$$

Remark 8.3.3. We do not expect every classical point in $\mathcal{E}_0$ corresponds to an irreducible Galois representation due to the endoscopy theory of automorphic forms. As we will be only interested in classical points that correspond to irreducible Galois representations, we do not loose information if we only consider $\mathcal{E}_0^{\mathrm{irr}}$.

Proposition 8.3.4. *Assume the truthfulness of Hypothesis 1.*

(i) For any $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}$, there is an associated Galois representation

$$\rho_{\boldsymbol{x}} : \mathrm{Gal}_{\mathbf{Q},\mathrm{S}_{\mathrm{bad}}} \xrightarrow{\rho_{\boldsymbol{x}}^{\mathrm{spin}}} \mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p) \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p)$$

that satisfies the properties in Hypothesis 1.

(ii) There is a universal determinant

$$\mathrm{Det}^{\mathrm{univ}} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$$

of dimension 2^g such that, for any $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}$, the specialisation $\mathrm{Det}^{\mathrm{univ}}|_{\boldsymbol{x}}$ (notation as in (8.1)) coincides with $\det \rho_{\boldsymbol{x}}$.

Proof. The first assertion is easy. Let $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}$. It corresponds to a p-stabilisation class $[\mu]^{(p)} \in H^{\mathrm{alg,tol},\leq h}_{\mathrm{tame,par},k}$. That is, there is a $\mathbb{T}^{\mathrm{tame}}$-eigenclass $[\mu] \in H^{\mathrm{alg,tol}}_{\mathrm{tame,par},k}$ such that $[\mu]^{(p)}$ is a p-stabilisation of $[\mu]$. By Hypothesis 1, the class $[\mu]$ is associated with a Galois representation with desired properties. Then, we define $\rho^{\mathrm{spin}}_{\boldsymbol{x}} := \rho^{\mathrm{spin}}_{[\mu]}$ and $\rho_{\boldsymbol{x}} := \rho_{[\mu]}$.

For the second assertion, we follow the proof of [Che04, Proposition 7.1.1] (see also [Che14, Example 2.32]). Consider the morphism

$$\Phi : \mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}}) \to \prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p, \quad f \mapsto (f(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}}.$$

Equipped $\prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p$ with the product topology, one sees that Φ is continuous. We claim that $\Phi(\mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}}))$ is homeomorphic to $\mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$ and is closed in $\prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p$. Indeed, since $\mathcal{G}^{\mathrm{cl}}$ is Zariski dense in the reduced space $\mathcal{E}_0^{\mathrm{oc}}$, the map Φ is injective. Apply [JN19, Corollary 5.4.4], we know that $\mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$ is compact and so $\Phi(\mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}}))$ is closed in $\prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p$.

On the other hand, we have a continuous map

$$\mathrm{Det} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p, \quad \sigma \mapsto (\det \rho_{\boldsymbol{x}}(\sigma))_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}}.$$

One checks easily that Det is a determinant of dimension 2^g, in fact, the determinant of a representation $\mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_{2^g}(\prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p)$. Condition (ii) and (iii) in Hypothesis 1 and $\mathrm{image}\,\Phi$ being closed in $\prod_{\boldsymbol{x}\in\mathcal{G}^{\mathrm{cl}}} \mathbf{C}_p$ imply that $\mathrm{image}\,\mathrm{Det} \subset \mathrm{image}\,\Phi$. Hence, we define

$$\mathrm{Det}^{\mathrm{univ}} := \Phi^{-1} \circ \mathrm{Det} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \mathscr{O}^+_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}}).$$

Since Φ is injective and Det is a determinant of dimension 2^g, $\mathrm{Det}^{\mathrm{univ}}$ is as desired. $\square$

Theorem 8.3.5. *There exists a subset $\mathcal{G}^{\mathrm{cl}}_{\heartsuit} \subset \mathcal{G}^{\mathrm{cl}}$ which is Zariski dense in $\mathcal{E}_0^{\mathrm{irr}}$, 2^g analytic functions $\alpha_1, ..., \alpha_{2^g} \in \mathscr{O}_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$ and 2^g analytic functions $F_1, ..., F_{2^g} \in \mathscr{O}_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$ such that*

$$(\mathcal{E}_0^{\mathrm{oc}}, \mathrm{Det}^{\mathrm{univ}}, \mathcal{G}^{\mathrm{cl}}_{\heartsuit}, \{\alpha_i : i = 1, ..., 2^g\}, \{F_i : i = 1, ..., 2^g\})$$

is a refined family of Galois representations.

Proof. For any p-adic weight $\kappa = (\kappa_1, ..., \kappa_g)$, define an ordering of functions on $\mathbf{Z}_p^\times$ via

$$(\alpha_1, ..., \alpha_{2^g}) := (0, \alpha'_g, ..., \alpha'_1, \alpha'_g + \alpha'_{g-1}, ..., \alpha'_g + \alpha'_1, \alpha'_{g-1} + \alpha'_{g-2}, ..., \alpha'_2 + \alpha'_1, ..., \alpha'_g + \cdots + \alpha'_1),$$

where $\alpha'_i = (g + 1 - i) + \kappa_i$ is the character $a \mapsto \kappa_i(a)a^{g+1-i}$ for every $a \in \mathbf{Z}_p^\times$. We can view α_j's as functions on $\mathcal{E}_0^{\mathrm{oc}}$ by composing with the weight map wt. Obviously from this definition, for any $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}$, the functions α_j's provides an ordering of the Hodge–Tate weight of $\rho_{\boldsymbol{x}}$ in Hypothesis 1 (ii).

Define

$$\mathcal{G}^{\mathrm{cl}}_{\heartsuit} := \left\{ \boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}} : \begin{array}{l} 0 = \alpha_1(\boldsymbol{x}) < \alpha_2(\boldsymbol{x}) < \cdots < \alpha_{2^g}(\boldsymbol{x}) \\ \text{eigenvalues of the crystalline Frobenius acting on } \mathbf{D}_{\mathrm{cris}}(\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}) \text{ are distinct} \end{array} \right\}.$$

Observe that $\mathcal{G}^{\mathrm{cl}}_{\heartsuit}$ is Zariski dense in $\mathcal{E}_0^{\mathrm{irr}}$ since $\mathcal{G}^{\mathrm{cl}}$ is Zariski dense in $\mathcal{E}_0^{\mathrm{irr}}$ and the first condition defining $\mathcal{G}^{\mathrm{cl}}_{\heartsuit}$ is an open condition on weights while the second condition is an open condition on the cuspidal eigenvariety $\mathcal{E}_0^{\mathrm{irr}}$. We claim that $\mathcal{G}^{\mathrm{cl}}_{\heartsuit}$ satisfies condition (iv) in Definition 8.1.14. That is, for any $C \in \mathbf{Z}_{>0}$, we have to show that the set

$$\mathcal{G}^{\mathrm{cl}}_{\heartsuit, C} := \left\{ \boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}_{\heartsuit} : \begin{array}{l} \alpha_{i+1}(\boldsymbol{x}) - \alpha_i(\boldsymbol{x}) > C(\alpha_i(\boldsymbol{x}) - \alpha_{i-1}(\boldsymbol{x})) \text{ for } i = 2, ..., 2^g - 1 \\ \alpha_2(\boldsymbol{x}) - \alpha_1(\boldsymbol{x}) > C \end{array} \right\}$$

satisfies that, for any basis of affinoid neighbourhoods $\mathcal{V}$ of $\boldsymbol{x}$, $\mathcal{V} \cap \mathcal{G}^{\mathrm{cl}}_{\heartsuit, C}$ is Zariski dense in $\mathcal{V}$. However, this follows from that the condition defining $\mathcal{G}^{\mathrm{cl}}_{\heartsuit, C}$ is an open condition on the weights.

Now, for any $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}_{\heartsuit}$, the associated representation $\rho_{\boldsymbol{x}}$ is crystalline at p. Let $\varphi_1(\boldsymbol{x}), ..., \varphi_{2^g}(\boldsymbol{x})$ be eigenvalues of the crystalline Frobenius $\varphi = \varphi_{\mathrm{cris}}$ acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$. The order of the eigenvalues φ_i's is defined so that it defines a non-critical refinement on $\rho_{\boldsymbol{x}}$. This is achievable by applying Proposition 8.1.11 (ii). Define

$$F_i(\boldsymbol{x}) := \varphi_i(\boldsymbol{x}) / p^{\alpha_i(\boldsymbol{x})} \in \mathbf{C}_p\,.$$

We claim that the collection $\{(F_i(\boldsymbol{x}))_{i=1,...,2^g}\}_{\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}_{\heartsuit}}$ glue to 2^g analytic functions $(F_1, ..., F_{2^g})$ in $\mathscr{O}_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}})$. Let $\lambda_{\boldsymbol{x}} : \mathbb{T}^{\mathrm{tame}} \to \overline{\mathbf{Q}}_p$ be the eigensystem corresponds to $\boldsymbol{x}$. Consider

$$p^{\vartheta} p^{\kappa_i'} F_i := \text{ image of the operator } U_{p,0}^{\iota(i)} \text{ in } \mathscr{O}_{\mathcal{E}_0^{\mathrm{irr}}}(\mathcal{E}_0^{\mathrm{irr}}),$$

where

$$(\kappa_1', ..., \kappa_{2^g}') = (0, \kappa_g, ..., \kappa_1, \kappa_g + \kappa_{g-1}, ..., \kappa_g + \kappa_1, \kappa_{g-1} + \kappa_{g-2}, ..., \kappa_2 + \kappa_1, ..., \kappa_g + \cdots + \kappa_1)$$

and $(\kappa_1, ..., \kappa_g) = \mathrm{wt}$ is the weight map. Then, Hypothesis 1 (ii) and Hypothesis 3 imply the desired result (see also[BC09, Proposition 7.5.13]). □

Remark 8.3.6. Recall that we have ordered the eigenvalues of the crystalline Frobenius φ so that they satisfy

$$(\varphi_1, ..., \varphi_{2^g}) = \varphi_1(1, \varphi_2', ..., \varphi_{g+1}', \varphi_2'\varphi_3', ..., \varphi_g'\varphi_{g+1}', ..., \varphi_2' \cdots \varphi_{g+1}').$$

On the other hand, recall that Weyl^H is a set of representatives of $\mathrm{Weyl}_H \setminus \mathrm{Weyl}_{\mathrm{GSp}_{2g}}$, where $\mathrm{Weyl}_H \simeq \mathbf{\Sigma}_g$. Observe that $\mathrm{diag}(\mathbb{1}_g, p\,\mathbb{1}_g)$ is stable under the action of $\mathbf{\Sigma}_g$, thus the action of Weyl^H on $T_{p,0}$ only depends on the action of $(\mathbf{Z}/2\,\mathbf{Z})^g$. Combining everything together, we have the relation

$$(F_1, ..., F_{2^g}) = F_1(1, F'_2, ..., F'_{g+1}, F'_2F'_3, ..., F'_gF'_{g+1}, ..., F'_2 \cdots F'_{g+1}).$$

In particular, $F_2, ..., F_{g+1}$ are divisible by F_1.

8.4 Local and global Galois deformations

8.4.1. We keep the notations in the previous subsection. Fix $\boldsymbol{x} \in \mathcal{G}^{\mathrm{cl}}_{\heartsuit}$ with $\mathrm{wt}(\boldsymbol{x}) = k = (k_1, ..., k_g)$ and we write

$$\rho_{\boldsymbol{x}} : \mathrm{Gal}_{\mathbf{Q}} \xrightarrow{\rho_{\boldsymbol{x}}^{\mathrm{spin}}} \mathrm{GSpin}_{2g+1}(\overline{\mathbf{Q}}_p) \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}(\overline{\mathbf{Q}}_p)$$

for the Galois representation attached to $\boldsymbol{x}$, given by Proposition 8.3.4. We fix a large enough finite field extension $k_{\boldsymbol{x}}$ of $\mathbf{Q}_p$ such that $k_{\boldsymbol{x}}$ contains the residue field at $\boldsymbol{x}$ and $\rho_{\boldsymbol{x}}^{\mathrm{spin}}$ takes values in $\mathrm{GSpin}_{2g+1}(k_{\boldsymbol{x}})$. We also assume that $k_{\boldsymbol{x}}$ contains all eigenvalues of the Frobenii.

Let now $\mathbf{AR}$ be the category of local artinian $k_{\boldsymbol{x}}$-algebras whose residue field is $k_{\boldsymbol{x}}$. We denote by $\mathbb{F}_{\bullet}^{\boldsymbol{x}}$ the refinement of $\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ induced by the refined family defined in Theorem 8.3.5. We also denote by $\delta = (\delta_1, ..., \delta_{2^g})$ the parameter attached to the triangulation associated with $\mathbb{F}_{\bullet}^{\boldsymbol{x}}$. Notice that the relation of the eigenvalues of crystalline Frobenius and the Hodge–Tate weight implies that the parameter δ satisfies

$$(\delta_1, ..., \delta_{2^g}) = \delta_1(1, \delta'_2, ..., \delta'_{g+1}, \delta'_2\delta'_3, ..., \delta'_g\delta'_{g+1}, ..., \delta'_2 \cdots \delta'_{g+1})$$

for some continuous characters $\delta'_2, ..., \delta'_{g+1}$ such that $\delta_i = \delta_1\delta'_i$ for all $i = 2, ..., g+1$.

8.4.2 (Local Galois deformations at p)**.** We shall consider two deformation problems at p:

(i) The deformation problem

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x}, \mathbb{F}_{\bullet}^{\boldsymbol{x}}, p} : \mathbf{AR} \to \mathbf{SETS},$$

sending each $A \in \mathbf{AR}$ to the isomorphism classes of representations $\rho_A^{\mathrm{spin}} : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GSpin}_{2g+1}(A)$ with a triangulation $\mathrm{Fil}_{\bullet} \mathbf{D}_{\mathrm{rig}}(\mathrm{spin} \circ \rho_A^{\mathrm{spin}})$ such that

- $\rho_A^{\mathrm{spin}} \otimes_A k_{\boldsymbol{x}} \simeq \rho_{\boldsymbol{x}}^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$;
- $(\mathrm{spin} \circ \rho_A^{\mathrm{spin}}, \mathrm{Fil}_{\bullet} \mathbf{D}_{\mathrm{rig}}(\mathrm{spin} \circ \rho_A^{\mathrm{spin}})) \in \mathscr{D}_{\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}, \mathbb{F}_{\bullet}^{\boldsymbol{x}}}(A)$ and write $\delta_A = (\delta_{A,1}, ..., \delta_{A,2^g})$ for the associated parameter;
- the parameter δ_A satisfies

$$\begin{aligned}&(\delta_{A,1}, ..., \delta_{A,2^g}) \\ &\quad = \delta_{A,1}(1, \delta'_{A,2}, ..., \delta'_{A,g+1}, \delta'_{A,2}\delta'_{A,3}, ..., \delta'_{A,2}\delta'_{A,g+1}, \delta'_{A,3}\delta'_{A,4}, ..., \delta'_{A,g}\delta'_{A,g+1}, ..., \delta'_{A,2} \cdots \delta'_{A,g+1})\end{aligned}$$

  for some continuous characters $\delta'_{A,2}, ..., \delta'_{A,g+1}$;
- $\det \mathrm{spin} \circ \rho_A^{\mathrm{spin}} = \det \rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$

(ii) The deformation problem

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f,p} : \mathbf{A}\textsc{r} \to \mathbf{S}\textsc{ets},$$

sending each $A \in \mathbf{A}\textsc{r}$ to the isomorphism classes of representations $\rho^{\mathrm{spin}}_A : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GSpin}_{2g+1}(A)$ such that

- $\rho^{\mathrm{spin}}_A \otimes_A k_{\boldsymbol{x}} \simeq \rho^{\mathrm{spin}}_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$;
- the (φ,Γ)-module $\mathbf{D}_{\mathrm{rig}}(\mathrm{spin}\circ\rho^{\mathrm{spin}}_A)$ is crystalline in the sense of [BC09, Definition 2.2.10] whose eigenvalues $(\varphi_{A,1},...,\varphi_{A,2^g})$ of the crystalline Frobenius satisfy

  $$(\varphi_{A,1},...,\varphi_{A,2^g}) = \varphi_{A,1}(1,\varphi'_{A,2},...,\varphi'_{A,g+1},\varphi'_{A,2}\varphi'_{A,3},...,\varphi'_{A,g}\varphi'_{A,g+1},...,\varphi'_{A,2}\cdots\varphi'_{A,g+1}),$$

  order chosen the same as for φ_i's;
- $\det \mathrm{spin}\circ\rho^{\mathrm{spin}}_A = \det \rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$

8.4.3. Consider

$$L'_p := \ker\left(H^1(\mathrm{Gal}_{\mathbf{Q}_p}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}}) \to H^1(\mathrm{Gal}_{\mathbf{Q}_p}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}} \otimes_{k_{\boldsymbol{x}}} \mathbf{B}_{\mathrm{cris}})\right),$$

where $\mathbf{B}_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods. It is well-known that L'_p defines the tangent space of the crystalline deformation problem for $\rho^{\mathrm{spin}}_{\boldsymbol{x}}$ with fixed determinant. Consequently, the tangent space $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f,p}(k_{\boldsymbol{x}}[\varepsilon])$, where ε is a variable such that $\varepsilon^2 = 0$, of $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f,p}$ defines a subspace of L'_p. Thus, we define

$$L_p := \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f,p}(k_{\boldsymbol{x}}[\varepsilon]) \subset L'_p. \tag{8.4}$$

8.4.4 (Local Galois deformations at N)**.** For any $\ell|N$, we consider the following deformation problem

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\ell} : \mathbf{A}\textsc{r} \to \mathbf{S}\textsc{ets}$$

sending each $A \in \mathbf{A}\textsc{r}$ to the isomorphism classes of representations $\rho^{\mathrm{spin}}_A : \mathrm{Gal}_{\mathbf{Q}_\ell} \to \mathrm{GSpin}_{2g+1}(A)$ such that

- $\rho^{\mathrm{spin}}_A \otimes_A k_{\boldsymbol{x}} \simeq \rho^{\mathrm{spin}}_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_\ell}}$;
- $\rho^{\mathrm{spin}}_A|_{I_\ell} \simeq \rho^{\mathrm{spin}}_{\boldsymbol{x}}|_{I_\ell} \otimes_{k_{\boldsymbol{x}}} A$
- $\det \mathrm{spin}\circ\rho^{\mathrm{spin}}_A = \det \rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_\ell}}$

Then, one sees that the tangent space $\mathscr{D}_{\boldsymbol{x},\ell}(k_{\boldsymbol{x}}[\varepsilon])$ of $\mathscr{D}_{\boldsymbol{x},\ell}$ is a $k_{\boldsymbol{x}}$-subspace of $H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}})$. We consequently define

$$L_\ell := \mathscr{D}_{\boldsymbol{x},\ell}(k_{\boldsymbol{x}}[\varepsilon]) \subset H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}}). \tag{8.5}$$

We learnt the following lemma from P. Allen.

Lemma 8.4.5. *Under the assumption of Hypothesis 2, we have*

$$L_\ell = H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}}).$$

Proof. Let

$$H^1_{\mathrm{unr}}(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) := \ker\left(H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) \to H^1(I_\ell, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}})\right)$$

By definition, we see that $H^1_{\mathrm{unr}}(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) \subset L_\ell$. Thus, it is enough to show that

$$H^1_{\mathrm{unr}}(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) = H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}).$$

First of all, observe that

$$H^1_{\mathrm{unr}}(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) = H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}/I_\ell, (\mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}})^{I_\ell})$$

by definition. Note that $\mathrm{Gal}_{\mathbf{Q}_\ell}/I_\ell \simeq \widehat{\mathbf{Z}}$. Hence, one deduces from the discussion in [Ser79, Chapter XIII, §1] that

$$\begin{aligned}\dim_{k_{\boldsymbol{x}}} H^1_{\mathrm{unr}}(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) &= \dim_{k_{\boldsymbol{x}}} H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}/I_\ell, (\mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}})^{I_\ell}) \\ &= \dim_{k_{\boldsymbol{x}}} H^0(\mathrm{Gal}_{\mathbf{Q}_\ell}/I_\ell, (\mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}})^{I_\ell}) \\ &= \dim_{k_{\boldsymbol{x}}} H^0(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}).\end{aligned}$$

By applying the local Euler characteristic, the desired equation will follow once we show

$$H^2(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}) = 0.$$

By Tate duality, it is equivalent to show

$$H^0(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}(1)) = 0.$$

Let L be the real extension of $\mathbf{Q}$ as in Hypothesis 2, we claim that for any place v in L sitting above ℓ, we have

$$H^0(\mathrm{Gal}_{L_v}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}(1)) = 0,$$

where $\mathrm{Gal}_{L_v} = \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/L_v)$ is the absolute Galois group of L_v. However, under the assumption of Hypothesis 2, the desired vanishing follows from [BLGGT14, Lemma 1.3.2] and the discussion in 8.2.6.

Finally, observe that the restriction map

$$\mathrm{Res} : H^0(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}(1)) \to H^0(\mathrm{Gal}_{L_v}, \mathrm{ad}^0\,\rho_{\boldsymbol{x}}^{\mathrm{spin}}(1))$$

is an injection since $k_{\boldsymbol{x}}$ is of characteristic zero so that

$$\mathrm{Corres} \circ \mathrm{Res} = \text{ multiplication by } [L_v : \mathbf{Q}_\ell]$$

is an injection. The assertion then follows. □

8.4.6 (Global Galois deformations)**.** Consider the following two global deformation functors:

(i) The deformation problem

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x}, \mathbb{F}^{\boldsymbol{x}}_\bullet} : \mathbf{A}\textsc{r} \to \mathbf{S}\textsc{ets},$$

sending each $A \in \mathbf{A}\mathrm{\mathbf{R}}$ to isomorphism classes of representations $\rho_A^{\mathrm{spin}} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \mathrm{GSpin}_{2g+1}(A)$ and triangulation $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(\mathrm{spin} \circ \rho_A^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ such that

- $\rho_A^{\mathrm{spin}} \otimes_A k_{\boldsymbol{x}} \simeq \rho_{\boldsymbol{x}}^{\mathrm{spin}}$
- $\det \mathrm{spin} \circ \rho_A^{\mathrm{spin}} = \det \rho_{\boldsymbol{x}}$
- $(\mathrm{spin} \circ \rho_A^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}, \mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{rig}}(\mathrm{spin} \circ \rho_A|_{\mathrm{Gal}_{\mathbf{Q}_p}})) \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}_\bullet^{\boldsymbol{x}},p}(A)$
- $\rho_A^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_\ell}} \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\ell}(A)$ for $\ell \in \mathtt{S}_{\mathrm{bad}}$

(ii) The deformation problem

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f} : \mathbf{A}\mathrm{\mathbf{R}} \to \mathbf{S}\mathrm{\mathbf{ETS}},$$

sending each $A \in \mathbf{A}\mathrm{\mathbf{R}}$ to isomorphism classes of representations $\rho_A^{\mathrm{spin}} : \mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}} \to \mathrm{GSpin}_{2g+1}(A)$ such that

- $\rho_A^{\mathrm{spin}} \otimes_{k_{\boldsymbol{x}}} k_{\boldsymbol{x}} \simeq \rho_{\boldsymbol{x}}^{\mathrm{spin}}$
- $\det \mathrm{spin} \circ \rho_A = \det \rho_{\boldsymbol{x}}$
- $\rho_A^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_p}} \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f,p}(A)$
- $\rho_A^{\mathrm{spin}}|_{\mathrm{Gal}_{\mathbf{Q}_\ell}} \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\ell}(A)$ for $\ell \in \mathtt{S}_{\mathrm{bad}}$.

Lemma 8.4.7. *Keep the above notations.*

(i) The deformation problems $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}_\bullet^{\boldsymbol{x}}}$ and $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f}$ are pro-representable. Denote by $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}_\bullet^{\boldsymbol{x}}}$ and $R^{\mathrm{univ}}_{\boldsymbol{x},f}$ the complete noetherian local rings that represent these two deformation functors respectively.

(ii) Suppose $\mathbb{F}_\bullet^{\boldsymbol{x}}$ is non-critical, then $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f}$ is a subfunctor of $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}_\bullet^{\boldsymbol{x}}}$.

Proof. Since $\rho_{\boldsymbol{x}}$ is absolutely irreducible, the first assertion follows from standard Galois deformation theory (see, for example, [KT17, §4] and [HT17, Proposition 3.7 & Proposition 3.8]). The second assertion is an immediate consequence of [BC09, Proposition 2.5.8]. Notice that our deformation problems are slightly different from the ones considered in *op. cit.* and [HT17]. In fact, one sees easily that our deformation problems are subfunctors of the deformation problems considered therein. Their results implies ours since $\mathrm{spin} : \mathrm{GSpin}_{2g+1} \to \mathrm{GL}_{2^g}$ is a closed immersion, the conditions we required on the relations of the parameters and the fixed determinant of the deformations are closed conditions and they are stable under isomorphisms, *i.e.*, they satisfy the definition of *deformation problems* (see, for example, [KT17, Definition 4.1]). ☐

8.4.8. The the Bloch–Kato Selmer group associated with $\mathrm{ad}^0 \rho_{\boldsymbol{x}}^{\mathrm{spin}}$ is defined to be

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_{\boldsymbol{x}}^{\mathrm{spin}}) := \ker \left(H^1(\mathrm{Gal}_{\mathbf{Q},\mathtt{S}_{\mathrm{bad}}}, \mathrm{ad}^0 \rho_{\boldsymbol{x}}^{\mathrm{spin}}) \xrightarrow{\mathrm{Res}} \prod_{\ell \in \mathtt{S}_{\mathrm{bad}} \cup \{p\}} \frac{H^1(\mathrm{Gal}_{\mathbf{Q}_\ell}, \mathrm{ad}^0 \rho_{\boldsymbol{x}}^{\mathrm{spin}})}{L_\ell} \right), \tag{8.6}$$

where L_ℓ are as defined in (8.4) and (8.5).

Proposition 8.4.9. *The tangent space $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f}(k_{\boldsymbol{x}}[\varepsilon])$ of $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},f}$ can naturally be identified with the Bloch–Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}})$.*

Proof. This is follows from standard Galois deformation theory (see, for example, [HT17, Proposition 3.7]) and the definition of L_p and L_ℓ (see (8.4) and (8.5)). □

8.5 The adjoint Bloch–Kato Selmer groups

8.5.1. We keep the notations and assumptions in the previous subsection. We further assume the following

- the refinement $\mathbb{F}^{\boldsymbol{x}}_{\bullet}$ of $\rho_{\boldsymbol{x}}$ satisfies (REG) and (NCR);[4]
- the representation $\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is not isomorphic to its twist by the p-adic cyclotomic character.

Lemma 8.5.2. *Denote by $\mathbb{T}_{\boldsymbol{x}} := \widehat{\mathscr{O}}_{\mathcal{E}_0^{\mathrm{oc}},\boldsymbol{x}}$. Then, for any ideal of cofinite length $\mathfrak{I} \subset \mathbb{T}_{\boldsymbol{x}}$ there exists a Galois representation*

$$\rho_{\mathfrak{I}} : \mathrm{Gal}_{\mathbf{Q},\mathrm{S}_{\mathrm{bad}}} \to \mathrm{GL}_{2^g}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I})$$

such that

(i) $\rho_{\mathfrak{I}} \otimes_{\mathbb{T}_{\boldsymbol{x}}} k_{\boldsymbol{x}} \simeq \rho_{\boldsymbol{x}}$

(ii) $\rho_{\mathfrak{I}}|_{\mathrm{Gal}_{\mathbf{Q}_p}} \in \mathscr{D}_{\rho_{\boldsymbol{x}}|_{\mathrm{Gal}_{\mathbf{Q}_p}},\mathbb{F}^{\boldsymbol{x}}_{\bullet},p}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I})$

Proof. The first assertion is a consequence of Theorem 8.1.5. The second assertion is a consequence of Theorem 8.1.16. □

Hypothesis 4. *Consider the Galois representation $\rho_{\mathfrak{I}}$ in Lemma 8.5.2 for any ideal of cofinite length $\mathfrak{I} \subset \mathbb{T}_{\boldsymbol{x}}$. We assume*

(i) The Galois representation $\rho_{\mathfrak{I}}$ factors as

$$\rho_{\mathfrak{I}} : \mathrm{Gal}_{\mathbf{Q},\mathrm{S}_{\mathrm{bad}}} \xrightarrow{\rho^{\mathrm{spin}}_{\mathfrak{I}}} \mathrm{GSpin}_{2g+1}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I}) \xrightarrow{\mathrm{spin}} \mathrm{GL}_{2^g}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I}).$$

(ii) The Galois representation $\rho^{\mathrm{spin}}_{\mathfrak{I}}|_{\mathrm{Gal}_{\mathbf{Q}_p}} \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet},p}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I})$.

(iii) The tame level structure $\Gamma^{(p)}$ implies that the Galois representation $\rho^{\mathrm{spin}}_{\mathfrak{I}}$ satisfies

$$\rho^{\mathrm{spin}}_{\mathfrak{I}}|_{\mathrm{Gal}_{\mathbf{Q}_\ell}} \in \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\ell}(\mathbb{T}_{\boldsymbol{x}}\,/\,\mathfrak{I})$$

for any $\ell | N$.

Remark 8.5.3. We remark that the above hypothesis is safe to assume:

[4] In fact, the condition (NCR) is already satisfied by the definition of $\mathcal{G}^{\mathrm{cl}}_{\heartsuit}$.

(i) The first two conditions are natural. When $g=1$, the conditions are trivial. When $g=2$, GSpin_5 is isomorphic to GSp_4. In this case, the proof of [GT05, Lemma 4.3.3] implies the conditions.

(ii) Roughly speaking, the third condition in the hypothesis means that the level structure on the automorphic side determines the ramification type on the Galois side. This condition is inspired by the Taylor–Wiles method. When $g=1$, the classical example is the work of R. Taylor and A. Wiles in [TW95]. In *loc. cit.*, they showed that if one considers the Hecke algebra on the space of weight-2 modular forms of a certain level, then the Galois representation with coefficients in the local Hecke algebra satisfies certain Galois deformation problem. For higher-rank groups, one sees, for example, such a relation in [GT05, §4.3] for GSp_4 and [CHT08, §3.4] for GL_n over CM fields.

Lemma 8.5.4. *Denote by $R_{\mathrm{wt}(\boldsymbol{x})}$ the complete local ring at $\mathrm{wt}(\boldsymbol{x})$ and so we have a natural homomorphism $R_{\mathrm{wt}(\boldsymbol{x})}\to\mathbf{Q}_p\to k_{\boldsymbol{x}}$, where the first map is given by quotienting the maximal ideal and the second map is the natural inclusion. Then, $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}$ admits an action of $R_{\mathrm{wt}(\boldsymbol{x})}$ and*

$$R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}\otimes_{R_{\mathrm{wt}(\boldsymbol{x})}}k_{\boldsymbol{x}}=R^{\mathrm{univ}}_{\boldsymbol{x},f}.$$

Proof. Let us first explain the action of $R_{\mathrm{wt}(\boldsymbol{x})}$ on $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}$. For any $A\in\mathbf{A}\mathrm{R}$, observe that we have a natural morphism

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}(A)\to\mathrm{Hom}_{\mathrm{cts}}(T_{\mathrm{GL}_g,1},A^\times),\quad \rho_A^{\mathrm{spin}}\mapsto((\delta'_{A,g+1})^{-1}|_{\mathbf{Z}_p^\times}-g,(\delta'_{A,g})^{-1}|_{\mathbf{Z}_p^\times}-(g-1),...,(\delta'_{A,2})^{-1}|_{\mathbf{Z}_p^\times}-1).$$

Under this map, the image of $\rho^{\mathrm{spin}}_{\boldsymbol{x}}$ is exactly $k=(k_1,...,k_g)$ by (8.2). Consequently, there is a natural morphism

$$\mathbf{Z}_p[\![T_{\mathrm{GL}_g,1}]\!]\to R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet},$$

which factors through $R_{\mathrm{wt}(\boldsymbol{x})}$.

Since the refinement $\mathbb{F}^{\boldsymbol{x}}_\bullet$ satisfies (REG), together with the relation of parameters and the condition of fixed determinant, the desired isomorphism follows from the constant weight lemma ([BC09, Proposition 2.5.4]), *i.e.*, the crystalline deformations of $\rho_{\boldsymbol{x}}$ are of constant Hodge–Tate weight, of which being the same as $\rho_{\boldsymbol{x}}$. □

Lemma 8.5.5. *Denote by $H^1_{\mathbb{F}^{\boldsymbol{x}}_\bullet}(\mathbf{Q},\mathrm{ad}^0\rho^{\mathrm{spin}}_{\boldsymbol{x}})$ the tangent space $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}(k_{\boldsymbol{x}}[\varepsilon])$ of $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}$. We have an exact sequence*

$$0\to H^1_f(\mathbf{Q},\mathrm{ad}^0\rho^{\mathrm{spin}}_{\boldsymbol{x}})\to H^1_{\mathbb{F}^{\boldsymbol{x}}_\bullet}(\mathbf{Q},\mathrm{ad}^0\rho^{\mathrm{spin}}_{\boldsymbol{x}})\to k^g_{\boldsymbol{x}}.$$

Proof. Following [BC09, Proposition 7.6.4], we expect an exact sequence

$$0\to H^1_f(\mathbf{Q},\mathrm{ad}^0\rho^{\mathrm{spin}}_{\boldsymbol{x}})\to H^1_{\mathbb{F}^{\boldsymbol{x}}_\bullet}(\mathbf{Q},\mathrm{ad}^0\rho^{\mathrm{spin}}_{\boldsymbol{x}})\to k^{2^g}_{\boldsymbol{x}}.$$

The first map is clear while the second map is defined as follows. For any $A\in\mathbf{A}\mathrm{R}$, we have

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}(A)\to\mathrm{Hom}_{\mathrm{cts}}(\mathbf{Q}_p^\times,A^\times)^{2^g},\quad \rho_A\mapsto(\delta_{A,1},...,\delta_{A,2^g}).$$

Composing with the derivative at 1, we obtain a morphism

$$\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}(A) \to A^{2g}.$$

That is, we obtain

$$\partial : \mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \widehat{\mathbb{G}^{2g}_m}.$$

The second map is then defined to be $\partial(k_{\boldsymbol{x}}[\varepsilon])$. Lemma 8.5.4 shows that $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}}) = \ker \partial(k_{\boldsymbol{x}}[\varepsilon])$.

Recall that the local condition of $\mathscr{D}^{\mathrm{spin}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ at p requires a relation of the parameters and a fixed determinant. Thus, the image of $\partial(k_{\boldsymbol{x}}[\varepsilon])$ lies in a subspace of dimension g, depending only on the continuous characters $\delta'_{A,2}$, ..., $\delta'_{A,g+1}$. □

Proposition 8.5.6. *Retain the notation in Lemma 8.5.2 and assume Hypothesis 4 holds.*

(i) There exists a canonical ring homomorphism $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \mathbb{T}_{\boldsymbol{x}}$.

(ii) If the adjoint Bloch–Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}})$ *vanishes, then the canonical map in (i) is an isomorphism* $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \simeq \mathbb{T}_{\boldsymbol{x}}$ *(an 'infinitesimal* $R = T$ *theorem').*

Proof. By Lemma 8.5.2 and Hypothesis 4, for any ideal $\mathfrak{I} \subset \mathbb{T}_{\boldsymbol{x}}$ of cofinite length, there is a canonical ring homomorphism

$$R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \mathbb{T}_{\boldsymbol{x}} / \mathfrak{I}.$$

This ring homomorphism is surjective due to the fact that the characteristic polynomials of the Frobenii under $\rho_{\mathfrak{I}}$ are given by the Hecke polynomials. Consequently, one obtains a canonical morphism

$$R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \mathbb{T}_{\boldsymbol{x}} = \varprojlim_{\mathfrak{I}\,:\,\text{cofinite length}} \mathbb{T}_{\boldsymbol{x}} / \mathfrak{I}$$

with dense image. Since $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ is complete, the canonical morphism $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \mathbb{T}_{\boldsymbol{x}}$ is surjective.

Finally, if $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}})$ vanishes, then the exact sequence in Lemma 8.5.5 implies that

$$\dim_{k_{\boldsymbol{x}}} H^1_{\mathbb{F}^{\boldsymbol{x}}_{\bullet}}(\mathbf{Q}, \mathrm{ad}^0\,\rho^{\mathrm{spin}}_{\boldsymbol{x}}) \le g.$$

Since $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ is a local noetherian ring, its Krull dimension is bounded by the dimension of its tangent space ([Stacks, Section 00KD]), *i.e.*, $\dim R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \le g$. Moreover, we also know from *loc. cit.* that the equality holds if and only if $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ is regular. However, since $\mathcal{E}^{\mathrm{oc}}_0$ is equidimensional and finite over $\mathcal{W}$, we know that $\dim \mathbb{T}_{\boldsymbol{x}} = \dim \mathcal{W} = g$. Therefore,

$$g \ge \dim R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \ge \dim \mathbb{T}_{\boldsymbol{x}} = g$$

and $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ is regular of dimension g. To conclude the proof, suppose $\mathfrak{a} = \ker(R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} \to \mathbb{T}_{\boldsymbol{x}})$ is non-zero and so we can identify $\mathbb{T}_{\boldsymbol{x}}$ with $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} / \mathfrak{a}$. Since $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}}$ is a regular local ring, it is a domain ([Stacks, Lemma 00NP]). We then obtain a contradiction

$$g = \dim R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} > \dim R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_{\bullet}} / \mathfrak{a} = \dim \mathbb{T}_{\boldsymbol{x}} = g.$$

□

Corollary 8.5.7. *Suppose Hypothesis 1, Hypothesis 2, and Hypothesis 4 hold. Assume the following also hold:*

- *The cuspidal automorphic representation $\pi_{\boldsymbol{x}}$ of $\mathrm{GL}_{2^g}(\mathbf{A}_L)$ associated with $\rho_{\boldsymbol{x}}$ as in Hypothesis 2 is regular algebraic and polarised (see, for example, [BLGGT14, §2.1]).*

- *The image $\rho_{\boldsymbol{x}}(\mathrm{Gal}_{L(\zeta_{p^\infty})})$ is enormous (see [NT20, Definition 2.27]).*

Then

(i) $\dim_{k_{\boldsymbol{x}}} H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho^{\mathrm{spin}}_{\boldsymbol{x}}) = 0$ *and*

(ii) $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet} \simeq \mathbb{T}_{\boldsymbol{x}}$.

Proof. By the discussion in 8.2.6, we have

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho^{\mathrm{spin}}_{\boldsymbol{x}}) \subset H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_{\boldsymbol{x}}).$$

However, the latter space vanishes by [NT20, Theorem 5.3] and so we conclude by Proposition 8.5.6. □

8.5.8. We conclude this thesis with another situation that one can also deduce the vanishing of the adjoint Bloch–Kato Selmer group. In this situation, one obtains a link between L^{adj} and the adjoint Bloch–Kato Selmer group, which then (conjecturally) justifies the name for L^{adj}.

Corollary 8.5.9. *Suppose Hypothesis 1 and Hypothesis 4 hold. Suppose the weight map is étale at $\boldsymbol{x}$ and suppose the canonical morphism $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet} \to \mathbb{T}_{\boldsymbol{x}}$ is an isomorphism. Then,*

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho^{\mathrm{spin}}_{\boldsymbol{x}}) = 0.$$

In particular, we have

$$\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} = \dim_{k_{\boldsymbol{x}}} H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho^{\mathrm{spin}}_{\boldsymbol{x}}).$$

Proof. Observe the following sequence of isomorphisms

$$\begin{aligned}
\Omega^1_{\mathbb{T}_{\boldsymbol{x}}/R_{\mathrm{wt}(\boldsymbol{x})}} \otimes_{\mathbb{T}_{\boldsymbol{x}}} k_{\boldsymbol{x}} &\simeq \Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}/R_{\mathrm{wt}(\boldsymbol{x})}} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}} k_{\boldsymbol{x}} \\
&\simeq \Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}/R_{\mathrm{wt}(\boldsymbol{x})}} \widehat{\otimes}_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}} R^{\mathrm{univ}}_{\boldsymbol{x},f} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},f}} k_{\boldsymbol{x}} \\
&\simeq \Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}/R_{\mathrm{wt}(\boldsymbol{x})}} \widehat{\otimes}_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}} R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet} \otimes_{R_{\mathrm{wt}(\boldsymbol{x})}} k_{\boldsymbol{x}} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},f}} k_{\boldsymbol{x}} \\
&\simeq \Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet}/R_{\mathrm{wt}(\boldsymbol{x})}} \otimes_{R_{\mathrm{wt}(\boldsymbol{x})}} k_{\boldsymbol{x}} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},f}} k_{\boldsymbol{x}} \\
&\simeq \Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},f}/k_{\boldsymbol{x}}} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},f}} k_{\boldsymbol{x}}.
\end{aligned}$$

Here, the first isomorphism follows from the assumption $R^{\mathrm{univ}}_{\boldsymbol{x},\mathbb{F}^{\boldsymbol{x}}_\bullet} \simeq \mathbb{T}_{\boldsymbol{x}}$ and the third and the final isomorphism follows from Lemma 8.5.4. Therefore, we have

$$\begin{aligned}\dim_{k_{\boldsymbol{x}}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho^{\mathrm{spin}}_{\boldsymbol{x}}) &= \dim_{k_{\boldsymbol{x}}} \mathrm{Hom}_{k_{\boldsymbol{x}}}(\Omega^1_{R^{\mathrm{univ}}_{\boldsymbol{x},f}/k_{\boldsymbol{x}}} \otimes_{R^{\mathrm{univ}}_{\boldsymbol{x},f}} k_{\boldsymbol{x}}, k_{\boldsymbol{x}}) \\ &= \dim_{k_{\boldsymbol{x}}} \mathrm{Hom}_{k_{\boldsymbol{x}}}(\Omega^1_{\mathbb{T}_{\boldsymbol{x}}/R_{\mathrm{wt}(\boldsymbol{x})}} \otimes_{\mathbb{T}_{\boldsymbol{x}}} k_{\boldsymbol{x}}, k_{\boldsymbol{x}}) \\ &= \dim_{k_{\boldsymbol{x}}} \Omega^1_{\mathbb{T}_{\boldsymbol{x}}/R_{\mathrm{wt}(\boldsymbol{x})}} \otimes_{\mathbb{T}_{\boldsymbol{x}}} k_{\boldsymbol{x}} \\ &\leq \mathrm{length}_{\mathbb{T}_{\boldsymbol{x}}} \Omega^1_{\mathbb{T}_{\boldsymbol{x}}/R_{\mathrm{wt}(\boldsymbol{x})}}.\end{aligned}$$

However, since the weight map is étale at $\boldsymbol{x}$, $\mathrm{length}_{\mathbb{T}_{\boldsymbol{x}}} \Omega^1_{\mathbb{T}_{\boldsymbol{x}}/R_{\mathrm{wt}(\boldsymbol{x})}} = 0$. We then conclude the result. $\square$

Remark 8.5.10. More generally, in light of the Bloch–Kato conjecture (Conjecture 1.2.2), we expect that, if $\boldsymbol{x}$ is a smooth point,

$$\mathrm{ord}_{\boldsymbol{x}} L^{\mathrm{adj}} = \dim_{k_{\boldsymbol{x}}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho^{\mathrm{spin}}_{\boldsymbol{x}}).$$

In particular, since $H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho^{\mathrm{spin}}_{\boldsymbol{x}})$ is expected to vanish, it seems fair to expect that, if $\boldsymbol{x}$ is a smooth point with small slope and at which L^{adj} is defined, the weight map is étale at $\boldsymbol{x}$. When $g = 1$, this is [Bel12, Theorem 2.16].

Appendix A

Log adic spaces

The notion of 'log adic spaces' was employed in the main body of this thesis in order to study the boundaries of toroidally compactified Siegel modular varieties. For the convenience of the readers, we briefly recall these notions in this appendix as well as some results that we used in the main body of this thesis, especially in the construction of the overconvergent automorphic sheaves and the construction of overconvergent Eichler–Shimura morphisms. More precisely, we succinctly review log adic spaces and their (pro)-Kummer étale sites in §A.1. In §A.2, we adopt the notion of *Banach sheaves* first introduced in [AIP15, §A] and prove a (generalised) projection formula.

A.1 Review of log adic spaces

In this section, let K be a complete field extension of $\mathbf{Q}_p$ and let $\mathcal{O}_K = \{x \in k : |x| \leq 1\}$ be its ring of integers.

Definition A.1.1. *Let X be a locally noetherian adic space over* $\mathrm{Spa}(K, \mathcal{O}_K)$.

*(i) A **pre-log structure** on X is a pair $(\mathscr{M}_X, \alpha)$ where $\mathscr{M}_X$ is a sheaf of monoids on $X_{\acute{e}t}$ and $\alpha : \mathscr{M}_X \to \mathscr{O}_{X_{\acute{e}t}}$ is a morphism of sheaves of (multiplicative) monoids. It is called a **log structure** if the induced morphism $\alpha^{-1}(\mathscr{O}^{\times}_{X_{\acute{e}t}}) \to \mathscr{O}^{\times}_{X_{\acute{e}t}}$ is an isomorphism. In this case, the triple $(X, \mathscr{M}_X, \alpha)$ is called a **log adic space**. If the context is clear, we simply say that X is a log adic space.*

*(ii) For a pre-log structure $(\mathscr{M}_X, \alpha)$ on X, the **associated log structure** is $({}^a\mathscr{M}_X, {}^a\alpha)$ where ${}^a\mathscr{M}_X$ is given by the pushout*

$$\begin{tikzcd} \alpha^{-1}(\mathscr{O}^{\times}_{X_{\acute{e}t}}) \arrow[r] \arrow[d] & \mathscr{M}_X \arrow[d] \\ \mathscr{O}^{\times}_{X_{\acute{e}t}} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & {}^a\mathscr{M}_X \end{tikzcd}$$

and ${}^a\alpha : {}^a\mathscr{M}_X \to \mathscr{O}_{X_{\acute{e}t}}$ is the induced morphism.

(iii) *A* ***morphism*** $f : (Y, \mathscr{M}_Y, \alpha_Y) \to (X, \mathscr{M}_X, \alpha_X)$ *of log adic spaces is a morphism* $f : Y \to X$ *of adic spaces together with a morphism of sheaves of monoids* $f^\sharp : f^{-1}\mathscr{M}_X \to \mathscr{M}_Y$ *such that the diagram*

$$\begin{tikzcd} f^{-1}\mathscr{M}_X \arrow[r, "f^\sharp"] \arrow[d, "f^{-1}\alpha_X"'] & \mathscr{M}_Y \arrow[d, "\alpha_Y"] \\ f^{-1}\mathscr{O}_{X_{\acute{e}t}} \arrow[r] & \mathscr{O}_{Y_{\acute{e}t}} \end{tikzcd}$$

commutes. Moreover, the log structure associated with the pre-log structure $f^{-1}\mathscr{M}_X \to f^{-1}\mathscr{O}_{X_{\acute{e}t}} \to \mathscr{O}_{Y_{\acute{e}t}}$ *is called the* ***pullback log structure****, denoted by* $f^*\mathscr{M}_X$. *We say that* f *is* ***strict*** *if* $f^*\mathscr{M}_X \xrightarrow{\sim} \mathscr{M}_Y$.

Definition A.1.2. *(i) Let* $(X, \mathscr{M}_X, \alpha)$ *be a locally noetherian log adic space as above. Let* P *be a monoid and let* P_X *denote the associated constant sheaf of monoids on* $X_{\acute{e}t}$. *A* ***chart of*** X ***modeled on*** P *is a morphism of sheaves of monoids* $\theta : P_X \to \mathscr{M}_X$ *such that* $\alpha(\theta(P_X)) \subset \mathscr{O}^+_{X_{\acute{e}t}}$ *and such that the log structure associated with the pre-log structure* $\alpha \circ \theta : P_X \to \mathscr{O}_{X_{\acute{e}t}}$ *is isomorphic to* $\mathscr{M}_X$. *We say that the chart is* ***fs*** *if* P *is fine and saturated.*

(ii) A locally noetherian log adic space is called an ***fs log adic space*** *if it étale locally admits charts modeled on fs monoids.*

(iii) Let $f : (Y, \mathscr{M}_Y, \alpha_Y) \to (X, \mathscr{M}_X, \alpha_X)$ *be a morphism between locally noetherian log adic spaces. A* ***chart*** *of* f *consists of charts* $\theta_X : P_X \to \mathscr{M}_X$ *and* $\theta_Y : Q_Y \to \mathscr{M}_Y$ *and a homomorphism* $u : P \to Q$ *such that the diagram*

$$\begin{tikzcd} P_Y \arrow[r, "u"] \arrow[d, "\theta_X"] & Q_Y \arrow[d, "\theta_Y"] \\ f^{-1}\mathscr{M}_X \arrow[r, "f^\sharp"] & \mathscr{M}_Y \end{tikzcd}$$

commutes. We say that the chart is ***fs*** *if both* P *and* Q *are fs. When the context is clear, we simply say that* $u : P \to Q$ *is a chart of* f.

Example A.1.3. Below we give two typical examples of locally noetherian fs log adic spaces.

(i) Let $n > 0$ be an integer. Consider the ***n*-dimensional unit disc**

$$\mathbb{D}^n := \mathrm{Spa}(K\langle T_1, \ldots, T_n\rangle, \mathcal{O}_K\langle T_1, \ldots, T_n\rangle),$$

equipped with the log structure associated with the pre-log structure induced by

$$\mathbf{Z}^n_{\geq 0} \to K\langle T_1, \ldots, T_n\rangle, \quad (a_1, \ldots, a_n) \mapsto T_1^{a_1} \cdots T_n^{a_n}.$$

Clearly, $\mathbb{D}^n$ is modeled on the fs chart $\mathbf{Z}^n_{\geq 0}$.

(ii) Let X be a smooth rigid analytic variety over K, viewed as an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ via [Hub13, (1.1.11)]. Let $D \subset X$ be a ***normal crossings divisor*** in the sense of [DLLZ19, Example 2.3.17]. Namely, $\iota : D \hookrightarrow X$ is a closed immersion such that, analytic locally, X and D are of the form $S \times \mathbb{D}^n$ and $S \times \{T_1 \cdots T_n = 0\}$, where S is a smooth connected rigid analytic variety and ι is the pullback of the natural inclusion $\{T_1 \cdots T_n = 0\} \hookrightarrow \mathbb{D}^n$. We equip X with the log structure

$$\mathscr{M}_X = \{f \in \mathscr{O}_{X_{\text{ét}}} \mid f \text{ is invertible on } X \smallsetminus D\}$$

with $\alpha : \mathscr{M}_X \to \mathscr{O}_{X_{\text{ét}}}$ being the natural inclusion. This is the ***divisorial log structure*** associated with the divisor D. This log structure agrees with the pullback of the log structure on $\mathbb{D}^n$ constructed above.

A.1.4. Log adic spaces in the example above are, in fact, *log smooth* log adic spaces. To recall the definition of log smoothness, we first set up some notation.

For any monoid P and any commutative ring R, we write $R[P]$ for the associated monoid algebra. Now, given a locally noetherian adic space X over $\mathrm{Spa}(K, \mathcal{O}_K)$ and a finitely generated monoid P, we let $(R\langle P\rangle, R^+\langle P\rangle)$ be the completion of $(R[P], R^+[P])$, for any affinoid open subspace $\mathrm{Spa}(R, R^+) \subset X$. By gluing the morphisms $\mathrm{Spa}(R\langle P\rangle, R^+\langle P\rangle) \to \mathrm{Spa}(R, R^+)$, we obtain a morphism $X\langle P\rangle \to X$. Moreover, we equip $X\langle P\rangle$ with the log structure modeled on the chart P; *i.e.*, the one locally induced by $P \to R\langle P\rangle$.

Definition A.1.5. *Let $f : Y \to X$ be a morphism between locally noetherian fs log adic spaces. We say that f is* ***log smooth*** *if étale locally f admits an fs chart $u : P \to Q$ such that*

(i) the kernel and the torsion part of the cokernel of $u^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ are finite groups of order invertible in $\mathscr{O}_X$; and

(ii) f and u induce a morphism $Y \to X \times_{X\langle P\rangle} X\langle Q\rangle$ of log adic spaces whose underlying morphism of adic spaces is étale.

A locally noetherian fs log adic space X is ***log smooth*** *if the structure morphism $X \to \mathrm{Spa}(K, \mathcal{O}_K)$ is log smooth, where $\mathrm{Spa}(K, \mathcal{O}_K)$ is equipped with the trivial log structure.*

A.1.6. The notion of 'Kummer étaleness' play an essential role in the main body of the thesis. Let us now recall its definition.

Definition A.1.7. *(i) An injective homomorphism $u : P \to Q$ of fs monoids is called* ***Kummer*** *if for every $a \in Q$, there exists some integer $n > 0$ such that $na \in u(P)$.*

(ii) A morphism (resp., finite morphism) $f : Y \to X$ of locally noetherian fs log adic spaces is called ***Kummer étale*** *(resp.,* ***finite Kummer étale****) if étale locally on X and Y, f admits an fs chart $u : P \to Q$ which is Kummer with $|Q^{\mathrm{gp}}/u^{\mathrm{gp}}(P^{\mathrm{gp}})|$ invertible on $\mathscr{O}_Y$, and such that f and u induce a morphism $Y \to X \times_{X\langle P\rangle} X\langle Q\rangle$ of log adic spaces whose underlying morphism of adic spaces is étale (resp., finite étale).*

(iii) If a Kummer étale (resp., finite Kummer étale) morphism is strict, we say it is ***strictly étale*** *(resp.,* ***strictly finite étale****).*

Remark A.1.8. By [DLLZ19, Lemma 4.1.10], if $f : Y \to X$ is a Kummer étale morphism between locally noetherian fs log adic spaces and if X admits a chart modeled on a sharp fs monoid P, then, étale locally on X and Y, the morphism f admits a Kummer fs chart $P \to Q$ with Q being sharp.

Definition A.1.9. *Let X be a locally noetherian fs log adic space.*

(i) The ***Kummer étale site*** *$X_{\text{két}}$ (resp.,* ***finite Kummer étale site*** *X_{fket}) of X is defined as follows. The underlying category is the full subcategory of the category of locally noetherian fs log adic spaces consisting of objects that are Kummer étale (resp., finite Kummer étale) over X. The coverings are given by the topological coverings.*

(ii) The ***structure sheaf*** *$\mathscr{O}_{X_{\text{két}}}$ (resp.,* ***integral structure sheaf*** *$\mathscr{O}^+_{X_{\text{két}}}$) on $X_{\text{két}}$ is defined to be the presheaf sending $U \mapsto \mathscr{O}_U(U)$ (resp., $U \mapsto \mathscr{O}^+_U(U)$). We also write $\mathscr{M}_{X_{\text{két}}}$ for the presheaf sending $U \mapsto \mathscr{M}_U(U)$. By [DLLZ19, Theorem 4.3.1, Proposition 4.3.4], these presheaves are indeed sheaves.*

Proposition A.1.10 ([DLLZ19, Proposition 3.1.10])**.** *Let X be an fs log adic space that is log smooth over $\mathrm{Spa}(K, \mathcal{O}_K)$. Then, étale locally on X, there exists a* ***toric chart*** *$X \to \mathbb{E} = \mathrm{Spa}(K\langle P\rangle, \mathcal{O}_K\langle P\rangle)$ for some sharp fs monoid P, i.e., a strictly étale morphism $K \to \mathbb{E} = \mathrm{Spa}(K\langle P\rangle, \mathcal{O}_K\langle P\rangle)$ that is a composition of rational localisations and finite étale morphisms.*

Proposition A.1.11 ([DLLZ19, Corollary 4.4.18])**.** *Let X be a connected locally noetherian fs log adic space and let ξ be a log geometric point (see [DLLZ19, Definition 4.4.2]). Then there is an equivalence of categories*

$$F_X : X_{\text{fkét}} \xrightarrow{\simeq} \pi_1^{\text{két}}(X, \xi) - \mathbf{FS\small{ETS}}$$

sending $Y \mapsto Y_\xi := \mathrm{Hom}_X(\xi, Y)$, where the $\pi_1^{\text{két}}(X, \xi) - \mathbf{FS\small{ETS}}$ denotes the category of finite sets equipped with a continuous action of the ***Kummer étale fundamental group*** *$\pi_1^{\text{két}}(X, \xi)$.*

Moreover, for any two log geometric points ξ and ξ', the fundamental groups $\pi_1^{\text{két}}(X, \xi)$ and $\pi_1^{\text{két}}(X, \xi')$ are isomorphic. Hence, we may omit 'ξ' from the notation whenever the context is clear.

Lemma A.1.12. *Assume K is of characteristic 0. Let X and Y be locally noetherian fs log adic spaces whose underlying adic spaces are smooth connected rigid analytic varieties over K. Suppose the log structures on X and Y are the divisorial log structures associated with the normal crossing divisors $D \subset X$ and $E \subset Y$ as in Example A.1.3 (ii). Let $U = X \smallsetminus D$ and $V = Y \smallsetminus E$. Suppose we have a finite Kummer étale surjective morphism $f : Y \to X$ such that $f^{-1}(U) = V$ and that $f|_V : V \to U$ is a finite étale Galois cover with Galois group G. Then f is a finite Kummer étale Galois cover with Galois group G.*

Proof. According to Proposition A.1.11, we have equivalences of categories

$$F_X : X_{\text{fkét}} \xrightarrow{\simeq} \pi_1^{\text{két}}(X) - \mathbf{FS\small{ETS}}$$

and

$$F_U : U_{\text{fét}} \xrightarrow{\simeq} \pi_1^{\text{ét}}(U) - \mathbf{FS\small{ETS}}\,.$$

We have to show that G is a finite quotient of $\pi_1^{\mathrm{k\acute{e}t}}(X)$ and, under the equivalence F_X, Y corresponds to the finite set G equipped with the natural $\pi_1^{\mathrm{k\acute{e}t}}(X)$-action.

By [DLLZ19, Proposition 4.2.1] and [Han20, Theorem 1.6], we have an equivalence of categories between $X_{\mathrm{fk\acute{e}t}}$ and $U_{\mathrm{f\acute{e}t}}$, under which Y corresponds to V. It also induces a natural isomorphism $\pi_1^{\mathrm{k\acute{e}t}}(X) \simeq \pi_1^{\mathrm{\acute{e}t}}(U)$ making the following diagram commutative.

$$\begin{tikzcd} X_{\mathrm{fk\acute{e}t}} \arrow[r, "\simeq"] \arrow[d, "\simeq"', "F_X"] & U_{\mathrm{f\acute{e}t}} \arrow[d, "\simeq"', "F_U"] \\ \pi_1^{\mathrm{k\acute{e}t}}(X) - \mathbf{FSets} \arrow[r, "\simeq"] & \pi_1^{\mathrm{\acute{e}t}}(U) - \mathbf{FSets} \end{tikzcd}$$

Since V corresponds to the finite set G under the equivalence F_U, we are done. □

A.1.13. One sees in the main body of the thesis that not only the notion of Kummer étaleness plays an important role, but the terminology of 'pro-Kummer étaleness' is also an essential player in the perfectoid method. Let us recall its definition.

Definition A.1.14. *Let X be a locally noetherian fs log adic space over* $\mathrm{Spa}(K, \mathcal{O}_K)$.

(i) The ***pro-Kummer étale site*** *$X_{prok\acute{e}t}$ of X is defined as follows. The underlying category is the full subcategory of pro-$X_{k\acute{e}t}$ consisting of cofiltered inverse limit $Y = \varprojlim_{i\in I} Y_i$ with $Y_i \in X_{k\acute{e}t}$ such that the transition morphisms $Y_i \to Y_j$ are finite Kummer étale and are surjective for sufficiently large i. Such an inverse limit if called a* ***pro-Kummer étale presentation*** *of Y. As for the coverings, we refer the readers to [DLLZ19, Definition 5.1.1, 5.1.2] for details.*

(ii) There is a natural projection of sites

$$\nu : X_{prok\acute{e}t} \to X_{k\acute{e}t}.$$

The ***structure sheaves*** *on $X_{prok\acute{e}t}$ are given by*

$$\mathscr{O}^+_{X_{prok\acute{e}t}} := \nu^{-1}\,\mathscr{O}^+_{X_{k\acute{e}t}}, \quad \mathscr{O}_{X_{prok\acute{e}t}} := \nu^{-1}\,\mathscr{O}_{X_{k\acute{e}t}}$$

and the ***completed structure sheaves*** *are given by*

$$\widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}} := \varprojlim_n \left(\mathscr{O}_{X_{prok\acute{e}t}}/p^n\right), \quad \widehat{\mathscr{O}}_{X_{prok\acute{e}t}} := \widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}}[1/p].$$

We also write $\mathscr{M}_{X_{prok\acute{e}t}} := \nu^{-1}(\mathscr{M}_{k\acute{e}t})$ together with a natural morphism $\alpha : \mathscr{M}_{prok\acute{e}t} \to \mathscr{O}_{X_{prok\acute{e}t}}$.

A.1.15. Similar to the pro-étale topology, the pro-Kummer étale topology admits a convenient basis consisting of the *log affinoid perfectoid objects.*

Definition A.1.16. *An object U in $X_{prok\acute{e}t}$ is called* ***log affinoid perfectoid*** *if it admits a pro-Kummer étale presentation $U = \varprojlim_{i\in I} U_i$ such that*

(i) There is an initial object $0 \in I$;

(ii) *Each $U_i = (\mathrm{Spa}(R_i, R_i^+)$ is affinoid and admits a chart modeled on a sharp fs monoid P_i such that each transition morphism $U_j \to U_i$ is modeled on a Kummer chart $P_i \to P_j$;*

(iii) *The affinoid algebra $(R, R^+) := \big(\varinjlim_{i\in I}(R_i, R_i^+)\big)^\wedge$ is a perfectoid affinoid algebra, where the completion is with respect to the p-adic topology;*

(iv) *The monoid $P := \varinjlim_{i\in I} P_i$ is n-**divisible**, for all $n \in \mathbf{Z}_{\geq 1}$. Namely, the n-th multiple map $[n] : P \to P$ is surjective for all $n \in \mathbf{Z}_{\geq 1}$.*

*Such a presentation $U = \varprojlim_{i\in I} U_i$ is called a **perfectoid presentation** of U.*

Proposition A.1.17 ([DLLZ19, Proposition 5.3.12])**.** *The log affinoid perfectoid objects in $X_{prok\acute{e}t}$ form a basis of the pro-Kummer étale site.*

Proposition A.1.18 ([DLLZ19, Theorem 5.4.3])**.** *Let $U \in X_{prok\acute{e}t}$ be a log affinoid perfectoid object, with the associated perfectoid space $\widehat{U} = \mathrm{Spa}(R, R^+)$. Then*

(i) *For each $n \in \mathbf{Z}_{\geq 1}$, we have $\mathscr{O}^+_{X_{prok\acute{e}t}}(U)/p^n \simeq R^+/p^n$, and it is canonically almost isomorphic to $(\mathscr{O}^+_{X_{prok\acute{e}t}}/p^n)(U)$.*

(ii) *For each $n \in \mathbf{Z}_{\geq 1}$ and $i \in \mathbf{Z}_{\geq 1}$, $H^i(U, \mathscr{O}^+_{X_{prok\acute{e}t}}/p^n)$ is almost equal to zero. Consequently, $H^i(U, \widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}})$ is almost equal to zero.*

(iii) *We have $\widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}}(U) \simeq R^+$ and $\widehat{\mathscr{O}}_{X_{prok\acute{e}t}}(U) \simeq R$. Moreover, $\widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}}(U)$ is canonically isomorphic to the p-adic completion of $\mathscr{O}^+_{X_{prok\acute{e}t}}(U)$.*

Example A.1.19. We recall the following example from [DLLZ19, §6]. Let P be a sharp fs monoid. Consider

$$\mathbb{E} := \mathrm{Spa}(\mathbf{C}_p\langle P\rangle, \mathcal{O}_{\mathbf{C}_p}\langle P\rangle)$$

equipped with the natural log structure modeled on chart P. (If $P = \mathbf{Z}^n_{\geq 0}$, then $\mathbb{E}$ is just the n-dimensional unit disc in Example A.1.3 (i).) For each $m \in \mathbf{Z}_{>0}$, let $\frac{1}{m}P$ denote the sharp fs monoid containing P such that the inclusion $P \hookrightarrow \frac{1}{m}P$ is isomorphic to the m-th multiple map $[m] : P \to P$. Define

$$\mathbb{E}_m := \mathrm{Spa}(\mathbf{C}_p\langle \frac{1}{m}P\rangle, \mathcal{O}_{\mathbf{C}_p}\langle \frac{1}{m}P\rangle)$$

equipped with the natural log structure modeled on the chart $\frac{1}{m}P$. If $m|m'$, there is a natural finite Kummer étale morphism $\mathbb{E}_{m'} \to \mathbb{E}_m$ modeled on the chart $\frac{1}{m}P \hookrightarrow \frac{1}{m'}P$. According to [DLLZ19, Proposition 4.1.6], the morphism $\mathbb{E}_m \to \mathbb{E}$ is actually finite Kummer étale Galois with Galois group

$$\Gamma_{/m} := \mathrm{Hom}\big((\frac{1}{m}P)^{\mathrm{gp}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty\big),$$

where $\boldsymbol{\mu}_\infty$ denotes the group of all roots of unity in $\mathbf{C}_p$. Let $P_{\mathbf{Q}_{\geq 0}} := \varinjlim_m(\frac{1}{m}P)$. It turns out

$$\widetilde{\mathbb{E}} := \varprojlim_m \mathbb{E}_m \in \mathbb{E}_{\mathrm{prok\acute{e}t}}$$

is a log affinoid perfectoid object, with associated perfectoid space

$$\widehat{\widetilde{\mathbb{E}}} = \mathrm{Spa}(\mathbf{C}_p\langle P_{\mathbf{Q}_{\geq 0}}\rangle, \mathcal{O}_{\mathbf{C}_p}\langle P_{\mathbf{Q}_{\geq 0}}\rangle).$$

A.1.20. Following [DLLZ19, Definition 6.1.2], a pro-Kummer étale cover $Y \to X$ is called a ***Galois cover with (profinite) Galois group*** G if there exists a presentation $Y = \varprojlim_i Y_i$ such that each $Y_i \to X$ is a finite Kummer étale cover with Galois group G_i and $G \simeq \varprojlim_i G_i$.

For example, $\widetilde{\mathbb{E}}$ is a Galois cover over $\mathbb{E}$ with profinite Galois group

$$\Gamma \cong \varprojlim_m \Gamma_{/m} = \varprojlim_m \mathrm{Hom}((\frac{1}{m}P)^{\mathrm{gp}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty) \cong \mathrm{Hom}(P^{\mathrm{gp}}_{\mathbf{Q}_{\geq 0}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty)$$

(see [DLLZ19, (6.1.4)]).

A.2 Banach sheaves and a (generalised) projection formula

In this section, we introduce the notion of 'Banach sheaves' on the Kummer étale topology of a log adic space, generalising the ones studied in [AIP15, §A] and [BP20, §2]. Then, for certain *admissible* Banach sheaves, we prove a projection formula which will be used to construct the overconvergent Eichler–Shimura morphism in Chapter 6.

A.2.1. Recall that a *small* $\mathbf{Z}_p$*-algebra* is a p-torsion free reduced ring R which is also a finite $\mathbf{Z}_p[\![T_1, ..., T_d]\!]$-algebra for some $d \in \mathbf{Z}_{\geq 0}$. It is a profinite flat $\mathbf{Z}_p$-module in the sense of [CHJ17, Definition 6.1]. In particular, there exists a set of elements $\{e_\sigma : \sigma \in \Sigma\}$ in R such that $R \simeq \prod_{\sigma\in\Sigma} \mathbf{Z}_p\, e_\sigma$ equipped with the product topology. This set of elements $\{e_\sigma : \sigma \in \Sigma\}$ is called a ***pseudobasis*** for R. Moreover, R is equipped with an adic profinite topology and is complete with respect to the p-adic topology.

Throughout this section, we keep the following notations:

- Let R be a fixed small $\mathbf{Z}_p$-algebra and let $\mathfrak{a}$ be a fixed ideal of definition containing p.
- All (log) adic spaces are assumed to be reduced and quasi-separated. In particular, X either stands for a locally noetherian reduced adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ or a locally noetherian reduced fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. In the second case, we use X_{an} to denote the underlying adic space of X.
- We adopt the notation of 'mixed completed tensors' $-\widehat{\otimes}'R$ and $-\widehat{\otimes}R$ as in Definition 3.1.5.

Lemma A.2.2. *(i) Let X be a locally noetherian adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Then the presheaf $\mathscr{O}_X^+\widehat{\otimes}'R$ (resp., $\mathscr{O}_X\widehat{\otimes}R$) sending any quasi-compact open subset $U \subset X$ to $\mathscr{O}_X^+(U)\widehat{\otimes}'R$ (resp., $\mathscr{O}_X(U)\widehat{\otimes}R$) is a sheaf. In particular, $\mathscr{O}_X\widehat{\otimes}R$ is a sheaf of Banach $\mathbf{C}_p$-algebras.*

(ii) Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Then the presheaf $\mathscr{O}^+_{X_{k\acute{e}t}}\widehat{\otimes}' R$ (resp., $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes} R$) sending any quasi-compact $U \in X_{k\acute{e}t}$ to $\mathscr{O}^+_{X_{k\acute{e}t}}(U)\widehat{\otimes}' R$ (resp., $\mathscr{O}_{X_{k\acute{e}t}}(U)\widehat{\otimes} R$) is a sheaf. In particular, $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes} R$ is a sheaf of Banach $\mathbf{C}_p$-algebras.

(iii) Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Then the presheaf $\widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}}\widehat{\otimes}' R$ (resp., $\widehat{\mathscr{O}}_{X_{prok\acute{e}t}}\widehat{\otimes} R$) sending any qcqs $U \in X_{prok\acute{e}t}$ to $\widehat{\mathscr{O}}^+_{X_{prok\acute{e}t}}(U)\widehat{\otimes}' R$ (resp., $\widehat{\mathscr{O}}_{X_{prok\acute{e}t}}(U)\widehat{\otimes} R$) is a sheaf. In particular, $\widehat{\mathscr{O}}_{X_{prok\acute{e}t}}\widehat{\otimes} R$ is a sheaf of Banach $\mathbf{C}_p$-algebras.

Proof. Choosing a presentation $R \simeq \prod_{\sigma\in\Sigma} \mathbf{Z}_p\, e_\sigma$ and using [CHJ17, Proposition 6.4], the statements reduce to the sheafiness of the corresponding structure presheaves. □

A.2.3. As already suggested by the title of this section, *Banach modules* are essential in this business. For completeness, we recall such a notion and several related terminologies by following [Buz07].

Let B be a Banach $\mathbf{Q}_p$-algebra and let B_0 be an open and bounded $\mathbf{Z}_p$-submodule.

(i) A topological B-module M is called a ***Banach B-module*** if there exists an open bounded B_0-submodule M_0 which is p-adically complete and separated such that $M = M_0[1/p]$.

(ii) Let J be an index set. Consider the B-module $B(J)$ consisting of sequences $\{b_j : j \in J\}$ which converge to 0 with respect to the filter in J of the complement of the finite subsets of J. It turns out $B(J)$ is a Banach B-module. Indeed, let $B_0(J)$ be the p-adic completion of the free B_0-module $\bigoplus_{j\in J} B_0$. Then we have $B(J) \simeq B_0(J)[1/p]$.

(iii) A topological B-module M is called an ***orthonormalisable Banach B-module*** (or, ***ON-able Banach B-module*** for short) if there exists a topological isomorphism $M \simeq B(J)$ for some index set J. A topological B-module M is called a ***projective Banach B-module*** if it is a direct summand (as a topological B-module) inside an orthonormalisable Banach B-module.

Definition A.2.4. *Let X be a locally noetherian adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$.*

*(i) A sheaf of topological $\mathscr{O}_X\widehat{\otimes} R$-modules $\mathscr{F}$ is called a **Banach sheaf of $\mathscr{O}_X\widehat{\otimes} R$-modules** if*

- *for every quasi-compact open subset $U \subset X$, $\mathscr{F}(U)$ is a Banach $\mathscr{O}_X(U)\widehat{\otimes} R$-module;*
- *there exists an affinoid open covering $\mathfrak{U} = \{U_i : i \in I\}$ of X such that for every $i \in I$ and every affinoid open subset $V \subset U_i$, the continuous restriction map*

$$\mathscr{F}(U_i) \otimes_{\mathscr{O}_X(U_i)} \mathscr{O}_X(V) \to \mathscr{F}(V)$$

induces a topological isomorphism

$$\mathscr{F}(U_i)\widehat{\otimes}_{\mathscr{O}_X(U_i)} \mathscr{O}_X(V) \to \mathscr{F}(V)$$

*where the completion is with respect to the p-adic topology. Such a covering $\mathfrak{U}$ is called an **atlas** of $\mathscr{F}$.*

*(ii) A sheaf $\mathscr{F}$ as in (i) is called a **projective Banach sheaf of $\mathscr{O}_X\widehat{\otimes}R$-modules** if there exists an atlas $\mathfrak{U}=\{U_i : i\in I\}$ such that $\mathscr{F}(U_i)$'s are projective Banach $\mathscr{O}_X(U_i)\widehat{\otimes}R$-modules.*

(iii) A morphism between Banach sheaves of $\mathscr{O}_X\widehat{\otimes}R$-modules is a continuous map of sheaves of topological $\mathscr{O}_X\widehat{\otimes}R$-modules.

*(iv) Let $\mathscr{F}$ be a Banach sheaf of $\mathscr{O}_X\widehat{\otimes}R$-modules as in (i). An **integral model** of $\mathscr{F}$ is a subsheaf $\mathscr{F}^+$ of $\mathscr{O}_X^+\widehat{\otimes}'R$-modules such that*

- *for every quasi-compact open $U\subset X$, $\mathscr{F}^+(U)$ is open and bounded in $\mathscr{F}(U)$;*
- *$\mathscr{F}=\mathscr{F}^+[1/p]$;*
- *there exists an atlas $\mathfrak{U}=\{U_i : i\in I\}$ of $\mathscr{F}$ such that, for every $i\in I$ and every affinoid open subset $V\subset U_i$, the canonical map*

$$\mathscr{F}^+(U_i)\widehat{\otimes}_{\mathscr{O}_X^+(U_i)}\mathscr{O}_X^+(V)\to\mathscr{F}^+(V)$$

is an isomorphism, where the completion is with respect to the p-adic topology.

A.2.5. Analogously, we also have a Kummer étale version of Banach sheaves. Such a notion is the one in which we are interested.

Definition A.2.6. *Let X be a locally noetherian fs log adic space of $(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})$.*

*(i) A sheaf of topological $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules $\mathscr{F}$ is called a **Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules** if*

- *for every quasi-compact open $U\in X_{k\acute{e}t}$, $\mathscr{F}(U)$ is a Banach $\mathscr{O}_{X_{k\acute{e}t}}(U)\widehat{\otimes}R$-module;*
- *there exists an Kummer étale covering $\mathfrak{U}=\{U_i : i\in I\}$ of X by affinoid U_i's such that for every Kummer étale map $V\to U_i$ with affinoid V, the continuous restriction map*

$$\mathscr{F}(U_i)\otimes_{\mathscr{O}_{X_{k\acute{e}t}}(U_i)}\mathscr{O}_{X_{k\acute{e}t}}(V)\to\mathscr{F}(V)$$

induces a topological isomorphism

$$\mathscr{F}(U_i)\widehat{\otimes}_{\mathscr{O}_{X_{k\acute{e}t}}(U_i)}\mathscr{O}_{X_{k\acute{e}t}}(V)\to\mathscr{F}(V)$$

*where the completion is with respect to the p-adic topology. Such a covering $\mathfrak{U}$ is called a **Kummer étale atlas** of $\mathscr{F}$.*

*(ii) A sheaf as in (i) is called a **projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules** if there exists a Kummer étale atlas $\mathfrak{U}=\{U_i : i\in I\}$ such that $\mathscr{F}(U_i)$'s are projective Banach $\mathscr{O}_{X_{k\acute{e}t}}(U_i)\widehat{\otimes}R$-modules.*

(iii) A morphism between Kummer étale Banach sheaves of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules is a continuous map of topological $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules.

(iv) *Let $\mathscr{F}$ be a Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules as in (i). An **integral model** of $\mathscr{F}$ is a subsheaf $\mathscr{F}^+$ of $\mathscr{O}^+_{X_{k\acute{e}t}}\widehat{\otimes}'R$-modules such that*

- *for every quasi-compact $U\in X_{k\acute{e}t}$, $\mathscr{F}^+(U)$ is open and bounded in $\mathscr{F}(U)$;*
- *$\mathscr{F}=\mathscr{F}^+[1/p]$;*
- *there exists a Kummer étale atlas $\mathfrak{U}=\{U_i : i\in I\}$ of $\mathscr{F}$ such that, for every $i\in I$ and every affinoid $V\in U_{i,k\acute{e}t}$, the canonical map*

$$\mathscr{F}^+(U_i)\widehat{\otimes}_{\mathscr{O}^+_{X_{k\acute{e}t}}(U_i)}\mathscr{O}^+_{X_{k\acute{e}t}}(V)\to\mathscr{F}^+(V)$$

is an isomorphism, where the completion is with respect to the p-adic topology.

A.2.7. Clearly, an analytic refinement of an atlas (resp., a Kummer étale refinement of a Kummer étale atlas) is also an atlas (resp., a Kummer étale atlas). Also notice that it is not true that a Banach sheaf (resp., Kummer étale Banach sheaf) on an affinoid adic space (resp., affinoid log adic space) is the sheaf associated with its global section. Nonetheless, we have the following result.

Lemma A.2.8. *Let (A,A^+) be a complete reduced Tate algebra over $(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})$ and let M be a projective Banach $A\widehat{\otimes}R$-module.*

(i) *Let $X=\mathrm{Spa}(A,A^+)$ be the associated adic space. Then the presheaf $M\widehat{\otimes}_A\mathscr{O}_X$ sending an affinoid open subset $\mathrm{Spa}(B,B^+)\subset X$ to $M\widehat{\otimes}_AB$ is a sheaf.*

(ii) *Suppose $X=\mathrm{Spa}(A,A^+)$ is equipped with an fs log structure. Then the presheaf $M\widehat{\otimes}_A\mathscr{O}_{X_{k\acute{e}t}}$ sending an affinoid open subset $\mathrm{Spa}(B,B^+)\in X_{k\acute{e}t}$ to $M\widehat{\otimes}_AB$ is a sheaf.*

Proof. It immediately reduces to the case where M is an orthonormalisable Banach $A\widehat{\otimes}R$-module; *i.e.*, $M\simeq(A\widehat{\otimes}R)(J)$ for some index set J. It then reduces to the case where $|J|=1$. Then the lemma follows from Lemma A.2.2. $\square$

Corollary A.2.9. *Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p,\mathcal{O}_{\mathbf{C}_p})$ and let $\mathscr{F}$ be a projective Banach sheaf of $\mathscr{O}_{X_{\mathrm{an}}}\widehat{\otimes}R$-modules with atlas $\mathfrak{U}=\{U_i : i\in I\}$. Suppose $\mathscr{F}$ admits an integral model $\mathscr{F}^+$. Consider the p-adically completed sheaf of $\mathscr{O}_{X_{k\acute{e}t}}$-modules $\mathscr{F}_{k\acute{e}t}$ associated with $\mathscr{F}$; namely,*

$$\mathscr{F}_{k\acute{e}t}:=\left(\varprojlim_m \mathscr{F}^+\otimes_{\mathscr{O}^+_{X_{\mathrm{an}}}}\mathscr{O}^+_{X_{k\acute{e}t}}/p^m\right)[\frac{1}{p}].$$

Then $\mathscr{F}_{k\acute{e}t}$ is a projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules with Kummer étale atlas $\mathfrak{U}=\{U_i : i\in I\}$, where each U_i is equipped with the induced log structure from X. Moreover, for every affinoid $V\in U_{i,k\acute{e}t}$, we have

$$\mathscr{F}_{k\acute{e}t}(V)\simeq\mathscr{F}(U_i)\widehat{\otimes}_{\mathscr{O}_{X_{\mathrm{an}}}(U_i)}\mathscr{O}_{X_{k\acute{e}t}}(V).$$

Proof. To prove the Corollary, we need the following lemma.

Lemma A.2.10. *Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ and let $\mathfrak{U} = \{U_i : i \in I\}$ be a Kummer étale covering of X by affinoid U_i's. Consider the full subcategory $\mathcal{B}_{\mathfrak{U}}$ of $X_{két}$ consisting of those affinoid $V \in X_{két}$ such that the map $V \to X$ factors through $V \to U_i \to X$ for some $i \in I$. Then $\mathcal{B}_{\mathfrak{U}}$ forms a basis for the site $X_{két}$.*

Proof. We have to prove that every $U \in X_{\text{két}}$ admits a covering by such V's and that $\mathcal{B}_{\mathfrak{U}}$ is closed under fibred products. Both statements are clear. □

Let $\mathcal{B}_{\mathfrak{U}}$ be the basis of $X_{\text{két}}$ associated with the covering $\mathfrak{U} = \{U_i : i \in I\}$ as in the lemma. It suffices to show that the assignment

$$V \mapsto \mathscr{F}(U_i) \widehat{\otimes}_{\mathscr{O}_{X_{\text{an}}}(U_i)} \mathscr{O}_{X_{\text{két}}}(V),$$

for every $V \in \mathcal{B}_{\mathfrak{U}}$ which factors through $V \to U_i \to X$, defines a sheaf on $\mathcal{B}_{\mathfrak{U}}$. (Notice that this assigment is independent of the choice of i and hence well-defined.) The sheafiness of this assignment follows from Lemma A.2.8 and the sheafiness of $\mathscr{F}$. □

Definition A.2.11. *Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ and let $\mathscr{F}$ be a projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{két}} \widehat{\otimes} R$-modules. Suppose it admits an integral model $\mathscr{F}^+$ and, for every $m \in \mathbf{Z}_{\geq 1}$, we write $\mathscr{F}^+_m := \mathscr{F}^+ / \mathfrak{a}^m$. We say that $\mathscr{F}$ is **admissible** if there exist*

- *a Kummer étale atlas $\mathfrak{U} = \{U_i : i \in I\}$ of X such that each $\mathscr{F}^+(U_i)$ is the p-adic completion of a free $\mathscr{O}^+_{X_{két}} \widehat{\otimes}' R$-module; and*
- *for every $m \in \mathbf{Z}_{\geq 1}$ and $d \in \mathbf{Z}_{\geq 1}$, a subsheaf $\mathscr{F}^+_{m,d} \subset \mathscr{F}^+_m$ which is a coherent $\mathscr{O}^+_{X_{két}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)$-module subject to the covering $\mathfrak{U}$,*

such that we have $\mathscr{F}^+ \simeq \varprojlim_m \mathscr{F}^+_m$ and $\mathscr{F}^+_m \simeq \varinjlim_d \mathscr{F}^+_{m,d}$ for every $m \in \mathbf{Z}_{\geq 1}$.

*Such a Kummer étale atlas is called an **admissible atlas** for $\mathscr{F}$.*

Lemma A.2.12. *Let $h : Y \to X$ be a finite Kummer étale morphism between locally noetherian fs log adic spaces over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Suppose $\mathscr{F}$ is an admissible Kummer étale Banach sheaf of $\mathscr{O}_{Y_{két}} \widehat{\otimes} R$-modules. Then $h_* \mathscr{F}$ is an admissible Kummer étale Banach sheaf of $\mathscr{O}_{X_{két}} \widehat{\otimes} R$-modules.*

Proof. Suppose $\mathfrak{U} = \{U_i : i \in I\}$ is an admissible atlas for $\mathscr{F}$ on Y. By Definition A.1.7 and [DLLZ19, Proposition 4.1.6], the finite Kummer étale morphism $h : Y \to X$ is, Kummer étale locally on X, isomorphic to a direct sum of isomorphisms. Therefore, one can find an affinoid Kummer étale covering $\{V_j : j \in J\}$ of X such that, for every $i \in I$ and $j \in J$, $U_i \times_X V_j$ is isomorphic to a disjoint union of finite copies of U_i's. Consequently, the Kummer étale covering $\mathfrak{V} = \{U_i \times_X V_j : i \in I, j \in J\}$ is a desired admissible atlas for $h_* \mathscr{F}$. □

A.2.13. If $\mathscr{F}$ is a Kummer étale Banach sheaf of $\mathscr{O}_{X_{\text{két}}} \widehat{\otimes} R$-modules with an integral structure $\mathscr{F}^+$. We write

$$\widehat{\mathscr{F}}^+ := \varprojlim_m \left(\mathscr{F}^+ \otimes_{\mathscr{O}^+_{X_{\text{két}}}} \mathscr{O}^+_{X_{\text{prokét}}} / p^m \right) \simeq \varprojlim_m \left(\mathscr{F}^+ \otimes_{(\mathscr{O}^+_{X_{\text{két}}} \widehat{\otimes}' R)} (\mathscr{O}^+_{X_{\text{prokét}}} \widehat{\otimes}' R) / p^m \right)$$

and $\widehat{\mathscr{F}} := \widehat{\mathscr{F}}^{+}[1/p]$. They are sheaves of $\widehat{\mathscr{O}}^{+}_{X_{\text{prokét}}}\widehat{\otimes}' R$-modules and $\widehat{\mathscr{O}}_{X_{\text{prokét}}}\widehat{\otimes} R$-modules, respectively. The main result of this section is the following.

Proposition A.2.14 (Generalised projection formula)**.** *Let X be a locally noetherian fs log adic space which is log smooth over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ and let $\mathscr{F}$ be a projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{\text{két}}}\widehat{\otimes} R$-modules. Suppose $\mathscr{F}$ is admissible. Then, for every $j \in \mathbf{Z}_{\geq 0}$, there is a natural isomorphism of Kummer étale Banach sheaves of $\mathscr{O}_{X_{\text{két}}}\widehat{\otimes} R$-modules*

$$\mathscr{F} \otimes_{\mathscr{O}_{X_{\text{két}}}} R^j \nu_* \widehat{\mathscr{O}}_{X_{\text{prokét}}} \xrightarrow{\sim} R^j \nu_* \widehat{\mathscr{F}}.$$

A.2.15. The strategy to prove the proposition is simple. Recall that for any ringed site, the projection formula holds for coherent sheaves (see, for example, [Stacks, Tag 01E6]). Thus, the projection formula holds for each $\mathscr{F}^{+}_{m,d}$ with respect to the ringed site $(X_{\text{két}}, \mathscr{O}^{+}_{X_{\text{két}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m))$. As $\mathscr{F}^{+} \simeq \varprojlim_m \varinjlim_d \mathscr{F}^{+}_{m,d}$, we would like to argue that the isomorphism still holds after passing to the limits. This is in general false. However, with the additional local information in the definition of admissibility, we can deduce the projection formula after passing to the limits and inverting p.

The following lemmas are needed in the proof Proposition A.2.14.

Lemma A.2.16. *Let X be a locally noetherian fs log adic space over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Let $\mathscr{H}$ be an $\widehat{\mathscr{O}}^{+}_{X_{\text{prokét}}}\widehat{\otimes} R$-module and let $\mathscr{H}_m := \mathscr{H} / \mathfrak{a}^m$ for every $m \in \mathbf{Z}_{\geq 1}$. Suppose*

- *$\mathscr{H} = \varprojlim_m \mathscr{H}_m$; and*
- *for every $m \in \mathbf{Z}_{\geq 1}$, there exists a sequence of finite free $\widehat{\mathscr{O}}^{+}_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)$-submodules $\{\mathscr{H}_{m,d} : d \in \mathbf{Z}_{\geq 0}\}$ of $\mathscr{H}_m$ such that $\mathscr{H}_m \simeq \varinjlim_d \mathscr{H}_{m,d}$.*

Then, for every $j \in \mathbf{Z}_{\geq 0}$, the natural map

$$R^j \nu_* \mathscr{H} \to \varprojlim_m R^j \nu_* \mathscr{H}_m$$

is an almost isomorphism.

Proof. We have to show the almost vanishing of the higher inverse limit $R^j \varprojlim_m \mathscr{H}_m$. Applying an almost version of [Sch13, Lemma 3.18], it suffices to show that, for every log affinoid perfectoid object $U \in X_{\text{prokét}}$, there are almost isomorphisms

$$R^1 \varprojlim_m \mathscr{H}_m(U)^a = 0$$

and

$$H^j(U, \mathscr{H}_m)^a = 0$$

for every $j \in \mathbf{Z}_{\geq 0}$. The first almost vanishing follows from the Mittag-Leffler condition. To obtain the second almost isomorphism, observe that

$$H^j(U, \mathscr{H}_m) \simeq \varinjlim_d H^j(U, \mathscr{H}_{m,d}).$$

and each $H^j(U, \mathscr{H}_{m,d})$ is almost zero by [DLLZ19, Theorem 5.4.3]. $\square$

Lemma A.2.17. *Let X be a locally noetherian fs log adic space which is log smooth over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. If $\mathscr{G}$ is an projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}} \widehat{\otimes} R$-modules, then, for every $j \in \mathbf{Z}_{\geq 0}$, the sheaf $R^j \nu_* \widehat{\mathscr{G}}$ is also a projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}} \widehat{\otimes} R$-modules.*

Proof. By considering a Kummer étale atlas for $\mathscr{G}$ and writing $R \simeq \prod_{\sigma \in \Sigma} \mathbf{Z}_p \, e_\sigma$, we immediately reduce to the case where

- X is affinoid and admits a toric chart $X \to \mathbb{E} = \mathrm{Spa}(\mathbf{C}_p\langle P \rangle, \mathcal{O}_{\mathbf{C}_p}\langle P \rangle)$ for some sharp fs monoid P;
- $R = \mathbf{Z}_p$ and $\mathfrak{a} = (p)$;
- $\mathscr{G}$ is globally projective; *i.e.*, $\mathscr{G}(X)$ is a projective Banach $\mathscr{O}_{X_{\mathrm{k\acute{e}t}}}(X)$-module and for every affinoid $U \in X_{\mathrm{k\acute{e}t}}$, we have a natural isomorphism

$$\mathscr{G}(X) \widehat{\otimes}_{\mathscr{O}_{X_{\mathrm{k\acute{e}t}}}(X)} \mathscr{O}_{X_{\mathrm{k\acute{e}t}}}(U) \xrightarrow{\sim} \mathscr{G}(U).$$

We further reduce to the case where $\mathscr{G}$ is globally orthonormalisable; namely, $\mathscr{G} \simeq \mathscr{O}_{X_{\mathrm{k\acute{e}t}}}(J)$ for some index set J. Let $\mathscr{G}^+$ be the p-adic completion of the free $\mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}}$-module $\bigoplus_J \mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}}$ and let $\mathscr{G}^+_m := \mathscr{G}^+ / p^m \simeq \bigoplus_J \mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}} / p^m$. By Lemma A.2.16, we have a natural almost isomorphism

$$R^j \nu_* \widehat{\mathscr{G}}^+ \xrightarrow{\sim} \varprojlim_m R^j \nu_* \widehat{\mathscr{G}}^+_m$$

where $\widehat{\mathscr{G}}^+_m = \widehat{\mathscr{G}}^+ / p^m \simeq \bigoplus_J \mathscr{O}^+_{X_{\mathrm{prok\acute{e}t}}} / p^m$.

We claim that, in this case, the sheaf $R^j \nu_* \widehat{\mathscr{G}}$ is isomorphic to $\left(\wedge^j (\mathscr{O}_{X_{\mathrm{k\acute{e}t}}})^n\right)(J)$ for some $n \in \mathbf{Z}_{\geq 1}$. For this, we follow the strategy as in the proof of [DRW22, Lemma A.2.1].

Consider the collection $\mathcal{B}_X$ used in the proof of [DRW22, Lemma A.2.1]. In particular, for every $V \in \mathcal{B}_X$, the map $V \to X$ admits a Kummer chart $P \to P'$ which is isomorphic to the m-th multiple map $[m] : P \to P$. Moreover, the injection $P \to P'$ induces an injection $\Gamma' \to \Gamma$, where Γ and Γ' are the profinite Galois groups as in A.1.20. If we fix an identification $\Gamma \simeq \widehat{\mathbf{Z}}(1)^n$, the injection $\Gamma' \to \Gamma$ can be identified with the m-th multiple map $[m] : \widehat{\mathbf{Z}}(1)^n \to \widehat{\mathbf{Z}}(1)^n$.

By the calculation in [DRW22, Lemma A.2.1], we obtain an almost injection

$$\left(\wedge^j (\mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}} / p^m(V))^n\right)^a \simeq H^j(\Gamma, \mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}} / p^m(V))^a \hookrightarrow H^j_{\mathrm{prok\acute{e}t}}(V, \mathscr{O}^+_X / p^m)^a$$

with cokernel killed by p. Taking direct sum and then inverse limit, we obtain an almost injection

$$\varprojlim_m \oplus_J \left(\wedge^j (\mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}} / p^m(V))^n\right)^a \hookrightarrow \varprojlim_m \oplus_J H^j_{\mathrm{prok\acute{e}t}}(V, \mathscr{O}^+_X / p^m)$$

with cokernel killed by p. Inverting p, we obtain an isomorphism

$$\left(\wedge^j (\mathscr{O}^+_{X_{\mathrm{k\acute{e}t}}}(V))^n\right)(J) \simeq \varprojlim_m \oplus_J H^j_{\mathrm{prok\acute{e}t}}(V, \mathscr{O}^+_X / p^m)$$

However, note that the sheaf

$$R^j\nu_*\widehat{\mathscr{G}} \simeq \left(\varprojlim_m R^j\nu_*\widehat{\mathscr{G}}^+_m\right)[\frac{1}{p}]$$

is just the sheafification of $W \mapsto \varprojlim_m \oplus_J H^j_{\text{prokét}}(W, \mathscr{O}^+_X/p^m)$. Consequently, $R^j\nu_*\widehat{\mathscr{G}}$ coincides with the sheaf $\left(\wedge^j(\mathscr{O}^+_{X_{\text{két}}})^n\right)(J)$ which is clearly an ON-able Banach sheaf of $\mathscr{O}_{X_{\text{két}}}\widehat{\otimes}R$-modules. □

Proof of Proposition A.2.14. We split the proof into three steps.

Step 1. We first verify that both $\mathscr{F}\otimes_{\mathscr{O}_{X_{\text{két}}}} R^j\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}}$ and $R^j\nu_*\widehat{\mathscr{F}}$ are projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{\text{két}}}\widehat{\otimes}R$-modules.

Indeed, the statement for $\mathscr{F}\otimes_{\mathscr{O}_{X_{\text{két}}}} R^j\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}}$ follows from the locally finite freeness of $R^j\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}}$ (see [DRW22, Lemma A.2.1]) and the statement for $R^j\nu_*\widehat{\mathscr{F}}$ follows from Lemma A.2.17. In fact, we can be more precise. Consider an affinoid Kummer étale covering $\mathfrak{U} = \{U_i : i \in I\}$ satisfying:

- $\mathfrak{U}$ is an admissible atlas of $\mathscr{F}$;
- each U_i admits a toric chart $U_i \to \mathrm{Spa}(\mathbf{C}_p\langle P_i\rangle, \mathcal{O}_{\mathbf{C}_p}\langle P_i\rangle)$ for some sharp fs monoid.

Then, by the proof of [DRW22, Lemma A.2.1] and Lemma A.2.17, we see that $\mathfrak{U}$ is a Kummer étale atlas for both $\mathscr{F}\otimes_{\mathscr{O}_{X_{\text{két}}}} R^j\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}}$ and $R^j\nu_*\widehat{\mathscr{F}}$. (In fact, they are both orthonormalisable on each U_i.) For the rest of the proof, we fix such a cover $\mathfrak{U}$.

Step 2. We construct two natural morphisms

$$\Psi : \mathscr{F}^+ \otimes_{\mathscr{O}^+_{X_{\text{két}}}\widehat{\otimes}'R} R^j\nu_*\left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}}\widehat{\otimes}'R\right) \to \varprojlim_m R^j\nu_*\widehat{\mathscr{F}}^+_m$$

and

$$\Theta : R^j\nu_*\widehat{\mathscr{F}}^+ \to \varprojlim_m R^j\nu_*\widehat{\mathscr{F}}^+_m.$$

where

$$\widehat{\mathscr{F}}^+_m = \mathscr{F}^+_m \otimes_{\mathscr{O}^+_{X_{\text{két}}}} \widehat{\mathscr{O}}^+_{X_{\text{prokét}}} = \widehat{\mathscr{F}}^+/\mathfrak{a}^m.$$

There is clearly such a map Θ. It remains to construct Ψ.

For every $m \in \mathbf{Z}_{\geq 1}$ and $d \in \mathbf{Z}_{\geq 0}$, we write

$$\widehat{\mathscr{F}}^+_{m,d} := \mathscr{F}^+_{m,d} \otimes_{\mathscr{O}^+_{X_{\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)\right).$$

By the usual projection formula for ringed sites (see, for example, [Stacks, Tag 01E6]), we obtain a canonical morphism

$$\Psi_{m,d} : \mathscr{F}^+_{m,d} \otimes_{\mathscr{O}^+_{X_{\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} R^j\nu_*\left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)\right) \to R^j\nu_*\widehat{\mathscr{F}}^+_{m,d}.$$

Taking direct limit with respect to d, followed by taking inverse limit with respect to m, we obtain a canonical morphism

$$\Psi' : \varprojlim_m \left(\mathscr{F}^+_m \otimes_{\mathscr{O}^+_{X_{\text{két}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m) \right) \right) \to \varprojlim_m R^j\nu_* \widehat{\mathscr{F}}^+_m.$$

On the other hand, we have natural morphisms

$$\begin{tikzcd}
\mathscr{F}^+ \otimes_{\mathscr{O}^+_{X_{\text{két}}} \widehat{\otimes}' R} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{proét}}} \widehat{\otimes}' R \right) \arrow[r] \arrow[ddr, bend right, "\Psi''"'] & \mathscr{F}^+ \otimes_{\mathscr{O}^+_{X_{\text{két}}} \widehat{\otimes}' R} \varprojlim_m \left(R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m) \right) \right) \arrow[d, equal] \\
& (\varprojlim_m \mathscr{F}^+_m) \otimes_{\mathscr{O}^+_{X_{\text{két}}} \widehat{\otimes}' R} \varprojlim_m \left(R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m) \right) \right) \arrow[d] \\
& \varprojlim_m \left(\mathscr{F}^+_m \otimes_{\mathscr{O}^+_{X_{\text{két}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m) \right) \right)
\end{tikzcd}$$

Composing with Ψ', we obtain the desired morphism

$$\Psi : \mathscr{F}^+ \otimes_{\mathscr{O}^+_{X_{\text{két}}} \widehat{\otimes}' R} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{X_{\text{prokét}}} \widehat{\otimes}' R \right) \to \varprojlim_m R^j\nu_* \widehat{\mathscr{F}}^+_m.$$

Step 3. For simplicity, we write $\mathscr{G}_i$, $\mathscr{G}^+_i$, and $\mathscr{G}^+_{i,m}$ for $\mathscr{F}|_{U_i}$, $\mathscr{F}^+|_{U_i}$, and $\mathscr{F}^+_{i,m}|_{U_i}$, respectively. Since $\mathscr{G}^+_{i,m}$ is a free $\mathscr{O}^+_{U_{i,\text{két}}} \otimes (R/\mathfrak{a}^m)$-module, we can express $\mathscr{G}^+_{i,m}$ as a filtered direct limit of finite free submodules $\mathscr{G}^+_{i,m,\alpha}$.

We can repeat the construction in Step 2 to $\mathscr{G}^+_i$, $\mathscr{G}^+_{i,m}$, and $\mathscr{G}^+_{i,m,\alpha}$. In particular, we obtain maps

$$\Psi_i : \mathscr{G}^+_i \otimes_{\mathscr{O}^+_{U_{i,\text{két}}} \widehat{\otimes}' R} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \widehat{\otimes}' R \right) \to \varprojlim_m R^j\nu_* \widehat{\mathscr{G}}^+_{i,m}$$

and

$$\Theta_i : R^j\nu_* \widehat{\mathscr{G}}^+_i \to \varprojlim_m R^j\nu_* \widehat{\mathscr{G}}^+_{i,m}$$

where

$$\widehat{\mathscr{G}}^+_{i,m} = \mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}} \widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} = \widehat{\mathscr{G}}^+_i / \mathfrak{a}^m.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}^+|_{U_i} \otimes_{(\mathscr{O}^+_{U_{i,\text{két}}} \widehat{\otimes}' R)} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \widehat{\otimes}' R \right) \arrow[r, "\Psi|_{U_i}"] \arrow[d, "\simeq"'] & \varprojlim_m R^j\nu_* \widehat{\mathscr{F}}^+_m|_{U_i} \arrow[d, "\simeq"] & R^j\nu_* \widehat{\mathscr{F}}^+|_{U_i} \arrow[l, "\Theta|_{U_i}"'] \arrow[d, "\simeq"] \\
\mathscr{G}^+_i \otimes_{(\mathscr{O}^+_{U_{i,\text{két}}} \widehat{\otimes}' R)} R^j\nu_* \left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \widehat{\otimes}' R \right) \arrow[r, "\Psi_i"] & \varprojlim_m R^j\nu_* \widehat{\mathscr{G}}^+_{i,m} & R^j\nu_* \widehat{\mathscr{G}}^+_i \arrow[l, "\Theta_i"']
\end{tikzcd}$$

The square on the left is commutative because the cofiltered systems $\{\mathscr{F}^+_{m,d}|_{U_i}\}$ and $\{\mathscr{G}^+_{i,m,\alpha}\}$

are cofinal to each other. By Lemma A.2.16, $\Theta_i = \Theta|_{U_i}$ is an almost isomorphism. This implies that $\Theta[1/p]$ is an isomorphism of projective Kummer étale Banach sheaves of $\mathscr{O}_{X_{\text{két}}}\widehat{\otimes}R$-modules.

We claim that Ψ_i also becomes an isomorphism after inverting p. By construction, Ψ_i factors as the composition of

$$\Psi_i'' : \mathscr{G}_i^+ \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\widehat{\otimes}' R} R^j\nu_*\left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}}\widehat{\otimes}' R\right) \to \varprojlim_m \left(\mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(R^j\nu_*\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)\right)\right)$$

and a canonical isomorphism Ψ_i' given by the composition

$$\begin{tikzcd}
\varprojlim_m \left(\mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(R^j\nu_*\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)\right)\right) \arrow[d, equal] \arrow[ddd, bend left, "\Psi_i'"] \\
\varprojlim_m \left(\varinjlim_\alpha \mathscr{G}^+_{i,m,\alpha} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(R^j\nu_*\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)\right)\right) \arrow[d, "\simeq"] \\
\varprojlim_m \varinjlim_\alpha R^j\nu_*\left(\mathscr{G}^+_{i,m,\alpha} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)\right)\right) \arrow[d, equal] \\
\varprojlim_m \varinjlim_\alpha R^j\nu_*\widehat{\mathscr{G}}^+_{i,m,\alpha} \arrow[r, "\simeq"] & \varprojlim_m R^j\nu_*\widehat{\mathscr{G}}^+_{i,m}
\end{tikzcd}$$

where the second isomorphism follows from the fact that each $\mathscr{G}^+_{i,m,\alpha}$ is a finite free $\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)$-module.

It remains to prove that Ψ_i'' becomes an isomorphism after inverting p. Recall that U_i admits a toric chart $U_i \to \mathrm{Spa}(\mathbf{C}_p\langle P\rangle, \mathcal{O}_{\mathbf{C}_p}\langle P\rangle)$ for some sharp fs monoid P. By choosing an identification $\Gamma := \mathrm{Hom}(P^{\mathrm{gp}}_{\mathbf{Q}_{\geq 0}}/P^{\mathrm{gp}}, \boldsymbol{\mu}_\infty) \simeq \widehat{\mathbf{Z}}(1)^n$, [DRW22, Lemma A.2.1] yields an isomorphis $R^j\nu_*\widehat{\mathscr{O}}_{U_{i,\text{prokét}}} \simeq \wedge^j(\mathscr{O}_{U_{i,\text{két}}})^n$.

On one hand, by [DRW22, Proposition A.2.3], we have

$$\begin{aligned}
\mathscr{G}_i^+ \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\widehat{\otimes}' R} R^j\nu_*\left(\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}}\widehat{\otimes}' R\right)[\frac{1}{p}] &= \mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}\widehat{\otimes} R} R^j\nu_*\left(\widehat{\mathscr{O}}_{U_{i,\text{prokét}}}\widehat{\otimes} R\right) \\
&\simeq \mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}\widehat{\otimes} R} \left((R^j\nu_*\widehat{\mathscr{O}}_{U_{i,\text{prokét}}})\widehat{\otimes} R\right) \\
&= \mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}} R^j\nu_*\widehat{\mathscr{O}}_{U_{i,\text{prokét}}} \\
&\simeq \mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}} \wedge^j\left(\mathscr{O}_{U_{i,\text{két}}}\right)^n
\end{aligned}$$

On the other hand, if we write $R/\mathfrak{a}^m \simeq \oplus_{\sigma\in\Sigma_m} \mathbf{Z}/p^\sigma$, then we have

$$R^j\nu_*\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m) \simeq \oplus_{\sigma\in\Sigma_m} R^j\nu_*(\mathscr{O}^+_{U_{i,\text{prokét}}}/p^\sigma).$$

By [DRW22, Lemma A.2.1 (iii)], there is an almost injection

$$\wedge^j(\mathscr{O}^+_{U_{i,\text{két}}}/p^\sigma)^n \hookrightarrow R^j\nu_*(\mathscr{O}^+_{U_{i,\text{prokét}}}/p^\sigma)$$

whose cokernel is killed by p. This yields an almost injection given by the composition

$$\begin{tikzcd}
\mathscr{G}^+_i \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}} \left(\wedge^j(\mathscr{O}^+_{U_{i,\text{két}}})^n\right) \arrow[r, equal] \arrow[dddr, bend right] & \varprojlim_m \mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(\wedge^j(\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m))^n\right) \arrow[d, equal] \\
& \varprojlim_m \mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(\oplus_{\sigma\in\Sigma_m} \wedge^j(\mathscr{O}^+_{U_{i,\text{két}}}/p^\sigma)^n\right) \arrow[d, hook] \\
& \varprojlim_m \mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(\oplus_{\sigma\in\Sigma_m} R^j\nu_*(\mathscr{O}^+_{U_{i,\text{prokét}}}/p^\sigma)\right) \arrow[d, "\simeq"] \\
& \varprojlim_m \mathscr{G}^+_{i,m} \otimes_{\mathscr{O}^+_{U_{i,\text{két}}}\otimes_{\mathbf{Z}_p}(R/\mathfrak{a}^m)} \left(R^j\nu_*\widehat{\mathscr{O}}^+_{U_{i,\text{prokét}}} \otimes_{\mathbf{Z}_p} (R/\mathfrak{a}^m)\right)
\end{tikzcd}$$

with cokernel killed by p.

Consequently, both sides of Ψ''_i are isomorphic to $\mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}} \left(\wedge^j(\mathscr{O}_{U_{i,\text{két}}})^n\right)$ after inverting p, and one checks that $\Psi''_i[1/p]$ is just the identity map on $\mathscr{G}_i \otimes_{\mathscr{O}_{U_{i,\text{két}}}} \left(\wedge^j(\mathscr{O}_{U_{i,\text{két}}})^n\right)$. This finishes the proof. □

Corollary A.2.18. *Let X be a locally noetherian fs log adic space which is log smooth over $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. Let $\mathscr{F}$ be an admissible projective Kummer étale Banach sheaf of $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-modules, with the corresponding integral structure $\mathscr{F}^+$. Suppose $\mathscr{F}^+$ is equipped with an $\mathscr{O}^+_{X_{k\acute{e}t}}\widehat{\otimes}'R$-linear action of a finite group G. This induces an $\mathscr{O}_{X_{k\acute{e}t}}\widehat{\otimes}R$-linear action of G on $\mathscr{F}$. Then the subsheaf of G-invariants $\mathscr{F}^G$ also satisfies the generalised projection formula. More precisely, we have a natural isomorphism*

$$\mathscr{F}^G \otimes_{\mathscr{O}_{X_{k\acute{e}t}}} R^i\nu_*\widehat{\mathscr{O}}_{X_{prok\acute{e}t}} \xrightarrow{\sim} R^i\nu_*\widehat{\mathscr{F}^G}$$

Proof. By Proposition A.2.14, we have an isomorphism

$$\mathscr{F} \otimes_{\mathscr{O}_{X_{\text{két}}}} R^i\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}} \xrightarrow{\sim} R^i\nu_*\widehat{\mathscr{F}}.$$

Taking the G-invariants, we obtain an isomorphism

$$\mathscr{F}^G \otimes_{\mathscr{O}_{X_{\text{két}}}} R^i\nu_*\widehat{\mathscr{O}}_{X_{\text{prokét}}} \xrightarrow{\sim} \left(R^i\nu_*\widehat{\mathscr{F}}\right)^G.$$

It remains to show $\left(R^i\nu_*\widehat{\mathscr{F}}\right)^G \simeq R^i\nu_*\widehat{\mathscr{F}^G}$. Indeed, consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X_{\text{prokét}}}[G]-\textbf{Mod} \arrow[r, "\nu_*"] \arrow[d, "(-)^G"'] & \mathscr{O}_{X_{\text{két}}}[G]-\textbf{Mod} \arrow[d, "(-)^G"] \\
\mathscr{O}_{X_{\text{prokét}}}-\textbf{Mod} \arrow[r, "\nu_*"] & \mathscr{O}_{X_{\text{két}}}-\textbf{Mod}
\end{tikzcd}$$

Notice that the higher right derived functors of both of the vertical arrows vanish as G is a finite group and the base field is of characteristic zero. Now, applying the standard Grothendieck spectral sequence argument to both compositions $\nu_* \circ (-)^G$ and $(-)^G \circ \nu_*$, we obtain the desired commutativity of $R^i\nu_*$ and $(-)^G$. $\square$

Bibliography

[AB59] Maurice Auslander and David Buchsbaum. 'On Ramification Theory in Noetherian Rings'. In: *American Journal of Mathematics* 81.3 (1959).

[AIP15] Fabrizio Andreatta, Adrian Iovita and Vincent Pilloni. 'p-adic families of Siegel modular cuspforms'. In: *Annals of Mathematics* 181.2 (2015), pp. 623–697. ISSN: 0003486X.

[AIP18] Fabrizio Andreatta, Adrian Iovita and Vincent Pilloni. 'Le Halo Spectral'. In: *Ann. Sci. ENS* 51.3 (2018).

[AIS10] Fabrizio Andreatta, Adrian Iovita and Glenn Stevens. *On overcnvergent modular forms*. Preprint. 2010.

[AIS14] Fabrizio Andreatta, Adrian Iovita and Glenn Stevens. 'Overconvergent modular sheaves and modular forms for $\mathrm{GL}_{2/F}$'. In: *Israel Journal of Mathematics* 201 (2014), pp. 299–359. DOI: `https://doi.org/10.1007/s11856-014-1045-8`.

[AIS15] Fabrizio Andreatta, Adrian Iovita and Glenn Stevens. 'Overconvergent Eichler–Shimura isomorphisms'. In: *Journal of the Institute of Mathematics of Jussieu* 14.2 (2015), pp. 221–274. DOI: `10.1017/S1474748013000364`.

[Ami64] Yvette Amice. 'Interpolation p-adique'. In: *Bulletin de la Société Mathématique de France* 92 (1964), pp. 117–180. DOI: `10.24033/bsmf.1606`.

[AS08] Avner Ash and Glenn Stevens. *p-adic deformations of arithmetic cohomology*. Preprint. Available at: `http://math.bu.edu/people/ghs/preprints/Ash-Stevens-02-08.pdf`. 2008.

[Bar18] Daniel Barrera Salazar. 'Overconvergent cohomology of Hilbert modular varieties and p-adic L-functions'. In: *Annales de l'Institut Fourier* 68.5 (2018), pp. 2177–2213. DOI: `10.5802/aif.3206`.

[BC09] Joël Bellaïche and Gaëtan Chenevier. *Families of Galois representations and Selmer groups*. Astérisque 324. Société mathématique de France, 2009.

[Bel12] Joël Bellaïche. 'Critical p-adic L-functions'. In: *Inventiones mathematicae* 189 (2012), pp. 1–60. DOI: `https://doi.org/10.1007/s00222-011-0358-z`.

[Bel21] Joël Bellaïche. *The Eigenbook: Eigenvarieties, families of Galois representations, p-adic L-functions*. Pathways in Mathematics. Birkhäuser Basel, 2021. DOI: `10.1007/978-3-030-77263-5`.

[Ber96] Pierre Berthelot. *Cohomologie rigide et cohomologie rigide à supports propres*. Preprint. Available at: https://perso.univ-rennes1.fr/pierre.berthelot/publis/Cohomologie_Rigide_I.pdf. 1996.

[BG17] Daniel Barrera and Shan Gao. 'Overconvergent Eichler-Shimura isomorphisms for quaternionic modular forms over $\mathbf{Q}$'. In: *International Journal of Number Theory* 13.10 (2017), pp. 2487–2504. DOI: https://doi.org/10.1142/S1793042117501494.

[BG21] Daniel Barrera and Shan Gao. 'Overconvergent Eichler–Shimura isomorphisms for unitary Shimura curves over totally real fields'. In: *Israel Journal of Mathematics* (2021). DOI: https://doi.org/10.1007/s11856-021-2143-z.

[BHW19] Christopher Birkbeck, Ben Heuer and Chris Williams. *Overconvergent Hilbert modular forms via perfectoid modular varieties*. Preprint. Available at: https://arxiv.org/abs/1902.03985, 2019.

[BK07] Spencer Bloch and Kazuya Kato. 'L-Functions and Tamagawa Numbers of Motives'. In: *The Grothendieck Festschrift Volume I*. Ed. by P. Cartier, L. Illusie, N.M. Katz, G. Laumon, Y.I. Manin and K.A. Ribet. Vol. 86. Progress in Mathematics. Birkhäuser, Boston, MA., 2007, pp. 333–400.

[BL20] Baskar Balasubramanyam and Matteo Longo. *p-adic adjoint L-function and ramification locus of the Hilbert eigenvariety*. Preprint. Available at: https://arxiv.org/abs/2011.05237. 2020.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty and Richard Taylor. 'Potential automorphy and change of weight'. In: *Annals of Mathematics* (2014). DOI: https://doi.org/10.4007/annals.2014.179.2.3.

[BP20] George Boxer and Vincent Pilloni. *Higher Coleman theory*. Preprint available at: https://perso.ens-lyon.fr/vincent.pilloni/HigherColeman.pdf. 2020.

[BS73] Armand Borel and Jean-Pierre Serre. 'Corners and Arithmetic Groups'. In: *Commentarii mathematici Helvetici* 48 (1973), pp. 436–483.

[Buz07] Kevin Buzzard. 'Eigenvarieties'. In: *L-Functions and Galois Representations*. Ed. by David Burns, Kevin Buzzard and Jan Nekovář. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007, pp. 59–120. DOI: 10.1017/CBO9780511721267.004.

[Che04] Gaëtan Chenevier. 'Familles p-adiques de formes automorphes pour GL_n'. In: *Journal für die reine und angewandte Mathematik* 2004.570 (2004), pp. 143–217. DOI: https://doi.org/10.1515/crll.2004.031.

[Che14] Gaëtan Chenevier. 'The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings'. In: *Automorphic Forms and Galois Representations*. Ed. by Fred Diamond, Payman L. Kassaei and Minhyong Kim. Vol. 1. London Mathematical Society Lecture Note Series. Cambridge University Press, 2014, pp. 221–285. DOI: 10.1017/CBO9781107446335.008.

[CHJ17] Przemyslaw Chojecki, David Hansen and Christian Johansson. 'Overconvergent Modular Forms and Perfectoid Shimura Curves'. In: *Documenta Mathematica 2017* vol. 22 (2017), ISSN 1431–0643. DOI: `10.25537/dm.2017v22.191-262`.

[CHT08] Laurent Clozel, Michael Harris and Richard Taylor. 'Automorphy for some l-adic lifts of automorphic mod l Galois representations'. In: *Publications Mathématiques de l'IHÉS* 108 (2008), pp. 1–181. DOI: `10.1007/s10240-008-0016-1`.

[CM98] Robert F. Coleman and Barry Mazur. 'The Eigencurve'. In: *Galois Representations in Arithmetic Algebraic Geometry*. Ed. by A. J. Scholl and R. L.Editors Taylor. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998, pp. 1–114. DOI: `10.1017/CBO9780511662010.003`.

[Col95] Robert F. Coleman. 'Classical and overconvergent modular forms'. In: *Journal de théorie des nombres de Bordeaux* 7.1 (1995), pp. 333–365.

[Col97] Robert F. Coleman. 'p-adic Banach spaces and families of modular forms'. In: *Inventiones mathematicae* 127 (1997), pp. 417–479. DOI: `10.1007/s002220050127`.

[Con99] Brian Conrad. 'Irreducible components of rigid spaces'. In: *Annales de l'institut Fourier* 49.2 (1999), pp. 473–541.

[Del71] Pierre Deligne. 'Travaux de Shimura'. In: *Séminaire Bourbaki : vol. 1970/71, exposés 382-399*. Séminaire Bourbaki 13. Springer-Verlag, 1971.

[DLLZ19] Hansheng Diao, Kai-Wen Lan, Rouchuan Liu and Xinwen Zhu. *Logarithmic adic spaces: some fundamental results*. Preprint. Available at: `https://arxiv.org/pdf/1912.09836.pdf`. 2019.

[DRW22] Hansheng Diao, Giovanni Rosso and Ju-Feng Wu. *Perfectoid overconvergent Siegel modular forms and the overconvergent Eichler–Shimura morphism*. Preprint. Available at: `https://arxiv.org/abs/2106.00094`. 2022.

[Far11] Laurent Fargues. 'La filtration canonique des points de torsion des groupes p-divisibles'. In: *Annales scientifiques de l'École Normale Supérieure* 4e série, 44.6 (2011), pp. 905–961. DOI: `10.24033/asens.2157`.

[FC90] Gred Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag Berlin Heidelberg, 1990. DOI: `10.1007/978-3-662-02632-8`.

[FGL08] Laurent Fargues, Alain Genestier and Vincent Lafforgue. *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*. Vol. 262. Progress in Mathematics. Birkhäuser Basel, 2008. DOI: `https://doi.org/10.1007/978-3-7643-8456-2`.

[FH91] William Fulton and Joe Harris. *Representation Theory: a first course*. Vol. 129. Graduate Texts in Mathematics. Springer, New York, 1991. DOI: `https://doi.org/10.1007/978-1-4612-0979-9`.

[GT05] Alain Genestier and Jacques Tilouine. 'Systèmes de Taylor-Wiles pour GSp_4'. In: *Formes automorphes (II) - Le cas du groupe* GSp(4). Ed. by Jacques Tilouine, Henri Carayol, Michael Harris and Marie-France Vignéras. Astérisque 302. Société mathématique de France, 2005, pp. 177–290.

[Han12] David Hansen. *Pairings on modules of analytic distributions*. Unpublished notes. Available at: `http://www.davidrenshawhansen.com/pairing.pdf`. 2012.

[Han15] David Hansen. *Iwasawa theory of overconvergent modular forms, I: Critical-slope p-adic L-functions*. Preprint. Available at: `https://arxiv.org/abs/1508.03982`. 2015.

[Han17] David Hansen. 'Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality'. In: *Journal für die reine und angewandte Mathematik* (730 2017), pp. 1–64. DOI: `https://doi.org/10.1515/crelle-2014-0130`.

[Han20] David Hansen. 'Vanishing and comparison theorems in rigid analytic geometry'. In: *Compositio Mathematica* 156.2 (2020), pp. 299–324.

[Hid02] Haruzo Hida. 'Control theorems of coherent sheaves on Shimura varieties of PEL type'. In: *Journal of the Institute of Mathematics of Jussieu* 1.1 (2002), pp. 1–76. DOI: `10.1017/S1474748002000014`.

[Hid16] Haruzo Hida. 'Arithmetic of Adjoint L-Values'. In: *p-Adic Aspects of Modular Forms*. Ed. by Baskar Balasubramanyam, Haruzo Hida, A Raghuram and Jacques Tilouine. 2016, pp. 185–236.

[Hid86] Haruzo Hida. 'Iwasawa modules attached to congruences of cusp forms'. In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 19.2 (1986), pp. 231–273. DOI: `10.24033/asens.1507`.

[HJ17] Robert Harron and Andrei Jorza. 'On symmetric power $\mathcal{L}$-invariants of Iwahori level Hilbert modular forms'. In: *American Journal of Mathematics* 139.6 (2017), pp. 1605–1647. DOI: `doi:10.1353/ajm.2017.0040`.

[HT17] David Hansen and Jack Thorne. 'On the GL_n-eigenvariety and a conjecture of Venkatesh'. In: *Selecta Mathematica* 23 (2017), pp. 1205–1234. DOI: `https://doi.org/10.1007/s00029-017-0303-0`.

[Hub13] Roland Huber. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*. Vol. 30. Aspects of Mathematics. Springer-Verlag, 2013.

[Ill02] Luc Illusie. 'An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology'. In: *Cohomologies p-adiques et applications arithmétiques (II)*. Ed. by Berthelot Pierre, Fontaine Jean-Marc, Illusie Luc, Kato Kazuya and Rapoport Michael. Astérisque 279. Société mathématique de France, 2002, pp. 271–322.

[JN19] Christian Johansson and James Newton. 'Extended eigenvarieties for overconvergent cohomology'. In: *Algebra and Number Theory* 13.1 (Feb. 2019), pp. 93–158. ISSN: 1937-0652. DOI: `10.2140/ant.2019.13.93`.

[Kat73] Nicholas M. Katz. 'p-adic Properties of Modular Schemes and Modular Forms'. In: *Modular Functions of One Variable III*. Ed. by Willem Kuijk and Jean-Pierre Serre. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 69–190. ISBN: 978-3-540-37802-0.

[Kim06] Walter Kim. 'Ramification Points on the Eigencurve and the Two Variable Symmetric Square p-adic L-Function'. PhD thesis. 2006.

[Kis03] Mark Kisin. 'Overconvergent modular forms and the Fontaine–Mazur conjecture'. In: *Inventiones mathematicae* 153 (2003).

[KS20] Arno Kret and Sug Woo Shin. *Galois representations for the general symplectic group*. To appear in *Journal of the European Mathematical Society*. Preprint available at: `https://arxiv.org/abs/1609.04223`. 2020.

[KT17] Chandrashekhar B. Khare and Jack A. Thorne. In: *American Journal of Mathematics* 139.5 (2017), pp. 1205–1273. DOI: `10.1353/ajm.2017.0030`.

[Lan12] Kai-Wen Lan. 'Toroidal compactifications of PEL-type Kuga families'. In: *Algebra Number Theory* 6.5 (2012), pp. 885–966. ISSN: 1937-0652. DOI: `10.2140/ant.2012.6.885`.

[Lan13] Kai-Wen Lan. *Arithmetic Compactifications of PEL-Type Shimura Varieties*. Princeton University Press, 2013.

[Laz65] Michel Lazard. 'Groupes analytiques p-adiques'. In: *Publications Mathématiques de l'IHÉS* 26 (1965), pp. 5–219.

[Liu15] Ruochuan Liu. 'Triangulation of refined families'. In: *Commentarii Mathematici Helvetici* 90.4 (2015), pp. 831–904. DOI: `10.4171/CMH/372`.

[Lüt74] Werner Lütkebohmert. 'Der Satz von Remmert-Stein in der nichtarchimedischen Funktionen'. In: *Mathematische Zeitschrift* 139 (1974), pp. 69–84. DOI: `https://doi.org/10.1007/BF01194146`.

[Mun84] James R. Munkres. *Elements of Algebraic Topology*. Addison Wesley Publishing Company, 1984.

[NT20] James Newton and Jack Thorne. 'Adjoint Selmer groups of automorphic Galois representations of unitary type'. In: *Journal of the European Mathematical Society* (2020). DOI: `https://doi.org/10.17863/CAM.55603`.

[Par10] Jeehoon Park. 'p-Adic family of half-integral weight modular forms via overconvergent Shintani lifting'. In: *manuscripta mathematica* 131 (2010), pp. 355–384. DOI: `https://doi.org/10.1007/s00229-009-0323-y`.

[Pil12] Vincent Pilloni. 'Sur la théorie de Hida pour le groupe GSp_{2g}'. In: *Bulletin de la Société Mathématique de France* 140.3 (2012), pp. 335–400. DOI: `10.24033/bsmf.2630`.

[Pil13] Vincent Pilloni. 'Overconvergent modular forms'. In: *Annales de l'Institut Fourier* 63.1 (2013), pp. 219–239. DOI: `10.5802/aif.2759`.

[Pin88] Richard Pink. 'Arithmetical Compactification of Mixed Shimura Varieties'. PhD thesis. Universität Bonn, 1988.

[PS16] Vincent Pilloni and Benoît Stroh. 'Cohomologie cohérente et représentations Galoisiennes'. In: *Annales mathématiques du Québec* 40.1 (2016), pp. 167–202. ISSN: 2195-4763. DOI: 10.1007/s40316-015-0056-0.

[PT15] Stefan Patrikis and Richard Taylor. 'Automorphy and irreducibility of some l-adic representations'. In: *Compositio Mathematica* 151.2 (2015), pp. 207–229. DOI: 10.1112/S0010437X14007519.

[Sch13] Peter Scholze. 'p-adic Hodge theory for rigid–analytic varieties'. In: *Forum of Mathematics, Pi* 1 (2013). DOI: 10.1017/fmp.2013.1.

[Sch15] Peter Scholze. 'On torsion in the cohomology of locally symmetric varieties'. In: *Annals of Mathematics* 182.3 (2015), pp. 945–1066.

[Sch90] Anthony J. Scholl. 'Motives for modular forms.' In: *Inventiones mathematicae* 100.2 (1990), pp. 419–430.

[Sen88] Shankar Sen. 'The Analytic Variation of p-Adic Hodge Structure'. In: *Annals of Mathematics* 127.3 (1988), pp. 647–661. ISSN: 0003486X.

[Ser73] Jean-Pierre Serre. 'Formes modulaires et fonctions zêta p-adiques'. In: *Modular Functions of One Variable III*. Ed. by Willem Kuijk and Jean-Pierre Serre. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 191–268. ISBN: 978-3-540-37802-0.

[Ser79] Jean-Pierre Serre. *Local Fields*. Vol. 67. Graduate Texts in Mathematics. Springer-Verlag New York, 1979. DOI: 10.1007/978-1-4757-5673-9.

[SGA1] Alexander Grothendieck and Michèle Raynaud. *Revêtements Étales et Groupe Fondamental: Séminaire de Géométrie Algébrique du Bois Marie 1960/61 (SGA 1)*. Springer Berlin, Heidelberg, 1971. DOI: https://doi.org/10.1007/BFb0058656.

[Sou81] Christophe Soulé. 'On higher p-adic regulators'. In: *Algebraic K-Theory Evanston 1980*. Ed. by Eric M. Friedlander and Michael R. Stein. Berlin, Heidelberg: Springer Berlin Heidelberg, 1981, pp. 372–401. ISBN: 978-3-540-38646-9.

[Stacks] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

[Ste94] Glenn Stevens. *Rigid Analytic Modular Symbols*. Preprint. Available at: http://math.bu.edu/people/ghs/research.html. 1994.

[Str10] Benoît Stroh. 'Compactification de variétés de Siegel aux places de mauvaise réduction'. In: *Bulletin de la Société Mathématique de France* 138.2 (2010), pp. 259–315.

[SW13] Peter Scholze and Jared Weinstein. 'Moduli of p-divisible groups'. In: *Cambridge Journal of Mathematics* 1.2 (2013). DOI: https://dx.doi.org/10.4310/CJM.2013.v1.n2.a1.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*. Annals of Mathematics Studies. Princeton University Press, 2020.

[Tay91] Richard Taylor. 'Galois representations associated to Siegel modular forms of low weight'. In: *Duke Math. J.* 63.2 (1991), pp. 281–332. DOI: 10.1215/S0012-7094-91-06312-X.

[TW95] Richard Taylor and Andrew Wiles. 'Ring-Theoretic Properties of Certain Hecke Algebras'. In: *Annals of Mathematics* 141.3 (1995).

[Urb05] Eric Urban. 'Sur les représentations p-adiques associées aux représentations cuspidales de $\mathrm{GSp}_{4/\mathbb{Q}}$'. In: *Formes automorphes (II) - Le cas du groupe* GSp(4). Ed. by Jacques Tilouine, Henri Carayol, Michael Harris and Marie-France Vignéras. Astérisque 302. Société mathématique de France, 2005, pp. 151–176.

[Urb11] Eric Urban. 'Eigenvarieties for reductive groups'. In: *Annals of Mathematics* 174.3 (2011), pp. 1685–1784.

[Wei05] Rainer Weissauer. 'Four dimensional Galois representations'. In: *Formes automorphes (II) - Le cas du groupe* GSp(4). Ed. by Jacques Tilouine, Henri Carayol, Michael Harris and Marie-France Vignéras. Astérisque 302. Société mathématique de France, 2005.

[Wu21] Ju-Feng Wu. 'A pairing on the cuspidal eigenvariety for GSp_{2g} and the ramification locus'. In: *Documenta Mathematica* 26 (2021), pp. 675–711. DOI: https://doi.org/10.25537/dm.2021v26.675-711.

[Wu22] Ju-Feng Wu. *On adjoint Bloch–Kato Selmer groups for* GSp_{2g}. Preprint. Available at: https://arxiv.org/abs/2107.12929. 2022.