Misconceptions in the Transition from Calculus to Real Analysis

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Abstract

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Marc-Olivier Ouellet

Misconceptions about limits in introductory Calculus such as the infamous "a function never reaches its limit" have been thoroughly studied in previous research. However, their resolution is rarely documented. Our objective is to contribute to the understanding of the "vanishing" of common misconceptions about limits as students progress from Calculus to Analysis. In addition, we investigate the possibility that early Calculus misconceptions may influence the learning of Real Analysis in such a way that new, related misconceptions are developed about more advanced concepts. To this end, we created a questionnaire devised to uncover seven of the well-documented Calculus misconceptions, as well as three conjectured misconceptions related to introductory Analysis concepts. The questionnaire was administered to ten students actively enrolled in a first or second Real Analysis course. To analyze participants' answers, we introduced a model of misconception classification which includes six levels. Using this model, we identified consistent incorrect reasonings indicating the possibility that instruction after elementary Calculus has not contributed to the resolution of some misconceptions. We observed that certain students' answers exhibited what we refer to as "transitional behavior" from one level to another and discuss what this may mean in terms of overcoming misconceptions. In addition, we identified one instance of a student's learning of Real Analysis potentially being influenced by their Calculus misconceptions. Finally, we briefly considered the presence of misconceptions about fundamental mathematics, such as logical argumentation and mathematical notation, and new misconceptions that students may develop as they learn more advanced mathematics.

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To me, this thesis represents the result of years of perseverance and hard work. I cannot say that working through a pandemic made the journey any easier, but it was incredibly enriching, nonetheless. I would not have been able to do it without the contribution of the many people who supported and guided me throughout this experience.

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Now that this study is completed, I can only be grateful for this wonderful opportunity and for the people who made it possible, whether their contribution was direct or indirect.

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Chapter 1 Introduction

1.1 Overview of the research

Most mathematics instructors are familiar with the common misconception that limits are a boundary that a function "approaches without crossing[1]". As such, it is unsurprising to find an abundance of research on the nature, causes, and the extent of this particular misconception. In addition, prior research (see Chapter 2) has identified a large array of different concepts that are often misunderstood by mathematics students. The natural question that comes to mind is: when and how do students who hold these misconceptions develop or assimilate the correct mathematical concepts? Research on this question, however, seems to be scarce. This gap in the mathematics education literature inspired us to design a study on the vanishing and the evolution of the documented post-secondary mathematics misconceptions.

Our use of the word "misconception" is as an umbrella term which encompasses a wide variety of cognitive frameworks. Any conception of a mathematical notion which differs from the formally accepted definitions is considered a misconception. In other words, a conception which is lacking only a few details to be accurate and one which is fully built on erroneous assumptions will both be referred to as a misconception. Considering the potentially misleading nature of such a broad definition, we propose a classification system for misconceptions which involves six levels. Taking inspiration from the literature and our experience as educators, we precisely define the typical behavior that would correspond to each of these six misconception levels, and construct a spectrum of understanding ranging from the lack of a conception to the expert conception. As such, we do not consider those typical misconceptions to be a one-size-fits-all. Rather, students' understanding of any given concept might lie on different levels of this spectrum.

The misconceptions that were chosen to be investigated in this study are the following:

- Limits can never be reached[2]
- The implicit monotonicity of convergent functions and sequences
- $0.\dot{9} < 1$[3]
- The limit only as a dynamic process and never as a mathematical object
- Cluster points are equivalent to limits
- Infinity as a number
- EA and AE statements

These seven misconceptions are well-documented in previous research and provide an important theoretical background for the purpose of this study.

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[1] We will use this very informal language that students and sometimes instructors use to mean that the function does not take the value of its limit at any point. In mathematical notation, if $\lim_{x\to\infty} f(x) = L, \forall\, x, f(x) \neq L$.

[2] In the same sense as mentioned in footnote #1.

[3] The dot refers to an infinite decimal expansion.

We hypothesize that in the case where students hold misconceptions beyond their Calculus studies and into their learning of Real Analysis, the past elementary misconceptions might influence their learning in such a way that they develop related and more advanced misconceptions regarding concepts in Analysis. Previous research (Driver & Easley, 1978; Resnik, 1983; Dubinsky & Yiparaki, 2000; Przenioslo, 2004) has documented that misconceptions can sometimes persist through instruction. Considering that mathematics is a subject that builds on prior knowledge, it is logical to believe that misconceptions might influence further learning and foster the development of new misconceptions. Hence, our study also involves an inquiry into misconceptions on topics in Real Analysis, inspired by our own experience and that may relate, in one way or another, to the common Calculus misconceptions that we listed above. We consider three such misconceptions, which are:

- Assuming a link between boundedness and convergence of sequences
- Infinity as a supremum
- Supremum is the same as maximum

Therefore, considering the seven misconceptions that relate to Calculus, and the three that might arise in Real Analysis, we set the objectives of this study as follows:

- Uncover unsolved misconceptions about topics learned in Calculus.
- Inquire into the vanishing of misconceptions related to Calculus.
- Investigate the presence of misconceptions related to topics in Real Analysis that may replace or evolve from previous Calculus misconceptions.

As such, we gathered 10 student-participants, on a voluntary basis, who were enrolled in an Analysis course. Throughout this thesis, when mentioning concepts of Real Analysis, we are referring to concepts which are introduced in the first and second Real Analysis classes (Analysis I and Analysis II), in which the student-participants were enrolled.

Our initial idea for a research tool was to use task-based interviews (Goldin, 2000). However, the arrival of the global COVID-19 pandemic prevented us from using any method that involved proximity with people, including interviews. We considered using online meetings to go forward with our initial plan, but this idea was discarded to avoid overstressing the participants in these particularly demanding times (we gathered data in the early times of the pandemic). The reality of the pandemic disrupted most people's personal, academic and professional lives, and we decided to devise a research tool that would contribute as little as possible to this disruption. To this end, we chose to create a questionnaire comprised of 26 questions to gather data that would allow us to address our research objectives. The questionnaire was administered through the online platform Moodle, in a dedicated page that could be accessed only by the participating students and by the research team. The students were instructed to take the survey at a time of their convenience and to respond to it using only their own understanding of the concepts emphasized in the questions. Once all the participants had completed the questionnaire, we began sorting through their answers and analysing their potential misconceptions.

The analysis process consists in a careful inspection of the students' answers to the questionnaire, with particular attention to reasonings that correlate to the misconceptions that are considered in this study. The answers that we found to involve faulty reasonings were evaluated using a set of criteria specifically selected to help us determine which misconception level corresponds most closely to each answer (Table 4.41). Then, we considered each participating student's questionnaire in its entirety to identify reoccurring thought patterns that would support the presence of misconceptions, or conflicting reasonings that would indicate otherwise. The entirety of a participant's conceptions was then considered together to determine the presence of misconceptions and the effects that these may have on their understanding of concepts in Analysis.

1.2 Summary of the results and conclusions

The data gathered from the questionnaire demonstrates that Analysis students' knowledge of elementary Calculus can still be lacking. We identified several common misconceptions in the participants of this study and evidence of other misconceptions which we initially did not expect. In addition, we observed some students who had, sometimes serious, issues with mathematical logic and notation.

For the misconception designated as "a limit cannot be reached," we identified five students (out of the 10 participants) whose answers hinted at them holding this misconception. One such student outright claimed they believed functions to never reach their limits.

Six out of the 10 students who were surveyed provided evidence that they believed that functions cannot reach their asymptotes. Certain students explained that they believe the difference between limits at infinity and asymptotes to be that limits can be reached while asymptotes cannot. We surmise that this misconception is due to students picturing basic examples of functions (such as $\frac{1}{x}$) that have horizontal asymptotes and generalizing from this image.

A single student provided answers that imply they hold the misconception "$0.\dot{9} < 1$." In their answer, the student made some peculiar claims about the nature of these two "distinct" numbers and the role played by infinity in infinite series. From their answer, it is clear they hold some misconception about the nature of numbers and their understanding of infinity. However, the origin of this misconception is difficult to identify only from their answer. We surmise that this student's conception of real numbers might be affected by "noise" from their current studies in Real Analysis.

Three students gave answers implying that limits are exclusively dynamical processes, as opposed to also being considered as (static) mathematical objects. These students used the Sum Law for limits at infinity with functions that diverge. This incorrect use of the Sum Law indicates not only that the students do not recall the conditions that need to be met to use this theorem, but also that their understanding of limits as static mathematical objects is weak.

Seven out of the 10 participating students provided answers where they used infinity as a number or alluded to infinity being a valid value for a limit. Three of these students claimed that the Sum Law could be used for functions that diverge to infinity, therefore performing arithmetic on infinity and using it in mathematical equations. Other students provided a variety of different reasonings where they suggested to use the symbol for infinity as we would any real number, or simply claimed that a certain limit was *equal* to infinity. This particular misconception can be revealed in many different ways since infinity is a crucial concept to Calculus and Analysis. The potential causes for these misconceptions and students' different uses of infinity are discussed individually for each student in Chapter 5.

In general, we found that the participants in this study understood quite well the meaning of the universal and existential quantifiers. Furthermore, most students appeared to properly understand mathematical syntax. However, 3 students provided answers to the questionnaire that pointed towards their misunderstanding of mathematical notation. Their answers appeared to imply that the syntax of mathematical statements caused misunderstandings.

Although students' understandings of, and misconceptions around mathematical logic were not purposefully investigated, certain argumentation errors were observed. We found several instances of students misusing mathematical logic and providing arguments that were either illogical, redundant, or simply invalid for other reasons. We claim these errors to be significant – being a source of, or strongly contributing to students' misunderstanding of key concepts or developing misconceptions about these key concepts. We identified logical fallacies in the reasonings of students of Real Analysis, which, given that it is a proof-based course, makes us question what students "really" understand of the theorems and their proofs that are at the heart of the course.

1.3 Structure of the thesis

The next chapter, Chapter 2, is a review of the literature in the field of didactics and about misconceptions. We situate our research around what has already been done and discuss how prior research informs our decisions and the design of this study.

In Chapter 3, we present our theoretical framework. We discuss in depth the assumptions that are necessary for this study. We thoroughly define the terms that are often used in this thesis. We also expand on the misconceptions that are investigated and discuss their possible causes and effects. Moreover, we describe how each misconception may be observed. Lastly, we propose a model of a classification system for misconceptions based on the severity of the conceptual misunderstandings observed.

Chapter 4 details the methodology used for this research. As previously mentioned, our research tool consists in a questionnaire that the students may complete from home on the online platform Moodle. We discuss the recruitment of participants, the administration of the data-gathering tool and the implications of having resorted to an online questionnaire on the integrity of the data collected. We also explain how each question allows us to identify misconceptions, and therefore how the choice of the tool, despite its disadvantages, allowed us to achieve our research objectives. We close this chapter with a presentation of the analysis procedure that is used to transform the completed questionnaires into usable data.

The fifth chapter is a presentation of the gathered and processed data. We discuss how certain answers provided by the student participants may reveal some of the expected seven misconceptions. In addition, we discuss the possibility that further misconceptions may be caused by those misconceptions, especially as they relate to Analysis. We also consider the possibility that the students hold certain misconceptions that were not anticipated. Chapter 5 closes with a discussion of the advantages and disadvantages of the research tool.

Chapter 6 consists of a series of discussions around our results. We open this chapter with a detailed explanation of a conjecture that relates to our classification table presented in Chapter 4 (Table 4.41). We then dive into deeper analyses of three students who appear to hold several misconceptions. We discuss how each of the apparent misconceptions affect those students' response patterns in the context of this study and how they may influence further learning. Finally, we discuss the special case of the effects that misunderstanding mathematical language and logic can have on learning.

In Chapter 7, we present our conclusions to this research. We discuss the findings of the study as they relate to our research goals, and we suggest further research that could advance our knowledge of mathematical misconceptions.

Chapter 2 Literature Review

This chapter provides context for the purpose of this study. We review the past work that has been done in the field of misconceptions research as it relates to mathematics and more specifically to some topics learned in elementary Calculus.

In Section 2.1, we present the findings of past research regarding misconceptions about topics in Calculus. We take inspiration from these studies and articles to surmise which misconceptions are most likely to still be present in Real Analysis students' understandings of elementary concepts.

In Section 2.2, we discuss the concept of cognitive conflict and its role in learning. We believe that exposing students to problems that could challenge their misconceptions is an effective way to reveal said misconceptions and help students overcome them. Past research regarding conceptual conflict informs us on the validity of this method in student learning.

2.1 Examples of misconceptions that have been observed in previous research

To investigate the vanishing of misconceptions, we must identify misconceptions that have been observed in past research and that are likely to be observed in the participants of this study. Issues with learning the concept of limits have been considered from many points of view. For instance, Tall & Vinner (1981) have discussed the different ways in which students' concept images can include the false idea of a sequence being strictly bounded by its limit at infinity ("the sequence never crosses its limit"). They describe how certain informal explanations that occur in class or appear in textbooks, can introduce this idea in students' conceptual framework of sequences and limits. The issue of misleading vocabulary is raised by Cornu (1980), Davis & Vinner (1986), Sierpińska (1990) and Monaghan (1991), among others. More specifically, these authors discuss the role that the dynamic connotation and the non-mathematical meanings of the terms "tends to," "approaches," "goes to," and other similar phrases, can have on students' conceptions of limits. Monaghan (1991) found that the phrases "tends to" and "approaches" are usually perceived by students as synonymous, most likely due to their similar everyday definitions. The term "limit" can cause issues since its non-mathematical meaning is usually synonymous to "boundary." Transposing this interpretation into a mathematical context can introduce a misconception of limits which involves only strictly monotone functions or sequences.

Przenioslo (2004) has observed several different student conceptions regarding limits, some of which include the criterion of functions and sequences being strictly monotone. The author specifically mentions the misconception where a certain value g is a limit if and only if the sequence (or function) approaches monotonically the asymptote $y = g$. This specific misconception raises a question: Can misconceptions about asymptotes interact with a student's understanding of limits? Considering that horizontal asymptotes are defined as the limit at infinity of a function, we hypothesize that students who hold some version of the misconception where sequences (or functions) are inherently monotone may also assume that asymptotes must be approached without being crossed.

The concept of infinity is famously difficult to grasp for inexperienced students. A number of misconceptions regarding infinity have been observed, discussed and described (e.g., Tall, 2001; Sierpińska, 1987; Sierpińska, 1990; Lara-Chavez & Hitt, 1999). For instance, Sierpińska (1987) has described epistemological obstacles in the learning of limits, including four attitudes regarding infinity. Those attitudes towards mathematical objects were found to be serious obstacles to learning crucial concepts such as limits, and Sierpińska asserts that mental conflict might be a starting point to overcome those attitudes. Sierpińska (1990) adds to the issues relating to infinity by discussing mathematical notation, and the different views that can be created from it. In this article, she discusses the case where students assume that a limit at infinity is equivalent to the value of a sequence when *evaluated* at infinity. This misconception can be supported by the occasional abuse of notation of using the symbol ∞ in mathematical equations. As a result, students might then assume that infinity is a real number, or at least can be viewed and used as such in the context of elementary Calculus. In addition, certain textbooks used in Real Analysis courses (e.g., D'Angello & Seyfried, 2000) introduce the concept of extended real numbers early in the text (Chapter 2) and they allow infinite upper and lower bounds. For students who are not experienced with the concept of boundedness, using extended real numbers might contribute to misconceptions on infinity. Sequences diverging to infinity might be considered convergent, and the algebraic manipulation of diverging sequences might become particularly confusing. These misconceptions are bound to have a tremendous impact on student learning and our study investigates these issues directly.

A concrete example of the effect that misconceptions can have on students' understanding of mathematics is the non-mathematical distinction between the numbers $0.\dot{9}$ (this notation represents the number 0 followed by infinitely many 9's after the decimal point) and 1. Students who misunderstand the concepts of limits of sequences or limits of partial sums to some extent, perceive these notations as representing different numbers. This specific problem has been explored by many researchers including Tall & Schwarzenberger (1978), Davis & Vinner (1986), Sierpińska (1990) and Lara-Chavez & Hitt (1999). The assumption that there is a distinction between these numbers can be caused by several sources. Tall & Schwarzenberger (1978) discuss the link between this misconception, the decimal expansion of real numbers, and sequences of real numbers. In their article, they discuss how decimal numbers can be expressed as sequences of approximations, in which case certain rational numbers and all irrational numbers would constitute an infinite sequence. Here lies the link between misconceptions regarding limits of sequences and the actual values of decimal numbers: a student who only views limits as an infinite process (and never as objects) might not understand the equivalence among the different notations. Calculus students must also understand limits as static objects that represent the approach of a function or sequence, and not only as the process of the approach itself. Students, however, see the notation representing the periodic expansion $0.\dot{9}$ as a sequence that is actively approaching 1, and not as a static value that is indistinguishable from 1. Lara-Chavez & Hitt (1999) discuss a similar issue in terms of potential and actual infinity. For a student who exclusively understands the concept of potential infinity, the number $0.\dot{9}$ might appear to be smaller than 1. This student's conceptualization of those numbers and of limits of sequences cannot be fully accurate unless they include the idea of an actual infinity.

Hah Roh (2008) observed a specific misconception in their study, which jeopardizes the crucial concept of uniqueness of limits. They observed students conceptualizing limits of sequences as a value which has infinitely many terms in its neighborhoods. This definition corresponds to a cluster point (also called accumulation point), which is distinct from limits for many reasons. The most important distinction between these objects is the fact that limits are always unique, while a single sequence may have multiple cluster points. We intend to investigate students' understanding of limits and cluster points, especially since the chosen participants are actively enrolled in a Real Analysis course, which typically introduces the concept of cluster points.

The last misconception that inspired this study relates to the distinction between universal and existential mathematical statements. Dubinsky & Yiparaki's (2000) study on the matter has shown that students of varied backgrounds and experience with mathematics seem to be unable to use mathematical conventions to interpret logical statements. In many cases, the students would produce valid answers to statements that are given in common English, but severely misunderstand the meanings of quantified mathematical statements. Some students would see no difference between two statements where the $\forall$ and $\exists$ quantifiers appear in a different order. In this thesis, we refer to this type of statements as AE and EA statements. Furthermore, Selden & Selden (1995) found that undergraduate students' ability to "unpack" mathematical statements is extremely low. The misconception that interests us relates to the following question: do students know the distinction between statements where the universal quantifier comes before the existential one, and those where the existential quantifier appears first. In addition, the misconception of equating the meaning of these two types of statements and Selden & Selden's (1995) article raise the more general question: can university students accurately understand logical mathematical statements? It goes without saying that a proper understanding of mathematical syntax is necessary to learn any kind of advanced mathematics, hence our interest in this misconception.

In the literature referred above, we identified evidence of 7 misconceptions that appear to be common and widespread:

- A limit can never be reached
- The implicit monotonicity of convergent sequences and functions
- $0.\dot{9} < 1$
- The limit only as a dynamic process and never as a mathematical object
- Cluster points are equivalent to limits
- Infinity as a number
- EA and AE statements

Each of these misconceptions are defined in detail in Chapter 3. From the literature and our own experiences teaching Calculus and Analysis, we assume that students can hold several of these misconceptions at the same time. We use these misconceptions as the basis for the creation of a questionnaire that will aim at identifying each of them individually.

Our interest in these misconceptions and their evolution in students as they progress with their learning of more advanced mathematics rests on our assumption that if held for a very long time, they can seriously impede on students' ability to learn more advance topics in mathematics. Past studies have shown that even students who successfully pass their classes can hold misconceptions and carry them forth (e.g., Driver & Easley, 1978; Resnik, 1983; Przenioslo, 2004). In fact, the following quote taken from Przenioslo (2004) precisely motivates our interest in the misconceptions held by students of Real Analysis:

"Often far removed from the accepted concept definitions, these convictions were not sufficiently or at all corrected by taking university Analysis courses. In fact, new incorrect associations were added to those developed at the university." (p. 129)

Therefore, in our study we explore whether students who have successfully passed their Calculus classes still hold elementary misconceptions and surmise how and if those misconceptions can hinder their learning of Real Analysis.

In addition, research regarding the actual vanishing of misconceptions is very scarce. Many authors discuss the nature of learning and possible explanations for students overcoming their misconceptions (e.g., Driver & Easley, 1978; Davis & Vinner, 1986; Smith, diSessa & Roschelle, 1993). However, very few studies involve actual observations of the mechanisms by which misconceptions are overcome. Hence why our main objective with this study is to propose a model of classification for the severity of students' misconceptions and consider the possibility that the overcoming of misconceptions results in the acquiring of more accurate knowledge and understanding, therefore progressing towards an expert understanding of mathematical concepts.

2.2 On the topic of cognitive conflict

Tall & Vinner (1981) describe concept images as "the total cognitive structure that is associated with the concept." They observe cognitive conflict as a result of students' concept images disagreeing with formal definitions. Certain authors believe that such a conflict may be a starting point for overcoming misconceptions, including Hewson & Hewson (1984) and Sierpińska (1987). These researchers argue that controlled conceptual conflict might be a viable strategy to induce a change in student's conceptions. They consider presenting students with different perspectives that may be conflicting with their current understanding of mathematical concepts, therefore encouraging the students to think critically and reshape their own conceptions. We agree with this theory especially since students' understanding of mathematical concepts is often shaped by routine exercises (e.g., Selden, Selden, Hauk & Mason, 1999; Lithner, 2000; Lithner, 2004; Hardy, 2009). Cornu (1981) states that exposing students to non-routine exercises can help researchers and educators alike uncover these misconceptions, which may slide 'under the radar' without those conflicts. We chose to use the idea of cognitive conflict to devise our research tool. Misconceptions are to be revealed through exposure to exercises that challenge them.

On the other hand, Smith, diSessa & Roschelle (1993) argue, from the perspective of constructivism (which asserts that learning builds onto prior knowledge), that misconceptions do not simply vanish and get replaced. The belief that incorrect conceptions are simply discarded and replaced has a lot of evidence against it. For instance, Davis & Vinner (1986) have observed students who held onto multiple different conceptions simultaneously having to recall the correct ones in given circumstances. The basis for our hypothesis relies on this point of view. We claim that students' conceptual frameworks need to be transformed into concept images (in the sense of Tall & Vinner, 1981) that involve more detailed elements, procedural and propositional knowledge, and fewer incorrect elements. We do believe that cognitive conflict can serve as a mechanism to prompt the acquisition of more accurate conceptions. Our research tool is designed with the intent of provoking the first step in this 'transformation'; causing cognitive conflict in students who hold misconceptions in order to reveal misconceptions.

In the next chapter, we introduce our theoretical framework. In the first section we discuss the definitions, assumptions and assertions that are necessary for this study. In the second section, we describe in depth the misconceptions that are being investigated. In the third and last section of this chapter, we propose a model for classifying misconceptions which we use for the results and analysis of the data.

Chapter 3 Theoretical Framework

In Section 3.1, we describe the primary assumptions and assertions that are necessary for this study. We also define the vocabulary that is used in this thesis. In addition, we explain three possible sources for common misconceptions and provide examples for them.

Section 3.2 describes the choice of specific misconceptions that this research investigates. We explore different perspectives on each misconception based on the literature and our own experience. We also discuss the possible sources for such misconceptions and the harmful effects they might have on a student's mathematical education.

Finally, in Section 3.3, we posit a model for classifying misconceptions. We suggest six categories to which a student's understanding of a given concept might correspond. These six categories are organized hierarchically according to "how accurate a given (mis)conception is", with the final level being "expert conception," which can be understood as a lack of a misconception.

3.1 The concept of misconception in mathematics education

The first supporting theory that we use as a basis for our study is that of constructivism. The main claims of this theory are that all learning is dependent on prior knowledge, and the rejection of the so-called "blank slate theory." We consider the misconceptions explored in this study to be a result of improper or incomplete learning paired with previous knowledge that is acquired in non-mathematical settings. As suggested by prior research (e.g., Driver & Easley; Resnik, 1983; Davis & Vinner, 1986; Przenioslo, 2004), students start building their understanding of mathematical concepts before these are formally introduced. Their previous conceptions often remain and perhaps coexists with new ones, even after further instruction. In addition, Przenioslo (2004) asserts that continued instruction may add new misconceptions over the pre-existing ones, without contributing to their improvement. As discussed in the introduction, our goal is to investigate whether students' typical Calculus misconceptions, as described in the literature, persist through their mathematical education, and influence their understanding of concepts in Real Analysis.

Let us address the meaning of the term "misconception," which we are constantly using throughout this thesis. Countless words have been used to describe a conception which is not fully formed: misconception (Davis & Vinner, 1986; Hah Roh, 2008;), preconception (Clement, J. 1982), alternative conception (Hewson & Hewson, 1984), alternative framework (Driver & Easley, 1978), primitive idea (Lara-Chavez & Hitt, 1999), concept image (Tall & Vinner, 1981; Prezniosło, 2004), naïve theories (Resnik, 1983). These terms refer to a similar idea: the entirety of a person's knowledge and understanding of a mathematical concept, which may not be complete or in accordance with the accepted mathematical definitions and properties. In this study we choose to use the word *misconception* as an umbrella term which encompasses a number of these phrases. We posit that a wrongful understanding of a concept can happen at varying levels and have deeper or weaker effects on a student's ability to conceptualize mathematical objects and use them in an academic setting. In Section 3.3 we detail 6 levels of misconceptions. We reuse some of this vocabulary, but we attach a carefully chosen definition that will remain consistent in this paper. To correspond to a specific level, a student's conception should be *consistent* across their mathematical framework. That is, if exposed to different situations that involve the same concept, a given student might provide reasonings that are lacking in similar ways. We choose to consider students' conceptions as more than simply "correct" or "incorrect." Rather, we consider students' misconceptions to lie on a *spectrum* of understanding. Moreover, we consider that students' conceptions are *mutable* and *changing* and may *progress* or *regress* on this gradient as they study mathematics.

These misconceptions can originate from three possible sources: epistemological obstacles (Bachelard, 1938; Sierpińska, 1987; Cornu, 1991), cognitive conflict (Tall & Vinner, 1981; Sierpinska, 1990) and didactic obstacles (Lara-Chavez & Hitt, 1999; Hardy, 2009). Epistemological obstacles and cognitive conflict affect the human mind. They are respectively related to the concepts themselves and to the stages of cognitive development of individuals. They can appear as unjustified beliefs that influence the student's ideas relating to mathematical concepts. Epistemological obstacles can be typically observed through the historical development of a concept. For example, the apparent human intuitive representation of the concept of infinity and the limitations to visualize the concept. Cognitive conflict refers to inconsistent knowledges and that are often constructed because of different experiences. For example, issues with the concept of limits can be caused simply by the words used to designate it. In linguistic communities, the term "limit" is synonymous to "boundary" in non-mathematical contexts. Students often struggle to separate the mathematical and non-mathematical meanings. To be functional and fluent in "everyday life" and in mathematics, one needs to be able to hold the two "conflicting" meanings and know when each applies.

Didactic (or institutional) obstacles result from teaching approaches (Hardy, 2009). For example, Lithner (2000) shows how the exercises to which students are exposed in Calculus courses result in the routinization of knowledge that is non-mathematical in nature, or mathematically incomplete (see also Broley, 2020). Furthermore, Lara-Chavez & Hitt (1999) have observed that misconceptions held by teachers can be passed down to their students. Didactic misconceptions also result from didactic approaches in textbooks (Raman, 2004).

In addition to these different sources of students' misconceptions, we also recognize that learning is a complex process influenced by a variety of experiences that occur inside and outside the classroom and prior, during and after formal instruction. We believe, therefore, that identifying the actual source of a misconception, in general or for a particular student, might often be a daunting task, if not an impossible one.

3.2 Our choice of concepts

Different types of misconceptions that occur in the learning of limits are well-documented by previous research (see Chapter 2 for details). Based on the literature and our own experience as mathematics students and educators, we have selected seven of these well-documented misconceptions to use as the underlying structure of this study and as the background to design and analyse students' responses to the questions we posed to them. In what follows, we describe each of these seven misconceptions and some of their possible sources.

3.2.1 A limit can never be reached.

This common misconception has been observed and described by many authors (Davis & Vinner, 1986; Tall & Vinner, 1981; Sierpińska, 1990; Lara-Chavez & Hitt, 1999; Mamona-Downs, 2001) throughout the years. From the literature and anecdotal personal experience, we believe this misconception is most often found when considering limits at infinity, as opposed to considering limits at a point. The actual incorrect conception can be described as follows: A function which has a limit L when x tends to infinity cannot be equal to its limit at any point. In mathematical terms:

$$if \lim_{x \to \infty} f(x) = L, then\ f(x) \neq L\ for\ every\ x$$

This type of incorrect understanding of the concept of limit can be attributed to many different sources.

Many mathematics students are exposed to the concept of limit before they become comfortable with formal mathematical notation. As such, the definition of limit might be lost to some students, and they may rely on informal definitions, particular drawings, and examples. Some mathematics educators might be familiar with the phrase "a limit is a numerical value which a function approaches but never reaches." This is an example of an informal way of describing limits which is inherently flawed and contributes to the perpetuation of this misconception.

Some may believe that a possible fix for this misconception is learning the analytical definition of limits:

$$\lim_{x \to \infty} f(x) = L\ if\ \ \forall\, \varepsilon > 0, \exists\, M > 0\ such\ that\ |f(x) - L| < \varepsilon\ whenever\ x > M$$

Although we agree that a deep understanding of this definition would prevent students from holding this misconception, in practice, this is difficult to learn. As mentioned above, most students are exposed to the concept of limits before they are comfortable with rigorous definitions, the basis of Analysis, and analytical notation. Since it is not explicitly mentioned that the case $f(x)\text{-}L=0$ is included in this definition, students might unconsciously rule-out this specific case and consider functions as ever-approaching and never-reaching of the limit (that is, they interpret that it is always the case that $0 < |f(x) - L|$).

Another possible source of this misconception is what Tall and Vinner (1981) have referred to as "concept images." They defined this idea in the following way:

"We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (p. 152)

They describe this phenomenon as being independent from formal definitions. As mentioned in Section 3.1, for our purpose, we will consider as misconception a concept image which carries any kind of erroneous elements (and may carry some correct elements). Among the various examples of students holding erroneous concept images of limits, we find those who consider the function $f(x) = \frac{1}{x}$ as the template example of a function which has a limit as x goes to infinity. This example, as 'the' example, often misleads students to develop and further affirm the misconception in question. These students retrieve (consciously or unconsciously) this example whenever they must respond to a question which involves the concept of limit at infinity. A possible mistake which can arise from this comparison is assuming that a function which does equal its limit at some values of x does not have a limit. For example, the function $f(x) = \frac{\sin(x)}{x}$ may cause issues for some students whose concept image of limits is heavily influenced by the case of $f(x) = \frac{1}{x}$.

Another possible cause of this misconception is that of the colloquial (non-mathematical) use of the word limit. A web search yields the following definition for limit: "something that bounds, restrains, or confines."[4] It is plausible for a mathematics student to transpose this common use of the word limit into a mathematical context and this behavior could contribute to causing such a misconception (Tall & Vinner, 1981).

3.2.2 The implicit strict monotonicity of convergent sequences and functions

This misconception could be considered as a special case of the previous one. As discussed by Davis and Vinner (1986), this misconception is characterized by the association of notions of convergence and monotone convergence. Although these are distinct concepts, typical examples used in the classroom and Calculus mathematical textbooks of converging sequences (and functions) are often monotone (see the examples analysed by Lithner, 2000, 2004). Students associate the typical examples of monotone convergent sequences and functions with the notion of convergence itself thus assuming that converging sequences and functions must be monotone.

This misconception and the previous one might have a harmful effect on a student's understanding of asymptotes. Students who hold these misconceptions might conceive horizontal asymptotes only as the line which a function approaches monotonically, further assuming that an oscillating or otherwise non-monotone function cannot have an asymptote. This creates a disparity between the concepts of limits (at infinity) and asymptotes and may induce inconsistent response patterns. In addition, this arbitrary criterion (an asymptote is a line that is approached monotonically) applies for vertical asymptotes. Students may assume that horizontal asymptotes must also obey this rule.

[4] Definition from Merriam-Webster dictionary: https://www.merriam-webster.com/dictionary/limit

3.2.3 $0.\dot{9} < 1$

Note: The above "dot" notation refers to an infinite number of 9's following the decimal point.

This misconception has appeared in the literature (Tall & Schwarzenberger, 1978; Sierpińska, 1990; Lara-Chavez & Hitt, 1999) with different aspects that all boil down to one idea: the equivalence in the representation of numbers; or in other terms, the confusion between the representation of a number and the concept of number. Students who hold this misconception associate some representations with unique numbers; a manifestation of this misconception is the assumption that a 0 followed by infinitely many 9's after the decimal point is a different number than 1. It can be shown using geometric series that, indeed, $0.\dot{9} = 1$ but less experienced post-secondary mathematics students might not see this fact immediately, if at all.

A possible cause of this misconception relates to infinity. The idea of "infinitely many 9's" is likely to create confusion in anyone who does not have a solid mathematical understanding of the concept of infinity. Sierpińska (1987) has described the different possible attitudes taken by students regarding infinity in the context of sequences. All four of those attitudes include incorrect assumptions and rely on an incorrect understanding of the behavior of sequences. For example, some students consider $0.\dot{9}$ as the last term of a sequence and 1 as its limit. We assume that it is conceivable that a university student might adopt one of the attitudes described by Sierpińska (1987), and thus incorrectly conclude that $0.\dot{9} < 1$.

3.2.4 The limit only as a dynamic process and never as a mathematical object.

Students who hold this misconception think of limits only as a process or, similarly, they associate the concept of limit only with the algorithmic technique to find its value. Students may get used to being presented a function or a sequence and using the methods they have learned to find a numerical value for a limit without understanding what that result means. As such, it is rather unsurprising to find students whose understanding of limits is that of an ongoing process that halts when they find the answer to their question.

This misconception may get in the way of doing algebra on limits as this requires thinking of them as static mathematical objects.

Tall and Vinner (1981) among other authors have touched upon the role of vocabulary misconceptions such as the one described here as well as in 3.2.1 and 3.2.2. Words such as "approaches" and "going to" are an integral part of the mathematical vocabulary, so much so that we may get desensitized to the dynamic connotation and the non-mathematical meanings that these words carry. When less experienced students are exposed to these words in a mathematical context, more often than not, without any formal explanation, they apply (as well-documented in the literature) alternative, non-mathematical definitions to these words. Although this erroneous conception of mathematical words might not interfere with the learning of the methods to find limits that are taught in most calculus classes, Tall and Vinner (1981) assert that these misconceptions may have harmful effects on a student's learning of rigorous definitions later in their mathematical education.

Moreover, mathematics educators sometimes use different words to refer to the same principle. For example, the words “approach,” “going to, ” “converges” and “tend to” are used interchangeably by mathematicians to refer to the behaviour of a function or a sequence. It is not clear what is the effect of this ‘loose’ use of words in students’ understanding or in students’ misconceptions.

3.2.5 Cluster points are equivalent to limits

These two concepts have a lot in common and students have been observed to confuse them (Hah Roh, 2008). As a reminder, the epsilon definition of the limit of a sequence reads as follows:

$$\lim_{n\to\infty} x_n = L \text{ } if \ \forall\, \varepsilon > 0, \exists\, M > 0 \text{ } such \text{ } that \ |x_n - L| < \varepsilon \text{ } whenever \text{ } n > M.$$

Hah Roh (2008) uses the following definition for cluster points:

“a point x is called a cluster point of the sequence $\{x_n\}$ if for every $\varepsilon > 0$ there are infinitely many values of n with $|x_n - x| < \varepsilon$.”

Thus, every limit is a cluster point, but the converse is not necessarily true. A sequence can have multiple cluster points but may have at most one limit. Students’ identification of cluster points with limits creates conflict with the uniqueness of limits.

Intertwined with this misconception is students’ difficulties with mathematical logic. Students in elementary Calculus courses and introductory Analysis courses have not been exposed or have not had the time to internalize the elements of mathematical logic. Considering $A \Rightarrow B$ as equivalent to $B \Rightarrow A$ is a well-documented misconception. While we don’t explore this misconception in this study, it is important to realize how it may be, as others, intertwined with the ones we chose.

3.2.6 Infinity as a number

Infinity is a concept that can be challenging for students to learn. Our choice to investigate this concept is motivated by the various reasons why infinity can be misunderstood by students and how this type of misconception can negatively affect their understandings of calculus. We conjecture that misconceptions about infinity can be carried over from early calculus into more advanced mathematics. This misconception focuses on one of the many ways students can misunderstand infinity, that is by considering it as a real number.

Several authors have discussed the challenges that come with using infinity in mathematics, such as Sierpińska (1987) who addressed the cognitive and epistemological aspect of this issue, and Lara-Chavez and Hitt (1999) who discussed the didactic nature of this misconception. In one way or another, most mathematics students have been exposed to the symbol for infinity being used as a number, for example: $\frac{\infty}{\infty}$. When learning about indeterminate forms, this expression is often used to abbreviate the case of a function expressed as the quotient of two functions in which both the numerator and the denominator diverge to infinity. Mathematicians do not confuse formal meaning and abuse of notation. However, students who are significantly less versed in formal meaning are likely to take those notation abuses as proper mathematical writing and ascribe meaning that seem reasonable to them. In that context, it is difficult to internalize that $\frac{\infty}{\infty}$ is not equal to 1 and that this notation does not mean the same thing as the quotient of two non-zero real numbers.

The informalization of notation before formal notation is internalized and understood may have several consequences on students' understanding: for example, hindering students' understanding of algebraic structures, of rules of logic, of limits as mathematical objects.

This misconception is mainly documented for students who have not been exposed to the concept of extended real numbers. For our study, the students are enrolled in Real Analysis classes and therefore, may have been introduced to this concept. In certain textbooks (e.g., Dangello & Seyfried, 2000), it is explicitly stated that infinity can be considered a bound in the extended real numbers. Students should be mindful of the difference between the real numbers and the extended real numbers and mention that they are using the latter if they consider infinity as a valid bound. We also consider that this concept might be a source of confusion to students who don't have a solid understanding of infinity and may introduce the misconception "infinity as a number."

3.2.7 EA and AE statements

By EA, we denote statements of the type "There exists… such that for all…", and by AE, statements of the type "for all… there exists…". It is well-documented in the literature that students struggle with the different meaning that results from the order of the quantifiers (Selden, Selden, Hauk, and Mason, 1999). This hinders students' understanding of mathematical definitions and theorems that rely heavily on a deep understanding of quantifiers: for example, the formal definition of limits.

Dubinsky and Yiparaki (2000) found that students are likely to interpret statements written in common English as AE, no matter how the statements are phrased.

Misinterpretations of mathematical statements using quantifiers can happen in a multitude of ways. One of the possible forms that this misconception can take is a student assuming that AE and EA statements are equivalent.

We include in this misconception any consistent misunderstandings of formal mathematical language. Regardless of whether a statement includes rigorous notation, every mathematical statement follows a syntax which is logically constructed. It is possible for students to misunderstand statements due to their lack of knowledge of the notation involved, but also due to their misunderstanding of the logic that serves as the foundation of mathematical syntax.

3.2.8 Further misconceptions related to Analysis

In this study, we consider three misconceptions that we surmise may arise during one's instruction in Real Analysis. These misconceptions have been chosen based on our experience as educators.

- Assuming an implication between boundedness and convergence of sequences

From the Monotone Convergence Theorem, we know that a monotone sequence of real numbers has a limit if and only if it is bounded. In other words, in the case of monotone sequences, the concepts of convergence and boundedness are equivalent. If a student mistakenly assumes that convergent sequences must be monotone, then they might be assuming that boundedness always implies convergence for any sequence. We expect this type of misconception from students who have trouble understanding convergence or who may rely on monotone examples to inform their conception of convergent sequences. This misconception would parallel the one we refer to as "implicit monotonicity of convergent functions and sequences," where a student assumes that a function that is convergent at infinity must be monotone. We suspect that this misconception might be transferred from Calculus into Analysis with a similar misunderstanding of convergence for the case of sequences.

- Infinity as a supremum

Much like the "infinity as a number" misconception that was discussed above, this misconception may be observed in students who assume that infinity is a valid supremum in the real numbers. We would expect this misconception to appear in students who also believe that infinity is a valid limit value. This would demonstrate a misunderstanding of the role of infinity in mathematics and/or of the definition of supremum. Moreover, certain textbooks (e.g., D'Angello & Seyfried, 2000) introduce the notion of extended real numbers quite early in the text (Chapter 2), in which infinite bounds may be allowed. This notion may compound the misconception of infinity as a number and jeopardize the learning of other concepts, such as boundedness and completeness.

- Supremum is the same as a maximum

Anecdotal experience shows us that students often consider these two concepts as equivalent. This misconception may originate in a misunderstanding of the definition of supremum or in a poor understanding of the structure of real numbers and of mathematical implication.

3.3 Model of a classification system for misconceptions

This section describes the model of a classification system that we have developed to analyse the students' conceptions. The goal of this system is to effectively label student's answers according to the depth of their understanding of the concept. We thus define six levels of misconceptions by taking inspiration from the literature and from our own experience.

- **Absence of a concept**: The student has received no instruction and has no preconceived ideas about the concept at hand. Some attempts can be made at interpreting the words. Generally, the student will admit outright that they have no knowledge of the concept at hand.

For example, an individual who has never studied mathematics, being exposed to the concept of combinatorics, may not have a preconception of the topic. The individual might not even recognize the words that are being used and hence cannot have, nor build, a misconception.

- **Preconception**: This is a preconceived idea which is formed without formal instruction. The learner recognizes their lack of formal understanding and draws meaning from the context. Generally, this level of misconception is not very stable or robust. It may be built based on the student's everyday life or on prior mathematical knowledge. The misconception they build is somewhat of a guess. In addition, the reasoning may be inconsistent from one step to the other as the student has no structure or framework to refer to.

  For example, the concept of sets is not completely foreign to someone who has not yet studied this mathematical object. An individual who has not been instructed about the concept of sets might have a vague idea of what the concept is, such as a "collection of objects." The individual may ascribe meaning to words such as "union" and "intersection" from the meaning they have in their everyday life. The individual's approach to working with the mathematical object of sets has no consistency or structure.

- **Alternative conceptions**: Alternative conceptions differ significantly from the formal definition of the concept itself but are stable, consistent, and resistant to change, as opposed to preconceptions. Students who hold onto alternative conceptions are aware of the existence of the concept and have built a framework for themselves. The student can explain their thought process which differs significantly from the formal definitions and accepted concepts. The method used may follow a consistent structure, but it has incorrect elements to it. The method may be incomplete (lacking details and formalism), but it is consistent and used with a certain degree of logic.

  For example, students use functions as an equation $y = f(x)$ from high school mathematics and are rarely (if ever) exposed to the formal definition of functions as being subsets of the Cartesian product. This is a type of alternative conception which begins in school but is internalized and difficult to break down once students start studying Real Analysis. This preconception is built by habit (routinization). Introducing a new, abstract definition to a basic concept that the students believe they understand well (as they have had success with their current conception in the past) can be confusing, and the student might prefer their conceptualization of functions over the abstract and formal one. This definition is new to them, and they often fail to see the need for it[5].

[5] This has didactic sources, in addition to cognitive or epistemological ones. An example from another domain: in high school, students are often introduced to algebraic manipulations to solve linear equations with examples that don't require them. Those examples can be so simple that arithmetic reasoning is sufficient to solve them.

- **Incomplete conception**: This is a conjecture arising from incomplete instruction. The student's concept image (in the sense of Tall & Vinner, 1981) is composed of isolated knowledge elements and they have completed their conceptions with conjectured details that agree with the student's experience of the concept at hand. The conjecture can be correctly applied to some situations, but it is not generalizable. The student can express their understanding of the concept using informal vocabulary, but their explanation is incomplete and not rigorous. The solutions provided by the student to most problem-solving situations are based on basic definitions, and the steps may lack continuity. The logic may be flawed as the student is conjecturing from past knowledge. Some seemingly arbitrary rules or guidelines may be used. The solution is often a procedural recreation of a simpler kind of problem. Only the basic notions are considered. The basics may be well understood but the more complex notions are omitted.

  For example, students may have a graphical understanding of continuity of functions. The intuitive[6] (and visual) explanation of a continuous function may go as follows: "a function which you can draw without lifting your pen from the page." This explanation fosters graphical understanding of the notion, which is not only insufficient, but also misleading. Similar to the situation of notation abuse that was discussed above, students do not necessarily understand the abuse of language and fail to understand the metaphoric role of the pen and the drawing in the thinking of continuity. If a student who only understands continuity of functions through this argument were to be exposed to a function that cannot be drawn (such as the Dirichlet function), they would not be able to argue in a way that feels natural to them and would be forced to conjecture from their experience to complete the problem or their understanding of the situation. Such a student may be able to intuitively guess that the Dirichlet function is not continuous, but they would find themselves incapable of justifying this. This misconception fits into this category as the student may have some correct understanding of the concept of continuity but this understanding is incomplete and fails to support the student's work in several situations. Their current knowledge could be sufficient for very basic situations or problems, but as soon as the student is exposed to more advanced problems, they may get stuck or end up conjecturing from their past and insufficient knowledge.

- **Unrefined conception**: The student's conception is robust but not rigorous. The students' conception is close to the formal definitions but lacking relevant details which can cause mistakes in certain contexts. The student is confident in their understanding of the concept and can apply it with consistency. This is a knowledge system that is consistent but lacking some elements to be expert (see the last level below). A student may be able to explain the concept using some formal vocabulary and abstract concepts, but the explanation lacks rigor and maturity. The method the student may employ to solve problems shows a good general understanding of the concept. Some details are omitted which may not affect their work in the problems they mostly have to deal with in introductory Analysis. This is not the abstract formalism that is expected from an expert.

[6] In this thesis, we use the term "intuitive" to refer to reasonings or explanations that are sourced in (possibly non-mathematical) prior knowledge. We consider an "intuitive" reasoning to be mostly built on surface thoughts and to not involve a deep consideration for formal definitions.

For example, a student might have a strong conception of continuity and differentiability, without drawing a proper connection between the two. Such a student might be able to use the two concepts separately but fail to understand that differentiability directly implies continuity. This unrefined conception might not prevent this student from succeeding in introductory Real Analysis. Since this connection becomes more relevant past the first Real Analysis course this student's misconception might eventually prove to be a hinderance to their learning, even though it might not prevent them from succeeding initially.

- **Expert conception**: This is a near identical conception to the formal accepted notion, used properly and showing a deep understanding of mathematical reasoning. The methods used are backed by a deep understanding of the formal definitions and consistent use of logic. A student who expertly understands a concept shows a mastery of the skills required to apply those concepts to specific situations. They can explain the connections and relations between the concept at hand and other concepts.

  For example, a student has an expert knowledge of the quadratic formula, if they not only remember the formula, but also know how to apply it, how it is derived and how it relates to other concepts (e.g., the vertex of a parabola, the number of zeros). We would consider such a student to have an expert conception of the quadratic formula. We recognize that expert understanding of any mathematical concept can take years to be acquired.

Previous literature has used some of the terms we chose here to refer to misconceptions. We use the word "misconception" as an umbrella term which includes the above categories and any conception that may describe the transition from one level to another. These six misconception levels are constructed hierarchically with the first three being expected to occur in students who have received little to no formal instruction about a given concept, and the last three being expected to be more common in students who have received formal instruction.

It is our conjecture that misconceptions do not simply disappear, but progress through these levels until a student's understanding of a concept corresponds closely to the "expert conception" level. We also believe that said progress is not linear. We assert that every individual's experience with misconceptions is different and may progress through the spectrum at a different pace. As such, we also conjecture that students will, at some point in their mathematical education, exhibit what we call "transitional behavior," where one person's conception might contain elements or features from more than one category simultaneously. We discuss this conjecture in-depth in Section 6.1.

In the next chapter, we explain the methodology of this research. We discuss the ethical ramifications of studies that include human participants as well as the process used to find volunteers. We break down our data-gathering tool, a questionnaire, and explain the purpose of each question in relation to our research goals. We then describe the method used to analyse the completed questionnaires.

Chapter 4 Methodology

In this chapter, we present a detailed description of the research design, the research tools, the choice of participants and the methods used to gather and analyse data. Initially, we considered task-based interviews (Goldin, 2000) as the tool to gather data for this study. However, the COVID-19 pandemic made person-to-person contact significantly more difficult and forced us to consider different approaches. We considered using online tools to conduct face-to-face interviews but discarded this idea on account of university students being stressed and overworked due to these extraordinary academic, professional, and personal circumstances. After some deliberations, we decided that a questionnaire hosted on a secure online platform would be an appropriate way to survey our chosen population given the overall constraints imposed by the pandemic. Even though this kind of questionnaire has some limitations, we consider it an acceptable compromise given the exceptional circumstances.

In Section 4.1, we describe the design of the research tool. Considering that our goals are to uncover unsolved misconceptions about Calculus and Analysis and to observe their evolutions, it is natural that the research tool should involve a substantial degree of student participation. The research tool was chosen to coincide with this goal, and the decision process that led to the development of a questionnaire is described in this section. We also explain how the survey was built and distributed to the participating students.

In Section 4.2, we address the usual concerns that come with studies involving human subjects, namely the choice of participants, the recruitment process, and ethical concerns.

In Section 4.3, we provide a thorough breakdown of the questionnaire itself. We explain the purpose of each question, detailing the mathematical topic at hand, the possible misconceptions that can be expected from students regarding this topic and the source of inspiration for each of these questions.

Lastly, Section 4.4 describes the crucial step of categorizing and analysing the participants' responses to the questionnaire. Since every question was open-ended, breaking down each answer into categorizable data has proven to be a very lengthy and challenging process. However, the unique model for classifying misconception levels that was described in Section 3.3 proved to be an effective tool.

4.1 Design of the questionnaire

In this first section, we explain the steps that led to our choice of data-gathering tool. A questionnaire has advantages and disadvantages, and this section explains why we chose it, and the means we took to mitigate the disadvantages. The main purpose of the questionnaire was to reveal students' conceptions about chosen mathematical concepts. Our choice of these mathematical concepts aligns with well-known and studied misconceptions. The questions had to reflect this purpose, and they were thus built with the intent of "pushing" the student participants to express their thought processes and to reflect on their own understanding of mathematical concepts. We will also discuss how the questionnaire was administered and which technological tools were used.

4.1.1 Choosing a questionnaire as a data-gathering tool

Questionnaires have several advantages, some of which are very in line with our goals for this specific research. We wanted a tool that would allow students to express their understanding of mathematical concepts while limiting outside factors. The main advantage that a survey has over other data-gathering tools is allowing the participants to take their time and thoroughly justify their answers to each question without the distractions and stress that may be imposed by an interviewer. This is ideal for our purpose since, in some cases, misconceptions can be deeply rooted. A quickly answered question might be insufficient for us to detect a misconception or to determine the source of the misconception. We thus encouraged the participants to take all the time they needed to answer each question to the best of their abilities, and the parameters of our online questionnaire also supported that.

Participating in a study that questions knowledge and understanding of mathematical concepts can be quite stressful to students, regardless of the tool that is used. We assume a questionnaire that has a very generous time limit can help reduce some of this stress. Not only does reducing stress contribute to the accuracy of the students' answers, but it also aligns with the importance of ensuring that our study is ethical.

There are, however, some issues with this choice of data-gathering tool. The main problem we have identified is the risk of disingenuity from the participants. Since there is no one supervising the students as they are taking the questionnaire, there is nothing preventing them from "cheating". To discourage dishonesty, we have made very clear to the participants that there is no incentive for them to use external sources to answer the questions. Their identities are known only to the research team. Their university professors would not even know that they volunteered to participate, even less so their answers to the questions. We made sure to thoroughly inform them about the purpose of the research, therefore emphasizing the importance of honesty. With this information, we believe that the participants had no incentive or reasons to use alternative sources of information while completing the questionnaire.

On the other hand, what would happen if a participant did cheat? It is reasonable to assume that a cheating student would answer at least some of the questions with very accurate definitions and justifications. Since the interest of this study is to investigate students' misconceptions, well-formulated answers are of no interest to us and are therefore disregarded. For more details about the analysis method, see Section 4.4.

4.1.2 The choice of medium

The way that we provide the volunteers with the survey is arguably as important as the survey itself. We used the platform Moodle as a medium for the questionnaire. This platform is usually used as a virtual classroom that allows professors to share documents, videos, assignments, quizzes, and a lot more with their students. We created a course on Moodle that would exclusively be used for the purpose of this study. The questionnaire was set up using the Moodle quiz tool.

Moodle allows us to carefully regulate the parameters of the questionnaire. We made it so students could only access one question at a time; if they wanted to proceed to the next question, they had to answer the previous one, and they could not go back to modify their answers. This eliminated the potential for subsequent questions to hint at the answers of the previous ones. Therefore, the students would answer according to their own understanding of the question and the concept and not with the information provided by the questions themselves.

The Moodle quiz tool requires that we impose a time limit for the completion of the questionnaire. Given the number of questions and their content, we made an informed guess[7] and set the time limit to 3 hours. For a relatively quick student, we estimated that one hour would be sufficient to complete the questionnaire. Allowing 3 hours would give a clear signal that there was no need to rush through the questions, relieving participants' stress and hopefully ensure they would take their time to answer honestly and thoroughly, perhaps even taking a break.

4.2 Context and participants

The chosen population for this study was mathematics students who were taking Analysis I or Analysis II at a large urban university in North America. These classes are, respectively, the first and second real analysis courses that undergraduate students in certain mathematics and statistics programs are required to take. We chose this population for a few reasons. We remind the reader of our research goals:

- Uncover unresolved misconceptions about topics learned in Calculus.
- Study the vanishing of misconceptions related to Calculus.
- Investigate the presence of misconceptions related to topics in Real Analysis that may replace or evolve from previous Calculus misconceptions.

As such, our interest was to observe which well-known and studied Calculus misconceptions (related to limits, functions, and sequences), if any, students still hold after having successfully completed their Calculus courses and while taking their analyses courses. Furthermore, we aimed at revealing if new misconceptions have arisen in addition to or replacing previous ones.

We recruited the participants on a voluntary basis. We contacted the professors teaching Analysis I and II in the fall semester of 2020 and in the winter semester of 2021, requesting that they send a pre-written email (see Appendix A.4) to their students. The email sent to the students explained the purpose of the study, the reason for our request and what kind of student involvement was required for the study. Interested students were able to directly email us and volunteer to participate. It was specified in the initial email that participating or refusal to participate would not affect their grade in the course and that their identity would remain confidential. After the willing students contacted us, we would then follow up (see Appendix A.5) with more details about their participation and with the consent form (see Appendix A.1).

As with any study involving human participants, there is a very strict protocol to ensure that the process of gathering data is ethical. Before the recruitment process began, I, as the primary researcher, completed all the required certifications to ensure that I was qualified to conduct this kind of research. The entire protocol for the recruitment and distribution of the questionnaire was submitted for review by the university's Human Research Ethics Committee which then granted permission to conduct this research with student participants.

[7] In a similar way in which professors 'guess-estimate' how much time students need to complete a quiz or exam.

The study required that the students be made aware of any risks that they may be subjected to and that they formally consent by signing an official consent form. This consent form included every piece of information that could affect their willingness to participate, such as the purpose of the study, the procedures, the risks and benefits, the conditions of participation and the details regarding confidentiality. The students were made aware that withdrawal from the study at any point was permitted and that any information whose use they were not comfortable with would be discarded as per their request. They received the consent form a few days before they were given access to the questionnaire. To simplify the process of signing the consent form, a copy of it was built into the questionnaire, as the very first question that the participants had to answer. To proceed with the questionnaire, they had to sign the consent form as their answer to the first question.

4.3 Breakdown of the questionnaire

In this section, we provide a detailed breakdown of the questionnaire itself. We first explain the misconceptions that were considered for this study and the overall organization of the questionnaire as it relates to those misconceptions. Next, we address the questions one-by-one, explaining how they are expected to reveal interesting conceptions, both from early calculus concepts, and from more advanced analysis concepts.

Before we began crafting the questionnaire, we identified the topics that we wanted to investigate, and categorized them. We then chose in which order the mathematical concepts should be explored and the best strategies to compel students to clearly explain their understanding of the topic at hand. From the literature, we identified 7 common misconceptions that we could use as the basis for our questionnaire (see Chapter 3):

1. A limit cannot be reached
2. The implicit monotonicity of convergent sequences and functions
3. $0.\dot{9} < 1$
4. The limit only as an infinite process and never as a mathematical object.
5. Cluster points are equivalent to limits
6. Infinity as a number
7. EA and AE statements

To anyone who has a little bit of experience with mathematics, it is easy to notice that some of these misconceptions have significant overlap. For example, misconceptions 1, 3, 4, 5 and 6 all relate to the concept of infinity. Therefore, some questions are built with the potential to uncover multiple misconceptions.

Although one would expect that students would overcome these misconceptions by the time they complete their calculus courses, we conjecture (see Section 1.2) that some may carry over from calculus to analysis. The questionnaire was constructed with a few goals in mind. First, we wanted to detect if the misconceptions are indeed still present in the participants. The first 21 questions are built with this goal in mind. We strived, however, to propose questions as open-ended as we could, to avoid leading students' answers. Thus, it is possible that participants might provide an answer that subverts our expectations. Second, we are interested in exploring in what sense misconceptions affect the students' answers to mathematical problems. Some questions invite the students to think a little bit more deeply about a given concept. We wanted the students to question their own understanding of those mathematical objects and methods with which they are expected to be competent. Finally, we designed the last 5 questions of the survey with the intent to uncover some new, more advanced versions of the previous misconceptions, or, potentially, new misconceptions. It is our conjecture that a basic calculus misconception that is not overcome by the student before they continue their mathematical journey can cause issues with the learning of the more advanced concepts. Based on our teaching experiences, we identified three misconceptions for which we have anecdotal evidence of their presence in analysis students:

1. Assuming an implication between boundedness and convergence of sequences
2. Infinity as a supremum
3. Supremum is the same as a maximum

Now that the general organization is explained, here is, question by question, a breakdown of the questionnaire that was administered to our participants. The questions are given verbatim and in italics.

Question 1: Consent form
The details of the consent form have been explained in the previous section (see also Appendix A.1).

Question 2: This question is to inform me about what you have been taught in your past calculus courses. Answer "yes" if you have learned about the topics, answer "no" if you haven't.

a. *Limits of functions*
b. *Continuity of functions*
c. *Limits of sequences*
d. *Convergence/Divergence of sequences*
e. *Convergence/Divergence of infinite series.*

Some topics can be more or less detailed in a calculus class, depending on many outside factors. We want to make sure that the information gathered from each students' questionnaire accurately reflects their understanding of a concept that has been taught to them. If a student were to answer "no" to any of these topics, we would disregard their apparent misconceptions related to such topics as we are not interested, in this study, in misconceptions that exist in the absence of formal instruction.

Question 3: Consider the statement: $\lim_{x\to\infty} f(x) = L, then\ f(x) \neq L\ for\ every\ x$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If possible, give one or more examples to explain your choice, your thinking and/or why you are not sure if this is true or false

d) Would your answer change if $\lim_{x\to c} f(x) = L$*? Please explain your answer with as much detail as you can*

This question is expected to reveal the well-documented misconception that a function cannot take the value of its limit ("a function does not reach its limit" or "[the graph of] a function cannot intersect [the graph of] its limit function") (Tall & Vinner, 1981; Davis & Vinner, 1986; Sierpińska, 1987; Monaghan, 1991; Mamona-Downs, 2001; Kidron & Zehavi, 2002; Hah Roh, 2008). We expect that some students may agree with the statement, as the most common concept image (in the sense of Tall & Vinner, 1981) that is associated with a limit at infinity is that of a strict monotone approach.

Also, a student could answer "false" while still providing an incorrect justification. We believe that such a case is possible if a student has a misconception about infinity and considers that the function would reach its limit at infinity. Sierpińska (1987) has classified different conceptions of infinity, some of which are prone to be misleading. We believe that it is possible that a student would respond with this type of justification.

The last sub-question's purpose is to prompt students into questioning the difference between limits at infinity and limit at a point. We expected that most students would correctly identify that the statement is false as any continuous function is a good counterexample. We consider, however, the possibility that a student's answer to d) contradicts their answer to a).

Question 4: Consider the sequence $a_n = (-1)^n$

Then, $\lim_{n\to\infty} a_n = 1$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

Question 5: Consider the sequence $a_n = (-1)^n$

Then, $\lim_{n\to\infty} a_n = -1$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

Question 6: Consider the sequence $a_n = (-1)^n$

Then, the sequence $\{a_n\}$ *diverges.*

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

Questions 4-6 are meant to encourage students to think carefully about their answers as they relate to their answers to the previous questions. We expected that most if not all participants would answer "false" to questions 4.a) and 5.a) since the sequence is quite simple.

It is possible for question 6 to reveal the additional misconception that any divergent subsequence must be unbounded. Although this is not one of our hypothesized misconceptions, we acknowledge that this is a possibility.

Moreover, if a student were to answer with "true" for both questions 4.a) and 5.a), this could indicate that the student applies the definition of accumulation points to the concept of limits.

Question 7: Consider $f(x) = \frac{\sin(x)}{x}$ *as x goes to infinity. You may refer to the graph below if needed.*

Then, f(x) converges to 0 as $x \to \infty$.

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain what is confusing you in this question.

Question 8: Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, the limit of f(x) is 0 as $x \to \infty$.

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain why the question is confusing to you.

Question 9: Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, f(x) tends to 0 as $x \to \infty$.

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know", please explain what is confusing you in this question.

Question 10: Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, f(x) approaches 0 as $x \to \infty$.

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain what is confusing you in this question

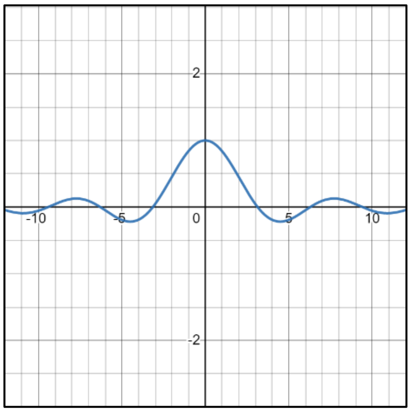
Question 11: Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, y=0 is an asymptote.

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

c) If you answered "I don't know" to part a), please explain what is confusing you in this question.



Note: the graph was provided once for each of questions 7-11, we included it only once in this case to avoid redundancy.

Questions 7 through 11 are different versions of the same question (different notation or different words to represent the same concepts). This block of questions was heavily inspired by John Monaghan's article "Problems with the Language of Limits (1991)." Monaghan (1991) explains that mathematics educators tend to use the words approach, tend, limit and converges interchangeably. Many other authors (Cornu, 1980; Tall & Vinner, 1981; Davis & Vinner, 1986; Sierpińska, 1987; Monaghan, 1991; Mamona-Downs, 2001; Hah Roh, 2008) have discussed this and its impact on students' learning. While it is true that these words mean the same thing in a mathematical context, it is also true that they have other, distinct meanings in other contexts and particularly, in students' everyday lives. Students' difficulties in ascribing mathematical meaning to words that have meaning in a context they are already familiar with, and how this leads to misconceptions, has been thoroughly studied by mathematics education researchers, especially by Cornu (1980), Tall & Vinner (1981), Davis & Vinner (1986), Sierpińska (1990), Monaghan (1991), Mamona-Downs (2001) and Hah Roh (2008).

We posed question 11 to investigate students' concept of asymptote. Anecdotal evidence suggests that students may not understand the relation between asymptotes and limits. In particular, if a student mistakenly believes that an asymptote must be a line which is strictly monotonically approached, this misconception could affect their understanding of limits as a whole. Students' confusion around the concepts of limits and asymptotes has been discussed by Hah Roh (2008), and we wish to explore this idea further.

Question 12: Consider the following expression: $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$

Then, the sequence converges to $0.\dot{9}$

Note: the $0.\dot{9}$ *notation refers to a 0 and infinitely many 9's after the decimal point.*

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

Question 13: Consider the following expression: $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$

Then, the sequence converges to 1.

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

With the previous two questions, we aimed at uncovering whether students successfully identify that $0.\dot{9}$ and 1 are conceptually the same number. These two questions were also inspired by Monaghan (1991). There are a few thought processes that can lead to different answers. A student might answer "true" to both questions, which would be correct if it is properly justified. It is expected that some students might list the first few terms of the sequence and, as a result, answer "true" to question 12 but for the wrong reasons. Their justifications for the following question would enlighten us about their understanding of partial sums.

Tall & Vinner (1981) have described this misconception as an issue between the modes of representation of numbers. Students could mistakenly consider $0.\dot{9}$ and 1 as different numbers for the simple reason that they are written differently.

These two questions allow for different correct answers if the students justify them properly. The numbers $0.\dot{9}$ and 1 are indistinguishable. Students may find a distinction due to a misconception about the limit of sequences being exclusively a dynamic process and not a static mathematical object.

Question 14: Consider the following expression: $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

In calculus, the symbol for infinity is often used as a placeholder for a limit (Sierpińska, 1990). This can be quite misleading to a less experienced student, as they might assume that the infinity symbol can be used in the same way that we use real numbers. In the notation for infinite series, it is common to use the symbol for infinity to refer to a lack of an upper bound (such as the left-hand side of the equation above). This may lead students to answer "false" to this question, if they are unaware of the formal notation or if they assume that infinity refers to the upper bound as opposed to a placeholder for the limit of partial sums.

Question 15: Consider the following conversation between two fictitious students, nicknamed A and B. The two students were asked to find the limit of the sequence $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$

A: Alright, I think we should start by computing the first few terms of the sequence to see better what the pattern is.

B: Good idea!

The students notice that the sequence is as follows: {0.9, 0.99, 0.999, 0.9999, etc.}

A: Just by looking at the pattern, it feels obvious that the limit is $0.\dot{9}$ *I don't really know how to prove it though.*

B: Actually, I think the limit is 1. When you look at the terms, they get closer and closer to 1 without ever reaching it, whereas at infinity, the sequence would reach 0.9˙ and the limit is supposed to never be reached.

A: But you can't reach infinity, you can't find the value of the "infinity-th" term.

B: I think we can reach infinity, look at it this way: $\sum_{k=1}^{\infty} 9(\frac{1}{10})^k$ *This is just an infinite series; we just need to figure out what it converges to using the usual tests.*

****

In the questions below, please, write as much as you can to clearly explain what you are thinking.

a) What do you think about A's statement: "Just by looking at the pattern, it feels obvious that the limit is $0.\dot{9}$*"?*

b) In your opinion, what does B mean when they say: "at infinity, the sequence would reach $0.\dot{9}$*"?*

c) What is $\lim_{n\to\infty} a_n$*? Clearly explain and justify your answer.*

d) If you were to help A and B to solve the problem, what would you tell each of them regarding their reasoning?

With question 15, we wanted to provide the participants with different thought processes that they may of may not agree with. The fictitious conversation is riddled with common misconceptions that we are trying to observe. If the participants agree with the reasoning of the fictitious students, their explanation may be quite revealing about their (mis)understanding of these concepts.

Sub-question a) prompts the participants to share their thoughts about the intuitive solution provided by the fictitious students. Agreeing with the statement might solidify any misconceptions exposed by question 12 and 13.

Sub-question b) has a similar role to a), but regarding infinity. Do the participants understand that infinity cannot be reached? Do they understand that computing an infinite sum corresponds to finding a limit?

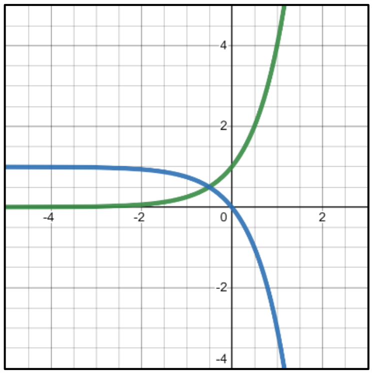
The goal of sub-question c) is to observe if their answer will be any different from questions 12 and 13. It is the same question, but now that they were provided with different thought processes, it is possible that their answers might change.

Finally, sub-question d) aims at allowing the participants to express their disagreements with the fictitious reasonings. If they noticed some issues, they could write their opinions on which train of thought should be followed, and which can be misleading in the context of a problem-solving exercise. We expected that their opinions about the fictitious conversation could reveal some interesting misconceptions.

Question 16: Consider the following conversation between two fictitious students, nicknamed C and D. You may use the graph below if needed.

The two students were asked to find $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$

For $f(x) = 4^x$ *(green curve) and* $g(x) = 1 - 2^{2x}$ *(blue curve)*



C: We should compute the two limits individually first, to see if they converge or not.

D: Good idea. Alright, clearly $\lim_{x\to\infty} 4^x$ *must be infinity, right? If x gets bigger and bigger, than so does* 4^x.

C: For sure, and I think it's similar for $\lim_{x\to\infty} 1-2^{2x}$. *As x gets bigger,* -2^{2x} *gets smaller and completely dominates the 1, so this would be negative infinity. But what's the sum of the two, then? Would it be zero since we add infinity to negative infinity?*

D: No, we can't do that. Infinity isn't like any number; we can't do algebra with it.

C: But look at the graphs, when we consider any x, f(x) and g(x) always sum to 1, it's like they cancel out. So, the sum of the limits must be 1 too, doesn't it?

D: I'm not sure. Maybe if we manipulate the functions a little bit, we could find something. Right, so $g(x) = 1 - 2^{2x}$, *but* $2^{2x} = 4^x$. *So,* $g(x) = 1 - 4^x$

C: So then, the function f(x) and the -4^x *part of g(x) would cancel out, and we're left with just 1.*

****

a) Do you agree with D when they say: "Infinity isn't like any number; we can't do algebra with it."?

b) What is $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$

c) If you were to help C and D to solve the problem, what would you tell each of them regarding their reasoning?

It is frequent for students to consider limits only, or exclusively, as an ongoing process (Tall & Vinner, 1981; Hah Roh, 2008) and not be able to conceive them as static, mathematical objects. Thinking of limits only as a process may result in the perception (misconception) that one can operate algebraically even when the object (the limit) does not exist. In the question, fictitious student C argues that limits are an ongoing process. They argue that at every point, f(x) and g(x) cancel out and yield 1, which means that the limit should do the same.

We expect that most if not all participating students will agree with the statement in sub-question a). However, we expect that they might not follow this statement in the following sub-questions.

To address sub-question b), students need to state that the question is itself wrongly posed – it has no answer. Their answer and justifications may uncover their perceptions of limits (as processes vs. objects) and help us identify the source of these perceptions.

Similarly to question 15, the last sub-question allows the students to express their disagreements with the logic of the fictitious students, and perhaps provide a different thought process.

Question 17: A student says that it is always true that $\lim_{x \to c} f(x) = f(c)$.

a) Is the student right? Clearly explain and justify your answer.

If you answer no to part a), continue with questions b) and c):

b) give one or two examples when the statement is not true. Can you give examples where the statement is true?

c) What conditions are necessary for the statement $\lim_{x \to c} f(x) = f(c)$ *to be correct?*

Sub-question a) aims at separating the students who know about continuity from those who don't. If a student answers that the equality is never correct, then the student might believe that a function can never reach its limit.

If a student answers that the equality is always correct, this would indicate that the student may view infinity as a number. The function may diverge to infinity and f(c) may be undefined, making the symbol "=" and the expression "f(c)" wrongly used.

It is also possible that a student agrees that the equality is correct simply by forgetting about the continuity condition, in which case their justification might enlighten us about their potential misconceptions.

Sub-questions b) and c) will help us determine which participants truly understand the concept of continuity when it comes to limits, allowing us to focus on the participants who may have a flawed understanding.

Question 18: Group the following functions according to any criteria of your choosing. Choose criteria that reflect your understanding of limits at infinity

a) $f(x) = \frac{1}{x}$
b) $f(x) = 4$
c) $f(x) = \frac{cos\,(x)}{x}$
d) $f(x) = -(e^x)$
e) $f(x) = sin\,(x)$
f) $f(x) = ln\,(x)$

Question 19:

a) Clearly explain your reasoning for why you put specific functions in specific groups.

b) What conditions must functions satisfy to be put in each group?

c) Although the question didn't give you the option to put the functions in multiple groups, are there functions that can belong in multiple groups?

Questions 18 and 19 are the only two questions that appeared on the same page in the Moodle quiz, as they fully depend on one another. These questions aim at identifying what types of convergence/divergence are fundamentally different for the participants. Their explanations for their choice of groups may give us some insight into their understanding of convergence and divergence as x goes to infinity.

For example, if option e) is grouped with other diverging functions, this could indicate that the student does not perceive “unbounded” and “divergent” as fundamentally different concepts, hence potentially showing some issues with the concept of infinity. If the student simply made a group for every diverging function, this case may not reveal any misconceptions.

A student who does not believe that a limit can be reached may be tempted to classify b) separately from other monotonically convergent functions, thus creating a group only for constant functions.

Separating monotone convergent functions from oscillating convergent functions could show that the student understands that a function can cross its limit. Of course, each of these examples depends heavily on the justifications provided by the students.

Question 20: When is the following equality correct?

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

Similarly to question 16, this question's goal is to identify which students know that limits cannot be manipulated like a linear transformation. We can expect some students to answer that this equality is always true. Such a response could arise due to a misconception about the behaviour of limits, or simply due to the student misremembering that the existence of both limits is a necessary condition.

Question 21: Consider the following definition.

$$\lim_{x\to\infty} f(x) = L \text{ if } \forall\ \varepsilon > 0, \exists\ N > 0 \text{ such that } \forall\ x > N, |f(x) - L| < \varepsilon$$

a) Re-write this statement in common English, and briefly explain its meaning.

b) Do you recognize what this statement defines?

From question 21 onwards, we aimed at revealing misconceptions that might arise from students' initial exposure to analytical thinking and the methods in analysis. We were also interested in misconceptions that might have carried over from the ones that we expected to observe in answers to the first twenty questions. Question 21 is especially focused on the understanding of quantifiers and the formal definition of limits. We expected that analysis students would be able to "translate" this statement to common English, but the actual meaning of it might be lost to them. In such a case, we can expect that the students do not recognize the definition of a limit. We are interested in students' understanding of this statement because properly understanding its meaning is at the core of overcoming several misconceptions we are studying.

Question 22: Consider the following two statements.

1. $\forall\ a > 0, \exists\ b > 0 \text{ such that } a > b$

2. $\exists b > 0, such\ that\ \forall\ a > 0, a > b$

a) Assuming that a and b are taken in the set of real numbers, are the two statements true or false?

b) Are the two statements equivalent? Give an example or a counterexample

This question was inspired by Dubinsky & Yiparaki's (2000) study titled "On Student Understanding of EA and AE Quantification."

This question investigates specifically the understanding of the universal and existential quantifiers. Our student participants are most likely not familiar with these specific examples and therefore need to rely only on their understanding of mathematical notation and real numbers to answer the question. We expect that students who hold misconceptions about the existential and universal quantifiers might mistakenly assume that these two statements are equivalent.

Question 23: Consider the following statement:

sup(A)=1 where A⊂R

Recall the following two definitions:

- *An upper bound b of A is called a supremum of A if, for all upper bounds z of A,* $b \leq z$
- *c ∈A is called the maximum if ∀a ∈A,* $c \geq a$

a) Explain in your own words what the statement means.

b) Can you give an example of a set A which satisfies this statement.

c) Consider the example you gave in b), is 1 the maximum of A? Clearly explain your reasoning.

This question aims at exploring students' potential misconceptions around the concepts of maximum and supremum. We want to question the students' understanding of these two concepts without relying on their memory of the definitions, which is why the definitions are provided. We conjectured that despite the definitions being given, students would rely on an incorrect meaning they have ascribed to the words maximum and supremum.

Sub-questions b) and c) may allow us to precisely identify this misconception, although that depends heavily on the choice of example and on the related explanation.

Question 24: Recall the following definitions:

- *An upper bound b of A is called a supremum of A if, for all upper bounds z of A,* $b \leq z$
- $c \in A$ *is called the maximum if* $\forall a \in A$, $c \geq a$

True or false? A is any non-empty subset of the real numbers.

a) If s is the supremum of A, then s is also the maximum.

b) If s is the maximum of A, then s is also the supremum,

c) It is possible for A to have a supremum and to NOT have a maximum.

d) It is possible for A to have a maximum and to NOT have a supremum.

Question 24 complements the goal of the previous question. A student who misunderstands the difference between a maximum and a supremum might answer "true" to the first two statements and "false" to the last two. Any other incorrect answer pattern may be hiding a different kind of misconception that should be investigated further.

Question 25: Consider any sequence which is bounded.

a) Does the sequence have a maximum?

b) Does the sequence have a supremum?

c) What can you infer about the limits of the subsequences of this sequence?

d) Is the sequence convergent?

This question tackles a few misconceptions. First, sub-questions a) and b) aim at identifying the same misconception as the two previous questions.

Sub-question c) leads the students into using the Bolzano-Weierstrass theorem. To properly answer this question, students need to understand the concepts of boundedness and subsequences. There is a chance that a student is unfamiliar with the theorem, in which case the answer to this question may provide information about the student's understanding of these concepts.

Sub-question d) has the goal of differentiating between convergence and boundedness. There have been a few instances in this questionnaire where students had the opportunity of differentiating these two concepts. This question directly inquires about their difference, an answer of "yes" will strongly hint at the presence of a misconception of either or both convergence and boundedness.

Question 26: Consider the set {A= ln(n):n=1,2,3, ...}

Recall the following definition:

- *An upper bound b of A is called a supremum of A if, for all upper bounds z of A,* $b \leq z$

True or false?

a) sup(A)=∞

b) This set does not have a supremum, nor an infimum.

c) The sequence $a_n = ln\ (n)$ *is monotone.*

The final question tackles the misconception that infinity can be a supremum. This idea is related to the misconception that infinity is "like" a number in Calculus and Analysis. Moreover, a similar idea can be observed when students are first introduced to limits. Using the infinity symbol to compute a limit (Sierpińska, 1990) or claiming that "the limit is infinity" are misconceptions about infinity which can be carried over into Analysis. The notation in sub-question a) is incorrect, and the students' answers may indicate that they see no issue with this use of infinity. In this case, the early misconception "the limit is infinity" and the statement "the supremum is infinity" are directly analogous and we expect that a student who is comfortable with the former might be more prone to agree with the latter.

In addition, certain introductory Real Analysis textbooks introduce the concept of extended real numbers quite early, where infinite bounds are allowed. If students get confused by this concept, they may assume that infinity is a valid supremum even if we don't mention the extended real numbers.

Sub-question c) inquires about the difference between monotonicity and convergence. We provide the students with a divergent monotone sequence, and we expect them to either state the difference between those concepts or be inconsistent with their answers throughout the three sub-questions.

4.4 Methods of data analysis

In this section, we explain how the data gathered was classified and interpreted for analysis. The participants' answers as they were received on Moodle were not in a state in which we could easily observe particular thought patterns. We had to organize the data and we had to operationalize our system of classification. The details of the data analysis are explained in this section.

The first step was to create a table including every answer for every question. This table was the primary tool that was used to analyse the data. Next, we classified each answer to each question one-by-one according to the misconception levels that are detailed in Section 3.4. The question is: how do we recognize each misconception level from answers to the questionnaire? To answer this question, we devised an operationalization table. There are 6 criteria by which we analysed the student's misconceptions:

- Vocabulary: A student who has a deep understanding of a concept should be able to explain their reasoning using abstract vocabulary and formal justifications.

- Replicability: The student's reasoning or conception should be applicable to different situations and yield similar results.

- Validity: The student's reasoning should yield results which are correct.

- Conceptual understanding: Mathematical concepts usually rely on conditions and details. A student with a good conceptual understanding will include every necessary element of the concept in their reasoning and demonstrate a proper understanding of the conditions and details.

- Procedural understanding: If the student's answer involves problem solving, it should include a method that is accepted as mathematically adequate. If the method requires a theorem, the student should mention it. If the student sees fit to include an example in their reasoning, the example should be properly provided and relevant to the context.

- Logical progression: The student's reasoning must be grounded in mathematical logic and thoroughly follow its rules. A good reasoning should involve a set of steps that logically follow from one another.

We evaluated each student's answer against each of these criteria. We then compared our analysis to the entries on this table and decided which level corresponds most accurately to the student's answers. Here is the detailed operationalization table:

	Vocabulary	Replicability	Validity	Conceptual understanding	Procedural understanding	Logical progression
Absence	No abstract explanations or definitions	None	None	None	None	None
Preconception	Incorrect use of formal vocabulary No abstract explanations or definitions	Very low	Very low	Very weak. May include interpretations which are not mathematical in nature	None	Inaccurate and lacking continuity. May contain contradictions.
Alternate conception	Occasional incorrect use of the formal vocabulary No abstract explanations or definitions	Average or high	Low	Potentially strong conception of inaccurate notions	Very weak	Consistent and used with a certain degree of logic
Incomplete conception	Basic vocabulary Occasional abstract explanations and definitions	Low or average	Average	Weak conceptual understanding of accurate notions	Strong for the more basic problems/ situations Weak otherwise	Occasional flaws in the logic May follow seemingly arbitrary rules/ guidelines
Unrefined conception	Instances of both formal and informal vocabulary Occasional abstract explanations and definitions	High	High	Strong but may be lacking details	Strong but may be lacking details	Strong logic. The method follows clear and robust guidelines
Expert conception	Formal vocabulary Abstract explanations and definitions	Very high	Very high	Very strong	Very strong	Very strong logic

Table 1 : Operationalization of the misconception levels

Table 1: Operationalization of the misconception levels

We associated a color to each misconception level and then color-coded phrases, sentences, or entire answers using the criteria above (Table 4.41). It is possible that some answers could belong in more than one category. In such cases, we made consistent, conscious choices of what we believed was the “closest” level.

We chose to also designate certain answers as corresponding to two levels simultaneously. It is our belief that misconceptions are not black or white. Student responses may have traces of different misconception levels in a single answer. It is also possible that a student shows hints of different misconception levels in their answers to different questions, in which case, it is our conjecture that such a student might be in a transitional state from one level to the next. Signs of such a situation may be identified using Table 4.41. A student who is actively improving their understanding of a mathematical concept might provide answers for which the six criteria above correspond to a combination of two (or more) misconception levels.

In the next chapter, we present our results and analysis. We discuss in detail every answer to the questionnaire that has indications of misconceptions. We also assign a misconception level to students’ conceptions based on Table 4.41.

Chapter 5 Results and analysis

In this chapter, we discuss the results and the analysis of the data collected in the context of this study. As mentioned in the introduction and further discussed in Chapter 4, we envisioned and planned the data gathering using a task-based interview tool. However, the COVID-19 pandemic derailed this plan and imposed limitations in the data collection approach. We transformed the task-based interview protocol into a survey that students answered on an online platform, in writing. Thus, the degree of 'depth' of students' answers varies significantly between students and questions. The first step in our analysis, consisted therefore in carefully considering each individual answer and identifying which ones offered interesting or meaningful hints that would allow us to interpret students' thinking relevant to this research.

The first section of this chapter is an overview of the data and the first step of the analysis referred to above. We explain the choices we made to analyse and discuss certain answers and not others.

In Section 5.2, we focus especially on the answers that are of interest to us in one way or another. This section discusses the students' answers that seem to correspond to the misconceptions that have been observed in students of their level and that we presented in Section 3.3. We will discuss the way in which we identified and labeled those misconceptions – on the basis of Table 4.41 – and to what extent each student seems to misunderstand the concepts. This section is separated in nine subsections. The first seven subsections focus on each of the seven hypothesized misconceptions highlighted in Section 3.3. In Subsection 5.2.8, we analyse the participants' understanding of vocabulary in the context of mathematical statements. The last subsection, 5.2.9, addresses misconceptions that could arise further on in a student's education, especially as it relates to analysis.

The third section of this chapter, 5.3, focuses on unexpected misconceptions. Upon analysing the participants' answers, we determined that certain students made mistakes which can be attributed to misconceptions we didn't anticipate.

Lastly, we end the chapter with a discussion of some advantages and disadvantages of the data collection tool.

5.1 Overview of the data

In this kind of study, the data gathered depends heavily on students' level of engagement. Even if a student volunteers to participate in the study, it is not guaranteed that their answers to the questionnaire will provide information that would allow the researchers to address their research questions. While we would have probably gained significant insight into students (mis)conceptions if we had had the chance to interact with them face-to-face, the reality of the pandemic prevented this[8]. We conformed ourselves with gathering written responses to open-ended questions and reflecting, based on students' answers, our experiences as students and teachers and our reading of previous relevant literature, on what misconceptions could be gleaned from students written work

[8] Data was gathered during the early months of the pandemic. While online tools were available and we could have conducted face-to-face interviews, we discarded this idea considering how stressed and overworked university students were as they navigated extraordinary academic, professional, and personal circumstances.

and what their source could be. Several answers, however, were discarded as we considered them imprecise, too brief, too vague, or too confusing to allow us to support any serious analysis or reflection. Answers were also discarded if we didn't consider them contributing to our research questions, for one or more reasons.

An example of the type of answers that we discarded are those that we describe as "expert conception;" answers that refer to rigorous mathematical logic and proven theorems, close to what one could find in a textbook. These answers did not provide us with insights into any misconceptions students may hold regarding the concept at hand. Students 2 and 7, for example, provided near expert responses to most of the questionnaire. Although some of their answers will be discussed for various reasons in the following sections, most of their answers were thorough, rigorous and did not show any hint of the expected misconceptions. This does not mean that these students do not have any misconceptions, it only means that from the provided answer, we are unable to identify any gaps in these students' knowledge of the concept at hand.

As mentioned above, some students responded to the questionnaire in a manner that does not provide us with enough data to conduct a meaningful analysis. For example, Student 3 provided very short and vague answers to most of the questions, and outright skipped some other questions. This student's participation hardly contributed to the goals of this study. We also noticed that for some of the participants, the answers to the last few questions were shorter and less detailed than the earlier ones. Although we cannot know for sure why that is, it is conceivable that the participants grew tired of responding to the questionnaire after some time. In any case, it is a fact of the data gathered that the answers to the last few questions provide less relevant data for us.

In some cases, students might be answering a question about a concept which they are unfamiliar with. This can still be interesting to analyse and discuss as the very first levels of our model of misconception classification mostly apply to students who have not formally learned the mathematical concept. However, it is important that we, as researchers, are aware if a concept is especially foreign to each participant. As such, as presented in Chapter 4, we directly asked the participants if they have been exposed to some of the more advanced concepts that are included in the questionnaire. Thus, we know that Student 4, for example, has not formally learned the concepts and processes related to sequences, and infinite series. Students 6 and 8 also mentioned that they were unfamiliar with the convergence/divergence of infinite series. These gaps in these students' mathematical education will be considered when discussing their misconceptions.

Finally, we note that some of the students' misconceptions (or plain lack of knowledge about certain mathematical concepts) became an obstacle to our inquiry. The first of these "obstacles" turned out to be the students' misuse of, or outright disregard for, the rules of mathematical logic. For example, students using examples to prove a universal statement. This type of errors may have prevented us from noticing conceptual issues in certain questions since any misconceptions regarding the concept at hand would be hidden behind logical fallacies. Another "obstacle" we encountered in students' answers relates to their struggles with mathematical language and notation. We believe that some of our questions may have been unsuccessful in revealing misconceptions because of participants' unfamiliarity with the formal mathematical notation. These students would provide answers that were completely unrelated to the question – we assume because they did not properly understand the question itself. In some cases, however, their answers were still helpful to us in identifying misconceptions related to notation and quantifiers, but they "failed" at revealing other misconceptions we intended to uncover. These issues are discussed further in Section 5.3.

5.2 Observed misconceptions from the hypothesized options

As mentioned in the introduction to this chapter, we organize the first seven subsections of 5.2 around the seven misconceptions listed in Section 3.3. The last two subsections, however, refer to the participants' understanding of vocabulary in the context of mathematical statements and misconceptions that could arise further on in a student's mathematical education, especially as it relates to analysis. For conciseness, the questions are not restated in this chapter. We recommend that the reader keeps the question statements (see Appendix A.2) close as they read this chapter. If the reader requires additional information about the questions, they may refer to Section 4.3.

Finally, using the operationalization table (Table 4.41), we analyse the student's answer to determine which misconception level it corresponds most accurately to. As a reminder, the six criteria used in the operationalization table are vocabulary, replicability, validity, conceptual understanding, procedural understanding, and logical progression (see Section 4.4, page 42).

5.2.1 A limit can never be reached

To explore students' holding of this misconception, we consider questions 3, 15, 17, 18 and 19. Other questions have not provided sufficient data regarding this misconception.

In total, when considering the above questions, we surmise that at least 5 students out of 10 hold the misconception "a limit cannot be reached": Students 1, 6, 8, 9 and 10. The other students have not provided any answers which could indicate the presence of this misconception in any significant way.

As mentioned before, some participating students' poor understanding of mathematical logic and notation creates 'noise' in their answers as we try to reflect on whether they hold a misconception or not. We sift through the noise to isolate the students' reasonings and reflect on their understanding of the concept.

STUDENT 1 – INCOMPLETE/UNREFINED MISCONCEPTION

In answering question 15, Student 1 relies on comparing the sequence at hand to a basic function that has a limit as x goes to infinity.

Student 1's answer to question 15.d): "Personally, I like using the example 1/x, since it is pretty simple. 1/x when x goes to infinity, it goes to 0, although it never reaches it. Same thing would apply here, it gets closer and closer to 1, although it never reaches it."

First, note that in their answer to question 15, the student never mentions that functions should never reach their limit. They simply use a function that does not reach its limit as a comparison for a sequence that also respects this criterion. We cannot know for sure if this student believes that functions never take the value of their limits. Student 1's chooses to compare the sequence at hand ($a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$) with a monotone function ($\frac{1}{x}$) which is a very commonly used basic example to illustrate limits as x goes to infinity. This strategy is not inherently problematic, but it may not yield accurate results in more advanced situations.

Student 1's vocabulary through their answers to question 15 is generally quite good. There are a few imprecisions, but their reasonings are easy to understand. The strategy to compare a situation with a more basic and familiar one has high replicability. The issue with reproducing this reasoning is that it may yield accurate results in some situations and not in others. This student's reasoning is valid, but it is incomplete. Student 1 does accurately point the fictitious student towards the correct answer, which is 1, but they do not identify that the two choices are equivalent. Their reasoning has average validity. Their conceptual understanding of limits seems to be strong, but we do not have enough data within question 15 to determine if this student truly understands that functions (or sequences) can reach their limits. This student's procedural understanding appears to be strong for the basic cases. Lastly, this student's logic is strong. Considering a basic case can be a good starting step when searching for the solution to a harder problem. However, this student seems to rely too heavily on this comparison, and it could eventually become misleading. Student 1's conception of limits in the context of question 15 appears to be transitioning from "incomplete conception" to "unrefined conception."

STUDENT 6 – INCOMPLETE CONCEPTION

We consider student 6's groupings in questions 18 and 19. We will focus on two of their groupings which provide insight into their interpretation of a function "reaching" its limit.

The student created one group (group 1) with only function b) $f(x) = 4$, and another group (group 2), in which they put a) $f(x) = \frac{1}{x}$, c) $f(x) = \frac{\cos(x)}{x}$, and f) $f(x) = \ln(x)$

Student 6's answer to question 19.a): "Group 1 has functions with the limit, as x goes to infinite, equal to a real finite number which they reached. Group 2 has functions with limit 0, which [the function] never really reaches, only approaches."

Student 6 admitted that they did not remember the behavior of the function $\ln(x)$. The student claims that they believe the limit to be 0 and we assume that they believe that $ln(x) \neq 0 \; \forall \, x$, justifying the erroneous placement of the function in group 2.

The student chose to create two distinct groups of convergent functions according to whether the functions reached their limits. We cannot know for certain why the student chose the criterion of "reaching the limit" and what they mean by this phrase. From a mathematical perspective, "reaching the limit" would normally mean that there are one or more values of x in the domain for which $f(x) = L$. Whether this is for one, infinitely many, or all values is irrelevant from the perspective of the definition of limit; in any case, the difference between $f(x) \; and \; L$ is less than any $\varepsilon > 0$ if x is large enough. We cannot know if the student understands this and simply put the oscillating function in this group because they don't know its behaviour (amounting to a mistake similar to including *ln(x)* in this group). Or, alternatively, they might know the behaviour of $f(x) = \frac{\cos(x)}{x}$, but use the phrase "reaching the limit" as meaning something other than its mathematical meaning. If in the latter case, we surmise that the student means the function stabilizes at the value of the limit. In mathematical terms, we would say: $\exists\ a$ for which $f(x) = L\ \forall\ x > a$. When studying limits of functions, this property is not a significant one. Regarding the study of the concept of limit, a function that satisfies this is no more or less interesting than a function that has a limit L as x goes to infinity. This interpretation is supported by their criterion for group 1, of which the only element is a constant function.

This student's vocabulary is quite basic. There is some evidence that their interpretation of "reach" is incorrect, but they do not provide a definition or an explanation for their use of the word. This leaves the criteria governing their groupings vague and subjective. We surmise that this student's classification system has low replicability. The first reason for this assessment is the shear imprecision of their criteria. Moreover, the three groups formed by student 6 seem to be tailored only to this set of functions and may not apply to other sets. Student 6's conceptual understanding of limits as x goes to infinity seems to be lacking as illustrated by their inconsistent interpretation of a function reaching its limit.

As far as identifying a function's limit, this student's procedural understanding is average. Their only mistake is with the natural logarithm function, which they admit not remembering. However, they incorrectly identify its limit by misusing L'Hôpital's rule, which points towards a lacking procedural understanding of the theorem. Student 6's conceptual understanding of limits as x goes to infinity seems to be weak as illustrated by their inconsistent interpretation of a function reaching its limit. The logic employed by student 6 in these two questions, 18 and 19, is difficult to understand. They created one group for diverging functions and two groups for convergent functions; one for functions converging to 0 and one for functions that converge to other real numbers. They also specify that the two groups separate functions that reach their limit from those that don't. Each of these criteria on their own could have been a logical way to group functions, but mixed, it creates inconsistencies. For example, the function $f(x) = 0$ could belong in both groups, and the function $f(x) = \frac{1}{x} + 1$ seems to belong in neither. Their logical progression seems to be flawed and to follow arbitrary guidelines. Student 6's conception of limits at infinity seems to be at the "incomplete conception" level.

STUDENT 8 – INCOMPLETE CONCEPTION

Student 8 shows signs of "the limit cannot be reached" misconception in questions 12 and 15.

Student 8's answer to question 12.b): "When we say that it converges to a certain number, from what I understand, it means the value [to which] the sequence goes towards to. In this case, it will never reach 1 because we keep adding 9s in the decimal [expansion], but it will approach to 1 as n goes to infinity."

The above quote gives us some insight into Student 8's understanding of limits and convergence. It seems that they use an intuitive version of the definition of limits to drive their arguments. Their idea of convergence seems to imply an arbitrarily small distance between the terms of the sequence and the value of the limit, which they correctly determine to be 1. However, their apparent conception also applies to the number $0.\dot{9}$ itself. Student 8 makes mention of the sequence terms never reaching its limit. This comment hints at a potential misconception about sequences (or functions) approaching and never reaching their limits. This hint is supported by the following quote:

Student 8's answer to question 15.c): "Because, like B said, I think the idea of a limit is essentially a value that can't really be reached but it's a value that [the sequence] gets closer and closer to."

This quote is exactly what we would expect from someone who holds this misconception. Student 8 states very clearly that a limit cannot be reached and therefore, in the context of this question, they agree with the fictitious student B. Furthermore, in part d) of their answer, Student 8 states that they have some doubts in common with the two fictitious students, thus confirming their lack of understanding of the limit concept.

This student's answers have instances of both formal and informal mathematical vocabulary. They can use mathematical words quite well and make their understanding clear, but there are cases where Student 8's answers are vague. This misconception is unfortunately highly replicable. This student has a set of guidelines to define what a limit is, and it is possible to use it in many different situations and get similar results. The main issue being that these guidelines are incorrect, which makes the reasoning's validity quite low.

Instead of relying on formal definitions, this student's conception of limits seems partially built from erroneous criteria, indicating that their conceptual understanding may be weak. Student 8's procedural understanding is weak since they were not able to identify the limit in part c) of question 15. Student 8's logical progression is strong. They do follow incorrect rules, but their logic is consistent. This student's level of understanding appears to stand at the "incomplete conception" level.

STUDENT 9 – INCOMPLETE CONCEPTION

Student 9 provides a somewhat correct but still imprecise answer to question 17.

Student 9's answer to question 17:

"a) Yes and no, it's approximately that but it never actually reaches c. So, it shouldn't be an equal sign.

b) It could be discontinuous, so [to] the left of c it's one number and [to] the right of c it's another, but it's not […] f(c).

c) When it's both f(c) from right and left."

From their answer to question 17.a), we surmise that Student 9 hold this section's titular misconception.

One interpretation to their dual answer could be that the negative answer ("no, it never actually reaches c") reflects the student's understanding of limit. With the positive answer ("yes"), they acknowledge that this is what mathematicians (their teacher, the textbook) say about limits: they use the equal sign. They recognize the conventional use of the equal sign in this situation, but they "disagree" with it. The student does not accept $\lim_{x \to c} f(x)$ as an object that can be identical to $f(c)$.

An alternative interpretation could be that their affirmative and negative reply might imply that there would be some situations for which the equality holds, and others where it doesn't. This would mean that they don't understand universal statements. Moreover, the student does provide a counterexample in part b) in the form of a discontinuous function, which shows that they somewhat have a grasp of the necessary condition for the statement to be true, although they fail to formally recall it in part c).

In either case, it seems the reason they disagree with the equality is that they understand $x \to c$ as implying that x is never c and $\lim_{x \to c} f(x)$ as implying that f(x) is never f(c). The fact that the student can give an example of a discontinuous function to support their response to 17.a) but cannot recall the conditions under which the statement is true is very telling. Continuity is a key topic in Calculus and normally given significant emphasis in class and exercises. We take this as a sign that this student's misconception about limits (the limit cannot be reached) is so strong that it overrides the well-studied case of continuous functions.

This student's vocabulary is very basic. They avoid using formal vocabulary and choose to describe the functions using vague terms. The replicability of their conception of limits is quite high. They have formed a concept in their mind which holds true for every limit and can be applied to other situations – despite that conception having low validity. Their conceptual understanding of limits seems weak. Their idea of a limit goes against the formal definition and, in this case, causes this student to provide an incorrect answer to a question. We cannot evaluate this student's procedural understanding of limits from their answer to question 17. However, their logical progression is flawed. They clearly show that they do not understand how to prove and disprove universal statements.

Based on their answer to question 17, we assess their conception of limits to be at the "incomplete conception" level.

STUDENT 10 – INCOMPLETE CONCEPTION

Student 10's answer to question 3 resembles a collection of facts about limits which have very little to do with the question at hand.

Student 10's answer to question 3.b): "For the limit of a function to converge to L when x-> inf, it means that as x grows, the limit converges to L. However, the limit at a certain point might be something."

Student 10's answer to question 3.c): "As an example, I'm thinking of: Limit of n/(n+1) as n -> inf. As n goes to infinity, the limit converges to 1. However, as n goes to c element of N, it will converge to 1/(1+1/c), which could be 2/3 if c=2"

Their answer to 3.b) appears to rely on a circular argument and it does not justify their answer to part a), for which they answered "true." Their claim that the limit at a point might exist is correct, but irrelevant to the question. We deduced from the context that in part c), they consider the sequence $a_n = \frac{1}{1+\frac{1}{n}} = \frac{n}{n+1}$; this example is coherent with their answer to 3.a). In addition, they consider examples of sequences as the variable n approaches a finite value c. Their example is barely related to the question since it boils down to evaluating a sequence at a point.

The lack of explanation for their answer to question 3 and the fact that they provide examples that are unrelated to the question leads us to believe that this student's conceptual understanding of limits of functions of real numbers is quite weak. Given the absence of any justification to their answer, we surmise that the validity and replicability of their reasoning is quite low. The student's procedural understanding of the concepts of limits, however, seems good to the extent of their own conception. Their choice of example is irrelevant to the question, but it is consistent with their answer to 3.a), and they correctly identified its limit. The student's vocabulary is basic. Their use of logic is quite weak since they failed to justify their response with logical arguments. We consider this student's misconception at the "incomplete conception" level as it stands. The student is clearly familiar with limits of functions, but their understanding of the concept does not seem accurate from this answer. They seem to have deep gaps in their knowledge of limits, and our only hint that they might hold the misconception discussed is the fact that they mistakenly answered "true" to part a).

5.2.2 The implicit strict monotonicity of convergent functions and sequences.

In reflecting on students holding of this misconception, we consider responses to questions 3, 11, and 15. Other questions have not provided sufficient data supporting students' holding of this misconception.

Question 11 was specifically posed to reveal misconceptions about strict monotonicity. While questions 3 and 15 were designed to reveal the misconception that a limit cannot be reached (see subsection 5.2.1), it is not surprising for students' answers to shed light on misconceptions about strict monotonicity as these two misconceptions share a great deal of similarities (see Section 3.3). As a reminder, the reader may find the questionnaire in the appendix.

Based on answers to these questions, we conjecture that at least 6 students (out of the 10 participants) potentially hold, to some degree, the misconceptions at hand: Students 1, 4, 5, 6, 8, and 10. The other students' answers do not indicate the presence of this misconception in any significant way.

STUDENT 5 – UNREFINED CONCEPTION

Student 5 answered that the statement in question 3 is false. However, in their answer to question 3.b), they state: “That may be true for a function [that] has an asymptote equal to L.”

This sentence reveals that Student 5 believes that a function can never reach / cross / be-equal-to an asymptote. This indicates that the student associates asymptotes and monotonicity. However, in part b), the student provides the example of the function $f(x) = \frac{sin(x)}{x}$ to justify responding that the statement is false. Evidently, this student has a rather good conceptual understanding of limits, but they have developed some misconception around the concept of asymptotes. Moreover, this misconception reveals that the student does not understand the relation between the two concepts: limits at infinity and asymptotes. Their answer also implies that a function must have a strictly monotone approach to its limit for the behavior to qualify as asymptotic.

In addition, when working on question 11, Student 5 wrote: “The "definition" I have in mind [for asymptotes] is more of a picture that is letting me down here.”

Student 5 mentions in their answer that they recall identifying asymptotes in cases where the function was bounded by that asymptote, and that this case where a function is oscillating confuses them. Extrapolating from this student’s answer to part b), where they use the function $\frac{1}{x}$ as an example of a function which has an asymptote, it is probable that the aforementioned picture is a similar one to this example. In other words, this student relies on an example of a basic case to inform their identification (their working definition) of asymptotes. This student is subject to a similar phenomenon to the other five students considered in this subsection 5.2.2, all for whom their conception of asymptotes is lacking a rigorous definition and thus, they rely on intuition and basic examples. However, we may say that Student 5’s misconception is weak – in the sense that they have not fully replaced the concept by an incorrect one: they recognize and do so consciously, that there is a definition, a concept, that they don’t remember. The misconception level that would be appropriate for Student 5 would be the “unrefined conception” level.

The analysis of our six criteria is near identical for all the other students in this section (1, 4, 6, 8 and 10). Student 5, however, shows that their misconception can easily be overcome since they are fully aware that their response is incomplete. They suggest an intuitive answer, and they admit not trusting this intuition as it does not rely on formal definitions.

The student’s answers have instances of both formal and informal vocabulary. They use accurate terminology, and they seem to properly understand the mathematical terms that they use. Evidently, their answer has low validity: the statement in question 3 is false even for functions with asymptotic behavior. However, their reasoning has high replicability since the student could obtain similar results in other situations. Student 5’s conceptual understanding of limits, asymptotes and monotonicity is overall quite strong. The exception being that they have mistakenly assumed a conceptual difference between limits at infinity and horizontal asymptotes, implying that to have an asymptote, the function must be monotone. Although this misconception can cause issues with certain types of problems, we would conjecture that it can be easily resolved by simply exposing the student to the definition of asymptotes. The student’s answers to parts c) and d) demonstrate a strong understanding of the methods and procedures for limits. Lastly, this student’s logical progression is strong. Even with a conceptual error, the student clearly follows the rules of mathematics they assume to be true and stay consistent with their reasoning. Overall, this student’s misconception regarding monotonicity, limits and asymptotes is rather shallow. Their misconception seems to be at the "unrefined conception" level; the student's answers are lacking in some aspects of our criteria, namely the validity, while they are strong in other aspects, namely the logical progression and procedural understanding.

STUDENTS 1, 4, 8 AND 10 – INCOMPLETE CONCEPTION

Students 1, 4, 8 and 10 have all provided very similar answers and reasonings to question 11: $y = 0$ cannot be an asymptote since the function f(x) periodically crosses it. These students consider that for a function to have a horizontal asymptote, it must be bounded by it. Here are quotes from those four students' answers to question 11.b):

Student 1: "However, in our case, y=0 is not an asymptote as f(x) crosses y=0 multiple times."

Student 4: "an asymptote is not reached but is approached as x tends to a certain value."

Student 8: "an asymptote is a line that the function approaches towards, but it will never be able to reach it."

Student 10: "Hence, as seen on the graph, it goes down the x-axis and above the x-axis, while the function should never touch an asymptote."

Student 10 provides further insight into their understanding of asymptotes, in their answer to question 15.d): "The output 1 is an asymptote of the sequence. Hence, it converges to it without ever actually being this value."

The phenomenon that we seem to be observing here is one where the students have incomplete conceptions of what an asymptote is, and they conjecture certain completing elements that would fill in the gaps in their knowledge. In this case, the students have constructed a working definition of asymptote: a line that must be approached strictly monotonically. We notice that this kind of reasoning is highly replicable but has low validity. It is expected for these students to repeat this exact mistake in different kinds of situations. Their conceptual understanding of asymptotes is weak, they clearly do not rely on any rigorous definitions, and they extrapolate criteria from their own intuition. Their procedural understanding would be very strong for the most basic problems (especially monotone functions), but their methods would quickly collapse when exposed to slightly more advanced situations. Lastly, these students' logical progression is flawed but consistent. It is evident that they follow seemingly arbitrary guidelines. Overall, this common misconception is at the "incomplete conception" level. These students have clearly been instructed about the concept of asymptotes and they have a basic but adequate image of what an asymptote is. The main issue in this case is that their conception of asymptotes includes certain false elements which might have been caused by an intuitive extrapolation of basic examples.

STUDENT 6 – INCOMPLETE/UNREFINED CONCEPTION

Student 6's response to question 11 has similarities with both previous cases. First, they explicitly admit not remembering the definition of asymptote. Moreover, the student provides extensive explanations for two possibilities: the first possibility being that asymptotes can be crossed, and the second being that they cannot. Consider the following quote which the student provides at the very end of their answer to question 11.c):

“I think this would not be considered a “normal” asymptote, […] maybe we called it a special kind of asymptote or maybe we didn’t (I can’t recall correctly)”

In their answer to question 15.b), Student 6 provides us with information about their understanding of the concept of asymptotes: “But limits can be reached, they don’t have to be an unattainable number. (I believe that this might be what [differentiates] limits [from] asymptotes: asymptotes can be where a function converges to but cannot reach.”

Student 6 is very clear on their conception: limits can be reached, but asymptotes cannot. It is plausible that this student has constructed this criterion from examples they have been exposed to.

In their hesitation, the student suggests the possibility that horizontal asymptotes are classified in different categories. Student 6 seems to be trying to complete a concept that they have not properly understood. This is not unlike the reasoning that we have observed in students 1, 4, 8 and 10, with the distinction that Student 6 acknowledges forgetting the definition. Student 6 displays a few different mechanisms that suggest an incomplete conception. For example, their conjecture that asymptotes have different classifications might indicate that they are attempting to fill in the gaps in their conception with rules which seem logical to them. On the other hand, Student 6’s reasoning contains elements that would point towards the “unrefined conception” level: they admit forgetting the formal definition, and they provide the correct answer as one of their possible scenarios. Therefore, we consider this student’s misconception regarding asymptotes at the threshold between the “incomplete” and “unrefined” levels, since their reasoning has elements that would correspond to both.

5.2.3 $0.\dot{9} < 1$

Questions 12 and 13 were expected to reveal this misconception above all others. We noticed that most students either avoided the expected conundrum, or simply solved the problem as they would in an examination and provided satisfactory answers. We will consider only Student 6 as potentially holding this misconception.

STUDENT 6 – PRECONCEPTION/INCOMPLETE CONCEPTION

Student 6’s answer provided some phrases that hint that they may hold the misconception in question. Although their answers do not precisely state that they believe $0.\dot{9}$ to be a number smaller than 1, they use the expression $0.\dot{9}$ to represent a number that is fundamentally different than 1. Note that Student 6 has specified that they were not formally taught about the convergence/divergence of infinite series.

Student 6’s answer to question 13.b): “Since the 0.9dot notation is an irrational number, the sequence cannot converge to a number that isn’t finite”

We make a guess about what the student means: numbers with infinite decimal expansion are irrational. We investigate this issue further in Subsection 5.2.6. Regardless of what exactly they meant in their answer, we consider that in describing $0.\dot{9}$ in a way that is fundamentally different than 1, Student 6 shows that they do not understand that these two numbers are indistinguishable. Moreover, in question 13.b), the student states that they should have answered false to question 12.a) and true to question 13.a), reinforcing this idea that they perceive a difference between those two numbers. We surmise that this student's misconception is that any number which has an infinite decimal expansion is irrational, which would then imply that $0.\dot{9}$ is a different number than 1, which would then lead to this subsection's titular misconception.

Student 6's vocabulary through questions 12 and 13 is quite poor. They often misuse certain important mathematical terms in a way that makes it difficult to understand their thought process. We judge that the replicability of their reasoning is also very low. The student identified the value 1 as a potential limit only once it was suggested to them, and even then, they did not draw a satisfactory connection between the two representations of the number. Due to the many conceptual mistakes, the validity of this response is very low. The student's conceptual understanding of numbers and convergence is also very weak, but we recall that they have admitted not having been instructed about infinite series and having forgotten limits of sequences. Hence, they do not have the tools to produce a resolution method to support their argument. Lastly, Student 6's logical progression is weak but consistent. Despite the lack of instruction, the student follows (their own) logical rules. This student's misconception about the relationship between the numbers $0.\dot{9}$ and 1 shares a lot of traits in common with our definition of a "preconception." Student 6 appears to be lacking a significant amount of knowledge to address this question. However, Student 6 still appears to be attempting to use their previous knowledge and consistent rules to answer the questions. We would consider their approach to be an "incomplete conception," especially as it relates to their conceptual understanding, since the student shows a very weak understanding of the concept at hand; that of the limit of a sequence. Student 6's conception appears to have some criteria befitting of a "preconception," and others of an "incomplete conception." It is therefore our conjecture that Student 6's understanding of the limits of sequences of partial sums and of the equivalence of different modes of representations of numbers is in transition towards the "incomplete conception" level but is held back by their poor knowledge of real numbers.

5.2.4 The limit only as a dynamic process and never as a mathematical object

In our reflection on students holding this misconception, we consider questions 14 and 16. Other questions have not provided sufficient data supporting students' holding of this misconception.

We conjecture that at least three students may hold this misconception based on their answers to these two questions: students 1, 4 and 8. The other students have not provided any answers which could indicate the presence of this misconception in any significant way.

STUDENTS 4 AND 8 – INCOMPLETE CONCEPTION

Consider the following two quotes from students 4 and 8's answers to question 16.c):

Student 4: “I think that manipulating the functions first [gives] a better idea to get a sense of where they are actually going.”

Student 8: “Then, D proceeded with some manipulations, which I think it is also valid because of the limits’ properties which allows us to add them together, as both have x approaching infinity. It leaves to 1”

These two students agree that manipulating the functions is a good starting point. This reasoning disregards the fact that the limits of f(x) and g(x) are mathematical objects, and that algebraically manipulating them together requires both limits to exist, which is not the case. They suggest manipulating the functions first, ignoring the meaning that the limit operator gives to the expression. While the misconception at hand is not explicitly displayed in the students’ answers, we surmise that any student who understands the fact that limits are static mathematical objects would not have proposed this kind of solution.

Both students use basic vocabulary to explain their reasoning. Their solution has high replicability but quite low validity. Meaning they are likely to reproduce this type of incorrect reasoning in other situations. The conceptual understanding of these two students is weak. Student 8 mentions being unsure of their answer and Student 4 only suggests a starting point, rather than a complete solution. These students’ procedural understanding might be quite good for cases where the limits exist. We conjecture that these students’ conceptions are at the “incomplete conception” level.

STUDENT 1 – INCOMPLETE CONCEPTION

Student 1’s answer to question 16 shares many similarities to the answers provided by students 4 and 8, discussed above.

Student 1’s answer to question 16.b): “[…] my first reflex would have been to calculate the limit as x goes to infinity of $(f(x) + g(x))$ together, which is what C and D [did]. So, with this reasoning I would have to say the limit is 1 as well.”

Indeed, the algebraic manipulation of the functions to simplify the expression to make the calculations “easier” is not an acceptable strategy for this specific problem. This student seems to disregard the fact that the limits f(x) and g(x) do not exist, and therefore cannot be algebraically manipulated.

In their answer to question 14, Student 1 responds that the equation is false. The student’s justification for this answer corresponds very closely to this misconception, as shown by the following quote:

Student 1’s answer to question 14.b): “The second part is the allure of the series as it goes to infinity, so we consider how it looks before infinity, rather than just plugging in the values to find the value at infinity.”

In their answer to question 14.b), Student 1 explains that their conception of limits is that of the appearance of a function, rather than a fixed value. This student seems to imply that the limit itself is a representation of a function's behavior, rather than an explicit value that the function approaches. Student 1 is correct in claiming that the limit informs us on the behavior of a function (or series in this case) at infinity. However, they seem to ignore or not know the meaning of the limit operator in the expression. The issue with this type of misconception is that it makes manipulating mathematical expressions involving limits very confusing. Manipulating what the student seems to perceive as a description of the series is much more abstract than manipulating a static mathematical object. In addition to being a serious obstacle in problem solving, it can create further issues with more advanced concepts, as is illustrated by Student 1's incorrect answer to question 14.a) (they answered that the equality is false).

Student 1's answers to questions 14 and 16 has instances of both formal and informal vocabulary. They make their conception of limits and infinite sums clear even though it is incorrect. We claim that their interpretation of limits has low replicability since their conception of these mathematical objects is quite vague, and it cannot be applied to many mathematical problems efficiently. For example, describing the allure of a series is not something that can be used in any problem-solving situations, since it is not a properly defined concept. Their conception has low validity since the student claims that the two sides of the expression are unequal. This student's conceptual understanding of limits is weak. They explain their thought process quite clearly in their answer to question 14 and they do not seem to understand that limits are mathematical objects in themselves. These questions do not require any methodical problem-solving, thus we cannot assess this student's procedural understanding of limits. The logic used in their explanation is heavily hindered by their misconception of limits. We surmise that this student's misconception is at the "incomplete conception" level.

5.2.5 Cluster points are equivalent to limits

We look at questions 4, 5 and 6, which involve the concepts of cluster points (also called accumulation points) and limit points in the context of sequences. Other questions have not provided sufficient data supporting students' holding of this misconception. Most participants have provided very clear and rigorous answers. We will discuss the answers provided by Student 1.

STUDENT 1 – UNREFINED CONCEPTION

Student 1 accurately determines that the sequence diverges. Although their explanation has several vocabulary mistakes and inaccuracies, their logic is sound, and it seems that this student does not confuse cluster points and limits.

Student 1's answer to 4.b): "I cannot recall which property this follows however, simply that the series never reaches or approaches more a number than another, so it is undefined"

We believe that this quote from Student 1 represents quite well their thought process for questions 4, 5 and 6, especially since they reiterate similar ideas in questions 5 and 6. The student's logic relies on the sequence approaching one single value, which is the primary difference between a cluster point and a limit point. We believe that at this stage in Student 1's mathematics education, they should be able to provide rigorous arguments to justify their answer and explain their understanding of cluster points and limit points. Instead, they provide a reasoning which is vague and intuitive rather than based on proven criteria or definitions.

This student's vocabulary is very basic and vague. Their thought process has high validity and replicability: they accurately respond that this sequence diverges and if they were to apply this reasoning to other situations, they would most likely arrive at an equally valid answer. This student's conceptual understanding is quite weak as their reasoning relies mostly on intuition. However, since that intuition is largely correct, the student is still able to arrive at a satisfying conclusion, even with many theoretical elements missing. It is difficult to judge the student's procedural understanding since they do not provide any methodical resolutions or examples. Lastly, their logical progression is quite strong. They clearly have a set of rules or guidelines that they follow, and they avoid contradicting themselves. This student conception of cluster points and limit points appears to be at the "unrefined conception" level due to their hesitation and lack of rigor.

5.2.6 Infinity as a number

We now consider questions 3, 12, 13, 14, 15, 16 and 20. Other questions have not provided sufficient data supporting students' holding of this misconception.

We are addressing as many as seven questions in this subsection because infinity is a concept that is prevalent in mathematics, especially in Calculus. Even though many of these questions do not address this misconception directly, it is expected for students who are accustomed to using infinity as a number to do so in many different situations. We surmise that 7 of the 10 participants potentially hold this misconception: students 1, 4, 5, 6, 7, 8, and 9. Students 2, 3 and 10 have not provided any answers which could indicate the presence of this misconception in any significant way.

STUDENT 1 – UNREFINED CONCEPTION

In Subsection 5.2.4, we observed Student 1's answer to question 14 and determine that they might hold a misconception about limits. In this section, we will reflect on the student's use of infinity. Here is the quote that was previously used to illustrate this student's misconceptions:

Student 1's answer to question 14.b): "The second part is the allure of the series as it goes to infinity, so we consider how it looks before infinity, rather than just plugging in the values to find the value at infinity."

It's difficult to understand precisely what the student meant with the phrase "plugging in the values to find the value at infinity" or with the phrase "how it looks before infinity." We surmise that this peculiar use of vocabulary might hide a misconception (or several) about infinity, and about the symbol for infinity, which is supported by their incorrect answer to question 14.a) (they answered false). In any case, we consider that the fact that they think of infinity as something that has a "before" and that they can find the value at infinity hints to the misconception at hand: they conceive infinity as something like a real number.

When focusing on this student's conception of infinity, we can notice that their use of vocabulary is poor. They misuse the word "infinity," and some phrases are vague. Unfortunately, we cannot evaluate Student 1's conception against any other of our criteria since their answer is mainly focused on limits, as opposed to infinity. However, from their suggested use of the symbol for infinity, we conjecture that this student's conception of infinity is at the "unrefined conception" level.

STUDENT 4 – INCOMPLETE CONCEPTION

Let us reflect on Student 4's answer to question 3 and why their conception of infinity may be flawed.

Student 4's answer to question 3.b): "I'm thinking about the fact that when x gets closer to infinity, then f(x) gets closer to the limit."

Student 4 includes the idea of a variable getting closer to infinity which implies the distance between x and infinity getting shorter. Their vocabulary suggest that they don't understand well-enough the difference between limits at a point and limits at infinity, and thus, they treat infinity as a point (as a number). From this sentence alone, we cannot ascertain whether the student simply misused common mathematical vocabulary or if they truly consider infinity as a reachable quantity. However, Student 4 then states the following:

Student 4's answer to question 3.b): "We extrapolate to think that x is going to reach that limit at some point."

This kind of attitude towards infinity has been described by Sierpińska (1987) and looking at this student's answer, this seems to be akin to the "intuitive indefinitist" attitude. Sierpińska (1987) describes this attitude as assuming that all sequences are finite and that the limit is the last term. Sierpińska was referring to sequences, but since we are talking about infinity and considering the idea of reaching it, we can apply this notion to functions of real numbers: the limit is the last value of the function. This student uses the idea that x will eventually reach infinity, at what point the function will have reached its limit and therefore be equal to it. It is worth noting that the student goes on to use very rigorous analytical arguments to justify their answer to 3.a).

Simply based on their analytical answer, it seems that this student has a very good understanding of limits and can use rigorous arguments to answer this question. However, they decided to use a non-conventional reasoning at first, which might indicate that the student might still hold a misconception regarding infinity. First, let's consider the vocabulary. The student seems to properly understand the words that are used in their explanation, the only exception being the phrase "gets closer to infinity." The most problematic parts of this student's reasoning are the validity and the conceptual understanding. Clearly, the thought process highlighted above has low validity, namely the idea that a function will reach its limit *at* infinity. We surmise, although there's no strong evidence from the student's answers, that their procedural understanding of infinity is weak. Their reasoning depends on an interpretation of infinity which is intuitive, and invalid. Student 4's misconception seems to be at the "incomplete conception" levels.

STUDENTS 5, 7, 9 – INCOMPLETE CONCEPTION

Question 16 required the participants to analyse a fictional conversation between two students trying to solve a problem.

Every participant gave a similar answer to 16.a): they agree with student D that we cannot perform algebra or arithmetic on infinity in the same way we do it with variables. However, even with this correct answer, several students gave contradictory answers to parts b) and c) of question 16. To avoid redundance, we will group some students who provided similar reasonings together for this reflection.

Student 5's answer to question 16.b): "It is 1. From a theorem learned in calculus / analysis, this is equivalent to the limit of the addition."

Student 7's answer to question 16.c): "D can use the fact that the sum of the limits is the limit of the sums and since $f(x) + g(x) = 1$, the limit clearly converges to 1."

Student 9's answer to question 16.c): "you can add two functions together, if you find them convergent, then each is also convergent."

Let us first preface this section by acknowledging that it is likely that none of the participants truly believe that infinity can be used as a real number, as is supported by their answers to part a). However, even if they mention the fact that infinity is not a number, the students mentioned above still used infinity as if it were a real number. The most evident issue in this question is the equivalence of the limit of a sum of functions and the sum of the limits of functions. This equivalence is only true for functions whose limits exist, which is the point of contention in question 16: the limits do not exist. The students have forgotten (or have not learned) the conditions under which such equivalence is true. A deeper, or less evident, issue in this question is the correct identification that one of the sides of the equality does not exist. All three of these students have ignored the fact that both functions diverge to infinity in favor of finding a numerical answer to the question. To these students, it seems that the limit of each of these functions is infinity, rather than non-existent, and they don't hesitate in performing algebraic manipulations on them.

These three students' vocabulary is decent in their answers to question 16. They all use mathematical words properly, but they avoid using more abstract explanations and rely on basic terminology. To evaluate the replicability of their reasoning, we ask ourselves "can these students apply their incorrect version of the Sum Law in different situations and get similar results?" We claim that indeed this reasoning can be applied in most (all) situations. However, its validity is low.

These students' conceptual understanding of limits seems quite weak. They all correctly remembered the part of the theorem that is algorithmic in nature and forgot the conditions that allow the use of this theorem. This shows a lack of understanding of the rules of calculus and especially those regarding divergent functions and thus, by extension, infinity. Their procedural understanding is strong but incomplete. They all seem comfortable with algebraic manipulations in the context of limits, but they are not aware of the conditions under which these are valid. Lastly, the logical progression used by these three students differs from one another. Student 5's logic is weak but not completely inconsistent. Their answers to parts b) and c) agree with each other, but somewhat disagree with part a) as they mistakenly used infinity as if it were a real number. Student 7's logic is inconsistent. They successfully identified that some operations can be inconclusive but failed to apply their knowledge to this situation and immediately suggested a solution that contradicts their argument to part a). Student 9's logic is also quite weak. They used the Sum Law backwards, which shows that they attempted to include the conditions of the theorem in their solution but failed to do so accurately. Students 5, 7 and 9 have an "incomplete conception" of limit laws, which caused them to incorrectly manipulate infinity.

STUDENT 6 – ALTERNATE/INCOMPLETE CONCEPTION

Student 6's answer to question 3 has issues that are similar to the ones we discussed above, regarding Student 4. We consider some quotes from Student 6's answer that hint at a misconception regarding infinity:

Student 6's answer to question 3.c): "For example, the limit of $f(x) = x$ as x goes to infinity is infinity"

The quote above is phrased in a way that suggests that infinity is a valid limit for a diverging function. This is an example of a mistake which is difficult to attribute to one specific source. It is plausible that these errors are simple misuses of vocabulary, and that this student's conception of infinity is simply unrefined. However, it is also plausible that this student mistakenly uses infinity as a real number and those quotes are products of this misconception. We surmise it is the former, as shown by the following quote:

Student 6's answer to question 3.c): "Furthermore, if the limit is infinite, the f(x) keeps going towards infinity but never actually reaches infinity as it isn't a finite number"

By this explanation, it seems that the student accurately understands the concept of infinity in the context of a limit.

Before we begin looking at Student 6's answer to questions 12 and 13, we recall that this student claimed not being formally taught about the convergence/divergence of infinite series. Moreover, they admit not remembering much regarding limits of sequences. However, even by attributing this student's misconceptions to their lack of formal instruction regarding these topics, Student 6 made some interesting comments regarding infinity and real numbers.

Student 6's answer to question 12.b): "[…] the fact that it is converging to a non-rational number means that this number continues infinitely, and therefore it is not a finite number. Thus, can a function or sequence converge to a non-finite number, or does that mean it is diverging?"

This quote was taken from Student 6's answer to question 12, but they reiterate some elements of it in their answer to question 13 as well. The first glaring issue in their reasoning is the stated equivalence between irrational numbers and infinity. The question becomes: does the student believe that irrational numbers are infinite, or do they simply misuse the term infinity? We cannot find a definitive answer to this question, but Student 6's answer hints at both. In question 13, the student revisits their answer to the previous question and states that the answer to question 12 is false, since a sequence cannot converge towards an infinite number. The student might have properly understood that sequences that go to infinity diverge, but mistakenly identifies certain finite values as behaving like infinity due to some misconception regarding numbers with infinite decimal expansions. On the other hand, it is also plausible that Student 6 uses the word "infinite" incorrectly. If the student means "irrational" whenever they use the term "infinite", then their misconception might be related to infinite series and the concept of convergence rather than infinity itself. This explanation is also supported by their admitted lack of instruction about infinite series.

Student 6 shared an interesting conception of infinity in their answer to question 16. They correctly answered that the mathematical expression in the question is undefined but justified their answer using some imprecise arguments.

Student 6's answer to question 16.a): "I agree in the sense that it isn't like any number, but we can use infinity in algebra as an idea of a number. 3+infinite if infinite, 3/infinite is 0, etc"

Student 6's answer to question 16.b): "If we use l'Hôpital's rule just to check what would happen, the x as an exponent would not come down, therefore this just isn't defined"

This student seems to understand how limits to infinity can converge and how they can diverge. But the notation used to explain their reasoning is flawed as it uses the symbol for infinity in arithmetic expressions. Their general understanding of the role of infinity in the reasoning process used to determine limits seems sound. The provided explanation is intuitive and simplified, but generally valid. This student seems to properly understand that infinity cannot be used as any real number, but the idea that the symbol of infinity is a placeholder that means "unbounded" seems to escape them.

The second quote illustrates how theorems can be misused. The fact that Student 6 attempts to use L'Hôpital's rule to solve this problem tells us that they see an indeterminate form in the mathematical expression, when there isn't one. In this case, it stands to reason that Student 6 assumes that the expression is the indeterminate form usually referred to as "$\infty - \infty$." In question 16, however, we are faced with a sum of two limits and therefore, L'Hôpital's rule does not apply. Much like the previous three students, Student 6 is substituting the symbol ∞ for the limits of the functions. In other words, they are considering that the limit of those functions is infinity, instead of non-existent.

This student's vocabulary is basic and lacks formal vocabulary. They misuse several mathematical terms, and their explanations can be quite vague. The validity of their reasonings is quite low since their answers often misuse the concept of infinity and lead them to an incorrect answer. The replicability of this conception of infinity is quite high as their use of what they call "an idea of a number" is not a significant hinderance to problem solving. However, if their misconception that irrational numbers are infinite is any deeper than a misuse of vocabulary, this conception might yield results that vary widely. It might also hold the student back from learning concepts using either irrational numbers, or infinity. The quotes above appear to show an important lack of conceptual understanding of infinity (and of real numbers). The procedural understanding of the concept is also very weak, especially since they incorrectly identified an indeterminate form and the conclusion they drew from their use of L'Hôpital's rule is a logical fallacy. For the same reason, Student 6's logical progression is weak. Although they misuse several concepts through their solution, the steps that are taken to arrive at an answer aren't completely illogical, but they are severely affected by the student's misuses of infinity. For these reasons, we surmise that Student 6's conception of infinity is transitioning from "alternate conception" to "incomplete conception." Their admitted lack of instruction on some mathematical concepts which heavily rely on infinity might contribute to this student's apparent misconception, but they also appear to be gradually honing their conception of it through their mathematics education.

STUDENT 8 – UNREFINED CONCEPTION

Let us look at Student 8's answers to questions 14 and 15.

Student 8's answer to question 14.b): "The way I see it is that they are the same because both will add up until an infinity value."

Student 8's answer to question 15.c): "So, if we add 9s infinitely, then we can get 0.9 conceptually (with infinite 9s) even though we can't really get that practically"

A possible explanation for Student 8's answer to question 14 is that they assume that since there are infinitely many terms in the series, it must diverge to infinity. This interpretation would suggest that they may be holding a version of this section's titular misconception in which student assumes that infinite sums must diverge because they have infinitely many terms. Alternatively, the student might be explaining the equality by stating that both sides have infinitely many terms which are identical. This is a naïve but valid way to answer this question.

In question 15, Student 8 nuances their conception of infinity and the idea of a sequence reaching it. In their reasoning, we believe that they are drawing a difference between a sequence's limit and its terms. When the student states that the sequence gets to $0.\dot{9}$ *conceptually*, we believe that they are implying the value of the sequence's limit. When they state that the sequence cannot get to $0.\dot{9}$ *practically*, we believe that the student means that none of the sequence's terms will take that value. If our interpretation of the student's answer is correct, their conception of infinity in the context of sequences is generally quite good. They are implying that there is no infinity-th term and that we cannot use infinity as a real number. The issue with their answer which warrants further consideration is the phrasing of their explanation which can be ambiguous. We interpret their answer in a certain manner but since this answer is unclear, we cannot determine for certain if Student 8's conception of infinity is as sound as it seems.

Overall, the only one of our six criteria which this student seems to have significant issues with is the vocabulary. Their answers can be ambiguous, but if our interpretations are correct, their conception of infinity in the context of sequences is quite good. Therefore, we will place their conception at the "unrefined conception" level.

5.2.7 EA and AE statements

Throughout the questionnaire, certain students provided incorrect answers to certain questions not only or exclusively because they hold misconceptions about limits (one of the 6 misconceptions discussed above), but also because some questions require an understanding of formal mathematical notation and/or an understanding of English words and phrases as they represent mathematics logic concepts (e.g., "for all"). In many cases, students obviously ascribe incorrect meaning to phrases, words, and notation and therefore, misinterpreted the questions.

We consider questions 3 and 22. Other questions have not provided sufficient data supporting students' holding of this misconception.

From the students' answers, we conjecture that three students may hold a misconception about mathematical notation or have issues understanding it: students 1 and 6. The other students have not provided any answers which could indicate the presence of this misconception in any significant way.

STUDENT 1 – INCOMPLETE CONCEPTION

Question 22 asks the participants to differentiate two formal mathematical statements. Both statements involve the same quantifiers, but the way they are phrased makes their meaning different. Student 1 provides some insight regarding common issues that students may experience when learning to read formal statements involving quantifiers.

Student 1's answer to question 22.a): "I do think the equation should be read as superior or equal to b, as they are both said to be superior to 0, therefore, there would be a possibility that they would take the same smallest value."

The student claims that both statements are equivalent and true, which can indicate that Student 1 has issues with statements using quantifiers. The nature of those issues is unfortunately unclear, but there are some plausible explanations. They might incorrectly assume that quantifiers can change places in a statement without changing its meaning. They might also lack experience with such statements and misunderstand their meaning, regardless of their conception of the rules of mathematical syntax. However, their explanation of the first statement being true suggest that they understand some formal mathematical statements and that their mistake is specifically with the second statement. Next, the above quote suggests changing the statement for it to read "$a \geq b$." This modification would slightly change both statements: the first one would remain true, but it would make it trivial since choosing $a = b$ would always be correct and the second would remain false, and it would not modify its meaning as per the definition of $\forall$. Therefore, this student's suggestion to change the statements would be mostly inconsequential. Moreover, their argument for this change is inconsistent with their own answer earlier in the question. Before this quote, Student 1 claimed that it is always possible to find a positive value which is smaller than another, and therefore that there is no smallest value. Their argument that both a and b could take "the same smallest value" contradicts that claim.

Student 1's vocabulary is quite good. However, the replicability of their conception of quantifiers and mathematical statements is very low. If they assume that quantifiers can be moved within a statement, they might unknowingly be changing the meaning of the statement. They might also simply have issues understanding formal language, which clearly would hinder their ability to respond consistently to mathematical questions. The students' suggestion to modify the statements has low validity since it is inconsequential. Moreover, they claim that the statements are equivalent which is also invalid. From Student 1's suggestion to modify the statements and their incorrect response that both statements are true, it seems that their conceptual understanding of formal mathematical statements is weak. Lastly, this student's logic is weak as is justified by their suggestion of irrelevant changes and their trivial examples at the very end of their answer. This student's conception of quantifiers and formal language seems to be at the "incomplete conception" level.

STUDENT 6 – INCOMPLETE CONCEPTION

In their answer to question 3.b), Student 6 stated the following:

"But, in any case, for cases which aren't constant functions, the function will vary as x increases and therefore, the function cannot be equal to L on its whole domain without being a constant function"

It seems that Student 6 interprets the statement "$f(x) \neq L\ for\ every\ x$" as meaning "f(x) is not a constant function". This would explain why they answered that the statement is true for every function that isn't constant. Later in their explanation, this student provides several examples to justify their reasoning and reiterates that the statement must be true for any non-constant function since they cannot be equal to their limit at every x. The issue that we are highlighting with this student's answer is that formal mathematical notation can be a major obstacle to student learning.

We believe it is likely that Student 6 has a misconception about the formal language used in mathematical statements. Considering our criteria, it seems that the main issues are their vocabulary, the replicability of their reasoning, and the conceptual understanding of mathematical symbols and phrases. In their explanation, the student uses very basic vocabulary and expresses their own personnal understanding of the statement, which happens to be incorrect. This misunderstanding cannot yield consistent results since every mathematical statement will use similar notation but in different contexts (hence the low replicability), and a weak understanding of the conventions will induce severe misunderstandings. From this answer to question 3, we judge that Student 6's misconception of mathematical language at the "incomplete conception" level. They seem to understand what each individual symbol and word mean, but not the meaning when used together.

5.2.8 Synonymous terms considered distinct

The terms which may create confusion in students are the following: "tends to", "goes to", "converges to", and "the limit is." They are often used interchangeably by mathematics educators and in textbooks without much attention to students' struggles to understand their mathematical meaning. Therefore, we decided to analyse our data to investigate whether students consider these words to be inherently different and whether they attach alternate definitions to them.

We will consider questions 4 through 10. Other questions have not provided sufficient data supporting students' holding of this misconception.

We conjecture that students 1 and 6 may hold this misconception, based on their answers to these questions. The other students have not provided any answers which could indicate the presence of this misconception in any significant way.

STUDENT 1 – ALTERNATE CONCEPTION

Student 1's answers to questions 4, 5 and 6 use very basic vocabulary. They use the word "series" instead of "sequence" several times, and they seem to misunderstand the meaning of the word "approach" as evidenced by the following quote:

Student 1's answer to question 5.b): "Therefore, we cannot determine which number the limit approaches as it approaches both equally."

Student 1's explanations to these three questions are generally quite good, but we believe it is possible that they hold this misconception based on their use of vocabulary. The student's misuse of the word "limit" may simply be a communication mistake instead of a misconception, but the recurring misuse of the word "approach" seems to point towards an actual misunderstanding of its meaning. In this case, instead of using the word "approaching" as meaning that the sequence has a limit over its domain, the student seems to be using it as meaning that the function has accumulation points. They imply that the sequence periodically approaches two different values. In using this word in such a way, the student shows that they consider the terms "approaching" and "having a limit" as conceptually different. The main issue in this case is not the concept of limit or of accumulation point, it is the vocabulary which the student uses to describe one or the other.

Their vocabulary is basic and lacks formal explanations and definitions. On more than one occasion, Student 1 incorrectly uses the word "limit", and as discussed above they seem to misunderstand the meaning of the word "approach" in a mathematical context (or ascribe a different meaning to it). This student's conception of this word, although it has low validity, seems quite robust. They seem to hold an alternate definition for the word "approach," but they use it consistently and efficiently. It is conceivable that this student could replicate this use of the word to different situations and obtain similar results, hence it has a high replicability. Their conceptual understanding is weak since they seem to consider the phrases "having a limit" and "approaches" as different. Since this vocabulary doesn't intervene in the student's problem solving, they seem to have a strong procedural understanding of limits and how to identify them in the context of sequences. Lastly, this student's logical progression is strong. They have an intuitive definition for those words which they use consistently and without any significant leap in logic. Therefore, this student's conception of common mathematical vocabulary is at the "alternate conception" level.

STUDENT 6 – UNREFINED CONCEPTION

Student 6 correctly answered questions 7, 8, 9 and 10 but at the very end, they realized that these questions were more about the vocabulary rather than the problem solving, and they provided some insight into their conception.

Student 6's answer to question 10.b): "[…] because the definition of limit is stricter than simply saying it tends towards or approaches, we can also say that f(x) tends towards 0 or approaches 0 as x goes to infinity. The limit equality is the stricter definition of the 3 although we often speak of limits as tending towards and approaching. […] Just by looking at the graph, we say that it approaches or tends towards 0, but to actually have the limit equal 0, we need to calculate it or use theorems to prove it."

The quote above summarizes Student 6's explanation very well. They first argue that the only one of these terms which is properly defined is "the limit is." They seem to believe that terms such as "approaches" or "tends to" are a similar but somewhat a weaker version of it. They consider that if a function has a limit, it is correct to claim that the function approaches it, but they seem to believe that the converse is not necessarily true. This reasoning is interesting because it provides, to a certain extent, a critique of the education that this student has received. Terms such as "approaches" and "tends to" are used often in informal explanations but we, instructors, may be assuming that students understand them in their mathematical meaning without necessarily taking the time to formally address this. Considering the whole picture, we believe that it is natural for a student to give more importance to the terms that are formally defined as opposed to the ones that are often used informally.

This student's vocabulary throughout their explanations is good, and their explanations are clear. They also explain how they interpret the key words that were the focus of questions 7 through 10, which includes some incorrect elements. Indeed, this student believes that these words are not quite synonymous. Their interpretation of those words is made clear by the student, which means that they would apply the same meaning to these words in different situations. Their reasoning has high replicability. Unfortunately, it is also slightly invalid. If this student was exposed to a statement phrased such as "a function approaches a value as x goes to infinity", they might incorrectly assume that the phrase does not provide enough information about the limit of the function. Their conceptual understanding of the vocabulary seems good, apart from the issues that were discussed above. We argue that Student 6's logical progression is very strong. They have a clear understanding of the role of each of those terms and they use them consistently. Their explanations as to why they attribute certain meanings to those words makes sense. This student's conception about the meanings of the phrases "the limit is", "approaches", "tends to" and "converges to" appears to be at the "unrefined conception" level.

5.2.9 Further misconceptions related to Analysis

The last few questions of the questionnaire are related to more advanced concepts that are usually taught in Analysis classes. Since these concepts do not relate closely to the seven misconceptions discussed in this chapter, we decided to analyse the answers to questions 23 to 26 in this separate subsection. The goal of these questions is to investigate the relationship between misconceptions which may arise earlier in a student's education and the ones which could appear as students are exposed to more advanced mathematics. It is our conjecture that misconceptions related to the concepts of limits and infinity can affect how a student learns Real Analysis and give rise to different but related misconceptions.

Each of these questions started with a disclaimer requesting that only students who have completed or are actively enrolled in an Analysis course should answer the last part of the questionnaire. While many students attempted to answer these questions, only Student 1's answer to question 23 was detailed enough for us to carry out an analysis.

STUDENT 1 – ALTERNATE CONCEPTION

We reproduce below Student 1's response to question 23:

23.a)

"The maximum upper bound of A is equal to 1. That means that the maximum value in A is 1 and all the other values are either equal or inferior to 1."

23.b)

"A={-1,1,-1,1,-1,1}"

23.c)

"Yes, it is the ultimate maximum of the function, as we know some functions have a few maximums, but the sup gives the highest maximum"

In their answer to question 23.a), the first mistake that appears is the phrase "maximum upper bound." We assume that the student meant "least upper bound," given the context of the question. The second error is including the value of the supremum in the set A, which is not necessary. Both mistakes could be thought to be "typos," but Student 1's response to part c) confirms that this student's answers were purposeful, and that they most likely have a misconception regarding the supremum of a set. Their explanation implies that they consider the supremum as equivalent to the "absolute maximum" of a function (or of a set of real numbers). This would also explain their misuse of the terms "maximum upper bound": they might assume that an upper bound is equivalent to a maximum.

This student's vocabulary is severely hindered by their apparent misconception. They are misusing mathematical terms in a way that seems to be rooted in their incorrect interpretation of supremum and upper bounds. This misconception is highly replicable. They seem to hold an alternate definition for the concepts mentioned above and it is conceivable that they may be applied to many different situations and yield consistent results.

From their answer to question 23, Student 1's conceptual understanding of the supremum seems to be weak. From the example they provided in part b), this student's procedural understanding of the supremum seems decent, although possibly accidentally so. However, if this student does have misconceptions about the concepts of supremum and upper bound, they are inevitably an obstacle to their problem-solving. Lastly, this student's logic is decent. Their main issue with those concepts seems to lie within their understanding of the definitions. This student has built for themselves alternate definitions of these concepts, but they remain consistent and logical with those definitions. This student appears to have an "alternate conception" of supremum and upper bounds.

5.3 Unexpected misconceptions

The questionnaire was constructed with the goal of exposing seven misconceptions well-documented in previous research. Certainly, neither us nor previous studies claim that these seven misconceptions are all that there is. It is therefore unsurprising to have found issues in students' answers that do not directly correspond to any of these seven misconceptions. This section will focus on misconceptions which we did not anticipated, but we judged significant enough to be discussed in this study.

5.3.1 Graphical resolution methods

Several questions included graphs to help the students visualize the situation without relying on their own recollection of functions. Certain students took this opportunity to illustrate and justify their reasoning only using visual representations, which in many cases can be insufficient and/or misleading. Among the questions which included a picture, we will consider questions 7 through 10 and question 16.

Our usual means of analysis do not apply to this type of incorrect reasoning. Table 4.41 was constructed with the intent to reflect on students' understanding of specific mathematical concepts. Since resolution methods cannot be considered "concepts," we refrain from using Table 4.41.

We reflect on the answers provided by students 4 and 8. The other students did not seem to rely on visual representations to justify their thought processes.

STUDENT 4

Let us first consider Student 4's answer to question 7. This question was the first of a block of five questions investigating common vocabulary in a situation where certain terms are synonymous. In these five questions, the graph of the function $f(x) = \frac{\sin(x)}{x}$ was provided.

Student 4's answer to question 7.b): "I can see that the "waves" get closer and closer to the x axis as x gets further from 0. This means that I can take any subsequence of this function (pi/2, 5pi/2, 9pi/2…) and it will be converging to 0."

This student's train of thought seems to begin with the graphical representation, but it quickly pivots into an attempt at a more rigorous analytical argument. The phrasing of this answer implies that since the graphical representation follows the expected behavior of a convergent function, then the subsequences must also converge. Student 4 is correct in stating that every subsequence of a convergent sequence also converges. However, they used this fact as a conclusion which they derived from a graphical argument, as opposed to using it to analytically justify the convergence of the function.

STUDENT 8

We will now reflect on Student 8's use of graphical representations. On four occasions, they have relied heavily, if not exclusively on the provided graph to derive their answers. Below are quotes from Student 8's answers to all four of these questions, in order.

Student 8's answer to question 7.b): "We can see from the graph that as x goes to infinity, the oscillation of the function becomes smaller and smaller, and f(x) is going towards the x-axis, where f(x)=0. Therefore, the limit of this function is 0, as x approaches infinity."

Student 8's answer to question 8.b): "From the graph, we can see that the oscillation becomes smaller and smaller, as x goes to infinity. It goes towards the x-axis, where f(x)=0. Therefore, the limit of this function is 0."

Student 8's answer to question 10.b): "The function does seem to approach 0 as x goes to infinity as the f(x) keeps decreasing in its oscillation as x increases. It does not seem that it's going to increase again, therefore it approaches 0."

Student 8's answer to question 16.c): "But I am little bit confused, because when I look at the graph, it doesn't seem as though the limit is 1."

Their answers to questions 7, 8 and 10 are extremely similar and they entirely depend on what the student can see on the provided graph. In this case, they are correct in their answers that indeed, this function converges to 0 as x goes to infinity and since the different terms used in these questions are synonymous in this context, their answer is correct for all three of them. However, their only justification for this answer is entirely dependent on the visual support.

In their answer to question 16.b), Student 8 correctly identified that the sum is undefined, but their answer to question 16.c) is inconsistent and contradictory. Their answer to question 16.c) is mostly a description of the steps taken by the fictitious students to solve the problem, paired with their own opinions of those steps. Student 8 failed to identify any mistakes in the fictitious conversation, which leads them to agree with the arguments put forward by the fictitious students C and D. However, since those arguments contradict their response that the sum is undefined, it seems to make Student 8 doubt their answer, and resort to using visual arguments. The quote above seems to suggest that their interpretation of the graph agrees with their answer that the sum is undefined but disagrees with the reasoning employed by the fictitious students. This opposition appears to confuse the student and gets them stuck without a satisfactory justification.

Student 8 does not provide any evidence that they could correctly justify their answers to questions 7 through 10 without visual arguments, and their use of graphical representations in question 16 appears to be a last resort to justify their reasoning. However, since Student 8 elected to use graphical arguments to answer these questions, it stands to reason that they do not understand that such arguments are insufficient.

5.3.2 Logical fallacies

Across this research, we have noticed that an overarching issue with students' responses to the questionnaire is poor understanding of the rules of mathematical logic. In this section, we will reflect on certain participants' answers that include examples of logical fallacies and how it may have influenced their answer. There are instances of logical fallacies by the participants of this study which are not discussed in this subsection. That is because some student responses were too vague for us to attribute their mistakes to incorrect logic, or their seemingly incorrect logic was used as illustrative rather than argumentative.

Our usual means of analysis do not apply to this type of incorrect reasoning. Table 4.41 was constructed with the intent to reflect on students' understanding of specific mathematical concepts. We will therefore simply reflect on the specific logical fallacy used by the students without attempting a formal analysis of the results.

INCORRECT USE OF IMPLICATION – STUDENT 1

We have discussed Student 1's answer to question 3 through the lens of the misconception "limits cannot be reached." In their answer, the student misuses logic in the context of a one-sided implication. As a reminder, question 3 is a true or false question with the statement being "$Let \lim_{x \to \infty} f(x) = L, then\ f(x) \neq L\ for\ every\ x$." Student 1 agreed with the statement, and their justification included the following quote:

Student 1's answer to question 3.b): "The limit indicates that when x goes to infinity, f(x) goes to L. However, that does not imply that $f(x) = L$, but rather that f(x) approaches L when observed to infinity."

If we set A to be the statement "$\lim_{x \to \infty} f(x) = L$," and B to be "$f(x) \neq L\ for\ every\ x$," we observe that Student 1 justifies the statement $A \Rightarrow B$ with the assertion $A \nRightarrow \neg B$. The rest of this student's answer to question 3 does not provide any further insight on their understanding of mathematical logic, but it also fails to provide a satisfactory justification to their affirmation, indicating that the student possibly believes their answer to be acceptable. This misuse of logic seems to be an incorrect version of a contrapositive argument. We speculate that the student was not attempting to perform a formal logical proof, but rather argue using their own understanding of limits. The result is an argument which does not explain the student's stance on the statement.

The source of this incorrect use of mathematical logic cannot be identified with so little information. Student 1's answer to question 3 raises the possibility that they have not been exposed to mathematical logic sufficiently to have it internalized and therefore fails to apply such principles in situations where that would be appropriate. Alternatively, it is possible that Student 1 has not formally learned the rules of mathematical logic, which would explain their failure to recognize the non-equivalence of their justification and the question statement.

MISUNDERSTANDING OF UNIVERSAL STATEMENTS – STUDENTS 6 AND 9

Both students have employed statements which hints at them not properly using or understanding the meaning of universal statements in mathematics.

Student 6's answer to questions 3.a): "true (unless f(x) is a constant function)"

Student 9's answer to question 17.b): "Yes and no, it's approximately that, but it never actually reaches c. So, it shouldn't be an equal sign."

Statements in mathematics are either true or false. These students demonstrate a certain lack of mathematical maturity by answering that the statement can be both true and false depending on the situation. In both cases, the students' answers to the question are incorrect; their answers have been explored in Subsection 5.2.1. However, if they were correct in stating that the statement is true only in specific conditions, their response should have been "false," followed by an explanation. It would have been acceptable, and perhaps showed a greater deal of understanding of universal statements, if the students had denied the statement, and then provided clear conditions for it to be true.

CIRCULAR ARGUMENTS – STUDENTS 4, 6 AND 10

Circular arguments occur when the argument depends on the truthfulness of the conclusion. This type of argumentation is invalid since the justifications are not based on agreed-upon facts. Rather, they depend on repetition and on the assumption that the conclusion is correct. Three students have used such arguments: Students 4, 6 and 10. We consider those arguments and discuss why they are not logically valid.

Student 4's answer to question 7.b): "I can see that the "waves" get closer and closer to the x axis as x gets further from 0. This means that I can take any subsequence of this function (pi/2, 5pi/2, 9pi/2…) and it will be converging to 0."

We interpret Student 4's thinking paraphrasing what they mean: "the graph shows a function that converges to 0, therefore all subsequences converge to 0, then the function converges to 0." We have discussed the invalidity of graphical arguments earlier in this chapter. Here, we comment on the fact that this student's very first statement depends on the truth of the conclusion, which invalidates their entire argumentation. The graphical behavior described by Student 4 is itself a result of the convergence of the function. However, the student does not appear to know that visual arguments are invalid, and therefore might be using circular logic "accidentally". This is possibly an example of how certain misconceptions can affect multiple concepts. In this case, this student's mistaken assumption that graphical arguments are valid may be leading them to using circular logic.

Student 10's answer to question 3.b): "For the limit of a function to converge to L when x-> inf, it means that as x grows, the limit converges to L. However, the limit at a certain point might be something."

Their answer to question 3.a) is "true," which means that the conclusion for this particular question would be "$\lim_{x \to \infty} f(x) = L, then\ f(x) \neq L\ for\ every\ x$," or, equivalently but in more colloquial terms, "the function does not reach (does not take the value of) its limit." Not only does Student 10 simply repeat their first statement as an argument, but they immediately oppose it with a statement that does not relate to the question. This student simply does not justify their answer to question 3.a) and their explanation consists in two identical statements which the student appears to have drawn a logical connection between. In this case, Student 10 does use circular logic by simply reiterating a fact, thus not really explaining anything.

Student 6's answer to question 3.b): "The limit of f(x) as x goes to infinity means that as x approaches infinity, the function tends towards L. It does not necessarily reach L."

The rest of Student 6's answer to this question concerns their misunderstanding of the question (see Subsection 5.2.7) and pivots away from the above quote's focus. Considering their incorrect interpretation of the question, it is difficult to identify what the intended conclusion for this student's reasoning was. From context, we believe that the student's goal was to argue that a function with limit L as x goes to infinity does not have to reach its limit. If this was indeed their goal, we would find that the primary argument supporting this conclusion is an attempt at explaining the concept of a function converging as x goes to infinity. This explanation of the concept of convergence is circular and it does not support their conclusion in any way. Their conclusion is simply a fact which follows from the definition of limits.

5.4 Advantages and limitations of the research tool

We begin by discussing the advantages of the method. As mentioned in Chapter 4, the data gathering process unfortunately began during a global pandemic. Using a questionnaire that could be distributed remotely guaranteed that the research team and the participants could avoid physical contact and therefore ensure the safety of everyone involved. Moreover, we noticed that the participants took significantly more time than expected to complete the questionnaire. On average, students took one hour and 58 minutes to complete all 26 questions. Although we expected participants to complete the survey in approximately one hour, the students were permitted up to three hours to allow them to thoroughly detail their answers, and to lessen any stress that could be related to their involvement in this study. Since the participants used almost double the expected time, we believe that the format of the questionnaire contributed to their engagement with the process.

In addition, the way in which we set up the online survey allowed the participants to complete the questionnaire at a time of their convenience. Rather than setting a date and time, the participant could begin the survey any time within a reasonable margin.

Finally, this research tool allowed for a quick and efficient distribution and collection of the surveys. The students completed the questionnaire without the supervision of the research team and since every step of the distribution and collection process was automated, we simply downloaded the completed surveys and were able to begin the analysis of the data immediately.

Regarding some of the limitations of the tool: when recruiting participants, our goal was to have approximately 15 of them. From previous experiences both seeking students' participation and analysing answers in similar studies, we thought 15 would be an achievable goal. Unfortunately, we were only able to gather 10 participants. In the past, we recruited student participants by meeting classes in person and explaining the importance of their participation in research that aims to better understand learning processes. As mentioned above, the pandemic prevented the standard approach to recruitment, and this may be the cause of not meeting the aimed number. In addition, most participants did not answer every question and the answers grew shorter and more imprecise as the questionnaire progressed. We can assume that the students got tired after a certain point and elected to write quicker answers and to give less thought and less detail into their reasonings. This could have been easily prevented in a task-based interview model where the interviewer sits quietly as witness to the students' work but intervenes in accordance with a strict protocol to encourage students to solve the problems and to share their thinking in doing so. This approach would have likely significantly reduced the brief, vague or imprecise answers. In this sense, we recognize the severe limitations of a written online questionnaire but accept the reality of the situation: a pandemic and graduate work with a limit to be completed.

In the next chapter, we introduce a conjecture about the severity of student misconceptions and the process that leads to the overcoming of misconceptions. Then, we discuss three students' apparent misconceptions in depth. We also reflect on the effects that holding onto the misconceptions explored in this study might have on the learning of advanced mathematics, and especially Real Analysis. We conclude the next chapter with a discussion on fundamental misconceptions.

Chapter 6 Discussions

In this chapter, we discuss our results and analysis, and reflect on misconceptions and how they may affect the participants of the study.

Section 6.1 is about our conjecture that the severity of a misconception lies on a spectrum. We surmise that as students learn mathematics, their understanding and assimilation of concepts progresses through the six levels described in Section 3.4. We further surmise that this progression is not linear, thus, students are often in a transitional state between two levels. We detail this conjecture and provide some examples of transitional behavior between two misconceptions levels, taken from the participants' answers.

Section 6.2 consists in a deeper look into specific students' misconceptions. Some participants appear to hold one or multiple of the hypothesized misconceptions based on their response to the questionnaire; this section is concerned with those students and the reoccurrence of erroneous reasonings in their answers.

In Section 6.3 we reflect on the effects that the misconceptions observed in this study can have on the learning of advanced mathematics. Since we conjecture that misconceptions can be carried through one's mathematics education, we believe that they can have a harmful effect on the learning of more advanced topics. We discuss the possible effects that holding the seven misconceptions explored in this study can have as one progresses through the learning of different, more advanced concepts of mathematics.

The last section of this chapter, 6.4, consists in a discussion about misconceptions that are expected to have harmful effects on the learning of most topics and concepts of mathematics. We discuss the roles that mathematical notation and mathematical logic have on learning, and especially how a lack of understanding of the formalities of language and argumentation can hinder one's education.

6.1 Gradient in misconception levels

As mentioned previously, we believe that misconceptions are not a one-size-fits-all misunderstanding of mathematical concepts, but rather that students hold onto misconceptions at different depths. We conjecture that common misconceptions as described in the literature might be present in multiple individuals, but it might be experienced in a multitude of ways. Where one student might consistently return to their alternate conception of a mathematical concept, another one might simply need to see a formal definition to completely dismiss their erroneous beliefs. The six levels of misconceptions considered in this study are part of this misconception gradient, which implies that students might experience a transitional behaviour between two levels at some point in their mathematics education.

The first subsection states and discusses this conjecture in detail. The second subsection consists in an observation of possible transitional behaviour from the students who participated in this study.

6.1.1 Conjecture about transitioning through misconception levels

Our conjecture requires a few assumptions:

- Students have misconceptions.
- The depth of a given student's misconception must eventually corresponds closely to one of the first five levels modelled in Section 3.4 (the "expert conception" level consisting of the lack of a misconception – the student has assimilated the formal mathematical concept at hand).
- The process of overcoming a misconception is gradual and multi-faceted.

With these assumptions in mind, our conjecture goes as follows:

Students eventually express traits corresponding to two or more of the misconception categories simultaneously as they are gradually progressing from one level to the next.

Before considering examples from the participants of this study, let us consider a fictitious student as a thought experiment. We consider a student who initially believes that a function's limit at infinity acts as a "boundary" that cannot be crossed or reached. We would expect this student's conception of limits to be incomplete[9]. We expect students holding this misconception to be competent with the basic problems, but their problem-solving skills may fail them once faced with more advanced examples. As this student progresses through their mathematics education, we expect their conception of limits to eventually improve. We might notice that their conceptual understanding of limits develops as they are exposed to more advanced functions with behaviors that might challenge their previous conception. They might get rid of the arbitrary rules they used to follow, but their conception might still be missing some key elements. For example, while the student might demonstrate proficiency with basic examples, they will lack experience to provide rigorous answers to more advanced problems. However, their work may no longer include false claims. This fictitious example corresponds to a student who is transitioning between the "incomplete conception" and the "unrefined conception" levels. Their overall understanding of limits might still include some incorrect elements, but their increasing experience puts them in transition towards the next level. Using table 4.41 to describe this transition, this student's conception of limits might have the vocabulary and validity of an unrefined conception, but the replicability and procedural understanding corresponding more closely to that of an incomplete conception. Their conceptual understanding could be getting stronger but still involve minor imprecisions. Their logical progression may not follow arbitrary guidelines anymore, but it may still have minor flaws.

[9] This is a conjecture arising from incomplete instruction. The student's concept image (in the sense of Tall & Vinner, 1981) is composed of isolated knowledge elements and they have completed their conceptions with conjectured details that agree with the student's experience of the concept at hand. The conjecture can be correctly applied to some situations, but it is not generalizable.

We expect someone transitioning from the incomplete level to the unrefined level to get rid of the arbitrary rules or guidelines that they have so far mistakenly followed but still lack rigour in their problem-solving approaches and justifications. Their understanding of the concept might still be missing some key elements, but it does not contain (anymore) blatantly false claims. Where someone holding an incomplete conception has significant issues with more advanced problems, and someone holding an unrefined conception shows relative competence with similar problems, a student transitioning between these levels would be able to produce a solution which may contain some (minor) errors and lack rigour.

6.1.2 Examples of participants exhibiting transitional behavior

This subsection contains concrete examples of student answers which seem to correspond to more than one misconception level. We discuss in more detail the reasons that justify the placement of the students' conceptions in a transitional state between two levels. The answers are discussed individually, deeper analyses of specific students are presented in the next section of this chapter.

STUDENT 5 – FROM INCOMPLETE CONCEPTION TO UNREFINED CONCEPTION

In this example, we are considering Student 5's possible misconception that functions with asymptotes must be strictly monotone. In our reflection, we consider their answers to questions 3 and 11.

Student 5's answer to question 3: "a) It is not true.

b) I first thought about it with examples. That may be true for a function who has an asymptote equal to L."

Student 5's answer to question 11: "a) I do not know. I would rather say it is true than not.

b) I have considered asymptote in the context that the function was bounded by that asymptote. For example, 1/x clearly has an asymptote at x = 0 and y = 0.

c) I am confused by the fact that this particular function goes over and below the "asymptote" (if it is one). I do not recall the exact definition of an asymptote, since it goes back to the beginning of CEGEP. The "definition" I have in mind is more of a picture that is letting me down here."

As previously mentioned in this chapter, an "incomplete conception" is generally characterized by a weak understanding of a mathematical notion which includes incorrect elements. Those incorrect elements can be a construction of the student made to fill in the gaps of a concept which is not understood in its entirety. The misconception observed in Student 5's answers appear to exhibit this characteristic. However, the student admits being unsure of their own conception and relying on their mental image of what an asymptotic function can look like when graphed.

In their answers, Student 5 claims that a function cannot cross its asymptote. In this case, it appears that Student 5's issue with the concept of asymptote is simply their failure to recall the definition.

Based on their answer to question 3.a), we believe that had they remembered the definition of an asymptote, Student 5 might have provided a very rigorous and accurate answer to question 11.

For this misconception, we would expect a student with an "incomplete conception" to simply claim that the function $f(x) = \frac{\sin(x)}{x}$ used in question 11 does not have a horizontal asymptote, with the justification being that this function periodically equals the value of the asymptote. Student 5 admits to their forgetfulness and even declares that their mental image of an asymptote is unsatisfactory to them. It is plausible that this student simply requires the correct definition to provide a near perfect answer. For this reason, we surmise that this student's conception of asymptote is more advanced than an "incomplete conception." Since their forgetting of the definition caused an important conceptual mistake and caused Student 5 to respond with incorrect claims about the functions at hand, we also surmise that their conception is not complete enough to qualify as an "unrefined conception." Student 5's conception might be transitioning from one level to the other, and perhaps a simple reminder of the definition of asymptote is all that is required.

One criterion from table 4.41 corresponds to an "incomplete conception" for Student 5: the validity. The validity is average since this misconception caused them to respond to question 3.b) with an incorrect claim. This student's conception does contain some arbitrary details which appear to be based on a graphical argument. However, since they admit not trusting this mental image, we could consider their logical progression to be in between the "incomplete" and "unrefined" conception levels. We consider this student's replicability, conceptual understanding, and procedural understanding to be at the "unrefined conception" level. The replicability and procedural understanding are considered unrefined because Student 5's forgetfulness prevented them from providing an answer to question 11, although they could have followed their (incorrect) instinct and answered with "false". Finally, this student's vocabulary is very clear, and they use formal mathematical terms at a level which we consider "expert." For these reasons, we conjecture that Student 5's misconception about asymptotes is in transition towards the "unrefined conception" level.

STUDENT 6 – FROM PRECONCEPTION TO INCOMPLETE CONCEPTION

We are considering Student 6's possible misconception that the numbers $0.\dot{9}$ and 1 are not equivalent. We conjecture that Student 6's misconception is in transition towards the "incomplete conception" level but is held back by some preconceptions they might have regarding related topics. Student 6 has declared not having received any formal instruction regarding the convergence/divergence of infinite series, and they admit to forgetting about limits of sequences.

A preconception is usually characterized by an uninstructed guess or assumption about a mathematical topic. Preconceptions can be caused by a non-mathematical definition of a term which also has a formal mathematical definition. As mentioned earlier in this chapter, incomplete conceptions are usually based in accurate notions but involve student-constructed conjectures which can be invalid. These two levels are not defined successively: the “alternate conception” level separates them. This level is characterized by a more robust conception of an inaccurate notion. We surmise that Student 6’s conception has more in common with the “preconception” and “incomplete conception” levels even though they are not successive. This example shows that the progress that a student makes in their learning is personal and can happen in a non-linear path. It is our conjecture that Student 6’s conception is improving in such a way that it might never be considered a true “alternate conception.” Let us reflect on this possibility by considering the following quotes taken from Student 6’s answers to the questionnaire:

Student 6, to question 12.b): “[…] we are always ending up adding one extra 9 at the end of the decimal series. Which means it would converge to that 0.9dot notation. […] the fact that it is converging to a non-rational number means that this number continues infinitely, and therefore it is not a finite number. Thus, can a function or sequence converge to a non-finite number, or does that mean it is diverging? I think it is still converging because it is getting more precise with every additional 9, but it is not 100% clear for me.”

Student 6, to question 13.b): “Since the 0.9dot notation is an irrational number, the sequence cannot converge to a number that isn’t finite. Thus, I think that the question 12, would be false, and this would be true because this sequence does amount to 0.9dot which itself is converging towards 1 (where 1 is a finite number)”

As we have discussed in Chapter 5, this student’s misconception appears to be caused by multiple different sources. The most striking error in this student’s reasoning is their claim that $0.\dot{9}$ is an irrational number, and thus that it is infinite. This statement alone appears to demonstrate that Student 6 holds misconceptions regarding basic sets of numbers and the meaning of finite/infinite. When it comes to the methods used by the student to identify the most accurate answer to questions 12 and 13, their use of the suggested sequence of partial sums is very unsophisticated and appears to be hindered by the previously mentioned misconception.

We conjecture that this student’s misconception regarding the connection between infinity and numbers with infinitely many non-zero decimal numbers does not origin from any formally taught notions. Rather, we suspect that this idea is a construction of the student in its entirety. It is clear from this answer that the concept of finite values is not appropriately understood by Student 6 which causes them to attribute the labels of “finite” and “non-finite” according to the mode of representation of numbers. This interpretation would correspond to a “preconception” of irrational/periodic numbers and infinity. However, the student’s attempted use of sequences contains some very basic but correct elements. Notably, the student claims that sequences do not converge to infinity. In this case the student assumes that an irrational number is infinite, but we believe that this mistake is caused by their misconception on numbers, and not on a misunderstanding of convergence. Their arguments which relate to sequences and convergence are very basic and intuitive and do not contain any formal or rigorous manipulations. This use of sequences reflects a very weak understanding of a notion used in a correct context. We conjecture that this student’s misconception that the numbers $0.\dot{9}$ and 1 are not equivalent is in transition towards the "incomplete conception" level, but it is held back by the issues underlined in this subsection.

We identified four of the evaluation criteria that seem to correspond to a preconception: the vocabulary, the replicability, the validity, and the logical progression. The vocabulary used by the student is not only quite basic, but often employs mathematical terms incorrectly. The very low validity of their reasoning has been explained above, but the replicability is also incredibly low since their interpretation of the "finiteness" of a value depends on arbitrary factors that cannot yield consistent results. The logic employed by this student is also very weak. Their claim in question 13 that sequences cannot converge to $0.\dot{9}$ contradicts their earlier claim that adding 9's into the decimal expansion makes the value more precise. This student's procedural understanding of these two questions corresponds more closely to the "alternate conception" level since they attempt to create a solution, but it is entirely based in intuition and employs very little rigorous mathematical methods. Finally, their conceptual understanding of sequences appears to be at the "incomplete conception" level. They do understand what it means for a sequence to converge, but their understanding of the concept is severely hindered by other misconceptions. For these reasons, we conjecture that Student 6's misconception that the numbers $0.\dot{9}$ and 1 are not equivalent is transitioning towards the "incomplete conception" level.

6.2 A more in-depth look at some participants

In this section, we reflect into the case of certain participants who appeared to hold multiple (sometimes contradictory) misconceptions. More importantly, we reflect on how these potential misconceptions work together as a deterrent to each of these students' understanding of fundamental mathematical principles and how they may hold back these students from appropriately learning more advanced mathematical concepts.

6.2.1 Student 1

We begin our reflection on Student 1's misconceptions with the well-studied example of "a function never reaches its limit." If we recall from Chapter 5, we have found some indications that Student 1 might hold this misconception to a certain degree. Their answer to question 3.a), which directly confronts this misconception, appears to support this possibility. Their justification in question 3.b) is an incorrect use of mathematical implications and it does not justify their answer logically. This prevents us from getting useful insight into their actual understanding of the statement. The example that Student 1 provides in question 3.c) also does not help us in our endeavours since they simply provide the example of the function $f(x) = \frac{1}{x}$. This example is strictly monotone, and it is consistent with Student 1's answer to question 3.a) (true), but it does not act to justify their point of view, it simply serves to illustrate their response. As we have discussed in Subsection 5.2.1, Student 1's answer to question 15.d) weakly suggests that they might hold this misconception. We claim that their answer includes a weak allusion to this misconception because, once more, they simply use the function $f(x)=\frac{1}{x}$ as an illustrative example and they claim that it does not reach its limit.

On the other hand, there are some comments made by Student 1 which directly refute this misconception, and therefore their answer to question 3.a). An example which directly indicates that Student 1 does not believe that functions cannot reach their limits is their answer to question 8. Student 1 directly states that $\lim_{x\to\infty}\frac{\sin(x)}{x}=0$. Their reasoning to arrive at this answer is mostly intuitive and not rigorous, but it is logical and shares some similarities with the logic behind the squeeze theorem. The question included a picture of the graph of this function so the students could see that the function periodically crosses its limit at infinity. Student 1's answers to questions 7, 9 and 10 are consistent with this conception of limits.

In this instance, we ask ourselves: if Student 1 can accurately determine the limit to infinity of an oscillating function, why did they respond "true" to question 3.a)? We initially suspected that Student 1 might have misread or misunderstood the question. However, any alternative interpretation of the question would be inconsistent with some of their answers to questions 3.a) through 3.d). We therefore surmise that Student 1 correctly read and interpreted the question but provided an answer to question 3.b) that does not respect the rules of mathematical logic. Another possible explanation for Student 1's contradictory answer is that earlier in the questionnaire, when they had not yet been exposed to any pre-constructed examples of functions, they failed to produce an example that would disagree with the statement in question 3. In such a case, Student 1 might have chosen to agree with the statement since they could not come up with an example that would contradict it. When they began responding to question 7, they were exposed to an example of a function directly contradicting the statement of question 3, and successfully identified the limit. If this explanation is accurate, this would be an occurrence of a cognitive obstacle caused by a lack of experience. This student might understand the concept of limit much better than they were able to show in question 3, but due to their insufficient mathematical maturity, they provided an answer which is conceptually incorrect.

Student 1's answer to question 11 informs us that they do not recall the definition of an asymptote. They appear to incorrectly distinguish limits at infinity and horizontal asymptotes since their answer to question 11 differs from their answers to questions 7 through 10. Student 1 appears to hold the misconception that asymptotes are values which are strictly monotonically approached by a function.

Student 1's answer to question 3.d) shows an understanding of limits at a point that appears to be at the "unrefined conception" level. Their answer is clear and correct: the statement "$\lim_{x\to c}f(x)=L, then\ f(x)\neq L\ for\ every\ x$" is false. To disprove this statement, all that is needed is a counterexample which the student provides in the form of a continuous function around the limit point. Student 1 also provides an example for which the statement is true, which is unnecessary to support their answer. We believe that Student 1 suggests continuity as a sufficient (but unnecessary) condition to disprove the statement and further examples that justify it not because they misunderstand the concept, but rather because they are uncertain of how much justification is needed. This type a behavior corresponds to a conception which is improving towards the “unrefined conception” level. This student’s knowledge of necessary and sufficient arguments appears to be lacking, but they successfully provide accurate justifications.

There is some anecdotal evidence that Student 1 struggles with mathematical notation in their answer to question 22, which is all about quantifiers and mathematical language. As we discussed in Subsection 5.2.7, we suspect that Student 1 might hold onto an incomplete conception of the rules and logic behind formal mathematical notation. The only other question that involves quantifiers is question 21, which asks the students to recognize the epsilon definition of limit. Student 1 correctly translates the statement in common English, indicating that they know the meaning of every symbol, including the universal and existential quantifiers. However, they do not identify that the statement defines a limit, and instead respond that it is the definition of a neighborhood. This may be caused by the simple lack of recognition of the statement, paired with some “noise” from their current studies in Analysis. Alternatively, the student might know the words that correspond to each symbol, but they might not understand the meaning through the syntax of the mathematical statement, and they might be guessing, from memory, what the statement refers to. Their answer to question 22 supports this explanation since they incorrectly claimed that both statements are equivalent, and incorrectly set a fixed value for the universal quantifier to illustrate their claim. We believe that Student 1 has an accurate knowledge of the English equivalent to mathematical quantifiers and statements but has issues with the logic behind the syntax which prevents them from accurately understanding statements written using formal notation. This conception would correspond to an “incomplete conception,” as stated in Subsection 5.2.7.

We now discuss Student 1’s understanding of limits as mathematical objects. We are interested in Student 1’s understanding that expressions such as “$\lim_{x\to\infty} f(x)$” are static objects which can only be manipulated if they fulfill strict conditions. Their answer to question 16 suggests that Student 1 does not understand (or remember) these conditions. They claim not remembering the “trick” to solve expressions such as $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$ when both functions diverge. This comment informs us that the student is solely focus on finding an answer to the question and does not recognize that the summation has no solution due the divergence of the functions. Student 1 recognizes that this question is asking to perform arithmetic with infinity, yet they still choose to agree with the fictitious students and provide “1” as the answer of the sum. We believe that once more, this contradiction is caused by Student 1’s lack of experience. Student 1’s answer to question 20 is consistent with our observations. They simply reply that they do not recall the correct conditions for the Sum Law to hold, and they incorrectly guess that continuity must be it. These answers strongly suggest that Student 1 does not understand that performing arithmetic with limits requires the limits to exist. Alternatively, it is possible that Student 1 understands that the Sum Law holds only for existing limits, but they might view infinity as a possible limit value. This possibility is supported by their answers to questions 18 and 19.

Questions 18 and 19 asks the students to group functions according to their behavior as x goes to infinity. Student 1 creates groups in the following way:

“Group 1: Limit x goes to infinity equals 0

Group 2: Limit x goes to infinity equals c (a constant)

Group 3: Limit x goes to infinity equals minus infinity

Group 4: Divergent limit

Group 5: Limit x goes to infinity equals infinity”

Student 1’s justifications for these groupings reveal that they do not understand that a function diverging to infinity implies that the limit does not exist (and therefore cannot be subjected to algebraic manipulation). When asked if their groups have intersections, Student 1 answered with the following quote: “No, since the answer of the limit can only be one value (we cannot have the limit being equal to 0 and minus infinity at the same time.” This claim implies that functions that go to infinity are not considered divergent. This is consistent with the previously discussed issues from their answer to question 16.

We now consider Student 1’s answers to the last few questions, which are about concepts related to Analysis. We have discussed in Subsection 5.2.9 how Student 1’s answer to question 23 appears to suggest that they hold a misconception where the concepts of supremum and absolute maximum are confused. In addition to this apparent misconception, their answer to question 26 includes a statement that misuses infinity, much as it has been highlighted previously.

Student 1’s answer to question 26: “[…] I do think the sup exists here and is equal to infinity.”

This quote appears to be a direct translation of Student 1’s potential misconception that infinity can be a limit value. We see a very similar application of the same misconception where Student 1 claims that infinity is the supremum of the set $A = \{\ln n \,|n = 1, 2, 3, \dots\}$. We conjecture that their earlier misconception regarding infinity as a limit value has directly transferred to their current studies in Analysis and changed into a similar yet more advanced version of the same misconception. This is a great example of how misconceptions developed in earlier mathematics classes (not only calculus, but earlier high school classes as well), if not overcome, can carry over and affect how one might learn more advanced concepts.

Alternatively, this answer might come from their studies in Real Analysis. Certain textbooks (such as Dangello & Seyfried, 2000) introduce the concept of extended real numbers in the context of discussing bounds. Using the extended real numbers, infinity can be used as a bound, and therefore as a supremum. However, the set of extended real numbers was not mentioned neither in the question, nor in Student 1’s answer. They simply claimed infinity to be the value of the supremum. It is plausible that this student has retained only a fraction of the notions related to extended real numbers and assumes that a supremum can take the “value” of infinity, without being mindful of the conditions that are required to make this claim. It is possible that Student 1 simply defaults to using the extended real numbers, in which case their main issue with this concept would be to mention the number system that they use in their answer.

We close our deeper dive into the case of Student 1 with some speculations about further concepts that could be affected by the misconceptions observed in their answers. The first and possibly most impactful misconception is Student 1's apparent misunderstandings of formal mathematical notation. The use of quantifiers and logical syntax only becomes more predominant in university mathematics classes after Calculus. It is undeniable that an incorrect understanding of the rules of mathematical notation is a significant obstacle to learning advanced and abstract mathematical concepts. Furthermore, the seemingly "minor" misconception concerning taking infinity as the value of a limit seems to have a significant impact on the student's learning of the concept of supremum; we surmise that if not overcome, it will continue to negatively impact the student's learning of other advanced mathematical concepts that heavily rely on the concept of limits.

6.2.2 Student 6

We begin our reflection on Student 6's misconceptions with a discussion on formal mathematical language. Observing Student 6's answers to questions 3 and 22, we notice that they misunderstand mathematical notation. We have previously discussed (see Subsection 5.2.7) their answer to question 3 and determined that they misinterpreted the conclusion "$then, f(x) \neq L \; for \; every \; x$" as meaning "then, the function is not constant." With this interpretation of the statement, Student 6's answer is consistent and shows a good understanding of functions and limits. However, the misunderstanding of a statement that does not involve many logical operators or quantifiers is a testament to a deeper issue with this student's learning of mathematical notation. This student's apparent issues with formal statements are supported by their short and incorrect answer to question 22.b): "They are equivalent. An example would be a=3, b=2." Much like Student 1, this student claims that the statements are equivalent regardless of the order in which the existential and universal quantifiers appear and provide a trivial example which suggests a misunderstanding of the latter. Furthermore, Student 6 failed to identify the epsilon definition of limit in question 21, and instead responded that the statement defines "the Lower Upper Bound." Again, similarly to Student 1, this could suggest a lack of experience with such statements, paired with noise from their current studies of infima and suprema. It appears that Student 6 has a good knowledge of each logical symbol, but incomplete understanding of the syntax behind mathematical statements which causes them to incorrectly interpret rather basic questions.

Student 6 has an interesting conception of the common words explored in questions 7 through 10. They initially did not notice that the questions were different until they reached question 10. This could indicate that Student 6 considers these terms to be synonymous. They provided acceptable answers to each question. However, as they realized that the phrasing slightly varied, they decided to address the indirectly asked question "are these four questions equivalent?" In their response, they reveal that they conceive the phrases in the question (approaches, goes to, etc.) as different from the phrase "the limit is." Furthermore, they assume that the term "limit" is stronger than the alternative vocabulary. In addition, Student 6 states that the terms "approach" and "tend to" can be used to refer to a purely visual interpretation of a function. Conversely, using the phrase "the limit is" requires the use of formal resolution methods. Although this interpretation is not entirely correct since these words can be used interchangeably in the context of these questions, we are interested in the source of such a misconception. We conjecture that this misconception is didactic in nature; not in the sense that the student was taught to grant less importance to certain words, but rather in the sense that they were not exposed to a formal discussion of what those terms mean and that they are interchangeable. In the absence of such formal discussions and explanations, students might associate a non-mathematical meaning to those terms. On the other hand, students who mistakenly adopt the non-mathematical meaning in the mathematical context, can develop other misconceptions, such as the implicit strict monotonicity of limits. The assumption that the word "limit" is viewed as stronger than the alternative terms has also been observed by Cornu (1980) and Monaghan (1991). For these reasons and the ones provided in Subsection 5.2.8, we surmise that Student 6 holds an unrefined conception of this mathematical vocabulary which is didactic in nature.

We now highlight an issue which has been observed in many participants, including Student 6: incorrectly claiming that asymptotes are values that the function approaches but never crosses. Student 6 provided this exact explanation in their answer to question 11. However, this issue appears to be exclusively for asymptotes since Student 6 asserts the following in their answer to question 15: "I believe that this might be what [differentiates] limits [from] asymptotes: asymptotes can be where a function converges to but cannot reach, whereas a limit can be reached." Since this condition of "unreachability" is incorrect, we conjecture that it is a construction of the student that is built through exposure to examples that respect this criterion. In Subsection 5.2.2, we surmised that Student 6's misconception about asymptotes is at the threshold between the "incomplete conception" and "unrefined conception" levels. This decision is mainly due to their awareness that they forgot the definition of asymptote. This student has a predisposition to actively seek out the definition and improve their understanding of the concept, and therefore is transitioning towards an "unrefined conception."

Although Student 6 appears to properly understand that a function can reach its limit, their answers to questions 18 and 19 include peculiar details relating to this concept. As a reminder, question 18 asked the students to group a set of functions according to criteria of their choice. Student 6 chose the following criteria:

"Group 1 has functions with the limit as x goes to infinity equal to a real finite number which they reached. Group 2 has functions with the limit as 0, which [they] never really reach only approach. Group 3 has functions which limits don't exist as x goes to infinity."

In Subsection 5.2.1, we suggested two alternatives for their inconsistent choice of groups: They may not know the behavior of the function $f(x) = \frac{\cos(x)}{x}$ (which they put in group 2), or they may hold an alternative definition for the word "reach." We believe the latter explanation to be more likely. As a rebuttal to the former alternative, Student 6 has provided correct reasonings to questions 7 through 10, which all involved the similar function $f(x) = \frac{\sin(x)}{x}$. Student 6 uses the term "reach" several times through the questionnaire, none of which are precise enough to confirm that their conception of this term aligns with the accepted mathematical meaning. Hence, we conjecture that

Student 6 has an incomplete conception about the word "reach" meaning a stabilization of a function onto the value of its limit at infinity.

In their answer to question 16, Student 6 provides a very intuitive and simplified conception of infinity which allowed them to correctly identify that the sum of the limits of two diverging functions is undefined. Extrapolating from their explanations, they seem to imply that infinity can be simulated into arithmetic equations for the purpose of identifying limits but cannot be used as a result or as the value of a limit. If our interpretation of their response is exact, their conception is naïve but not particularly harmful. Student 6's main issues with infinity appear in their answers to questions 12 and 13, which have been discussed in Subsection 5.2.6. We discussed the apparent misconception held by Student 6 which claims that irrational and periodic numbers behave like infinity, and therefore cannot be convergence points for sequences or series. Student 6's misconception appears to lie with the definitions of "finite" and "infinity." More specifically, there seems to be a discrepancy in this student's conception of "infinity" and "non-finite." Consider the two following quotes:

Student 6's answer to question 13.b): "Since the 0.9dot notation is an irrational number, the sequence cannot converge to a number that isn't finite."

Student 6's answer to question 15.b): "[…] infinity is not a real number […]"

These two quotes are directly contradictory. The first quote stipulates that irrational numbers are not finite, the second quote asserts that infinity is not a real number. Therefore, for those two quotes to agree, Student 6 must believe that irrational numbers are not real numbers. Alternatively, and as stated above, perhaps this student's conception of "infinite" and "non-finite" are different. We conjecture that they perceive "infinite" as the concept represented by the symbol ∞, and the term "non-finite" as referring to anything that has a never-ending component, such as irrational numbers. We also conjecture that this misconception might originate in the basic sets of real numbers, and in the different representations of numbers. They might not have internalized the fact that even with infinitely many decimal values, irrational (and periodic) numbers are still points on the real number line, and therefore ascribe to the same mathematical rules as other real numbers. Alternatively, this misconception might be caused by noise from their current studies. Real numbers are discussed extensively in Real Analysis, and it may be possible that Student 6's answers were negatively influenced by this discussion, if new information was not properly understood.

We now discuss the effects that holding such misconceptions can have on Student 6's learning of mathematics. The two most harmful issues observed in Student 6's answers are their apparent lack of understanding of formal mathematical statements and their misconception of irrational and periodic numbers being infinite. The former is obviously a hinderance to learning since it prevents the student from accurately understanding a multitude of definitions and theorems in every area of mathematics. We even speculate that this issue might play a role in other of Student 6's misconceptions. The latter is also a severe obstacle to learning because it can cause conflicts between this student's different conceptions. An example can be observed in Student 6's answer to question 13: their idea of irrational numbers being infinite and their understanding that sequences can only converge towards finite values disagree, which creates confusion. Other examples of similar situations include $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ or $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)} = \frac{\pi}{4}$. Both previous examples include irrational numbers which are respectively the limit of a sequence and the value of a power series. When Student 6 eventually gets shown those results or similar ones, their understanding might be held back by this misconception which makes them believe that sequences cannot converge towards irrational numbers. In addition, this misconception may represent a significant obstacle to Student 6's learning of other concepts related to the set of real numbers, such as the density or openness of subsets. Their misconception about the words "goes to," "approaches," "tends to," and "reaches" could be an obstacle to learning if these words appear often in textbooks, of if their teacher uses them often. Otherwise, it may simply serve as an incentive for Student 6 to use rigorous language.

6.2.3 Student 8

We begin our reflection on Student 8's potential misconceptions with a discussion on their answer to question 15.c): "Because, like B said, I think the idea of a limit is essentially a value that can't really be reached but it's a value that it gets closer and closer to." At first glance, one might claim that Student 8 holds the common misconception "a function never reaches its limit." However, this specific answer from Student 8 is in direct contradiction with an earlier statement they made in their answer to question 3.b): "It would seem that the function can equal to its limit for some value x. I think it's true that not all x will result in f(x) = L, but I think it is possible that f(x) can be the same as its limit, as x approaches infinity." Through the rest of the questionnaire, Student 8 does not mention anything about functions' ability to reach their limits. However, they respond to questions 7 through 10 accurately without being confused or troubled by the function periodically crossing its limit (even with the graph provided). Our conjecture for this incoherence in Student 8's conception is more behavioral rather than mathematical. We propose that Student 8's understanding is better represented by their answer to question 3, but that their mistake in their answer to question 15 was caused by suggestion. Question 15 presents a sequence which monotonically approaches its limit. In the text for the question, a fictitious student (named B) claims that a sequence should never reach its limit, and this statement is not disputed in the question. We surmise that Student 8 was influenced into agreeing with student B by the text in the question and by the fact that the sequence at hand respects this arbitrary criterion. Therefore, we conjecture that Student 8 does not believe that functions cannot reach their limits and that their apparent misconception from their answer to question 15 does not originate from their own understanding of the concept. We surmise that their conception of limits is at the "unrefined conception," level since their understanding appears accurate, but their hesitation which caused the mistake in question 15 shows that their conception is not robust and can be improved.

Student 8's answers to questions 7 through 10 are inconsistent due to their own admitted doubts about the synonymity of the words used in those questions. The inconsistency appears in question 9 where they acknowledge not knowing if "tends to" is equivalent to "the limit is." Similarly, to students 1 and 6, this might be caused by a lack of formal definitions for commonly used words.

Perhaps this student has heard the phrase “tends to” more rarely than the other alternatives in the same context, which would explain their hesitation specifically with question 9. This misconception is very superficial and is unlikely to represent any significant hinderance to Student 8’s understanding of limits. Therefore, we believe that this misconception is at the “unrefined conception” level.

Question 11 reveals the same misconception that has been observed in students 1, 6 and many others, which is the incorrect assumption that asymptotes are horizontal lines which a function approaches without ever reaching. This shows the lack of understanding of the definition of horizontal asymptotes, and perhaps an effect of overexposure to examples that share common arbitrary traits, such as monotonicity. Student 8 appears to firmly believe this misconception, as opposed to Student 6 who admitted being unsure.

An overarching issue observed in their answers to questions 7 through 11 is Student 8’s exclusively visual arguments. In Subsection 5.3.1, we discussed how the presence of graphs in the questions may have influenced their decision to solely rely on graphical arguments, which are generally considered invalid, especially in the context of limits at infinity. Since Student 8 did not provide any other reasonings for their answer, we cannot confidently claim that they would have been able to provide a satisfactory solution for these questions without the graph. In this instance, we believe that the image of the graph might have influenced Student 8 into providing a visual argument over a more formal or rigorous solution, regardless of their mathematical skills.

Student 8’s answer to question 16 includes a few mistakes, including an inconsistency regarding the answer to the fictitious students’ problem. In question 16.b), Student 8 accurately determines that the sum is undefined. However, in their answer to 16.c), their answer pivots into agreeing with the fictitious students’ logic without providing any additional reasoning. Student 8 concludes that the answer must be 1, in accordance with the fictitious conversation, but in contradiction to their earlier response. Moreover, Student 8 states that the problem involves an indeterminate form, which is incorrect. Similarly to our conjecture regarding their answer to question 15.c) discussed earlier, we believe that this inconsistency in Student 8’s answers in caused by the influence that the fictitious conversation has on their thought patterns. The student begins with the correct answer but appears to get confused by statements which they are incapable of refuting and abandon their initial instincts. We conjecture that Student 8’s issue with mathematical concepts here is exclusively with indeterminate forms since they misidentified a situation. Their doubts and hesitations do not appear to be caused by a weak conception, but rather by the suggestion of other incorrect reasonings. Their misidentification of an indeterminate form is telling on their conception of limits. In this situation, Student 8 was considering the sum of two limits that did not exist. Student 8 most likely thought that they were in the presence of the “$\infty - \infty$” indeterminate form. This informs us that they might instinctively view diverging limits as “equal” to infinity, even if they conceptually understand that the limit does not exist. This misconception appears to be quite weak considering the correct answer that was provided in question 15.b). We surmise that their understanding of the concepts explored in question 15 lie at the “unrefined conception” level.

Overall, Student 8's misconceptions appear to be quite weak. Their mistakes seem to be caused by superficial misunderstandings that most likely will have little to no effect on their learning of new mathematical concepts. For example, their hesitation on the meaning of the phrase "tends to" or their misconception about asymptotes can cause unwanted mistakes in certain specific situations, but they will most likely not prevent Student 8 from learning to use new mathematical objects or theorems. Their use of visual arguments might be an obstacle for problem-solving. We have no evidence that Student 8 believes graphical arguments to be valid, but if that is the case, Student 8 might have significant issues with problem-solving that can only be resolved if they understand the need for rigor.

6.3 Expected misconceptions and their effect on mathematics learning

It is undeniable that misconceptions can have a significant effect on one's learning experience. It is our and other researchers' (Sierpińska, 1987, 1990) conjecture that misconceptions are inevitable and a natural part of the learning process. Students must begin their education from a point where their ideas of mathematical concepts are undeveloped and feeble. Through formal teaching or other means of instruction, students must slowly build their knowledge and understanding of abstract mathematical objects. Misconceptions may arise at any point in one's education and they may originate prior to instruction, but it is the eventual dismissal of erroneous views that truly characterizes the learning process and the acquiring of expert conceptions. In this study, we illustrate the possibility that misconceptions can remain even after the students acquire the necessary qualifications to continue their mathematics education past the basic Calculus class. In the case where those misconceptions remain, what obstacle could they represent to the learning of more advanced mathematics?

We believe that pre-existing misconceptions can hinder one's learning in multiple ways. The first way, we will refer to as concept-to-concept. If one concept serves as a building block for further mathematical concepts, holding a misconception related to the first one will most likely influence the learning of the next. For example, horizontal asymptotes are typically defined as a horizontal line $y = L$ such that $\lim_{x\to\infty} f(x) = L$ (or, equivalently, the limit to $-\infty$). Since the definition of a horizontal asymptote depends on the understanding of limits, a student who has a misconception regarding limits at infinity might transpose this misconception into their conceptualization of asymptotes. For instance, if a student initially believes that a function may never cross or reach its limit, this misconception might follow into the student's learning of asymptotes. If the student then learns that limits can be crossed, perhaps their misconception about asymptotes will be improved simultaneously, or perhaps it won't. Several of the participants to this study claimed that asymptotes and limits at infinity are different because the former cannot be crossed while the latter can. We conjecture that those students might have, at some point, held the misconception "a limit cannot be reached," which was eventually overcome, while leaving a misconception about asymptotes as a by-product.

We can speculate about further issues that can arise due to concept-to-concept influence of misconceptions; mathematics is learned incrementally with more advanced knowledge depending on elementary knowledge. An example that carries immense importance in the study of functions is the progression limits → continuity → differentiability. Each of these concepts depend on the previous one and each of them is developed into a more detailed array of concepts that have potential to cause misconceptions.

Another possible way that a misconception can affect further learning relates to the cause of the misconception. Misconceptions can arise due to some learning behavior or strategy. An example is the learning strategy of associating "basic" images into one's conception, which holds the student back from accurately understanding the concept in its entirety. Very basic examples are necessary to understand the meaning of the concept itself, but they rarely, if ever, show the whole picture. A student who is used to visualizing mathematical concepts might be influenced by this behavior and lose some important information or oversimplify the concept. For example, we have observed students who associate a function which has a limit at infinity to the basic case of the function $f(x) = \frac{1}{x}$. This function is monotone, and it is likely to introduce the misconception that functions or sequences need to be strictly monotone to have a limit, if the student exclusively relies on this image. Holding reference images in one's conception is not necessarily purely misleading. In fact, Alcock & Simpson (2004, 2005) have observed that visualization can be beneficial, on the condition that students draw links between their visual representations and the algebraic definitions. Therefore, we believe that purely relying on images of elementary cases to inform one's understanding of a concept is likely to cause a multitude of different misconceptions, since those elementary cases are insufficient to complement the algebraic definitions. This behavior can be reconciled if the student is inclined to develop their conception to include algebraic representations of the concept. Tall & Vinner (1981) suggest another example that illustrates how certain strategies can cause issues further in one's education: issues with limits and continuity can bring up conflicts when learning about differentiability. The concept image that accurately served them previously is now a hinderance since, for example, it is quite challenging to visualize a function that is continuous everywhere but differentiable nowhere (Tall & Vinner, 1981).

We observed several instances of students using incorrect definitions of mathematical concepts. One could think that simply recalling the correct definition or informing the students of their mistakes to be sufficient to accelerate their learning. Davis & Vinner (1986) described the possibility that students hold multiple conceptualizations simultaneously, and simply fail to recall the correct ones every single time.

6.4 The special case of fundamental misconceptions

In Chapter 5, we analyse students' responses that did not necessarily relate to the seven misconceptions we set out to uncover. In particular, we discussed students' misunderstandings, misconceptions, and lack of knowledge around mathematical logic and mathematical notation. We classify the related misconceptions as "fundamental misconceptions." We use this term due the importance that a proper understanding of these concepts has over every field of mathematics. It is expected from students who are enrolled in a Real Analysis class to be able to read, write and reason around logical connectors and quantifiers, not only to learn new concepts but also to be able to construct proofs that respect the mathematical conventions, and that are logically valid.

Logical fallacies include any kind of argument which is invalid due to misuse of mathematical logic. At the level of the students who participated in this study, the most prevalent type of logic is deductive. Most importantly is the understanding of truth tables and the notions of implication and equivalence. Student 1 has provided one instance of incorrect use of implication, where they assumed the equivalence of two statements whereas one statement was simply implied by the other. This is a typical naïve mistake that is usually observed in students who lack experience with mathematical statements. The reason for Student 1's incorrect use of mathematical logic is up to speculation. Regardless, at the education level in question, we would expect students to know the basic rules of logical implication. The results of our study show that several of the participants struggle with this. Furthermore, many students used circular arguments to explain their answers to different questions.

Also, we have observed students who appear to misunderstand the meaning of universal statements. And the concept of a mathematical statement itself. Certain students' answers suggest that they don't understand, or don't know, that statements are either true of false. This implies a general lack of understanding for the rules that dictate mathematics in general, and makes us wonder: what do these students understand of the theorems they are exposed to in their university courses?

The next fundamental issue that has been observed is that of clear misunderstandings of notation. Not only have we observed several misunderstandings of statements which involve quantifiers, but also students misinterpreting statements that did not involve any abstract notation. The standing conjecture from our observations is that some students of Analysis appear to know what each symbol mean individually, but they appear to have issues understanding the syntax and the logic that is used to construct those statements, thus failing to understand their meaning. Considering that any definition, theorem, and proof in mathematics makes use of rigorous notation, a proper understanding of the syntax is a crucial part of the knowledge that is required to learn and succeed in higher mathematics.

Lastly, we discuss the students' apparent misconception that graphical arguments are valid or sufficient. Visual representations of mathematical situations can be a useful tool to students. Alcock & Simpson (2004) specifically discuss the validity of graphical arguments and how they can be used efficiently. However, they also explain how certain students use graphs to solve mathematical problems due to their incapacity or unwillingness to provide definition-based arguments. Alcock & Simpson (2004) explain the ongoing debate about the importance of graphical arguments in mathematics. The observations that were made regarding this topic in this study were of students relying purely on graphical representations to infer answers, thus providing little to no valid arguments for their answer.

Our observations about the fundamental nature of mathematical argumentation and notation have raised the following question: should these issues be emphasized in a classroom setting? We acknowledge that many universities have classes dedicated to mathematical logic. In some cases, the basics of truth tables and some crucial rules of logic are taught in Analysis courses, along with definitions for quantifiers and instruction on the mathematical language. However, these lectures' purpose is to provide students with the tools to understand the content of the course, instead of fostering a deep understanding of the fundamental pillars of mathematics. The validity of certain types of arguments, and the details behind mathematical syntax are rarely addressed in formal learning situations. Directly addressing these issues in a prerequisite course might be a solution to strengthen university students' understanding of concepts that they will be using in every single mathematics class.

In the next chapter, we discuss the conclusions of this study. We review the research and reflect on our achievement of our research goals. We also discuss the extent of students' misconceptions of real numbers, even as they are learning Real Analysis. We close the last chapter of this thesis with a suggestion for further research.

Chapter 7 Conclusions

The first section of this chapter, 7.1, is a brief review of our research. We reiterate the research goals and address to what extent we succeeded in achieving them. We also discuss the limitations of our research tool.

Section 7.2 is a discussion on further considerations, especially as they relate to our assumptions regarding Real Analysis students' knowledge of real numbers.

We finish this chapter and this thesis by suggesting an idea for supplementary research.

7.1 Review of the research

Our objectives are three-fold. First, we want to uncover unresolved misconceptions of elementary Calculus in students of Real Analysis. Second, we inquire the evolution of these misconceptions and their potential improvement. Finally, we investigate the possibility that past unresolved misconceptions might affect one's learning and produce analogous, new misconceptions about Real Analysis concepts. To achieve these goals, we devised a questionnaire with the intent of assessing the participating students' conceptions of elementary Calculus and Real Analysis.

This choice of tool allowed us to reflect on students' understanding of seven common misconceptions of elementary Calculus, three misconceptions of Real Analysis, and even unexpected misunderstandings of mathematical logic and notation. However, the extent to which this research tool allowed us to perceive the student's conceptions is quite limited since cognitive frameworks can be much deeper than what is discernable with a questionnaire. Moreover, since our research design has a high dependency on student involvement and honesty, we cannot naively assume that what they wrote accurately represents their understanding nor that our interpretation of their writings is in itself accurate. To circumvent these limitations, we avoid making concrete claims, and instead of considering certitude, we remain in the realm of conjecture.

The answers provided by the students included several instances of conceptual mistakes, procedural mistakes, logical inconsistencies, and overall lack of a robust conception for many elementary concepts of Calculus. These observations contribute to our first and second research goals. We have uncovered indications that some of the participating students may hold misconceptions of Calculus, even after successfully completing[10] a Calculus course. Their answers correlated with the expected thought processes of students who hold these misconceptions, potentially indicating their presence in some participants of this study. This speaks to our first goal.

Furthermore, using table 4.41, we were able to locate sufficiently detailed answers on a gradient of misconception and identify phrases in the answers that we take as hints that certain conceptions are possibly in transition from one level to the next in the gradient. This addresses our second research goal, and we conclude that the vanishing or improvement of misconceptions may correspond to a

[10] In the sense of passing the course

gradual change of one's understanding of mathematical concepts in such a way that the corresponding misconception level grows closer to the "expert conception" level.

For our third objective, we have identified hints of previous misconceptions influencing the learning of concepts in Real Analysis only in Student 1's answers to questions 23 and 26. Student 1 seems to hold two of our hypothesized misconceptions of Real Analysis, namely confusing supremum and maximum and claiming infinity to be a valid supremum. The latter is an example of a misconception we investigated: using infinity as a real number. We surmise that this is a misconception that Student 1 developed as a student of Calculus, that they have not yet overcome it, and that it gets in the way of learning new conceptions (supremum). This could represent an example of earlier misconceptions contributing to the development of others in more advanced fields of mathematics. However, other explanations are possible. For instance, a textbook that is commonly used in Real Analysis courses (D'Angello & Seyfried, 2000) introduces the concept of the extended real numbers very early on (in Chapter 2), which allows infinity and negative infinity to be upper and lower bounds. This may cause confusion in students and could be a possible explanation for Student 1's mistake.

In addition to our findings specifically related to our research goals, we found hints in students' answers that point to them holding misconceptions about fundamental mathematics, particularly around notation and logic. Several instances of students misunderstanding formal mathematical notation were identified. We surmise that these students have a fairly good understanding of the meaning of each symbol, including existential and universal quantifiers, but a poor understanding of the syntax and semantics in the use of such symbols. We claim, however, that fluency in the syntax and semantics of mathematical language is key in the learning of fundamental Real Analysis concepts. Most definitions, theorems and proofs rely on a deep understanding of mathematical formalism, and logical deduction is the very basis of mathematical proofs. Our observations are a warning for instructors and curriculum developers who may take for granted that students in Real Analysis understand the meaning in mathematical sentences – a call to reflect on how to improve students' fluency in mathematical language.

This research is our contribution to knowledge about post-secondary mathematical misconceptions. We hope to bring awareness to university educators that students come into their mathematics class with pre-built (mis)conceptions that can affect their learning of new concepts. Furthermore, our study illustrates that even after instruction, it is possible for elementary misconceptions to remain. We hope that this thesis feeds the discussion about the challenges of learning and teaching mathematics in a way that is beneficial for students and instructors.

7.2 Further considerations – real numbers

It is assumed that prior to enrolling in a Real Analysis course, students have received sufficient instruction regarding real numbers to be able to understand the material and concepts taught in the course. Our results reveal that this might not always be the case. We have discovered that certain students' knowledge of real numbers is quite poor. Misunderstanding the value of numbers based on their representation constitutes the basis of one of the misconceptions we investigated: "$0.\dot{9} < 1$." Our results revealed that certain students' misconception of number representation can go much further. More specifically, the distinction between numbers that have an infinite decimal expansion, and irrational numbers seems to be a source of misunderstandings for certain students. One student's apparent misconception went as far as confusing "infinite decimal expansion" with "infinite," which then confused them further into assuming sequences (or series) cannot converge to irrational numbers. Of course, this is one instance of a particularly flagrant misconception of real numbers, but the student in question was enrolled in a Real Analysis class during their participation to this study. It is therefore plausible that multiple students holding such misconceptions may eventually go through Real Analysis classes.

We judge it is important for instructors to be aware that their assumptions regarding student knowledge of real numbers might be incorrect. This study has revealed that Real Analysis students may have misconceptions of real numbers, and we surmise that they might be severely hindered in their ability to learn more advanced concepts as a result. Further research should investigate Real Analysis students' conceptions of real numbers and the effects that a poor knowledge of them may have on their learning of advanced mathematical concepts.

7.3 Further research into mathematical misconceptions

Out of our many observations, the most unexpected were students misunderstanding basic notation and using logical fallacies to justify their reasonings. At the university level, we consider this knowledge to be crucial for learning. Some institutions offer specialized courses in this topic and others offer mathematics courses that are built around the learning of proof methods and thus contribute to the development of accurate conceptions of logic. However, no such course is required to enroll in the Real Analysis course in the university we considered in this study, and this is frequent across North American universities.

While it is usual for Real Analysis courses and textbooks to include a brief overview of truth tables and the basics of mathematical proofs, we surmise based on personal experience, the observations of our study, and previous research (e.g., Broley, 2020) that undergraduate students do not receive sufficient instruction on mathematical logic, and therefore are hindered in their ability to properly understand the topics introduced in their Analysis classes.

We believe it is key to further investigate students' misconceptions around elementary mathematical logic as they progress in their university studies. Furthermore, we believe it is important to develop, and test, teaching designs that can foster an earlier exposure to the syntax and semantics of mathematical language and study the impact this may have on students' understanding of advanced mathematical concepts.

Such a study could rely, for example, on the work of researchers such as Selden and Selden, who have made extensive research on the topic of mathematical statements (1995) and on undergraduate students' abilities to solve non-routine problems (1994, 1999).

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Appendix

A.1 Consent form

UNIVERSITÉ Concordia UNIVERSITY

NOTE: This will appear in the first page of the questionnaire in Moodle. The participant will not be able to continue with the questionnaire until they check the "I agree" box. All information in this consent form will be sent to the participant in advance (at least 24 hours), via email, so they can read and ask questions (also via email).

To: participant

Subject line: Research study: INFORMATION AND CONSENT

Dear student,

Thank you for contacting me about the research study I am carrying out as part of my master's program.

Below you will find information regarding this study and your consent to participate. If you have any questions about your participation, please, contact me by replying to this email.

As explained below, the questionnaire I hope you can answer is in a Moodle site. To give you access to it, I will need your full name, netname and student ID. As explained below, all information that can serve to identify you will NOT be linked to the answers you provide to the questionnaire and I will not share with your instructors whether you have chosen to participate or not.

Many thanks for offering to participate. I would not be able to complete my studies if it weren't for students willing to help with the research!

Best regards,

Marc-Olivier Ouellet

Master's student

Mathematics & Statistics

Concordia University

Research Study Title: The Vanishing of Misconceptions About Limits and Their Possible Replacements in Advanced Mathematics

Researcher: Marc-Olivier Ouellet

Researcher's Contact Information:

Email address: MA_UELL@live.concordia.ca

Faculty Supervisor: Nadia Hardy, Galia Dafni

Faculty Supervisor's Contact Information:

Email address: nadia.hardy@concordia.ca, galia.dafni@concordia.ca

You are being invited to participate in the research study mentioned above. The text below provides information about what participating would mean. Please read it carefully before deciding if you want to participate or not. If there is anything you do not understand, or if you want more information, please ask the researcher.

A. PURPOSE

The purpose of the research is to identify some well-known common misconceptions about limits and to investigate how and when they are overcome. A secondary purpose is to find out if basic misconceptions might be replaced by new, similar misconceptions about more advanced topics.

B. PROCEDURES

If you participate, you will be asked to complete a questionnaire. You will be allowed 3 hours to complete the questionnaire, it is more than enough time to do so and to take some breaks if you feel like you need them (without breaks, it would take you approximately **45 minutes** to answer all questions). The 3 hours timer will begin when you start the questionnaire at a time of your convenience.

The questionnaire has similar structure to a traditional exam, but significantly shorter. The questionnaire will test your knowledge and understanding of certain key concepts involving limits and is expected to require around 45 minutes to complete.

In total, participating in this study will take at most two sessions of 3 hours – but likely a lot less as answering the questionnaire without breaks would not take more than 45 minutes.

C. RISKS AND BENEFITS

You might face certain risks by participating in this research. These risks include: Mild stress when completing the questionnaire because of its similarity to exams.

Potential benefits include: Contributing valuable knowledge to the scientific community. Improving the quality of post-secondary mathematics education. This research is not intended to benefit you personally.

D. CONFIDENTIALITY

The researcher will gather the following information as part of this research: Your name, email address, netname and student ID.

The researcher will not allow anyone to access the information, except people directly involved in conducting the research. The researcher will only use the information for the purposes of the research described in this form.

The information gathered will be coded. That means that the information will be identified by a code. The researcher will have a list that links the code to your name.

The researcher will protect the information by keeping the codes confidential and keeping the data in a locked computer.

The researcher intends to publish the results of the research. However, it will not be possible to identify you in the published results as the data will be encoded to protect your identity, and individual-level data with identifiers will not be published.

The researcher will destroy the information five years after the end of the study.

Your instructors will not have access to the questionnaire, and they will not know who is participating. Hence, your grades cannot be impacted by your participation in any way. Research supervisor Dr. Hardy and the researcher are the only ones who have access to the Moodle site, and the researcher will be the only one who will be managing the data provided by your answers to the questionnaire.

F. CONDITIONS OF PARTICIPATION

You do not have to participate in this research. It is purely your decision. If you do participate, you can stop at any time. You can also ask that the information you provided not be used, and your choice will be respected. If you decide that you don't want us to use your information, you must tell the researcher before January 15th 2021. Contact the researcher via email if you choose that you do not want your information to be used, or if you want to request your information to be destroyed.

There are no negative consequences for not participating, stopping in the middle, or asking the researcher not to use your answers to the questionnaire.

G. PARTICIPANT'S DECLARATION

The text above and the consent below will appear as you enter the questionnaire. You will not be able to continue with the questionnaire until you check the "I agree" box which indicates your consent to participate.

I have read and understood the conditions of my participation in this research study. I have had the chance to ask questions and any questions I had have been answered. I agree to participate in this research under the conditions described.

I agree [CHECK BOX]

If you don't agree, exit the questionnaire and let the researcher know via email that you will not participate.

If you have questions about the scientific or scholarly aspects of this research, please contact the researcher. Their contact information is on page 1. You may also contact their faculty supervisors.

If you have concerns about ethical issues in this research, please contact the Manager, Research Ethics, Concordia University, 514.848.2424 ex. 7481 or oor.ethics@concordia.ca.

A.2 Questionnaire

1- Consent form (see appendix A.1)

2- This question is to inform me about what you have been taught in your past calculus courses. Answer "yes" if you have learned about the topics, answer "no" if you haven't.

f. Limits of functions
g. Continuity of functions
h. Limits of sequences
i. Convergence/Divergence of sequences
j. Convergence/Divergence of infinite series.

3- Consider the statement : $\lim_{x \to \infty} f(x) = L, then\ f(x) \neq L\ for\ every\ x$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If possible, give one or more examples to explain your choice, your thinking and/or why you are not sure if this is true or false

d) Would your answer change if $\lim_{x \to c} f(x) = L$? Please explain your answer with as much detail as you can

4- Consider the sequence $a_n = (-1)^n$

Then, $\lim_{n \to \infty} a_n = 1$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

5- Consider the sequence $a_n = (-1)^n$

Then, $\lim_{n \to \infty} a_n = -1$

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

6- Consider the sequence $a_n = (-1)^n$

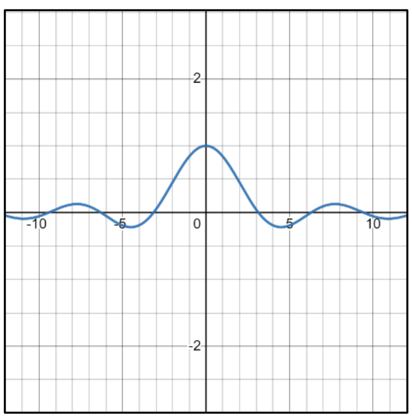
Then, the sequence $\{a_n\}$ diverges.

a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

7- Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, f(x) converges to 0 as $x \to \infty$.



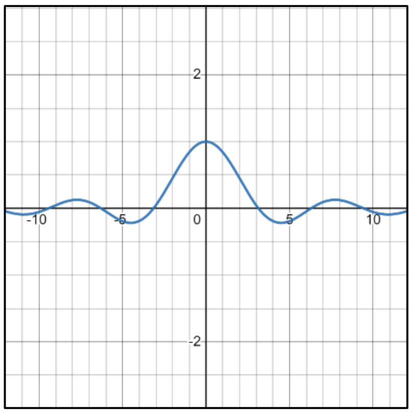
a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain what is confusing you in this question.

8- Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, the limit of f(x) is 0 as x→∞.



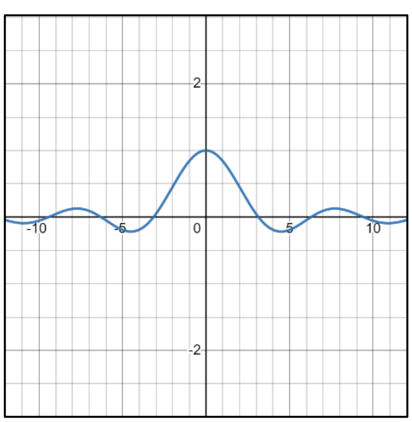
a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain why the question is confusing to you.

9- Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, f(x) tends to 0 as x→∞.



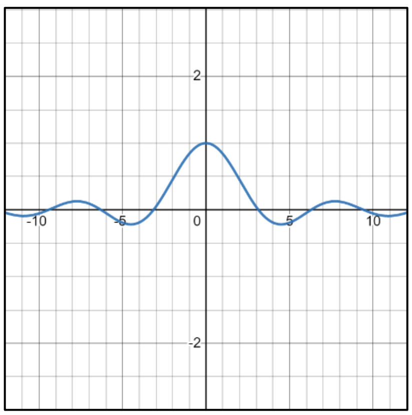
a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know", please explain what is confusing you in this question.

10- Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, f(x) approaches 0 as $x \to \infty$.



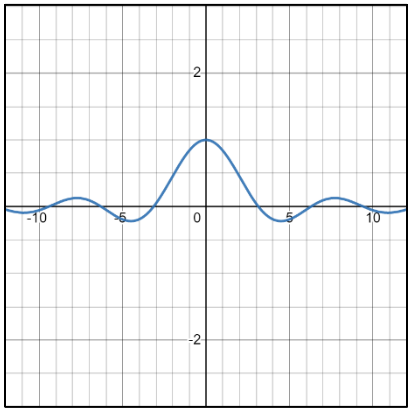
a) Is the statement true or false (write "I don't know" if you are not sure)

b) Clearly explain your choice, and how you are thinking about this (your thought process)

c) If you answered "I don't know" to part a), please explain what is confusing you in this question

11- Consider $f(x) = \frac{\sin(x)}{x}$ as x goes to infinity. You may refer to the graph below if needed.

Then, $y = 0$ is an asymptote.



a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

c) If you answered "I don't know" to part a), please explain what is confusing you in this question.

12- Consider the following expression: $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$
Then, the sequence converges to $0.\dot{9}$

Note: the $0.\dot{9}$ notation refers to a 0 and infinitely many 9's after the decimal point.

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

Note: For students 1-5, this question was miswritten. The version those students received read : "$a_n = \sum_{k=1}^{n}(\frac{9}{10})^k$"

13- Consider the following expression: $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$
Then, the sequence converges to 1.

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

Note: For students 1-5, this question was miswritten. The version those students received read : "$a_n = \sum_{k=1}^{n}(\frac{9}{10})^k$"

14- Consider the following expression: $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

a) Is the statement true or false (write "I don't know" if you are not sure).

b) Clearly explain your choice, and how you are thinking about this (your thought process).

15- Consider the following conversation between two fictitious students, nicknamed A and B. The two students were asked to find the limit of the sequence $a_n = \sum_{k=1}^{n} 9(\frac{1}{10})^k$

A: Alright, I think we should start by computing the first few terms of the sequence to see better what the pattern is.

B: Good idea!

The students notice that the sequence is as follows: {0.9, 0.99, 0.999, 0.9999, etc.}

A: Just by looking at the pattern, it feels obvious that the limit is $0.\dot{9}$. I don't really know how to prove it though.

B: Actually, I think the limit is 1. When you look at the terms, they get closer and closer to 1 without ever reaching it, whereas at infinity, the sequence would reach $0.\dot{9}$ and the limit is supposed to never be reached.

A: But you can't *reach* infinity, you can't find the value of the "infinity-th" term.

B: I think we can reach infinity, look at it this way: $\sum_{k=1}^{\infty} 9(\frac{1}{10})^k$ This is just an infinite series; we just need to figure out what it converges to using the usual tests.

****

In the questions below, please, write as much as you can to clearly explain what you are thinking.

a) What do you think about A's statement: "Just by looking at the pattern, it feels obvious that the limit is $0.\dot{9}$"?

b) In your opinion, what does B mean when they say: "at infinity, the sequence would reach $0.\dot{9}$"?

c) What is $\lim_{n\to\infty} a_n$? Clearly explain and justify your answer.

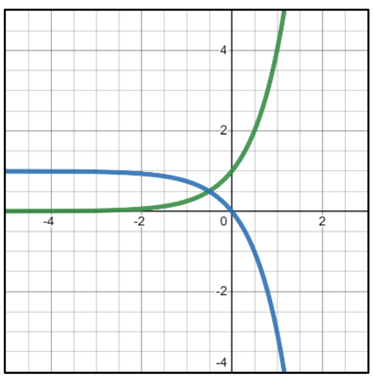
d) If you were to help A and B to solve the problem, what would you tell each of them regarding their reasoning?

Note: For students 1-5, this question was miswritten. The version those students received read : "$a_n = \sum_{k=1}^{n} (\frac{9}{10})^k$"

16- Consider the following conversation between two fictitious students, nicknamed C and D. You may use the graph below if needed.

The two students were asked to find $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$

For $f(x) = 4^x$ (green curve) and $g(x) = 1 - 2^{2x}$ (blue curve)



C: We should compute the two limits individually first, to see if they converge or not.

D: Good idea. Alright, clearly $\lim_{x\to\infty} 4^x$ must be infinity, right? If x gets bigger and bigger, than so does 4^x.

C: For sure, and I think it's similar for $\lim_{x\to\infty} 1 - 2^{2x}$. As x gets bigger, -2^{2x} gets smaller and completely dominates the 1, so this would be negative infinity. But what's the sum of the two, then? Would it be zero since we add infinity to negative infinity?

D: No, we can't do that. Infinity isn't like any number; we can't do algebra with it.

C: But look at the graphs, when we consider any x, f(x) and g(x) always sum to 1, it's like they cancel out. So, the sum of the limits must be 1 too, doesn't it?

D: I'm not sure. Maybe if we manipulate the functions a little bit, we could find something. Right, so $g(x) = 1 - 2^{2x}$, but $2^{2x} = 4^x$. So, $g(x) = 1 - 4^x$

C: So then, the function f(x) and the -4^x part of g(x) would cancel out, and we're left with just 1.

****

a) Do you agree with D when they say: "Infinity isn't like any number; we can't do algebra with it."?

b) What is $\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$

c) If you were to help C and D to solve the problem, what would you tell each of them regarding their reasoning?

17- A student says that it is always true that $\lim_{x\to c} f(x) = f(c)$.

a) Is the student right? Clearly explain and justify your answer.

If you answer no to part a), continue with questions b) and c):

b) give one or two examples when the statement is not true. Can you give examples where the statement is true?

c) What conditions are necessary for the statement $\lim_{x\to c} f(x) = f(c)$ to be correct?

18- Group the following functions according to any criteria of your choosing. Choose criteria that reflect your understanding of limits at infinity

g) $f(x) = \frac{1}{x}$
h) $f(x) = 4$
i) $f(x) = \frac{\cos(x)}{x}$
j) $f(x) = -(e^x)$
k) $f(x) = \sin(x)$
l) $f(x) = \ln(x)$

19- a) Clearly explain your reasoning for why you put specific functions in specific groups.

b) What conditions must functions satisfy to be put in each group?

c) Although the question didn't give you the option to put the functions in multiple groups, are there functions that can belong in multiple groups?

20- When is the following equality correct?

$$\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

21- Consider the following definition.

$$\lim_{x \to \infty} f(x) = L \; if \; \forall \, \varepsilon > 0, \exists \, N > 0 \; such \; that \; \forall \, x > N, |f(x) - L| < \varepsilon$$

a) Re-write this statement in common english, and briefly explain its meaning.

b) Do you recognize what this statement defines?

22- Consider the following two statements.

1. $\forall \, a > 0, \exists \, b > 0 \; such \; that \; a > b$

2. $\exists b > 0, such \; that \; \forall \, a > 0, a > b$

a) Assuming that a and b are taken in the set of real numbers, are the two statements true or false?

b) Are the two statements equivalent? Give an example or a counterexample

23- Consider the following statement:

$sup(A) = 1$ where $A \subset R$

Recall the following two definitions:

- An upper bound b of A is called a *supremum* of A if, for all upper bounds z of A, $b \leq z$
- $c \in A$ is called the *maximum* if $\forall \, a \in A$, $c \geq a$

a) Explain in your own words what the statement means.

b) Can you give an example of a set A which satisfies this statement.

c) Consider the example you gave in b), is 1 the maximum of A? Clearly explain your reasoning.

24- Recall the following definitions:

- An upper bound b of A is called a *supremum* of A if, for all upper bounds z of A, $b \leq z$
- $c \in A$ is called the *maximum* if $\forall\ a \in A, c \geq a$

True or false? A is any non-empty subset of the real numbers.

a) If s is the supremum of A, then s is also the maximum.

b) If s is the maximum of A, then s is also the supremum,

c) It is possible for A to have a supremum and to NOT have a maximum.

d) It is possible for A to have a maximum and to NOT have a supremum.

25- Consider any sequence which is bounded.

a) Does the sequence have a maximum?

b) Does the sequence have a supremum?

c) What can you infer about the limits of the subsequences of this sequence?

d) Is the sequence convergent?

26- Consider the set $A = \{\ln(n) : n = 1, 2, 3, \ldots\}$

Recall the following definition:

- An upper bound b of A is called a *supremum* of A if, for all upper bounds z of A, $b \leq z$

True or false?

a) $sup(A) = \infty$

b) This set does not have a supremum, nor an infimum.

c) The sequence $a_n = \ln(n)$ is monotone.

A.3 Student answers

Note: If a student's answer includes text of the form "FileXXX," this indicates that the student included an image to support their answer. You may find the corresponding files in appendix A.4. Similarly, if an answer is exclusively the text "Blank," this indicates that the question was left unanswered.

	Student 1*	Student 2*	Student 3*	Student 4*	Student 5*	Student 6	Student 7	Student 8	Student 9	Student 10
Q2	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **Yes** Continuity of functions **yes** Limits of sequences **No** Convergence/ Divergence of sequences **No** Convergence/ Divergence of infinite series. **No**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **No**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **no**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**	Limits of functions **yes** Continuity of functions **yes** Limits of sequences **yes** Convergence/ Divergence of sequences **yes** Convergence/ Divergence of infinite series. **yes**
Q 3	a) True b) Here, we are observing the function f(x) for x goes to infinity. The limit indicates that when x goes to infinity, f(x) goes to L. However, that does not imply that f(x)=L, but rather that f(x) approaches L when observed to infinity. c) For example, if f(x)=1/x, then the limit of f(x)	a) False. b) I thought of the constant function as a counter example right away so there is at least one function where f(x) can equal its limit. c) If f(x) = 5 (i.e. a constant function) then lim f(x) = 5 and f(x) = 5 for every x. So the statement should be false.	a) True b) For example, let f(x)=1/x. Therefore, lim f(x) = 0, but 1/x cannot be zero for every x d) No. For example let f(x)=1/x and c=0, therefore, lim f(x) = infinity, but 1/x can be inifinity	a) False b) I'm thinking about the fact that when x gets closer to infinity, then f(x) gets closer to the limit. We extrapolate to think that x is going to reach that limit at some point. Also, when learning about limits of sequences, we know that a monotone convergent sequence will converge to its	a) It is not true. b) I first thought about it with examples. That may be true for a function who has an assymptote equal to L. But it is not for a constant function. c) A constant function (f(x) = L). Also, (1/x)sinx (L = 0). d) No, the constant	a) true (unless f(x) is a constant function) b) The limit of f(x) as x goes to infinite means that as x approaches infinite, the function tends towards L. It does not necessarily reach L. But, in any case, for cases which aren't constant functions, the function will vary as x	a) False b) & c) We can easily refute the statement with a simple counter-example, take a function F: R to R where f(x) = L for all x element of R, then f(x) = L for every x and the limit of f(x) as x approaches infinity is also L. Furthermore, by keeping the definition of the limit in	a) False b) I'm not sure, but I would think it depends on the function. It would seem that the function can equal to its limit for some value x. I think it's true that not all x will result in f(x) = L, but I think it is possible that f(x) can be the same as its limit, as x approaches infinity.	a) False b) In my mind, as x goes to infinity, the function goes to L. But, it doesn't mean that for every x the function gives back L. Unless the function is a constant one. c) A positive linear function explains it. At f(x) =o but as x goes to infinity, it goes to L	a) True b) For the limit of a function to converge to L when x-> inf, it means that as x grows, the limit converges to L. However, the limit at a certain point might be something. c) As an example, I'm think of :

	where x goes to infinity, the limit approaches 0 (L), but never reaches it, therefore, f(x) does not equal L for any x. d)Yes, in the case this limit could be f(x)=L. Here we consider the limit of f(x) where x goes to c, a value. This means that we consider the limit around a point, so if the function is continuous, f(x)=L. For example, if f(x)=x and c=0, then the limit of f(x) as x goes to 0 is equal to 0 (L), which is equal to the value of f(x), so f(x)=L. However, if the function is not continuous, say f(x)= {x for]-inf,0[U]0,inf[5 for {0} Then the limit of f(x) as x	d) A constant function (f(x) = L) has the same value of L for every x so the statement can't be true		supremum and that the supremum can be inside the set (if the set is closed) therefore I think f(x) can be the limit. c) Let f(x)=c where c is constant, then Lim(f(x)) as x tends to infinity is c. So f(x)=L. d) My answer would not change because of the idea that monotone convergent sequences will converge to the supremum of the set they are in.	function example is still valid. For continuous functions, we also have that the limit when x goes to c is f(c).	increases and therefore, the function cannot be equal to L on its whole domain without being a constant function. c) For example, the limit of f(x)=x as x goes to infinite is infinite, but, if we take for example x=3, than, f(x)=3 and not equal to infinite, therefore f(x) is not equal to L on every x. Furthermore, if the limit is infinite, then f(x) keeps going towards infinite but never actually reaches infinite as it isn't a finite number. But, on the other hand, if we look at a constant function, for example f(x)=3, then limit as f(x) goes to infinite	mind, it is easy to see why the proposed statement is not generally true. d) No, a constant function would still be a great counter example	c) For example, a function that oscillates, but decreases as x goes to infinity can have a limit of 0, and actually have f(x)=0 at some point for some x. In that case, that statement would be false. d) If instead of approaching infinity, it approaches a value c, then I think that the statement would still be false. Because when it approaches to a specific value of a function, then as long as there is no discontinuity, then it should be able to reach the limit for some x.	d) no, because the same logic as b)	Limit of /(n+1) as n -> inf. As n goes to infinity, the limit converges to 1. However, as n goes to c element of N, it will converge to 1/(1+1/c), which could be 2/3 if c=2. d) As answered above, while c varies, the limit is dependent of the value that c will take.

	goes to zero would be 0, and so f(x) does not equal L (f(x)=5 does not equal L=0).					is 3 and f(x) = 3 on every single x, therefore this is a special case that shows that the statement is false for constant function. Yet, the statement remains true for other cases even in the function does not tend towards infinite but rather tends towards a finite number, since by the fact that they aren't constant function they must vary, and by such, they cannot be equal to the limit at every x. (I've been trying to find an example for a function with a finite number as a limit and I know there are common ones, but I can't think of one at the moment.) d) My answer would not				

						change. Again, it would be true for the same reasons as a function varies therefore it cannot be equal to its limit at every x (again, except for constant functions).					
Q 4	a)False b)This limit is divergent. If we expand the series starting from 1, we get an= {-1,1,-1,1,-1,1,-1,1,-1,1,-1, ...}. Therefore, we cannot define the limit, as we keep going from -1 to 1 and vice versa. I cannot recall which property this follows however, simply that the series never reaches or approaches more a number than another, so it is undefined	a) False. b) {An} = (-1,1,-1,1,-1,1,...) Graphically, this sequence is jumping between -1 and 1 so intuitively, the sequence cannot converge to 1. More analytically, the sequence can be decomposed into 2 constant subsequences . Where the odd valued subsequence converges to -1 and the even valued subsequence converges to	a)false. The sequence is divergence. n) (an) = {1,-1,1,-1,1,-1,...}	a) False b) The sequence can take 2 distinct constant values, 1 and -1. Therefore I can create two constant subsequences where an = 1 when n is even an = -1 when n is odd Knowing that for a sequence to converge to a single limit (accumulation point) we need all the subsequences to converge to that same limit, I think that it is wrong	a) It is not true. b) We can break the sequence into two subsequences . The odd numbers and the even numbers. One subsequence converges to -1. The other one converges to 1. Since there are two subsequential limits, the sequence can not converge	a) false b) As can be seen from the uploaded pdf, the sequence (drawing 1) oscillates between -1 and 1, which I believe was called an oscillating limit (but very very unsure). Since it isn't oscillating and decreasing its range of oscillation at the same time like in drawing 2, there isn't a 1 limit that it reaches. By comparison, in drawing 1 (which would be the limit involved in the	a) False b) The series is divergent, the limit does not exist. Although, you could have a subsequence with limit 1, as well as a subsequence with limit -1	a) False b) I believe that is false, because n can be an even or odd number. An even number yields to a value of 1, and an odd number yields a value of -1. The sequence will then be composed of values that alternate between 1 and -1, as n goes to infinity. There is no defined limit.	a) False b) the sequence is : -1, 1, -1, 1, hence the limit doesn't exist because it's not convergent to 1	a) False b) The limit is divergent since a_n is the sequence (-1,1,-1,1,-1,1...) which means there are two subsequences : 1) (-1,-1,-1,-1,...) 2) (1,1,1,1,1,1...) Hence the sequence has two convergent subsequences

		1. For a sequence to converge, all of its subsequences must converge to the same limit as the sequence. Therefore, An cannot converge since you can find subsequences of An that have different limits		to say that the limit of this sequence can be stated like this		true or false), the sequence will always oscillate between -1 and 1 without having that "distance between -1 and 1" decrease, versus drawing 2, where that "distance" decreases and where a limit would be able to be found. Therefore, the limit for this particular sequence (an=(-1)^n) would not have a limit (DNE). See FileS6-1					, making it divergent.
Q 5	a)False b) As previously mentioned, when expanding the an series, we notice that it is a constant shift between -1 and 1. Therefore, we cannot determine on which number	a) False b) By same reasoning as Q4. (For a limit to converge, all of its subsequences must also converge to the same limit as the sequence itself.)	a) false. The sequence is a divergence b) (an) = {1,-1,1,-1,1,-1,...}	a) False b) Again, this sequence has two constant subsequences that converge to different points whether n is odd or even. We cannot say that the limit of this sequence is -1 since when n is	a) False b) Again, we can break the sequence into two subsequences (odd and even). This gives us two different subsequential limits. Therefore, it	a) false b) See reasoning for question 4. This is the same sequence as question 4 and, following the same reasoning, the sequence does not have a limit	a) False b) For the same reason as the previous question, although to be more precise, there is exist no 'n0' element of the naturals where for all Epision>0, |a n-L|<Episilon	a) False b) As the question before, because n can be odd or even, making the sequence alternate between 1 and -1, the limit is not defined, it's not 1 or -1. However, I think that	a) False b) same logic, the sequence is as follow : -1,1,-1,1 .. hence it doesn't converge to -1. The limit DNE	a)False b) As explained on question 4, the sequence has two convergent subsequences , hence it diverges. The limit might as well be equal to 1 or -

	the limits approaches as it approaches both equally. It is undefined			even, the subsequence converges to 1. (Same thought process as before).	cannot converge. In simpler words, we can see that as n grows, the sequence still alternates between -1 and 1. It does not depend on the size of the number but the category (odd or even, infinity is neither).		when n>n0 That is, for L=1 or L=-1, no matter how large n0 may be, Episilon would have to be greater than 2 for the inequality to always hold...	would be true if we restricted the n to only odd values.		1, hence proving its divergence.
Q 6	a) True b)Since an goes constantly from -1 to 1, we cannot determine on which number the series approaches, as it goes equally to both. Since we cannot set a number to which the series approaches, then it diverges	a) True b) {An} only takes on values of -1 or 1. By same logic as before, you can find 2 subsequences with different limits so the sequence itself doesn't converge and so it diverges. I suppose if one is not convinced by that argument, you could argue that graphically, the sequence does not	a) True. b) (an)={1,-1,1,-1,1,-1}	a) True b) The definition of a divergent sequence from this call is of a sequence that does not converge to a unique limit. Therefore this sequence diverges. (However, the word divergence does intuitively give the idea that thigs need to get "further" from each other, like the function xsin(x) which	a) That is true. b) There are two subsequential limits. Therefore it must diverge	a) true b) See reasoning for question 4 and 5, since the limit does not exist, it must diverge.	a) True b) The subsequence a2n has limit 1 and the subsequence a(2n+1) has limit -1, hence the sequence is not convergent and thus must be divergent.	a) True b) The sequence never reaches a limit, as it alternates between the values 1 and -1. As n goes to infinity, it will never converge to the same point.	a) True b) it diverges because the sequence is as follow : -1,1,-1,1, as n goes to infinity. Hence, it doesn't converge to a specific number, but 2. Hence, it diverges	a) True b) Since a_n = (-1,1,-1,1,-1,1,...), then it has two convergent subsequences 1) (-1,-1,-1,-1,-1,...) 2) (1,1,1,1,1,...) Hence diverging.

<table>
<tr><td></td><td></td><td>approach a particular value since it just jumps between 1 and -1</td><td></td><td>seems to be expanding. But that's just a thought.)</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Q 7</td><td>a) I am more inclined to say True, although I am not 100 percent sure of this answer.
b) I would say the limit goes to zero, since the limit of 1/x as x goes to infinity is zero, and since sin(x) is divided by x, then f(x) would also approach zero. The confusing part about this function is the top part, as sin(x) on its own is a cyclic function, therefore would be divergent. But since it is divided by x that goes to zero, then the entire function would go to zero. When thinking about it for longer, it makes sense</td><td>a) True
b) Let g(x) = 1/x. I know that lim g(x) as x approaches infinity is equal to 0. (Dividing 1 by an infinitely large number is approximately 0). The range of f is contained in the interval (-1,1) so as x approaches infinity, the f's denominator will "overpower" its numerator resulting in the limit being 0</td><td>a) True.
b) According to graph, when x increase or decrease infinitely, f(x) tends to be zero</td><td>a) True
b) I can see that the "waves" get closer and closer to the x axis as x gets further from 0. This means that I can take any subsequence of this function (pi/2, 5pi/2, 9pi/2...) and it will be converging to 0.</td><td>a) It is true.
b) We see two main functions in the function that are multiplied. Sinx is always between -1 and 1. And 1/x obviously grows smaller as x increases. If we multiply something that gets smaller and smaller by something that almost stays constant (between -1 and 1), then it gets smaller and smaller.
Also, we could use the Squeeze theorem to show that (but the idea would be similar to the explanation)</td><td>a) true
b) This is a great example of the kind of sequence I was trying to explain in my drawing in number 4's pdf. Obviously, this is a function and not a sequence, but the same idea applies. Since the function is oscillating around 0, but that the distance between each "wave peak" is decreasing as x increases after x=0 (and symmetrically on the negative side as well). As x goes to infinite, the function tends towards zero since the oscillations become closer and closer to 0</td><td>a) True
b) -1/x < sin(x)/x < 1/x, note -1/x & 1/x goes to 0 as x approaches infinity, then by squeeze them. sin(x)/x also goes to 0 as x approaches infinity.</td><td>a) True
b) We can see from the graph that as x goes to infinity, the oscillation of the function becomes smaller and smaller, and f(x) is going towards the x-axis, where f(x)=0. Therefore, the limit of this function is 0, as x approaches infinity.</td><td>a) True
b) By squeezing theorem -1 < sin(x)< 1 then -1/x < sinx/x< 1/x as x goes to infinity, it gives zero . Hence, the function goes to zero</td><td>a)True
b) as sin(x) grows to infinity, the function sin(x) is bounded by (-1,1). Given epsilon > 0, we know that |sin(x)-0| < epsilon. Since sin(x) is bounded by -1 or 1, we can establish that
|sin(x)-0|/|x| = |1-0|/|x|=|-1-0|/|x| =1/x<epsilon
Establishing the function sin(x)/x < 1/x < 1/x, the we know that 1/x converges to 0 has n goes to infinity, which means that sin(x)/x also goes to infinity.</td></tr>
</table>

	that it approaches 0 when x goes to infinity, since the denominator gets bigger and the nominator varies constantly between the same values, so f(x) would become smaller and smaller as x goes to infinity and would tend to 0					on both sides (squeezing it symmetrically from the positive and negative y directions).				
Q 8	a) False b) We need to specify which limit here we are talking about. If we are talking about the limit of x going to -infinity or infinity, then yes the limit of f(x) is equal to 0 (as explained on the previous page, we have to consider the 1/x part that goes to zero as x goes to -infinity or infinity).	a) True b) I'm not sure if I misread the last question, but I thought it was asking the same as this question. I was under the impression "f converges to 0" and the "limit of f is 0" were synonymous in this context	a) True. b) 0 is accumulation point	a) I don't know b) The statement does not clearly state that we are talking about the limit to infinity. For example, as x goes to 0, there seems to be a limit which is equal to 1. c) I would think that there are more than one limit depending on the point we are approaching.	a) This is true if we consider the limit as x goes to infinity. b) There is no difference with number 7, if we are talking about the limit as x goes to infinity. Saying that f(x) converges to 0 as x goes to infinity or that the limit of f(x) as x goes to infinity is 0 is the same thing to the best of my knowledge. Is is not that the limit is the notation to	See FileS6-2	a) True b) The wording of the question may have changed slightly, but appears to essentially state the same thing as before, so my reasoning is unchanged.	a) True b) From the graph, we can see that the oscillation becomes smaller and smaller, as x goes to infinity. It goes towards the x-axis, where f(x)=0. Therefore, the limit of this function is 0.	a) True b) Squeeze theorem as explained in 7	a) True b)We can say that given epsilon > 0, $\|sin(x)-0\|/\|x\| <$ 1/x where 1/x converges to 0. Hence, the limit of f(x) goes to 0.

	However, if were talking about a specific value, so x goes to c, then the limit would be a number. For example, for x goes to 0, the limit of f(x)=1.				refer to the convergence? However, if it is not what we meant, then saying "the limit" does not necessarily mean we are considering the case when x goes to infinity.					
Q 9	a)True b)(***I may have read wrong previous question 7, as I answered exactly what I am about to answer for this question, it might have been written - infinity on question 7 and I misread, I am sorry. It is the same answer though, if x goes to minus infinity or to infinity, the limit of f(x) goes to zero, and it is the same logic (following the limit of 1/x))	a) True b) My logic is that as x goes to infinity, "f tends to 0" if and only if "the limit of f is 0" if and only if "f converges to 0." So again, I was under the impression the terms are synonymous in this context	a) True. b) The graph shows as x goes to infinity, f(x) tends to be zero	a) False b) Again, I would say that f(x) tends to 0 as x tends to infinity. But there are also other limits like the limit as x tends to 0 or as x tends to any other point. Because this is a continuous well defined function, it might seem trivial to talk about these limits but I believe they still exist.	a) This is true if we mean that it tends to 0 as x goes to infinity. b) These were used alternatively in my math classes if I recall well. "Convergence" , "tend to", "goes to" means the same thing to me.	a) true b) This is the same question as 7 and 8, therefore I have the same reasoning. Hopefully I am not missing a different word on meaning, if so, I'm sorry!	a) True b) Again the wording seems to have changed but appears to state the same as before. My reasoning is left unchanged.	a) I don't know b) I believe that its limit is 0 for the reasons that I've said previously. I'm guessing by "tends to", we are referring that the function is going towards a specific value, no matter the value of x that is in question, then I think that yes, it tends to 0. c) I'm not sure if "tends to" is the same as saying the limit.	a) True b)same thing as previous questions / squeeze theorem ..	a) True b) Given epsilon > 0, We can establish that |sin(x)-0|/|x|<1/x and 1/x converges to 0. Hence the limit converges.

	The top part of the function is cyclic, so that part only would give a divergent limit. But the denominator, as x goes to infinity, 1/x goes to 0. Therefore, when put together, f(x) goes to zero as x goes to infinity									
Q 10	a)True b)This is the exact same question as 9, I will briefly re-explain: Since sin(x) is cyclic, it is a divergent limit, but 1/x goes to zero when x goes to infinity, so when put together, f(x) goes to zero when x goes to infinity	a) True b) same reasoning as previous 3 questions. Synonymous terms	a) True. b) The graph shows that f(x) tends to be zero, while x approach infinity.	a) False b) Same thought process, f(x) approaches 0 as x approaches infinity	a) I am not sure about this one, but would most likely say it is true. b) Approaches means that it gets closer to. Thus it must mean the same thing as converging to or going to. c) It has rarely been used in my math classes to talk about the converge of f(x) (I do not remember hearing it in this context). However, I recall it being used to say	a) True b) Just realized that the last question asked if it tends towards and now if it approaches 0. I still think that they are both true. And that the limit also is equal to 0. (all questions still true) and again with the same oscillating reasoning. But, as for if these questions are the same, Since the limit is equal to 0, then because	a) True b) The wording is changed but appears to state the same as before. My reasoning is unchanged.	a) True b) The function does seem to approach 0 as x goes to infinity as the f(x) keeps decreasing in its oscillation as x increases. It does not seem that it's going increase again, therefore it approaches 0.	Blank	a) True b)As the function sin(x) goes to infinity, it is bounded by -1 and 1. Hence, the output value will always be in the range [-1,1]. Therefore, given epsilon >0, we can say |sin(x)-0|/|x| = |1-0|/|x|= |-1-0|/|x|= 1/x < epsilon

					"as x goes to c" --> "as x approaches c".	the definition of limit is more strict than simply saying it tends towards or approaches, we can also say that f(x) tends towards 0 or approaches 0 as x goes to infinite. The limit equality is the stricter definition of the 3 although we often speak of limits as tending towards and approaching. Therefore if the limit as x goes to infinite is EQUAL to 0, then we can say that it approaches or tends towards 0. But, someone is telling me that function f(z) approaches to 0 or tends towards 0 as x goes to infinite, I wouldn't say that the limit is equal to 0, without properly				Since it is know that 1/x converges to 0, we can establish that the limit of f(x) as the limit goes to infinity will also converge to 0.

						calculating the limit using calculus and theorems. Therefore, if I can re-explain the question where we are asked the limit with the equal sign, I would also add that using calculus and theorems, the limit of f(x) as x goes to infinite is = to 0. Just by looking at the graph, we say that it approaches or tends towards 0, but to actually have the limit equal 0, we need to calculate it or use theorems to prove it.				
Q 11	a) False b) I am not sure of the exact definition of an asymptote, but to my knowledge, its a linear function that f(x) would	a) True b) f(x) oscillates around the x-axis. Every time f(x) reaches a local min it increases until it reaches the next local max and then	a) True. c) The graph seems asymptote at y=0	a) False b) If I remember well from Calculus, an asymptote is not reached but is approached as x tends to a certain value (not necessarily	a) I do not know. I would rather say it is true than not. b) I have considered asymptote in the context that the function was bounded by that	a) I don't know b) I don't quite know the definition of an asymptote by heart, so I don't know if it qualifies as an asymptote or only acts as one. From what I can	a) True b) As x approaches infinity, the function will go to zero	a) False b) Because from what I remember, an asymptote is a line that the function approaches towards but it will never be able to reach it. In this case,	Blank	a) False b) Since sin(x) is bounded by -1 and 1, it has both a negative and positive ouput when taking the function alone (without

	approach but never reach (so f(x) would never equal y). A good example for an asymptote if for f(x)=1/x as x goes to infinity, then y=0 is an asymptote as 1/x approaches 0 when going to infinity, but never reaches zero. However, in our case, y=0 is not an asymptote as f(x) crosses y=0 multiple times	decreases until it reaches the next local min. The absolute value of each local min/max is always smaller than the previous one. So as x approaches infinity the local min/max will become increasingly smaller, becoming approximately zero. Thus y=0 serves as an asymptote since f(x) approaches it as x approaches infinity. *I forget the exact definition of asymptote.		infinity). For example for the function f(x)=-x/(x-1), there is an asymptote at -1 when x tends to infinity or - infinity	asymptote. For example, 1/x clearly has an asymptote at x = 0 and y = 0. c) I am confused by the fact that this particular function goes over and below the "asymptote" (if it is one). I do not recall the exact definition of an asymptote, since it goes back to the beginning of CEGEP. The "definition" I have in mind is more of a picture that is letting me down here	recall, an asymptote is an "invisible" line which a function approaches but never really touches. Thus, by this definition, y=0 would not be an asymptote, because the function crosses y=0 constantly. Yet, if the definition is only that it approaches this line, then, it would be called an asymptote because the function does end up approaching the line y=0. c) I answered I don't know because I can't recall the exact definition of an asymptote, but if I were to go with my reasoning and what I think I recall, I think this would not be considered a "normal"		this function does approach towards 0. However, I think at some point x, it will also reach the value f(x)=0. Because it can be reached, then y=0 is not considered to be an asymptote.		the denominator of x). Hence, as seen on the graph, it goes down the x-axis and above the x-axis, while the function should never touch an symptote. Hence, we can say that it is not an asymptote.

						asymptote, but I recall hearing about it in math 364 and maybe we called it a special kind of asymptote or maybe we didn't (I can't recall correctly). Yet, by the definition that the function does not touch an asymptote, obviously this couldn't be considered an asymptote				
Q 12	a)False b)I first expanded the series to obtain a simplification. From that simplification, I did the limit as n goes to infinity, which gave me 9. So the sequence converges to 9. (Please refer to the calculations I have put in attachment FileS1	a) I don't know b) Just a disclaimer, I learned convergence/divergence of series in cal 2 in cegep but I don't quite remember anything of it and I haven't taken MATH 365 yet so I'm not too familiar with the topic. However, I am comfortable with computing sums, so	a) false b)a1=0.9, a2=1.71, a3=2.439... not even close to 0.9999999	a) False b) an = 9/10 = 0.9 when n=1 an = 9/10 + 81/100 = 1.71 when n = 2 therefore it cannot converge to 0.9 dot	a) It is false. b) This is a geometric series. From the formula to solve geometric series, a_n = (1-(9/10)^n)/(1-(9/10)) = 10(1-(9/10)^n). As n goes to infinity, we multiply (9/10) infinitely many times, this part goes to zero. We are left with 1 that goes to 0.	a) True b) See pdf FileS6-3, but it seems like the more we add on, we are always ending up adding one extra 9 at the end of the decimal series. Which means it would converges to that 0.9dot notation. I am unsure about this one, because first of all I don't remember	a) False b) The geometric sum, (1/10)^k from 1 to n, converges to 1/(1-1/10)=1.11111111... and 9 times that is 10	a) I don't know b) I feel like it would make more sense to say that it converges to 1 instead of saying it converges to 0.9 (with infinitely many 9s), although I can see where this statement would come from. When we say that it converges to a certain number, from what I	a) I don't know b) if n is finite number, then yes it would converge but I am not sure	a) True b) By the properties of a summation, we can write : ∑ 9(1/10)^k = 9 *∑ (1/10)^k If we expand the summation, we can see the terms are 1/10 + 1/10^2 + 1/10^3 +

		based on my knowledge of computing sums, my answer to a) would be **False** because if n=infinty, then the sum is equal to 1/(1-0.9) - 1 = 9. And if one of the values of the sequence at infinity is approximately 9, then it does not seem logical that the sequence converges to a much smaller number of 0.999999.			Thus, we get 10*1 = 10. However, I am not sure if there was a mistake in the question: maybe the intent was to get the sum of 9(1/10)^n. Then, this would create more confusion to me. I believe the answer would be that 0.9999999 (infinitely many 9) = 1 from the geometric series formula, so these could be used alternatively	much about sequence limits, and secondly, I am unfamiliar with the notation, and even if there is a notation to it, the fact that it is converging to a non-rational number means that this number continues infinitely, and therefore it is not a finite number. Thus, can a function or sequence converge to a non-finite number, or does that mean it is diverging? I think it is still converging because it is getting more precise with every additional 9, but it is not 100% clear for me.		understand, it means the value where the sequence goes towards to. In this case, it will never reach 1 because we keep adding 9s in the decimal, but it will approach to 1 as n goes to infinity.		1/10^4 + ... + 1/10^n . which means the summation (without the coefficient of 9) is 0.11111111...111 Thus if we multiply the coefficient of 9 the the output of the summation, the result will be 9 * 0.11111111...111 = 0.9999999...999
Q 13	a)False b)This is the same	a) I don't know b) same answer as	a) false b) same to Q12 answer	a) False	a) Again, this is false. It	a) True b) From this question, I	a) False b) As explained	a) True b) I would say that it does	a) I don't know	a) False

	calculations and train of thoughts as the previous question, since it is the exact same serie. I have attached the calculations, which gives that the series converges to 9, not 1. FileS1	before. I would say false because the sequence evaluated at infinity would have a value that is approximately 9		b) Same argument an = 0.9 when n =1 an = 1.71 when n = 2 an = 2.439 when n = 3 and this is an increasing sequence	converges to 10. b) The intent was probably to have 9(1/10)^k so I will answer that question instead. This would then be true. From the formula of geometric series, having infinitely many 9 or writing 1 would be equivalent (so I believe). Using the epsilon definition of a limit, we could always find a n_0 that would satisfy the epsilon interval (or neighborhood) .	think I will go with my reasoning from the last few sentences of question 12's explanation. Since the 0.9dot notation is an irrational number, the sequence cannot converge to a number that isn't finite. Thus, I think that the question 12, would be false, and this would be true because this sequence does amount to 0.9dot which itself is converging towards 1 (where 1 is a finite number).	before, the sequence converges to 10	converges to 1 based on my previous answer. I think it's more accurate to say that it converges to 1, because as n increases, we will keep adding a 9 after the decimal. We don't know when we will stop. Therefore, even though it does approach 0.9 with infinitely many 9s, the limit of this sequence is 1, therefore it converges to 1.	b) it would give n(n+1)/2, it depends on what n is ?	b) We can develop a_n to be a_n = (0.9 , 0.99, 0.999, ... , 0.9999...99) If we consider the neighborhood of points U = (x-E, x+E) for E>0, the sequence must be in the range of points centered at x (where x would be 1) for every epsilon given. Hence as n grows to infinity, the decimal number 0.9999...99 will also grow closer to 1. Hence I would say that the limit converges to 1 because a_n is in the neighborhood U for every E>0 given.
Q 14	a)False b)First part is the infinite sum of ak. The second part is the allure of the series as it	a) I don't know	a) True. b) if ∑ ak is convergent, then a limit with a constant is	a) True b) Since infinity is not an integer, writing the sum the first way implies	a) This is true. b) The way this was taught to me, anytime we write infinity without the limit, it is	a) True b) I think that this is a similar method as having an improper integral and instead, taking	a) False b) It is possible that the limit simply does not exist	a) True b) The left hand side is adding up the values of a sequence from k=1 to infinity.	a) True b) because it equates	a) True b) on the RHS of the equation, we let the n be a variable.

	goes to infinity, so we consider how it looks before infinity, rather than just plugging in the values to find the value at infinity		still the constant	directly the second notation. (I think.)	solely to shorten it. To make the idea of a sum to infinity, we need to consider the limit. I believe the limit is the "definition" of the sum to infinity.	the limit as t goes to the improper boundary and having a change of variables such that the integral can become proper, and thus easier to calculate. Therefore, I think it is true, but sequences is definitely not my forte. (pdf for a better explanation of the method for improper integrals in case it isn't clearer explained in words) FileS6-4		The right hand side would be finding the limit of the sequence from k=1 to n, where n goes to infinity. The way I see it is that they are the same because both will add up until an infinity value.		Hence the summation depends of the value n. Because of the limit, the variable n is denoted to be infinity, hence making n -> infinity. Therefore, on LHS, we establish that the summation is an infinite summation and on the RHS, we let the summation be finite with a variable n, however this variable tends to infinity, making the summation be an infinite one, which means the equality remains.
Q 15	a) I realize now that I was not looking at the series properly in questions 12 and 13, the ak goes to 1. Since the first terms give 0.9, 0.99, 0.999, ...,	a) That's usually a good starting point when trying to prove something. Perhaps an induction proof could help demonstrate A's suspicion.	Blank	I will answer as though the sequence was "sum of k from 0 to n of 9/(10)^k" a) The pattern does seem obvious that we are adding a 9 to the next	a) Patterns can be deceiving. What feels obvious to me is that 0.9 (with infinite 9) is the same concept as 1. b) He means that as n goes	a) I am sceptical of A's statement because to actually find the limit of a sequence or a function, one should not go with one's feeling, but rather find	a) The conclusion is reasonable for a finite n number of terms, where there would be n trailing nines after zero. b) I believe he means that 'at	a) I think by looking at the pattern is a good start, that's what I do too. But, I think just looking at the pattern isn't sufficient enough to say that the limit is	a) that it's intuitive b) that it's not true because as you increase k, it started at 0,9 but it's increasing towards 1	a) The summation is 0.9 (with the dot) but the limit is not, since we need to take in consideration the neighborhood of points. The sequence will

	there is always another nine, which is why it is obvious to think that the series when to infinity is 0.9 to infinity. b) When you compute the infinite series at infinity, the value you obtain is 0.9 to infinity. c) It is the limit of an as n approaches infinity, which is equal to 1. I do not know how to prove this, I just know that that is the answer. d)I would look at this situation differently than student A. Personally, I like using the example 1/x, since it is pretty simple. 1/x when x goes to infinity, it goes to 0, although it never reaches it. Same thing would apply here, it gets closer and	b) I think B means that if you were to evaluate the sum by adding up every single term until you reach the "infinity'th" term you would get a value of 0.999999999999. c) (see photo FileS2-1) I would say the limit is 1, but I'm not too sure how this would work because I simplified the sum using an approximation so I have doubts on how correct that would be		term of the decimal expansion. b) I think B means that if we add an infinity of 9's to the end of the decimal expansion, that means that at infinity, the limit is 0."infinity 9's" c) I think that the limit is 1 because there is no number between 0.9dot and 1. The way to write 0.9dot as a fraction is actually 1/1. As opposed to the number 0.3 periodic where there could be a 0.333...33334, 0.33333....3335,...,0.33333...33339. d) I would ask them if 0.9dot is a rational number, and if so to write it as a ratio of two integers. If not, I would ask them to prove that 0.9dot is	to infinity, it gets closer to 0.9 (with infinite 9) = 1. I do not think infinity can be reached. Infinity seems like a concept representing numbers growing bigger and bigger. c) I would say it is 1 (and 0.9 with infinite 9). I think both mean the same thing. d) That they are both saying the same thing. One is looking at the pattern to find the limit, the other is using a formula. B has a better reasoning because the formula is based on a proof that this is how we obtain the limit while A goes with his gut feeling of the pattern. I would tell them to use the neighborhood	theorems or use calculus or series definitions to figure out the limit. b) I think he means to say that this function would reach an "finite point" and thus that it can't be a limit. But limits can be reached, they don't have to be an unattainable number. (I believe that this might be what distincts limits to asymptotes: asymptotes can be where a function converges to but cannot reach, whereas a limit can be reached, or cannot be reached but the reachability of a limit is not important, but rather the function's behavior is.	infinity', even if you add an extra 9, would still have an infinite amount of trailing 9s after zero? c) Previously, I neglected that since k starts at 1, the first term is 1/10 rather than 1. so the limit should be calculated as follows (1/(1-(1/10))-1 = 1/9, the geometric sum minus the first missing term, and then 9*(1/9) = 1. d) For all Epsilon greater than 0, |.9999... - 1|<episilon holds for n sufficiently large, hence 0.9999... is 1	0.9 (with infinite many 9s). b) What B means, from what I am getting, is if we keep adding 9s, then at some point, infinitely many 9s will be reached even though it's not necessarily countable. c) I believe that the limit of this sequence is 1. Because, like B said, I think the idea of limit is essentially a value that can't really be reached but it's a value that it gets closer and closer to. So, if we add 9s infinitely, then we can get 0.9 conceptually (with infinite 9s) even though we can't really get that practically. It makes more sense to me to	c) as the sequence tends to infinity, where does it converges to d) I would tell B that what he said at the end is not true because (9/10)^k isn't the same as 9(1/10)^k specially in series. And to A, to think of infinity as a concept and not a rational number	end up being in the neighborhood U = (x-E,x+E) for E>0 for all E where x=1. b) That the infinith term is 0.9 (with the dot). If we keep the sequence, we see that it does {0.9, .99, 0.999, ... } hence reaching the 0.9 (with the dot). c) I think it is 1, since it will be in the neighborhood of points of x=1 for every E>0. d) The output 1 is an asymptote of the sequence. Hence, it converges to it without ever actually being this value,

	closer to 1, although it never reaches it			irrational. From high school, we learned that numbers with periodic decimals are rational, but I would imagine they could find a better reasoning. Also, I don't believe the limit is supposed to never be reached is a valid argument because 0.9dot is 1. Not because we won't reach 1, but because there is nothing between 0.9dot and 1.	definition and see if they can prove each their limit. I think B is wrong when saying you can "reach" infinity (A is right to tell him that does not make sense) and also that you can not reach the limit. It is possible to reach a limit, the important is that the sequence gets closer to the limit as n grows.	c) The limit of the sequence as n goes to infinite is 1. I am sure there is a method or a theorem or a known series for which I should explain this, but I don't remember it. d) I would tell A not to focus on his feeling, but rather to focus on finding a real method of proving his theory. I think that the most important thing for him to realize at the moment is that he needs to view limits and such as things to prove methodically, not just because it feels like something. I would also find a different function where the limit feels like something, but it actually something else and show him how to		say that 1 is its limit. d) They said the doubts that I had in mind too when thinking about a limit. But I guess I would just ask them other questions like what is defined as infinity, is this countably finite, and maybe that will provide them more ways to think about this.		meaning that it is where the sequence converges to. We can see it as an accumulation point (I just learned about this topic, I'm not 100% sure it actually is)

						prove that one so he has a clear example of what I mean. As for B, I would tell her that it is a good idea to focus on methodology to find the actual limit, but I would also tell her to re-read her definition of limit, because a limit can be reached. I would also give her as example, limit as x goes to infinite of a constant function is equal to that constant and the actual finite number of that function is constantly reached.				
Q 16	a)Yes, it is not a number that I can plug in to my calculator and obtain an answer. b) I am not sure of the answer of this	a) yes I agree b) Does not exist c) the question is asking for the sum of 2 limits. Not the limit of the	a) agree	a) I do agree that we can not use infinity the same way as numbers because two functions might reach infinity " at a different rate ".	a) Yes, infinity is a concept. It is not a real number. b) It is 1. From a theorem learned in calculus / analysis, this	a) I agree in the sense that it isn't like any number, but we can use infinity in algebra as an idea of a number. 3+infinite if	a) Infinity isn't a number on its own and there are some operations that are inconclusive when dealing with infinity,	a) Yes I agree. b) I think it's undefined. c) I think that the initial reasoning of C was good. So to check	a) he is kinda right b) it's 1 c) you can add two function	a) Yes, I agree. It is a concept, not a number. b) I would say indeterminate value (inf - inf),

	limit, as my first reflex would have been to calculate the limit as x goes to infinity of (f(x)+g(x)) together, which is what C and D have come to. So with this reasoning I would have to say the limit is 1 as well. c) The issue here is that we are adding infinity and minus infinity. I do not remember what the trick is when there is addition of two infinities, because I know that you cannot cancel them out. So I guess I would stick to my previous answer and say that their reasoning is correct and that the answer is 1	sum of two functions. These two things are not necessarily the same. Lim(f(x)+g(x)) = Lim(4^x + (1-2^(2x))) = Lim(1) = 1 Lim f + Lim g does not equal Lim(f+g) = 1 because f and g both don't converge to an explicit x in R		We can therefore not "cancel out" their limits. b) The limit as x tends to infinity is 1. c) I think that manipulating the functions first is a better idea to get a sense of where they are actually going. Infinity is not a number.	is equivalent to the limit of the addition. Adding them together gives 1. The limit of 1 as x goes to infinity is 1. c) They are wrong at the start. But they end up using a theorem that was proven, which works. The end of their reasonning is good. The first part however, is flawed when they talk about infinity cancelling out, since these are not real numbers.	infinite, 3/infinite is 0, etc. b) It isn't defined since when you subtract infinite from infinite (again "algebra" like in a) you don't get a defined answer because infinite is not a real number therefore we cannot subtract them. If we use l'hopital's rule just to check what would happen, the x as an exponent would not come down, therefore this just isn't defined. c) I would tell them to go through the problem methodically without trying to cancel things out visually (C) and to look at infinite like a number for algebraic	such as infinity - infinity = 0, which is not generally true. So I have to agree with the idea he is pushing forward, but not the wording. b) 1 c) C has to use more rigorous reasoning, although his intuition is on the right track. D can use the fact that the sum of the limits is the limit of the sums and since f(x)+g(x)=1, the limit clearly converges to 1	individually if they converge or not first seems like a good idea. Then, they checked that f(x) and g(x) goes to infinity and minus infinity respectively. which is an indeterminate form. Then, D proceeded with some manipulations, which I think it is also valid because of the limits' properties which allows us to add them together, as both have x approaching infinity. It leaves to 1. But I am little bit confused, because when I look at the graph, it doesn't seem as though the limit is 1.	together, if you find them convergent, then each is also convergent so yes, since f(x) + g(x) = 4^x+1-4^x = 1 then as x goes to infinity it's 1	because I don't think you can actually do algebra between the limits before evaluation them separately. c) Since we are evaluating the limits to tend to infinity, we cannot say that the limits sum up to 1 . Since they both equate inf and -inf, the value is indetermiante.

						purposes without treating it like a number but more of an idea of a number. You can't subtract many from many and get a real idea of how much there is but if you subtract 3 from many, you still have many because it is a measurement of a quality. (D)					
Q 17	a)No, because if f(x) is no continuous at c, then limit of f(x) will not equal f(c) b)For example, if f(x)= x for]-inf,0[U]0,inf[5 for {0} Then for c=0, the limit of f(x)= 0 and f(0)=5, so limit of f(x) does not always equal f(c)	a) The student is wrong, there could be a discontinuity at c. f could have a jump discontinuity at c. b) see photo FileS2-2. In the first example if c = 0 then the limit from the right and the limit from the left are not both equal to f(c). Same idea for	a) correct	a) No. For discontinuous functions, it is obvious that a limit to a point does not mean that the function is defined at that point. b) Let f(x) = x^2 when x is not 0 f(x) = 1 when x is 0 An example when the statement is true is f(x)=(x^2)	a) No. b) It is only true when f is continuous at c. For example, 1/x as x goes to 0, f(0) is not defined (1/0 is undefined). Another example would be f(x) = x^2 for x not 1 and f(x) = 0 for x = 1. Then the limit would be 1, from both sides, but f(1) = 0.	a) No, the limit of a function is not necessarily reached by that function, therefore you must look at the behavior of a function, not just at it's formula and plug and play. b) 1- f(x)=1/x : limit as x goes to 0 of 1/x is still 0, but f(x)= 1/x is not defined at x=0 the statement is true for	a) No, not necessarily b) f(x)=1 for all x element of R, except for x=1, f(x)=0 for x=1, then the limit of f(x) as x approaches 1 is 1, but f(1)=0 c) The function must be continuous	a) I don't think the student is right, because f(c) might not be defined on the function. b) For example, if we have a step function with a discontinuity at x=0, with f(x)=1 for the interval 0 to 2, then the limit as x approaches to 0 for f(x) is 1. However, the limit is not equal to f(c)	a) Yes and no, it's approximately that, but it never actually reaches c. So it shouldn't be an equal sign b). it could be discontinuous, so be the left of c it's one number and the right of c it's another, but it's not directly f(c).	a) No, I would not say "always" true b) not true : let f = (-1)^n -> the limit diverges but evaluating at f(c) will give either 1 or -1 true : let f(x)=x (always be true) c) The conditions

	c)The statement is correct for f(x) continuous	c=1 in example 2. c) the limit from the right needs to be equal to the limit from the left. And the limit at c needs to be equal to f(c)		c) I think the function needs to be continuous at every point.	Any other discontinuity would yield the same result. c) f must be continuous.	constant functions: f(x)=3, then limit as f(x) goes to 7 is equal to f(7)=3 c) The function needs to be bounded or constant and it needs to be continuous.		because there is a hole there. But I think this is true if there are no such discontinuities. c) I think for this statement to be correct, we must have the condition that the function is continuous.	c) when it's both f(c) from right and left	would be that the function does not "oscillate" between value (I can't think of a way of formulating it more scientifically)
Q 18	a)F(x)=1/x Group 1 b)F(x)=4 Group 2 c)F(x)=cos(x)/ xGroup 1 d)F(x)= -(e^x) Group3 e)F(x)=sin(x) Group 4 f)F(x)=ln(x) Group 5	a)F(x)=1/x Group 2 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 2 d)F(x)= -(e^x) Group4 e)F(x)=sin(x) Group3 f)F(x)=ln(x) Group 4	a)F(x)=1/x Group1 b)F(x)=4 Group2 c)F(x)=cos(x)/x Group2 d)F(x)= - (e^x) Group3 e)F(x)=sin(x) Group4 f)F(x)=ln(x) Group1	a)F(x)=1/x Group 2 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 2 d)F(x)= -(e^x) Group3 e)F(x)=sin(x) Group5 f)F(x)=ln(x) Group 4	a)F(x)=1/x Group 1 b)F(x)=4 Group 2 c)F(x)=cos(x)/ x Group 1 d)F(x)= -(e^x) Group 3 e)F(x)=sin(x) Group 4 f)F(x)=ln(x) Group 3	a)F(x)=1/x Group 2 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 2 d)F(x)= -(e^x) Group 3 e)F(x)=sin(x) Group 3 f)F(x)=ln(x) Group 2	a)F(x)=1/x Group 2 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 3 d)F(x)= -(e^x) Group 4 e)F(x)=sin(x) Group 5 f)F(x)=ln(x) Group 4	a)F(x)=1/x Group 3 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 4 d)F(x)= -(e^x) Group 2 e)F(x)=sin(x) Group 5 f)F(x)=ln(x) Group 2	a)F(x)=1/x Group 3 b)F(x)=4 Group 1 c)F(x)=cos(x)/ x Group 3 d)F(x)= -(e^x) Group 4 e)F(x)=sin(x) Group 2 f)F(x)=ln(x) Group 2	a)F(x)=1/x Group 1 b)F(x)=4 Group 2 c)F(x)=cos(x)/ x Group 1 d)F(x)= -(e^x) Group 3 e)F(x)=sin(x) Group 2 f)F(x)=ln(x) Group 3
Q 19	Group 1: Limit x goes to infinity equals 0 Group 2: Limit x goes to infinity equals c (a constant)	a) and b) Group 1) has only the constant function, so functions in group one are pretty easy to visualize at	Blank	a) I put the functions according to their limit at infinity. b) Group 1: lim to infinity = 4	a) Group 1 was functions that go to 0 because they get smaller and smaller (can not be constant). Group 2 was	Group 1 has functions with the limit as x goes to infinite equal to a real finite number which they reached.	a & b) Group 1: constant functions, the limit of the function as x approaches any point c from the domain is	a) Group 1: constants Group 2: functions related to log functions	a) /b) Group 3 : goes to zero. Group1 : goes to a number that isn't zero	Group 1 : converges to y=0 or/and asymptotic to y=0 Group 2 : Real-valued

	Group 3: Limit x goes to infinity equals minus infinity Group 4: Divergent limit Group 5: Limit x goes to infinity equals infinity a) I set the groups according to their limit to infinity, so the options are 0, -Infinity, Infinity, c, divergent b)See the groups description above c)No, since the answer of the limit can only be one value (we cannot have the limit being equal to 0 and minus infinity at the same time	infinity since it has the same value every in its domain. Group 2) has only functions that have x as their denominators. The numerator value of these 2 functions have a very small absolute value so, intuitively, dividing something infinitely many times will result in nothing/zero. Group 3) Sin (x) clearly doesn't have a limit since it oscillates about the x-axis without ever losing amplitude, so its limit cannot exist. Group 4) These functions approach positive/negative infinity as x approaches infinity.		Group 2: lim to infinity = 0 Group 3: lim to infinity = -infinity Group 4: lim to infinity = infinity Group 5: lim to infinity = DNE c) No, but I might have put all functions with a positive limit to infinity that is not 0 nor infinity in Group 1 and had another group for functions for negative limit at infinity that are not 0 nor -infinity.	for the constant function (equal to its limit). Group 3 was for functions that go to infinity as x goes to infinity. Group 4 was for divergent functions (that alternate between values and do not get closer to any value). b) They must have the characteristic written above. c) No, I have tried to create disjoint groups. However, if we considered their value as x when to -infinity, then -e^x could go in the first group because it goes to 0. Lnx would need its own new group, because it is not defined in negative values.	Group 2 has functions with the limit as 0, which is limit it never really reaches only approaches. Group 3 has functions which limits don't exist as x goes to infinite. function f(x) is the function I was hesitating between group 1 and 2 for placing because I don't know what the limit is and I don't remembers what the graph looks like either, but using l'hopital's rule, it tends to 0, so maybe it should actually be in group 2. (I changed it because I think it should actually be group 2, but it was originally in group 1)	always the same Group 2: By using the standard definition of the limit, the limit of the function as x approaches infinity can be easily found Group 3: By using squeeze's theorem, the limit can be easily found Group 4: The function is divergent as it goes to infinity as x approaches infinity Group 5: The function is divergent, as the limit does not exist c) Yes, my categorization is somewhat subjective and not very rigorous...	Group 3: other functions that are not log or trigonometric functions Group 4: related to trigonometric functions Group 5: raw trigonometric functions b) If the functions are not directly related to each other, then I tried to separate them into different groups. Otherwise, if they have a link, usually I put them in the same group. c) Yes, some of them could be put into the same group. For example c and e.	Group 2 : goes to infinity Group 4: goes to - infinity c) group 2 and 4 could be in the same group : DNE Group 1 and 3 could be in the same group : to c	limit (with a limited choice of answers) like f(x)=4 will be always 4, sin(x) is bounded by -1 and 1 Group 3: Diverges to infinity or -infinity

		c) Perhaps you could put group 2 functions in group 1 since they also have an explicit limit in R. You could also argue that sin(x)/x can be put in group 3 since it also oscillates, but I do not think it is appropriate since group 3 functions do not converge.								
Q 20	I do not know the correct answer for this question, but my guess is for continuous functions f(x) and g(x)	both f and g need to converge to x element of R union {-inf,+inf}. In addition, f + g cannot have an indeterminate form such as inf-inf for example.	Incorrect	Yes, I think this is correct if the limit exists and the function is continuous. But it might also work if the function is not continuous, I am not sure because I can't find a counterexampl e.	When both limit exist	When both functions are continuous and bounded at c	when the limit of f(x) and g(x) both exist	When c is the same value everywhere.	if both the limit of f(x) and g(x) exist	I think the following equality is correct when c is a real-value (c is element of R)
Q 21	a)The limit of f(x) when x goes to infinity is equal to L if for all epsilon superior 0, there exists N	a) the limit of f as x approaches infinitity equals L if for every epsilon greater than 0, there	a) for all e >0, there is a N >0 such that for all x > N, the absolute	a) The limit as x tends to infinity of a function of x is defined if as close as the function gets	a) We say that the limit of f(x) as x goes to infinity is L if for any open interval centered at L,	a) The limit as x goes to infinite of function f(x) is equal to L if on all of epsilon larger than 0,	a) the limit of f(x) as x approaches infinity is equal to L if for all epsilon greater than 0, there	a) The limit as n approaches infinity of f(x) is equal to L if for any small positive value E, there exists	a) if the limit exist, then there is an epsilone containing the interval of f(x)- L	Blank

	superior to 0 such that, for all x superior to N, the absolute value of the f(x) minus L (the answer of the limit) is inferior to epsilon. This means that for the limit, there exists a neighborhood (x-epsilon, x+epsilon) to which the limit exists b)This statement defines neighborhoods	exists N greater than 0 such that for every x greater than N the function f evaluated at x minus L is in the interval (f(x)-L-epsilon,f(x)-L + epsilon). It means that for any x you can pick any positive epsilon to create a neighbourhood of f(x) where abs(f(x)-L) will fall in this neighbourhood. As x approaches infinity, f evaluated at x will approach L. So for any epsilon you can find an N smaller than x such that f(x) will be closer to L than f(N). b) The definition of the limit of a function.	value of f(x) - L less than e b) Limit of a function. (But, x is to c)	to that limit, there is always a point closer. b) This is the definition of the limit of a function.	there is an element on the domain such that every subsequent element is in said interval. b) A limit.	there exists an N larger than 0 such that on all of x larger than N, the absolute value of f(x) minus L is smaller than epsilon. b) This statement defines the Lower Upper Bound.	exists an N greater than 0 such that, for all x greater than N, the absolute difference between f(x) and L is less than epsilon. Essentially, no matter how small epsilon is, the difference between f(x) and its limit is smaller than epsilon, for n sufficiently large.	a a positive value of N such that for any x that is greater than N, the absolute value of the difference of f(x) and L is less than that small positive value E. It means that for a limit to exist, the difference between the function at that x value and the L has to be very small. b) existence of a limit	b) yes	

Q 22	a)I believe the statements are true, because there can always be found a number that is inferior to another, so it is possible for b to be inferior to a. However, I do think the equation should be read a superior or equal to b, as they are both said to be superior to 0, therefore, there would be a possibility that they would take the same smallest value. b) Yes, set a =2 and b=1, satisfies 1., since a superior to 0 and b superior to 0, and 2 is superior to 1 so a superior to b holds Satisfies 2 as well, since b is superior to 0, there is a superior to	1. True 2. False	Blank	a) 1. The first one is true 2. The second one is false b) No. 1. If I chose any a bigger than 0, (let's chose something very very small so that it matters) for example a=0.00000000001 then there is always a b smaller than that but bigger than 0, let's say b=0.0000000000000001. 2. There is no number in the positive reals that is smaller than all numbers. Even if I chose b= 0.0000000000000000000000000000000000000001 there exists a smaller, say a=0.0000000000000000000000000000000	a) 1 is true. 2 is false. b) They are not equivalent. For 1, we mean that for any real number, I can find a bigger number: for example, if I give you 2, you can find 3, which is bigger than 2. For 2, we mean that there would be a number that is bigger than all the other real numbers. That does not exist to the best of my knowledge.	a) true b) They are equivalent. An example would be a=3, b=2.	a) Both are true b) 1. for all a>0, there exist a b=a/2 such that a>b 2. there exist b=a/2>0 such that for all a>0, a>b	a) True b) They are not equivalent.	blank	Blank

	zero and a superior to b			000000000000 000000000000 000000000000 000000000000 000000000001						
Q 23	a) The maximum upper bound of A is equal to 1. That means that the maximum value in A is 1 and all the other values are either equal or inferior to 1. b)A={-1,1,-1,1,-1,1} c)Yes it is the ultimate maximum of the function, as we know some functions have a few maximums, but the sup gives the highest maximum	a) The lowest upper bound of A is 1. So the smallest value in R that has all values of A smaller than it. b) A =[0,1] c) Yes it is. 1>=a for every x in A making it the maxim of A. [1,inf) are all upper bounds for A and clearly 1<=z where z element of [1,inf).	a) The supremum of set A is 1, where A is a subset of R b) A = {0,1} c) Yes	a) All elements of A are smaller or equal to 1 b) (0,1) c) No, because 1 is not element of A. If I found the square brackets on on my keyboard I could write a set where 1 is the supremum and the maximum of A.	a) The smallest upper bound (meaning that it is bigger than all the numbers contained in A) of A is 1. b) The set containing only 1. c) Yes. Since it is the only number, it is the smallest upper bound, but also the maximum (biggest number). A better example could be the set of all the numbers of the form 1-(1/n). Then the maximum would be impossible to find (not exist?). But the supremum would be 1.	a) the supremum of A is equal to 1 where A is contained in R means that 1 is the lowest upper bound possible contained in A. b) f(x) = 1, [0,1] c) Yes 1 is also the maximum of A	a) the supremum is the lowest upper bound of A, meaning any number smaller than the supremum is not an upper bound of A The maximum of A is its greatest point b) A=[0,1], sup(A)=1 and the maximum of A is also 1 as 1 is in A c) Yes, because 1 is in A and no other element of A can be greater or else that would contradict sup(A)=1	a) The upper bound is a supremum if among the upper bounds of the set A, it's the smallest one	a) that the the highest upper bound is 1 b) A= (1,1,1,1,1,1,1,1)	Blank

Q 24	a)True b)True c)True d)False	a) False b)True c) True d) False	a) True b) True c) False d) False	a) No, not for an open subset. b) Yes, the definition of supremum is contained in the definition of maximum. c) Yes, if A is open. b) No, the definition of supremum is contained in the definition of maximum.	a) False. I gave an example previously for which this is the case. b) True in the real numbers. That implies that we can find the maximum. Then it will automatically be the smallest upper bound. c) Yes, number of the form 1-(1/n) for n in the natural numbers, has no maximum, but supremum is 1. d) Yes, but not in the real numbers (because the reals are complete).	a) true b) false c) false d) true	a) False, s also has to be an element of A b) True, there can be no lower bound than the maximum if it exists c) True d) False	a) False b) True c) Yes d) False	Blank	Blank

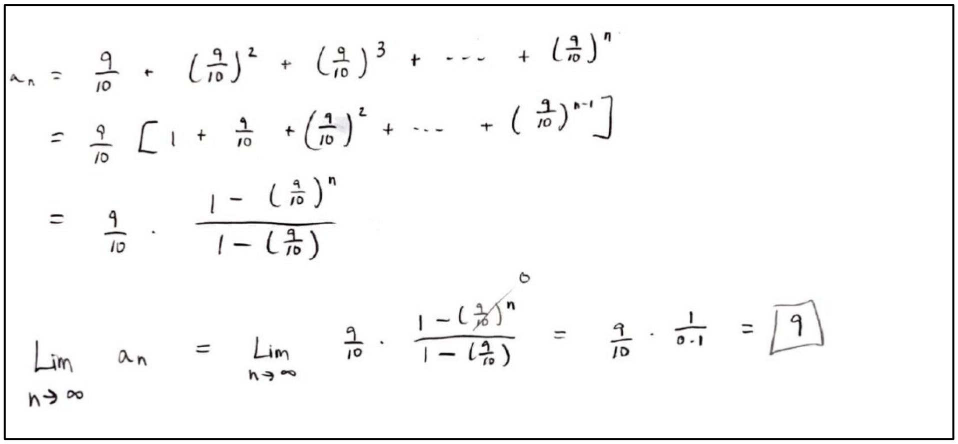
Q 25	a) Not necessarily, take a set of a constant term b) If the sequence is bounder it means it automatically has a sup c) The subsequences will also be bounded, since the sequence is bounded. And since the limit is bounded, I believe the subsequences will converge d)Yes since it is bounded	a) No b) Yes c) The limits of the subsequence must be the same as the limits of the sequence. d) Yes	a) Yes b) Yes	a) No, not if the accumulation point of the sequence is outside the sequence bound b) Yes c) If the sequence is monotone increasing, it converges to the supremum so the limit as n goes to infinity of Xn is the supremum. Therefore all the subsequences converge to that same limit. d) If the sequence is monotone increasing and has an upper bound, definitely. If the sequence is monotone decreasing and has a lower bound, definitely. If not, all the subsequences need to converge to the same point	a) Yes, the elements of a sequence are countable, we can take the biggest element. b) Yes, it is its maximum. c) They must be between the bounds. At least one exists. d) Maybe. (-1)^n is not.	a) Yes b) not necessarily c) they will be contained in the limit of the sequence d) yes	a) Not necessarily, is the sequence also closed? b) Yes, as it is bounded c) Some subsequence(s) of this sequence must have a limit d) No you could have an=(-1)^n, that is a bounded sequence that does not converge	a) Yes b) Yes	Blank	Blank

				for the sequence to be convergent.						
Q 26	a) True b) False, since it is bounded from below it has necessarily at least one of sup or inf (I cannot recall what were the rules anymore, but my guess would be since bounded from below, would have an inf, although I do think the sup exists here and is equal to infinity) c) True, because it is continuously and constantly increasing	a) False b) False c) True	a) false b) false c) True	a) False b) False, it has a supremum c) True, ln(n+1) is greater than ln(?) for every n greater than 0.	a) True b) False c) True	a) true b) false c) yes it is monotone increasing	a) False b) True c) True	Blank	Blank	Blank

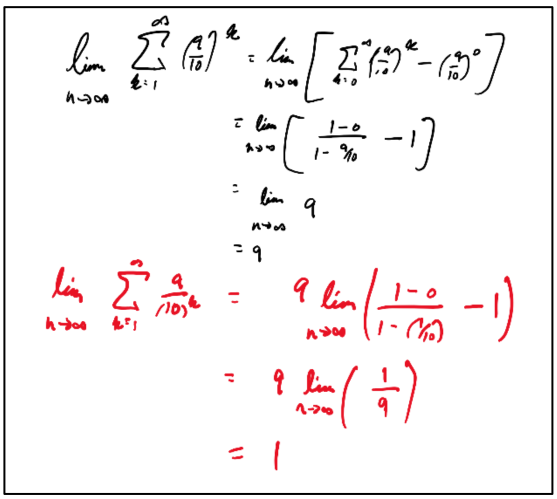
Table 2 : Student's answers to the questionnaire

A.4 Student files

FileS1



FileS2-1



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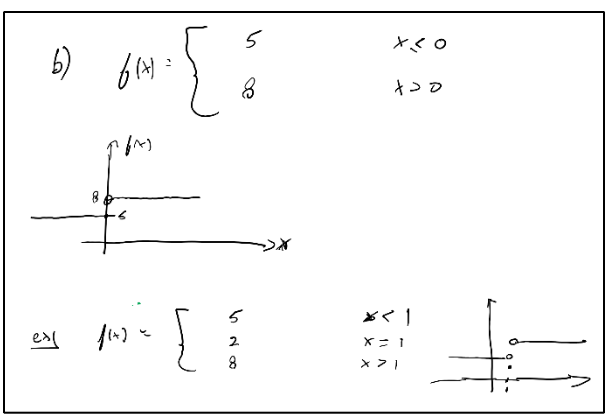
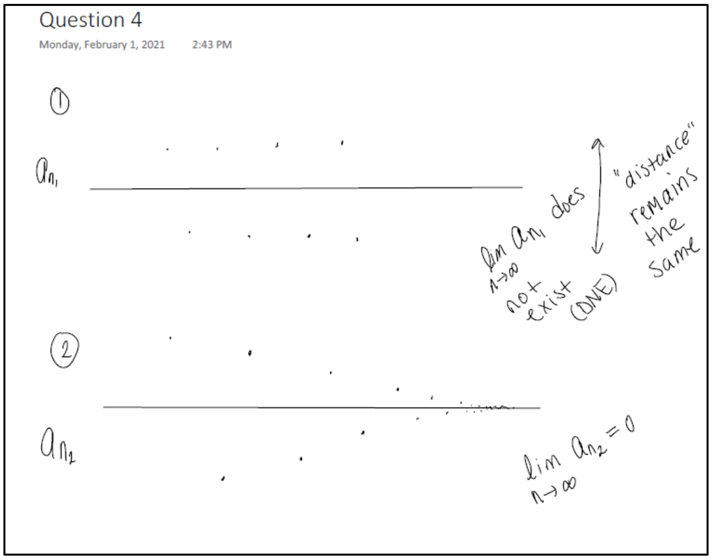


FileS6-1



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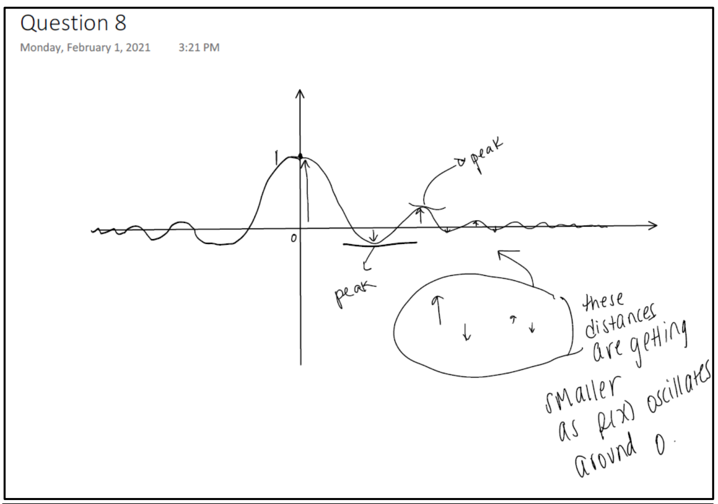
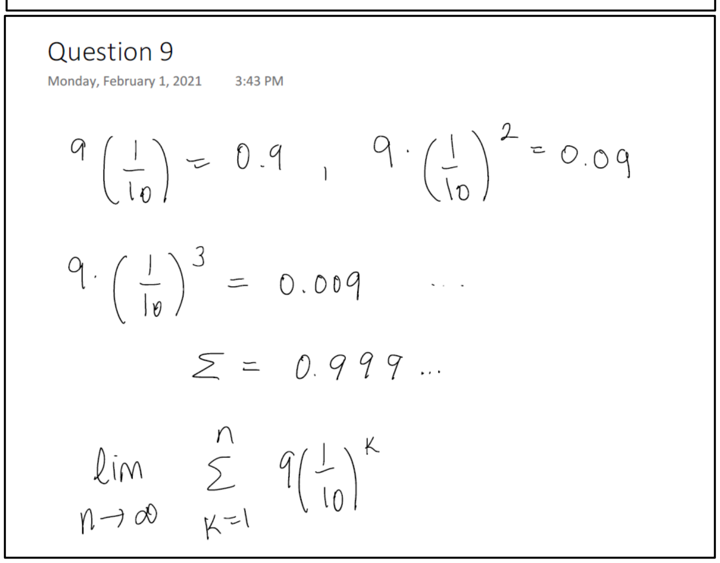
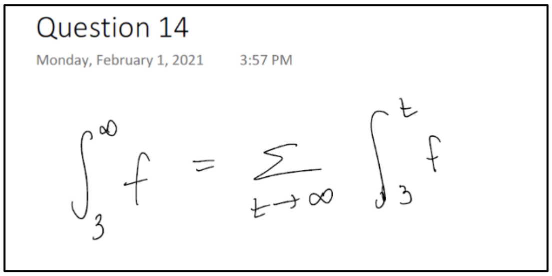


FileS6-3



FileS6-4 

A.5 Recruitment Email

THE FIRST EMAIL:

Dear student,

My name is Marc-Olivier Ouellet, and I am a Mathematics master's student at Concordia.

I am contacting you because you are taking MATH 364 with Dr. [redacted] this semester. I am wondering if you would be willing to participate in my research, which aims to better understand the challenges students face when taking undergraduate calculus and analysis courses.

Participation would require the completion of two questionnaires[11]: one in the next few weeks, and a second similar one later in the semester. There will not be a time window for you to complete the questionnaires, you can do them at a time of your convenience. However, you will have three hours to do it once you have started. It should take around 45 minutes to complete each questionnaire if completed in one sitting.

There is nothing to prepare! Your participation would be limited to the two questionnaires at the beginning and end of the semester and would help me out so much in completing my degree. I won't be able to do it without participation from students like you! Please note that participating (or refusing to participate) will not affect your grades and that your instructors will not know who is participating or have access to your questionnaires.

Please feel free to contact me with any questions.

To reach me, send an email to MA_UELL@live.concordia.ca

Hope to hear from you soon!

THE SECOND EMAIL:

[11] We initially considered having two questionnaires, but the method was later changed to only include one.

Dear student,

My name is Marc-Olivier Ouellet, and I am a Mathematics master's student at Concordia.

First, I would like to thank the students who have volunteered to take part in my research. This follow-up email is to invite any student willing to help me to participate in my study. I also take this opportunity to inform you that the participants' involvement in this research has been modified since my previous email.

Participation would require the completion of only one questionnaire, in a few weeks, near the end of the semester. There will not be a time window for you to complete the questionnaire, you can do them at a time of your convenience. However, you will have three hours to do it once you have started. Please note that if done without breaks, I expect the questionnaire to take no more than 40 minutes to complete. We will not judge or grade your performance in those questions and participating (or not participating) will not affect your grade in this course.

I won't be able to do it without participation from students like you!

Please feel free to contact me with any questions.

To reach me, send an email to MA_UELL@live.concordia.ca

Hope to hear from you soon!

A.6 Follow-up recruitment Email

THE EMAIL:

Hello students,

If you receive this email, it is because you have volunteered to participate in my study. This is to inform you that it is finally time to take the questionnaire. Like it was stated in my previous emails, the questionnaire should take between 45 minutes and one hour if you complete it in one sitting and is hosted on Moodle. You will have three hours from the time that you begin the questionnaire to allow you to take some breaks, if you decide that it is necessary.

I would like to request your honesty and integrity; it is very important for the study that you answer the questionnaire without any outside help. That includes the internet, old textbooks or help from friends.

Joined to this email is the consent form for participating in the study. This form will also be the very first question of the questionnaire, and you will be asked to type "I agree" to confirm your consent. Joining the form to this email is so that you can read it thoroughly and ask me any questions before beginning the questionnaire.

Finally, in order to give you access to the Moodle site, I will need your full name, your Netname, and you student ID.

I want to personally thank you for your help, it is because of the participation of students like you that I will be able to complete my degree.
Best regards,

Marc-Olivier Ouellet