APPLICATIONS OF MULTIPLE DIRICHLET SERIES IN ANALYTIC NUMBER THEORY

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Abstract

Applications of multiple Dirichlet series in analytic number theory

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We consider several classical problems in analytic number theory from the point of view of multiple Dirichlet series.

In Chapter 1, we review the required background and give an overview of the thesis.

In Chapter 2, we introduce multiple Dirichlet series and the relevant ideas used in later chapters.

In Chapter 3, we study the double character sum

$$S(X,Y) = \sum_{\substack{m \le X, \\ m \text{ odd}}} \sum_{\substack{n \le Y, \\ n \text{ odd}}} \left(\frac{m}{n} \right).$$

The asymptotic formula for this sum was obtained by Conrey, Farmer and Soundararajan. We recover this formula using our approach with an improved error term.

In Chapter 4, we use multiple Dirichlet series to give a conditional proof of the ratios conjecture for the family of real Dirichlet L-functions in some region of the shifts. As an application of our result, we compute the one-level density in our family for test functions whose Fourier transform is supported in $(-2, 2)$, including all lower-order terms.

In Chapter 5, we elaborate on an ongoing work of the author with Siegfred Baluyot. We compare two methods to come up with conjectural asymptotic formulas for moments in the family of real Dirichlet L-functions – the recipe developed by Conrey, Farmer, Keating, Rubinstein and Snaith, and the multiple Dirichlet series approach introduced by Diaconu, Goldfeld and Hoffstein. We consider shifted moments and show that the two methods are essentially equivalent, in that they give rise to the same terms which come from the functional equation of the individual L-functions.

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Contribution of Authors

Chapters 1 and 2 are original contributions of the author.

Chapter 3 is a slightly modified version of the author's paper [Če1], which be published in the Canadian Mathematical Bulletin.

Chapter 4 is a slightly modified version of the author's paper [Če1], which has been submitted for publication.

Chapter 5 is based on ongoing joint work of the author with Siegfred Baluyot (American Institute of Mathematics).

All contributors reviewed the final manuscript and approved of the contents.

Contents

Chapter 1

Introduction

In the thesis, we study various problems from analytic number theory from the point of view of multiple Dirichlet series. It is based on the author's results [Če1] and [Če2], supplemented with additional background and a piece of work in progress with Siegfred Baluyot.

Due to the nature of the thesis, there is a considerable amount of overlap between the first chapter and the introduction in each of the later chapters.

In the remaining part of the introduction, we outline the general theory and studied problems and conclude with a more thorough description of the later chapters.

Many problems in number theory can be stated in terms of finding precise estimates or asymptotics for various sums. In his memoir in 1859, motivated by the question of how many prime numbers there are up to a given bound, Riemann [Rie] found a way to express sums of arithmetic functions in terms of integrals, which enables us to use techniques from complex analysis. This formula was named after Perron [Per], who provided the first rigorous proof. The resulting integral involves a Dirichlet series, such as the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, or a Dirichlet L-function $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ corresponding to a Dirichlet character χ. Analytic information about the L-functions can provide us with answers to the original questions.

L-functions can be associated to many objects in number theory, such as elliptic curves, modular forms or Galois representations. They are believed to encode many important properties of the underlying objects.

For this reason, L-functions and their properties are studied on their own and form an important part of current research. Of particular importance is the location of their zeros, which is the subject of some of the most important conjectures, such as the (generalized) Riemann Hypothesis (GRH), or the Birch and Swinnerton-Dyer conjecture.

In 1973, Montgomery [Mon] investigated the pair correlation function of the zeros of the Riemann zeta function, and found out in a lucky conversation with Dyson that it is the same as the pair correlation of eigenvalues of random matrices in the Gaussian unitary ensemble (an ensemble of random $n \times n$ Hermitian matrices, in which the upper triangular entries are independent and identically distributed complex numbers with standard complex normal distribution, and the diagonal entries are independent identically distributed real numbers with the standard normal distribution, which are also independent of the upper-triangular entries). In the following decades, many more astonishing similarities between the distribution of values of L-functions and that of characteristic polynomials of random matrices were discovered. These connections remain mainly conjectural and motivate a large body of contemporary research. An example is the work of Keating and Snaith [KeSn], who conjectured precise asymptotic formulas for all moments of the Riemann zeta function.

The subject of this thesis is the adoption of Riemann's ideas to tackle problems about L-functions themselves, such as determination of their moments or the distribution of their zeros. An application of Perron's formula to a sum involving L-functions leads to an integral that involves a *multiple Dirichlet series*, that is a Dirichlet series in several variables (or, equivalently, a Dirichlet series whose coefficients are themselves L-functions). To tackle the original problems, we then need to find the analytic properties of the multiple Dirichlet series in question, in particular show that they can be meromorphically continued to a larger region and determine their poles and residues. As in the single-variable cases, the main terms in the resulting asymptotics come from the poles and their residues after an application of Cauchy's Residue Theorem. These terms can usually be computed, at least in the region of absolute convergence. It can then be shown that if the associated multiple Dirichlet series has a meromorphic continuation to a larger region, the results extend and agree with the predictions coming from random matrix theory or other models.

Obtaining this meromorphic continuation is the hardest part of the process. We have a heuristic due to Bump, Chinta, Friedberg and Hoffstein (see [BFH1] and

[CFH]), according to which the multiple Dirichlet series should satisfy some functional equations, which come from the functional equations of the individual L-functions in the coefficients. Making these heuristics rigorous and finding the correct objects to study is the subject of a large theory developed by Bump, Chinta, Diaconu, Friedberg, Hoffstein, Whitehead and others (see the references in Chapter 2).

1.1 Perron's formula

Given any sequence of complex numbers $\{a(n)\}_{n=1}^{\infty}$ that grows at most polynomially with n, Perron's formula enables us to express its partial sums as an integral. In particular, it says that

$$\sum_{n \leq X} a(n) = \frac{1}{2\pi i} \int_{(c)} A(s) \cdot \frac{X^s}{s} ds, \tag{1.1.1}$$

where we integrate over the vertical line in the complex plane $\mathrm{Re}(s) = c$, $A(s)$ is the Dirichlet series defined by

$$A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{1.1.2}$$

and c is large enough so that $A(s)$ converges absolutely on the line of integration. We also note the technical condition that (1.1.1) holds in this form only if X is not a positive integer, because otherwise the term $a(X)$ should be replaced by $\frac{a(X)}{2}$. We will further assume that $X \notin \mathbb{Z}$ (and the situation can always be modified to allow integer X as well).

This formula is very useful in analytic number theory, because for many interesting sequences $\{a(n)\}$, the Dirichlet series $A(s)$ can be shown to have nice analytic properties that enable us to obtain an asymptotic formula for the integral on the right-hand side.

A common strategy is to shift the integral as much to the left as possible, as then $|X^s| = X^{\mathrm{Re}(s)}$ becomes small, and capture the contribution of the residues coming from the possible poles of $A(s)$. To be able to do this, we need to obtain a meromorphic continuation of the Dirichlet series $A(s)$, determine its poles and compute their residues.

In practice, we can use the smoothed version of the formula, which quickly follows after an application of Mellin inversion. For a smooth function $f(x)$, we have

$$\sum_{n\geq 1} a(n)f(n/X) = \frac{1}{2\pi i}\int_{(c)} A(s)X^s\mathcal{M}f(s)ds, \tag{1.1.3}$$

where $\mathcal{M}f(s)$ denotes the Mellin inversion of f defined by

$$\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx. \tag{1.1.4}$$

Note that we recover (1.1.1) if f is the characteristic function of the interval $[0,1]$, as then $\mathcal{M}f(s) = 1/s$. The advantage of (1.1.3) is that if $f(x)$ is smooth, $\mathcal{M}f(s)$ decays faster than any polynomial, so as opposed to the non-smooth version, the integral is absolutely convergent on the line of integration.

1.2 Riemann zeta function and the distribution of prime numbers

The first example of a Dirichlet series is the Riemann zeta function initially defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}. \tag{1.2.1}$$

Euler showed that it can be expressed as an infinite product over primes as

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.2.2}$$

which can be viewed as an analytic form of the fundamental theorem of arithmetic and explains its importance in analytic number theory. Taking logarithmic derivatives of (1.2.2), we obtain the formula

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s}, \tag{1.2.3}$$

where the von Mangoldt function $\Lambda(n)$ is 0 if n is not a prime power, and $\log p$ if $n = p^k$. Substituting this into Perron's formula, we obtain

$$\sum_{n\leq X}\Lambda(n) = \frac{1}{2\pi i}\int_{(2)} -\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{X^s}{s}ds, \tag{1.2.4}$$

and the integral can be computed if we have sufficient analytic information about $\zeta(s)$ and its logarithmic derivative. This expression provides the first step in most proofs of the prime number theorem, which gives an approximate expression for the number of primes up to X. We sketch the remaining steps of the proof below.

Riemann showed that $\zeta(s)$ has a meromorphic continuation to the whole $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1, and that it satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.2.5}$$

relating the values of ζ at s and $1 - s$.

We can thus shift the integral in (1.2.4) to the left and capture the contribution of the poles of $\frac{\zeta'(s)}{\zeta(s)}$. These poles come from the poles and zeros of $\zeta(s)$, and the pole of $1/s$ at $s = 0$. Thus shifting the integral all the way to $\mathrm{Re}(s) = -\infty$, we obtain the (for now formal) identity

$$\sum_{n\leq X}\Lambda(n) = X - \sum_{\rho}\frac{X^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)}, \tag{1.2.6}$$

where the sum runs over all zeros ρ of $\zeta(s)$. This identity is called the explicit formula because it gives a *precise* expression for the sum over primes. Note that the left-hand side is a step function, but we have a series of smooth functions on the right-hand side, so it cannot be absolutely convergent. However, after a correct interpretation, the formula can be justified rigorously and it explains the connection between the zeros of $\zeta(s)$ and the distribution of prime numbers. The real parts of the zeros are particularly important, as $|X^{\rho}| = X^{\mathrm{Re}(\rho)}$. An equivalent statement to the prime number theorem is that in (1.2.6), X is the main term and the sum over zeros is $o(X)$, i.e., it goes into the error term.

Let us now briefly talk about the zeros of $\zeta(s)$. It follows from the Euler product (1.2.2), which is absolutely convergent for $\mathrm{Re}(s) > 1$, that $\zeta(s)$ has no zeros in this region, and the functional equation implies that other than the trivial zeros at negative even integers, there are no zeros with $\mathrm{Re}(s) < 0$. It follows that all the nontrivial zeros, of which there are infinitely many, lie in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$, and the prime number theorem is equivalent to there being no zeros on the line $\mathrm{Re}(s) = 1$.

If $N(T)$ denotes the number of zeros in the critical strip whose imaginary part lies between 0 and T, then it can be proved using the argument principle and the functional equation that

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + O(\log T). \tag{1.2.7}$$

Thus there are infinitely many non-trivial zeros, and the average gap between them has size $\frac{2\pi}{\log T}$.

The Riemann hypothesis (RH) asserts that all zeros in this strip lie on the critical line $\mathrm{Re}(s) = 1/2$. It would imply that the error term in the prime number theorem is $O(X^{1/2+\varepsilon})$ for any $\varepsilon > 0$. Note that this is essentially best possible, due to the fact that there are infinitely many zeros in the critical strip, and the functional equation implies that they are symmetric about the critical line.

Another important problem is that of estimating the size of $\zeta(s)$ on the critical line. The main conjecture is the Lindelöf hypothesis, which says that for any $\varepsilon > 0$,

$$|\zeta(1/2 + it)| \ll_\varepsilon |t|^\varepsilon. \tag{1.2.8}$$

This is a consequence of RH, but has some interesting applications by itself. These include very strong zero-density estimates, which are bounds for the number of zeros in the critical strips off the critical line, or applications to more delicate questions about the primes, such as their distribution in short intervals.

1.3 L-functions

L-functions are more general Dirichlet series with nice properties similar to those of the Riemann zeta function, such as having an Euler product, a meromorphic continuation and a functional equation (see Chapter 5 of [IwKo] for a precise definition). We will mainly work with Dirichlet L-functions $L(s, \chi)$, where χ denotes a Dirichlet character, i.e., a completely multiplicative function that is periodic modulo n for some n, and satisfies $\chi(a) = 0$ if and only if $(a, n) > 1$. The L-function is defined by

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}, \tag{1.3.1}$$

whose Euler product is

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}. \tag{1.3.2}$$

Dirichlet L-functions have a meromorphic continuation to $\mathbb{C}$, and are holomorphic if χ is a non-principal character (i.e., takes other values than just 0 or 1).

If χ is primitive, $L(s,\chi)$ satisfies a functional equation similar to (1.2.5) that relates the values at s and $1-s$. In particular, for a primitive Dirichlet character χ of conductor n, we have

$$\left(\frac{\pi}{n}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)=\varepsilon(\chi)\left(\frac{\pi}{n}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s+a}{2}\right)L(1-s,\overline{\chi}), \tag{1.3.3}$$

where $a=0$ if χ is even (i.e. $\chi(-1)=1$), $a=1$ if χ is odd (i.e. $\chi(-1)=-1$), and the sign $\varepsilon(\chi)$ is

$$\varepsilon(\chi)=\frac{\tau(\chi,1)}{i^a\sqrt{n}}, \tag{1.3.4}$$

where $\tau(\chi,q)$ is the Gauss sum, defined for all (possibly non-primitive) characters via

$$\tau(\chi,q)=\sum_{j \pmod n}\chi(j)e^{2\pi ijq/n}. \tag{1.3.5}$$

In Section 4.10, we will prove a functional equation that is valid for the L-functions of *all* Dirichlet characters, not only primitive. For any Dirichlet character χ modulo n, we have

$$L(s,\chi)=\varepsilon(\chi)\frac{\pi^{s-1/2}}{n^s}\cdot\frac{\Gamma\left(\frac{1+a-s}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)}K(1-s,\chi), \tag{1.3.6}$$

where

$$K(s,\chi)=\sum_{q\geq 1}\frac{\tau(\chi,q)}{q^s}, \tag{1.3.7}$$

and $a=0$, $\varepsilon(\chi)=1$ if χ is even, or $a=1$, $\varepsilon(\chi)=-i$ for χ odd.

If χ is primitive, then $\tau(\chi,q)=\bar{\chi}(q)\tau(\chi,1)$, so we recover the usual functional equation for primitive characters.

The generalized Riemann hypothesis (GRH) asserts that all non-trivial zeros of $L(s,\chi)$ lie on the line $\mathrm{Re}(s)=1/2$, and it implies the generalized Lindelöf hypothesis which states that for $\mathrm{Re}(s)\geq 1/2$,

$$|L(s,\chi)|\ll|ns|^{\varepsilon}. \tag{1.3.8}$$

If $N(T,\chi)$ denotes the number of zeros of $L(s,\chi)$ in the critical strip with imaginary parts between 0 and T, where χ is a primitive character of conductor n, then

$$N(T,\chi)=\frac{T}{2\pi}\log\left(\frac{nT}{2\pi e}\right)+O(\log T). \tag{1.3.9}$$

L-functions can be associated to many other important objects in number theory, such as number fields, elliptic curves or automorphic forms. We will not work with these in this thesis, we refer the interested reader to Chapter 5 of [IwKo].

1.4 Families of L-functions

In the recent decades, a new perspective direction of research has emerged, consisting of considering L-functions as part of a larger family rather than on their own. This idea was pioneered by Katz and Sarnak in 1999 (whose motivation stems from Deligne's proof of the Riemann hypothesis in the function fields setting [Del]), who conjectured that every natural family (an informal term that is not precisely defined) of L-functions has an associated symmetry type (unitary, symplectic, (odd and even) orthogonal) related to the classical compact matrix groups. This connection between random matrices and L-functions has been investigated since the work of Montgomery [Mon]. It is now predicted that random matrices can be used to model many other statistics of L-functions in families, such as their moments [KeSn] or the local distribution of zeros [KaSa1], [KaSa2].

Let us now give examples of some families and their symmetries. See also Section 4 of [KaSa2] for other examples.

1. The zeta function $\zeta(1/2+it)$ or any other individual L-function can be considered as a family parametrized by the continuous parameter t. This family has a unitary symmetry.

2. The family of Dirichlet L-functions associated to (primitive) Dirichlet characters modulo Q, with $Q \to \infty$. This family has a unitary symmetry.

3. Throughout the thesis, we will focus on the family of real Dirichlet L-functions, associated to (primitive) real characters. This family has a symplectic symmetry.

4. The family of elliptic curves, ordered by their conductor or discriminant. This family has an orthogonal symmetry. We can also consider subfamilies according to the sign of their functional equations, whose symmetry is even orthogonal (if the sign is $+1$) or odd orthogonal (if the sign is -1).

5. The family of holomorphic cusp forms of weight k, which are newforms of level N, with $N \to \infty$. This family has an orthogonal symmetry, or even or odd orthogonal if we consider the subfamily with a fixed sign of the functional equation.

6. The family of quadratic twists of a fixed cusp form. This family has an orthogonal symmetry.

Some of these families have a natural counterpart in the function fields setting, which is also intensely studied. This context is interesting, because the symmetry groups have an actual geometric meaning in terms of a monodromy group.

1.5 Moments of L-functions

An important part of current research is the study of moments of zeta or other L-functions.

For the Riemann zeta function, we consider the moments

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k}. \tag{1.5.1}$$

For $k = 1$, Hardy and Littlewood [HaLi] showed that $I_1(T) \sim T \log T$, and Littlewood [Lit] proved that $I_1(T) = TP_1(\log T) + O(T^{3/4+\varepsilon})$, where $P_1(x)$ is a polynomial of degree 1. Similarly, for $k = 2$, Ingham [Ing] obtained the leading order term $I_2(T) \sim \frac{1}{2\pi^2} T(\log T)^4$, and Heath-Brown [Hea1] proved that $I_2(T) = TP_4(\log T) + O(T^{7/8+\varepsilon})$, where $P_4(x)$ is a computable polynomial of degree 4.

No higher moments of $\zeta(s)$ were computed so far. It is conjectured that

$$I_k(T) \sim c_k T(\log T)^{k^2}, \tag{1.5.2}$$

where the constants c_k were first precisely determined by Keating and Snaith [KeSn] and partly come from random matrix theory. Later, Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS] developed a recipe which predicts a more precise asymptotic

$$I_k(T) = TP_{k^2}(\log T) + O(T^{1-\delta}), \tag{1.5.3}$$

where $P_{k^2}(x)$ is a (computable) polynomial of degree k^2, and $\delta > 0$.

The recipe is much more general and is also able to predict moments in other families in L-functions. In Section 5.1, we give an example of its application to the family of real Dirichlet L-functions, where we consider the family of Kronecker symbols $\{\chi_d\} = \left(\frac{d}{\cdot}\right)$, parametrized by fundamental discriminants d. The conjecture is that

$$\sum_{d \leq X}^{*} L(1/2, \chi_d)^k = XQ_{k(k+1)/2}(\log X) + O(X^{1-\delta}). \tag{1.5.4}$$

Here the sum runs over fundamental discriminants, and $Q_{k(k+1)/2}$ is a polynomial of degree $k(k+1)/2$.

The precise size of the error term in (1.5.3), (1.5.4) and the analogues in other families remains an interesting question. In fact, the original conjecture is that we can take any $\delta < 1/2$ in all families, justified by the "square-root cancellation philosophy" which comes from probability theory. However, a secondary term can arise in some cases, as was later conjectured by Diaconu, Goldfeld and Hoffstein [DGH] and confirmed by Diaconu [Dia] and Diaconu and Whitehead [DiWh] using multiple Dirichlet series.

In the family of real Dirichlet L-functions, Jutila [Jut] computed the first moment with a power-saving error term, and proved the asymptotic formula for the second moment. Soundararajan [Sou] then obtained the full main term for the second and third moments with a power saving error term.

The third moment in this family was also later computed by Diaconu, Goldfeld and Hoffstein [DGH] with an improved error term using multiple Dirichlet series. The authors also used multiple Dirichlet series to give another justification for the conjectured asymptotics for moments of the Riemann zeta function and of real Dirichlet L-functions. In particular, they proved that if certain multiple Dirichlet series have a meromorphic continuation beyond a particular point, then they obtain an asymptotic formula for these moments which agree with the main terms conjectured by the recipe. The multiple Dirichlet series heuristic predicts that there are additional secondary terms. For the third moment, the authors conjectured that there is a secondary term of order $O(X^{3/4})$, which was later confirmed by Diaconu [Dia] and Diaconu and Whitehead [DiWh]. They also mentioned the possible presence of infinitely many secondary terms for higher moments, a conjecture that was lately made precise by Diaconu, Paşol and Popa [DPP].

1.6 Distribution of zeros of L-functions

Due to their intimate connection with the distribution of primes, the study of zeros of the Riemann zeta function is of great interest. The Riemann hypothesis is by many considered as the most important unsolved problem.

The zeros of other L-functions bear a similar importance. Those of Dirichlet L-functions govern the distribution of prime numbers in arithmetic progressions, and the multiplicity of the zero of an elliptic curve L-function at the central point is by the Birch and Swinnerton-Dyer conjecture equal to the algebraic rank of the group of rational points on the elliptic curve.

Even though we are very far from a proof of the Riemann Hypothesis, there are several unconditional results about the distribution of the zeros, which can serve as a substitution in some cases. Such results concern for example the proportion of zeros lying on the critical line, or zero-density estimates which bound the number of zeros off the critical line.

In 1973, Montgomery [Mon] studied the distribution of zeros of $\zeta(s)$ on the critical line under the assumption of the Riemann hypothesis. In a lucky conversation, Dyson noticed that the distribution is the same as in the pair correlation of eigenvalues of random matrices in the Gaussian unitary ensemble. This observation led to several decades of fruitful research, with many more connections between the distribution of values of L-functions and of characteristic polynomials of random matrices found or conjectured, including the mentioned work [KeSn] leading to the conjectures (1.5.2).

Katz and Sarnak [KaSa1], [KaSa2] assumed GRH and considered a more refined statistic, the $n-$level density of low-lying zeros (those close to the real line) in various families of L-functions. They conjecture that every natural family of L-functions has an associated symmetry group which determines the behaviour of normalized zeros near the central point. See also the nice survey of Conrey [Con].

In the family of real Dirichlet L-functions, the one-level density is defined by

$$D(X;h) = \frac{1}{X^*} \sum_{d \leq X}^{*} \sum_{\gamma_d} h\left(\frac{\gamma_d \log X}{2\pi}\right), \tag{1.6.1}$$

where X^* denotes the number of positive fundamental discriminants up to X, $\rho_d = 1/2 + i\gamma_d$ runs over the zeros of $L(s,\chi_d)$, and $h(x)$ is a test function. In view of (1.3.9), the normalization factor $\frac{\log X}{2\pi}$ ensures that the average spacing of zeros of $L(s,\chi_d)$

around $s = 1/2$ with $d \approx X$ is 1. In Chapter 4, we consider a smoothed version for a slightly different family. According to the conjectures in [KaSa1], [KaSa2],

$$D(X;h) \sim \int_{-\infty}^{\infty} h(u) W_{\mathrm{Sp}}(u) du, \tag{1.6.2}$$

where $W_{\mathrm{Sp}}(u) = 1 - \frac{\sin(2\pi u)}{2\pi u}$ is the one-level scaling density of the eigenvalues near 1 for the group of unitary symplectic matrices. Özlük and Snyder [ÖzSn] proved that (1.6.2) holds if the Fourier transform $\hat{h}$ is supported in $(-1,1)$, or in $(-2,2)$ under GRH, and extending the range of the support is a major problem.

There is an abundance of similar results in the literature for other families (see the references in Chapter 4).

1.7 Ratios conjectures

In 1993, Farmer [Far] conjectured the asymptotic formula

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)} dt \sim T \frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} + T^{1-\alpha-\beta} \frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}, \tag{1.7.1}$$

where $s = 1/2 + it$, and $\alpha, \beta, \gamma, \delta$ are shifts of size $\approx \frac{1}{\log T}$. He noticed that this conjecture would have many important consequences, including Montgomery's pair correlation conjecture or Levinson's formula for the mollified second moment of $\zeta(s)$ on the critical line (which in turn implies the Riemann Hypothesis by the work of Bettin and Gonek [BeGo]).

Conrey, Farmer and Zirnbauer [CFZ2], [CFZ1] generalized Farmer's conjecture and the recipe from [CFKRS] to produce the so-called ratios conjectures, which predict an asymptotic formula for the ratios of products of shifted L-functions averaged over a family. These conjectures have a vast amount of applications in computing local or global statistics. To illustrate the power of these conjectures, Conrey and Snaith [CoSn] wrote in their nice paper that: "From the ratios conjectures not only can you obtain all $n-$level correlations, but also essentially any local or global statistic."

We now state the conjecture for the family of real Dirichlet characters with one L-function in the numerator and denominator.

Conjecture 1.7.1. *Let $\varepsilon > 0$. Then uniformly for all $\alpha, \beta \in \mathbb{C}$ with $-1/4 < \mathrm{Re}(\alpha) <$*

$1/4$, $\frac{1}{\log X} \ll \mathrm{Re}(\beta) < 1/4$, *and* $|\mathrm{Im}(\alpha)|, |\mathrm{Im}(\beta)| \ll_\varepsilon X^{1-\varepsilon}$, *we have*

$$\begin{aligned}\sum_{d\le X}^{*} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\beta,\chi_d)} &= \frac{X}{2\zeta(2)}\frac{\zeta(1+2\alpha)}{\zeta(1+\beta+\alpha)}P_D(\alpha,\beta)+ \\ &+ \frac{X^{1-\alpha}}{2\zeta(2)(1-\alpha)}\cdot\frac{\pi^\alpha\Gamma\left(\frac14-\frac{\alpha}{2}\right)}{\Gamma\left(\frac14+\frac{\alpha}{2}\right)}\cdot\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\beta)}P_D(-\alpha,\beta)+O(X^{1/2+\varepsilon}),\end{aligned} \tag{1.7.2}$$

where d runs over fundamental discriminants, and

$$P_D(\alpha,\beta) = \prod_p \left(1+\frac{1-p^{\beta-\alpha}}{(p^{1+\alpha+\beta}-1)\,(p+1)}\right). \tag{1.7.3}$$

The general form of the conjectures allow more shifted L-functions in the numerator or denominator, and hold for other families of L-functions as well. Similarly as in the moments conjectures, it is possible that the error term isn't as small as predicted.

The conditions for the real parts of the shifts are mainly to ensure convergence of the Euler product $P_D(\alpha,\beta)$. The important and difficult part is taking $\mathrm{Re}(\beta)$ as small as possible, which can make the denominator close to zero.

The usefulness of the conjecture stems from the fact that it holds uniformly in the considered range of the shifts, so we can differentiate with respect to some variables without affecting the error term (see Section 4.7). Differentiating with respect to α and setting $\alpha=\beta$, we obtain a conjectured asymptotic formula for the sum of shifted logarithmic derivatives:

Conjecture 1.7.2. *Let $\varepsilon>0$. Then uniformly for $r\in\mathbb{C}$ with $\frac{1}{\log X} \ll \mathrm{Re}(r) < 1/4$, $\mathrm{Im}(r) \ll X^{1-\varepsilon}$, we have*

$$\begin{aligned}\sum_{d\le X}^{*} \frac{L'(1/2+r,\chi_d)}{L(1/2+r,\chi_d)} &= \frac{X}{2\zeta(2)}\left(\frac{\zeta'(1+2r)}{\zeta(1+2r)}+\sum_p \frac{\log p}{(p+1)(p^{1+2r}-1)}\right) \\ &- \frac{X^{1-r}}{2\zeta(2)(1-r)}\cdot\frac{\pi^r\Gamma\left(\frac14-\frac{r}{2}\right)}{\Gamma\left(\frac14+\frac{r}{2}\right)}\zeta(1-2r)P_D(-r,r)+O(X^{1/2+\varepsilon}),\end{aligned} \tag{1.7.4}$$

where the sum runs over fundamental discriminants, and $P_D(\alpha,\beta)$ is as in (1.7.3).

Among many applications of the ratios conjectures, the formula (1.7.4) enables us to compute the one-level density without any restriction on the support of the Fourier transform of the test function. See the paper of Conrey and Snaith [CoSn] for many applications.

An important feature of the ratios conjectures is that not only they predict the main order in the distribution (as the Katz-Sarnak predictions), but also contain all lower-order terms (see the introduction of Chapter 4 for more details).

1.8 Overview of the thesis

In the next chapter, we introduce the theory of multiple Dirichlet series and outline the ideas that are used throughout the rest of the thesis. We briefly describe the theory that has been developed by Bump, Chinta, Diaconu, Friedberg, Goldfeld, Hoffstein, Whitehead and others in the last few decades, who considered the multiple Dirichlet series related to moments in various families, such as the real Dirichlet characters or quadratic twists of higher order L-functions, and came up with techniques that allowed to obtain interesting results on moments and non-vanishing of quadratic twists of various L-functions at the central point.

We consider the simplest example of a double Dirichlet series related to the first moment of real Dirichlet characters, or to the double character sum (1.8.1) studied in Chapter 3. This Dirichlet series was also studied by Blomer [Blo], who proved that it satisfies some subconvexity estimates. Blomer showed that it possesses a group of functional equations which gives a meromorphic continuation to the whole $\mathbb{C}^2$. We relate the method of Blomer with the more general theory developed by the aforementioned authors.

Apart from the rigorous results, part of the theory are very useful heuristics developed by Bump, Chinta, Friedberg and Hoffestein in [BFH1], [CFH], which tell us which properties we can expect the multiple Dirichlet series to have. In particular, they provide us with an expected group of functional equations, which can also be used to determine the poles, and is useful in predicting the results. We introduce these heuristics that are also used in later chapters.

Chapter 3 is based on the manuscript [Če1] which will be published in the Canadian Mathematical Bulletin. We consider the double character sum

$$S(X,Y) = \sum_{\substack{m\leq X,\\ m \text{ odd}}} \sum_{\substack{n\leq Y,\\ n \text{ odd}}} \left(\frac{m}{n}\right) \tag{1.8.1}$$

and its smooth version

$$S(X,Y;\varphi,\psi) = \sum_{m,n \text{ odd}} \left(\frac{m}{n}\right) \varphi(m/X)\psi(n/Y). \tag{1.8.2}$$

An asymptotic formula for $S(X,Y)$ was found by Conrey, Farmer and Soundararajan [CFS], whose motivation was the study of some off-diagonal terms that arose in the computation of the mollified moments in [Sou]. Their result (3.1.3) is surprising because it involves a non-smooth function of X/Y. For us, the main motivation was a "proof of concept" that the method based on the study of multiple Dirichlet series is useful in solving such problems, in particular that the complicated function of X/Y does arise from the residues of a double Dirichlet series. A second motivation is that a similar sum arises in the computation of the one-level density in the family of real Dirichlet L-functions, with the difference that one of the variables runs over primes.

The main results of this chapter are theorems 3.1.1 and 3.1.2, where we obtain asymptotics for $S(X,Y)$ and $S(X,Y,\varphi,\psi)$ with improved error terms. We also obtain an interesting form of the main term (see Section 3.7).

Chapter 4 is based on the manuscript [Če2], which is submitted for publication. We assume GRH and use multiple Dirichlet series to prove a form of the ratios conjecture 1.7.1 for some range of the shifts α, β. This is the first result of this kind in the number fields setting. Recently, Bui, Florea and Keating [BFH1] and Florea [Flo] obtained similar results in the function fields setting using a different method.

The main results in this chapter are theorems 4.1.1 and 4.1.2, together with its corollaries stated as theorems 4.1.3 and 4.1.4. The strength of the result depends on how far we can meromorphically continue a certain triple Dirichlet series. An interesting feature is that this only limits the real parts of the shifts, but not their imaginary parts. In particular, we show that the result remains true if we allow the imaginary parts to grow as fast as any power of X, which is better than originally conjectured.

In our setting, we focus on shifts β with $\mathrm{Re}(\beta) > \varepsilon$, not allowing it to go to 0 with X. However, this still remains useful in some of the applications. We use our results to prove Corollary 4.1.5, where we compute the one-level density in our family for test functions whose Fourier transform is supported in $(-2, 2)$, including all lower order terms with a power saving error, and explain how a further meromorphic continuation of the considered triple Dirichlet series would lead to an extended support.

The sections 4.10, 4.11 and 4.12, which appeared as appendices to the original manuscript, were included into the main text.

In Chapter 5, we elaborate on an ongoing project joint with Siegfred Baluyot. We consider the two ways to conjecture asymptotic formulas for moments in the family of real Dirichlet L-functions and notice that the two methods are very similar. In particular, it shows up that considering the shifted moments, the recipe from

[CFKRS] and the multiple Dirichlet series approach from [DGH] give rise to exactly the same terms, which also arise from the same source – the functional equation of the individual L-functions.

We also sketch the multiple Dirichlet series heuristic that leads to the prediction of secondary terms in the moments of real Dirichlet L-functions. It would be very interesting to modify the recipe to predict these secondary terms as well, and the multiple Dirichlet series process might hint on how to achieve this.

Notation remarks

As the chapters are based on different manuscripts, the notation in each chapter may slightly differ. If the meaning isn't apparent or can differ from previous use, the notation is made precise in the relevant chapter. In some cases, the notation has been modified from the original manuscripts to be more concise throughout the thesis.

Chapter 2

Multiple Dirichlet series

In this chapter, we give a basic overview of the theory of multiple Dirichlet series and outline the techniques used in later chapters. For further references, see the expository papers [Bum], [BFH1], [CFH], and the references therein.

Multiple Dirichlet series naturally arise after an application of Perron's formula when studying most of the problems mentioned in the previous chapter. In particular, they arise in two related situations:

1. Estimating sums involving L-functions, including moments, the ratios conjectures or computation of the n-level density.

2. Estimating multiple sums, such as $S(X, Y)$ in (1.8.1).

For example, considering the first moment in our family of real Dirichlet characters, Perron's formula gives

$$\sum_{d \leq X}^{*} L(1/2, \chi_d) = \frac{1}{2\pi i} \int_{(2)} A_D(1/2, w) \cdot \frac{X^w}{w} dw, \tag{2.0.1}$$

where

$$A_D(s, w) = \sum_{d \geq 1}^{*} \frac{L(s, \chi_d)}{d^w}. \tag{2.0.2}$$

The subscript D here and beyond refers to the family parametrized by fundamental discriminants. It is now clear that the knowledge of the analytic properties of $A_D(s, w)$ would enable us to compute the integral in (2.0.1). In particular, we would like to shift the integral as far to the left as possible, for which we need to obtain a meromorphic continuation of $A_D(s, w)$, find its poles and determine the residues. The continuation can be obtained by showing that $A_D(s,w)$ (or a related series) satisfies some functional equations and iteratively apply them to the region of absolute convergence, together with a continuation principle from multivariable complex analysis (Bochner's Tube Theorem, see [Boc] and Section 4.12). For a nice illustration of this process, see for example Section 2 of [Bum].

We can see that the functional equation of the individual L-functions in the coefficients gives a functional equation of the whole multiple Dirichlet series. Indeed, using the fact that for a fundamental discriminant $d \geq 1$, χ_d is an even primitive character with conductor d, we have

$$\begin{aligned} A_D(s,w) &= \sideset{}{^*}\sum_{d\geq 1} \frac{L(s,\chi_d)}{d^w} = \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \sideset{}{^*}\sum_{d\geq 1} \frac{L(1-s,\chi_d)}{d^{s+w-1/2}} \\ &= \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} A(1-s, s+w-1/2). \end{aligned} \tag{2.0.3}$$

Note how the presence of the conductor in the functional equation of $L(s,\chi_d)$ affected the form of the functional equation of $A_D(s,w)$. We would have a different functional equation if we considered, for example, quadratic twists of a higher degree automorphic L-function.

We can also obtain another useful expression for $A_D(s,w)$ by expanding the Dirichlet series and exchanging sums:

$$A_D(s,w) = \sideset{}{^*}\sum_{d\geq 1} \frac{L(s,\chi_d)}{d^w} = \sideset{}{^*}\sum_{d\geq 1} \sum_{n\geq 1} \frac{\left(\frac{d}{n}\right)}{d^w n^s} = \sum_{n\geq 1} \frac{L_D\left(w, \left(\frac{\cdot}{n}\right)\right)}{n^s}, \tag{2.0.4}$$

where

$$L_D(w,\chi) = \sideset{}{^*}\sum_{d\geq 1} \frac{\chi(d)}{d^w} \approx \frac{L(w,\chi)}{L(2w,\chi^2)} \tag{2.0.5}$$

(note that the $\approx$ would be an equality if the sum ran over square-free integers, but this approximation is enough for our purpose; see Lemma 4.2.23 for a precise expression).

From (2.0.4), we see that $A_D(s,w)$ has a pole at $w=1$ coming from the terms with $n = \square$, in which case $\left(\frac{\cdot}{n}\right)$ is a principal character. Moreover, the functional equation (2.0.3) gives us a second pole at $w = 3/2 - s$, which also becomes $w = 1$ in the case when $s = 1/2$. Thus $A_D(1/2, w)$ has a double pole at $w = 1$, which is exactly where the main term of the first moment comes from!

Note that if we considered a shifted moment, replacing the point $s = 1/2$ by $s = 1/2 + \alpha$ with a small, nonzero α, we would have two simple poles instead of a double pole, so the computations of the residues would be easier.

The expression (2.0.4) is useful for another reason. Notice that if there wasn't the subscript D in the L-function on the right-hand side, we could again use the functional equation for $L\left(w, \left(\frac{\cdot}{n}\right)\right)$ and obtain a new functional equation for $A_D(s,w)$.

To get rid of the subscript, we can try to replace $A_D(s,w)$ by $A(s,w)$ defined by

$$A(s,w) = \sum_{n\geq 1} \frac{L(s,\chi_n)}{n^w}, \tag{2.0.6}$$

where $\chi_n = \left(\frac{n}{\cdot}\right)$. We can also consider the multiple Dirichlet series associated to higher moments

$$A_k(s,w) = \sum_{n\geq 1} \frac{L(s,\chi_n)^k}{n^w} \tag{2.0.7}$$

However, more difficulties arise: $\left(\frac{n}{\cdot}\right)$ is not always a character, and even if it was, it is not always primitive, so $L(s,\chi_n)$ doesn't satisfy a nice functional equation.

Ignoring these problems, Bump, Chinta, Friedberg and Hoffstein [BFH1], [CFH] came up with heuristics to see which properties we can expect $A(s,w)$ to have. In these heuristics, we assume that all the characters χ_n are primitive with conductor n, that the quadratic reciprocity holds perfectly in the form $\chi_n(m) = \chi_m(n)$, and we don't write the gamma factors in the functional equations, thus writing $L(s,\chi_n) \approx n^{1/2-s}L(1-s,\chi_n)$. These heuristics provide the expected functional equations and poles, which gives us a good idea of what answer to expect in (2.0.1) or its analogues. We thus have

$$A_k(s,w) = \sum_n \frac{L(s,\chi_n)^k}{n^w} \approx \sum_n \frac{L(1-s,\chi_n)^k}{n^{w+ks-k/2}} \approx A_k(1-s, w+ks-k/2), \tag{2.0.8}$$

and

$$\begin{aligned} A_k(s,w) &= \sum_n \frac{L(s,\chi_n)^k}{n^w} = \sum_{n,m_1,\ldots,m_k} \frac{\chi_n(m_1\ldots m_k)}{n^w(m_1\ldots m_k)^s} \\ &\approx \sum_{n,m_1,\ldots,m_k} \frac{\chi_{m_1\ldots m_k}(n)}{n^w(m_1\ldots m_k)^s} \approx \sum_{m_1,\ldots,m_k} \frac{L(w,\chi_{m_1\ldots m_k})}{(m_1\ldots m_k)^s} \\ &\approx \sum_{m_1,\ldots,m_k} \frac{L(1-w,\chi_{m_1\ldots m_k})}{(m_1\ldots m_k)^{s+w-1/2}} \approx A_k(s+w-1/2, 1-w). \end{aligned} \tag{2.0.9}$$

If $k \leq 3$, these two functional equations generate a finite group, thus enabling us (in theory) to obtain a meromorphic continuation to the whole $\mathbb{C}^{k+1}$ and compute the first three moments. On the other hand, the group is infinite for $k \geq 4$, so it only gives a meromorphic continuation to part of $\mathbb{C}^{k+1}$ in these cases. Furthermore, the polar lines accumulate, giving rise to a curve of essential singularities beyond which $A_k(s,w)$ probably can't be continued. It is proved in [DGH] that if we were able to obtain the continuation of $A_k(s,w)$ past the point $(s,w) = (1/2, 1)$, we could compute the k-th moment at the central point and the result would agree with the conjecture from the recipe.

In Chapter 5 we make the connection between the two methods more precise, showing that exactly the same terms arise in both settings if considering the shifted moments. This requires only the use of the functional equation (2.0.8), which is rigorous if we only consider the family of primitive characters. However, it is (2.0.9) that provides us with extra meromorphic continuation.

Note that if $k = 1$, we also have $A(s,w) \approx A(w,s)$, which together with (2.0.8) or (2.0.9) generates a bigger group than just (2.0.8) and (2.0.9).

Making the above heuristics precise is one of the major goals of the general theory. Dealing with the quadratic reciprocity is usually simple by introducing some extra characters modulo 8 to capture some congruence conditions and removing some bad primes (or places over general number fields), but the hard part is dealing with the non-primitive characters.

The usual way to do this is to consider a modified multiple Dirichlet series of the form

$$Z(s,w) = \sum_{d\geq 1} \frac{L(s,\chi_{d_0})^k a_k(s,d)}{d^w}, \tag{2.0.10}$$

where d_0 is the square-free part of d and $a_k(s,d)$ are some carefully chosen weights, whose purpose is to make the functional equations true, and enable to perform the exchange of summations required in (2.0.9). There are now several general methods to obtain these weights, including :

1. They arise naturally in the coefficients of some Eisenstein series of half-integral weights, which play a similar role as the theta functions in the classical theory. This method was used by Goldfeld and Hoffstein in [GoHo], see also Section 3.4 of [CFH].

2. They can be obtained after solving a combinatorial problem that arises from some natural assumptions (such as them being finite Dirichlet polynomials with an Euler product). See [BFH2] for a reference.

3. The Chinta-Gunnels averaging method introduced in [ChGu1], [ChGu2].

See also [Bum] or [DPP] for further discussion and references.

In [DGH], the authors computed the third moment using this method and developed a sieving technique that allows them to go to the original family of primitive characters.

In Chapter 4, we introduce another method to rigorously prove some functional equations of the non-modified multiple Dirichlet series associated to the ratios conjectures. It is based on (1.3.6), the functional equation valid for all Dirichlet L-functions, even those associated to non-primitive characters. This enables us to get the functional equation of the multiple Dirichlet series in a very straightforward manner, with the difference that there is a different multiple Dirichlet series on the other side, whose coefficients contain the shifted Gauss sums (see (4.5.10) for the precise formulation).

In addition to the above mentioned methods, Blomer [Blo] considered the multiple Dirichlet series associated to the first moment and found an elegant way to modify it so that it satisfies the functional equation. This method is interesting because the extra coefficients come out very naturally from taking into account the extra Euler factors in the L-functions of non-primitive characters. This doesn't seem to be the case in the general process outlined for example in [CFH]. On the other hand, Blomer didn't consider the L-functions associated to higher moments, and it would be of great interest to see if his method generalizes.

We finish this chapter by comparing Blomer's computations with the more general theory and showing that they lead to the same result.

Let us begin with Blomer's computations. Consider the slightly modified series

$$A(s,w) = \sum_{n \text{ odd}} \frac{L_2(s,\chi_n)}{n^w}, \tag{2.0.11}$$

where the subscript 2 means that the Euler factor at 2 is removed, $n = n_0 n_1^2$ with n_0 squarefree and χ_n is the Kronecker symbol of conductor n_0 if $n \equiv 1 \pmod 4$, or modulo $4n$ if $n_0 \equiv 3 \pmod 4$. This is the same series that arises after applying Perron's formula to the double character sum $S(X,Y)$ and we will work with it in Chapter 3. Note that

$$L_2(s,\chi_n) = L_2(s,\chi_{n_0}) \prod_{p \mid n_1} \left(1 - \frac{\chi_{n_0}(p)}{p^s}\right), \tag{2.0.12}$$

with χ_{n_0} primitive, so

$$
\begin{aligned}
A(s,w) &= \sum_{\substack{n_0 \text{ odd},\\ \mu^2(n_0)=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w} \sum_{n_1 \text{ odd}} \frac{1}{n_1^{2w}} \prod_{p|n_1} \left(1 - \frac{\chi_{n_0}(p)}{p^s}\right) \\
&= \sum_{\substack{n_0 \text{ odd},\\ \mu^2(n_0)=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w} \sum_{n_1 \text{ odd}} \frac{1}{n_1^{2w}} \sum_{d|n_1} \frac{\mu(d)\chi_{n_0}(d)}{d^s} \\
&= \sum_{\substack{n_0 \text{ odd},\\ \mu^2(n_0)=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w} \sum_{d \text{ odd}} \frac{\mu(d)\chi_{n_0}(d)}{d^{s+2w}} \sum_{n_1 \text{ odd}} \frac{1}{n_1^{2w}} \\
&= \sum_{\substack{n_0 \text{ odd},\\ \mu^2(n_0)=1}} \frac{L_2(s,\chi_{n_0})\zeta_2(2w)}{n_0^w L_2(s+2w,\chi_{n_0})}.
\end{aligned}
\tag{2.0.13}
$$

Now all the L-functions $L(s,\chi_{n_0})$ satisfy a nice functional equation since the characters are primitive. Recall that we expect a functional equation under the transformation $(s,w) \mapsto (1-s, s+w-1/2)$. This transformation leaves $s+2w$ intact, so the factor $L(s+2w,\chi_{n_0})$ in the denominator remains unchanged, but $2w$ is sent to $2s+2w-1$. Thus Blomer adds an extra factor $\zeta_2(2s+2w-1)$ to compensate for $\zeta_2(2w)$ in the last expression of (2.0.13), and considers

$$
Z(s,w) = \zeta_2(2s+2w-1)A(s,w). \tag{2.0.14}
$$

He then shows that $Z(s,w)$ (after adding some extra characters modulo 8) satisfies some transformation properties of the expected form. See Lemma 2 in [Blo] for an exact formulation.

Let us now show that Blomer's series is the same as that obtained by the other method. We consider

$$
Z(s,w) = \sum_{n \text{ odd}} \frac{L_2(s,\chi_{n_0})a(s,n)}{n^w}, \tag{2.0.15}
$$

where (see (3.15) in [CFH])

$$
a(s,n) = \sum_{e_1e_2|n_1} \frac{\mu(e_1)\chi_{n_0}(e_1)}{e_1^s e_2^{2s-1}}. \tag{2.0.16}
$$

As mentioned earlier, these coefficients can be obtained either by solving a combinatorial problem [BFH2], or from the Fourier expansion of an Eisenstein series of half-integral weight [GoHo].

Note that by replacing e_2 by $n_1/(e_1e_2)$, we obtain

$$a(s,n) = n_1^{1-2s} \sum_{e_1e_2|n_1} \frac{\mu(e_1)\chi_{n_0}(e_1)}{e_1^{1-s}e_2^{1-2s}} = n_1^{1-2s} a(1-s,n), \tag{2.0.17}$$

which implies

$$L(s,\chi_{n_0})a(s,n) = \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left(n_0n_1^2\right)^{s-1/2} L(1-s,\chi_{n_0})a(1-s,n). \tag{2.0.18}$$

This gives the functional equation for $Z(s,w)$ of the expected form!

To see that (2.0.14) and (2.0.15) are the same, we have

$$\begin{aligned} Z(s,w) &= \sum_{n \text{ odd}} \frac{L_2(s,\chi_{n_0})a(s,n)}{n^w} \\ &= \sum_{\substack{n_0 \text{ odd}, \\ \mu(n_0)^2=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w} \sum_{n_1 \text{ odd}} \frac{1}{n_1^{2w}} \sum_{e_1e_2|n_1} \frac{\mu(e_1)\chi_{n_0}(e_1)}{e_1^s e_2^{2s-1}} \\ &= \sum_{\substack{n_0 \text{ odd}, \\ \mu(n_0)^2=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w} \sum_{e_1,e_2,n_1 \text{ odd}} \frac{\mu(e_1)\chi_{n_0}(e_1)}{e_1^{s+2w}e_2^{2s+2w-1}n_1^{2w}} \\ &= \zeta_2(2w)\zeta_2(2s+2w-1) \sum_{\substack{n_0 \text{ odd}, \\ \mu(n_0)^2=1}} \frac{L_2(s,\chi_{n_0})}{n_0^w L_2(s+2w,\chi_{n_0})}. \end{aligned} \tag{2.0.19}$$

Chapter 3

Mean value of real Dirichlet characters using a double Dirichlet series

3.1 Introduction

We study the double character sum

$$S(X,Y) := \sum_{\substack{m\leq X,\\ m \text{ odd}}} \sum_{\substack{n\leq Y,\\ n \text{ odd}}} \left(\frac{m}{n}\right). \tag{3.1.1}$$

This sum was studied by Conrey, Farmer and Soundararajan in [CFS], where the authors give an asymptotic formula valid for all large X and Y.

If $Y = o(X/\log X)$, then the main term of $S(X,Y)$ comes from the terms where n is a square, and the error term can be estimated using the Pólya-Vinogradov inequality. In particular, we get that in this range,

$$S(X,Y) = \frac{2}{\pi^2} XY^{1/2} + O(Y^{3/2}\log Y + Y^{1/2+\varepsilon} + X\log Y), \tag{3.1.2}$$

and similarly for $X = o(Y/\log Y)$.

Conrey, Farmer and Soundararajan showed that there is a transition in the behavior of $S(X,Y)$ when X,Y are of similar size. In particular, they proved the following asymptotic formula, which is valid for all large X,Y:

$$S(X,Y) = \frac{2}{\pi^2} X^{3/2} C\left(\frac{Y}{X}\right) + O\left(\left(XY^{7/16} + YX^{7/16}\right)\log(XY)\right), \tag{3.1.3}$$

where

$$C(\alpha) = \alpha + \alpha^{3/2}\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1/\alpha}\sqrt{y}\sin\left(\frac{\pi k^2}{2y}\right)dy. \tag{3.1.4}$$

The size of the main term in this formula is $XY^{1/2}+YX^{1/2}$, so it is always larger than the error term. The result is interesting, because $C(\alpha)$ is a non-smooth function. For a heuristic explanation why such functions arise in this type of problems, see the first section in [Pet] and the references therein.

Conrey, Farmer and Soundararajan also gave the following asymptotic estimates for $C(\alpha)$:

$$C(\alpha) = \sqrt{\alpha} + \frac{\pi}{18}\alpha^{\frac{3}{2}} + O\left(\alpha^{5/2}\right) \quad \text{as } \alpha \to 0, \tag{3.1.5}$$

and

$$C(\alpha) = \alpha + O\left(\alpha^{-1}\right) \quad \text{as } \alpha \to \infty. \tag{3.1.6}$$

To prove (3.1.3), Conrey, Farmer and Soundararajan applied the Poisson summation formula and estimated the sums of Gauss sums which appeared in the computation. Similar techniques were used in the work of Gao and Zhao to compute the mean value in other families of characters, such as cubic and quartic Dirichlet characters [GaZh2], and some quadratic, cubic and quartic Hecke characters [GaZh1]. Gao used similar methods to compute the mean value of the divisor function twisted by quadratic characters [Gao].

Our approach is to rewrite $S(X,Y)$ as a double integral by using the inverse Mellin transform twice. The integral will then involve the double Dirichlet series

$$A(s,w) = \sum_{m \text{ odd}}\sum_{n \text{ odd}}\frac{\left(\frac{m}{n}\right)}{m^w n^s},$$

which was studied by Blomer [Blo], who showed that it admits a meromorphic continuation to the whole $\mathbb{C}^2$ and determined the polar lines. We then shift the integrals to the left and compute the contribution of the residues. The quality of the error term depends whether we assume the truth of the Riemann Hypothesis because the zeros of $\zeta(s)$ appear in the location of the poles of $A(s,w)$, and also in the contribution of the residues.

An interesting feature of our proof is that the 3 polar lines from which our main term arises naturally correspond to the contribution of squares (the polar lines $s=1$ and $w=1$), and the transition term where the non-smooth function appears (the polar line $s+w=3/2$).

A more general theory of multiple Dirichlet series has been developed by Bump, Chinta, Diaconu, Friedberg, Goldfeld, Hoffstein and others. We refer the reader interested in the theory and its applications to the expository articles [Bum], [BFH1], [CFH], the paper [DGH] or the book [BFG].

To state our results, we first define the smooth sum

$$S(X,Y;\varphi,\psi) = \sum_{m,n \text{ odd}} \left(\frac{m}{n}\right) \varphi(m/X)\psi(n/Y), \tag{3.1.7}$$

where φ, ψ are nonnegative smooth functions supported in $(0,1)$.

If we denote by $\mathcal{M}f$ the Mellin transform of f (see (3.2.7)), the main result is the following:

Theorem 3.1.1. *Let $\varepsilon > 0$. Then for all large X, Y, we have*

$$S(X,Y;\varphi,\psi) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X};\varphi,\psi\right) + O_\varepsilon(XY^\delta + YX^\delta), \tag{3.1.8}$$

where $\delta = \varepsilon$, and

$$\begin{aligned} D(\alpha;\varphi,\psi) =& \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right)\alpha^{1/2} + \mathcal{M}\psi(1)\mathcal{M}\varphi\left(\frac{1}{2}\right)\alpha}{2} + \\ &+ \frac{1}{i\sqrt{\pi}} \int_{(3/4)} \left(\frac{\alpha}{2\pi}\right)^s \cdot \mathcal{M}\varphi\left(\frac{3}{2} - s\right)\mathcal{M}\psi(s)\Gamma\left(s - \frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)ds. \end{aligned}$$

If we assume the Riemann Hypothesis, then we can take $\delta = -1/4 + \varepsilon$.

We can remove the smooth weights and obtain the following asymptotic formula for $S(X,Y)$, which improves the error term in (3.1.3):

Theorem 3.1.2. *Let $\varepsilon > 0$. Then for all large X, Y, we have*

$$S(X,Y) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X}\right) + O_\varepsilon(XY^{1/4+\varepsilon} + YX^{1/4+\varepsilon}), \tag{3.1.9}$$

where

$$D(\alpha) = \sqrt{\alpha} + \alpha - \frac{1}{i\sqrt{\pi}} \int_{(3/4)} \left(\frac{\alpha}{2\pi}\right)^s \cdot \frac{\Gamma\left(s-\frac{3}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)}{s} ds. \tag{3.1.10}$$

We show in Section 3.7 that $D(\alpha) = C(\alpha)$, so our main term agrees with that of Conrey, Farmer and Soundararajan.

Let us also remark that a similar asymptotic can be obtained if the integers m, n were restricted to lie in a congruence class modulo 8 by working with a suitable combination of the twisted double Dirichlet series, as defined in (3.3.3).

3.2 Preliminaries and notation

Throughout the chapter, ε will denote a sufficiently small positive number, different at each occurrence, and all implied constants are allowed to depend on ε.

We follow the notation of [Blo]. For integers m, n, we denote by $\chi_m(n)$ the Kronecker symbol

$$\chi_m(n) = \left(\frac{m}{n}\right).$$

Assume that m is odd and write it as $m = m_0 m_1^2$ with m_0 squarefree. Then χ_m is a character of conductor $|m_0|$ if $m \equiv 1 \pmod 4$ and $|4m_0|$ if $m \equiv 3 \pmod 4$. We denote by $\psi_1, \psi_{-1}, \psi_2, \psi_{-2}$ the four Dirichlet characters modulo 8 given by the Kronecker symbol $\psi_j(n) = \left(\frac{j}{n}\right)$. We also let

$$\tilde{\chi}_m = \begin{cases} \chi_m, & \text{if } m \equiv 1 \pmod 4, \\ \chi_{-m}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

With this notation, quadratic reciprocity tells us that for odd positive integers m, n,

$$\chi_m(n) = \tilde{\chi}_n(m). \tag{3.2.1}$$

The fundamental discriminants m correspond to primitive real characters of conductor $|m|$. In such cases, the completed L-function is

$$\Lambda(s, \chi_m) = \left(\frac{|m|}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi_m),$$

where $a = 0$ or 1 depending on whether the character is even or odd, i.e., whether $\chi_m(-1) = 1$ or -1, and we have the functional equation

$$\Lambda(s, \chi_m) = \Lambda(1-s, \chi_m). \tag{3.2.2}$$

All primitive real characters can be uniquely written as $\chi_{m_0}\psi_j$ for some positive odd squarefree integer m_0 and $j \in \{\pm 1, \pm 2\}$.

If m is not a fundamental discriminant, then χ_m is a character of conductor $m_0 \mid 4m$, and we have

$$L(s, \chi_m) = L(s, \chi_{m_0}) \cdot \prod_{p \mid \frac{|m|}{m_0}} \left(1 - \frac{\chi_{m_0}(p)}{p^s}\right). \tag{3.2.3}$$

A subscript 2 of an L-function means that the Euler factor at 2 is removed, so in particular

$$L_2(s,\chi) = \sum_{n \text{ odd}} \frac{\chi(n)}{n^s}. \tag{3.2.4}$$

We now record two estimates that will be used later.

The first estimate holds for any s with $\mathrm{Re}(s) \geq 1/2$:

$$\sum_{\substack{m \leq X, \\ m \text{ odd}}} |L_2(s, \chi_m \psi_j)| \ll_\varepsilon X^{1+\varepsilon} |s|^{\frac{1}{4}+\varepsilon}. \tag{3.2.5}$$

It follows after applying Hölder's inequality on the bound for the fourth moment, proved by Heath-Brown [Hea2, Theorem 2].

The second is conditional under RH, and it says that for any fixed $\sigma > 1/2$, we have

$$\left| \frac{1}{\zeta(\sigma + it)} \right| \ll_\varepsilon (1 + |t|)^\varepsilon. \tag{3.2.6}$$

It follows from [CaCh, Theorem 2].

For a function $f(x)$, we denote by $\mathcal{M}f(s)$ its Mellin transform, which is defined as

$$\mathcal{M}f(s) = \int_0^\infty f(x) x^{s-1} dx, \tag{3.2.7}$$

when the integral converges. If $\mathcal{M}f$ is analytic in the strip $a < \mathrm{Re}(s) < b$, then the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s} \mathcal{M}f(s) ds, \tag{3.2.8}$$

where the integral is over the vertical line $\mathrm{Re}(s) = c$, and $a < c < b$ is arbitrary.

We will use the following estimate for the Gamma function, which is a consequence of Stirling's formula: for a fixed $\sigma \in \mathbb{R}$ and $|t| \geq 1$, we have

$$|\Gamma(\sigma + it)| \asymp e^{-|t|\frac{\pi}{2}} |t|^{\sigma - 1/2}. \tag{3.2.9}$$

We will also use the formula

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{2^s \sin(\pi s/2) \Gamma(1-s)}{\sqrt{\pi}}. \tag{3.2.10}$$

We write the functional equation for the Riemann zeta function as

$$\zeta(s) = \chi(s) \zeta(1-s), \tag{3.2.11}$$

where

$$|\chi(\sigma+it)| \ll_\sigma (1+|t|)^{1/2-\sigma}. \tag{3.2.12}$$

We will also use the estimate

$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 dt \ll T^{1+\varepsilon}, \tag{3.2.13}$$

which is true for any $\sigma \geq 1/2$.

3.3 Outline of the proof and double Dirichlet series

Applying Mellin inversion to $S(X,Y;\varphi,\psi)$ twice, we obtain

$$S(X,Y;\varphi,\psi) = \left(\frac{1}{2\pi i}\right)^2 \int\limits_{(\sigma)}\int\limits_{(\omega)} A(s,w)X^wY^s\mathcal{M}\varphi(w)\mathcal{M}\psi(s)dwds, \tag{3.3.1}$$

where for $\mathrm{Re}(s)=\sigma$ and $\mathrm{Re}(w)=\omega$ large enough, we have the absolutely convergent double Dirichlet series

$$A(s,w) = \sum_{m \text{ odd}}\sum_{n \text{ odd}} \frac{\left(\frac{m}{n}\right)}{m^w n^s} = \sum_{m \text{ odd}} \frac{L_2(s,\chi_m)}{m^w}. \tag{3.3.2}$$

We use the results of Blomer to meromorphically continue $A(s,w)$ to the whole $\mathbb{C}^2$, shift the two integrals to the left and compute the contribution of the crossed polar lines.

We now cite and sketch the proof of Lemma 2 in [Blo]. For two characters ψ,ψ' of conductor dividing 8, we define

$$Z(s,w;\psi,\psi') := \zeta_2(2s+2w-1)\sum_{m,n \text{ odd}} \frac{\chi_m(n)\psi(n)\psi'(m)}{m^w n^s}, \tag{3.3.3}$$

which converges absolutely if $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ are large enough, and we let

$$Z(s,w) := Z(s,w;\psi_1,\psi_1) = \zeta_2(2s+2w-1)A(s,w). \tag{3.3.4}$$

We also denote

$$\mathbf{Z}(s,w;\psi) = \begin{pmatrix} Z(s,w;\psi,\psi_1) \\ Z(s,w;\psi,\psi_{-1}) \\ Z(s,w;\psi,\psi_2) \\ Z(s,w;\psi,\psi_{-2}) \end{pmatrix}, \quad \mathbf{Z}(s,w) = \begin{pmatrix} \mathbf{Z}(s,w,\psi_1) \\ \mathbf{Z}(s,w,\psi_{-1}) \\ \mathbf{Z}(s,w,\psi_2) \\ \mathbf{Z}(s,w,\psi_{-2}) \end{pmatrix}.$$

Theorem 3.3.1. *The functions* $Z(s,w;\psi,\psi')$ *have a meromorphic continuation to the whole* $\mathbb{C}^2$ *with a polar line* $s+w=3/2$*. There is an additional polar line at* $s=1$ *with residue* $\mathrm{res}_{(1,w)}Z(s,w)=\zeta_2(2w)/2$ *if and only if* $\psi=\psi_1$*, and an additional polar line* $w=1$ *with residue* $\mathrm{res}_{(s,1)}Z(s,w)=\zeta_2(2s)/2$ *if and only if* $\psi'=\psi_1$*.*

The functions $(s-1)(w-1)(s+w-3/2)Z(s,w;\psi,\psi')$ *are polynomially bounded in vertical strips, meaning that for fixed* $\mathrm{Re}(s)$ *and* $\mathrm{Re}(w)$*,* $(s-1)(w-1)(s+w-3/2)Z(s,w;\psi,\psi')$ *is bounded by a polynomial in* $\mathrm{Im}(s),\mathrm{Im}(w)$*. The functions satisfy functional equations relating* $\mathbf{Z}(s,w)$ *with* $\mathbf{Z}(w,s)$*, and* $\mathbf{Z}(s,w)$ *with* $\mathbf{Z}(1-s,s+w-1/2)$*.*

Remarks:

(i) Blomer gives explicit 16×16 matrices A and $B(s)$, such that $\mathbf{Z}(s,w)=A\cdot\mathbf{Z}(w,s)$, and $\mathbf{Z}(s,w)=B(s)\cdot\mathbf{Z}(1-s,s+w-1/2)$, we will use the explicit form in (3.5.2) to compute the residues on the polar line $s+w=3/2$.

(ii) We can also iterate the two functional equations and obtain others, for example relating $\mathbf{Z}(s,w)$ with $\mathbf{Z}(1-s,1-w)$. Blomer also gives an almost explicit form of this case.

(iii) For us, a polar line means that if we fix one of the variables, the resulting function of the other variable has a pole on the corresponding line with the given residue. What we state doesn't exactly hold at the points $(1/2,1)$ and $(1,1/2)$, where two of the polar lines intersect, but we will not need to know the exact behavior at these points.

Proof sketch. We write the Dirichlet series for $Z(s,w;\psi,\psi')$ in two ways.

First, writing $m=m_0m_1^2$ with $\mu^2(m_0)=1$, we have

$$\begin{aligned}Z(s,w;\psi,\psi') &= \zeta_2(2s+2w-1)\sum_{\substack{m_0 \text{ odd},\\ \mu^2(m_0)=1}}\frac{L_2(s,\chi_{m_0}\psi)\psi'(m_0)}{m_0^w}\times\\ &\qquad\times\sum_{m_1\text{ odd}}\frac{1}{m_1^{2w}}\prod_{p|m_1}\left(1-\frac{\chi_{m_0}\psi(p)}{p^s}\right) \qquad (3.3.5)\\ &= \zeta_2(2s+2w-1)\sum_{\substack{m_0 \text{ odd},\\ \mu^2(m_0)=1}}\frac{L_2(s,\chi_{m_0}\psi)\psi'(m_0)\zeta_2(2w)}{m_0^wL_2(s+2w,\chi_{m_0}\psi)}.\end{aligned}$$

If ψ is non-trivial, the right-hand side converges absolutely in the region

$$\{(s,w) : \mathrm{Re}(w) > 1 \text{ and } \mathrm{Re}(s+w) > 3/2\},$$

the second condition comes from using the functional equation in the numerator when $\mathrm{Re}(s) < 1/2$. When ψ is the trivial character, the summand corresponding to $m_0 = 1$ is $\frac{\zeta_2(s)\zeta_2(2w)}{\zeta_2(s+2w)}$, so there is a pole at $s = 1$ with residue $\zeta_2(2w)/2$. Note that the other potential polar lines coming from $\frac{\zeta_2(2w)}{L_2(s+2w,\chi_{d_0}\psi)}$ are outside of the considered region.

The second way to write $Z(s,w;\psi,\psi')$ is by exchanging summations and using the quadratic reciprocity. We obtain

$$\begin{aligned} Z(s,w;\psi,\psi') &= \zeta_2(2s+2w-1) \sum_{m,n \text{ odd}} \frac{\chi_m(n)\psi(n)\psi'(m)}{m^w n^s} \\ &= \zeta_2(2s+2w-1) \sum_{n \text{ odd}} \frac{L_2(w,\tilde{\chi}_n\psi')\psi(n)}{n^s}. \end{aligned} \tag{3.3.6}$$

We can again write $n = n_0 n_1^2$ with $\mu^2(n_0) = 1$ and obtain a series that is absolutely convergent in the region

$$\{(s,w) : \mathrm{Re}(s) > 1 \text{ and } \mathrm{Re}(s+w) > 3/2\},$$

unless ψ' is the trivial character, in which case there is a pole at $w = 1$ coming from the summands when n is a square, and the residue is $\zeta_2(2s)/2$.

Note that (3.3.6) gives a link between $Z(s,w;\psi,\psi')$ and $Z(w,s;\psi',\psi)$, which gives us a functional equation relating $\mathbf{Z}(s,w)$ with $\mathbf{Z}(w,s)$. To finish the proof and obtain the meromorphic continuation to the whole $\mathbb{C}^2$, we use the functional equation in the numerator of (3.3.5), which gives a functional equation relating $\mathbf{Z}(s,w)$ and $\mathbf{Z}(1-s, s+w-1/2)$, where the change in the second coordinate comes from the conductor in the functional equation for $L(s,\chi_m)$. Notice that this change of variables interchanges $2s+2w-1$ and $2w$, leaves $s+2w$ fixed, and maps the line $w = 1$ to $s + w = 3/2$, which becomes a new polar line.

We can iterate the two transformations coming from (3.3.5) and (3.3.6) and obtain a function meromorphic on a tube region of the form $\{(s,w) : \mathrm{Re}(s)^2 + \mathrm{Re}(w)^2 > c\}$ for some c. During this process, we obtain some additional potential polar lines, but these will be canceled by the gamma factors coming from the functional equations. To obtain a continuation to the region $\{(s,w) : \mathrm{Re}(s)^2 + \mathrm{Re}(w)^2 \le c\}$, we use Bochner's

Tube theorem from multivariable complex analysis, which states that a function that is holomorphic on a tube region can be continued to its convex hull (see [Boc]).

The proof that the function is polynomially bounded in vertical strips is similar to the proof of Proposition 4.11 in [DGH]. □

We will also use the following estimate, which is Theorem 2 in [Blo].

Theorem 3.3.2. *For any $Y_1, Y_2 \geq 1$ and characters ψ, ψ' modulo 8, we have*

$$\int_{-Y_1}^{Y_1} \int_{-Y_2}^{Y_2} |Z(1/2+it, 1/2+iu; \psi, \psi')|^2 du dt \ll (Y_1 Y_2)^{1+\varepsilon}. \tag{3.3.7}$$

In the next two sections, we are going to shift the two integrals in (3.3.1) to the left and compute the contribution of the crossed polar lines. By Theorem 3.3.1, the polar lines of $A(s,w) = \frac{Z(s,w)}{\zeta_2(2s+2w-1)}$ are the following:

- The polar lines of $Z(s,w)$, which give us the main term in Theorem 3.1.2:
  - the line $s = 1$ with residue $\mathrm{res}_{(1,w)} A(s,w) = \frac{\zeta_2(2w)}{2\zeta_2(2w+1)}$,
  - the line $w = 1$ with residue $\mathrm{res}_{(s,1)} A(s,w) = \frac{\zeta_2(2s)}{2\zeta_2(2s+1)}$,
  - the line $s + w = 3/2$, whose residue will be computed in Lemma 3.5.1.
- Zeros of $\zeta_2(2s+2w-1) = \zeta(2s+2w-1)\,(1-2^{1-2s-2w})$, which are the lines $s + w = \frac{\rho+1}{2}$, where ρ is such that $\zeta(\rho) = 0$, or $s + w = \frac{k\pi i}{\log 2} + \frac{1}{2}$ for some $k \in \mathbb{Z}$. All these satisfy $\mathrm{Re}(s+w) < 1$, and even $\mathrm{Re}(s+w) \leq \frac{3}{4}$ if we assume RH.

We will see that the main term comes from the polar lines of $Z(s,w)$, while the polar lines coming from the zeros of $\zeta_2(2s+2w-1)$ determine how far to the left we will be able to shift the integrals, so they give us our error term.

3.4 Contribution of the polar lines $s = 1$ and $w = 1$

In this section, we shift the integrals to the left and compute the contribution of the polar lines $s = 1$ and $w = 1$. We begin with $\sigma = 2$ and $\omega = 2$ in (3.3.1), where everything converges absolutely.

Then we move the inner integral to the line $\mathrm{Re}(w) = 3/4 + \varepsilon$, so we obtain

$$\begin{aligned} S(X,Y;\varphi,\psi) = \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \int_{(3/4+\varepsilon)} & A(s,w) X^w Y^s \mathcal{M}\varphi(w)\mathcal{M}\psi(s) dw ds \\ & + \frac{1}{2\pi i}\int_{(2)} XY^s \mathcal{M}\varphi(1)\mathcal{M}\psi(s) \mathrm{res}_{(s,1)} A(s,w) ds. \end{aligned} \tag{3.4.1}$$

This shift of integrals is justified by the fast decay of the Mellin transform and polynomial boundedness of $A(s,w)$ in vertical strips.

Now we compute the second integral in (3.4.1), which equals

$$\frac{\mathcal{M}\varphi(1)X}{2\pi i}\int_{(2)} \frac{Y^s \mathcal{M}\psi(s)\zeta_2(2s)}{2\zeta_2(2s+1)} ds. \tag{3.4.2}$$

We again estimate this integral using the residue theorem. The integrand has the following poles:

- At $s = 1/2$ with residue

$$\frac{Y^{1/2}\mathcal{M}\psi\left(\frac{1}{2}\right)}{8\zeta_2(2)} = \frac{Y^{1/2}\mathcal{M}\psi\left(\frac{1}{2}\right)}{\pi^2}.$$

- Zeros of

$$\zeta_2(2s+1) = \left(1 - \frac{1}{2^{2s+1}}\right)\zeta(2s+1).$$

  These are at the points $s = \frac{\rho-1}{2}$, where $\zeta(\rho) = 0$, and

$$s = \frac{k\pi i}{\log 2} - \frac{1}{2},\ k \in \mathbb{Z}.$$

  These poles have $\mathrm{Re}(s) < 0$ and if we assume RH, they all have $\mathrm{Re}(s) \leq -1/4$.

Therefore, we have the following:

$$\begin{aligned} &\frac{\mathcal{M}\varphi(1)X}{2\pi i}\int_{(2)} \frac{Y^s \mathcal{M}\psi(s)\zeta_2(2s)}{2\zeta_2(2s+1)} ds \\ &\qquad = \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right)XY^{1/2}}{\pi^2} + \frac{\mathcal{M}\varphi(1)X}{2\pi i}\int_{(\delta)} \frac{Y^s \mathcal{M}\psi(s)\zeta_2(2s)}{2\zeta_2(2s+1)} ds. \end{aligned} \tag{3.4.3}$$

Depending whether we assume RH or not, we take $\delta = -\frac{1}{4} + \varepsilon$ or $\delta = \varepsilon$, bound the integral trivially (we use (3.2.6) when $\delta = -1/4 + \varepsilon$) and get

$$\frac{\mathcal{M}\varphi(1)X}{2\pi i} \int_{(2)} \frac{Y^s \mathcal{M}\psi(s)\zeta_2(2s)}{2\zeta_2(2s+1)} ds = \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right) XY^{1/2}}{\pi^2} + O\left(XY^{\delta}\right). \tag{3.4.4}$$

Using this in (3.4.1), we obtain

$$\begin{aligned} S(X,Y;\varphi,\psi) =& \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right) XY^{1/2}}{\pi^2} + \\ &+ \left(\frac{1}{2\pi i}\right)^2 \int_{(2)} \int_{(3/4+\varepsilon)} A(s,w)X^w Y^s \mathcal{M}\varphi(w)\mathcal{M}\psi(s) dw ds + O(XY^{\delta}), \end{aligned} \tag{3.4.5}$$

Note that when $\varphi = \psi = 1_{[0,1]}$, the first term is $\frac{2}{\pi^2} XY^{1/2}$ and corresponds to the contribution when n is a square.

Next, we exchange the integrals and shift the integral over $\mathrm{Re}(s) = 2$ to $\mathrm{Re}(s) = 3/4$, crossing the polar line at $s = 1$. The computation of the residues coming from this polar line is completely analogous to the previous case, and the result is stated in the following theorem:

Theorem 3.4.1. *Let $\varepsilon > 0$. Then we have:*

$$\begin{aligned} \sum_{m \text{ odd}} \sum_{n \text{ odd}} \left(\frac{m}{n}\right) \varphi\left(\frac{m}{X}\right) \psi\left(\frac{n}{Y}\right) =& \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right) XY^{1/2} + \mathcal{M}\psi(1)\mathcal{M}\varphi\left(\frac{1}{2}\right) YX^{1/2}}{\pi^2} + \\ &+ \left(\frac{1}{2\pi i}\right)^2 \int_{(3/4)} \int_{(3/4+\varepsilon)} A(s,w)X^w Y^s \mathcal{M}\varphi(w)\mathcal{M}\psi(s) dw ds + O_{\varepsilon}\left(YX^{\delta} + XY^{\delta}\right), \end{aligned} \tag{3.4.6}$$

where $\delta = \varepsilon$. If we assume the Riemann Hypothesis, then we can take $\delta = -1/4 + \varepsilon$.

3.5 Contribution of the polar line $s + w = 3/2$

Before further shifting the integrals, we need to compute the residues on the polar line $s + w = 3/2$, which is done in the following lemma.

Lemma 3.5.1. *For all $s \in \mathbb{C}$,*

$$\mathrm{res}_{\left(s, \frac{3}{2} - s\right)} Z(s,w) = \frac{\sqrt{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma\left(s - \frac{1}{2}\right) \zeta(2s-1)}{2(2\pi)^s}. \tag{3.5.1}$$

Proof. We use the functional equation (28) in [Blo], from which it follows that

$$Z(1-u,u+v-1/2) = \frac{\pi^{-u+\frac{1}{2}}\Gamma\left(\frac{u}{2}\right)}{(4^{1-u}-4)\,\Gamma\left(\frac{1-u}{2}\right)} \cdot \Bigg(-4^u Z(u,v;\psi_1,\psi_1)+$$
$$+ (4^u-2)\,Z(u,v;\psi_1,\psi_{-1}) + \left(2^u - 2^{1-u}\right)\left(Z(u,v;\psi_1,\psi_2)+Z(u,v;\psi_1,\psi_{-2})\right)\Bigg). \tag{3.5.2}$$

Under the change of variables $(s,w)=(1-u,u+v-1/2)$, the line $v=1$ transforms to the line $s+w=3/2$. Since $v=1$ is a polar line of $Z(u,v;\psi,\psi')$ if and only if $\psi'=\psi_1$, the residue comes only from the first term in the parenthesis on the right-hand side of (3.5.2), and is given by

$$\mathrm{res}_{\left(1-u,u+\frac{1}{2}\right)} Z(u,v;\psi_1,\psi_1) = \frac{\pi^{-u+\frac{1}{2}}\Gamma\left(\frac{u}{2}\right)(-4^u)\,\zeta_2(2u)}{2\,(4^{1-u}-4)\,\Gamma\left(\frac{1-u}{2}\right)} = \frac{\pi^{-u+\frac{1}{2}}\Gamma\left(\frac{u}{2}\right)\zeta(2u)}{2\cdot 4^{1-u}\Gamma\left(\frac{1-u}{2}\right)},$$

so we have

$$\mathrm{res}_{\left(s,\frac{3}{2}-s\right)} Z(s,w) = \frac{\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(2-2s)}{2\cdot 4^s\Gamma\left(\frac{s}{2}\right)} = \frac{\sqrt{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{2(2\pi)^s},$$

where the last equality follows after using the functional equation for $\zeta(s)$ and the formula (3.2.10). □

We are now ready to prove Theorem 3.1.1:

Proof of Theorem 3.1.1. We move the integral from (3.4.6) further to the left. According to the discussion at the end of Section 3.3, we know that except the line $s+w=3/2$, the integrand has no poles with $\mathrm{Re}(s+w)\geq 1$, or with $\mathrm{Re}(s+w)>3/4$ if we assume RH. Hence we have

$$\left(\frac{1}{2\pi i}\right)^2 \int\limits_{(3/4)}\int\limits_{(3/4+\varepsilon)} A(s,w)X^wY^s\mathcal{M}\varphi(w)\mathcal{M}\psi(s)dwds$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int\limits_{(3/4)}\int\limits_{(\delta')} A(s,w)X^wY^s\mathcal{M}\varphi(w)\mathcal{M}\psi(s)dwds+ \tag{3.5.3}$$
$$+\frac{1}{2\pi i}\int\limits_{(3/4)} X^{\frac{3}{2}-s}Y^s\mathcal{M}\varphi\left(\frac{3}{2}-s\right)\mathcal{M}\psi\left(s\right)\mathrm{res}_{\left(s,\frac{3}{2}-s\right)}A(s,w)ds,$$

where $\delta' = 1/4 + \varepsilon$, or ε under RH. Using Lemma 3.5.1 and (3.3.4), we have

$$\mathrm{res}_{\left(s, \frac{3}{2}-s\right)} A(s,w) = \frac{\mathrm{res}_{\left(s,\frac{3}{2}-s\right)} Z(s,w)}{\zeta_2(2)} = \frac{\sqrt{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{2\zeta_2(2)(2\pi)^s}.$$

Therefore the second integral on the right-hand side of (3.5.3) is

$$\frac{2X^{\frac{3}{2}}}{i\pi^{\frac{5}{2}}} \int_{(3/4)} \left(\frac{Y}{2\pi X}\right)^s \mathcal{M}\varphi\left(\frac{3}{2}-s\right)\mathcal{M}\psi(s)\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)ds. \tag{3.5.4}$$

Hence we have

$$\begin{aligned} S(X,Y;\varphi,\psi) &= \frac{\mathcal{M}\varphi(1)\mathcal{M}\psi\left(\frac{1}{2}\right)XY^{1/2} + \mathcal{M}\psi(1)\mathcal{M}\varphi\left(\frac{1}{2}\right)YX^{1/2}}{\pi^2} \\ &\quad + \frac{2X^{\frac{3}{2}}}{i\pi^{\frac{5}{2}}} \int_{(3/4)} \left(\frac{Y}{2\pi X}\right)^s \mathcal{M}\varphi\left(\frac{3}{2}-s\right)\mathcal{M}\psi(s)\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)ds \\ &\quad + \left(\frac{1}{2\pi i}\right)^2 \int_{(3/4)}\int_{(\delta')} A(s,w)X^wY^s\mathcal{M}\varphi(w)\mathcal{M}\psi(s)dwds + O\left(XY^\delta + YX^\delta\right) \\ &= \frac{2}{\pi^2}\cdot X^{3/2}\cdot D\left(\frac{Y}{X};\varphi,\psi\right) + O\left(XY^\delta + YX^\delta\right), \end{aligned} \tag{3.5.5}$$

where the last equality follows after trivially bounding the second integral, and using $X^{\delta'}Y^{\frac{3}{4}} \ll XY^\delta + YX^\delta$. □

3.6 Removing the smooth weights

In this section, we show how to remove the smooth weights from Theorem 3.1.1 and prove Theorem 3.1.2. We choose the weights $\varphi = \psi$ to be a smooth function which is 1 on the interval $\left[\frac{1}{U}, 1-\frac{1}{U}\right]$ for some U to be chosen later, and 0 outside of $(0,1)$, and which satisfies

$$|\mathcal{M}\varphi(\sigma+it)| \ll_{j,\sigma} \frac{U^{j-1}}{1+|t|^j} \tag{3.6.1}$$

for all $j \geq 1$. Then using the Pólya-Vinogradov inequality (see (3.1) in [CFS]), we have

$$|S(X,Y) - S(X,Y;\varphi,\psi)| \ll \frac{X^{3/2}+Y^{3/2}}{U}\log(XY). \tag{3.6.2}$$

Now we need to estimate the dependence on U of the error term in the computation of $S(X,Y;\varphi,\psi)$. These come from the following:

- The error from the polar lines $s = 1$ and $w = 1$;
- The error from the shifted integral;
- The difference of the main terms.

The error from the polar lines $s = 1$ is

$$\ll XY^{\sigma} \left| \int_{(\delta)} \frac{\mathcal{M}\psi(s)\zeta_2(2s)}{\zeta_2(2s+1)} ds \right|, \tag{3.6.3}$$

where $\delta = \varepsilon$, or $-1/4 + \varepsilon$, and similarly for the error from the polar line $w = 1$. We have $\frac{1}{\zeta_2(2\delta+2it+1)} \ll 1$ in both cases by (3.2.6).

We can shift the integral (3.6.3) to any vertical line $\mathrm{Re}(s) = \sigma$ with $1/4 \geq \sigma \geq \delta$, and bound the integral using the Cauchy-Schwarz inequality as

$$\begin{aligned} &\ll \int_{(\sigma)} |\mathcal{M}\psi(s)\zeta(2s)| ds \\ &\ll \left(\int_{(\sigma)} \frac{|\zeta(1-2s)|^2}{(1+|s|)^{1+\varepsilon}} ds \right)^{\frac{1}{2}} \left(\int_{(\sigma)} |\mathcal{M}\psi(s)\chi(2s)|^2 (1+|s|)^{1+\varepsilon} ds \right)^{\frac{1}{2}} \\ &\ll U^{j-1} \int_{(\sigma)} (1+|s|)^{2-4\sigma-2j+\varepsilon} ds \\ &\ll U^{1/2-2\sigma+\varepsilon}, \end{aligned} \tag{3.6.4}$$

where the first integral on the second line converges by (3.2.13), and we took $j = 3/2 - 2\sigma + \varepsilon$.

A similar computation for the polar line $w = 1$ gives the same result with X, Y interchanged, so the error from these terms is

$$(XY^{\sigma} + YX^{\sigma})U^{1/2-2\sigma+\varepsilon}. \tag{3.6.5}$$

For the error coming from the shifted integral, we need to estimate

$$\left| \int_{(\sigma)} \int_{(\omega)} A(s,w)\mathcal{M}\varphi(w)\mathcal{M}\psi(s) X^w Y^s dw ds \right|. \tag{3.6.6}$$

We have

$$A(s,w) = \frac{Z(s,w)}{\zeta_2(2s+2w-1)}, \tag{3.6.7}$$

so the integral is

$$\ll X^{\omega}Y^{\sigma}\int_{(\sigma)}\int_{(\omega)}|Z(s,w)\mathcal{M}\varphi(w)\mathcal{M}\psi(s)|\,dwds \tag{3.6.8}$$

provided $\sigma+\omega>1+\varepsilon$ or $>3/4+\varepsilon$ if we assume RH. We take $\sigma=\omega=1/2+\varepsilon$ and estimate the double integral using the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}
&\int_{(\frac{1}{2}+\varepsilon)}\int_{(\frac{1}{2}+\varepsilon)}|Z(s,w)\mathcal{M}\varphi(w)\mathcal{M}\psi(s)|\,dwds\\
&\ll\left(\int\limits_{(\frac{1}{2}+\varepsilon)}\int\limits_{(\frac{1}{2}+\varepsilon)}\frac{|Z(s,w)|^2}{(1+|s|)^{1+\varepsilon}(1+|w|)^{1+\varepsilon}}dwds\right)^{\frac{1}{2}}\times\\
&\times\left(\int\limits_{(\frac{1}{2}+\varepsilon)}\int\limits_{(\frac{1}{2}+\varepsilon)}|\mathcal{M}\varphi(w)\mathcal{M}\psi(s)|^2(1+|s|)^{1+\varepsilon}(1+|w|)^{1+\varepsilon}dwds\right)^{\frac{1}{2}} \qquad (3.6.9)\\
&\ll U^{j_1+j_2-2}\left(\int\limits_{(\frac{1}{2}+\varepsilon)}\int\limits_{(\frac{1}{2}+\varepsilon)}(1+|s|)^{1+\varepsilon-2j_1}(1+|w|)^{1+\varepsilon-2j_2}dwds\right)^{1/2}\\
&\ll U^{\varepsilon},
\end{aligned}$$

where the integral on the second line converges by (3.3.7), and we took $j_1=j_2=1+\varepsilon$. It follows that the error from the shifted integral is

$$\ll (XY)^{1/2+\varepsilon}U^{\varepsilon}. \tag{3.6.10}$$

The error from the difference of the main terms is

$$\begin{aligned}
E(X,Y;\varphi,\psi)&=\frac{2}{\pi^2}X^{3/2}\left(D\left(\frac{Y}{X};\varphi,\psi\right)-D\left(\frac{Y}{X}\right)\right)\\
&\ll XY^{1/2}\left|\mathcal{M}\varphi(1)\mathcal{M}\psi(1/2)-2\right|+YX^{1/2}\left|\mathcal{M}\psi(1)\mathcal{M}\varphi(1/2)-2\right|\\
&+X^{3/2}\int_{(3/4)}\left|\left(\frac{Y}{2\pi X}\right)^{s}\Gamma\left(s-\frac{1}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)\right|\times\\
&\qquad\times\left|\mathcal{M}\varphi(3/2-s)\mathcal{M}\psi(s)-\frac{1}{s(3/2-s)}\right|ds\\
&\ll XY^{1/2}\left|\mathcal{M}\varphi(1)\mathcal{M}\psi(1/2)-2\right|+YX^{1/2}\left|\mathcal{M}\psi(1)\mathcal{M}\varphi(1/2)-2\right|\\
&+(XY)^{3/4}\int_{(3/4)}(1+|t|)^{-\frac{1}{4}}\left|\zeta(2s-1)\right|\left|\mathcal{M}\varphi\left(\frac{3}{2}-s\right)\mathcal{M}\psi(s)-\frac{1}{s\left(\frac{3}{2}-s\right)}\right|ds.
\end{aligned} \tag{3.6.11}$$

We estimate the Mellin transforms in the following lemma:

Lemma 3.6.1. *For $s=\sigma+it$, $0<\sigma<1$ and φ, ψ as above, we have*

$$\mathcal{M}\varphi(1)\mathcal{M}\psi(1/2)=2+O\left(U^{-1/2}\right), \tag{3.6.12}$$

and

$$\left|\mathcal{M}\varphi(3/2-s)\mathcal{M}\psi(s)-\frac{1}{s(3/2-s)}\right| \ll \frac{|\mathcal{M}\varphi(3/2-s)|}{U^{\sigma}}+\frac{1}{|s|U^{3/2-\sigma}}. \tag{3.6.13}$$

Proof. For (3.6.12), we have

$$\begin{aligned}
&\mathcal{M}\varphi(1)\mathcal{M}\psi(1/2)=\int_0^\infty \varphi(u)du\int_0^\infty \psi(v)v^{-1/2}dv\\
&=\left(\left(\int\limits_0^{\frac{1}{U}}+\int\limits_{1-\frac{1}{U}}^{1}\right)\varphi(u)du+1-\frac{2}{U}\right)\left(\left(\int\limits_0^{\frac{1}{U}}+\int\limits_{1-\frac{1}{U}}^{1}\right)\frac{\psi(v)}{\sqrt{v}}dv+\int\limits_{\frac{1}{U}}^{1-\frac{1}{U}}\frac{1}{\sqrt{v}}dv\right)\\
&=\left(1+O\left(\frac{1}{U}\right)\right)\left(O\left(\frac{1}{\sqrt{U}}\right)+2\left(\sqrt{1-\frac{1}{U}}-\frac{1}{\sqrt{U}}\right)\right)=\\
&=2+O\left(U^{-1/2}\right).
\end{aligned} \tag{3.6.14}$$

For (3.6.13), we first use the triangle inequality to get

$$\begin{aligned}
&\left|\mathcal{M}\varphi(3/2-s)\mathcal{M}\psi(s)-\frac{1}{s(3/2-s)}\right|\\
&\le |\mathcal{M}\varphi(3/2-s)|\left|\mathcal{M}\psi(s)-\frac{1}{s}\right|+\left|\frac{1}{s}\right|\left|\mathcal{M}\varphi(3/2-s)-\frac{1}{3/2-s}\right|.
\end{aligned} \tag{3.6.15}$$

Now we have

$$\begin{aligned}
\left|\mathcal{M}\psi(s)-\frac{1}{s}\right| &\le \int_0^\infty |\psi(u)-1|\,u^{\sigma-1}du\\
&=\int_0^{\frac{1}{U}} |\psi(u)-1|\,u^{\sigma-1}du+\int_{1-\frac{1}{U}}^{1} |\psi(u)-1|\,u^{\sigma-1}du\\
&\ll \frac{1}{U^{\sigma}},
\end{aligned} \tag{3.6.16}$$

and similarly

$$\left|\mathcal{M}\varphi(3/2-s)-\frac{1}{3/2-s}\right| \le \int_0^\infty |\varphi(u)-1|\,u^{1/2-\sigma}du \ll \frac{1}{U^{3/2-\sigma}}. \tag{3.6.17}$$

□

Using this Lemma in (3.6.11), we get

$$E(X,Y;\varphi,\psi) \ll \frac{XY^{1/2}+YX^{1/2}}{\sqrt{U}} + \left(\frac{XY}{U}\right)^{3/4}. \tag{3.6.18}$$

Putting everything together, the error in both cases is

$$\frac{X^{\frac{3}{2}+\varepsilon}+Y^{\frac{3}{2}+\varepsilon}}{U} + \frac{XY^{\frac{1}{2}}+YX^{\frac{1}{2}}}{\sqrt{U}} + \left(\frac{XY}{U}\right)^{\frac{3}{4}} + (XY^{\sigma}+YX^{\sigma})U^{\frac{1}{2}-2\sigma+\varepsilon} + (XY)^{\frac{1}{2}}U^{\varepsilon}, \tag{3.6.19}$$

where $1/4 \geq \sigma \geq \delta$. In both cases, the best choice is $\sigma = 1/4$. Then we can take $U = \min\{X,Y\}$ and obtain the error

$$XY^{1/4+\varepsilon} + YX^{1/4+\varepsilon} \tag{3.6.20}$$

in the range $Y^{2/5} \leq X \leq Y^{5/2}$. In the remaining range, the result follows from (3.1.2) and the asymptotic expansion of $C(\alpha)$ (3.1.5) and (3.1.6), together with the fact that $C(\alpha) = D(\alpha)$ as will be proved in the next section.

3.7 Proving that $C(\alpha) = D(\alpha)$.

In this section, we show that $C(\alpha) = D(\alpha)$. Recall that

$$D(\alpha) = \sqrt{\alpha} + \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(3/4)} \left(\frac{\alpha}{2\pi}\right)^{s} \cdot \frac{\Gamma\left(s-\frac{3}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)}{s} ds. \tag{3.7.1}$$

We shift the integral to the left, capturing the pole at $s = \frac{1}{2}$, which contributes

$$2\sqrt{\pi}\left(\frac{\alpha}{2\pi}\right)^{1/2} \cdot \frac{-\sin(\pi/4)\zeta(0)}{1/2} = \alpha^{1/2}.$$

The horizontal integrals vanish by (3.2.9) and a convexity estimate for $\zeta(s)$, so

$$\begin{aligned} D(\alpha) &= \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(-1/4)} \left(\frac{\alpha}{2\pi}\right)^{s} \cdot \frac{\Gamma\left(s-\frac{3}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(2s-1)}{s} ds \\ &= \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(1/4)} \alpha^{-s} \cdot \frac{(2\pi)^{s}\Gamma\left(-s-\frac{3}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(-2s-1)}{s} ds. \end{aligned} \tag{3.7.2}$$

This integral is an inverse Mellin transform, so we rewrite $C(\alpha)$ using Mellin inversion. We have

$$\begin{aligned} C(\alpha) &= \alpha + \alpha^{3/2}\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1/\alpha}\sqrt{y}\sin\left(\frac{\pi k^2}{2y}\right)dy \\ &= \alpha + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1}\sqrt{u}\sin\left(\frac{\pi k^2\alpha}{2u}\right)du \\ &= \alpha + \frac{2}{\pi}\cdot\frac{1}{2\pi i}\int_{(c)}\alpha^{-s}\mathcal{M}f(s)ds, \end{aligned} \tag{3.7.3}$$

where

$$f(x) = \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1}\sqrt{u}\sin\left(\frac{\pi k^2 x}{2u}\right)du. \tag{3.7.4}$$

For $0 < \mathrm{Re}(s) < 1$, we have

$$\begin{aligned} \mathcal{M}f(s) &= \int_0^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1}\sqrt{u}\sin\left(\frac{\pi k^2 x}{2u}\right)du\; x^{s-1}dx \\ &= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1}\sqrt{u}\int_0^{\infty}\sin\left(\frac{\pi k^2 x}{2u}\right)x^{s-1}dx\; du, \end{aligned} \tag{3.7.5}$$

which isn't obvious as the double integral doesn't converge absolutely, but we will justify the interchange of summation and integrals in Lemma 3.7.1. We can now make a change of variables $y = \frac{\pi k^2 x}{2u}$ and obtain

$$\begin{aligned} \mathcal{M}f(s) &= \sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{1}\sqrt{u}\int_0^{\infty}\sin(y)y^{s-1}dy\;\left(\frac{2u}{\pi k^2}\right)^{s}du \\ &= \left(\frac{2}{\pi}\right)^{s}\zeta(2+2s)\int_0^{1}u^{s+1/2}du\int_0^{\infty}\sin(y)y^{s-1}dy \\ &= \frac{2^s\zeta(2+2s)\Gamma(s)\sin\left(\frac{\pi s}{2}\right)}{\pi^s(s+3/2)}, \end{aligned} \tag{3.7.6}$$

which holds for $0 < \mathrm{Re}(s) < 1$, so we can take $c = 1/4$ in (3.7.3). It therefore suffices to show that

$$-2\sqrt{\pi}\frac{(2\pi)^s\Gamma\left(-s-\frac{3}{2}\right)\sin\left(\frac{\pi s}{2}\right)\zeta(-2s-1)}{s} = \frac{2}{\pi}\cdot\frac{2^s\zeta(2+2s)\Gamma(s)\sin\left(\frac{\pi s}{2}\right)}{\pi^s(s+3/2)}. \tag{3.7.7}$$

The functional equation for the zeta function gives

$$\zeta(2s+2) = \pi^{2s+3/2}\cdot\frac{\Gamma\left(-s-\frac{1}{2}\right)}{\Gamma(s+1)}\zeta(-2s-1), \tag{3.7.8}$$

so using $s\Gamma(s) = \Gamma(s+1)$ and $(-s-3/2)\Gamma(-s-3/2) = \Gamma(-s-1/2)$ gives the result.

It remains to justify the interchange of the order of summation and integrations in (3.7.5):

Lemma 3.7.1. *If* $0 < \mathrm{Re}(s) < 1$, *it holds that*

$$\begin{aligned}&\int_0^\infty \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin\left(\frac{\pi k^2 x}{2u}\right) du\ x^{s-1} dx \\ &= \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \int_0^\infty \sin\left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx\ du.\end{aligned} \tag{3.7.9}$$

Proof. We have

$$\begin{aligned}&\int_0^\infty \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin\left(\frac{\pi k^2 x}{2u}\right) du\ x^{s-1} dx \\ &= \lim_{A\to\infty} \int_{1/A}^A \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin\left(\frac{\pi k^2 x}{2u}\right) du\ x^{s-1} dx.\end{aligned} \tag{3.7.10}$$

We can now interchange the integrals and summation, because

$$\sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \int_{1/A}^A \left| \sqrt{u} \sin\left(\frac{\pi k^2 x}{2u}\right) x^{s-1} \right| dx\ du \ll A, \tag{3.7.11}$$

so

$$\begin{aligned}&\lim_{A\to\infty} \int_{1/A}^A \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin\left(\frac{\pi k^2 x}{2u}\right) du\ x^{s-1} dx \\ &= \lim_{A\to\infty} \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \int_{1/A}^A \sin\left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx\ du.\end{aligned} \tag{3.7.12}$$

To insert the limit inside the sum and integral, we use the Dominated convergence theorem with the bound $\sqrt{u} |\int_{1/A}^A \sin\left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx| \leq K\sqrt{u}$ for an absolute constant K independent of u and A in our range, which holds because $\int_0^\infty \sin(y) y^{s-1} dy$ converges. □

Chapter 4

The Ratios conjecture for real Dirichlet characters and multiple Dirichlet series

4.1 Introduction

In [Mon], Montgomery studied the pair correlation of zeros of the Riemann zeta function, and Dyson famously observed that the resulting density matches that of the pair correlation of eigenvalues in a GUE ensemble of random matrices. Since then, it is believed that random matrix theory can be used to model many statistics of L-functions, such as moments [CFKRS], or the distribution of low-lying zeros [KaSa1], [KaSa2].

Farmer [Far] conjectured that

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}dt \sim T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}, \quad (4.1.1)$$

where $s = 1/2 + it$, and $\alpha, \beta, \gamma, \delta \asymp \frac{1}{\log T}$, and noticed that it has many implications including Montgomery's pair correlation conjecture.

In [CFZ2], [CFZ1], based on the conjecture above, Conrey, Farmer and Zirnbauer came up with the ratios conjectures, which give a general recipe to predict asymptotic formulas for the sum of ratios of products of shifted L-functions. These are very powerful as they are able to predict many other local or global statistics, which agree with the predictions coming from random matrix theory (see [CoSn] for some applications). The conjectured asymptotics are expected to hold with a very strong error term, so, unlike random matrix theory, they are also able to predict lower order terms.

We will study the family of quadratic Dirichlet L-functions with one shift in the numerator and in the denominator. For a fundamental discriminant d, we denote by χ_d the Kronecker symbol $\left(\frac{d}{\cdot}\right)$. In this case, the ratios conjecture has the form

$$\begin{aligned} \sideset{}{^*}\sum_{d\le X} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\beta,\chi_d)} = \sideset{}{^*}\sum_{d\le X} \Bigg(& \frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\beta)} A_D(\alpha,\beta) \\ & + \left(\frac{\pi}{d}\right)^{\alpha} \frac{\Gamma(1/4-\alpha/2)\zeta(1-2\alpha)}{\Gamma(1/4+\alpha/2)\zeta(1-\alpha+\beta)} A_D(-\alpha,\beta) \Bigg) + O(X^{1/2+\varepsilon}), \end{aligned} \tag{4.1.2}$$

where the star indicates that the sums run over fundamental discriminants, and

$$A_D(\alpha,\beta) = \prod_p \left(1-\frac{1}{p^{1+\alpha+\beta}}\right)^{-1} \left(1-\frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\beta}}\right). \tag{4.1.3}$$

The originally conjecture asserts that this formula holds uniformly in α,β with $|\mathrm{Re}(\alpha)| < 1/4$, $\frac{1}{\log X} \ll \mathrm{Re}(\beta) < 1/4$, and $\mathrm{Im}(\alpha), \mathrm{Im}(\beta) \ll X^{1-\varepsilon}$. We, conditionally under the generalized Riemann hypothesis (GRH), prove the conjecture in a smaller range of the shifts.

We are not aware of any other results towards the proof of this conjecture over number fields. See the recent work of Bui, Florea and Keating [BFK2] for similar results over function fields, using a different method.

We now briefly describe the recipe of Conrey, Farmer and Zirnbauer. The L-functions in the numerator are replaced by Dirichlet polynomials using the approximate functional equation, while those in the denominator are expanded into a full Dirichlet series. To obtain the main terms in the conjecture, the sums are completed, only the diagonal terms are retained, and they are replaced by their average over the family.

Our strategy is to use Mellin inversion (or Perron's formula), which leads to an integral that involves a triple Dirichlet series $A(s,w,z)$. The main terms arise naturally from residues of $A(s,w,z)$, which gives another evidence for the validity of the heuristics of Conrey, Farmer and Zirnbauer. An advantage of our approach is that there are no non-diagonal terms to be bounded. The error term depends on how far we can meromorphically continue the triple Dirichlet series, which we obtain from showing that $A(s,w,z)$ satisfies some functional equations.

Let us now state our results, which are conditional under GRH. Theorem 4.1.1 is weaker than Theorem 4.1.2, its purpose is mainly to illustrate the method of the proof and to illustrate how the main terms arise from the residues of the multiple Dirichlet series. The difference in the results can be explained on the level of functional equations satisfied by the associated multiple Dirichlet series, which is explained in Section 4.2.1.

Let

$$R_D(X,\alpha,\beta;f) = \sideset{}{^*}\sum_{d\geq 1} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\beta,\chi_d)} f(d/X), \tag{4.1.4}$$

where the sum runs over positive fundamental discriminants. We also let $f(x)$ be a smooth, fast-decaying weight function.

To simplify some of the formulas, we denote by $\Gamma_e(s)$ and $\Gamma_o(s)$ the ratios of gamma factors that appear in the functional equation for even or odd characters, so that

$$\Gamma_e(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \qquad \Gamma_o(s) = \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}. \tag{4.1.5}$$

Theorem 4.1.1. *Assume GRH, and let* $0 < |\mathrm{Re}(\alpha)| < \mathrm{Re}(\beta) < 1/2$. *Then*

$$\begin{aligned} R_D(X,\alpha,\beta;f) &= \frac{X\mathcal{M}f(1)\zeta(1+2\alpha)}{2\zeta(2)\zeta(1+\alpha+\beta)} P_D(1/2+\alpha,1/2+\beta) \\ &\quad + \frac{X^{1-\alpha}\mathcal{M}f(1-\alpha)\zeta(1-2\alpha)\pi^{\alpha}\Gamma_e(1/2+\alpha)}{2\zeta(2)\zeta(1-\alpha+\beta)} P_D(1/2-\alpha,1/2+\beta) \\ &\quad + O_{\alpha,\beta}(X^{1-\mathrm{Re}(\alpha)/2-\mathrm{Re}(\beta)/2+\varepsilon}), \end{aligned} \tag{4.1.6}$$

where

$$P_D(z,w) = \prod_p \left(1 + \frac{1-p^{z-w}}{(p^{z+w}-1)(p+1)}\right). \tag{4.1.7}$$

The error term in this result is not uniform in the shifts α, β. It is possible to obtain a uniform result using a similar estimate as in Section 4.5.5, which would introduce a factor of size $|\alpha|^{c+\varepsilon}|\beta|^{\varepsilon}$ for some $c > 0$ into the error term, limiting the size of the imaginary parts. Note also that this theorem doesn't hold in the range $\alpha = \beta$, which is important for some applications. Both of these aspects can be improved by making the sum run over all quadratic characters, not only the primitive ones, as shown in Theorem 4.1.2.

The main terms from (4.1.6) agree with those in (4.1.2) (adjusted for the smooth weights). We remark that the computations leading to them are the same as those in the recipe of Conrey, Farmer and Zirnbauer, but they come from a different source. While in the heuristic, one obtains the main terms by discarding the non-diagonal terms assuming that they only contribute into the error, in our case they come from the residues of certain triple Dirichlet series.

It turns out that we can get a better range of the shifts if we extend our family to contain all characters, including non-primitive ones. The reason is that in this case, the associated triple Dirichlet series has an extra functional equation. We will change notation from Section 4.5 on and denote by χ_n the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. We denote by $L_{(2)}(s, \chi_n)$ the L-function with the Euler factor at 2 removed. Then we get the following result:

Theorem 4.1.2. *Assume GRH and let $\varepsilon > 0$. Then for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > \varepsilon$, $1 + \mathrm{Re}(\beta) > \mathrm{Re}(\alpha)$, we have*

$$\begin{aligned}
&\sum_{n \text{ odd}} \frac{L_{(2)}(1/2+\alpha, \chi_n)}{L_{(2)}(1/2+\beta, \chi_n)} f(n/X) \\
&= X\mathcal{M}f(1)\frac{\zeta_{(2)}(1+2\alpha)}{2\zeta_{(2)}(1+\alpha+\beta)}\prod_{p>2}\left(1+\frac{p^{\alpha-\beta}-1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta}-1)}\right) \\
&+ X^{1-\alpha}\mathcal{M}f(1-\alpha)\pi^{\alpha}\left(\Gamma_o\left(\frac{1}{2}+\alpha\right)+\Gamma_e\left(\frac{1}{2}+\alpha\right)\right)\frac{\zeta(1-2\alpha)P\left(\frac{3}{2}-\alpha+\beta\right)}{\zeta(2)\zeta(1-\alpha+\beta)(6-2^{\alpha-\beta+1})} \\
&+ O\left((1+|\alpha|)^{\varepsilon}|\beta|^{\varepsilon}X^{N(\alpha,\beta)+\varepsilon}\right),
\end{aligned} \tag{4.1.8}$$

where

$$P(z) = \prod_p \left(1 + \frac{1}{(p^{z-1/2}-1)(p+1)}\right), \tag{4.1.9}$$

and

$$N(\alpha, \beta) = \max\left\{1 - 2\mathrm{Re}(\alpha), 1 - 2\mathrm{Re}(\beta), \frac{1}{2} - \mathrm{Re}(\alpha), \frac{1}{2} - \mathrm{Re}(\beta), -\frac{5}{2}\right\}. \tag{4.1.10}$$

The condition $1+\mathrm{Re}(\beta) > \mathrm{Re}(\alpha)$ is to ensure convergence of the product $P(3/2-\alpha+\beta)$, and the $-5/2$ in the error term comes from our definition of $\tilde{S}_j$ in Section 4.5.5.

In this case, the error term is uniform in α, β. Let us emphasize that the imaginary parts to grow as fast as any power of X, which is better than the original conjecture.

This result also allows us to differentiate with respect to α and take $\alpha = \beta = r$, thus obtain

Theorem 4.1.3. *Assume GRH and Let* $\mathrm{Re}(r) > \varepsilon$. *Then*

$$\begin{aligned}\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} f(n/X) &= \frac{X\mathcal{M}f(1)}{2}\left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2}\frac{\log p}{p(p^{1+2r}-1)}\right)\\ &- X^{1-r}\mathcal{M}f(1-r)\pi^r\left(\Gamma_o\left(\frac{1}{2}+r\right)+\Gamma_e\left(\frac{1}{2}+r\right)\right)\frac{\zeta(1-2r)}{4}\\ &+ O\left(1+|r|^{\varepsilon}X^{N(r)+\varepsilon}\right),\end{aligned} \tag{4.1.11}$$

where

$$N(r) = \max\{1-2\mathrm{Re}(r), 1/2-\mathrm{Re}(r), -5/2\}. \tag{4.1.12}$$

At this point, we can sieve out the non-primitive characters and obtain an asymptotic formula for the sum running over square-free integers:

Theorem 4.1.4. *Assume GRH and let* $\varepsilon < \mathrm{Re}(r) < 1/4$. *Then*

$$\begin{aligned}&\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{\mu^2(n)L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} f(n/X)\\ &= \frac{2X\mathcal{M}f(1)}{3\zeta(2)}\left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2}\frac{\log p}{(p+1)(p^{1+2r}-1)}\right)\\ &- X^{1-r}\mathcal{M}f(1-r)\pi^r\left(\Gamma_o\left(\frac{1}{2}+r\right)+\Gamma_e\left(\frac{1}{2}+r\right)\right)\frac{\zeta(1-2r)}{4\zeta_{(2)}(2-2r)}\\ &+ O\left(|r|^{\varepsilon}X^{1-2\mathrm{Re}(r)+\varepsilon}\right).\end{aligned} \tag{4.1.13}$$

As opposed to Theorems 4.1.2 and 4.1.3, but similarly as Theorem 4.1.1, we now only have primitive characters, so we can compare the result with the prediction. This is done in Section 4.11, where we show that the main terms in Theorem 4.1.4 agree with those coming from the recipe. However, in this case, the computations leading to Theorem 4.1.4 are different from those in the heuristic.

As an application of our results, we compute the one-level density in our family of quadratic Dirichlet characters. For an even Schwartz function $h(x)$ whose Fourier transform $\hat{h}$ is supported in $[-a, a]$ for some $a > 0$, the one-level density is defined by

$$D(X; h) = \frac{1}{F(X)} \sum_{\substack{n\geq 1\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right) \sum_{\gamma_n} h\left(\frac{\gamma_n \log X}{2\pi}\right), \tag{4.1.14}$$

where γ_n runs over the imaginary parts of the non-trivial zeros of $L(s, \chi_n)$, and

$$F(X) = \sum_{\substack{n \geq 1, \\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right). \tag{4.1.15}$$

By the conjectures of Katz and Sarnak [KaSa1], [KaSa2],

$$D(X; h) \sim \int_{-\infty}^{\infty} h(u) \left(1 - \frac{\sin(2\pi u)}{2\pi u}\right) du, \tag{4.1.16}$$

where the kernel of the integral governs the distribution of eigenvalues close to 1 in a symplectic ensemble of random matrices. This has been proved for $a < 1$ and for $a < 2$ under GRH by Özlük and Snyder [ÖzSn], and increasing the support further is a notoriously difficult problem. The ratios conjecture (4.1.2) implies that the above asymptotic holds for arbitrarily large a, and it allows us to compute all lower order terms up to square-root error. In [Mil1], Miller showed unconditionally that for limited a, the ratios conjecture prediction agrees with reality, including lower order terms. See also [Mil2], [MiMo], [GJM+], [HMM] for similar results in other families, [BFK1] for related work over function fields, or [DHJ], where the ratios conjecture is applied to compute the one-level density in two families of elliptic curves.

On the other hand, Fiorilli and Miller [FiMi] computed the one-level density in the family of all Dirichlet characters modulo q, including lower order terms beyond square-root. They discovered a term not predicted by the ratios conjecture, concluding that the conjectured error is essentially best possible.

As a consequence of Theorem 4.1.4, we compute the one-level density provided $a < 2$, thus recovering the results of Özlük and Snyder, including lower order terms with a power-saving error.

Corollary 4.1.5. *Assume GRH and let h be as above. Then*

$$\begin{aligned}
&\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right) \sum_{\gamma_n} h\left(\frac{\gamma_n \log X}{2\pi}\right) \\
&= \frac{2X}{\log X} \int_{-\infty}^{\infty} h(u) \Bigg\{ \frac{2\mathcal{M}f(1)}{3\zeta(2)} \left(\frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i u}{\log X}\right) + \sum_{p>2} \frac{\log p}{(p+1)(p^{1+4\pi i u/\log X} - 1)} \right) \\
&- \frac{e^{-2\pi i u} \mathcal{M}f\left(1 - \frac{2\pi i u}{\log X}\right) \pi^{\frac{2\pi i u}{\log X}} \left(\Gamma_o\left(\frac{1}{2} + \frac{2\pi i u}{\log X}\right) + \Gamma_e\left(\frac{1}{2} + \frac{2\pi i u}{\log X}\right)\right) \zeta\left(1 - \frac{4\pi i u}{\log X}\right)}{4\zeta_{(2)}\left(2 - \frac{4\pi i u}{\log X}\right)} \Bigg\} du \\
&- \frac{1}{\log X} \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right) \log\left(\frac{\pi}{n}\right) \int_{-\infty}^{\infty} h(u) du \\
&- \frac{X\mathcal{M}f(1)}{3\zeta(2)\log X} \int_{-\infty}^{\infty} h(u) \left(\frac{\Gamma_o'}{\Gamma_o}\left(\frac{1}{2} + \frac{2\pi i u}{\log X}\right) + \frac{\Gamma_e'}{\Gamma_e}\left(\frac{1}{2} + \frac{2\pi i u}{\log X}\right) \right) du \\
&+ O\left(X^{1/2+a/4+\varepsilon}\right).
\end{aligned} \tag{4.1.17}$$

This implies that if $a < 2$,

$$D(X; h) = \int_{-\infty}^{\infty} h(u) \left(1 - \frac{\sin(2\pi u)}{2\pi u}\right) du + O\left(\frac{1}{\log X}\right). \tag{4.1.18}$$

The proof is given in Section 4.9, where we also explain that any improvement of the error term in Theorem 4.1.4, which is directly related to a meromorphic continuation of certain triple Dirichlet series, would allow us to increase the support.

Our strategy to prove Theorems 4.1.1 and 4.1.2 is to rewrite the sums as integrals using Mellin inversion (a smooth version of Perron's formula), and then investigate the analytic properties of the relevant triple Dirichlet series. We show that they have two poles, whose residues give rise to the two main terms. The error then depends on how far our triple Dirichlet series can be meromorphically extended, which in turn depends on whether they satisfy certain functional equations.

A general theory of multiple Dirichlet series has been developed by Bump, Chinta, Diaconu, Friedberg, Hoffstein and others (see for example [BFG], [Blo], [DGH], [GoHo], or the expository papers [Bum], [BFH1], [CFH] for an introduction to the theory). In [DGH], the authors prove that if certain multiple Dirichlet series admit a meromorphic continuation beyond a certain point, then the moments of the Riemann zeta function and quadratic Dirichlet L-functions satisfy the asymptotics predicted by random matrix theory.

According to the heuristics developed in [BFH1] and [CFH] that we present in Section 4.2.1, our triple Dirichlet series are expected to satisfy two functional equations. To prove these rigorously, one usually twists the L-functions in the coefficients of the triple Dirichlet series by certain weights, carefully chosen so that the equations are satisfied. For more details, see [CFH] for an exposition of the general theory, or Blomer's work [Blo], where the situation is "simple" enough, so the author gives very explicit results and computations.

We use a different method in this chapter. In the case of fundamental discriminants, we only have one functional equation, which is essentially due to the fact that

$$L_D(s,\chi) = \sum_{d\geq 1}^{*} \frac{\chi(d)}{d^s} \tag{4.1.19}$$

doesn't satisfy any relation between $L_D(s,\chi)$ and $L_D(1-s,\chi)$. This explains why we obtain a weaker result in this case.

When we include the non-primitive characters, we are summing over a nicer set, so both functional equations potentially hold. However, we have to deal with the L-functions of non-primitive characters – this is usually done by inserting the extra weights mentioned above. Instead, we introduce a functional equation that is valid for all Dirichlet characters, but where on the other side, one has a different Dirichlet series whose coefficients are twisted Gauss sums (see Proposition 4.2.3 and Section 4.10). It then becomes straightforward to prove the functional equations for our triple Dirichlet series, but with a different triple Dirichlet series on the other side.

4.2 Preliminaries

Throughout the chapter, ε will denote a small positive number, that may be different at each appearance. All implied constants can depend on ε.

For an odd positive integer n, χ_n denotes the quadratic Dirichlet character given by the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. The character χ_n is primitive for square-free n, and it is even for $n \equiv 1 \pmod 4$ and odd for $n \equiv 3 \pmod 4$. We will also work with the Kronecker symbol $\left(\frac{k}{\cdot}\right)$, which is periodic only if $k \equiv 0, 1 \pmod 4$. However, we often use the fact that if n is odd, then $\left(\frac{k}{n}\right) = \left(\frac{4k}{n}\right)$, and $\left(\frac{4k}{\cdot}\right)$ is a Dirichlet character modulo $4k$ for any odd k. We also denote by ψ_1, ψ_{-1} the principal and non-principal character modulo 4, by ψ_2, ψ_{-2} the quadratic characters modulo 8 given by the Kronecker symbols $\psi_j(n) = \left(\frac{4j}{n}\right)$, and by ψ_0 the primitive principal character, so that $\psi_0(n) = 1$ for all $n \in \mathbb{Z}$.

In sections 4.2.1, 4.3 and 4.4, χ_d denotes the Kronecker symbol $\left(\frac{d}{\cdot}\right)$ for a fundamental discriminant d.

For a Dirichlet character χ modulo n, we define the shifted Gauss sums

$$\tau(\chi, q) = \sum_{j \pmod n} \chi(j) e(jq/n), \tag{4.2.1}$$

where we use the standard notation $e(x) = e^{2\pi i x}$.

If χ is a primitive character, then

$$\tau(\chi, q) = \bar{\chi}(q)\tau(\chi, 1). \tag{4.2.2}$$

Lemma 4.2.1. *Let χ_1, χ_2 be two Dirichlet characters modulo n_1 and n_2, respectively, and assume that $(n_1, n_2) = 1$. Then for $\chi_1\chi_2$ considered as a Dirichlet character modulo $n_1 n_2$, we have*

$$\tau(\chi_1\chi_2, q) = \chi_1(n_2)\chi_2(n_1)\tau(\chi_1, q)\tau(\chi_2, q). \tag{4.2.3}$$

Proof. By the Chinese remainder theorem, we can write any $j \in \mathbb{Z}/(n_1 n_2 Z)$ uniquely as

$$j = j_1 n_2 \overline{n_2} + j_2 n_1 \overline{n_1}, \tag{4.2.4}$$

where $n_1\overline{n_1} \equiv 1 \pmod{n_2}$, $n_2\overline{n_2} \equiv 1 \pmod{n_1}$, $j_1 \in \mathbb{Z}/n_1\mathbb{Z}$, and $j_2 \in \mathbb{Z}/n_2\mathbb{Z}$. Hence, we have

$$\begin{aligned} \tau(\chi_1\chi_2, q) &= \sum_{j \pmod{n_1 n_2}} \chi_1(j)\chi_2(j) e\left(\frac{qj}{n_1 n_2}\right) \\ &= \sum_{j_1 \pmod{n_1}} \sum_{j_2 \pmod{n_2}} \chi_1(j_1)\chi_2(j_2) e\left(\frac{q j_1 \overline{n_2}}{n_1}\right) e\left(\frac{q j_2 \overline{n_1}}{n_2}\right) \\ &= \chi_1(n_2)\chi_2(n_1)\tau(\chi_1, q)\tau(\chi_2, q). \end{aligned} \tag{4.2.5}$$

□

Lemma 4.2.2. *1. If $\ell \equiv 1$ (mod 4), then*

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \begin{cases} 0, & \text{if } q \text{ is odd,} \\ -2\tau\left(\left(\frac{\cdot}{\ell}\right), q\right), & \text{if } q \equiv 2 \pmod 4, \\ 2\tau\left(\left(\frac{\cdot}{\ell}\right), q\right), & \text{if } q \equiv 0 \pmod 4. \end{cases} \tag{4.2.6}$$

2. If $\ell \equiv 3 \pmod 4$, then

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \begin{cases} 0, & \text{if } q \text{ is even,} \\ -2i\tau\left(\left(\frac{\cdot}{\ell}\right), q\right), & \text{if } q \equiv 1 \pmod 4, \\ 2i\tau\left(\left(\frac{\cdot}{\ell}\right), q\right), & \text{if } q \equiv 3 \pmod 4. \end{cases} \tag{4.2.7}$$

Proof. Quadratic reciprocity gives for $\ell \equiv 1 \pmod 4$

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \tau\left(\left(\frac{\cdot}{\ell}\right)\psi_1, q\right), \tag{4.2.8}$$

and for $\ell \equiv 3 \pmod 4$

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \tau\left(\left(\frac{\cdot}{\ell}\right)\psi_{-1}, q\right). \tag{4.2.9}$$

The result then follows from Lemma 4.2.1. □

For quadratic characters, we also define $G(\chi_n, q)$ by

$$\begin{aligned} G(\chi_n, q) &= \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right)\tau(\chi_n, q) = \\ &= \begin{cases} \tau(\chi_n, q), & \text{if } n \equiv 1 \pmod 4, \\ -i\tau(\chi_n, q), & \text{if } n \equiv 3 \pmod 4, \end{cases} \end{aligned} \tag{4.2.10}$$

whose advantage is that they are multiplicative in n, so that for $(m, n) = 1$, we have

$$G(\chi_{mn}, q) = G(\chi_m, q)G(\chi_n, q). \tag{4.2.11}$$

Moreover, if p is an odd prime and $p^a \,||\, q$, we have by [Sou, Lemma 2.3]

$$G\left(\left(\frac{\cdot}{p^k}\right), q\right) = \begin{cases} \varphi(p^k), & \text{if } k \leq a,\ k \text{ even}, \\ 0, & \text{if } k \leq a,\ k \text{ odd}, \\ -p^a, & \text{if } k = a+1,\ k \text{ even}, \\ \left(\frac{qp^{-a}}{p}\right)p^a\sqrt{p}, & \text{if } k = a+1,\ k \text{ odd}, \\ 0, & \text{if } k \geq a+2, \end{cases} \tag{4.2.12}$$

For primitive Dirichlet characters of conductor d, we have

$$L(s, \chi) = \varepsilon(\chi)\left(\frac{\pi}{d}\right)^{s-1/2}\Gamma_{e/o}(s)L(1-s, \bar{\chi}), \tag{4.2.13}$$

where $\varepsilon(\chi) = \frac{a\tau(\chi,1)}{\sqrt{d}}$, and $a = 1$ and $\Gamma_{e/o} = \Gamma_e$ if χ is even, or $a = -i$ and $\Gamma_{e/o} = \Gamma_o$ if χ is odd.

We now introduce a functional equation valid for all Dirichlet characters χ modulo n.

Proposition 4.2.3. *Let χ be any character modulo n. Then we have*

$$L(s,\chi) = \varepsilon(\chi)\frac{\pi^{s-1/2}}{n^s}\Gamma_{e/o}(s)K(1-s,\chi), \tag{4.2.14}$$

where

$$K(s,\chi) = \sum_{q=1}^{\infty}\frac{\tau(\chi,q)}{q^s}, \tag{4.2.15}$$

$$\Gamma_{e/o}(s) = \begin{cases} \Gamma_e(s), & \text{if } \chi \text{ is even, or} \\ \Gamma_o(s), & \text{if } \chi \text{ is odd,} \end{cases} \tag{4.2.16}$$

and

$$\varepsilon(\chi) = \begin{cases} 1, & \text{if } \chi \text{ is even, or} \\ -i, & \text{if } \chi \text{ is odd.} \end{cases} \tag{4.2.17}$$

Note that if χ is a primitive character, then we can use (4.2.2) and recover (4.2.13) from (4.2.14).

Proof. Follow one of the usual proofs of the functional equation that uses Poisson summation. The application of the Poisson summation leads to some Gauss sums, and for a primitive character χ, one uses (4.2.2) to change the Gauss sums back to characters. Skipping this last step and leaving the Gauss sums unchanged gives the proof.

We include the details in Section 4.10. □

We denote by $L_{(k)}(s,\chi)$ the L-function with Euler factors of $p \mid k$ removed, so

$$L_{(k)}(s,\chi) = L(s,\chi)\prod_{p|k}\left(1 - \frac{\chi(p)}{p^s}\right) \tag{4.2.18}$$

We now record some useful estimates that hold under GRH. First is the Lindelöf bound: for $\mathrm{Re}(s) \geq 1/2$,

$$|L(s,\chi_n)| \ll |sn|^{\varepsilon}. \tag{4.2.19}$$

Next, if n is squarefree so that χ_n is primitive, and for $\mathrm{Re}(r) > \varepsilon$, we also have

$$\left| \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} \right| \ll \log^2(|r|n). \tag{4.2.20}$$

Finally, for $\mathrm{Re}(s) > 1/2 + \varepsilon$, we have

$$\frac{1}{|L(s, \chi_n)|} \ll |sn|^{\varepsilon}. \tag{4.2.21}$$

Let

$$L_D(s, \chi) = \sum_{d \geq 1}^{*} \frac{\chi(d)}{d^s} \tag{4.2.22}$$

be the Dirichlet series where the sum runs over fundamental discriminants.

Lemma 4.2.4. *For a Dirichlet character χ, we have*

$$L_D(s, \chi) = \left(\frac{1}{2} + \frac{\chi(4)}{2 \cdot 4^s} + \frac{\chi(8)}{8^s} \right) \frac{L(s, \chi\psi_1)}{L(2s, \chi^2\psi_1)} + \left(\frac{1}{2} - \frac{\chi(4)}{2 \cdot 4^s} \right) \frac{L(s, \chi\psi_{-1})}{L(2s, \chi^2\psi_1)}. \tag{4.2.23}$$

Proof. Any integer d is a fundamental discriminant if either

a) $d \equiv 1 \pmod 4$ and is squarefree, or

b) $d = 4m$ with $m \equiv 3 \pmod 4$ and m is squarefree, or

c) $d = 4m$ with $m \equiv 2 \pmod 4$ and m is squarefree.

We will compute the contribution of each of these three sets separately.

For part a), we have:

$$\begin{aligned} \sum_{d \equiv 1 \ (\mathrm{mod}\ 4)} \frac{\chi(d)\mu^2(d)}{d^s} &= \frac{1}{2} \sum_{d \geq 1} \frac{\chi(d)\mu^2(d)(\psi_1(d) + \psi_{-1}(d))}{d^s} \\ &= \frac{1}{2} \prod_p \left(1 + \frac{\chi(p)\psi_1(p)}{p^s} \right) + \frac{1}{2} \prod_p \left(1 + \frac{\chi(p)\psi_{-1}(p)}{p^s} \right) \\ &= \frac{1}{2} \left(\frac{L(s, \chi\psi_1)}{L(2s, \chi^2\psi_1)} + \frac{L(s, \chi\psi_{-1})}{L(2s, \chi^2\psi_1)} \right). \end{aligned} \tag{4.2.24}$$

Part b) gives:

$$\begin{aligned} \sum_{\substack{d = 4m, \\ m \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{\chi(d)\mu^2(m)}{d^s} &= \frac{\chi(4)}{2 \cdot 4^s} \sum_{m \geq 1} \frac{\chi(m)\mu^2(n)(\psi_1(m) - \psi_{-1}(m))}{m^s} \\ &= \frac{\chi(4)}{2 \cdot 4^s} \left(\frac{L(s, \chi\psi_1)}{L(2s, \chi^2\psi_1)} - \frac{L(s, \chi\psi_{-1})}{L(2s, \chi^2\psi_1)} \right). \end{aligned} \tag{4.2.25}$$

The condition in part c) is equivalent to $d = 8m$ with m odd and squarefree. Hence we obtain

$$\begin{aligned}\sum_{\substack{d=8m,\\ m \text{ odd}}} \frac{\chi(d)\mu^2(m)}{d^s} &= \frac{\chi(8)}{8^s}\sum_{m\geq 1}\frac{\chi(m)\psi_1(m)\mu^2(m)}{m^s} \\ &= \frac{\chi(8)}{8^s}\frac{L(s,\chi\psi_1)}{L(2s,\chi^2\psi_1)}.\end{aligned} \tag{4.2.26}$$

Adding the three parts gives the result. □

For a function $f(x)$, we denote by $\mathcal{M}f(s)$ its Mellin transform defined by

$$\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{4.2.27}$$

for these s where the integral converges. If $f(x)$ is smooth, $\mathcal{M}f(s)$ decays faster than any polynomial in vertical strips.

If $\mathcal{M}f(s)$ is analytic for $a < \mathrm{Re}(s) < b$, then the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i}\int_{(c)} x^{-s}\mathcal{M}f(s)ds, \tag{4.2.28}$$

where the integral is taken over a vertical line $\mathrm{Re}(s) = c$ and $a < c < b$ is arbitrary.

The following estimate is a consequence of Stirling's formula: for a fixed $\sigma \in \mathbb{R}$ and $|t| \geq 1$, we have

$$|\Gamma(\sigma + it)| \asymp e^{-|t|\frac{\pi}{2}}|t|^{\sigma-1/2}. \tag{4.2.29}$$

We have Legendre's duplication formula

$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s) \tag{4.2.30}$$

and Euler's reflection formula

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}. \tag{4.2.31}$$

We will also use the formula

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{2^s\sin(\pi s/2)\Gamma(1-s)}{\sqrt{\pi}}. \tag{4.2.32}$$

Recall that $\Gamma_o(s)$ and $\Gamma_e(s)$ denote the ratios of the gamma factors that appear in the functional equation for even or odd characters, so

$$\Gamma_e(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \tag{4.2.33}$$

and

$$\Gamma_o(s) = \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}. \tag{4.2.34}$$

Lemma 4.2.5. *We have*

$$\Gamma_o(s) + \Gamma_e(s) = \frac{2^{s+1/2}\Gamma(1-s)\cos\left(\frac{\pi s}{2} - \frac{\pi}{4}\right)}{\sqrt{\pi}}. \tag{4.2.35}$$

Proof. We have

$$\Gamma_o(s) + \Gamma_e(s) = \frac{\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{s}{2}\right) + \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right)}. \tag{4.2.36}$$

Using the reflection formula in the numerator and the duplication formula in the denominator, this equals

$$\frac{2^{s-1}\sqrt{\pi}}{\Gamma(s)}\left(\frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)}\right), \tag{4.2.37}$$

and another application of the reflection formula in the denominator gives

$$\frac{2^{s-1}\Gamma(1-s)\sin(\pi s)}{\sqrt{\pi}}\left(\frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)}\right). \tag{4.2.38}$$

The lemma follows after using the identity

$$\sin(\pi s)\left(\frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)}\right) = 2\sqrt{2}\cos\left(\frac{\pi s}{2} - \frac{\pi}{4}\right). \tag{4.2.39}$$

□

A key tool from multivariable complex analysis that we use is Bochner's Tube Theorem [Boc]. For a set $U \subset \mathbb{R}^n$, we define $T(U) = U + i\mathbb{R}^n \subset \mathbb{C}^n$. Then we have the following theorem:

Theorem 4.2.6 (Bochner's Tube Theorem)**.** *Let $U \subset \mathbb{R}^n$ be a connected open set and $f(z)$ be a function that is holomorphic on $T(U)$. Then $f(z)$ has a holomorphic continuation to the convex hull of $T(U)$.*

A more general version of the theory of domains of holomorphy allows us to show that in the situation of Bochner's tube theorem, some of the properties of $f(z)$ also hold for its holomorphic continuation. An example that we include in Section 4.12 is Theorem 4.12.5, which we use to bound the continuation of some triple Dirichlet series in vertical strips. See also sections 4.3 and 4.4 of [DGH].

4.2.1 Overview of the proofs

For simplicity, let us now assume our sums run over all positive integers. Our goal is to find an asymptotic formula for

$$\sum_{n\leq X}\frac{L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)}. \tag{4.2.40}$$

We insert a smooth weight into the sum and use Mellin inversion to rewrite it as an integral. Thus we obtain

$$\sum_{n\geq 1}\frac{L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)}f(n/X)=\frac{1}{2\pi i}\int_{(c)}A^*(s,1/2+\alpha,1/2+\beta)X^s\mathcal{M}f(s)ds, \tag{4.2.41}$$

where $A^*(s,w,z)$ is the triple Dirichlet series

$$A^*(s,w,z)=\sum_{n\geq 1}\frac{L(w,\chi_n)}{L(z,\chi_n)n^s}=\sum_{m,n,k\geq 1}\frac{\mu(k)\chi_n(k)\chi_n(m)}{n^s m^w k^z}. \tag{4.2.42}$$

To be able to evaluate the integral, we need to investigate the analytic properties of $A^*(s,w,z)$. Assuming GRH and using (4.2.19) and (4.2.21), the series is absolutely convergent (up to a simple pole at $w=1$) in the region

$$\{(s,w,z)\in\mathbb{C}^3:\mathrm{Re}(s)>1,\ \mathrm{Re}(s+w)>3/2,\ \mathrm{Re}(z)>1/2\}, \tag{4.2.43}$$

where the second condition comes after using the functional equation for the L-function in the numerator if $\mathrm{Re}(w)<1/2$. We remark that we get the same region without assuming GRH, as we only need the Lindelöf bound on average (see for example [Blo] for details).

Comparing the integral with the prediction of the ratios conjecture, we may expect that $A^*(s,w,z)$ has a meromorphic continuation to $\mathrm{Re}(s)>1/2$, with poles at $s=1$ and $s=1-\alpha=3/2-w$.

We now present a heuristic of Bump, Friedberg and Hoffstein (see [BFH1]) suggesting that $A^*(s,w,z)$ satisfies some functional equations that can be used to obtain a meromorphic continuation, and that it has the predicted poles. In these heuristics, we assume that all characters are primitive, and that the quadratic reciprocity holds in the form $\chi_n(m)\approx\chi_m(n)$. We also don't write the gamma factors in the functional equations, so they have the form

$$L(s,\chi_n)\approx n^{1/2-s}L(1-s,\chi_n). \tag{4.2.44}$$

Using this functional equation in w, we obtain the first heuristic functional equation for $A^*(s,w,z)$:

$$A^*(s,w,z) \approx \sum_n \frac{L(w,\chi_n)}{L(z,\chi_n)n^s} \approx \sum_n \frac{L(1-w,\chi_n)}{L(z,\chi_n)n^{s+w-\frac{1}{2}}} \approx A^*(s+w-1/2, 1-w, z). \tag{4.2.45}$$

On the other hand, we can expand the $L-$functions into Dirichlet series, first sum over the n variable, and use the functional equation in s. Thus we obtain the second heuristic functional equation:

$$\begin{aligned} A^*(s,w,z) &\approx \sum_{m,n,k} \frac{\mu(k)\chi_n(k)\chi_n(m)}{n^s m^w k^z} \approx \sum_{m,k} \frac{\mu(k)L(s,\chi_{mk})}{m^w k^z} \\ &\approx \sum_{m,k} \frac{\mu(k)L(1-s,\chi_{mk})}{m^{s+w-\frac{1}{2}}k^{s+z-\frac{1}{2}}} \approx A^*\left(1-s, s+w-\frac{1}{2}, s+z-\frac{1}{2}\right). \end{aligned} \tag{4.2.46}$$

We also see from this computation that there is a pole at $s=1$ coming from the terms with $mk=\square$, and (4.2.45) then gives the pole at $s=3/2-w$. We can also use (4.2.45) to see a pole at $w=1$, which becomes a pole at $s=3/2-w$ after (4.2.46).

An important aspect of our result is the admissible range of the parameters α and β, and the error term. This depends on the region of meromorphic continuation of $A^*(s,w,z)$, which we obtain from the functional equations and a careful application of Bochner's Tube Theorem.

Let us now be more precise. For Theorem 4.1.1, we rewrite $R_D(X,\alpha,\beta;f)$ as an integral as

$$R_D(X,\alpha,\beta;f) = \frac{1}{2\pi i}\int_{(2)} A_D\left(s, \frac{1}{2}+\alpha, \frac{1}{2}+\beta\right) X^s \mathcal{M}f(s)ds, \tag{4.2.47}$$

where $A_D(s,w,z)$ is the triple Dirichlet series

$$\begin{aligned} A_D(s,w,z) &= \sum_{d\geq 1}{}^{*} \frac{L(w,\chi_d)}{L(z,\chi_d)d^s} = \sum_{d\geq 1}{}^{*} \sum_{m,k\geq 1} \frac{\mu(k)\chi_d(k)\chi_d(m)}{k^z m^w d^s} \\ &= \sum_{m,k\geq 1} \frac{\mu(k)L_D\left(s, \left(\frac{\cdot}{mk}\right)\right)}{m^w k^z}. \end{aligned} \tag{4.2.48}$$

Since all characters are even and primitive, it is straightforward to get the functional equation in w. However, after exchanging summations, we obtain $L_D(s,\chi)$ instead of $L(s,\chi)$ in the heuristic in (4.2.46), so we don't have the functional equation in s.

This explains the weaker results for this family. The details are written in Section 4.3.

For Theorem 4.1.2, we similarly obtain

$$\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}(1/2+\alpha,\chi_n)}{L_{(2)}(1/2+\beta,\chi_n)} f\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(2)} A\left(s,\frac{1}{2}+\alpha,\frac{1}{2}+\beta\right) X^s \mathcal{M}f(s)ds, \tag{4.2.49}$$

where

$$\begin{aligned} A(s,w,z) &= \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}(w,\chi_n)}{L_{(2)}(z,\chi_n)n^s} = \sum_{\substack{k,m,n\geq 1,\\ k,m,n \text{ odd}}} \frac{\mu(k)\chi_n(k)\chi_n(m)}{k^z m^w n^s} \\ &= \sum_{\substack{m,k\geq 1,\\ m,k \text{ odd}}} \frac{L\left(s,\left(\frac{4mk}{\cdot}\right)\right)}{m^w k^z}. \end{aligned} \tag{4.2.50}$$

In this case, we can potentially obtain both functional equations, but we have to deal with the presence of non-primitive characters. A key tool here is the functional equation in Proposition 4.2.3, which is valid for all Dirichlet characters. After applying it, we obtain a relation between A and some different triple Dirichlet series, whose coefficients involve Gauss sums. We elaborate on this in Section 4.5.

4.3 The triple Dirichlet series for fundamental discriminants

As in (4.2.47) we can write $R_D(X,\alpha,\beta;f)$ as

$$R_D(X,\alpha,\beta;f) = \frac{1}{2\pi i}\int_{(c)} A_D(s,w,z)X^s\mathcal{M}f(s)ds, \tag{4.3.1}$$

where

$$A_D(s,w,z) = \sum_{d\geq 1}^{*} \frac{L(w,\chi_d)}{L(z,\chi_d)d^s}, \tag{4.3.2}$$

and $w = 1/2+\alpha$, $z = 1/2+\beta$. In this section, we investigate the analytic properties of the triple Dirichlet series $A_D(s,w,z)$ that allow us to shift the integral and compute the main terms coming from the two poles of $A_D(s,w,z)$.

4.3.1 Region of absolute convergence

Using (4.2.19), (4.2.21) and the functional equation for $L(w,\chi_d)$ if $\mathrm{Re}(w)<1/2$, up to a simple pole at $w=1$, the triple Dirichlet series $A_D(s,w,z)$ converges absolutely in the region

$$R_0=\left\{(s,w,z):\ \mathrm{Re}(s)>1,\ \mathrm{Re}(s+w)>\frac{3}{2},\ \mathrm{Re}(z)>\frac{1}{2}\right\}, \tag{4.3.3}$$

and is polynomially bounded in vertical strips in this region, i.e., away from the possible poles, we have

$$|A_D(s,w,z)| \ll_{\mathrm{Re}(s),\mathrm{Re}(w),\mathrm{Re}(z)} ((1+|s|)(1+|w|)(1+|z|))^c \tag{4.3.4}$$

for some constant c. By exchanging summations and using Lemma 4.2.4, we have

$$\begin{aligned} A_D(s,w,z) &= \sum_{m,k\geq 1}\sideset{}{^*}\sum_{d\geq 1}\frac{\mu(k)\chi_d(k)\chi_d(m)}{d^s m^w k^z} = \sum_{m,k\geq 1}\frac{\mu(k)L_D\left(s,\left(\frac{\cdot}{mk}\right)\right)}{m^w k^z} \\ &= \sum_{m,k\geq 1}\left(\frac{1}{2}+\frac{\left(\frac{4}{mk}\right)}{2\cdot 4^s}+\frac{\left(\frac{8}{mk}\right)}{8^s}\right)\frac{\mu(k)L\left(s,\left(\frac{\cdot}{mk}\right)\psi_1\right)}{m^w k^z L\left(2s,\left(\frac{\cdot}{mk}\right)^2\psi_1\right)} \\ &\quad+\sum_{m,k\geq 1}\left(\frac{1}{2}-\frac{\left(\frac{4}{mk}\right)}{2\cdot 4^s}\right)\frac{\mu(k)L\left(s,\left(\frac{\cdot}{mk}\right)\psi_{-1}\right)}{m^w k^z L\left(2s,\left(\frac{\cdot}{mk}\right)^2\psi_1\right)}. \end{aligned} \tag{4.3.5}$$

This expression converges absolutely in the region

$$\begin{aligned} R_1=\Big\{(s,w,z):\ \mathrm{Re}(s)>\frac{1}{4},\ \mathrm{Re}(w)>1,\ \mathrm{Re}(z)>1,& \\ \mathrm{Re}(s+w)>\frac{3}{2},\ \mathrm{Re}(s+z)>\frac{3}{2}\Big\},& \end{aligned} \tag{4.3.6}$$

except there is a pole at $s=1$ coming from the terms in the first sum when $mk=\square$. Moreover, $(s-1)A_D(s,w,z)$ is polynomially bounded in vertical strips in R_1. Bochner's tube theorem allows us to meromorphically continue $A_D(s,w,z)$ to the convex hull of R_0 and R_1, which is

$$R_2=\left\{(s,w,z):\ \mathrm{Re}(s)>\frac{1}{4},\ \mathrm{Re}(z)>\frac{1}{2},\ \mathrm{Re}(s+w)>\frac{3}{2},\ \mathrm{Re}(s+z)>\frac{3}{2}\right\}, \tag{4.3.7}$$

and by Proposition 4.12.5, $(s-1)A_D(s,w,z)$ is polynomially bounded in vertical strips in R_2.

4.3.2 Residue at $s = 1$

We see from expression (4.3.5) that $A_D(s, w, z)$ has a pole at $s = 1$ coming from the terms with $mk = \square$. In this case, we have

$$\frac{L\left(s, \left(\frac{\cdot}{mk}\right)\psi_1\right)}{L\left(2s, \left(\frac{\cdot}{mk}\right)^2 \psi_1\right)} = \frac{\zeta(s)}{\zeta(2s)} \prod_{p|2mk} \frac{p^s}{p^s+1}. \tag{4.3.8}$$

If mk is an odd square, then $mk \equiv 1 \pmod 8$, so

$$\left(\frac{1}{2} + \frac{\left(\frac{4}{mk}\right)}{8} + \frac{\left(\frac{8}{mk}\right)}{8}\right) \prod_{p|2mk} \frac{p}{p+1} = \frac{3}{4} \cdot \frac{2}{3} \prod_{p|mk} \frac{p}{p+1} = \frac{1}{2} \prod_{p|mk} \frac{p}{p+1}, \tag{4.3.9}$$

and the same holds if mk is an even square. Therefore, we have

$$\mathrm{res}_{s=1} A_D(s, w, z) = \frac{1}{2\zeta(2)} \sum_{mk=\square} \frac{\mu(k)}{m^w k^z} \prod_{p|mk} \frac{p}{p+1}. \tag{4.3.10}$$

This is the same expression as in the heuristic computation (2.21) in [CoSn], and can be written as

$$\mathrm{res}_{s=1} A_D(s, w, z) = \frac{\zeta(2w)}{2\zeta(2)\zeta(z+w)} P_D(z, w), \tag{4.3.11}$$

where

$$\begin{aligned} P_D(z, w) &= \prod_p \left(1 - \frac{1}{p^{z+w}}\right)^{-1} \left(1 - \frac{1}{(p+1)p^{2w}} - \frac{p}{(p+1)p^{z+w}}\right) \\ &= \prod_p \left(1 + \frac{1 - p^{z-w}}{\left(p^{z+w} - 1\right)(p+1)}\right). \end{aligned} \tag{4.3.12}$$

4.3.3 Functional equation and meromorphic continuation

We now use the functional equation of $L(w, \chi_d)$ to obtain a functional equation for $A_D(s, w, z)$, which gives us a meromorphic continuation beyond the region R_2. All of the characters χ_d are even and primitive of conductor d, so (4.2.13) gives

$$\begin{aligned} A_D(s, w, z) &= \sum_{d\geq 1}^{*} \frac{L(w, \chi_d)}{L(z, \chi_d) d^s} = \pi^{w-1/2}\Gamma_e(w) \sum_{d \geq 1}^{*} \frac{L(1-w, \chi_d)}{L(z, \chi_d) d^{s+w-1/2}} \\ &= \pi^{w-1/2}\Gamma_e(w) A_D(s + w - 1/2, 1 - w, z). \end{aligned} \tag{4.3.13}$$

This functional equation provides a meromorphic continuation to the region

$$R_3 = \left\{(s, w, z) : \ \mathrm{Re}(s+w) > \frac{3}{4}, \ \mathrm{Re}(z) > \frac{1}{2}, \ \mathrm{Re}(s) > 1, \ \mathrm{Re}(s+w+z) > 2\right\}, \tag{4.3.14}$$

and gives rise to a new pole of $A_D(s, w, z)$ at $s = 3/2 - w$ with residue

$$\mathrm{res}_{s=3/2-w} A_D(s, w, z) = \pi^{w-1/2}\Gamma_e(w)\frac{\zeta(2-2w)}{2\zeta(2)\zeta(1+z-w)} P_D(z, 1-w). \tag{4.3.15}$$

Bochner's tube theorem allows us to meromorphically continue $A_D(s, w, z)$ to the convex hull of R_2 and R_3, which is the region

$$R_4 = \Bigg\{(s, w, z) : \ \mathrm{Re}(s) > \frac{1}{4}, \ \mathrm{Re}(z) > \frac{1}{2}, \ \mathrm{Re}(s+w) > \frac{3}{4}, \ \mathrm{Re}(2s+w) > \frac{7}{4}, \\ \mathrm{Re}(s+z) > \frac{3}{2}, \ \mathrm{Re}(2s+w+z) > 3, \ \mathrm{Re}(s+w+z) > 2\Bigg\}. \tag{4.3.16}$$

Moreover, $(s-1)(w-1)(s+w-3/2)A_D(s, w, z)$ is polynomially bounded in vertical strips in the region R_3, and by Proposition 4.12.5 also in the region R_4.

4.4 Proof of Theorem 4.1.1

In this section, we prove Theorem 4.1.1. Assume that $-1/2 < \mathrm{Re}(\alpha) < 1/2$, $0 < \mathrm{Re}(\beta) < 1/2$, and $\mathrm{Re}(\beta) > |\mathrm{Re}(\alpha)|$. We have

$$R_D(X, \alpha, \beta; f) = \frac{1}{2\pi i}\int_{(2)} A_D(s, w, z) X^s \mathcal{M}f(s) ds, \tag{4.4.1}$$

where $w = 1/2 + \alpha$, $z = 1/2 + \beta$. Let

$$M(\alpha, \beta) = 1 - \frac{\mathrm{Re}(\alpha)}{2} - \frac{\mathrm{Re}(\beta)}{2}. \tag{4.4.2}$$

By (4.3.16), we can shift the integral to $\mathrm{Re}(s) = M(\alpha, \beta) + \varepsilon$, and our assumptions about $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$ ensure that we cross the poles at $s = 1$, $s = 1 - \alpha$, and that the Euler products $P_D\left(\frac{1}{2}+\beta, \frac{1}{2}\pm\alpha\right)$ converge absolutely.

$$\begin{aligned} R_D(X, \alpha, \beta; f) &= X\mathcal{M}f(1)\mathrm{res}_{s=1} A_D(s, 1/2+\alpha, 1/2+\beta) \\ &+ X^{1-\alpha}\mathcal{M}f(1-\alpha)\mathrm{res}_{s=1-\alpha} A_D(s, 1/2+\alpha, 1/2+\beta) \\ &+ \frac{1}{2\pi i}\int_{(M(\alpha,\beta)+\varepsilon)} A_D(s, 1/2+\alpha, 1/2+\beta)\mathcal{M}f(s) X^s ds. \end{aligned} \tag{4.4.3}$$

Since $A_D(s,w,z)$ is polynomially bounded in vertical strips, we can bound the last integral and obtain

$$\begin{aligned} R_D(X,\alpha,\beta;f) = &\frac{X\mathcal{M}f(1)\zeta(1+2\alpha)}{2\zeta(2)\zeta(1+\alpha+\beta)} P_D(1/2+\alpha, 1/2+\beta) \\ &+ \frac{X^{1-\alpha}\mathcal{M}f(1-\alpha)\zeta(1-2\alpha)\pi^{\alpha}\Gamma_e(1/2+\alpha)}{2\zeta(2)\zeta(1-\alpha+\beta)} P_D(1/2-\alpha, 1/2+\beta) \\ &+ O_{\alpha,\beta}(X^{M(\alpha,\beta)+\varepsilon}). \end{aligned} \tag{4.4.4}$$

4.5 The triple Dirichlet series for all characters

We now proceed with proving the results for the family with non-primitive characters. In view of (4.2.49), we begin by studying the properties of the triple Dirichlet series $A(s,w,z)$.

4.5.1 Region of absolute convergence

By (4.2.19) and (4.2.21), the initial series

$$A(s,w,z) = \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}(w,\chi_n)}{L_{(2)}(z,\chi_n)n^s}, \tag{4.5.1}$$

up to a simple pole at $w=1$, converges absolutely in the region

$$S_0 = \{(s,w,z) : \mathrm{Re}(s) > 1,\ \mathrm{Re}(s+w) > 3/2,\ \mathrm{Re}(z) > 1/2\}. \tag{4.5.2}$$

We can exchange summations and get

$$A(s,w,z) = \sum_{\substack{m,n,k\geq 1,\\ m,n,k \text{ odd}}} \frac{\chi_n(m)\chi_n(k)\mu(k)}{n^s m^w k^z} = \sum_{\substack{m,k\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k)L\left(s, \left(\frac{4mk}{\cdot}\right)\right)}{m^w k^z}, \tag{4.5.3}$$

which converges absolutely in the region

$$S_1 = \{(s,w,z) : \mathrm{Re}(w) > 1,\ \mathrm{Re}(z) > 1,\ \mathrm{Re}(s+w) > \tfrac{3}{2},\ \mathrm{Re}(s+z) > \tfrac{3}{2}\}, \tag{4.5.4}$$

except the pole at $s=1$ coming from the summands with $mk = \square$. The convex hull of S_0 and S_1 is

$$S_2 = \{(s,w,z) : \mathrm{Re}(z) > 1/2,\ \mathrm{Re}(s+w) > 3/2,\ \mathrm{Re}(s+z) > 3/2\}. \tag{4.5.5}$$

4.5.2 Pole and residue at $s = 1$

We see from (4.5.3) that $A(s, w, z)$ has a pole at $s = 1$ coming from the summands where $mk = \square$. In this case, we have

$$L\left(s, \left(\frac{4mk}{\cdot}\right)\right) = \zeta(s) \prod_{p|4mk} \left(1 - \frac{1}{p^s}\right), \tag{4.5.6}$$

so denoting by $a(n)$ the multiplicative function with $a(p^k) = 1 - \frac{1}{p}$, we have

$$\text{res}_{s=1} A(s, w, z) = \sum_{\substack{mk=\square,\\ m,k \text{ odd}}} \frac{\mu(k)a(4mk)}{m^w k^z} = \frac{1}{2} \sum_{\substack{mk=\square,\\ m,k \text{ odd}}} \frac{\mu(k)a(mk)}{m^w k^z}. \tag{4.5.7}$$

We can write the last sum as an Euler product, slightly abusing notation by writing p^k for the prime factors of k, and similarly for m. We thus obtain

$$\begin{aligned}
&\frac{1}{2} \prod_{p>2} \sum_{\substack{m,k \geq 0,\\ m+k \text{ even}}} \frac{\mu(p^k)a(p^{m+k})}{p^{mw+kz}} \\
&= \frac{1}{2} \prod_{p>2} \left(\sum_{\substack{m \geq 0,\\ m \text{ even}}} \frac{a(p^m)}{p^{mw}} - \sum_{\substack{m \geq 0,\\ m \text{ odd}}} \frac{a(p^{m+1})}{p^{z+mw}} \right) \\
&= \frac{1}{2} \prod_{p>2} \left(1 + \left(1 - \frac{1}{p}\right) \cdot \frac{p^{-2w}}{1 - p^{-2w}} - \left(1 - \frac{1}{p}\right) \frac{p^{-z-w}}{1 - p^{-2w}} \right) \\
&= \frac{\zeta_{(2)}(2w)}{2\zeta_{(2)}(z+w)} \prod_{p>2} \left(1 - \frac{1}{p^{1+w-z}(p^{z+w} - 1)} + \frac{1}{p(p^{z+w} - 1)} \right) \\
&= \frac{\zeta_{(2)}(2w)}{2\zeta_{(2)}(z+w)} \prod_{p>2} \left(1 + \frac{p^{w-z} - 1}{p^{1+w-z}(p^{z+w} - 1)} \right).
\end{aligned} \tag{4.5.8}$$

Setting $w = 1/2 + \alpha$, $z = 1/2 + \beta$ gives

$$\text{res}_{s=1} A(s, 1/2 + \alpha, 1/2 + \beta) = \frac{\zeta_{(2)}(1 + 2\alpha)}{2\zeta_{(2)}(1 + \alpha + \beta)} \prod_{p>2} \left(1 + \frac{p^{\alpha-\beta} - 1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta} - 1)} \right). \tag{4.5.9}$$

4.5.3 Functional equation in s

To get a functional equation for $A(s, w, z)$, we use expression (4.5.3) together with the functional equation from Proposition 4.2.3. Since $\left(\frac{4mk}{\cdot}\right)$ is an even Dirichlet character modulo $4mk$ for any $m, k \geq 1$, we obtain

$$
\begin{aligned}
A(s,w,z) &= \sum_{\substack{m,k\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k) L\left(s, \left(\frac{4mk}{\cdot}\right)\right)}{m^w k^z} \\
&= \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{4^s\Gamma\left(\frac{s}{2}\right)} \sum_{\substack{m,k\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k) K\left(1-s, \left(\frac{4mk}{\cdot}\right)\right)}{m^{s+w} k^{s+z}} \qquad (4.5.10)\\
&= \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{4^s\Gamma\left(\frac{s}{2}\right)} C(1-s, s+w, s+z),
\end{aligned}
$$

where $C(s, w, z)$ is the triple Dirichlet series

$$
C(s,w,z) = \sum_{\substack{m,k\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k) K\left(s, \left(\frac{4mk}{\cdot}\right)\right)}{m^w k^z} = \sum_{\substack{m,k,q\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k)\tau\left(\left(\frac{4mk}{\cdot}\right), q\right)}{q^s m^w k^z} \qquad (4.5.11)
$$

4.5.4 Region of convergence of $C(s, w, z)$

By (4.5.5) and the functional equation (4.5.10), $C(s, w, z)$ is initially defined in the region

$$
P = \{(s,w,z) : \ \mathrm{Re}(s+z) > 3/2, \ \mathrm{Re}(w) > 3/2, \ \mathrm{Re}(z) > 3/2\}. \qquad (4.5.12)
$$

To extend this region, we exchange the summations in $C(s, w, z)$ and get

$$
C(s,w,z) = \sum_{\substack{m,k,q\geq 1,\\ m,k \text{ odd}}} \frac{\mu(k)\tau\left(\left(\frac{4mk}{\cdot}\right), q\right)}{q^s m^w k^z} = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{\substack{\ell\geq 1,\\ \ell \text{ odd}}} \frac{\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right)}{\ell^w} \sum_{k|\ell} \frac{\mu(k)}{k^{z-w}}, \qquad (4.5.13)
$$

where we also substituted $mk = \ell$. Our goal now is to study the properties of the inner Dirichlet series, that is, the sum over ℓ.

We denote by $a_t(\ell)$ the multiplicative function with $a_t(p^k) = 1 - \frac{1}{p^t}$ and use Lemma

4.2.2, so we can rewrite (4.5.13) as

$$\begin{aligned}C(s,w,z) = &\sum_{\substack{\ell\equiv 1 \pmod 4,\\ q\equiv 2 \pmod 4}} \frac{-2\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s} + \sum_{\substack{\ell\equiv 1 \pmod 4,\\ q\equiv 0 \pmod 4}} \frac{2\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s}\\
&+ \sum_{\substack{\ell\equiv 3 \pmod 4,\\ q\equiv 1 \pmod 4}} \frac{-2i\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s} + \sum_{\substack{\ell\equiv 3 \pmod 4,\\ q\equiv 4 \pmod 4}} \frac{2i\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s}\\
&= \frac{-2}{2^s}\sum_{\substack{\ell\equiv 1 \pmod 4,\\ q \text{ odd}}} \frac{\left(\frac{2}{\ell}\right)\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s} + \frac{2}{4^s}\sum_{\substack{\ell\equiv 1 \pmod 4,\\ q\geq 1}} \frac{\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s}\\
&+ \sum_{\substack{\ell\equiv 3 \pmod 4,\\ q\equiv 1 \pmod 4}} \frac{-2i\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s} + \sum_{\substack{\ell\equiv 3 \pmod 4,\\ q\equiv 3 \pmod 4}} \frac{2i\tau\left(\left(\frac{\cdot}{\ell}\right),q\right)a_{z-w}(\ell)}{\ell^w q^s}\end{aligned} \tag{4.5.14}$$

All of the terms in (4.5.14) can be written as combinations of terms of the form

$$C(s,w,z;\psi,\psi') := \sum_{\ell,q\geq 1} \frac{G\left(\left(\frac{\cdot}{\ell}\right),q\right)\psi(\ell)\psi'(q)}{\ell^w q^s}\sum_{k\mid\ell}\frac{\mu(k)}{k^{z-w}}, \tag{4.5.15}$$

where ψ,ψ' are characters modulo $b \mid 8$, and $\psi(2)=0$. In particular, we have

$$\begin{aligned}C(s,w,z) = &-2^{-s}\big(C(s,w,z;\psi_2,\psi_1) + C(s,w,z;\psi_{-2},\psi_1)\big)\\
&+ 4^{-s}\big(C(s,w,z;\psi_1,\psi_0) + C(s,w,z;\psi_{-1},\psi_0)\big)\\
&+ C(s,w,z;\psi_1,\psi_{-1}) - C(s,w,z;\psi_{-1},\psi_{-1}).\end{aligned} \tag{4.5.16}$$

In $C(s,w,z;\psi,\psi')$, all coefficients are multiplicative in ℓ. We write

$$C(s,w,z;\psi,\psi') = \sum_{q=1}^{\infty}\frac{\psi'(q)}{q^s}\cdot D(w,z-w,q;\psi), \tag{4.5.17}$$

where

$$D(w,t,q;\psi) = \sum_{\ell=1}^{\infty}\frac{G\left(\left(\frac{\cdot}{\ell}\right),q\right)\psi(\ell)a_t(\ell)}{\ell^w}. \tag{4.5.18}$$

We have the following lemma:

Lemma 4.5.1. *1. $D(w,t,q;\psi)$ has a meromorphic continuation to the region*

$$\{(w,t):\ \mathrm{Re}(w)>1,\ \mathrm{Re}(w+t)>1\}. \tag{4.5.19}$$

The only pole in this region is at $w=3/2$, which occurs if $q=\square$, ψ is a principal character and $t\neq 0$.

2. For $\mathrm{Re}(w) > 1+\varepsilon$ *and* $\mathrm{Re}(t+w) > 1+\varepsilon$*, away from the possible poles, we have*

$$|D(w,t,q,\psi)| \ll |w(t+w)|^{\varepsilon} q^{\max\{\varepsilon, \varepsilon-\mathrm{Re}(t)\}}. \tag{4.5.20}$$

Proof. We can write $D(w,t,q;\psi)$ as an Euler product

$$\begin{aligned} D(w,t,q;\psi) &= \prod_p \left(\sum_{k=0}^{\infty} \frac{G\left(\left(\frac{\cdot}{p^k}\right), q\right) \psi(p^k) a_t(p^k)}{p^{kw}} \right) \\ &= P_{p\nmid q}(w,t,q;\psi) P_{p|q}(w,t,q;\psi), \end{aligned} \tag{4.5.21}$$

where $P_{p\nmid q}$ is the product over odd primes not dividing q, and $P_{p|q}$ is the rest. Since $\psi(2) = 0$, we can only consider odd primes. Also note that $P_{p|q}$ is a finite product, and each factor has only finitely many terms by (4.2.12).

For an odd $p \nmid q$, we have

$$G\left(\left(\frac{\cdot}{p^k}\right), q\right) = \begin{cases} 1, & \text{if } k = 0, \\ \left(\frac{q}{p}\right)\sqrt{p}, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2, \end{cases} \tag{4.5.22}$$

so for $P_{p\nmid q}$, we have

$$\begin{aligned} P_{p\nmid q}(w,t,q;\psi) &= \prod_{p\nmid q} \left(1 + \frac{\left(\frac{4q}{p}\right)\psi(p)}{p^{w-1/2}} \right) \prod_{p\nmid q} \frac{1 + \frac{\left(\frac{4q}{p}\right)\psi(p)(1-p^{-t})}{p^{w-1/2}}}{1 + \frac{\left(\frac{4q}{p}\right)\psi(p)}{p^{w-1/2}}} \\ &= \frac{L\left(w - 1/2, \left(\frac{4q}{\cdot}\right)\psi\right)}{\zeta_{(4q)}(2w-1)} E(w,t,q;\psi), \end{aligned} \tag{4.5.23}$$

where

$$\begin{aligned} E(w,t,q;\psi) &= \prod_p \left(1 - \frac{\left(\frac{4q}{p}\right)\psi(p)}{p^{t+w-1/2}} \cdot \frac{1}{1 + \frac{\left(\frac{4q}{p}\right)\psi(p)\sqrt{p}}{p^w}} \right) \\ &= \frac{1}{L\left(t+w-1/2, \left(\frac{4q}{\cdot}\right)\psi\right)} \prod_p \left(\frac{1 - \frac{\left(\frac{4q}{p}\right)\psi(p)}{p^{t+w-1/2}} \cdot \frac{1}{1+\left(\frac{q}{p}\right)\psi(p)p^{1/2-w}}}{1 - \frac{\left(\frac{4q}{p}\right)\psi(p)}{p^{t+w-\frac{1}{2}}}} \right) \\ &= \frac{1}{L\left(t+w-\frac{1}{2}, \left(\frac{4q}{\cdot}\right)\psi\right)} \prod_p \left(1 + \frac{\left(\frac{4q}{p}\right)^2}{\left(p^{t+w-\frac{1}{2}} - \left(\frac{4q}{p}\right)\psi(p)\right)\left(p^{w-\frac{1}{2}} + \left(\frac{4q}{p}\right)\psi(p)\right)} \right). \end{aligned} \tag{4.5.24}$$

For $\mathrm{Re}(t+w) > 1+\varepsilon$, $\mathrm{Re}(w) > 1+\varepsilon$, the last Euler product is absolutely convergent and $\ll 1$. This finishes the proof of part (1) of the Lemma.

To prove part (2), we have to estimate the size of the remaining factors.

We have

$$\zeta_{(4q)}(2w-1)^{-1} = \zeta(2w-1)^{-1} \prod_{p|4q} \left(1 - \frac{1}{p^{2w-1}}\right)^{-1}, \tag{4.5.25}$$

and for $\mathrm{Re}(w) > 1/2+\varepsilon$, the product can be bounded by

$$\prod_{p|4q} \left(1 - \frac{1}{p^{2w-1}}\right)^{-1} = \prod_{p|4q} \left(1 + \frac{1}{p^{2w-1}-1}\right) \ll c^{\omega(4q)} \ll q^{\varepsilon}, \tag{4.5.26}$$

where c is a suitable constant (depending on ε), and the last bound follows from the elementary estimate $\omega(n) \ll \frac{\log n}{\log\log n}$ (here $\omega(n)$ denotes the number of prime factors of n).

It remains to bound $P_{p|q}(w,t,q;\psi)$. By (4.2.12), we can write it as

$$\begin{aligned} &\prod_{p^a||q} \left(1 + \sum_{k=1}^{\lfloor \frac{a}{2} \rfloor} \frac{\varphi(p^{2k})\psi(p^{2k})}{p^{2kw}}(1-p^{-t}) + \frac{G\left(\left(\frac{\cdot}{p^{a+1}}\right), q\right)\psi(p^{a+1})}{p^{(a+1)w}}(1-p^{-t})\right) \\ &\ll \prod_{p|q} \left(1 + (1-p^{-t}) \sum_{k=1}^{\infty} p^{2k(1-w)}\right), \end{aligned} \tag{4.5.27}$$

and the geometric series is $\ll 1$ for $\mathrm{Re}(w) > 1+\varepsilon$. We have

$$1 - p^{-t} \ll \begin{cases} 1, & \text{if } \mathrm{Re}(t) \geq 0, \\ p^{-\mathrm{Re}(t)}, & \text{if } \mathrm{Re}(t) < 0, \end{cases} \tag{4.5.28}$$

so the last expression in (4.5.27) is

$$\ll \prod_{p|q} \left(1 + c_1\left(1 + p^{-\mathrm{Re}(t)}\right)\right) \ll \begin{cases} q^{\varepsilon}, & \text{if } \mathrm{Re}(t) \geq 0, \\ q^{-\mathrm{Re}(t)+\varepsilon}, & \text{if } \mathrm{Re}(t) < 0, \end{cases} \tag{4.5.29}$$

where the bounds were obtained similarly as in (4.5.26). □

Using this Lemma, we can extend each of $C(s,w,z;\psi,\psi')$, and hence also $C(s,w,z)$ to the region

$$\{(s,w,z):\ \mathrm{Re}(w) > 1,\ \mathrm{Re}(z) > 1,\ \mathrm{Re}(s) + \min\{0, \mathrm{Re}(z-w)\} > 1\}. \tag{4.5.30}$$

Using the functional equation (4.5.10) enables us to extend $A(s,w,z)$ to

$$S_3 = \{(s,w,z): \ \mathrm{Re}(s+w) > 1, \ \mathrm{Re}(s+z) > 1, \ \mathrm{Re}(1-s) + \min\{0, \mathrm{Re}(z-w)\} > 1\}. \tag{4.5.31}$$

The convex hull of S_2 and S_3 contains

$$S_4 = \Bigg\{ (s,w,z) : \mathrm{Re}(s+2w) > 2, \ \mathrm{Re}(s+2z) > 2, \\ \mathrm{Re}(s+z) > 1, \ \mathrm{Re}(s+w) > 1, \ \mathrm{Re}(z) > 1/2 \Bigg\}. \tag{4.5.32}$$

4.5.5 Bounding $A(s,w,z)$ in vertical strips

We now give bounds for $|A(s,w,z)|$ in vertical strips, which are necessary to bound the error term coming from the shifted integral. To get the desired result, we need a bound of the form $|wz|^{\varepsilon}|s|^{K}$ for some constant K, so we can be a little wasteful in the exponent of s.

For the earlier defined regions S_j, we define

$$\tilde{S}_j = S_{j,\varepsilon} \cap \{(s,w,z) : \mathrm{Re}(s) > -5/2, \ \mathrm{Re}(w) > 1/2 - \varepsilon\}, \tag{4.5.33}$$

where $S_{j,\varepsilon} = S_j + \varepsilon\vec{v}$, with $\vec{v} = (1,1,1)$. Let also

$$p(s,w) = (s-1)(w-1)(s+w-3/2), \tag{4.5.34}$$

so that $A(s,w,z)p(s,w)$ is an analytic function in the considered regions. We also denote $\tilde{p}(s,w) = 1 + |p(s,w)|$.

We first give bounds in the regions of absolute convergence. We have

$$|p(s,w)A(s,w,z)| = \left| p(s,w) \sum_{\substack{n \geq 1, \\ n \text{ odd}}} \frac{L_{(2)}(w,\chi_n)}{L_{(2)}(z,\chi_n)n^s} \right| \ll \tilde{p}(s,w)|wz|^{\varepsilon}, \tag{4.5.35}$$

valid in $\tilde{S}_0$. Exchanging summations and using (4.2.19), we also get the bound

$$|p(s,w)A(s,w,z)| = \left| p(s,w) \sum_{\substack{m,k \geq 1, \\ m,k \text{ odd}}} \frac{\mu(k) L\left(s, \left(\frac{4mk}{\cdot}\right)\right)}{m^w k^z} \right| \\ \ll \tilde{p}(s,w)(1+|s|)^{\max\{\varepsilon, \frac{1}{2} - \mathrm{Re}(s) + \varepsilon\}}. \tag{4.5.36}$$

which holds in $\tilde{S}_1$. Using Proposition 4.12.5, we get the bound

$$|p(s,w)A(s,w,z)| \ll \tilde{p}(s,w)|wz|^{\varepsilon}(1+|s|)^{3+\varepsilon} \tag{4.5.37}$$

in the convex hull of $\tilde{S}_0, \tilde{S}_1$, which is $\tilde{S}_2$.

By (4.5.16) and Lemma 4.5.1, we have

$$|(w-3/2)C(s,w,z)| \ll (1+|w-3/2|)|wz|^{\varepsilon} \tag{4.5.38}$$

in the region

$$\{(s,w,z) : \mathrm{Re}(w) > 1+\varepsilon,\ \mathrm{Re}(z) > 1+\varepsilon,\ \mathrm{Re}(s)+\min\{0, \mathrm{Re}(z-w) > 1+\varepsilon\}, \tag{4.5.39}$$

so the functional equation (4.5.10) gives the bound

$$|p(s,w)A(s,w,z)| \ll \tilde{p}(s,w)|wz|^{\varepsilon}(1+|s|)^{3+\varepsilon} \tag{4.5.40}$$

in the region $\tilde{S}_3$. Using Proposition 4.12.5 once more gives us the final bound

$$|p(s,w)A(s,w,z)| \ll \tilde{p}(s,w)|wz|^{\varepsilon}(1+|s|)^{3+\varepsilon} \tag{4.5.41}$$

in the convex hull of $\tilde{S}_2$ and $\tilde{S}_3$, which is $\tilde{S}_4$.

Finally, dividing everything by $p(s,w)$, we obtain the following bound valid in $\tilde{S}_4$ and away from the poles of $A(s,w,z)$:

$$|A(s,w,z)| \ll |wz|^{\varepsilon}(1+|s|)^{3+\varepsilon}. \tag{4.5.42}$$

4.5.6 Residue of $A(s,w,z)$ at $s = 3/2 - w$

We have

$$C(s,w,z;\psi,\psi') = \sum_{q\geq 1} \frac{\psi'(q)}{q^s} \cdot D(w, z-w, q; \psi), \tag{4.5.43}$$

and if $\psi = \psi_1$, $q = \square$, and $z - w \neq 0$, $D(w, z-w, q; \psi)$ has a pole at $w = 3/2$. Using the notation from the proof of Lemma 4.5.1, we have

$$D(w,t,q;\psi_1) = \frac{L\left(w-1/2, \left(\frac{4q}{\cdot}\right)\psi_1\right)}{\zeta_{(4q)}(2w-1)} E(w,t,q;\psi_1) P_{p|q}(w,t,q;\psi_1). \tag{4.5.44}$$

We now compute the residue. The residue at $w = 3/2$ of $\frac{L\left(w-1/2, \left(\frac{4q}{\cdot}\right)\psi_1\right)}{\zeta_{(4q)}(2w-1)}$ is

$$\frac{1}{\zeta(2)} \prod_{p|4q} \frac{p}{p+1}. \tag{4.5.45}$$

If p is an odd prime and $p^{2a} \mid\mid q$, then by (4.2.12)

$$P_{p|q}\left(\frac{3}{2}, z-\frac{3}{2}, q; \psi_1\right) = \prod_{\substack{p|q,\\ p \text{ odd}}} \sum_{k=0}^{\infty} \frac{G\left(\left(\frac{\cdot}{p^k}\right), q\right)\psi(p^k) a_{z-\frac{3}{2}}(p^k)}{p^{\frac{3k}{2}}}$$
$$= \prod_{\substack{p|q,\\ p \text{ odd}}} \left(1 + \sum_{k=1}^{a} \frac{G\left(\left(\frac{\cdot}{p^{2k}}\right), q\right) a_{z-\frac{3}{2}}(p^{2k})}{p^{3k}} + \frac{G\left(\left(\frac{\cdot}{p^{2a+1}}\right), q\right) a_{z-\frac{3}{2}}(p^{2a+1})}{p^{3a+\frac{3}{2}}}\right)$$
$$= \prod_{\substack{p|q,\\ p \text{ odd}}} \left(1 + \left(1 - p^{\frac{3}{2}-z}\right) \sum_{k=1}^{a} \frac{\varphi(p^{2k})}{p^{3k}} + \left(1 - p^{\frac{3}{2}-z}\right) \frac{p^{2a}\sqrt{p}}{p^{3a+\frac{3}{2}}}\right)$$
$$= \prod_{\substack{p|q,\\ p \text{ odd}}} \left(1 + \frac{1 - p^{\frac{3}{2}-z}}{p}\right). \tag{4.5.46}$$

Finally, for $q = \square$ and $\psi = \psi_1$, we have

$$E\left(\frac{3}{2}, z-w, q; \psi\right)$$
$$= \frac{1}{L\left(z-\frac{1}{2}, \left(\frac{4q}{\cdot}\right)\right)} \prod_{p} \left(1 + \frac{\left(\frac{4q}{p}\right)^2}{\left(p^{z-\frac{1}{2}} - \left(\frac{4q}{p}\right)\right)\left(p + \left(\frac{4q}{p}\right)\right)}\right) \tag{4.5.47}$$
$$= \frac{1}{\zeta\left(z-\frac{1}{2}\right)} \prod_{p|4q} \left(1 - \frac{1}{p^{z-\frac{1}{2}}}\right)^{-1} \prod_{p\nmid 4q} \left(1 + \frac{1}{\left(p^{z-\frac{1}{2}} - 1\right)(p+1)}\right).$$

If we now denote

$$P(z) = \prod_{p} \left(1 + \frac{1}{\left(p^{z-\frac{1}{2}} - 1\right)(p+1)}\right), \tag{4.5.48}$$

the last expression in (4.5.47) equals

$$\frac{P(z)}{\zeta\left(z-\frac{1}{2}\right)} \prod_{p|4q} \left(1 - \frac{1}{p^{z-\frac{1}{2}}}\right)^{-1} \left(1 + \frac{1}{\left(p^{z-\frac{1}{2}} - 1\right)(p+1)}\right)^{-1}$$
$$= \frac{P(z)}{\zeta\left(z-\frac{1}{2}\right)} \prod_{p|4q} \frac{p^{z-1/2}(p+1)}{\left(p^{z-\frac{1}{2}} - 1\right)(p+1) + 1}, \tag{4.5.49}$$

Putting everything together, we find that

$$\begin{aligned}\mathrm{res}_{w=3/2} C(s,w,z;\psi_1,\psi') &= \frac{P(z)}{\zeta(2)\zeta\left(z-\frac{1}{2}\right)} \sum_{q\geq 1} \frac{\psi'(q^2)}{q^{2s}} \times \\ &\times \prod_{p|4q} \frac{p}{p+1} \cdot \frac{p^{z-1/2}(p+1)}{\left(p^{z-1/2}-1\right)(p+1)+1} \prod_{\substack{p|q,\\ p \text{ odd}}} \frac{p+1-p^{3/2-z}}{p}.\end{aligned} \tag{4.5.50}$$

We have

$$\frac{p}{p+1} \cdot \frac{p^{z-1/2}(p+1)}{(p^{z-1/2}-1)(p+1)+1} \cdot \frac{p+1-p^{3/2-z}}{p} = 1, \tag{4.5.51}$$

so

$$\mathrm{res}_{w=3/2} C(s,w,z;\psi_1,\psi') \ = \frac{P(z)}{\zeta(2)\zeta\left(z-\frac{1}{2}\right)} \cdot \frac{2^{z+1/2}}{3\cdot 2^{z-1/2}-2} \cdot L\left(2s,\psi'^2\right). \tag{4.5.52}$$

Using (4.5.16) gives

$$\begin{aligned}\mathrm{res}_{w=\frac{3}{2}} C(s,w,z) &= 4^{-s}\mathrm{res}_{w=\frac{3}{2}} C(s,w,z;\psi_1,\psi_0) + \mathrm{res}_{w=\frac{3}{2}} C(s,w,z;\psi_1,\psi_{-1}) \\ &= \frac{P(z)}{\zeta(2)\zeta(z-1/2)} \cdot \frac{2^{z+1/2}}{3\cdot 2^{z-1/2}-2} \left(4^{-s}\zeta(2s) + L(2s,\psi_1)\right) \\ &= \frac{P(z)\zeta(2s)}{\zeta(2)\zeta(z-1/2)} \cdot \frac{2^{z+1/2}}{3\cdot 2^{z-1/2}-2}.\end{aligned} \tag{4.5.53}$$

Note that this is also true in the case $w=z$, when there is no pole and the residue is 0.

The functional equation

$$A(s,w,z) = \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{4^s\Gamma\left(\frac{s}{2}\right)} C(1-s,s+w,s+z) \tag{4.5.54}$$

implies that

$$\begin{aligned}&\mathrm{res}_{s=3/2-w} A(s,w,z) \\ &= \frac{\pi^{1-w}\Gamma\left(\frac{w-1/2}{2}\right)}{\Gamma\left(\frac{3/2-w}{2}\right)} \cdot \frac{P(3/2-w+z)\zeta(2w-1)}{\zeta(2)\zeta(1-w+z)} \cdot \frac{2^{z+w-1}}{3\cdot 2^{z+1-w}-2}.\end{aligned} \tag{4.5.55}$$

Substituting $w=1/2+\alpha$, $z=1/2+\beta$, we obtain

$$\begin{aligned}&\mathrm{res}_{s=1-\alpha} A(s,1/2+\alpha,1/2+\beta) \\ &= \frac{\pi^{1/2-\alpha}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \cdot \frac{P(3/2-\alpha+\beta)\zeta(2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)} \cdot \frac{2^{\alpha+\beta}}{3\cdot 2^{1-\alpha+\beta}-2}.\end{aligned} \tag{4.5.56}$$

Using the functional equation

$$\zeta(2\alpha) = \pi^{2\alpha-1/2}\frac{\Gamma\left(\frac{1-2\alpha}{2}\right)}{\Gamma(\alpha)}\zeta(1-2\alpha), \tag{4.5.57}$$

the last expression becomes

$$\frac{\pi^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1-2\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)} \cdot \frac{P(3/2-\alpha+\beta)\zeta(1-2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)} \cdot \frac{2^{\alpha+\beta}}{3\cdot 2^{1-\alpha+\beta}-2}. \tag{4.5.58}$$

Now using the relation (4.2.32) gives

$$\pi^{\alpha-\frac{1}{2}}\cos\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\frac{1}{2}-\alpha\right)\frac{P\left(\frac{3}{2}-\alpha+\beta\right)\zeta(1-2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)} \cdot \frac{2^{\beta}}{3\cdot 2^{-\alpha+\beta}-1}, \tag{4.5.59}$$

and an application of Lemma 4.2.5 yields

$$\pi^{\alpha}\left(\Gamma_o\left(\frac{1}{2}+\alpha\right)+\Gamma_e\left(\frac{1}{2}+\alpha\right)\right)\frac{P\left(\frac{3}{2}-\alpha+\beta\right)\zeta(1-2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)} \cdot \frac{1}{6-2^{1+\alpha-\beta}}. \tag{4.5.60}$$

4.6 Proof of Theorem 4.1.2

We will now prove Theorem 4.1.2. We have

$$\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}\left(\frac{1}{2}+\alpha,\chi_n\right)}{L_{(2)}\left(\frac{1}{2}+\beta,\chi_n\right)} f\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(2)} A\left(s,\frac{1}{2}+\alpha,\frac{1}{2}+\beta\right)\mathcal{M}f(s)X^s ds. \tag{4.6.1}$$

By (4.5.32) and the definition of $\tilde{S}_4$ in (4.5.33), we can shift the integral to $\mathrm{Re}(s) = N(\alpha,\beta)+\varepsilon$, where

$$N(\alpha,\beta) = \max\left\{1-2\mathrm{Re}(\alpha), 1-2\mathrm{Re}(\beta), \frac{1}{2}-\mathrm{Re}(\alpha), \frac{1}{2}-\mathrm{Re}(\beta), -\frac{5}{2}\right\}. \tag{4.6.2}$$

We capture the residues at $s = 1$ (4.5.9) and $s = 1-\alpha$ (4.5.60), and then use (4.5.42) to estimate the error term. Thus if $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > \varepsilon$, we get

$$\begin{aligned}
&\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}(1/2+\alpha,\chi_n)}{L_{(2)}(1/2+\beta,\chi_n)} f(n/X) \\
&\qquad = X\mathcal{M}f(1)\mathrm{res}_{s=1}A\left(s,\frac{1}{2}+\alpha,\frac{1}{2}+\beta\right) \\
&\qquad + X^{1-\alpha}\mathcal{M}f(1-\alpha)\mathrm{res}_{s=1-\alpha}A\left(s,\frac{1}{2}+\alpha,\frac{1}{2}+\beta\right) \\
&\qquad + \frac{1}{2\pi i}\int_{(N(\alpha,\beta)+\varepsilon)} A(s,w,z)\mathcal{M}f(s)X^s ds \\
&\qquad = X\mathcal{M}f(1)\frac{\zeta_{(2)}(1+2\alpha)}{2\zeta_{(2)}(1+\alpha+\beta)}\prod_{p>2}\left(1+\frac{p^{\alpha-\beta}-1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta}-1)}\right) \\
&\qquad + X^{1-\alpha}\mathcal{M}f(1-\alpha)\pi^{\alpha}\left(\Gamma_o\left(\frac{1}{2}+\alpha\right)+\Gamma_e\left(\frac{1}{2}+\alpha\right)\right)\times \\
&\qquad \times\frac{P\left(\frac{3}{2}-\alpha+\beta\right)\zeta(1-2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)(6-2^{\alpha-\beta+1})} \\
&\qquad + O\left((1+|\alpha|)^{\varepsilon}|\beta|^{\varepsilon}X^{N(\alpha,\beta)+\varepsilon}\right).
\end{aligned} \tag{4.6.3}$$

4.7 Proof of Theorem 4.1.3

Fix r with $\mathrm{Re}(r) > \varepsilon$. We let

$$M_1(\alpha,\beta) = \mathcal{M}f(1)\frac{\zeta_{(2)}(1+2\alpha)}{2\zeta_{(2)}(1+\alpha+\beta)}\prod_{p>2}\left(1+\frac{p^{\alpha-\beta}-1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta}-1)}\right), \tag{4.7.1}$$

and

$$\begin{aligned}
M_2(\alpha,\beta) = \mathcal{M}f(1-\alpha)\pi^{\alpha}&\left(\Gamma_o\left(\frac{1}{2}+\alpha\right)+\Gamma_e\left(\frac{1}{2}+\alpha\right)\right)\times \\
&\times\frac{P\left(\frac{3}{2}-\alpha+\beta\right)\zeta(1-2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)(6-2^{\alpha-\beta+1})},
\end{aligned} \tag{4.7.2}$$

so that (4.1.8) is

$$\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L_{(2)}(1/2+\alpha,\chi_n)}{L_{(2)}(1/2+\beta,\chi_n)} f(n/X) = XM_1(\alpha,\beta) + X^{1-\alpha}M_2(\alpha,\beta) + E(X,\alpha,\beta), \tag{4.7.3}$$

where $E(X,\alpha,\beta) \ll X^{N(\alpha,\beta)+\varepsilon}$ is the error term. Note that the left-hand side and both $M_1(\alpha,\beta)$ and $M_2(\alpha,\beta)$ are analytic functions of α,β, so $E(X,\alpha,\beta)$ is analytic too.

To deduce Theorem 4.1.3, we fix $\beta = r$ with $\mathrm{Re}(\beta) > \varepsilon$, differentiate with respect to α, and set $\alpha = \beta = r$.

For the first term, we get

$$\frac{d}{d\alpha} X M_1(\alpha, \beta)\Big|_{\alpha=\beta=r} = \frac{X\mathcal{M}f(1)}{2}\left(\frac{\zeta'_{(2)}(1+2r)}{\zeta_{(2)}(1+2r)} + \sum_{p>2} \frac{\log p}{p(p^{1+2r}-1)}\right). \tag{4.7.4}$$

For the second term, we notice that due to the factor $\frac{1}{\zeta(1-\alpha+\beta)}$, only one term survives. We also note that $P(3/2) = \zeta(2)$, so we get

$$\begin{aligned}&\frac{d}{d\alpha} X^{1-\alpha} M_2(\alpha, \beta)\Big|_{\alpha=\beta=r} \\ &\quad = -X^{1-r}\mathcal{M}f(1-r)\pi^r \left(\Gamma_o\left(\frac{1}{2}+r\right) + \Gamma_e\left(\frac{1}{2}+r\right)\right)\frac{\zeta(1-2r)}{4}.\end{aligned} \tag{4.7.5}$$

Therefore we have

$$\begin{aligned}&\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'_{(2)}(1/2+r,\chi_n)}{L_{(2)}(1/2+r,\chi_n)} f(n/X) \\ &= \frac{X\mathcal{M}f(1)}{2}\left(\frac{\zeta'_{(2)}(1+2r)}{\zeta_{(2)}(1+2r)} + \sum_{p>2} \frac{\log p}{p(p^{1+2r}-1)}\right) \\ &- X^{1-r}\mathcal{M}f(1-r)\pi^r \left(\Gamma_o\left(\frac{1}{2}+r\right) + \Gamma_e\left(\frac{1}{2}+r\right)\right)\frac{\zeta(1-2r)}{4} \\ &+ \frac{d}{d\alpha} E(X,\alpha,\beta)\Big|_{\alpha=\beta=r}.\end{aligned} \tag{4.7.6}$$

Since $E(X, \alpha, \beta)$ is analytic in α, we can use Cauchy's integral formula to compute its derivative. We have

$$\frac{d}{d\alpha} E(X,\alpha,\beta) = \frac{1}{2\pi i}\int_{C_\alpha} \frac{E(X,z,\beta)}{(z-\alpha)^2} dz, \tag{4.7.7}$$

where C_α is a circle centered at α of radius ρ with $\varepsilon/2 < \rho < \varepsilon$. Then

$$\left|\frac{d}{d\alpha} E(X,\alpha,\beta)\right| \ll \frac{1}{\rho}\cdot \max_{z\in C_\alpha} |E(X,z,\beta)| \ll (1+|\alpha|)^\varepsilon |\beta|^\varepsilon X^{N(\alpha,\beta)+\varepsilon}. \tag{4.7.8}$$

Taking $\alpha = \beta = r$ and denoting $N(r) := N(r,r)$ gives

$$\left|\frac{d}{d\alpha} E(X,\alpha,\beta)\right| \ll |r|^\varepsilon X^{N(r)+\varepsilon}. \tag{4.7.9}$$

Now we recover the Euler factors at 2 that we removed from $A(s, w, z)$. For odd n, we have

$$L(1/2+r, \chi_n) = L_{(2)}(1/2+r, \chi_n) \left(1 - \frac{\left(\frac{2}{n}\right)}{2^{1/2+r}}\right)^{-1}, \tag{4.7.10}$$

so

$$\begin{aligned} &\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r, \chi_n)}{L(1/2+r, \chi_n)} f(n/X) \\ &= \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'_{(2)}(1/2+r, \chi_n)}{L_{(2)}(1/2+r, \chi_n)} f(n/X) - \log 2 \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{\left(\frac{2}{n}\right)}{2^{1/2+r} - \left(\frac{2}{n}\right)} f(n/X). \end{aligned} \tag{4.7.11}$$

Since for any $a \in \mathbb{Z}/8\mathbb{Z}$, partial summation gives

$$\sum_{n \equiv a \pmod 8} f(n/X) = \frac{X}{8} \mathcal{M}f(1) + O(1), \tag{4.7.12}$$

we get

$$\begin{aligned} &\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r, \chi_n)}{L(1/2+r, \chi_n)} f(n/X) \\ &= \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'_{(2)}(1/2+r, \chi_n)}{L_{(2)}(1/2+r, \chi_n)} f(n/X) - \frac{X\mathcal{M}f(1)}{2} \cdot \frac{\log 2}{2^{1+2r} - 1} + O(1). \end{aligned} \tag{4.7.13}$$

We also have

$$\frac{\zeta'(1+2r)}{\zeta(1+2r)} = \frac{\zeta'_{(2)}(1+2r)}{\zeta_{(2)}(1+2r)} - \frac{\log 2}{2^{2r+1} - 1}, \tag{4.7.14}$$

so using (4.7.6) and (4.7.14) in (4.7.13), we obtain

$$\begin{aligned} &\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r, \chi_n)}{L(1/2+r, \chi_n)} f(n/X) \\ &= \frac{X\mathcal{M}f(1)}{2} \left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2} \frac{\log p}{p(p^{1+2r} - 1)} \right) \\ &- X^{1-r} \mathcal{M}f(1-r) \pi^r \left(\Gamma_o\left(\frac{1}{2} + r\right) + \Gamma_e\left(\frac{1}{2} + r\right) \right) \frac{\zeta(1-2r)}{4} \\ &+ O\left(1 + |r|^{\varepsilon} X^{N(r)+\varepsilon}\right). \end{aligned} \tag{4.7.15}$$

4.8 Proof of Theorem 4.1.4

In this section, we assume that $\varepsilon < \mathrm{Re}(r) < 1/4$. We write all odd integers n as $n = n_0 n_1^2$ with n_0 square-free. Then we have

$$\begin{aligned}&\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{\mu^2(n) L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} f(n/X)\\ &\qquad = \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(d) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r,\chi_{nd^2})}{L(1/2+r,\chi_{nd^2})} f\left(\frac{nd^2}{X}\right).\end{aligned} \tag{4.8.1}$$

From

$$L(s,\chi_{nd^2}) = L(s,\chi_n) \prod_{p\mid d} \left(1 - \frac{\chi_n(p)}{p^s}\right), \tag{4.8.2}$$

we obtain

$$\frac{L'(1/2+r,\chi_{nd^2})}{L(1/2+r,\chi_{nd^2})} = \frac{L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} + \sum_{p\mid d} \frac{\chi_n(p)\log p}{p^{1/2+r} - \chi_n(p)}, \tag{4.8.3}$$

so (4.8.1) equals

$$\begin{aligned}&\sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(d) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} f\left(\frac{nd^2}{X}\right)\\ &+ \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(d) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} f\left(\frac{nd^2}{X}\right) \sum_{p\mid d} \frac{\chi_n(p)\log p}{p^{1/2+r} - \chi_n(p)}.\end{aligned} \tag{4.8.4}$$

Let us now consider the first sum. The terms with $d > \sqrt{X}$ here contribute $\ll |r|^{\varepsilon} X^{1/2}$ by (4.2.20). For the other terms, we use (4.1.11) (with X/d^2 instead of X). Writing the right-hand side of (4.1.11) as $XM_1(r) - X^{1-r}M_2(r) + E(X,r)$, this part of (4.8.4) contributes

$$\begin{aligned}&XM_1(r) \sum_{\substack{d\leq\sqrt{X},\\ d \text{ odd}}} \frac{\mu(d)}{d^2} - X^{1-r} M_2(r) \sum_{\substack{d\leq\sqrt{X},\\ d \text{ odd}}} \frac{\mu(d)}{d^{2-2r}} + \sum_{\substack{d\leq\sqrt{X},\\ d \text{ odd}}} \mu(d) E\left(\frac{X}{d^2}, r\right)\\ &= \frac{2XM_1(r)}{3\zeta(2)} - \frac{X^{1-r}M_2(r)}{\zeta_{(2)}(2-2r)} + O(|r|^{\varepsilon} X^{1-2r+\varepsilon}).\end{aligned} \tag{4.8.5}$$

Now we compute the second term in (4.8.4). Setting $s = \frac{1}{2} + r$ and exchanging summations gives

$$\begin{aligned}
&\sum_{p \text{ odd}} \frac{\log p}{p^s} \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(pd) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{\chi_n(p)}{1-\chi_n(p)/p^s} f\left(\frac{nd^2p^2}{X}\right) \\
&= \sum_{p \text{ odd}} \frac{\log p}{p^s} \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(pd) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \chi_n(p) f\left(\frac{nd^2p^2}{X}\right) \sum_{k=0}^{\infty} \frac{\chi_n(p)^k}{p^{ks}}.
\end{aligned} \tag{4.8.6}$$

We split the last sum depending on the parity of k.

For an even k, the sum equals

$$\sum_{p \text{ odd}} \frac{\log p}{p^{(k+1)s}} \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(pd) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \chi_n(p) f\left(\frac{nd^2p^2}{X}\right). \tag{4.8.7}$$

If we denote

$$\tilde{\chi}_p = \begin{cases} \chi_p & \text{if } p \equiv 1 \pmod 4, \\ \chi_p \psi_{-1} & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{4.8.8}$$

the inner sum is

$$\frac{1}{2\pi i} \int_{(c)} \frac{L_{(2)}(u, \tilde{\chi}_p) \mathcal{M}f(u) X^u}{d^{2u} p^{2u}} du \ll \frac{X^{1/2+\varepsilon}}{d^{1+2\varepsilon} p^{1+\varepsilon}} \tag{4.8.9}$$

where we shifted the integral to $c = 1/2 + \varepsilon$ and used Lindelöf's bound (4.2.19).

Using (4.8.9) and summing (4.8.7) over all even $k \geq 0$, we see that the contribution of even k in (4.8.6) is

$$\ll X^{1/2+\varepsilon} \sum_{k=0}^{\infty} \sum_{p\geq 1} \frac{\log p}{p^{(2k+1)s+1+\varepsilon}} \sum_{d\geq 1} \frac{1}{d^{1+2\varepsilon}} \ll X^{1/2+\varepsilon}. \tag{4.8.10}$$

Now we compute the contribution of odd values of k into (4.8.6). For an odd k, the summand equals

$$\begin{aligned}
&\sum_{p>2} \frac{\log p}{p^s} \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(pd) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \frac{\chi_n(p)^{k+1}}{p^{ks}} f\left(\frac{nd^2p^2}{X}\right) \\
&= \sum_{p>2} \frac{\log p}{p^{(k+1)s}} \sum_{\substack{d\geq 1,\\ d \text{ odd}}} \mu(pd) \sum_{\substack{n\geq 1,\\ n \text{ odd}' \\ p\nmid n}} f\left(\frac{nd^2p^2}{X}\right).
\end{aligned} \tag{4.8.11}$$

We write the innermost sum as a Mellin integral:

$$\sum_{\substack{n\geq 1,\\ n \text{ odd},\\ p\nmid n}} f\left(\frac{nd^2p^2}{X}\right) = \frac{1}{2\pi i}\int_{(c)} \frac{\zeta_{(2p)}(u)X^u\mathcal{M}f(u)}{d^{2u}p^{2u}}du$$
$$= \frac{X\mathcal{M}f(1)(1-1/p)}{2d^2p^2} + O\left(\frac{X^{1/2+\varepsilon}}{d^{1+2\varepsilon}p^{1+2\varepsilon}}\right), \tag{4.8.12}$$

where the last equation holds after shifting the integral to $c = 1/2+\varepsilon$. Summing the error term over all odd k, d and p contributes $\ll X^{1/2+\varepsilon}$, so it remains to compute the contribution of the main term. This gives

$$\frac{X\mathcal{M}f(1)}{2}\sum_{p>2}\frac{\left(1-\frac{1}{p}\right)\log p}{p^{2+(k+1)s}}\sum_{\substack{d\geq 1,\\ d \text{ odd}}}\frac{\mu(pd)}{d^2} = \frac{-2X\mathcal{M}f(1)}{3\zeta(2)}\sum_{p>2}\frac{\log p}{(p+1)p^{1+(k+1)s}}. \tag{4.8.13}$$

Summing this over odd values of k gives

$$\frac{-2X\mathcal{M}f(1)}{3\zeta(2)}\sum_{p>2}\frac{\log p}{(p+1)p^{1+s}}\sum_{k=0}^{\infty}\frac{1}{p^{(2k+1)s}} = \frac{-2X\mathcal{M}f(1)}{3\zeta(2)}\sum_{p>2}\frac{\log p}{p(p+1)(p^{2s}-1)} \tag{4.8.14}$$

Putting all together, we obtain the final result

$$\sum_{\substack{n\geq 1,\\ n \text{ odd}}}\frac{\mu^2(n)L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)}f(n/X)$$
$$= \frac{2X\mathcal{M}f(1)}{3\zeta(2)}\left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2}\frac{\log p}{(p+1)(p^{1+2r}-1)}\right) \tag{4.8.15}$$
$$- X^{1-r}\mathcal{M}f(1-r)\pi^r\left(\Gamma_o\left(\frac{1}{2}+r\right)+\Gamma_e\left(\frac{1}{2}+r\right)\right)\frac{\zeta(1-2r)}{4\zeta_{(2)}(2-2r)}$$
$$+ O\left(|r|^{\varepsilon}X^{1-2r+\varepsilon}\right).$$

4.9 Proof of Corollary 4.1.5

In this section, we show how the one-level density can be computed using the ratios conjecture. We follow Section 3 of [CoSn].

Let $h(x)$ be an even Schwartz function whose Fourier transform $\hat{h}$ is supported in the interval $[-a, a]$ for some $a > 0$. It follows that h has an analytic continuation to the whole $\mathbb{C}$ via Fourier inversion.

Recall that the one-level density is defined by

$$D(X;h) = \frac{1}{F(X)} \sum_{\substack{n\geq 1\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right) \sum_{\gamma_n} h\left(\frac{\gamma_n \log X}{2\pi}\right), \tag{4.9.1}$$

where γ_n runs over the imaginary parts of the non-trivial zeros of $L(s,\chi_n)$, and

$$\begin{aligned} F(X) &= \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{(c)} \frac{\zeta_{(2)}(s)}{\zeta_{(2)}(2s)} X^s \mathcal{M}f(s)ds \\ &= \frac{2X\mathcal{M}f(1)}{3\zeta(2)} + O(X^{1/4+\varepsilon}). \end{aligned} \tag{4.9.2}$$

By the residue theorem, we have

$$\begin{aligned} &F(X)D(X;h) \\ &= \frac{1}{2\pi i}\left(\int_{(c)} - \int_{(1-c)}\right) h\left(\frac{\log X}{2\pi i}(s-1/2)\right) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f(n/X) \frac{L'(s,\chi_n)}{L(s,\chi_n)} ds \end{aligned} \tag{4.9.3}$$

for any $1/2 < c < 1$.

Using the functional equation (4.2.13) and the fact that h is even, the integral over $(1-c)$ equals

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(c)} h\left(\frac{\log X}{2\pi i}\left(s-\frac{1}{2}\right)\right) \times \\ &\times \left(\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X}\right)\left(\log\left(\frac{\pi}{n}\right) + \frac{\Gamma_n'}{\Gamma_n}(s) - \frac{L'}{L}(s,\chi_n)\right)\right) ds, \end{aligned} \tag{4.9.4}$$

where $\Gamma_n(s)$ is $\Gamma_e(s)$ or $\Gamma_o(s)$, depending on the parity of χ_n.

Hence

$$\begin{aligned} &F(X)D(X;h) \\ &= \frac{1}{\pi i}\int_{(c)} h\left(\frac{\log X}{2\pi i}(s-1/2)\right) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f(n/X) \frac{L'(s,\chi_n)}{L(s,\chi_n)} ds \\ &- \frac{1}{2\pi i}\int_{(c)} h\left(\frac{\log X}{2\pi i}(s-1/2)\right) \sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n) f(n/X) \left(\log\left(\frac{\pi}{n}\right) + \frac{\Gamma_n'(s)}{\Gamma_n(s)}\right) ds. \end{aligned} \tag{4.9.5}$$

We now consider the first integral. We start with the following lemma:

Lemma 4.9.1. *For h as above, we have*

$$h\left(\frac{s\log X}{2\pi i}\right) \ll \frac{X^{a\cdot\mathrm{Re}(s)}}{|s|^2(\log X)^2}. \tag{4.9.6}$$

Proof. Using Fourier inversion and integrating by parts twice, we obtain

$$\begin{aligned} h\left(\frac{s\log X}{2\pi i}\right) &= \int_{-\infty}^{\infty} \hat{h}(t)e^{ts\log X}dt \\ &= \frac{1}{s^2(\log X)^2}\int_{-\infty}^{\infty} \hat{h}''(t)X^{ts}dt \\ &\ll \frac{X^{a\cdot\mathrm{Re}(s)}}{|s|^2(\log X)^2}. \end{aligned} \tag{4.9.7}$$

□

We now split the integral, writing

$$\frac{1}{\pi i}\int_{(c)} h\left(\frac{\log X}{2\pi i}(s-1/2)\right)\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n)f(n/X)\frac{L'(s,\chi_n)}{L(s,\chi_n)}ds =: I_1 + I_2, \tag{4.9.8}$$

where I_1 is the part of the integral with $|\mathrm{Im}(s)| \leq X$, and I_2 the part with $|\mathrm{Im}(s)| > X$.

For I_2, we set $c = 1/2+\varepsilon$, use Lemma 4.9.1 and (4.2.20), and obtain the bound

$$\begin{aligned} &\sum_{\substack{n\geq 1,\\ n \text{ odd}}} \mu^2(n)f\left(\frac{n}{X}\right)\log^2(n)\int_{\substack{\mathrm{Re}(s)=c\\ |\mathrm{Im}(s)|>X}} \log^2(|s|)\left|h\left(\frac{\log X}{2\pi i}(s-1/2)\right)\right|ds \\ &\ll X^{1+\varepsilon+a\varepsilon}\int_{|t|>X}\frac{\log^2(|t|)}{t^2}dt \ll X^{\varepsilon}. \end{aligned} \tag{4.9.9}$$

For I_1, we substitute $s = 1/2+r+it$ with $0 < r < 1/4$ and use Theorem 4.1.4. Thus I_1 is

$$\begin{aligned} &\frac{X}{\pi}\int_{-X}^{X} h\left(\frac{\log X}{2\pi i}(r+it)\right)\left\{\frac{2\mathcal{M}f(1)}{3\zeta(2)}\left(\frac{\zeta'(1+2r+2it)}{\zeta(1+2r+2it)} + \sum_{p>2}\frac{\log p}{(p+1)(p^{1+2r+2it}-1)}\right)\right. \\ &-\frac{\mathcal{M}f(1-r-it)\pi^{r+it}\left(\Gamma_o\left(\frac{1}{2}+r+it\right)+\Gamma_e\left(\frac{1}{2}+r+it\right)\right)\zeta(1-2r-2it)}{X^{r+it}4\zeta_{(2)}(2-2r-2it)} \\ &\left.+O(|t|^{\varepsilon}X^{1-2r+\varepsilon})\right\}dt. \end{aligned} \tag{4.9.10}$$

Using Lemma 4.9.1, we can bound the error term by

$$X^{1-2r+\varepsilon}\int_{-X}^{X} h\left(\frac{\log X}{2\pi i}(r+it)\right)|t|^{\varepsilon}dt \ll X^{1-2r+ar+\varepsilon}, \tag{4.9.11}$$

and this is $o(X)$ as long as $a < 2$. Setting $r = 1/4 - \varepsilon$ gives the error term in Corollary 4.1.5.

We remark here that this is the only restriction on a, so any improvement of the error term in Theorem 4.1.4 would allow us to extend the support of the Fourier transform of our test functions.

Now we compute the main terms of (4.9.10). Since the integrand is regular at $r + it = 0$, we can shift the line of integration to $r = 0$. We extend the range of integration from $-\infty$ to ∞ introducing an error of size X^{ε}, and get

$$\begin{aligned} &\frac{X}{\pi}\int_{-\infty}^{\infty} h\left(\frac{\log X}{2\pi}t\right)\left\{\frac{2\mathcal{M}f(1)}{3\zeta(2)}\left(\frac{\zeta'(1+2it)}{\zeta(1+2it)}\right) + \sum_{p>2}\frac{\log p}{(p+1)(p^{1+2it}-1)}\right.\\ &\left.\quad - \frac{X^{-it}\mathcal{M}f(1-it)\pi^{it}\left(\Gamma_o(1/2+it)+\Gamma_e(1/2+it)\right)\zeta(1-2it)}{4\zeta_{(2)}(2-2it)}\right\}dt\\ &= \frac{2X}{\log X}\int_{-\infty}^{\infty} h(u)\left\{\frac{2\mathcal{M}f(1)}{3\zeta(2)}\left(\frac{\zeta'}{\zeta}\left(1+\frac{4\pi iu}{\log X}\right) + \sum_{p>2}\frac{\log p}{(p+1)(p^{1+\frac{4\pi iu}{\log X}}-1)}\right)\right.\\ &\left.\quad - \frac{\mathcal{M}f\left(1-\frac{2\pi iu}{\log X}\right)\pi^{\frac{2\pi iu}{\log X}}\left(\Gamma_o\left(\frac{1}{2}+\frac{2\pi iu}{\log X}\right)+\Gamma_e\left(\frac{1}{2}+\frac{2\pi iu}{\log X}\right)\right)\zeta\left(1-\frac{4\pi iu}{\log X}\right)}{e^{2\pi iu}\cdot 4\zeta_{(2)}\left(2-\frac{4\pi iu}{\log X}\right)}\right\}du. \end{aligned} \tag{4.9.12}$$

We now compute the second integral in (4.9.5). We shift the line of integration to $c = 1/2$ and substitute $\frac{\log X}{2\pi i}\left(s - \frac{1}{2}\right) = u$, getting

$$\frac{1}{\log X}\int_{-\infty}^{\infty} h(u)\sum_{\substack{n\geq 1,\\ n \text{ odd}}}\mu^2(n)f\left(\frac{n}{X}\right)\left(\log\left(\frac{\pi}{n}\right) - \frac{\Gamma_n'}{\Gamma_n}\left(\frac{1}{2}+\frac{2\pi iu}{\log X}\right)\right)du. \tag{4.9.13}$$

The first part of Corollary 4.1.5 now follows from (4.9.11) with $r = 1/4 - \varepsilon$, (4.9.12), (4.9.13) and noticing that a similar computation as in Lemma 4.11.1 gives

$$\begin{aligned} &\sum_{\substack{n\geq 1,\\ n \text{ odd}}}\frac{\Gamma_n'(s)}{\Gamma_n(s)}\mu^2(n)f(n/X)\\ &= \frac{\Gamma_e'(s)}{\Gamma_e(s)}\sum_{n\equiv 1 \pmod 4}\mu^2(n)f(n/X) + \frac{\Gamma_o'(s)}{\Gamma_o(s)}\sum_{n\equiv 3 \pmod 4}\mu^2(n)f(n/X)\\ &= \left(\frac{\Gamma_e'(s)}{\Gamma_e(s)} + \frac{\Gamma_o'(s)}{\Gamma_o(s)}\right)\left(\frac{X\mathcal{M}f(1)}{3\zeta(2)} + O(\sqrt{X})\right). \end{aligned} \tag{4.9.14}$$

To prove the second part, we use the Laurent expansions $\zeta(s) = \frac{1}{s-1} + \ldots$, and

$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \ldots$, so (4.9.12) is

$$\frac{2X\mathcal{M}f(1)}{3\zeta(2)} \int_{-\infty}^{\infty} h(u) \left(\frac{e^{-2\pi iu} - 1}{2\pi iu} \right) du + O\left(\frac{X}{\log X} \right). \tag{4.9.15}$$

We have

$$\sum_{\substack{n \geq 1, \\ n \text{ odd}}} \mu^2(n) f\left(\frac{n}{X} \right) \log\left(\frac{\pi}{n} \right) = \frac{-2\mathcal{M}f(1) X \log X}{3\zeta(2)} + O(X), \tag{4.9.16}$$

so by (4.9.14), (4.9.13) is

$$-\frac{2X\mathcal{M}f(1)}{3\zeta(2)} \int_{-\infty}^{\infty} h(u) du + O\left(\frac{X}{\log X} \right). \tag{4.9.17}$$

Altogether, we obtain

$$D(X; h) = \int_{-\infty}^{\infty} h(u) \left(1 + \frac{e^{-2\pi iu} - 1}{2\pi iu} \right) du + O\left(\frac{1}{\log X} \right). \tag{4.9.18}$$

Finally, to see that this result agrees with the density conjecture of Katz and Sarnak, we use the fact that $h(u)$ is even. We can thus drop the last term in the integral and replace $\frac{e^{-2\pi iu}}{2\pi iu}$ by

$$\frac{1}{2} \left(\frac{e^{-2\pi iu} - e^{2\pi iu}}{2\pi iu} \right) = \frac{-\sin(2\pi u)}{2\pi u}, \tag{4.9.19}$$

which completes the proof.

4.10 Proof of the functional equation from Proposition 4.2.3

We now give the proof of the functional equation in Proposition 4.2.3. We only give the details in the case of even characters, and then explain the usual modification for odd characters. For a detailed proof of the classical functional equation, see for example [Gar].

Let χ be an even character modulo q. We define two theta functions

$$\theta_\chi(y) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 y}, \tag{4.10.1}$$

and

$$\theta_{\tau(\chi)}(y) = \sum_{n\in\mathbb{Z}} \tau(\chi, n) e^{-\pi n^2 y}. \tag{4.10.2}$$

Then using the fact that $\chi(n)$ is even, we have the integral representation of the L-function:

$$\int_0^\infty y^{\frac{s}{2}} \frac{\theta_\chi(y)}{2} \frac{dy}{y} = \sum_{n\geq 1} \chi(n) \int_0^\infty y^{\frac{s}{2}} e^{-\pi n^2 y} \frac{dy}{y} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi), \tag{4.10.3}$$

and similarly

$$\int_0^\infty y^{\frac{s}{2}} \frac{\theta_{\tau(\chi)}(y)}{2} \frac{dy}{y} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) K(s, \chi). \tag{4.10.4}$$

These integrals converge absolutely for s with $\mathrm{Re}(s) > 0$.

The two theta functions are related by the following functional equation, which follows after an application of the Poisson summation.

Lemma 4.10.1. *Let χ be a character modulo q. Then*

$$\theta_\chi(y) = \frac{1}{q\sqrt{y}} \theta_{\tau(\chi)}\left(\frac{1}{yq^2}\right). \tag{4.10.5}$$

Proof. We have

$$\theta_\chi(y) = \sum_{n\in\mathbb{Z}} \chi(n) e^{-\pi n^2 y} = \sum_{j \pmod q} \chi(j) \sum_{n\in\mathbb{Z}} e^{-\pi(qn+j)^2 y}. \tag{4.10.6}$$

The Fourier transform of the function in the inner sum is

$$\begin{aligned} \int_{-\infty}^\infty e^{-\pi(qt+j)^2 y} e^{-2\pi i t n} dt &= \frac{1}{q\sqrt{y}} \int_{-\infty}^\infty e^{-\pi u^2} e^{-2\pi i n\left(\frac{u\sqrt{y}}{q} - \frac{j}{q}\right)} du \\ &= \frac{e^{2\pi i j n/q}}{q\sqrt{y}} \cdot e^{\frac{-\pi n^2}{yq^2}}, \end{aligned} \tag{4.10.7}$$

so Poisson summation gives

$$\begin{aligned} \theta\chi(y) &= \frac{1}{q\sqrt{y}} \sum_{j \pmod q} \chi(j) \sum_{n\in\mathbb{Z}} e^{2\pi i j n/q} \cdot e^{\frac{-\pi n^2}{yq^2}} \\ &= \frac{1}{q\sqrt{y}} \sum_{n\in\mathbb{Z}} e^{\frac{-\pi n^2}{yq^2}} \sum_{j \pmod q} \chi(j) e^{2\pi i j n/q} \\ &= \frac{1}{q\sqrt{y}} \sum_{n\in\mathbb{Z}} \tau(\chi, n) e^{\frac{-\pi n^2}{yq^2}} \\ &= \frac{1}{q\sqrt{y}} \theta_{\tau(\chi)}\left(\frac{1}{yq^2}\right). \end{aligned} \tag{4.10.8}$$

□

Substituting $y \mapsto \frac{1}{yq^2}$ and using relation (4.10.5), we obtain

$$\begin{aligned} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) L(s,\chi) &= \int_0^\infty y^{\frac{s}{2}} \frac{\theta_\chi(y)}{2}\frac{dy}{y} \\ &= \int_0^\infty \left(\frac{1}{yq^2}\right)^{s/2} \frac{\theta_\chi\left(\frac{1}{yq^2}\right)}{2}\frac{dy}{y} \\ &= q^{-s}\int_0^\infty y^{\frac{1-s}{2}} \frac{\theta_{\tau(\chi)}(y)}{2}\frac{dy}{y} \\ &= \frac{\pi^{\frac{s-1}{2}}}{q^s}\Gamma\left(\frac{1-s}{2}\right) K(1-s,\chi), \end{aligned} \tag{4.10.9}$$

which holds for s with $0 < \mathrm{Re}(s) < 1$. However, since the left-hand side has a meromorphic continuation to the whole complex plane, the functional equation holds for all $s \in \mathbb{C}$. This finishes the proof for even characters.

For odd characters, the functions $\theta_\chi(y), \theta_{\tau(\chi)}(y)$ are identically 0, so we work with the following functions instead:

$$\tilde{\theta}_\chi(y) = \sum_{n\in\mathbb{Z}} n\chi(n)e^{-\pi n^2 y}, \tag{4.10.10}$$

and

$$\tilde{\theta}_\chi(y) = \sum_{n\in\mathbb{Z}} n\tau(\chi,n)e^{-\pi n^2 y}. \tag{4.10.11}$$

The sign in the resulting functional equation comes from the fact that the function $f(x) = xe^{-\pi x^2}$ is an eigenfunction of the Fourier transform with eigenvalue $-i$.

4.11 Ratios conjecture predictions for the family $\left(\frac{\cdot}{n}\right)$ for odd square-free n

In this section, we follow the recipe of Conrey, Farmer and Zirnbauer for the family of Dirichlet characters $\chi_n = \left(\frac{\cdot}{n}\right)$ with n odd and square-free, which is slightly different from the family of $\left(\frac{d}{\cdot}\right)$ for positive fundamental discriminants d usually considered in the literature. In our case, the characters χ_n are primitive and even if $n \equiv 1 \pmod 4$, or odd if $n \equiv 3 \pmod 4$. We therefore split the family according to the parity of the character as

$$\begin{aligned}\sum_{\substack{n\le X,\\ n \text{ odd}}} \frac{\mu^2(n)L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)} \\ = \sum_{\substack{n\le X,\\ n\equiv 1 \ (\text{mod } 4)}} \frac{\mu^2(n)L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)} + \sum_{\substack{n\le X,\\ n\equiv 3 \ (\text{mod } 4)}} \frac{\mu^2(n)L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)}.\end{aligned} \tag{4.11.1}$$

The approximate functional equation is

$$L(1/2+\alpha,\chi_n) \approx \sum_{m\le x} \frac{\chi_n(m)}{m^{1/2+\alpha}} + \left(\frac{n}{\pi}\right)^{-\alpha} \Gamma_{e/o}(1/2+\alpha) \sum_{m\le y} \frac{\chi_n(m)}{m^{1/2-\alpha}}, \tag{4.11.2}$$

where $xy \approx n/2\pi$, and $G_{e/o}$ is Γ_e or Γ_o, depending on the parity of χ_n.

We now consider the first sum in (4.11.1). We write the numerator using the approximate functional equation and the denominator as a Dirichlet series. The first part of the approximate functional equation then gives

$$\sum_{\substack{n\le X,\\ n\equiv 1 \ (\text{mod } 4)}} \mu^2(n) \sum_{m\le x} \frac{\chi_n(m)}{m^{1/2+\alpha}} \sum_{k\ge 1} \frac{\mu(k)\chi_n(k)}{k^{1/2+\beta}}. \tag{4.11.3}$$

We extend the sum over m all the way to infinity, and assume that the main contribution comes from the diagonal terms, which is computed in the following lemma:

Lemma 4.11.1. *For $b = 1$ or 3, we have*

$$\sum_{\substack{n\le X,\\ n\equiv b \ (\text{mod } 4)}} \mu^2(n)\chi_n(\ell) = \begin{cases} \frac{X}{2\zeta(2)} a(4\ell) + \textit{small} & \textit{if } \ell = \square, \\ \textit{small} & \textit{if } \ell \ne \square, \end{cases} \tag{4.11.4}$$

where

$$a(k) = \prod_{p|k} \frac{p}{p+1}. \tag{4.11.5}$$

Proof. Since n runs over odd integers, we may replace ℓ by 4ℓ. Then we have

$$\sum_{\substack{n\le X,\\ n\equiv b \ (\text{mod } 4)}} \mu^2(n)\chi_n(4\ell) = \sum_{n\le X} \mu^2(n)\chi_n(4\ell) \frac{\psi_1(n) \pm \psi_{-1}(n)}{2} \tag{4.11.6}$$

for an appropriate sign depending on b. By Perron's formula,

$$\sum_{n\le X} \mu^2(n)\chi_n(4\ell)\psi(n) = \frac{1}{2\pi i} \int_{(c)} A\left(s, \left(\frac{4\ell}{\cdot}\right)\psi\right) \frac{X^s}{s} ds, \tag{4.11.7}$$

where

$$A\left(s,\left(\frac{4\ell}{\cdot}\right)\psi\right)=\sum_{n=1}^{\infty}\frac{\mu^2(n)\left(\frac{4\ell}{n}\right)\psi(n)}{n^s}=\frac{L\left(s,\left(\frac{4\ell}{\cdot}\right)\psi\right)}{L\left(2s,\left(\frac{4\ell}{\cdot}\right)^2\psi^2\right)}. \tag{4.11.8}$$

$A\left(s,\left(\frac{4\ell}{\cdot}\right)\psi\right)$ has a pole at $s=1$ only if $\ell=\square$ and $\psi=\psi_1$ is the principal character, in which case we have

$$A\left(s,\left(\frac{4\ell}{\cdot}\right)\psi\right)=\frac{\zeta(s)}{\zeta(2s)}\prod_{p|4\ell}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}=\frac{\zeta(s)}{\zeta(2s)}a(4\ell). \tag{4.11.9}$$

Therefore the residue at $s=1$ is $\frac{a(4\ell)}{\zeta(2)}$, which gives the main term

$$\frac{X}{2\zeta(2)}a(4\ell). \tag{4.11.10}$$

□

By Lemma 4.11.1, the main term of (4.11.3) should be

$$\frac{X}{2\zeta(2)}\sum_{mk=\square}\frac{\mu(k)a(4mk)}{m^{1/2+\alpha}k^{1/2+\beta}}. \tag{4.11.11}$$

We split the sum depending on the parity of mk and expand into an Euler product:

$$\begin{aligned}\sum_{mk=\square}\frac{\mu(k)a(4mk)}{m^{1/2+\alpha}k^{1/2+\beta}}&=\frac{2}{3}\sum_{\substack{mk=\square,\\ mk\text{ odd}}}\frac{\mu(k)a(mk)}{m^{1/2+\alpha}k^{1/2+\beta}}+\sum_{\substack{mk=\square,\\ mk\text{ even}}}\frac{\mu(k)a(mk)}{m^{1/2+\alpha}k^{1/2+\beta}}\\
&=\frac{2}{3}\prod_{p>2}\sum_{m+k\text{ even}}\frac{\mu(p^k)a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}\\
&+\left(\sum_{\substack{m+k\text{ even},\\ m+k\geq 1}}\frac{\mu(2^k)a(2^{m+k})}{2^{m(1/2+\alpha)+k(1/2+\beta)}}\right)\prod_{p>2}\sum_{m+k\text{ even}}\frac{\mu(p^k)a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}\\
&=\frac{2}{3}\left(1+\sum_{\substack{m+k\text{ even},\\ m+k\geq 1}}\frac{\mu(2^k)}{2^{m(1/2+\alpha)+k(1/2+\beta)}}\right)\prod_{p>2}\sum_{m+k\text{ even}}\frac{\mu(p^k)a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}.\end{aligned} \tag{4.11.12}$$

The product over $p>2$ is the same as in the ratios conjecture for fundamental discriminants, so using (2.25) in [CoSn], it equals

$$\frac{2}{3}\cdot\frac{\zeta_{(2)}(1+2\alpha)}{\zeta_{(2)}(1+\alpha+\beta)}\cdot P_{D,2}(\alpha,\beta), \tag{4.11.13}$$

where

$$P_{D,2}(\alpha,\beta) = \prod_{p>2}\left(1 + \frac{p^{\alpha-\beta}-1}{p^{\alpha-\beta}(p+1)(p^{1+\alpha+\beta}-1)}\right) \tag{4.11.14}$$

The remaining factor in (4.11.12) is

$$1 + \sum_{\substack{m\geq 1,\\ m \text{ even}}} \frac{1}{2^{m(1/2+\alpha)}} - \sum_{\substack{m\geq 0,\\ m \text{ odd}}} \frac{1}{2^{m(1/2+\alpha)+1/2+\beta}} = \left(1 - \frac{1}{2^{1+\alpha+\beta}}\right)\left(1 - \frac{1}{2^{1+2\alpha}}\right)^{-1}, \tag{4.11.15}$$

so it recovers the missing factors in zeta's in (4.11.13). Therefore the first main term for $n \equiv 1 \pmod 4$ is

$$\frac{X}{3\zeta(2)}\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\beta)}P_{D,2}(\alpha,\beta). \tag{4.11.16}$$

The second part of the functional equation contributes

$$\Gamma_e(1/2+\alpha)\sum_{n\leq X}\mu^2(n)\left(\frac{n}{\pi}\right)^{-\alpha}\sum_{mk=\square}\frac{\mu(k)a(4mk)}{m^{1/2-\alpha}k^{1/2+\beta}}. \tag{4.11.17}$$

The inner sum is similar as above with α replaced by $-\alpha$, so this term gives

$$\frac{X^{1-\alpha}}{(1-\alpha)3\zeta(2)}\cdot\frac{\pi^\alpha\Gamma_e(1/2+\alpha)\zeta(1-2\alpha)}{\zeta(1-\alpha+\beta)}P_{D,2}(-\alpha,\beta). \tag{4.11.18}$$

Finally, the computation for $n \equiv 3 \pmod 4$ will be similar with Γ_e replaced by Γ_o in the second main term, so we obtain

Conjecture 4.11.2. *Let* $-1/4 < \mathrm{Re}(\alpha) < 1/4$, $\frac{1}{\log X} \ll \mathrm{Re}(\beta) < 1/4$, *and* $\mathrm{Im}(\alpha), \mathrm{Im}(\beta) \ll X^{1-\varepsilon}$. *Then*

$$\begin{aligned}\sum_{\substack{n\leq X,\\ n \text{ odd}}}\frac{\mu^2(n)L(1/2+\alpha,\chi_n)}{L(1/2+\beta,\chi_n)} &= \frac{2X}{3\zeta(2)}\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\beta)}P_{D,2}(\alpha,\beta)\\ &+ \frac{X^{1-\alpha}\pi^\alpha\left(\Gamma_e\left(\frac{1}{2}+\alpha\right)+\Gamma_o\left(\frac{1}{2}+\alpha\right)\right)\zeta(1-2\alpha)}{(1-\alpha)3\zeta(2)\zeta(1-\alpha+\beta)}P_{D,2}(-\alpha,\beta)\\ &+ O(X^{1/2+\varepsilon}).\end{aligned} \tag{4.11.19}$$

To obtain an asymptotic for the sum of logarithmic derivatives, we differentiate with respect to α, and set $\alpha = \beta = r$.

For the first term, we have

$$\frac{d}{d\alpha}\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\beta)}P_{D,2}(\alpha,\beta)\bigg|_{\alpha=\beta=r} = \frac{\zeta'(1+2r)}{\zeta(1+2r)} + P'_{D,2}(r,r), \tag{4.11.20}$$

where we noted that $P_{D,2}(r,r) = 1$, and

$$P'_{D,2}(r,r) = \sum_{p>2} \frac{\log p}{(p+1)(p^{1+2r}-1)}, \tag{4.11.21}$$

so the contribution of the first term is

$$\frac{2X}{3\zeta(2)} \left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2} \frac{\log p}{(p+1)(p^{1+2r}-1)} \right). \tag{4.11.22}$$

For the second term, we notice that only one term in the derivative survives due to the factor $\frac{1}{\zeta(1-\alpha+\beta)}$, so it equals

$$-\frac{X^{1-r}\pi^r \left(\Gamma_e\left(\frac{1}{2}+r\right) + \Gamma_o\left(\frac{1}{2}+r\right)\right)\zeta(1-2r)}{(1-r)3\zeta(2)} P_{D,2}(-r,r). \tag{4.11.23}$$

We are thus led to the following conjecture:

Conjecture 4.11.3. *For* $\frac{1}{\log X} \ll \mathrm{Re}(r) \ll \frac{1}{4}$, $\mathrm{Im}(r) \ll X^{1-\varepsilon}$, *we have*

$$\begin{aligned} &\sum_{n\le X} \frac{\mu^2(n) L'(1/2+r,\chi_n)}{L(1/2+r,\chi_n)} \\ &= \frac{2X}{3\zeta(2)} \left(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p>2} \frac{\log p}{(p+1)(p^{1+2r}-1)} \right) \\ &- \frac{X^{1-r}\pi^r \left(\Gamma_e\left(\frac{1}{2}+r\right) + \Gamma_o\left(\frac{1}{2}+r\right)\right)\zeta(1-2r)}{(1-r)3\zeta(2)} P_{D,2}(-r,r) \\ &+ O(X^{1/2+\varepsilon}). \end{aligned} \tag{4.11.24}$$

The two main terms match our result in Theorem 4.1.4, after using $\mathcal{M}f(s) \approx 1/s$ and noticing that

$$\begin{aligned} \frac{P_{D,2}(-r,r)}{3\zeta(2)} &= \frac{1}{3\zeta(2)} \prod_{p>2} \left(1 + \frac{1-p^{2r}}{(p+1)(p-1)}\right) \\ &= \frac{1}{3\zeta(2)} \prod_{p>2} \left(1-1/p^2\right)^{-1} \left(1-p^{2r-2}\right) \\ &= \frac{1}{4\zeta_{(2)}(2-2r)} \end{aligned} \tag{4.11.25}$$

4.12 Multivariable complex analysis

A general reference for the theory of multivariable complex analysis is [Hör].

Definition 4.12.1. *An open set $R \subset \mathbb{C}^n$ is a domain of holomorphy if there are no open sets $R_1, R_2 \subset \mathbb{C}^n$ such that $\emptyset \neq R_1 \subset R \cap R_2$, R_2 is connected and not contained in R, and for any holomorphic function f on R, there is a function f_2 holomorphic on R_2 such that $f = f_2$ on R_1.*

Open balls $B(c, r)$ centered in c of radius r are domains of holomorphy. The following is a generalization of vertical strips in $\mathbb{C}^n$.

Definition 4.12.2. *An open set $T \subset \mathbb{C}^n$ is a tube if there is an open set $U \subset \mathbb{R}^n$ such that $T = U + i\mathbb{R}^n = \{z \in \mathbb{C}^n : \ \mathrm{Re}(z) \in U\}$.*

The following is (a generalization of) Bochner's Tube Theorem [Boc].

Theorem 4.12.3. *A tube domain is a domain of holomorphy if and only if it is convex.*

We denote the convex hull of T by $\hat{T}$. In particular, every holomorphic function on T has a holomorphic continuation to $\hat{T}$.

The following is useful in showing that some properties of holomorphic functions extend to their analytic continuations.

Theorem 4.12.4. *Let $R_1 \subset \mathbb{C}^m, R_2 \subset \mathbb{C}^n$ be domains of holomorphy, and $f : R_1 \to \mathbb{C}^n$ a holomorphic map. Then*

$$R = f^{-1}(R_2) = \{z \in \mathbb{R}_1 : f(z) \in R_2\} \tag{4.12.1}$$

is a domain of holomorphy.

The following proposition is used to estimate the size of the meromorphic continuations of our triple Dirichlet series in vertical strips.

Proposition 4.12.5. *Assume that $T \subset \mathbb{C}^n$ is a tube domain, $g, h : T \to \mathbb{C}$ are holomorphic functions, and let $\tilde{g}, \tilde{h}$ be their holomorphic continuation to $\hat{T}$. If $|g(z)| \leq |h(z)|$ for all $z \in T$, and $h(z)$ is nonzero in T, then also $|\tilde{g}(z)| \leq |\tilde{h}(z)|$ for all $z \in \hat{T}$.*

Proof. Since $h(z)$ is nonzero, the function $f(z) = g(z)/h(z)$ is holomorphic in T, and hence has a holomorphic continuation $\tilde{f}$ to $\hat{T}$. By our assumptions, $|\tilde{f}(z)| \leq 1$ for all $z \in T$, so $T \subset \tilde{f}^{-1}(B(0, 1))$. However, by Theorem 4.12.4, $\tilde{f}^{-1}(B(0, 1))$ is a domain of holomorphy, so it is the whole $\hat{T}$. □

Chapter 5

Different ways to compute moments of real Dirichlet L-functions

In the influential paper of Conrey, Farmer, Keating, Rubinstein and Snaith [CFKRS] published in 2005, the authors conjectured precise asymptotic formulas for moments in various families of L-functions. They introduced the "recipe", a process that allows us to obtain these asymptotics, based on approximating the L-functions by a Dirichlet polynomial using the approximate functional equation, and then replacing all of the quantities by their expected values when averaged over the family.

Diaconu, Goldfeld and Hoffstein [DGH] then used a different method based on the investigation of multiple Dirichlet series to study moments of $\zeta(s)$ on the critical line, and in the family of real Dirichlet L-functions at the central point. They showed that if the associated multiple Dirichlet series has an analytic continuation past certain point, then the asymptotic formulas can be computed and the results agree with those predicted in [CFKRS].

Let us note that as opposed to [DGH], the authors of [CFKRS] consider shifted moments, that is moments slightly to the right of the critical line. These are easier to handle and give rise to many terms of different sizes. At the central point without the shifts, all these terms get combined together into the main term.

This chapter is based on ongoing joint work of the author and Siegfred Baluyot. We consider the shifted moments in the family of real Dirichlet characters and show that the two approaches not only give the same results, but the processes that lead to them are very much equivalent. In particular, we will see that in both approaches exactly the same terms arise, from a similar source: the functional equation of the individual L-functions. The multiple Dirichlet series approach becomes more intuitive due to the introduction of the shifts. The reason is that the main terms come from the residues at the poles of the associated multiple Dirichlet series, and while we have a single pole of higher order when computing the moments at the central point, it splits into many simple poles after introducing the shifts, so the computations of the residues become easier.

The recipe not only provides an asymptotic formula for the moments at the central point, but predicts a whole main term in terms of descending powers of $\log X$. On the other hand, the multiple Dirichlet series method as used in [DGH] only gives the leading order term. It is clear from our approach, where we consider the shifted moments, that the multiple Dirichlet series technique predicts exactly the same main terms as the recipe.

The approach of [CFKRS] leads to a very precise conjecture, which includes a square-root cancellation error term. However, [DGH] conjectured a secondary term of size $X^{3/4}$ for the third moment, coming from an additional pole of the multiple Dirichlet series, which was later confirmed by [Dia] and [DiWh]. Moreover, they predicted infinitely many secondary terms for higher moments (see also [DPP] for details). We explain in Section 5.3 how the multiple Dirichlet series approach predicts these secondary terms, by using an additional functional equation of the multiple Dirichlet series. It would be very interesting to see if this term can also be predicted by modifying the recipe.

In this chapter, we write χ_d for the Kronecker symbol $\left(\frac{d}{\cdot}\right)$, where d denotes a positive fundamental discriminant. We write the functional equation in the form

$$L(s, \chi_d) = X_d(s) L(1 - s, \chi_d), \tag{5.0.1}$$

with (recall that the characters χ_d are even if $d \geq 1$)

$$X_d(s) = d^{1/2-s} X(s) = d^{1/2-s} \cdot \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \tag{5.0.2}$$

We consider the family of real Dirichlet characters $\{\chi_d : \; d \le X\}$, and for the L-functions in this family, we write the approximate functional equation in the form

$$L(s, \chi_d) \approx \sum_{n \le \sqrt{X}} \frac{\chi_d(n)}{n^s} + X_d(s) \sum_{n \le \sqrt{X}} \frac{\chi_d(n)}{n^{1-s}}. \tag{5.0.3}$$

The upper bounds $n \le \sqrt{X}$ are slightly imprecise, but sufficient for our heuristic (we only apply the functional equation as part of a recipe, whose part is also completing all the sums and dropping these conditions).

Our goal is to find (conjectural) asymptotic formulas for the shifted moments

$$M(X; s_1, \ldots, s_k) = \sum_{d \le X}^{*} L(s_1, \chi_d) \ldots L(s_k, \chi_d), \tag{5.0.4}$$

where the sum runs over fundamental discriminants and the $s_j > 1/2$ are fixed real numbers, which should be thought of as $1/2 + \alpha_j$ with α_j small. Indeed, our ultimate goal is to find an asymptotic when all the $s_j = 1/2$, i.e., to be able to take $\alpha_j \to 0$.

In the following sections, we first compute $M(X; s_1, \ldots, s_k)$ using the recipe, and then express it using multiple Dirichlet series, comparing the steps of the two approaches. Our goal is not to arrive to the simplest formula for the moments, but rather to stop at the point when it is clear that both techniques give the same answer. Finally, in the last section of this chapter, we explain the multiple Dirichlet series heuristic that leads to the secondary terms.

5.1 The recipe prediction

The starting point of the recipe is replacing each L-function in (5.0.4) by the approximate functional equation (5.0.3). We thus obtain

$$M(X; s_1, \ldots, s_k) \approx \sum_{d \le X}^{*} \prod_{j=1}^{k} \left(\sum_{n \le \sqrt{X}} \frac{\chi_d(n)}{n^{s_j}} + X_d(s_j) \sum_{n \le \sqrt{X}} \frac{\chi_d(n)}{n^{1-s_j}} \right). \tag{5.1.1}$$

The general philosophy behind the recipe (which applies also for other families of L-functions) is to expand the product and replace each quantity in the various summands by its average over the family.

We now write a typical term that occurs after expanding the product in (5.1.1). Let $J \subset \{1, \ldots, k\}$ represent the subset of indices where we take the second part of the approximate functional equation, and define s_j^J by

$$s_j^J = \begin{cases} s_j & \text{if } j \notin J, \\ 1 - s_j & \text{if } j \in J. \end{cases} \tag{5.1.2}$$

Then we have a term of the form

$$\sum_{d \le X}^{*} \sum_{n_1, \dots, n_k \le \sqrt{X}} \frac{\chi_d(n_1 \dots n_k)}{n_1^{s_1^J} \dots n_k^{s_k^J}} \cdot \prod_{j \in J} X_d(s_j). \tag{5.1.3}$$

According to the recipe, we now replace the various quantities by their expectations when averaged over the family. Averaged over the fundamental discriminants up to X, $\chi_d(n_1 \dots n_k)$ is small unless $n_1 \dots n_k = \Box$, in which case

$$\frac{2\zeta(2)}{X} \sum_{d \le X}^{*} \chi_d(n_1 \dots n_k) \sim a(n_1 \dots n_k), \tag{5.1.4}$$

where $a(n) = \prod_{p|n} \frac{p}{p+1}$, and $\frac{X}{2\zeta(2)}$ is asymptotically the number of fundamental discriminants up to X (this follows for example from Perron's formula and Lemma 4.2.4). For the terms $X_d(s_j)$, we have

$$\sum_{d \le X}^{*} \prod_{j \in J} X_d(s_j) = \prod_{j \in J} X(s_j) \cdot \sum_{d \le X}^{*} d^{\frac{|J|}{2} - \sum_{j \in J} s_j} \sim \frac{X^{1 + \frac{|J|}{2} - \sum_{j \in J} s_j}}{2\zeta(2) \left(\frac{|J|}{2} - \sum_{j \in J} s_j \right)} \prod_{j \in J} X(s_j). \tag{5.1.5}$$

We also complete the sums (drop the conditions $n_j \le \sqrt{X}$), and obtain the terms

$$T_J(X; s_1, \dots, s_k) = \frac{X^{1 + \frac{|J|}{2} - \sum_{j \in J} s_j}}{2\zeta(2) \left(\frac{|J|}{2} - \sum_{j \in J} s_j \right)} \prod_{j \in J} X(s_j) \sum_{n_1 \dots n_k = \Box} \frac{a(n_1 \dots n_k)}{n_1^{s_1^J} \dots n_k^{s_k^J}}. \tag{5.1.6}$$

Note that the sum doesn't converge unless the $\mathrm{Re}(s_j)$ are large enough, so it should be interpreted as the meromorphic continuation of the absolutely convergent sums. Since the function $a(n)$ is multiplicative, this meromorphic continuation can be obtained in a usual fashion after considering the Euler product.

The conjecture then is that asymptotically as $X \to \infty$, the $k-$tk moment is the sum of all of the 2^k terms T_J:

$$M(X; s_1, \dots, s_k) \sim \sum_{J \subset \{1, \dots, k\}} T_J(X; s_1, \dots, s_k). \tag{5.1.7}$$

5.2 The multiple Dirichlet series approach

In this section, we compute $M(X; s_1, \dots, s_k)$ using Perron's formula. We obtain

$$M(X; s_1, \dots, s_k) = \frac{1}{2\pi i} \int_{(2)} Z(s_1, \dots, s_k, w) \frac{X^w}{w} dw, \tag{5.2.1}$$

where

$$Z(s_1, \dots, s_k, w) = \sideset{}{^*}\sum_{d \geq 1} \frac{L(s_1, \chi_d) \dots L(s_k, \chi_d)}{d^w}. \tag{5.2.2}$$

Our goal now is to shift the integral as much to the left as possible, and compute the contribution of the crossed poles. We will find the poles of $Z(s_1, \dots, s_k, w)$ (considered as a function of the single complex variable w) and compute the residues. The asymptotic formula could then be obtained rigorously provided that $Z(s_1, \dots, s_k, w)$ has a meromorphic extension. If we expand all L-functions in the coefficients of $Z(s_1, \dots, s_k, w)$ and first sum over the family (the d variable), we obtain

$$Z(s_1, \dots, s_k, w) = \sideset{}{^*}\sum_{d \geq 1} \sum_{n_1, \dots, n_k \geq 1} \frac{\left(\frac{d}{n_1 \dots n_k}\right)}{n^{s_1} \dots n^{s_k} d^w} = \sum_{n_1, \dots, n_k} \frac{L_D\left(w, \left(\frac{\cdot}{n_1 \dots n_k}\right)\right)}{n_1^{s_1} \dots n_k^{s_k}}, \tag{5.2.3}$$

where

$$L_D(s, \chi) = \sideset{}{^*}\sum_{d \geq 1} \frac{\chi(d)}{d^s} \tag{5.2.4}$$

as in (4.1.19). Note that this exchange of summation can be interpreted as the step from the recipe where we replace the quantities by their averages over the family.

As in (4.3.10) we find that if $n = \square$, $L_D\left(w, \left(\frac{\cdot}{n}\right)\right)$ has a pole at $w = 1$ with residue $\frac{a(n)}{2\zeta(2)}$. Hence $Z(s_1, \dots, s_k, w)$ has a pole at $w = 1$ with residue

$$\frac{1}{2\zeta(2)} \sum_{n_1 \dots n_k = \square} \frac{a(n_1 \dots n_k)}{n_1^{s_1} \dots n_k^{s_k}}. \tag{5.2.5}$$

This pole contributes to the integral (5.2.1) with the term

$$\frac{X}{2\zeta(2)} \sum_{n_1 \dots n_k = \square} \frac{a(n_1 \dots n_k)}{n_1^{s_1} \dots n_k^{s_k}}, \tag{5.2.6}$$

which is exactly equal to $T_\emptyset(X; s_1, \dots, s_k)$ from (5.1.6). In other words, it corresponds to taking the first parts in the approximate functional equations in the previous section.

We will obtain the other terms from functional equations that are satisfied by the multiple Dirichlet series $Z(s_1, \dots, s_k, w)$. For every $j \in \{1, \dots, k\}$, we have the functional equation σ_j "in the s_j variable":

$$\begin{aligned} Z(s_1, \dots, s_k, w) &= \sum_{d \geq 1}^{*} \frac{L(s_1, \chi_d) \dots L(s_k, \chi_d)}{d^w} \\ &= X(s_j) \sum_{d \geq 1}^{*} \frac{L(s_1, \chi_d) \dots L(1 - s_j, \chi_d) \dots L(s_k, \chi_d)}{d^{w + s_j - 1/2}} \\ &= X(s_j) Z(s_1, \dots, 1 - s_j, \dots, s_k, w + s_j - 1/2). \end{aligned} \tag{5.2.7}$$

Note that σ_j is an involution, and for two different $i, j \in \{1, \dots, k\}$, σ_i and σ_j commute. Thus for any $J \subset \{1, \dots, k\}$ we can iterate the σ_j for $j \in J$ and obtain the functional equation σ_J:

$$Z(s_1, \dots, s_k, w) = \prod_{j \in J} X(s_j) \cdot Z\left(s_1^J, \dots, s_k^J, w + \sum_{j \in J} s_j - \frac{|J|}{2}\right). \tag{5.2.8}$$

Under this functional equation, the pole at $w = 1$ becomes a pole at

$$w = 1 + \frac{|J|}{2} - \sum_{j \in J} s_j, \tag{5.2.9}$$

with residue

$$\operatorname{res}_{w = 1 + \frac{|J|}{2} - \sum_{j \in J} s_j} Z(s_1, \dots, s_k, w) = \prod_{j \in J} X(s_j) \cdot \frac{1}{2\zeta(2)} \sum_{n_1 \dots n_k = \square} \frac{a(n_1 \dots n_k)}{n_1^{s_1^J} \dots n_k^{s_k^J}}. \tag{5.2.10}$$

This pole contributes to the integral (5.2.1) with the term

$$\frac{X^{1 + \frac{|J|}{2} - \sum_{j \in J} s_j}}{2\zeta(2) \left(\frac{|J|}{2} - \sum_{j \in J} s_j\right)} \prod_{j \in J} X(s_j) \cdot \sum_{n_1 \dots n_k = \square} \frac{a(n_1 \dots n_k)}{n_1^{s_1^J} \dots n_k^{s_k^J}}, \tag{5.2.11}$$

which is exactly equal to the term $T_J(X; s_1, \dots, s_k)$, and adding the contribution of all the poles leads to the formula (5.1.7).

Note that this formula is true if $\operatorname{Re}(s_j)$ are large enough, in which case we can shift all the integrals (after possibly considering the smooth moments to make the integral absolutely convergent). This range of admissible $\operatorname{Re}(s_j)$ depends on how far we can meromorphically continue $Z(s_1, \dots, s_k, w)$.

5.3 A heuristic for the secondary terms

So far, we have used the functional equations of $Z(s_1, \ldots, s_k, w)$ in the variables s_j. However, from the expression (5.2.3), we see that there might also be an extra functional equation in the variable w. Let us now use a heuristic introduced by Bump, Diaconu, Friedberg and Hoffstein [BFH1], [CFH], and see what this equation might look like. In this heuristic, we drop the subscript D from the L-function and assume that $\left(\frac{\cdot}{n}\right)$ is always a primitive Dirichlet character modulo n. We also don't write the gamma factors $X(s)$ in the functional equation, writing it in the form $L\left(s, \left(\frac{\cdot}{n}\right)\right) \approx n^{s-1/2} L\left(1-s, \left(\frac{\cdot}{n}\right)\right)$. We thus obtain the heuristic functional equation σ_w

$$\begin{aligned} Z(s_1, \ldots, s_k, w) &\approx \sum_{n_1, \ldots, n_k} \frac{L\left(w, \left(\frac{\cdot}{n_1 \ldots n_k}\right)\right)}{n_1^{s_1} \ldots n_k^{s_k}} \approx \sum_{n_1, \ldots, n_k} \frac{L\left(1-w, \left(\frac{\cdot}{n_1 \ldots n_k}\right)\right)}{n_1^{s_1+w-1/2} \ldots n_k^{s_k+w-1/2}} \\ &\approx Z(s_1 + w - 1/2, \ldots, s_k + w - 1/2, 1-w). \end{aligned} \tag{5.3.1}$$

Note that it is not straightforward to obtain this equation directly from (5.2.3), because there is no relation between $L_D(w, \chi)$ and $L_D(1-w, \chi)$. To obtain a relation similar to σ_w, we must consider the family of all Dirichlet characters, parametrized for example by all odd integers, not only by fundamental discriminants. This poses other problems, such as the presence of non-primitive characters, as already described in Section 4.2.1.

Note that σ_w doesn't commute with the equations σ_j from the previous section. In fact the group generated by $\{\sigma_j : j \in \{1, \ldots, k\}\} \cup \{\sigma_w\}$ is infinite for $k \geq 4$, which is one of the reasons why this method only allows us to rigorously compute the moments for $k \leq 3$.

Let us now write σ_J as

$$Z(s_1, \ldots, s_k, w) \approx Z\left(s_1^J, \ldots, s_k^J, w + \sum_{j \in J} s_j - \frac{|J|}{2}\right). \tag{5.3.2}$$

Applying $\sigma_{\{1,\ldots,k\}}$ to the right-hand side of (5.3.1), we obtain

$$Z(s_1, \ldots, s_k, w) \approx Z\left(\frac{3}{2} - s_1 - w, \ldots, \frac{3}{2} - s_k - w, 1 - k + (k-1)w + \sum_{j=1}^{k} s_j\right). \tag{5.3.3}$$

We thus (heuristically) obtain a pole when

$$1 - k + (k-1)w + \sum_{j=1}^{k} s_j = 1. \tag{5.3.4}$$

We don't get anything new if $k = 1$ or 2, but for $k \geq 3$, we have a new pole at

$$w = \frac{k - \sum_{j=1}^{k} s_j}{k-1}. \tag{5.3.5}$$

If $k = 3$ and $s_j = 1/2$ for $j = 1, 2, 3$, this gives a term of size $X^{3/4}$ for $M(X; 1/2, 1/2, 1/2)$. This term was conjectured by [DGH], and later confirmed by Diaconu [Dia] over function fields, and by Diaconu and Whitehead [DiWh] over number fields.

For $k \geq 4$, this is just one of infinitely many conjectured secondary terms, which arise from other functional equations from the infinite group. A precise conjecture in this case was lately obtained by Diaconu, Paşol, Popa [DPP], who also made a notable progress for $k = 4$ in the function fields setting.

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