Additive Combinatorics

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ABSTRACT

Additive Combinatorics

An Extension of the Eventown Theorem

A set family F that is a subset of $2^{[n]}$, $[n]=\{1,...,n\}$ is said to have the Eventown property if all its component sets are even sized and the intersection of any two of these sets is even sized. The Eventown theorem states a bound for the size of F in this case, namely $|F| \leq 2^{[n/2]}$. The aim of the thesis is to discuss a generalization of the Eventown theorem through the lens of additive combinatorics.

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CONTENTS

1 Introduction

We consider a set family $\mathscr{F} \subset 2^{[n]}$. Here, $[n]$ just denotes the set of natural numbers $\{1, 2, 3, ..., n\}$ and $2^{[n]}$ being its power set. We impose on this set family the property that the intersection of any two members of $\mathscr{F}$ has size divisible by l. In symbols, $\forall\, A, B \in \mathscr{F}$, we have $l \mid \mid A \cap B \mid$. We are interested in bounds on the set $\mathscr{F}$. The well-known Eventown theorem states the following with respect to the case $l = 2$.

Theorem 1.1. *When $l = 2$, we have $\mid \mathscr{F} \mid \leq 2^{\lfloor n/2 \rfloor}$.*

Following the reference [1], the aim of this document is to show that in the general case for l, i.e $\mid \mathscr{F} \mid \leq 2^{\lfloor n/l \rfloor}$, we may establish the following theorem.

Theorem 1.2. *Let l be a positive integer. Then there exists k=k(l) such that for every positive integer n, the following holds. Let $\mathscr{F} \subset 2^{[n]}$ such that the intersection of any k distinct elements of $\mathscr{F}$ is divisible by l. Then $\mid \mathscr{F} \mid \leq 2^{\lfloor n/l \rfloor} + c$, where c = c(l,k) is a constant, and c = 0 if l | n and n is sufficiently large.*

Thus we aim to establish a bound that is dependent on l but is ultimately independent of l when l divides n and n is chosen appropriately.

2 Preliminaries

We establish defintions that will be useful.

- A family $\mathscr{F}$ is atomic, if there exist disjoint sets $A_1, \ldots, A_d \in [n]$ such that $\mathscr{F}$ is the family of all sets F satisfying any of the two following conditions: 1) $A_i \in F$ 2) $A_i \cap F = \phi$ for every $i \in [d]$ and F contains no element not covered by the set A_i. The sets $A_1 \ldots A_d$ are called the atoms of $\mathscr{F}$.
- $S(n,l)$ is the atomic family such that $d = \lfloor n/l \rfloor$ and all A_i such that $i \in [d]$ have size l.
- We also need to establish a measure on the size on a set family $\mathscr{F}$, for which we assign the dimension of the subspace $\langle \mathscr{F} \rangle$ spanned by the characteristic vectors of the sets in $\mathscr{F}$ over some field $\mathbf{F}$.
- Given $i, j \in [n]$, i and j are twins for $\mathscr{F}$ if 1) every $F \in \mathscr{F}$ either contains both i, j or neither of them, and 2) there is at least one $F \in \mathscr{F}$ such that $i, j \in F$. A set of coordinates $T \subset [n]$ is called a set of twins if 1) any pair of elements in T are twins, or $\mid T \mid = 1$. Being twins is an equivalence relation, and T is a maximal set of twins if it is a complete equivalence class of the twins relation.

We also establish some notation that is useful.

- $\mathbf{Z_l}$ is the ring of integers modulo l, and if p is a prime, we write the field $\mathbf{F_p}$.

- R is a commutative ring with unity.
- For any vector $v \in R^n$, $v(i)$ denotes the ith coordinate of v.
- The support of $v = \{i \in [n] : v(i) \neq 0\}$
- $\langle \mathscr{F} \rangle$ is the span, where $\mathscr{F} \subset R^n$
- If $A \subset [n]$ and $F \in R^n$, then $F|_A$ is the restriction of F to the coordinates in A.
- For vectors $v, w \in R^n$, let $v.w$ be the vector in R^n defined as $(v.w)(i) = v(i)w(i)$ for $i \in [n]$.

3 Atomicity and k closed families

Here, we establish a stronger form of Theorem 1.2 which will eventually help us prove it. To this end, we consider some more definitions. $\mathscr{F}$ is *non-reducible* if $\mathscr{F}$ does not vanish on any of the coordinates, i.e $\nexists$ i such that $v(i) = 0$ for all $v \in \mathscr{F}$. Define $\mathscr{F} \cdot \mathscr{F} = \{v \cdot w : v, w \in \mathscr{F}\}$, where $v \cdot w$ is the coordinate-wise product, i.e. $v.w = (v_1w_1, \ldots, v_nw_n)$. Define $v^k = v \cdot \ldots \cdot v$ and $\mathscr{F}^k = \mathscr{F} \cdot \ldots \cdot \mathscr{F}$, where the products contain k terms. Also define a norm $||v|| = \sum_{i=1}^n v(i)$. Note here that we do not mean a norm in the regular sense, but as a convenient way of describing the sum of coordinates. A set $\mathscr{F} \subset \mathbb{Z}_\ell^n$ is *k-closed* if $||v|| = 0$ for every $1 \leq i \leq k$ and $v \in \mathscr{F}^i$. Thus a coordinate wise product gives us new vectors if you take the product of the vector space with itself up to k times. If the sum of the coordinates of the vectors is 0, the vector space is k-closed.

Based on these definitions, we have the following observations: If $\mathscr{F} \subset \mathbb{Z}_l^n$ is k-closed, then $\langle \mathscr{F} \rangle$ is also k-closed. Also, if $\mathscr{F}, \mathscr{F}' \subset \mathbb{Z}_l^n$, then $\langle \mathscr{F} \cdot \mathscr{F}' \rangle = \langle \mathscr{F} \rangle \cdot \langle \mathscr{F}' \rangle$.

We note some important properties of twins. Let $\mathscr{F} \subset \{0,1\}^n$.

If $\mathscr{F}$ is non-reducible, then the maximal sets of twins for $\mathscr{F}$ form a partition of $[n]$. Also, for every $k \geq 1$, the family $\bigcup_{i=1}^k \mathscr{F}^i$ has the same sets of twins as $\mathscr{F}$.

We now show that if $\mathscr{F} \subset 2^{[n]}$ is such that the intersection of any k not necessarily distinct elements of $\mathscr{F}$ has size divisible by ℓ, then $|\mathscr{F}| \leq 2^{\lfloor n/\ell \rfloor}$, given k is sufficiently large with respect to ℓ. We also show that if $\mathscr{F}$ is close to being extremal, then $\mathscr{F}$ must be a subfamily of (an isomorphic copy of) $S(n, \ell)$, i.e. equal up to a permutation of [n].

Theorem 3.1. *Let ℓ be a positive integer, then there exists k such that the following holds. Let $\mathscr{F} \subset \{0,1\}^n$ such that $\mathscr{F}$ is k-closed over $\mathbb{Z}_\ell$. Then $|\mathscr{F}| \leq 2^{\lfloor n/\ell \rfloor}$. Also, if $|\mathscr{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, then $[n]$ can be partitioned into sets $A_1, \ldots, A_d, A'$ such that A_i is a maximal set of twins for $\mathscr{F}$ for $i \in [d]$, $|A_i| = \ell$, $|A'| \leq \ell - 1$, and $\mathscr{F}$ vanishes on A'.*

The Eventown theorem is based on sets has an analogous vector space version, where set intersection has vector space dot product as described above as the analogue.We consider an extension of this where scalar products are replaced with bilinear forms.

Lemma 3.2. *(Bilinear form lemma) Let $\mathbb{F}$ be a field, let $b_1, \ldots, b_n \in \mathbb{F}$, let z be the number of coefficients among $b_1, \ldots, b_n$ that are zero, and let $b : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ be the bilinear form defined as $b(v,w) = \sum_{i=1}^n b_i v(i) w(i)$ where $v = (v_1 \ldots v_n), w = (w_1 \ldots w_n)$. Let $V < \mathbb{F}^n$ such that $b(v,w) = 0$ for every $v, w \in V$. Then* $\dim(V) \leq \frac{1}{2}(n+z)$.

Proof. Consider a linear transformation $T : \mathbb{F}^n \to \mathbb{F}$ with $T(v) = (b_1 v_1, \ldots, b_n v_n)$. Let M be the $n \times n$ diagonal matrix with diagonal entries $b_1, \ldots, b_n$. Let $W = \{Mv : v, \in V\} < \mathbb{F}^n$. Then, $\dim(kerM) = z$, since T makes precisely z coordinates vanish. Hence, $\dim(W) \geq \dim(V) - z$. By definition, $W = Im(T)$, and W is orthogonal to V. Therefore, $\dim(V) + \dim(W) \leq n$, which implies $\dim(V) \leq \frac{1}{2}(n+z)$. □

We now show that if $\ell = p^\alpha$ is a prime power, and $\mathscr{F} \subset \{0,1\}^n$ is k-closed over $\mathbb{Z}_\ell$ for some large constant k, then most sets of maximal twins for $\mathscr{F}$ must have size divisible by ℓ, provided that the dimension of $\langle \mathscr{F} \rangle_p$ is large. First, we consider ℓ to be prime.

Lemma 3.3. *(Prime lemma) Let $V < \mathbb{F}_p^n$, let $A_1, \dots, A_d$ be a partition of $[n]$ into twins for V, and suppose that V is 2-closed. If $\dim(V) = d - h$, then at least $d - 2h$ of the numbers $|A_1|, \dots, |A_d|$ are divisible by p.*

Proof. For $i \in [d]$, let $b_i = |A_i|$ and let b be the bilinear form defined as in Lemma 3.2. Let $\phi : V \to \mathbb{F}_p^d$ be the linear map defined as $\phi(v)(i) = s$ if $v|_{A_i}$ is the constant s vector. Thus every vector in V takes the same value on every 'block'. Then ϕ is an injection, so $\dim(\phi(V)) = \dim(V) = d - h$. Also, for every $u, v \in V$, we have $||u \cdot v|| = b(\phi(u), \phi(v))$, so we have $b(x, y) = 0$ for every $x, y \in \phi(V)$, since V is 2-closed. But then by Lemma 3.2, if z is the number of zeros among $b_1, \dots, b_d$, then $\dim(\phi(V)) \leq \frac{1}{2}(d + z)$, so $z \geq d - 2h$. □

Lemma 3.4. *(Prime power lemma) Let p be a prime and $\alpha \in \mathbb{Z}^+$. Let $\mathscr{F} \subset \{0,1\}^n$ be $2(p+\alpha)$-closed over $\mathbb{Z}_{p^\alpha}$, let $\dim(\langle \mathscr{F} \rangle_p) = d$, and let $A_1, \dots, A_d, B$ be a partition of $[n]$ such that A_i is a set of twins, and $\dim(\langle \mathscr{F}|_B \rangle_p) \leq h$. Then at least $d - 2\alpha h$ of the numbers $|A_1|, \dots, |A_d|$ are divisible by p^α.*

Proof. Let $V = \langle \mathscr{F} \rangle_p$. We use induction on α. The case $\alpha = 1$ follows from Lemma 3.3. Let $\alpha > 1$. Then by our induction hypothesis, at least $k = d - (2\alpha - 2)h$ of the sets $A_1, \dots, A_d$ have size divisible by $p^{\alpha-1}$. Without loss of generality, let these sets be $A_1, \dots, A_k$. Also, let $B' = A_{k+1} \cup \dots \cup A_d \cup B$. Note that

$\dim(V|_{B'}) \leq \dim(V|_B) + (d-k) \leq h+d-k$. Therefore, V contains a subspace W such that $\dim(W) \geq d - \dim(V|_{B'}) \geq k-h$ and W vanishes on B'.

We remark that for every $w \in W$ there exists some $w' \in \langle \mathscr{F} \rangle_{p^\alpha}$ such that $w' \equiv w \pmod{p}$. Let β be the smallest number such that $\beta > \alpha$ and $\beta = 1 \pmod{p-1}$, then $\beta < \alpha + p$. As $\mathscr{F}$ is 2β-closed, for every $u, v \in W$ we have that $||(u')^\beta \cdot (v')^\beta||$ is divisible by p^α. However, note that $(w')^\beta \equiv w \pmod{p}$, and if $w(i) = 0$ (over $\mathbb{F}_p$) for some $i \in [n]$, then $p^\alpha \mid (w')^\beta(i)$.

For $i \in [k]$, let A_i' be a set of size $|A_i|/p^{\alpha-1}$, and let $A' = \bigcup_{i=1}^k A_i'$. Define the linear map $\phi : W \to \mathbb{F}_p^{A'}$ as follows. If $w \in W$, $i \in [k]$, and $w|_{A_i}$ is the constant s vector, then $\phi(w)|_{A_i'}$ is the constant s vector. Then we see that ϕ is an injection, so $\dim(W) = \dim(\phi(W))$. Also, for every $u, v \in W$, we have $||(u')^\beta \cdot (v')^\beta|| \equiv p^{\alpha-1}||\phi(u)\cdot\phi(v)|| \pmod{p^\alpha}$. So we must have $||x \cdot y|| = 0$ for every $x, y \in \phi(W)$. Let z be the number of sets among $A_1', \ldots, A_k'$, whose size is divisible by p. We can apply Lemma 3.3 to conclude that $z \geq k - 2(k - \dim(\phi(W))) \geq k - 2h \geq d - 2\alpha h$. As z is also the number of sets among $A_1, \ldots, A_k$ whose size is divisible by p^α. This concludes the proof. □

For the next lemma, we admit the following theorem from [1].

Theorem 3.5. *Let $\mathscr{F} \subset \{0,1\}^n$, let $\mathbb{F}$ be a field, and suppose that $\dim\langle \mathscr{F} \rangle = d$ and $\dim\langle \mathscr{F} \cdot \mathscr{F} \rangle = d + h$. Then $[n]$ can be partitioned into $d+1$ sets $A_1, \ldots, A_d, B$ such that A_i is a maximal set of twins for $\mathscr{F}$ for $i \in [d]$, and $\dim\langle \mathscr{F}|_B \rangle \leq 2h$.*

Lemma 3.6. *(Small dimension lemma) Let p be a prime, and $\alpha, t \in \mathbb{Z}^+$. Let $\mathscr{F} \subset \{0,1\}^n$ such that $\mathscr{F}$ is non-reducible and $2^{t+1}(p+\alpha)$-closed over $\mathbb{Z}_{p^\alpha}$. Let*

$A_1,\ldots,A_d$ be the unique partition of $[n]$ into maximal sets of twins, and let

$$B = \bigcup_{\substack{i\in[d] \\ |A_i|\not\equiv 0 \pmod{p^\alpha}}} A_i.$$

Then $\dim(\langle \mathscr{F}|_B\rangle_p) \le \frac{6n\alpha}{t}$.

Proof. Let $\mathscr{F}_0 = \mathscr{F}$, and for $i = 1,2,\ldots,t$, let $\mathscr{F}_i = \mathscr{F}_{i-1}\cdot\mathscr{F}_{i-1}$. Remark that $\mathscr{F}_{i-1} \subset \mathscr{F}_i$ and $\mathscr{F}_i$ is $2^{t+1-i}(p+\alpha)$-closed over $\mathbb{Z}_{p^\alpha}$, and $A_1,\ldots,A_d$ is also the unique partition of $[n]$ into maximal sets of twins for $\mathscr{F}_i$. As $\dim(\langle\mathscr{F}_r\rangle_p)$ is monotone increasing, there exists $0 \le r < t$ such that

$$\dim(\langle\mathscr{F}_{r+1}\rangle_p) \le \dim(\langle\mathscr{F}_r\rangle_p) + \frac{n}{t}.$$

Let $d' = \dim(\langle\mathscr{F}_r\rangle_p)$. Applying the theorem above, we get that $[n]$ can be partitioned into $d'+1$ sets $A_1,\ldots,A_{d'},C$ such that A_i is a maximal set of twins for $\mathscr{F}_r$, and $\dim(\langle\mathscr{F}_r|_C\rangle_p) \le \frac{2n}{t}$. But then as $\mathscr{F}_r$ is $2(p+\alpha)$-closed, we can apply the Primepower lemma to conclude that at least $q = d' - \frac{4n\alpha}{t}$ of the numbers $|A_1|,\ldots,|A_{d'}|$ are divisible by p^α. Without loss of generality, let $A_1,\ldots,A_q$ be the sets of twins whose sizes are divisible by p^α. Let $D = C\cup A_{q+1}\cup\cdots\cup A_{d'}$. Then $B\subset D$, and noting that $\dim(\langle\mathscr{F}_r|_{D\setminus C}\rangle_p) \le d'-q$, and $\mathscr{F}\subset\mathscr{F}_r$, we get the chain of inequalities

$$\dim(\langle\mathscr{F}|_B\rangle_p) \le \dim(\langle\mathscr{F}_r|_D\rangle_p) \le (d'-q) + \dim(\langle\mathscr{F}_r|_C\rangle_p) \le \frac{4n\alpha+2n}{t} \le \frac{6n\alpha}{t}.$$

This finishes the proof. □

We admit a final lemma we need for the proof of Theorem 3.1 by Odlyzko [4].

Lemma 3.7. *Let p be a prime and $V < \mathbb{F}_p^n$. Then $|V \cap \{0,1\}^n| \leq 2^{\dim(V)}$.*

Proof of Theorem 3.1. Write $\ell = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct primes. We show that $k = 2^{t+1} \max_{r \in [s]} (p_r + \alpha_r)$ suffices, where $t = 12\ell \sum_{r=1}^{s} \alpha_r$. We want to show the following. Let $\mathscr{F} \subset \{0,1\}^n$ such that $\mathscr{F}$ is k-closed over $\mathbb{Z}_\ell$.

(1) Then $|\mathscr{F}| \leq 2^{\lfloor n/\ell \rfloor}$.

(2) If $|\mathscr{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, then $[n]$ can be partitioned into sets $A_1, \dots, A_{\lfloor n/\ell \rfloor}, A'$ such that A_i is a maximal set of twins for $i \in [\lfloor n/\ell \rfloor]$, $|A_i| = \ell$, $|A'| \leq \ell - 1$, and $\mathscr{F}$ vanishes on A'.

We use induction on n. Let $n \leq 6s\ell$. Let $A_1, \dots, A_d, A'$ be a partition of $[n]$ such that A_i is a maximal set of twins for $\mathscr{F}$ for $i \in [d]$, and $\mathscr{F}$ vanishes on A'. Then $|\mathscr{F}| \leq 2^d$. As $k \geq n$, the characteristic vector of A_i is contained in $(\langle \mathscr{F} \rangle_\ell)^k$. Take $v \in \mathscr{F}$ such that $v|_{A_i}$ is the all 1 vector $\mathbf{1}$, and let J be the set of $j \in [d] \setminus \{i\}$ such that $v|_{A_j}$ is $\mathbf{1}$. For each $j \in J$, since A_i, A_j are maximal sets of twins, there is $u_j \in \mathscr{F}$ such that either $u_j|_{A_i} = \mathbf{1}$ and $u_j|_{A_j} = \mathbf{0}$, or the other way around. Let J_1 be the set of j of the first type, and J_2 the set of j of type two. Then the product of u_j over $j \in J_1$ and $(v - u_j)$ over $j \in J_2$ is the characteristic vector of A_i, as required. Now, since $\mathscr{F}$ is k-closed, ℓ divides $|A_i|$ for $i \in [d]$. But then $d \leq \lfloor n/\ell \rfloor$, and we are done with (1). Also, if $|\mathscr{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, we must have $d = \lfloor n/\ell \rfloor$, which is only possible if all the sets $A_1, \dots, A_d$ have size ℓ. Therefore, (2) also holds.

Now, on the other hand, let $n > 6s\ell$. First, suppose $\exists\, A \subset [n]$ such that ℓ divides $|A|$ and A is a set of twins for $\mathscr{F}$. Then the family $\mathscr{F}' = \mathscr{F}|_{[n]\setminus A}$ is also k-closed over $\mathbb{Z}_\ell$ and $|\mathscr{F}'| \geq \frac{1}{2}|\mathscr{F}|$. By our induction hypothesis, we have $|\mathscr{F}'| \leq 2^{\lfloor (n-\ell)/\ell \rfloor}$, so we get $|\mathscr{F}| \leq 2^{\lfloor n/\ell \rfloor}$, and (1) indeed holds. If $|\mathscr{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, then $|\mathscr{F}'| > 2^{\lfloor (n-\ell)/\ell \rfloor - 1}$, so by our induction hypothesis there exists a partition of $[n] \setminus A$ into sets $A_1, \ldots, A_{\lfloor (n-\ell)/\ell \rfloor}, A'$ satisfying (2) with respect to $\mathscr{F}'$. Setting $A_{\lfloor n/\ell \rfloor} = A$, the sets $A_1, \ldots, A_{\lfloor n/\ell \rfloor}, A'$ satisfy (2) with respect to $\mathscr{F}$. So to to finish the proof, it is enough to show that if $|\mathscr{F}| > 2^{\lfloor n/\ell \rfloor - 1}$, then $\mathscr{F}$ has a set of twins of size divisible by ℓ. Next, we show that if $I \subset [n]$ is large, then the dimension of $\langle \mathscr{F}|_I \rangle_p$ cannot be too small for any prime p.

We consider the following 'Coordinates' lemma: Let p be a prime and $I \subset [n]$ such that $|I| \geq \ell$. Then

$$|I| \leq \ell \dim(\langle \mathscr{F}|_I \rangle_p) + 3\ell.$$

Proof. Let $V = \langle \mathscr{F}|_I \rangle_p$ and $d = \dim(V)$. Then $|V \cap \{0,1\}^I| \leq 2^d$ by Lemma 3.7. Hence $\exists\, v \in \{0,1\}^I$ and $\mathscr{F}' \subset \mathscr{F}$ such that $w|_I = v$ for every $w \in \mathscr{F}'$, and $|\mathscr{F}'| \geq |\mathscr{F}|/2^d$. Let $0 \leq m \leq \ell - 1$ such that $||v|| \equiv m \pmod{\ell}$, and in every $w \in \mathscr{F}'$, replace the coordinates in I with m coordinates of 1 entries. This gives a family $\mathscr{F}'' \subset \{0,1\}^{n-|I|+m}$ such that $\mathscr{F}''$ is k-closed over $\mathbb{Z}_\ell$ and $|\mathscr{F}''| = |\mathscr{F}'| \geq |\mathscr{F}|/2^d$. Therefore, by our induction hypothesis, we have

$$2^{\lfloor n/\ell \rfloor - d - 1} < \frac{|\mathscr{F}|}{2^d} \leq |\mathscr{F}''| \leq 2^{\lfloor (n-|I|+m)/\ell \rfloor} < 2^{\lfloor n/\ell \rfloor + 2 - |I|/\ell}.$$

Comparing the left- and right-hand-side gives the required inequality $|I| \leq \ell d + 3\ell$. $\square$

Assume that $\mathscr{F}$ is non-reducible, because otherwise we are immediately done by applying our induction hypothesis. Let $A_1, \ldots, A_d$ be the unique partition of $[n]$ such that A_i is a maximal set of twins for $\mathscr{F}$. Let $r \in [s]$, and apply the Small dimension lemma to $\mathscr{F}$ with respect to the prime power $p_r^{\alpha_r}$. Let

$$B_r = \bigcup_{\substack{i \in [d] \\ |A_i| \not\equiv 0 \pmod{p_r^{\alpha_r}}}} A_i.$$

As $\mathscr{F}$ is $2^{t+1}(p_r + \alpha_r)$-closed, we get that $\dim(\langle \mathscr{F}|_{B_r} \rangle_{p_r}) \leq \frac{6n\alpha_r}{t}$. But then by the Coordinates lemma we also have $|B_r| \leq \frac{6n\alpha_r \ell}{t} + 3\ell$. Let $B = \bigcup_{r=1}^{s} B_r$, then $|B| \leq \sum_{r=1}^{s} |B_r| \leq 3s\ell + \frac{6n\ell}{t} \sum_{r=1}^{s} \alpha_r < n$, where the last inequality holds by the choice of t as well as the fact that $n > 6sl$. Note that B is the union of those maximal sets of twins A_i where $|A_i|$ is not divisible by ℓ. Therefore, as $|B| < n$ and $A_1, \ldots, A_d$ form a partition of $[n]$, there must exists $j \in [d]$ such that ℓ divides $|A_j|$. This concludes the proof. $\square$

4 Proof of the Main Result

In this section, we show how to deduce Theorem 1.2 from Theorem 3.1. We start with the following variant well-known Oddtown theorem.

Lemma 4.1. *(Oddtown lemma) Let ℓ, m, n be positive integers.*

Let $A_1, \dots, A_m, B_1, \dots, B_m \subset [n]$ such that $\ell \nmid |A_i \cap B_i|$ for $i \in [m]$, but ℓ divides $|A_i \cap B_j|$ for $i \neq j$. Then $m \leq sn$, where s is the number of distinct prime divisors of ℓ.

Proof. Write $\ell = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct primes. Let v_i and w_i be the characteristic vectors of A_i and B_i over $\mathbb{Q}$, respectively. Let $t = \lceil m/s \rceil$, then there exists $r \in [s]$ such that for at least t of the indices $i \in [m]$, we have that $|A_i \cap B_i|$ is not divisible by $p_r^{\alpha_r}$. Without loss of generality let these t indices be $1, \dots, t$. We show that $v_1, \dots, v_t$ are linearly independent (over $\mathbb{Q}$), which then implies $t \leq n$ and $m \leq sn$.

Suppose this is not the case, then $\exists\ c_1, \dots, c_t \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^{t} c_i v_i = 0$. We can assume that at least one of $c_1, \dots, c_t$ is not divisible by p_r, otherwise we can replace c_i with $c_i' = c_i / p_r$ for every $i \in [t]$. Let $k \in [t]$ be an index such that $p_r \nmid c_k$. Consider the equality

$$0 = \left\langle \sum_{i=1}^{t} c_i v_i, w_k \right\rangle = \sum_{i=1}^{t} c_i |A_i \cap B_k|.$$

We have $p_r^{\alpha_r} \mid c_i |A_i \cap B_k|$ if $i \neq k$, and $p_r^{\alpha_r} \nmid c_k |A_k \cap B_k|$, so $p_r^{\alpha_r} \nmid \langle \sum_{i=1}^{t} c_i v_i, w_k \rangle$, contradiction. ☐

We consider the following definition. $\mathscr{F} \subset 2^{[n]}$ is *weakly k-closed over* $\mathbb{Z}_\ell$ if the intersection of any k distinct elements of $\mathscr{F}$ is divisible by ℓ. Additionally, $\mathscr{F} \subset 2^{[n]}$ is *k-closed over* $\mathbb{Z}_\ell$ if the family formed by the characteristic vectors of the elements of $\mathscr{F}$ is k-closed over $\mathbb{Z}_\ell$. So $\mathscr{F}$ is k-closed over $\mathbb{Z}_\ell$ if and only if the intersection of any k not necessarily distinct elements of $\mathscr{F}$ is divisible by ℓ. We admit following lemma, for which the case $\ell = 2$ is found in [5].

Lemma 4.2. *(Weakly closed lemma) Let* ℓ, k *be positive integers, and let* s *be the number of distinct prime divisors of* ℓ*. Let* $\mathscr{F} \subset 2^{[n]}$ *such that* $\mathscr{F}$ *is weakly k-closed over* $\mathbb{Z}_\ell$*. Then there exists* $\mathscr{F}' \subset \mathscr{F}$ *such that* $|\mathscr{F}'| \geq |\mathscr{F}| - sk^2n$*, and* $\mathscr{F}$ *is k-closed over* $\mathbb{Z}_\ell$*.*

The Weakly closed lemma combined with Theorem 3.1 implies that if $\mathscr{F} \subset 2^{[n]}$ is weakly k-closed over $\mathbb{Z}_\ell$, then $|\mathscr{F}| \leq 2^{\lfloor n/\ell \rfloor} + sk^2n$. In order to improve the term sk^2n to a constant, we use the second part of Theorem 3.1

Proof of Theorem 1.2. Let $d = \lfloor n/\ell \rfloor$. Let $\mathscr{F} \subset \{0,1\}^n$ such that $\mathscr{F}$ is weakly k-closed over $\mathbb{Z}_\ell$. Then by Lemma 4.2, there exists $\mathscr{F}' \subset \mathscr{F}$ such that $|\mathscr{F}'| \geq |\mathscr{F}| - sk^2n$, and $\mathscr{F}$ is k-closed over $\mathbb{Z}_\ell$, where s is the number of distinct prime divisors of ℓ. If $|\mathscr{F}'| \leq 2^{d-1}$, then $|\mathscr{F}| \leq 2^{d-1} + sk^2n < 2^d$, where the last inequality holds if n is sufficiently large.

Suppose that $|\mathscr{F}'| > 2^{d-1}$ and $|\mathscr{F}| \geq 2^d$, otherwise we are done. Then by Theorem 3.1, $[n]$ can be partitioned into sets $A_1, \ldots, A_d, A'$ such that A_i is a maximal set of twins for $\mathscr{F}'$ for $i \in [d]$, $|A_i| = \ell$, $|A'| \leq \ell - 1$, and $\mathscr{F}'$ vanishes on A'.

Let $S \subset \{0,1\}^n$ be the atomic family containing all possible 2^d sets C such that $C \cap A_i \in \{\emptyset, A_i\}$ for every $i \in [d]$. Then $\mathscr{F}' \subset S$ and $|S \setminus \mathscr{F}'| \le sk^2 n$. Also, if n is large enough, $\forall\ i \in [d]$ we can find $k-1$ distinct sets $B_{i,1}, \dots, B_{i,k-1} \in \mathscr{F}'$ such that $A_i = \bigcap_{j=1}^{k-1} B_{i,j}$. $\mathscr{F}'$ contains a set of the form $A_i \cup A_a \cup A_b$ for some a, b, as the number of such sets in S is at least $\binom{d-1}{2} > sk^2 n$, let this set be $B_{i,1}$. Also $\mathscr{F}'$ contains $k-2$ sets that contain A_i but do not contain A_a and A_b as the number of such sets in S is $2^{d-3} > sk^2 n + k$. Let these $k-2$ sets be $B_{i,2}, \dots, B_{i,k-1}$. Then $A_i = \bigcap_{j=1}^{k-1} B_{i,j}$.

Let $F \in \mathscr{F} \setminus S$. $\forall\ i \in [d]$, we have $A_i \subset F$ or $A_i \cap F = \emptyset$, as the size of $A_i \cap F = B_{i,1} \cap \dots \cap B_{i,k-1} \cap F$ must be divisible by ℓ. Now, as $F \notin S$, we must have $F \cap A' \neq \emptyset$. But for any $H \subset A'$, where $H \neq \emptyset$, there are at most $k-1$ elements $F \in \mathscr{F} \setminus S$ such that $F \cap A' = H$, else we would have k distinct $F_1, \dots, F_k \in \mathscr{F}$ with $|F_1 \cap \dots \cap F_k| \equiv |H| \not\equiv 0 \pmod{\ell}$, which is a contradiction. So we see that $|\mathscr{F} \setminus S| \le k2^{|A'|} \le k2^{\ell-1}$, and if $\ell \mid n$ then $\mathscr{F} \subset S$. This concludes the proof. $\square$

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