On geometrical and analytical aspects of moduli spaces of quadratic differentials

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A Thesis
in
The Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy (Mathematics) at
Concordia University
Montréal, Québec, Canada

December 2022

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

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Entitled: **On geometrical and analytical aspects of moduli spaces of quadratic differentials**

and submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy (Mathematics)

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Abstract

On geometrical and analytical aspects of moduli spaces of quadratic differentials

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Concordia University, 2023

In this dissertation, we consider moduli spaces of meromorphic quadratic differentials in homological coordinates and applications of underlying deformation theory of Ahlfors-Rauch type.

At first, we derive variational formulas for objects associated with generalized $SL(2)$ Hitchin's spectral covers: Prym matrix, Prym bidifferential, Bergman tau-function. The resulting formulas are antisymmetric versions of Donagi-Markman residue formula. Then we adapt the framework of topological recursion to the case of double covers to compute higher-order variations.

Another application of the deformation theory lies within the symplectic geometry of the monodromy map of the Schrödinger equation on a Riemann surface with a meromorphic potential having second order poles. We discuss the conditions for the base projective connection, which induces its own set of Darboux homological coordinates, to imply the Goldman Poisson structure on the character variety. Using this result, we perform generalized WKB expansion of the generating function of monodromy symplectomorphism (the Yang-Yang function) and compute its leading asymptotics.

Finally, we relate these two studies by showing how the variational analysis on Hitchin's spectral covers could be applied towards the computation of higher asymptotics of the WKB expansion.

Acknowledgments

I would like to thank:

Dmitry Korotkin, for your guidance, support, and trust during my doctoral studies. I am fortunate to have a supervisor who is so encouraging and committed to my research. This work would not be possible without your advice, vast knowledge, and love for the profession that has always been an inspiration to me.

My parents, **Maria** and **Kirill** for your love and caring through all my life.

My grandfather, **Igor** for awakening my interest in mathematics and believing in me. I will always remember you.

I also thank the Department of Mathematics and Statistics, faculty members, and personnel for the exceptional work environment, friendliness and professionalism.

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Chapter 1

Introduction and results

Meromorphic quadratic differentials on Riemann surfaces and their moduli spaces play the fundamental role in algebraic geometry and the theory of integrable systems: from algebro-geometric construction of solutions of KdV and KP equations [14] to description of the combinatorial model of moduli spaces of curves [5].

The goal of this dissertation is to derive variational formulas of Ahlfors-Rauch type on moduli spaces of meromorphic quadratic differentials and to apply them to study 1) deformations of Hitchin's spectral covers and 2) symplectic geometry of the monodromy map of the Schwarzian equation.

Moduli spaces $\mathcal{Q}_{g,m}[\mathbf{k}]$ of quadratic differentials are defined as the set of isomorphism classes of pairs $(\mathcal{C}, Q)$, where $\mathcal{C}$ is a compact smooth complex curve of genus g with m distinct marked points $(z_j)_{j=1}^m$, and Q is a meromorphic quadratic differential on $\mathcal{C}$ with poles at z_j. We will also assume even orders $\mathbf{k} = (2k_j)_{j=1}^m$ of the poles, and that all zeroes of Q, denoted by x_i, are simple. While the latter assumption is generic, the condition of even-order poles is ruled by the applications. Spaces $\mathcal{Q}_{g,m}[\mathbf{k}]$ could be embedded into the moduli spaces of meromorphic Abelian differentials and inherit their coordinate systems together with variational formulas from there. Such approach was adopted in [35, 8] for holomorphic quadratic differentials and in [32] for differentials with first order poles. Here we generalize these results to include meromorphic differentials with poles of arbitrary even orders (Lemma 3.1.1, Proposition 3.1.1).

To define a set of homological coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$ we observe that equation

$$v^2 = Q \tag{1.0.1}$$

in the cotangent bundle $T^*\mathcal{C}$ defines a two-fold spectral cover $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, branched at zeroes of Q. Differential v is single valued on $\hat{\mathcal{C}}$ with double zeroes at the branch points x_i and $2m$ poles at the preimages $\{z_j^{(1)}, z_j^{(2)}\}$ of z_j with corresponding orders k_j. The covering surface $\hat{\mathcal{C}}$ is equipped with the natural involution map $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ that splits the homology group $H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m, \mathbb{Z})$ into even and odd subgroups

$$H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m, \mathbb{Z}) = H_+ \oplus H_-, \tag{1.0.2}$$

which are the $+1$ and -1 eigenspaces of the map, induced by the involution μ. Notice that the differential v is skew-symmetric with respect to $\mu : v(\mu(x)) = -v(x)$. Then the local coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$ are given by the basic cycles of the odd subgroup $s_k \in H_-$:

$$\mathcal{P}_{s_k} = \int_{s_k} v. \tag{1.0.3}$$

Denote by Ω the period matrix of $\mathcal{C}$ computed in some homology basis $\{a_\alpha, b_\alpha\}$ on $\mathcal{C}$. The dual basis of holomorphic differentials u_α is normalized via $\oint_{a_\alpha} u_\beta = \delta_{\alpha\beta}$.

Then, for example, variational formulas for the period matrix Ω with respect to homological coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$ look as follows:

$$\frac{\partial \Omega_{\alpha\beta}}{\partial \mathcal{P}_{s_k}} = \frac{1}{2} \oint_{s_k^*} \frac{u_\alpha u_\beta}{v}, \tag{1.0.4}$$

where s_k^* is a cycle dual to s_k with respect to the intersection pairing. Since the integration on the right-hand side is performed on the covering surface $\hat{\mathcal{C}}$, we deal with the pullbacks of u_α, u_β from $\mathcal{C}$ to $\hat{\mathcal{C}}$. These formulas strongly resemble the classical result in Teichmüller theory (Ahlfors-Rauch formula, see for example [1]) measuring variation of the period matrix under change of conformal structure of the Riemann surface defined by an arbitrary Beltrami differential.

First application of the deformation theory on $\mathcal{Q}_{g,m}[\mathbf{k}]$ is related to the theory of Hitchin's systems which were introduced in [23] as a dimensional reduction of the self-dual Yang-Mills equation. These systems together with their meromorphic generalizations [24] provide the widest class of integrable systems associated to a Riemann surface. Hamiltonians of such systems are given by meromorphic N-differentials arising in the definition of a spectral cover. We study variations on the moduli spaces of generalized $SL(2)$ Hitchin's spectral covers, naturally identified with the vector spaces of meromorphic quadratic differentials on a fixed Riemann surface $\mathcal{C}$. We denote these spaces by $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$.

The involution $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ of the canonical double cover induces a splitting of first cohomology group $H^{(1,0)}(\hat{\mathcal{C}})$ into even and odd subgroups implying that holomorphic Abelian differentials defined on the covering surface could be represented as a sum of two differentials – symmetric and skew-symmetric under the involution. While the symmetric element is a pullback from the base curve, the skew-symmetric differential is associated exclusively with the covering surface and is called Prym differential. The similar decomposition also applies to the canonical bidifferential [35] and meromorphic Abelian differentials. It turns out that only skew-symmetric differentials contribute to the variations under the assumption that the base curve is kept fixed.

Let $\{a^-_\alpha, b^-_\alpha\} \in H_-(\hat{\mathcal{C}}, \mathbb{Z})$ be generators of the odd part of homology group of cycles on $\hat{\mathcal{C}}$ with intersection index $a^-_\alpha \circ b^-_\beta = \frac{1}{2}\delta_{\alpha\beta}$. We denote a dual basis of *holomorphic Prym differentials* by u^-_α. The derivatives of the *Prym matrix* $\Omega^-_{\alpha\beta} = \oint_{b^-_\beta} u^-_\alpha$ with respect to local coordinates on the space $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ reproduce a formula analogous to the Donagi-Markman cubic [13]: denote by $A_\alpha = \oint_{a^-_\alpha} v$ integrals of the differential v over a^--cycles, remaining coordinates are defined by coefficients spanning the singular part of v near the poles. Then variation of the Prym matrix Ω^- representing a deformation of complex structure of the covering surface takes the following form:

$$\frac{\partial \Omega^-_{\alpha\beta}}{\partial A_\gamma} = -\pi i \sum_{x_i} \underset{x_i}{res} \left(\frac{u^-_\alpha u^-_\beta u^-_\gamma}{d\xi \, d(v/d\xi)} \right), \tag{1.0.5}$$

where the sum is over all branch points x_i in the base curve $\mathcal{C}$; ξ denotes a local coordinate near a branch point.

Variations with respect to the moduli representing the singular part of v are obtained by similar formulas with Prym holomorphic differentials u^-_γ replaced by *Prym second-kind and third-kind differentials* (Theorem 3.4.1). The formula (1.0.5) is a specialization of a more general case of $GL(n)$ spectral covers (see [4]). Its derivation, however, is different. While the $GL(n)$ case essentially relies on the generic assumption of simple zeros of v, in our case differential v has double zeroes at the branch points x_i and the formula (1.0.5) follows from the specific geometry of the double cover which is governed by its global involution automorphism.

We also derive variations of the objects depending on a point on the covering surface. Introduce the canonical (Bergman) bidifferential $\hat{B}(x,y)$ on $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$ and define the *Prym bidifferential* $B^-(x,y) := \hat{B}(x,y) - \mu_y^* \hat{B}(x,y)$ [35], where the notation μ_y^* means that we take the pullback with respect to the involution on the second factor in $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$. The variations of $B^-(x,y)$ are given in Theorem 3.4.2. The derivative with respect to A_γ looks as follows:

$$\frac{\partial B^-(x,y)}{\partial A_\gamma} = -\frac{1}{2} \sum_{x_i} \underset{x_i}{res} \left(\frac{u_\gamma^-(t) B^-(x,t) B^-(t,y)}{d\xi \, d(v/d\xi)} \right). \tag{1.0.6}$$

The *Bergman tau-function* τ_B was originally defined as a higher genus generalization of the Dedekind eta-function on elliptic surface. It appears in various context - from isomonodromy deformations to spectral geometry, Frobenius manifolds and random matrices, an extensive review was done in [31]. In the context of moduli spaces of quadratic differentials, it was considered in [35] in holomorphic case and in [5, 7] in presence of second order poles. We are able to extend its definition to the full space $\mathcal{Q}_{g,m}[\mathbf{k}]$ (Theorem 3.2.1) and derive its variational formulas on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ by the restriction of corresponding differential equations from $\mathcal{Q}_{g,m}[\mathbf{k}]$ (Theorem 3.4.4).

Denote by $(\tilde{z}_j)_{j=1}^n \subset (z_j)_{j=1}^m$ the set of double poles of Q, then $\pi^{-1}(\tilde{z}_j) = \{\tilde{z}_j^{(1)}, \tilde{z}_j^{(2)}\}$ are corresponding simple zeroes of v with residues denoted by $\tilde{r}_j$ and $-\tilde{r}_j$ respectively. The variation with respect to A_γ takes the following form

$$\frac{\partial \log \tau_B}{\partial A_\gamma} = \frac{5}{432} \sum_{x_i} \underset{x_i}{res} \left(\frac{u_\gamma^-}{\int_{x_i}^x v} \right) - \sum_{j=1}^n \frac{1}{48 \tilde{r}_j} \int_{\tilde{z}_j^{(2)}}^{\tilde{z}_j^{(1)}} u_\gamma^- . \tag{1.0.7}$$

We also modify the framework of topological recursion introduced in [16] to address the spaces of $SL(2)$ covers and use it to derive higher variations of the Prym matrix Ω^- (Proposition 3.5.1). This result particularly shows how the Prym bidifferential $B^-(x,y)$ defined on a single spectral curve generates the geometry of its neighborhood on the space $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$.

Another application of the variational formulas on $\mathcal{Q}_{g,m}[\mathbf{k}]$ lies within the symplectic aspects of the monodromy map of the Schwarzian equation. This study was initiated by S. Kawai [27] who established a relationship between the canonical symplectic structure on the cotangent bundle $T^*\mathcal{M}_g$ of the moduli space of curves and Goldman's bracket for the traces of monodromy matrices. Later in [8, 32] authors proposed an alternative approach to the symplectic geometry of the monodromy map involving moduli spaces of quadratic differentials in homological coordinates, in cases when Q is holomorphic or with first order poles. These works highly relied on the canonical identification between the moduli spaces of quadratic differentials and $T^*\mathcal{M}_{g,n}$. In this dissertation, we will generalize their results by considering quadratic differentials with second order poles where aforesaid identification is absent.

Introduce the linear second order equation on a Riemann surface $\mathcal{C}$ of genus g with n punctures in the form

$$\partial^2 \phi + U\phi = 0, \tag{1.0.8}$$

where U is a meromorphic potential on $\mathcal{C}$ with double poles at the punctures $(z_j)_{j=1}^n$. Invariance of the equation under a coordinate change implies that U transforms as a projective connection, while the solution ϕ locally transforms as $\frac{1}{2}$-differential [22]. To parametrize the space of all meromorphic potentials we represent the potential U as

$$U = \frac{1}{2} S - Q, \tag{1.0.9}$$

where S is a fixed projective connection on $\mathcal{C}$ with at most simple poles at z_j, while the quadratic differential Q, having double poles at z_j, varies. Let the asymptotics of Q near the poles be given by

$$Q(x) \sim \Big(\frac{r_j^2}{\xi_j^2} + O(\xi_j^{-1})\Big)(d\xi_j)^2. \tag{1.0.10}$$

The space of pairs $(\mathcal{C}, Q)$, such that all zeros of Q are simple, is denoted by $\mathcal{Q}_{g,n}$. This space is foliated into leaves $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ which correspond to fixed r_j's. We choose the cycles $\{a_\alpha^-, b_\alpha^-\} \in H_-(\hat{\mathcal{C}}, \mathbb{Z})$ with intersection index $a_\alpha^- \circ b_\beta^- = \frac{1}{2}\delta_{\alpha\beta}$, so that the integrals

$$A_\alpha = \oint_{a_\alpha^-} v, \quad B_\alpha = \oint_{b_\alpha^-} v \tag{1.0.11}$$

become local coordinates on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$. The intersection pairing defines the homological symplectic form on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$

$$\Omega_{\text{hom}} = \sum_\alpha 2\, dB_\alpha \wedge dA_\alpha. \tag{1.0.12}$$

The ratio $f = \phi_1/\phi_2$ of two linearly independent solutions of (1.0.8) solves the Schwarzian equation

$$\{f, \xi\} = S(\xi) - 2Q(\xi), \tag{1.0.13}$$

where ξ is an arbitrary local parameter on $\mathcal{C}$ and $\{\cdot\,,\cdot\}$ denotes the Schwarzian derivative. Analytical continuation of f along the cycles $\pi(\mathcal{C}\backslash(z_j)_{j=1}^n, x_0)$ determines a $PSL(2,\mathbb{C})$ monodromy representation of the fundamental group with the chosen basepoint x_0. The matrix corresponding to the monodromy around the pole z_j has the following diagonal form:

$$D_j = \begin{pmatrix} m_j & 0 \\ 0 & m_j^{-1} \end{pmatrix}. \tag{1.0.14}$$

We denote by $CV_{g,n}$ the $PSL(2)$ character variety corresponding to the monodromy representation. It is a classical result that the stratum $CV_{g,n}\{\mathbf{m}\}$ for fixed values m_j is a symplectic leaf with a Poisson structure given by the Goldman bracket [21]. Additionally assuming that Q is free from saddle trajectories (i.e., it is a "Gaiotto-Moore-Nietzke differential" [20]), the symplectic form on $CV_{g,n}\{\mathbf{m}\}$ that inverts the Goldman bracket could be written in terms of homological shear coordinates given by linear combinations of the logarithms of classical Thurston's shear coordinates [43]:

$$\Omega_G = \sum_\alpha 2\, d\rho_{a_\alpha^-} \wedge d\rho_{b_\alpha^-}. \tag{1.0.15}$$

The monodromy map

$$\mathcal{F}_{(S)} : \mathcal{Q}_{g,n}\{\mathbf{r}\} \to CV_{g,n}\{\mathbf{m}\} \tag{1.0.16}$$

with selected 2-forms essentially depends on the base projective connection S. Our first main result in this setting gives a criterion telling which projective connection S is *admissible* i.e., turning this map into a symplectomorphism: introduce the Bergman projective connection S_B defined in terms of the canonical bidifferential $B(x,y)$ on $\mathcal{C}$, which is normalized with respect to chosen Torelli marking in $H_1(\mathcal{C}, \mathbb{Z})$:

$$S_B(x) = \Big(B(x,y) - \frac{d\xi(x)d\xi(y)}{(\xi(x)-\xi(y))^2}\Big)\Big|_{y=x}, \tag{1.0.17}$$

where ξ is any local coordinate near point x. Using the canonical identification of the moduli space of quadratic differentials with simple poles and the cotangent bundle $T^*\mathcal{M}_{g,n}$, we can associate the family of quadratic differentials $S - S_B$ with the 1-form $\Theta_{(S-S_B)}$ on $\mathcal{M}_{g,n}$. Theorem 4.3.2 asserts that the map (1.0.16) is a symplectomorphism with $\mathcal{F}^*_{(S)}\Omega_G = -\Omega_{\text{hom}}$ if and only if the 1-form $\Theta_{(S-S_B)}$ is closed.

Let us take $S = S_B$. Choosing symplectic potentials on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$:

$$\theta_{hom} = \sum_\alpha \Big(B_\alpha dA_\alpha - A_\alpha dB_\alpha \Big) \tag{1.0.18}$$

and on $CV_{g,n}\{\mathbf{m}\}$:

$$\theta_G = \sum_\alpha \Big(\rho_{b_\alpha^-} d\rho_{a_\alpha^-} - \rho_{a_\alpha^-} d\rho_{b_\alpha^-} \Big) \tag{1.0.19}$$

we may consider the generating function of this symplectomorphism (the *Yang-Yang function* from [41]) given by

$$d\mathcal{G}_B = \mathcal{F}^*_{(S_B)}\theta_G - \theta_{hom}. \tag{1.0.20}$$

Although $\mathcal{G}_B$ is defined very implicitly, the WKB approximation of the equation (1.0.8) allows us to compute its asymptotic expansion. Let Q_1 be a fixed meromorphic differential on $\mathcal{C}$ with at most simple poles at the punctures $(z_j)_{j=1}^n$. For a small parameter $\hbar$ we consider second order equation on a Riemann surface $\mathcal{C}$ in the form

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q_1}{\hbar} - \frac{Q}{\hbar^2}\Big)\phi = 0 \tag{1.0.21}$$

and compute the leading asymptotics (terms G_{-1}, G_0 and G_1) of the WKB expansion of $\mathcal{G}_B$ as $\hbar \to 0^+$ (Theorem 4.4.1). Interestingly, the term G_0 contains Bergman tau-function $\log \tau_B$ restricted to the space $\mathcal{Q}_{g,n}\{\mathbf{r}\}$. Finally, we relate the WKB approximation of (1.0.21) with corresponding WKB approximation of the equation

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q}{\hbar^2}\Big)\phi = 0 \tag{1.0.22}$$

studied in [7], by performing calculus on the spaces $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ of Hitchin's SL(2) spectral covers defined by the quadratic differential Q having double poles with fixed biresidues. In particular, we show how the formula (1.0.7) appears in the term G_1 of the expansion. We also employ variational techniques on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ and $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ altogether to compute the next term G_2 (Proposition 5.1.3).

Outline

• In Chapter 2 we introduce the moduli spaces of meromorphic Abelian differentials on Riemann surfaces and define a set of homological coordinates consisting of absolute and relative periods of the meromorphic differential. Then we derive variational formulas of Ahlfors-Rauch type of the fundamental objects (Period matrix, normalized holomorphic differentials and Bergman bidifferential) associated to a Riemann surface.

• In Chapter 3 we explain most of the content of [30]. We consider the moduli spaces of meromorphic quadratic differentials with suitable system of local homological coordinates and obtain variational formulas of the objects related to the canonical double cover by pulling back corresponding formulas from the spaces of Abelian differentials. Then using a natural embedding of the spaces of generalized $SL(2)$ spectral covers into the spaces of quadratic differentials we derive variational formulas with respect to moduli of the former. We also accommodate the general theory of topological recursion to the case of $SL(2)$ spectral covers to derive higher-order variations.

• In Chapter 4 we provide proofs of the results contained in [29], which are devoted to symplectic properties of the spaces of quadratic differentials with second order poles. We consider a second order linear differential equation with meromorphic potential on a Riemann surface and prove a criterion which turns the monodromy representation of the equation into a symplectomorphism. Then we employ the WKB approximation to compute leading terms of the asymptotic expansion of the monodromy generating function (Yang-Yang function).

• Finally, in Chapter 5 we combine the results of Chapters 3 and 4 to derive higher asymptotics of the WKB expansion by varying the double cover.

Chapter 2

Spaces of Abelian differentials

This chapter is arranged as follows: in **Section 2.1** we define a set of homological coordinates on the space of meromorphic Abelian differentials given by the integrals over the elements of the relative homology group of underlying Riemann surface. In **Section 2.2** we introduce canonical tensor objects associated with a Riemann surface. In **Section 2.3** we define variations with respect to homological coordinates and derive variational formulas of canonical objects.

2.1 Coordinates on spaces $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ of meromorphic Abelian differentials

Consider a compact Riemann surface $\mathcal{C}$ of genus g and an arbitrary meromorphic differential v on $\mathcal{C}$ with m poles $z_1, \dots, z_m$ and r zeros $x_1, \dots, x_r$; we fix the multiplicities of poles and zeroes such that the divisor of v is given by

$$(v) = \sum_{i=1}^{r} n_i x_i - \sum_{j=1}^{m} k_j z_j, \quad n_i, k_j \geq 0, \tag{2.1.1}$$

where $\sum_{i=1}^{r} n_i - \sum_{j=1}^{m} k_j = 2g - 2$ is the degree of the canonical divisor.

The moduli space of pairs (Riemann surface $\mathcal{C}$ of genus g, meromorphic differential v with m poles, and fixed degrees (n_i, k_j) of divisor (v)) is denoted by $\mathcal{H}_g[n_1, \dots, n_r, -k_1, \dots, -k_m]$ (briefly $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$),

$$\dim \mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m] = 2g + r + m - 2. \tag{2.1.2}$$

This space is stratified according to the multiplicities (n_i, k_j). The corresponding strata may have several connected components. The classification of these connected components is given by [40] in holomorphic and by [9] in meromorphic case. In particular, for fixed multiplicities k_j of the poles the stratum of the space $\mathcal{H}_g[\mathbf{1}_r, -\mathbf{k}_m]$ having the highest dimension (on this stratum all the zeros of v are simple) is connected.

In the traditional framework of the theory of integrable systems the set of coordinates on the space $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ consists of moduli $\mathcal{M}_{g,m}$ of the Riemann surface punctured at poles $z_1, \dots, z_m$ and coefficients spanning the singular part of v near the poles; assuming that all a-periods of v vanish, these coordinates determine $\mathcal{C}$ and v uniquely. The issue with such coordinate system is that the coefficients of singular parts of v depend on the choice of local parameters near poles, that could also deform along with the complex structure of $\mathcal{C}$. In this work we use a system of homological coordinates, generalizing the one used in the theory of Hurwitz spaces, when the local coordinates are given by branch points of a covering of complex plane [37]. Such coordinate system is proven to be effective in describing the boundary geometry of moduli spaces of Abelian differentials on Riemann surfaces [34]. More applications lie within the study of Teichmüller flow [39] and determinants of flat Laplacians [38] on these spaces.

According to [39], system of homological local coordinates on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ is given by the integrals of v along the generators of

$$H_1(\mathcal{C} \setminus \{z_j\}_{j=1}^m, \{x_i\}_{i=1}^r), \tag{2.1.3}$$

which is the first homology group of the Riemann surface $\mathcal{C}$ punctured at *poles* $z_1, \dots, z_m$, relative to the set of *zeros* $x_1, \dots, x_r$ of v. We briefly denote this group by H_1.

Let us introduce a canonical basis of cycles (a_α, b_α) on $\mathcal{C}$; $t_2, \dots, t_m$ are small contours around poles $z_2, \dots, z_m$; $l_2, \dots, l_r$ are paths connecting the "first" zero x_1 with other zeros $x_2, \dots, x_r$. The chosen set of curves generate the homology group (2.1.3) and the integrals

$$A_\alpha := \oint_{a_\alpha} v\ , \qquad B_\alpha := \oint_{b_\alpha} v\ , \qquad \alpha = 1, \dots, g, \tag{2.1.4}$$

$$2\pi i\, r_j := \int_{t_j} v\ , \qquad \varrho_i := \int_{l_i} v\ , \qquad j = 2, \dots, m,\ i = 2, \dots, r\ . \tag{2.1.5}$$

serve as local coordinates on the space $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$.

Remark 2.1.1. If the differential v is exact: $v = df$, where f is a meromorphic function $\mathcal{C} \to \mathbb{C}P^1$, it is easy to see that all of its a, b and t–periods (residues) vanish, which particularly implies that $k_j \neq 1$. The space of such differentials coincide with Hurwitz space $Hur_g[\mathbf{n}_r, -\mathbf{k}_m]$, which is the moduli space of all branched coverings of genus g over a Riemann Sphere. The remaining non-vanishing homological coordinates in this case coincide with the critical values of function f i.e., with branch points of corresponding branch covering. Notice that in genus 0, when all a and b periods are absent, spaces $\mathcal{H}_0[\mathbf{n}_r, -\mathbf{k}_m]$ and $Hur_0[\mathbf{n}_r, -\mathbf{k}_m]$ are isomorphic. These are just spaces of rational functions on $\mathbb{C}P^1$ with fixed multiplicities of zeroes and poles.

2.2 Canonical objects on Riemann surfaces

On a compact Riemann surface $\mathcal{C}$ of genus g introduce a canonical basis of cycles (a_α, b_α) in $H_1(\mathcal{C}, \mathbb{Z})$. Denote by $u_\alpha(P)$ the basis of holomorphic 1-forms on $\mathcal{C}$ normalized by

$$\oint_{a_\alpha} u_\beta = \delta_{\alpha\beta}. \tag{2.2.1}$$

The period matrix of the surface $\mathcal{C}$ is given by

$$\Omega_{\alpha\beta} := \oint_{b_\alpha} u_\beta. \tag{2.2.2}$$

The prime form $E(x, y)$ (see [18, 19]) is an antisymmetric $-1/2$-differential with respect to both x and y. For a basepoint x_0 we define the Abel map $\mathcal{A}_\alpha(P) = \int_{x_0}^x u_\alpha$ from the Riemann surface $\mathcal{C}$ to its Jacobian. Let $\Theta[*](\mathbf{z})$ be the genus g theta-function corresponding to the period matrix Ω with some odd half-integer characteristic $[*]$. Introduce the holomorphic differential $h(x) = \sum_{\alpha=1}^g \Theta[*]_{z_\alpha}(0) u_\alpha(x)$. All zeros of this differential are double, therefore, we can correctly define square root of $h(x)$ and corresponding prime form on $\mathcal{C}$ by

$$E(x, y) = \frac{\Theta[*](\mathcal{A}(x) - \mathcal{A}(y))}{\sqrt{h(x)}\sqrt{h(y)}}\ , \tag{2.2.3}$$

this expression in fact is independent of the choice of the odd characteristic $[*]$.

The prime-form has the following properties (see [18], p.4):

- Under analytic continuation of y along the cycle a_α the prime-form remains invariant; under the analytic continuation along b_α it gains the factor

$$\exp(-\pi i\Omega_{\alpha\alpha} - 2\pi i \int_x^y u_\alpha) \ . \tag{2.2.4}$$

- On the diagonal $y \to x$ the prime-form has first order zero (and no other zeros or poles) with the following asymptotics:

$$E(\xi(x), \xi(y))\sqrt{d\xi(x)}\sqrt{d\xi(y)} = \tag{2.2.5}$$

$$= (\xi(y) - \xi(x)) \left(1 - \frac{1}{12} S_B(\xi(x))(\xi(y) - \xi(x))^2 + O((\xi(y) - \xi(x))^3\right) , \tag{2.2.6}$$

where the term S_B is called *Bergman projective connection* and $\xi(x)$ is an arbitrary local coordinate.

We recall that an arbitrary projective connection S transforms under change of the local coordinate $\eta \to \xi$ as follows:

$$S(\eta) = S(\xi) \left(\frac{d\xi}{d\eta}\right)^2 + \{\xi, \eta\} \tag{2.2.7}$$

where $\{\xi, \eta\} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2$ is the Schwarzian derivative. One can verify that the term S_B in (2.2.6) indeed transforms as (2.2.7) under change of the local coordinate. Another projective connection that will be used is associated to the differential v and given by the Schwarzian derivative:

$$S_v(\xi(x)) := \left\{\int_{x_1}^x v, \, \xi(x)\right\} (d\xi(x))^2. \tag{2.2.8}$$

Note that difference of any two projective connections is a quadratic differential on $\mathcal{C}$.

The key object in the theory of Riemann surfaces is fundamental meromorphic (Bergman) bidifferential $B(x, y)$, defined by

$$B(x, y) = \partial_x \partial_y \log E(x, y). \tag{2.2.9}$$

It is symmetric $B(x, y) = B(x, y)$ and holomorphic everywhere except for the second order pole on the diagonal $x = y$ with biresidue 1. While its a-periods with respect to both x and y vanish, the b-periods of $B(x, y)$ with respect to any of its arguments are given by the basic holomorphic differentials:

$$\oint_{b_\alpha} B(x, t) = 2\pi i u_\alpha(x). \tag{2.2.10}$$

Choosing some local coordinate ξ near the diagonal $\{x = y\} \subset \mathcal{C} \times \mathcal{C}$, we have the following local behavior of $B(x, y)$ as $y \to x$:

$$B(x, y) = \left(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{S_B(\xi(x))}{6} + O((\xi(x) - \xi(y))^2)\right) d\xi(x) d\xi(y). \tag{2.2.11}$$

If two canonical bases of cycles on $\mathcal{C}$, $\{a^\sigma_\alpha, b^\sigma_\alpha\}_{\alpha=1}^g$ and $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ are related by a matrix

$$\sigma = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbb{Z}) \ , \tag{2.2.12}$$

the corresponding period matrix, fundamental bidifferential and Bergman projective connection transform as follows ([19], p. 21):

$$\Omega^\sigma = (A\Omega + B)(C\Omega + D)^{-1}, \tag{2.2.13}$$

$$B^\sigma(x,y) = B(x,y) - \pi i \sum_{1\le\alpha\le\beta\le g} (u_\alpha(x)u_\beta(y) + u_\beta(x)u_\alpha(y)) \frac{\partial \ln\det(C\Omega+D)}{\partial\Omega_{\alpha\beta}}, \tag{2.2.14}$$

$$S_B^\sigma = S_B - 12\pi i \sum_{1\le\alpha\le\beta\le g} u_\alpha u_\beta \frac{\partial}{\partial\Omega_{\alpha\beta}} \log\det(C\Omega+D). \tag{2.2.15}$$

2.3 Variational formulas on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$

Cutting the Riemann surface $\mathcal{C}$ along the canonical cycles (a_α, b_α) which pass through a single point, chosen to be a "first" zero x_1, we get the simply connected fundamental polygon $\mathcal{C}_0$. Inside of $\mathcal{C}_0$ we also introduce branch cuts connecting poles of v with non-vanishing residues. These branch cuts are assumed to start at "first" pole z_1 connecting it with $z_2, \dots, z_m$; denote them by $\kappa_2, \dots, \kappa_m$. Let us introduce a local coordinate on $\mathcal{C}$ away from zeroes x_i and poles z_j, given by the integral

$$z(x) = \int_{x_1}^x v. \tag{2.3.1}$$

While $z(x)$ is a uniquely defined inside the simply connected domain $\mathcal{C}_0\backslash\{\kappa_j\}_{j=2}^m$, it is not globally defined on $\mathcal{C}$ itself, where it gains monodromies. Let $\mathcal{P}_k$ be a coordinate from the list (2.1.4-2.1.5).

We *define* the derivative of the basic holomorphic differentials with respect to $\mathcal{P}_k$ as follows:

$$\frac{\partial u_\alpha(x)}{\partial \mathcal{P}_k}\Big|_{z(x)} := v(x) \frac{\partial}{\partial \mathcal{P}_k}\Big|_{z(x)=const} \left\{ \frac{u_\alpha(x)}{v(x)} \right\} \tag{2.3.2}$$

where $u_\alpha(x)/v(x)$ is a meromorphic function on $\mathcal{C}$ with poles at x_i. Outside of the points in divisor (v) this function can be viewed as a function of $z(x)$ and $\mathcal{P}_k$; the derivative of this function with respect to $\mathcal{P}_k$ in the right-hand side of (2.3.2) is computed assuming that $z(x)$ is independent of $\mathcal{P}_k$.

More formally, consider the local universal family

$$\pi: \quad \mathcal{X} \to \mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]. \tag{2.3.3}$$

Then the set $(z := \int_{x_1}^x v, \mathcal{P}_1, \dots, \mathcal{P}_{2g+m+r-2})$ gives a system of local coordinates on $\mathcal{X} \setminus (v)$. A vicinity of a point $\{(\mathcal{C}, v), x\}$ in the level set $H_{z(x)} := \{t \in \mathcal{X}\ ,\ z(t) = z(x)\}$ is biholomorphically mapped onto a vicinity of the point $(\mathcal{C}, w)$ of $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ via the projection π. Then

$$((\pi|_{H_{z(P)}})^{-1})^* \left\{ \frac{u_\alpha}{v} \right\} \big|_{H_{z(P)}} \tag{2.3.4}$$

is a locally holomorphic function on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ and we denote

$$\frac{\partial}{\partial \mathcal{P}_k}\Big|_{z(x)=const} \left\{ \frac{u_\alpha(x)}{v(x)} \right\} := \frac{\partial}{\partial \mathcal{P}_k} \left[((\pi|_{H_{z(x)}})^{-1})^* \left\{ \frac{u_\alpha}{v} \right\} \big|_{H_{z(x)}} \right] . \tag{2.3.5}$$

Since the map $x \to z(x)$ is not globally defined on $\mathcal{C}$, the 1-forms $\frac{\partial u_\alpha(x)}{\partial \mathcal{P}_k}$ are local meromorphic differentials defined within $\mathcal{C}_0\backslash\{\kappa_j\}_{j=2}^m$. They do not necessarily correspond to global 1-forms on $\mathcal{C}$ itself.

Similarly, the derivatives of $B(x,y)$ with respect to the moduli are defined as follows:

$$\frac{\partial B(x,y)}{\partial \mathcal{P}_k}\Big|_{z(x),z(y)} := v(x)v(y)\frac{\partial}{\partial \mathcal{P}_k}\Big|_{z(x),z(y)}\left\{\frac{B(x,Q)}{v(x)v(y)}\right\}. \tag{2.3.6}$$

Derivatives of other tensor objects, depending on the point in $\mathcal{C}$ are defined by obvious analogy.

It is convenient introduce contours s_i generating the homology group (2.1.3):

$$\{s_i\}_{i=1}^{2g+r+m-2} = \Big\{\{a_\alpha, b_\alpha\}_{\alpha=1}^{g}, \{t_j\}_{j=2}^{m}, \{l_i\}_{i=2}^{r}\Big\}. \tag{2.3.7}$$

Then the homological coordinates on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ are defined as integrals of v over $\{s_i\}$:

$$\mathcal{P}_{s_i} = \int_{s_i} v\ , \qquad i = 1, \dots, 2g+r+m-2. \tag{2.3.8}$$

The homology group dual to (2.1.3) is the homology group of $\mathcal{C}$, punctured at *zeros* of v, relative to the set of *poles* of v:

$$H_1(\mathcal{C}\setminus\{x_i\}_{i=1}^{r}, \{z_j\}_{j=1}^{m}). \tag{2.3.9}$$

The dual basis $\{s_i^*\}$ is defined by the condition

$$s_i^* \circ s_j = \delta_{ij} \tag{2.3.10}$$

and is given by

$$\{s_i^*\}_{i=1}^{2g+r+m-2} = \Big\{\{-b_\alpha, a_\alpha\}_{\alpha=1}^{g}, \{\kappa_j\}_{j=2}^{m}, \{c_i\}_{i=2}^{r}\Big\}, \tag{2.3.11}$$

where $\kappa_2, \dots, \kappa_m$ are contours connecting the “first” pole z_1 with other poles $z_2, \dots, z_m$, respectively; $c_2, \dots, c_r$ are small circles around the zeros $x_2, \dots, x_r$.

Now we are in a position to formulate the following theorem, which gives variational formulas on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$ with respect to homological coordinates $\mathcal{P}_{s_i}$. This theorem was originally stated in [38] in holomorphic case and then extended in [26] to cover meromorphic Abelian differentials. The proof of this result is very instructional: techniques developed here will be adapted throughout the thesis.

Theorem 2.3.1. *[38, 26] For a basis $\{s_i\}_{i=1}^{2g+r+m-2}$ of H_1 and its dual basis $\{s_i^*\}_{i=1}^{2g+r+m-2}$ the following formulas hold on $\mathcal{H}_g[\mathbf{n}_r, -\mathbf{k}_m]$:*

$$\frac{\partial \Omega_{\alpha\beta}}{\partial \mathcal{P}_{s_i}} = \oint_{s_i{}^*} \frac{u_\alpha u_\beta}{v}, \tag{2.3.12}$$

$$\frac{\partial u_\alpha(x)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x)=const} = \frac{1}{2\pi i}\oint_{s_i{}^*} \frac{u_\alpha(t)B(x,t)}{v(t)}, \tag{2.3.13}$$

$$\frac{\partial B(x,y)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x),z(y)=const} = \frac{1}{2\pi i}\oint_{s_i{}^*} \frac{B(x,t)B(t,y)}{v(t)}, \tag{2.3.14}$$

$$\frac{\partial}{\partial \mathcal{P}_{s_i}}(S_B(x) - S_v(x))\Big|_{z(x)=const} = \frac{3}{\pi i}\oint_{s_i{}^*} \frac{B^2(x,t)}{v(t)}. \tag{2.3.15}$$

Proof. Let us prove first the variational formula (2.3.13) for the normalized holomorphic differential. We use the Abelian integral $z(x) = \int_{x_1}^{x} v$ as a local coordinate in a neighborhood of any point of $\mathcal{C}$ not coinciding with the zeros x_i and poles z_j of the differential v. Consider the derivative of $u_\alpha(x)$ with respect to $\varrho_i = \int_{l_i} v$ $(i \geq 2)$ assuming that the coordinate $z(x)$ is independent of ϱ_i. The proof of the corresponding variational formula follows the idea of the proof for the standard Rauch formula on the Hurwitz spaces (see f.e. Section 2.3 of [37]).

Let us at first study the behavior of the differential $\partial_{\varrho_i} u_\alpha(x)|_{z(x)}$ near the poles. Near a first order pole z_j of v the local coordinate could be chosen as

$$\zeta_j(x) = \exp\left(\frac{z(x)}{r_j}\right), \tag{2.3.16}$$

where r_j is the residue defined by (2.1.5). Near a pole z_j of order $k_j > 1$ the local coordinate is the solution to transcendental equation

$$\frac{1}{\zeta_j^{k_j-1}(x)} + r_j \ln \zeta_j(x) = z(x). \tag{2.3.17}$$

In either case, one can observe that $\frac{\partial \zeta_j}{\partial \varrho_i}\big|_{z(x)} = 0$, so expressing the differential u_α in local coordinate ζ_j by

$$u_\alpha(\zeta_j) = (a_0 + a_1\zeta_j + \dots) d\zeta_j \tag{2.3.18}$$

and differentiating with respect to ϱ_i, we have

$$\partial_{\varrho_i} u_\alpha(\zeta_j)|_{z(x)} = O(1) d\zeta_j, \tag{2.3.19}$$

meaning that $\partial_{\varrho_i} u_\alpha(x)|_{z(x)}$ is holomorphic at z_j.

Let us consider the local behavior of $\partial_{\varrho_i} u_\alpha(x)|_{z(x)}$ near zero x_i. We choose the local parameter near x_i to be

$$\xi_i(x) = (z(x) - z(x_i))^{1/(n_i+1)}, \tag{2.3.20}$$

where n_i is the multiplicity of x_i. We have

$$u_\alpha(\xi_i) = (a_0 + a_1\xi_i + \dots + a_{n_i}\xi_i^{n_i} + O({\xi_i}^{(n_i+1)})) d\xi_i\ . \tag{2.3.21}$$

Noticing that $z(x_i) = \varrho_i$, we differentiate this expansion with respect to ϱ_i for fixed $z(x)$:

$$\frac{\partial}{\partial \varrho_i}\{u_\alpha(x)\}|_{z(x)} = \left\{ a_0 \left(1 - \frac{1}{n_i+1}\right)\frac{1}{\xi_i^{n_i+1}} + a_1\left(1 - \frac{2}{n_i+1}\right)\frac{1}{\xi_i^{n_i}} + \dots \right. \tag{2.3.22}$$

$$\left. + a_{n_i-1}\left(1 - \frac{n_i}{n_i+1}\right)\frac{1}{\xi_i^2} + O(1) \right\} d\xi_i. \tag{2.3.23}$$

Consider the set of standard meromorphic differentials of second kind with vanishing a-periods: $w_i^{s+1}(x)$ with the only singularity at the point x_i of the form $\xi_i(x)^{-s-1} d\xi_i(x)$. The differential $\partial_{\varrho_i} u_\alpha(x)|_{z(x)}$ also has all vanishing a-periods, since the a-periods of u_α are constant. Therefore, it can be expressed in terms of these standard differentials as follows:

$$\frac{\partial}{\partial \varrho_i}\{u_\alpha(x)\}|_{z(x)} = a_0\left(1 - \frac{1}{n_i+1}\right) w_i^{n_i+1}(x) + a_1\left(1 - \frac{2}{n_i+1}\right) w_i^{n_i}(x) + \dots \tag{2.3.24}$$

$$+ a_{n_i-1}\left(1-\frac{n_i}{n_i+1}\right) w_i^2(x). \tag{2.3.25}$$

Now, the differentials $w_i^s(x)$ can be expressed in terms of Bergman bidifferential $B(x,y)$ as:

$$w_i^s(x) = \frac{(-1)^s}{(s-1)!}\frac{d^{s-2}}{d\xi_i^{s-2}(x)} B(x,y)\Big|_{y=x_i}. \tag{2.3.26}$$

Using (2.3.26) we can rewrite (2.3.24-2.3.25) in the following compact form:

$$\frac{\partial u_\alpha(x)}{\partial \varrho_i}\Big|_{z(x)} = \frac{1}{(n_i+1)(n_i-1)!}\left(\frac{d}{d\xi_i(y)}\right)^{n_i-1}\left\{\frac{B(x,y)u_\alpha(y)}{(d\xi_i(y))^2}\right\}\Big|_{y=x_i} \tag{2.3.27}$$

or, equivalently,

$$\frac{\partial u_\alpha(x)}{\partial \varrho_i}\Big|_{z(x)} = \underset{x_i}{res}\left\{\frac{u_\alpha(y)B(x,y)}{v(y)}\right\}\ , \tag{2.3.28}$$

which leads to (2.3.13) for $s = l_2, \dots, l_r$.

Let us now prove formulas (2.3.13) for $s = (a_\beta, b_\beta)$. For example, consider the derivative of u_α with respect to $B_\beta = \int_{b_\beta} v$.

Let us introduce the fundamental polygon $\mathcal{C}_0$ of $\mathcal{C}$ with the branch cuts $\kappa_2, \dots, \kappa_m$ between the poles z_1 and $z_j,\ j = 2, \dots, .m$. The map $z(x)$ is a single-valued holomorphic function on $\mathcal{C}_0\backslash\{\kappa_j\}_{j=2}^m$ away from poles. Consider an arbitrary point x in $\mathcal{C}_0$ which does not coincide with any zero or pole of v. Denote by T_{a_i}, T_{b_i} the deck transformations which correspond to the cycles a_i and b_i, respectively. The sides a_β^+ and a_β^- of the fundamental polygon $\mathcal{C}_0$ are identified by T_{b_β}. The deck transformation $T_{b_\beta}(x)$ of the point x corresponds to the analytic continuation of the function $z(x)$ along the b_β cycle, such that $z(x)$ is mapped to $z(x) + B_\beta$. Since the holomorphic differential u_α is globally defined on $\mathcal{C}$, it can be lifted to a holomorphic differential on $\mathcal{C}_0$ invariant with respect to the deck transformations. Let us write $u_\alpha(x) = f(z)dz$. As u_α is invariant under the deck transformations, we have

$$f(z + B_\beta) = f(z). \tag{2.3.29}$$

Differentiating this equality with respect to z, we get

$$\frac{\partial f(z+B_\beta)}{\partial z} = \frac{\partial f(z)}{\partial z}. \tag{2.3.30}$$

Differentiating (2.3.29) again with respect to B_β, while z is kept constant, we also mind that f implicitly depends on B_β:

$$\frac{\partial f(z+B_\beta)}{\partial z} + \frac{\partial f(z+B_\beta)}{\partial B_\beta} = \frac{\partial f(z)}{\partial B_\beta}. \tag{2.3.31}$$

Combining these formulas, we write

$$\frac{\partial f(z+B_\beta)}{\partial B_\beta}dz - \frac{\partial f(z)}{\partial B_\beta}dz = -\frac{\partial f(z)}{\partial z}dz. \tag{2.3.32}$$

Let us denote

$$\Phi(x) := \frac{\partial u_\alpha(x)}{\partial B_\beta}\Big|_{z(x)}. \tag{2.3.33}$$

Since the coordinate $z(x)$ is single valued on the fundamental polygon $\mathcal{C}_0\backslash\{\kappa_j\}_{j=2}^m$, the differential Φ is also single-valued and holomorphic on $\mathcal{C}_0\backslash\{\kappa_j\}_{j=2}^m$. Consider global function $(u_\alpha/v)(x)$ on $\mathcal{C}$

and corresponding Abelian differential $d(u_\alpha/v)(x)$, which in any local coordinate $\xi(x)$ is given by $df(\xi)$ if $u_\alpha(\xi) = f(\xi)d\xi$. Then we can rewrite (2.3.32) in a coordinate-independent form:

$$T_{b_\beta}[\Phi(x)] = \Phi(x) - d\Big(\frac{u_\alpha}{v}\Big)(x) \ . \tag{2.3.34}$$

By analogy, we can show that

$$T_{b_\gamma}[\Phi(x)] = \Phi(x) \ , \qquad \gamma \neq \beta \ , \tag{2.3.35}$$

and

$$T_{a_\gamma}[\Phi(x)] = \Phi(x) \ , \qquad \gamma = 1, \dots, g \ , \tag{2.3.36}$$

Additionally, differentiating the formulas (2.3.18), (2.3.21) with respect to B_β we conclude, that (2.3.33) is holomorphic at zeroes and poles of v. Therefore, the differential Φ can be viewed as a differential on $\mathcal{C}$, which is holomorphic everywhere except the cycle a_β, where it has the jump discontinuity given by $d\Big(\frac{u_\alpha}{v}\Big)$. Moreover, one can observe that all a-periods of $\Phi(x)$ vanish, with agree with the fact that all a-periods of the jump differential are also zero.

To write down an explicit formula for Φ, we recall the Plemelj formula on the complex plane $\mathbb{C}$. Let γ be a positively oriented simple closed curve in the complex plane, $f(x)$ is a holomorphic function defined in the tubular neighborhood of γ. The integral

$$F(y) := \frac{1}{2\pi i} \oint_\gamma f(x)(x-y)^{-2} dx \tag{2.3.37}$$

defines two holomorphic functions $F^l(y)$ and $F^r(y)$ which are restrictions of $F(y)$ to the interior and the exterior of γ, respectively. The boundary values of F^r and F^l on γ are related by the Plemelj formula $F^r(y) - F^l(y) = -f_y(y)$.

This observation allows to write immediately the formula for the differential Φ with discontinuity $-d\Big(\frac{u_\alpha}{v}\Big)$ on the cycle a_β and all vanishing a-periods:

$$\Phi(y) = \frac{1}{2\pi i} \oint_{a_\beta} \frac{u_\alpha(x) B(x,y)}{v(x)}. \tag{2.3.38}$$

The required discontinuity on the cycle a_β is implied by singularity structure of $B(x,y)$ on the diagonal and Plemelj formula; vanishing of all the a-periods follows from vanishing of all the a-periods of bidifferential $B(x,y)$.

The formula for differentiation with respect to A_β has the different sign due to the interchange of the roles of interior and exterior domains (due to the asymmetry of the intersection index between a_β and b_β ($a_\beta \circ b_\beta = -b_\beta \circ a_\beta = 1$).

Derivatives with respect to the residues $2\pi i\, r_j = \int_{t_j} v$ ($j \geq 2$) are proved by analogy: differential $\partial_{2\pi r_j} u_\alpha(x)|_{z(x)}$ has an additive jump on the contour κ_j, given by $-d\Big(\frac{u_\alpha}{v}\Big)$. Then the Plemelj formula on the complex plane and its generalization to a Riemann surface provide the formula (2.3.13) for $s = t_2, \dots, t_m$.

Integrating (2.3.13) over b-cycles and changing the order of integration, one gets (2.3.12). Formula (2.3.14) can be proved in the same manner as (2.3.13). Formula (2.3.15) follows from the variational formulas for the bidifferential $B(x,y)$ (2.3.14) by taking the limit $y \to x$ and using the singular structure of $B(x,y)$ on the diagonal in local coordinate $z(x)$ (in this local coordinate projective connection $S_v(z(x))$ vanishes).

$\square$

Remark 2.3.1. Derived variational formulas could be seen as an analog of classical results of Ahlfors-Rauch deformation theory. For example, variation of the Period matrix $\Omega_{\alpha\beta}$ in the direction of arbitrary Beltrami differential μ is given by

$$\delta_\mu \Omega_{\alpha\beta} = \iint_{\mathcal{C}} u_\alpha u_\beta \mu. \tag{2.3.39}$$

Then in related formulas (2.3.12) the role of "Beltrami differential" is played by the vector field $1/v$ localized on the contour s_i^*. The precise correspondence was discussed in Section 2.4 of [38].

Chapter 3

Spaces of quadratic differentials and SL(2) Hitchin's spectral covers

This chapter has the following structure: in **Section 3.1** we discuss geometry of the canonical double cover associated with a quadratic differential and define a set of homological coordinates on spaces of meromorphic quadratic differentials, given by the integrals over cycles lying in the odd part of the first homology group of covering surface. We also introduce main objects linked to the canonical double cover and derive their variations by restricting variational formulas on spaces of Abelian differentials obtained in Chapter 2. **Section 3.2** is devoted to the definition and main properties of Bergman tau-function. In **Section 3.3** we introduce a coordinate system on spaces of generalized $SL(2)$ spectral covers with fixed base. In **Section 3.4** we derive variational formulas of Prym matrix, Prym bidifferential and Bergman tau-function with respect to coordinates on spaces of $SL(2)$ spectral covers. We obtain these formulas by the pullback of the variational formulas on spaces of quadratic differentials. The resulting expressions provide explicit realizations of Donagi-Markman residue formulas. Finally, in **Section 3.5** we adapt the theory of Eynard-Orantin topological recursion to derive higher order variations.

3.1 Spaces $\mathcal{Q}_{g,m}[\mathbf{k}]$ of meromorphic quadratic differentials

3.1.1 Geometry of double cover

Denote by $\mathcal{Q}_{g,m}[\mathbf{k}]$ moduli space of pairs: a Riemann surface $\mathcal{C}$ of genus g and a meromorphic quadratic differential Q with simple zeroes and poles at $\{z_j\}_{j=1}^m$ of corresponding orders $\mathbf{k} = \{2k_j\}_{j=1}^m$. The degree $4g-4$ of the divisor class (Q) implies a quadratic differential has

$$r = 4g - 4 + 2\sum_{j=1}^{m} k_j \tag{3.1.1}$$

simple zeroes denoted by x_i. For all such quadratic differentials the equation

$$v^2 = Q, \tag{3.1.2}$$

in the cotangent bundle $T^*\mathcal{C}$ defines double covering $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, branched at zeroes of Q. The covering surface $\hat{\mathcal{C}}$ possesses a natural holomorphic involution $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$. Square-root differential $v = $"$\sqrt{Q}$" is a single-valued meromorphic Abelian differential on $\hat{\mathcal{C}}$ and skew-symmetric under the involution: $v(x^\mu) = -v(x)$.

Differential v has double zeroes at branch points, which will be denoted by same letters x_i. That follows from the following short observation: given ξ is a local coordinate near any zero x_i on $\mathcal{C}$ such that $Q = \xi(d\xi)^2$, one has locally: $v = \sqrt{Q} = \sqrt{\xi}d\xi = 2\hat{\xi}^2 d\hat{\xi}$, where $\hat{\xi} = \sqrt{\xi}$ is a coordinate near x_i on $\hat{\mathcal{C}}$. Then the Riemann-Hurwitz formula implies the genus of the covering surface $\hat{\mathcal{C}}$ equals

$$\hat{g} = 4g - 3 + \sum_{j=1}^{m} k_j. \tag{3.1.3}$$

Since the branch points of $\hat{\mathcal{C}}$ do not coincide with z_j we have $\pi^{-1}(z_j) = \{z_j^{(1)}, z_j^{(2)}\}$ and differential v has poles of order k_j at both $z_j^{(1)}$ and $z_j^{(2)}$ on $\hat{\mathcal{C}}$.

The dimension of $\mathcal{Q}_{g,m}[\mathbf{k}] = (\mathcal{C}, Q)$ consists of $3g-3$ modular parameters of $\mathcal{C}$, m positions of singularities, $2\sum_{j=1}^m k_j$ coefficients of singular parts and $3g-3$ parameters that form a holomorphic part of Q. Thus, the total dimension is

$$\dim \mathcal{Q}_{g,m}[\mathbf{k}] = 6g - 6 + m + 2\sum_{j=1}^{m} k_j. \tag{3.1.4}$$

We decompose the first homology group of $H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m)$ into

$$H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m) = H_+ \oplus H_-, \tag{3.1.5}$$

which are the $+1$ and -1 eigenspaces of the map, induced by the involution μ. $\dim(H_+) = 2g+m-1$ and $\dim(H_-) = 6g-6+2\sum_{j=1}^m k_j + m$. The canonical basis of $H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n)$ can be chosen as follows:

$$\{a_k, a_k^\mu, \tilde{a}_l, b_k, b_k^\mu, \tilde{b}_l, t_j, t_j^\mu\}, \quad k = 1, ..., g, \quad l = 1, ..., 2g-3+\sum_{j=1}^m k_j, \quad j = 1, ..., m. \tag{3.1.6}$$

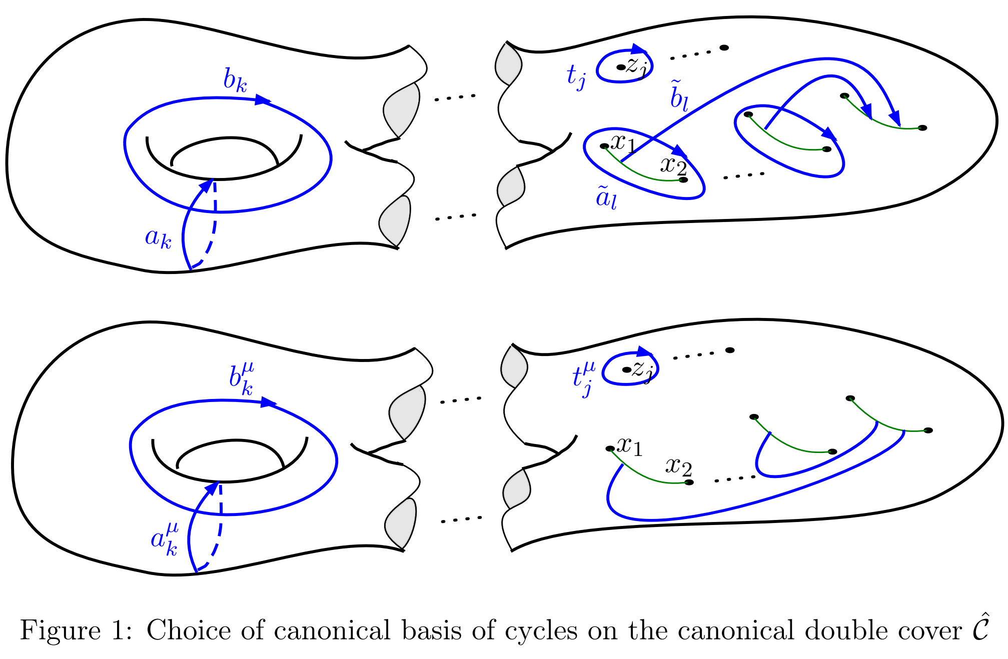

Figure 1: Choice of canonical basis of cycles on the canonical double cover $\hat{\mathcal{C}}$

Here $\{a_k, b_k, a_k^\mu, b_k^\mu\}$ is a lift of the canonical basis of cycles $\{a_k, b_k\}$ from $\mathcal{C}$ to $\hat{\mathcal{C}}$ such that

$$\mu_* a_k = a_k^\mu, \quad \mu_* b_k = b_k^\mu, \quad \mu_* \tilde{a}_l + \tilde{a}_l = \mu_* \tilde{b}_l + \tilde{b}_l = 0. \tag{3.1.7}$$

$\{t_j, t_j^\mu\}$ is a lift of a small positively-oriented loop t_j around z_j on $\mathcal{C}$. On double cover $\hat{\mathcal{C}}$, t_j denotes a positively-oriented loop encircling $z_j^{(1)}$, while t_j^μ is a small loop around $z_j^{(2)}$. In the group (3.1.5) there is a single relation given by

$$\sum_{j=1}^{m} (t_j + t_j^\mu) = 0. \tag{3.1.8}$$

The classes

$$a_k^+ = \frac{1}{2}(a_k + a_k^\mu), \quad b_k^+ = \frac{1}{2}(b_k + b_k^\mu), \quad t_j^+ = \frac{1}{2}(t_j + t_j^\mu) \tag{3.1.9}$$

generate the group H^+ with the intersection index

$$a_i^+ \circ b_k^+ = \frac{1}{2}\delta_{ik}, \tag{3.1.10}$$

while all other intersections are zero. The following cycles

$$a_k^- = \frac{1}{2}(a_k - a_k^\mu), \quad b_k^- = \frac{1}{2}(b_k - b_k^\mu), \tag{3.1.11}$$

$$a_l^- = \frac{1}{\sqrt{2}}\tilde{a}_l, \quad b_l^- = \frac{1}{\sqrt{2}}\tilde{b}_l, \tag{3.1.12}$$

$$t_j^- = \frac{1}{2}(t_j - t_j^\mu) \tag{3.1.13}$$

are the generators of the group H_-. Similarly, their intersection index is

$$a_i^- \circ b_k^- = \frac{1}{2}\delta_{ik} \tag{3.1.14}$$

and all other intersections are zero.

The dimension of H_- coincides with the dimension of $\mathcal{Q}_{g,m}[\mathbf{k}]$. We introduce the following set of *period* (homological) local coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$:

$$A_k = \oint_{a_k^-} v, \quad B_k = \oint_{b_k^-} v, \quad 2\pi i r_j = \oint_{t_j^-} v, \tag{3.1.15}$$

here $+r_j$ and $-r_j$ are residues of v near $z_j^{(1)}$ and $z_j^{(2)}$, respectively.

Lemma 3.1.1. *The parameters (3.1.15) provide a system of local coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$.*

Proof. The differential v has a double zero at each branch point of the covering $\hat{\mathcal{C}} \to \mathcal{C}$. Additionally, v has $2m$ poles which are split into pairs according to their orders. Therefore, the pair $(\hat{\mathcal{C}}, v)$ belongs to the stratum $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ of the moduli space of Abelian differentials on algebraic curves of genus $\hat{g}$ with $r = \hat{g} - 1 + \sum_{j=1}^m k_j$ zeroes of multiplicity 2, and $2m$ poles of multiplicities $\mathbf{k}_{2m} = (k_1, k_1, ..., k_m, k_m)$. The dimension of the space $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ is $2\hat{g} - 2 + r + 2m = 12g - 12 + 4\sum_{j=1}^m k_j + 2m$ and the space $\mathcal{Q}_{g,m}[\mathbf{k}]$ (of dimension $6g - 6 + 2\sum_{j=1}^m k_j + m$) forms a subspace of codimension $6g - 6 + 2\sum_{j=1}^m k_j + m$ in $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2n}]$ and can be described in terms of homological coordinates on $\mathcal{H}_{\hat{g}}(\mathbf{2}_r, -\mathbf{k}_{2n})$.

Let $(\hat{\mathcal{C}}, v)$ represent a point in $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$. Following Chapter 2, the set of homological coordinates on $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ consists of integrals of v along the cycles representing a basis in the relative homology group $H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m, \{x_i\}_{i=2}^r)$; such basis consists of the cycles (3.1.6) in $H_1(\hat{\mathcal{C}} \backslash \{z_j^{(1)}, z_j^{(2)}\}_{j=1}^m)$, and of the relative homology classes of $r - 1 = 4g - 5 + 2\sum_{j=1}^m k_j$ paths $l_2, \dots, l_r$ connecting x_1 with x_i, $i = 2, \dots, r$, and not intersecting the cycles (3.1.6). As a pair $(\hat{\mathcal{C}}, v)$ also belongs to $\mathcal{Q}_{g,m}[\mathbf{k}]$, surface $\hat{\mathcal{C}}$ is invariant under a holomorphic involution μ and $\mu^* v = -v$ (in particular, each zero x_i of v is invariant under μ). Clearly, $\int_{l_i} v = -\int_{l_i^\mu} v$, so the combination $l_i - l_i^\mu$ is a cycle on $\hat{\mathcal{C}}$ which is skew-symmetric under μ. Therefore, each coordinate $\int_{l_i} v$ on $\mathcal{Q}_{g,m}[\mathbf{k}]$ becomes a linear combination of the periods of v along the cycles (3.1.11 – 3.1.12) on $\hat{\mathcal{C}}$ with half-integer coefficients. Moreover, we have $\int_{a_j^\mu} v = -\int_{a_j} v$, $\int_{b_j^\mu} v = -\int_{b_j} v$ for each $j = 1, \dots, g$, and $\int_{t_i^\mu} v = -\int_{t_i} v$ for $i = 2, \dots, m$ (relation $\int_{t_1^\mu} v = -\int_{t_1} v$ would be dependable due (3.1.8)), which gives $2g + m - 1$ more vanishing linear combinations of homological coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$. The remaining $6g - 6 + 2\sum_{j=1}^m k_j + m$ homological coordinates associated with a basis in H_- therefore serve as local coordinates on $\mathcal{Q}_{g,m}[\mathbf{k}]$. □

3.1.2 Standard meromorphic objects

In this section we introduce basic meromorphic objects associated with the canonical double cover. The involution map $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ yields a decomposition of the first cohomology group into even and odd parts

$$H^{1,0}(\hat{\mathcal{C}}) = H^+(\hat{\mathcal{C}}) \oplus H^-(\hat{\mathcal{C}}). \tag{3.1.16}$$

We shall denote by

$$\{\hat{u}_k, \hat{u}_k^\mu, \hat{w}_l\}, \quad k = 1, ..., g, \qquad l = g+1, ..., \hat{g} \tag{3.1.17}$$

the basis of normalized holomorphic Abelian differentials on $\hat{\mathcal{C}}$, dual to the basis of a-cycles of (3.1.6). The differentials

$$u_k^+ = \hat{u}_k + \hat{u}_k^\mu \tag{3.1.18}$$

provide a basis in $H^+(\hat{\mathcal{C}})$. These differentials are invariant under involution and naturally isomorphic to the space of holomorphic differentials on $\mathcal{C}$. Differentials u_k^+ are normalized over a^+- cycles and vanish over a^-- cycles.

$$\dim(H^+(\hat{\mathcal{C}})) = g. \tag{3.1.19}$$

The space $H^-(\hat{\mathcal{C}})$ consists of holomorphic differentials on $\hat{\mathcal{C}}$ with a skew-symmetric property $u(x^\mu) = -u(x)$. Such elements are called *Prym holomorphic differentials*. The basis for $H^-(\hat{\mathcal{C}})$ is generated by

$$u_l^- = \begin{cases} \hat{u}_l - \hat{u}_l^\mu, & l = 1, ..., g, \\ \sqrt{2}\hat{w}_l, & l = g+1, ..., \hat{g}. \end{cases} \tag{3.1.20}$$

Notice, that differentials u_l^- are normalized over a^-- cycles and have vanishing a^+- periods.

$$\dim(H^-(\hat{\mathcal{C}})) = 3g - 3 + \sum_{j=1}^m k_j := g^-. \tag{3.1.21}$$

Integrating these differentials over corresponding even and odd parts of b - cycles on $\hat{\mathcal{C}}$ we obtain Period and *Prym matrices*

$$\Omega_{ij}^+ = \oint_{b_j^+} u_i^+, \quad \Omega_{ij}^- = \oint_{b_j^-} u_i^-. \tag{3.1.22}$$

Matrix Ω^+ is identified with the period matrix Ω of the base surface $\mathcal{C}$. Let us represent the Prym matrix in a block form as

$$\Omega^- = \begin{pmatrix} \Omega_1^- & \Omega_2^- \\ \Omega_2^{-t} & \Omega_3^- \end{pmatrix}, \tag{3.1.23}$$

where Ω_1^- is $g \times g$ matrix; Ω_2^- is a $g \times (g^- - g)$ matrix and Ω_3^- is a $(g^- - g) \times (g^- - g)$ matrix. Then the period matrix $\hat{\Omega}$ of the double cover $\hat{\mathcal{C}}$ in the basis $\{b_k, b_k^\mu, \tilde{b}_l\}$ can be expressed in terms of Ω^+ and Ω^- as follows:

$$\hat{\Omega} = \begin{pmatrix} \frac{\Omega^+ + \Omega_1^-}{2} & \frac{\Omega^+ - \Omega_1^-}{2} & \frac{\Omega_2^-}{\sqrt{2}} \\ \frac{\Omega^+ - \Omega_1^-}{2} & \frac{\Omega^+ + \Omega_1^-}{2} & -\frac{\Omega_2^-}{\sqrt{2}} \\ \frac{\Omega_2^{-t}}{\sqrt{2}} & -\frac{\Omega_2^{-t}}{\sqrt{2}} & \Omega_3^- \end{pmatrix}. \tag{3.1.24}$$

We proceed with bidifferentials and projective connections on double covers. Let $\hat{B}(x,y)$ denote the canonical (Bergman) bidifferential on $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$ normalized with respect to the a-cycles of homology basis (3.1.6). We put

$$B^+(x,y) := \hat{B}(x,y) + \mu_y^* \hat{B}(x,y), \tag{3.1.25}$$

$$B^-(x,y) := \hat{B}(x,y) - \mu_y^* \hat{B}(x,y), \tag{3.1.26}$$

where notation μ_y^* means that we take the pullback with respect to the involution on the second factor in $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$. While bidifferential $B^+(x,y)$ is the pullback of the canonical bidifferential $B(x,y)$ on $\mathcal{C} \times \mathcal{C}$ (normalized relative to the a-cycles on the base), bidifferential $B^-(x,y)$ is called the *Prym bidifferential* and was originally intoduced in [35] to study the Picard group of the compactification of the moduli space of quadratic differentials. It follows from the definitions that bidifferentials B^+ and B^- are symmetric in both arguments

$$B^\pm(x,y) = B^\pm(y,x) \tag{3.1.27}$$

and behave symmetrically and skew-symmetrically, respectively, under involution μ:

$$\mu_x^* B^+(x,y) = \mu_y^* B^+(x,y) = B^+(x,y), \tag{3.1.28}$$

$$\mu_x^* B^-(x,y) = \mu_y^* B^-(x,y) = -B^-(x,y). \tag{3.1.29}$$

These properties imply that

$$\oint_{a_k^+} B^+(x,y) = \oint_{a_k^-} B^+(x,y) = \oint_{b_k^-} B^+(x,y) = 0, \tag{3.1.30}$$

$$\oint_{a_k^+} B^-(x,y) = \oint_{a_k^-} B^-(x,y) = \oint_{b_k^+} B^-(x,y) = 0. \tag{3.1.31}$$

Near the diagonal $x = y$ on $\hat{\mathcal{C}} \times \hat{\mathcal{C}}$ one has

$$\hat{B}(x,y) = \Big(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6}\hat{S}_B(\xi(x)) + ...\Big) d\xi(x) d\xi(y), \tag{3.1.32}$$

as $y \to x$ for any local coordinate ξ on $\hat{\mathcal{C}}$. The term $\hat{S}_B$ transforms like a projective connection under the change of coordinates. It is called the Bergman projective connection. For $B^\pm(x,y)$ near the diagonal one has

$$B^\pm(x,y) = \Big(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B^\pm(\xi(x)) + ...\Big) d\xi(x) d\xi(y), \tag{3.1.33}$$

with two projective connections S_B^+ and S_B^- that are related by

$$S_B^{\pm}(x) = \hat{S}_B(x) \pm 6\mu_y^* \hat{B}(x,y)|_{y\to x}. \tag{3.1.34}$$

We call S_B^- the *Prym projective connection.* Note that $\hat{S}_B$ is holomorphic on $\hat{\mathcal{C}}$, while $S_B^{\pm}$ have double poles at branch points.

3.1.3 Variational formulas on $\mathcal{Q}_{g,m}[\mathbf{k}]$

It is convenient to introduce the periods $\mathcal{P}_{s_i} = \oint_{s_i} v$ for s_i being an element from the canonical basis of H_- :

$$\{s_i\}_{i=1}^{2g^-+m} = \Big\{\{a_k^-, b_k^-\}_{k=1}^{g^-}, \{t_j^-\}_{j=1}^m\Big\}. \tag{3.1.35}$$

The dual basis $\{s_i^*\}$ is defined by the condition

$$s_i^* \circ s_j = \delta_{ij} \tag{3.1.36}$$

and is given by

$$\{s_i^*\}_{i=1}^{2g^-+m} = \Big\{\{-2b_k^-, 2a_k^-\}_{k=1}^{g^-}, \{2\kappa_j^-\}_{j=1}^m\Big\}, \tag{3.1.37}$$

here κ_j^- is a 1/2 of the contour connecting poles $z_j^{(1)}$ with $z_j^{(2)}$ and skew-symmetric under the involution, not intersecting other contours. Such generator may be chosen as follows: connect $z_j^{(1)}$ with a branch point x_1 by an arc γ_j on a first copy of $\mathcal{C}$. Then join x_1 with $z_j^{(2)}$ on second copy of $\mathcal{C}$ by the arc $\mu(\gamma_j)$.

Choose a fundamental polygon $\hat{\mathcal{C}}_0$ of $\hat{\mathcal{C}}$ with vertex at x_1 and introduce a system of branch cuts inside $\hat{\mathcal{C}}_0$ in the following way: on base curve $\mathcal{C}$ connect first zero x_1 with a chosen first pole z_1 by a branch cut γ_1, then connect z_1 with the remaining poles $\{z_j\}_{j=2}^m$ by γ_j forming a tree graph G. Then we lift G to $\hat{\mathcal{C}}$ via π^{-1} and denote the corresponding lift by $\hat{G} = \pi^{-1}(G)$. Inside the simply connected domain $\hat{\mathcal{C}}_0 \backslash \hat{G}$ we define the "flat" coordinate

$$z(x) = \int_{x_1}^x v, \tag{3.1.38}$$

which can be used as local coordinate on $\hat{\mathcal{C}}$ outside of zeros and poles of v. We will also assume that generators κ_j^- agree with a system of cuts defined by the graph $\hat{G}$. Namely, $\kappa_j^- \cap \hat{G} = \{z_j^{(1)}, x_1, z_j^{(2)}\}$.

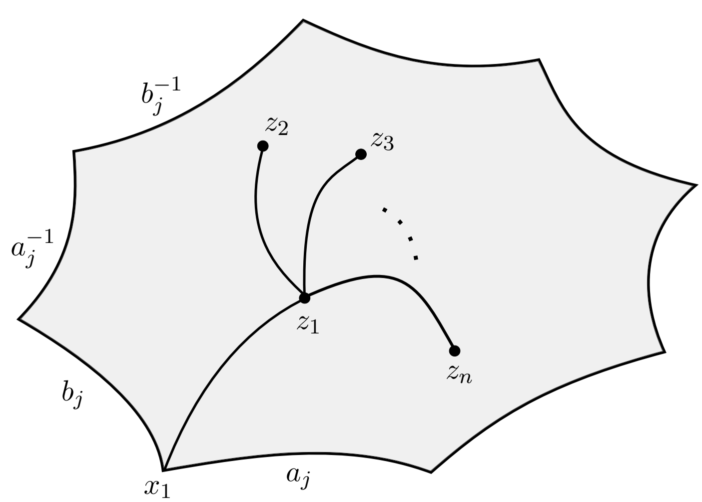

Figure 2: Tree graph G within the fundamental polygon of $\mathcal{C}$

Lemma 3.1.1 allows to derive variational formulas on $\mathcal{Q}_{g,m}[\mathbf{k}]$ by restricting the already known variational formulas on spaces of Abelian differentials $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ obtained in Theorem 2.3.1. While these formulas appear in [8, 35] for a holomorphic differential v, here we extend this result to arbitrary poles of v. The derivation of variations with respect to coordinates A_k, B_k follows the narrative [8, 35], the variations with respect to the residues r_j are new.

Proposition 3.1.1. *For a basis $\{s_i\}_{i=1}^{2g^-+m}$ of H_- and its dual basis $\{s_i^*\}_{i=1}^{2g^-+m}$ the following formulas hold on $\mathcal{Q}_{g,m}[\mathbf{k}]$:*

$$\frac{\partial \Omega_{ij}^{\pm}}{\partial \mathcal{P}_{s_i}} = \frac{1}{2} \oint_{s_i{}^*} \frac{u_i^{\pm} u_j^{\pm}}{v}, \tag{3.1.39}$$

$$\frac{\partial u_j^{\pm}(x)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x)=const} = \frac{1}{4\pi i} \oint_{s_i{}^*} \frac{u_j^{\pm}(t) B^{\pm}(x,t)}{v(t)}, \tag{3.1.40}$$

$$\frac{\partial B^{\pm}(x,y)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x),z(y)=const} = \frac{1}{4\pi i} \oint_{s_i{}^*} \frac{B^{\pm}(x,t) B^{\pm}(t,y)}{v(t)}. \tag{3.1.41}$$

$$\frac{\partial}{\partial \mathcal{P}_{s_i}} (S_B^{\pm} - S_v)(x)\Big|_{z(x)=const} = \frac{1}{4\pi i} \oint_{s_i{}^*} \frac{(B^{\pm}(x,t))^2}{v(t)}. \tag{3.1.42}$$

Proof. Consider, for example, the derivative of Prym bidifferential $B^-(x,y)$ with respect to $A_1 = \int_{a_1^-} v$, where $a_1^- = \frac{1}{2}(a_1 - a_1^{\mu})$ (see (3.1.11)). According to Theorem 2.3.1, variational formulas on the space $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ are given by:

$$\frac{\partial \hat{B}(x,y)}{\partial (\oint_{a_1^-} v)} = -\frac{1}{2\pi i} \oint_{t \in b_1^-} \frac{\hat{B}(x,t) \hat{B}(t,y)}{v(t)}, \tag{3.1.43}$$

where $\hat{B}(x,y)$ is the canonical bidifferential on $\hat{\mathcal{C}}$ normalized with respect to the a-cycles of the basis (3.1.6) on $\hat{\mathcal{C}}$ (consequently the periods of $\hat{B}(x,y)$ along cycles a_j^+, a_j^- in (3.1.9), (3.1.11) and (3.1.12) also vanish). Considering the period of $\oint_{a_k^{\mu}} v$ as an independent variable, Theorem 2.3.1 implies

$$\frac{\partial \hat{B}(x,y)}{\partial (\oint_{a_1^{\mu}} v)} = -\frac{1}{2\pi i} \oint_{t \in b_1^{\mu}} \frac{\hat{B}(x,t) \hat{B}(t,y)}{v(t)}. \tag{3.1.44}$$

Recall that $A_1 = \frac{1}{2}(\int_{a_1} v - \int_{a_1^{\mu}} v)$ and $\int_{a_1} v = -\int_{a_1^{\mu}} v = A_1$. Using the chain rule and taking into account the symmetry $v(t^{\mu}) = -v(t)$, we find

$$\frac{\partial \hat{B}(x,y)}{\partial A_1} = \frac{\partial \hat{B}(x,y)}{\partial A_1}\Bigg|_{\left(\oint_{a_1^{\mu}} v\right)=const} + \frac{\partial \hat{B}(x,y)}{\partial (\oint_{a_1^{\mu}} v)} \frac{\partial (\oint_{a_1^{\mu}} v)}{\partial A_1} = \tag{3.1.45}$$

$$= -\frac{1}{2\pi i} \oint_{t \in b_1} \frac{\left(\hat{B}(x,t)\hat{B}(t,y) + \hat{B}(x,t^{\mu})\hat{B}(t^{\mu},y)\right)}{v(t)}. \tag{3.1.46}$$

Along the same lines we also find

$$\frac{\partial \hat{B}(x^{\mu},y)}{\partial A_1} = -\frac{1}{2\pi i} \oint_{t \in b_1} \frac{\left(\hat{B}(x^{\mu},t)\hat{B}(t,y) + \hat{B}(x^{\mu},t^{\mu})\hat{B}(t^{\mu},y)\right)}{v(t)}. \tag{3.1.47}$$

We recall the definition of Prym bidifferential

$$B^-(x,y) = \hat{B}(x,y) - \hat{B}(x^\mu, y). \tag{3.1.48}$$

Subtracting (3.1.47) from (3.1.46) and rearranging the terms one obtains

$$\frac{\partial B^-(x,y)}{\partial A_1} = -\frac{1}{2\pi i}\oint_{t\in b_1} \frac{B^-(x,t)B^-(t,y)}{v(t)}. \tag{3.1.49}$$

The integrand is anti-symmetric and $b_1^- = \frac{1}{2}(b_1 - b_1^\mu)$, therefore,

$$\frac{\partial B^-(x,y)}{\partial A_1} = -\frac{1}{2\pi i}\oint_{t\in b_1^-} \frac{B^-(x,t)B^-(t,y)}{v(t)} \tag{3.1.50}$$

which implies (3.1.41) when $s_i = a_1^-$, $s_i^* = -2b_1^-$ (recall the normalization $a_i^- \circ b_j^- = \delta_{ij}/2$). Analogously one can verify (3.1.41) for any cycle from (3.1.11).

The computation of the derivatives with respect to the periods over the cycles (3.1.12) is more involved. Consider, for example, $A_{g+1} = \int_{a_{g+1}^-} v$, where $a_{g+1}^- = \frac{1}{\sqrt{2}}\tilde{a}_1$. Since this cycle is formed as a double cover of branch cut between two zeroes x_i of v, the coordinates $\{l_i\}_2^r$, given by integrals over the paths between x_1 and $\{x_i\}_2^r$ and restricted from $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ to $\mathcal{Q}_{g,m}[\mathbf{k}]$, depend linearly on $\oint_{a_{g+1}^-} v$. Therefore, applying the chain rule, we have that

$$\frac{\partial \hat{B}(x,y)}{\partial A_{g+1}} = \left.\frac{\partial \hat{B}(x,y)}{\partial A_{g+1}}\right|_{\left(\oint_{l_i} v\right)_{i=2}^r = const} + \sum_{i=2}^r \left(\frac{\partial}{\partial A_{g+1}}\int_{l_i} v\right)\frac{\partial \hat{B}(x,y)}{\partial(\oint_{l_i} v)} = \tag{3.1.51}$$

$$= -\frac{1}{2\pi i}\oint_{t\in b_{g+1}^-}\frac{\hat{B}(x,t)\hat{B}(t,y)}{v(t)} + \frac{1}{2\pi i}\sum_{i=2}^r\left(\frac{\partial}{\partial A_{g+1}}\int_{l_i} v\right)\oint_{t\in c_i}\frac{\hat{B}(x,t)\hat{B}(t,y)}{v(t)}, \tag{3.1.52}$$

where c_i is a small circle around zero x_i.

Similarly, after antisymmetrizing left-hand side, we get

$$\frac{\partial B^-(x,y)}{\partial A_{g+1}} = -\frac{1}{2\pi i}\oint_{t\in b_{g+1}^-}\frac{B^-(x,t)B^-(t,y)}{v(t)} + \tag{3.1.53}$$

$$+\frac{1}{2\pi i}\sum_{j=2}^r\left(\frac{\partial}{\partial A_{g+1}}\int_{l_j} v\right)\oint_{t\in c_j}\frac{B^-(x,t)B^-(t,y)}{v(t)}. \tag{3.1.54}$$

The first term in the right-hand side of (3.1.53) gives the necessary contribution. To show that the second term vanishes, we notice that this integral is nothing but the residue of the integrand at x_j. Skew-symmetry of the integrand with respect to involution μ implies that it expands by even powers in local coordinate $\hat{\xi}_i$ near branch point x_i and, therefore, has vanishing residue. Hence,

$$\oint_{t\in c_j}\frac{B^-(x,t)B^-(t,y)}{v(t)} = 0. \tag{3.1.55}$$

Similarly, one can prove (3.1.41) for any cycle from (3.1.12).

The dirivation of variations of $B^-(x,y)$ with respect to the residues $2\pi i r_j = \oint_{t_j^-} v$, for $t_j^- = \frac{1}{2}(t_j - t_j^\mu)$ (3.1.13), follows the exact logic of computation of the derivative with respect to A_1.

In the same way (3.1.41) is proved for the canonical bidifferential $B^+(x,y)$ by employing the formula (3.1.25). Variational formulas for the matrices $\Omega^\pm$, holomorphic Abelian differentials $u_j^\pm(x)$ and quadratic differentials $(S^\pm - S_v)(x)$ are derived from the corresponding variational formulas on $\mathcal{H}_{\hat{g}}[\mathbf{2}_r, -\mathbf{k}_{2m}]$ following the same steps in the proof above. □

3.2 Bergman tau-function

The Bergman tau-function on moduli spaces of differentials was originally defined as a higher genus generalization of the Dedekind eta function on elliptic surface. Starting from Hurwitz spaces [37] and moduli spaces of holomorphic Abelian differentials [38], it was extended to the case of Abelian differential with arbitrary divisor [26]; further generalizations cover moduli spaces of holomorphic N-differentials [33] and Hitchin's spectral covers [36]. In our framework we consider a moduli space of quadratic meromorphic differentials with simple zeroes. With this space we associate a naturally arising Bergman tau-function, also called Hodge tau-function in literature, since it is a holomorphic section of the determinant line bundle of the Hodge vector bundle. Beginning with the holomorphic case [35], the definition of the Bergman tau-function was expanded in [5, 7] to include meromorphic quadratic differentials with second order poles. Here we allow to have poles of arbitrary even orders.

For the purpose of the explicit definition of the tau-function we introduce a special system of local coordinates on the double cover called *distinguished*.

3.2.1 Distinguished local coordinates on $\mathcal{C}$ and $\hat{\mathcal{C}}$

The quadratic differential Q on the base curve $\mathcal{C}$ allows us to define the set of distinguished local coordinates on both surfaces $\mathcal{C}$ and $\hat{\mathcal{C}}$. Denote by $\{\tilde{z}_j\}_{k=1}^{n} \subset \{z_j\}_{j=1}^{m}$ a subset of poles of of order 2. Then the divisor of Q looks as follows:

$$(Q) = \sum_{i=1}^{r+m} d_i q_i \equiv \sum_{i=1}^{r} x_i - \sum_{j=1}^{n} 2\tilde{z}_j - \sum_{j=1}^{m-n} 2k_j z_j, \;\; k_j \geq 2. \tag{3.2.1}$$

The divisor of Abelian differential v on $\hat{\mathcal{C}}$ is given by

$$(v) = \sum_{i=1}^{r+2m} \hat{d}_i \hat{q}_i \equiv \sum_{i=1}^{r} 2x_i - \sum_{j=1}^{n} (\tilde{z}_j^{(1)} + \tilde{z}_j^{(2)}) - \sum_{j=1}^{m-n} k_j (z_j^{(1)} + z_j^{(2)}), \;\; k_j \geq 2. \tag{3.2.2}$$

- Near any point $x_0 \in \hat{\mathcal{C}}$ such that $\pi(x_0) \notin (Q)$ the local coordinates on $\mathcal{C}$ and $\hat{\mathcal{C}}$ can be chosen as

$$z(x) = \int_{x_0}^{x} v. \tag{3.2.3}$$

- Near a branch point x_i local parameters $\hat{\zeta}_i$ on $\hat{\mathcal{C}}$ and ζ_i on $\mathcal{C}$ are given by

$$\hat{\zeta}_i(x) = \left(\int_{x_i}^{x} v \right)^{\frac{1}{3}}, \quad \zeta_i(x) = \hat{\zeta}_i^2(x) = \left(\int_{x_i}^{x} v \right)^{\frac{2}{3}}. \tag{3.2.4}$$

- In the neighborhood of a double pole $\tilde{z}_j$ on $\mathcal{C}$ and corresponding simple poles $(\tilde{z}_j^{(1)}, \tilde{z}_j^{(2)})$ on $\hat{\mathcal{C}}$ the local coordinate is

$$\zeta_j(x) = \exp\Big(\frac{1}{\tilde{r}_j} \int_{x_1}^{x} v \Big), \tag{3.2.5}$$

where x_1 is a chosen first zero of v; $\tilde{r}_j$ is a residue of v on $\hat{\mathcal{C}}$ defined in (3.1.15).

• If $k_j \geq 2$ one has a pole of order $2k_j$ at z_j on $\mathcal{C}$ and corresponding poles $(z_j^{(1)}, z_j^{(2)})$ of order k_j on $\hat{\mathcal{C}}$ with nontrivial residues $\pm r_j$. The local coordinate on both $\mathcal{C}$ and $\hat{\mathcal{C}}$ in this case is defined from the following transcendental equations:

$$v = \left(\frac{1-k_j}{\zeta_j^{k_j}} + \frac{r_j}{\zeta_j} \right) d\zeta_j \tag{3.2.6}$$

or

$$\frac{1}{\zeta_j^{k_j-1}} + r_j \ln \zeta_j = \int_{x_1}^{x} v. \tag{3.2.7}$$

Remark 3.2.1. Definition of the tau-function depend on the choice of local coordinates near the poles z_j of Q. To define these coordinates uniquely we consider them inside the simply connected domain given by the fundamental polygon $\hat{\mathcal{C}}_0$ with a chosen system of cuts as discussed in Section 3.1.3.

3.2.2 Definition and properties

Denote by $E(x, y)$ the Prime form on $\mathcal{C}$, by $\mathcal{A}_x$ the Abel map with x as a base point and by K^x the vector of Riemann constants. Introduce two vectors $\mathbf{r}, \mathbf{s} \in \frac{1}{2}\mathbb{Z}^g$ such that

$$\frac{1}{2}\mathcal{A}_x((Q)) + 2K^x + \Omega\mathbf{r} + \mathbf{s} = 0 \tag{3.2.8}$$

and the following notations:

$$E(x, q_i) = \lim_{y \longrightarrow q_i} E(x, y)\sqrt{d\zeta_i(y)}, \tag{3.2.9}$$

$$E(q_i, q_j) = \lim_{x \longrightarrow q_i, y \longrightarrow q_j} E(x, y)\sqrt{d\zeta_i(x)}\sqrt{d\zeta_j(y)}, \tag{3.2.10}$$

where ζ_i is the distinguished local parameter on $\mathcal{C}$ near a point q_i from the list (3.2.1).

Consider the following multi-valued $g(1-g)/2$ - differential $C(x)$ on $\mathcal{C}$

$$C(x) = \frac{1}{W(x)} \left(\sum_{i=1}^{g} u_i(x) \frac{\partial}{\partial w_i} \right)^g \theta(w, \Omega) \Big|_{w=K^x}, \quad W(x) := det \left[\frac{d^{k-1}}{dx^{k-1}} u_j \right]_{1 \leq j,k \leq g} \tag{3.2.11}$$

here Ω is the period matrix of the base curve $\mathcal{C}$, $\{u_j\}_{j=1}^g$ are normalized holomorphic differentials on $\mathcal{C}$ and θ is the corresponding theta-function.

Definition 3.2.1. For a given choice of Torelli marking and tree graph G on $\mathcal{C}$ *the Bergman tau-function* τ_B is given by the following expression:

$$\tau_B(\mathcal{C}, Q) = C^{2/3}(x) \left(\frac{Q(x)}{\prod_{i=1}^{r+m} E^{d_i}(x, q_i)} \right)^{(g-1)/6} \prod_{i<j} E(q_i, q_j)^{d_i d_j / 24} \ e^{\frac{-\pi i}{6} <\Omega\mathbf{r}, \mathbf{s}> - \frac{2\pi i}{3} <\mathbf{r}, K^x>}. \tag{3.2.12}$$

The following properties of τ_B generalize the ones in [7] where only double poles of Q were considered:

• The expression (3.2.12) for τ_B does not depend on the point x although it seems that it does [38].
• Let the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z})$ denote symplectic transformations of the Torelli marking in $H_1(\mathcal{C})$. Then the tau-function transforms as

$$\tau_B \to \epsilon \det(C\Omega + D)\ \tau_B, \tag{3.2.13}$$

where $\epsilon^{48} = 1$; Ω is a Period matrix of the base curve [5], [35].
• The expression

$$\tau_B^{48} \prod_{k=1}^{n} (d\zeta_k(\tilde{z}_k))^4 \tag{3.2.14}$$

is invariant under the choice of local parameters ζ_k near $\tilde{z}_k$ [5].
• The function τ_B satisfies the following homogeneity property [26], [31]:

$$\tau_B(\mathcal{C}, \kappa Q) = \kappa^{\epsilon} \tau_B(\mathcal{C}, Q), \quad \epsilon = \frac{1}{48} \sum_{d_i \neq -2} \frac{d_i(d_i + 4)}{d_i + 2}. \tag{3.2.15}$$

• The expression for τ_B depends on the choice of the first zero x_1 and on the integration paths between x_1 and poles $\tilde{z}_i$ which are chosen in the complement of the tree graph $\hat{G}$; the change of the graph $\hat{G}$ within the fundamental polygon $\hat{\mathcal{C}}_0$ affect the coordinates ζ_i near $\tilde{z}_i$ by a factor of the form

$$\exp\left\{2\pi i \sum_{j,k} n_{jk} \frac{\tilde{r}_j}{\tilde{r}_k}\right\}, \tag{3.2.16}$$

where n_{jk} is a matrix of integers [26].

3.2.3 Differential equations

Consider the Bergman bidifferential $B^+(x, y)$ (3.1.25), which is just a pullback of the bidifferntial $B(x, y)$ to $\hat{\mathcal{C}}$, and its regularization near the diagonal by the differential $v(x)$:

$$B^+_{reg}(x, x) = \left(B^+(x, y) - \frac{v(x)v(y)}{(\int_x^y v)^2}\right)\Bigg|_{y=x}. \tag{3.2.17}$$

The differential equations for τ_B with respect to the coordinates (3.1.15) on $\mathcal{Q}_{g,m}[\mathbf{k}]$ are given by the following theorem:

Theorem 3.2.1. *Bergman tau-function τ_B defined by (3.2.12) satisfies the following system of equations on $\mathcal{Q}_{g,m}[\mathbf{k}]$:*

$$\frac{\partial \log \tau_B}{\partial A_j} = \frac{1}{4\pi i} \oint_{b_j^-} \frac{B^+_{reg}}{v}, \quad \frac{\partial \log \tau_B}{\partial B_j} = -\frac{1}{4\pi i} \oint_{a_j^-} \frac{B^+_{reg}}{v}, \tag{3.2.18}$$

for $j = 1, ..., g^-$

$$\frac{\partial \log \tau_B}{\partial\, (2\pi i \tilde{r}_k)} = -\frac{1}{4\pi i} \oint_{\kappa_k^-} \left(\frac{B^+_{reg}}{v} + \frac{1}{12\tilde{r}_k^2} v\right), \tag{3.2.19}$$

for $k = 1, ..., n$.

$$\frac{\partial \log \tau_B}{\partial\,(2\pi i r_k)} = -\frac{1}{4\pi i}\oint_{\kappa_k^-} \frac{B^+_{reg}}{v}, \tag{3.2.20}$$

for $k = n+1, ..., m$. Here $\tilde{r}_k$ are residues near simple poles, while r_k are residues near higher order poles.

This is a combination of the results outlined in [7] and [26]. While variational formulas (3.2.19) with respect to residues near simple poles were recently present in [7], the variations with respect to the periods A_j, B_j and residues near higher order poles could be obtained following the proof of Proposition 3.1.1, by symmetrizing the variational formulas for τ_B on moduli spaces of meromorphic Abelian differentials, studied in [26].

Remark 3.2.2. Local analysis shows that near $\tilde{z}_j^{(1)}$ and $\tilde{z}_j^{(2)}$ the expression $\frac{B^+_{reg}}{v}$ gain simple poles. Then the addition appearing in (3.2.19) regularize the integrand at the endpoints of the integration path. This issue does not emerge in case of higher order poles with $k_j \geq 2$, where $\frac{B^+_{reg}}{v}$ gain a zero of order $k_j - 2$ and equations (3.2.20) are valid.

Remark 3.2.3. Quadratic differential B^+_{reg} also admits the following representation in terms of projective connections:

$$B^+_{reg} = \frac{S^+_B - S_v}{6}. \tag{3.2.21}$$

Here S^+_B is the Bergman projective connection appearing in (3.1.33), S_v is Schwarzian projective connection defined by

$$S_v = \Big(\frac{v'}{v}\Big)' - \frac{1}{2}\Big(\frac{v'}{v}\Big)^2, \tag{3.2.22}$$

where $v' = (v/d\hat{\xi})'$ for any local coordinate $\hat{\xi}$ on $\hat{\mathcal{C}}$. Then the formulas (3.1.42) manifest compatibility conditions for differential equations of τ_B. Namely, if $s_i \neq s_j$ are two cycles in H_- and $\mathcal{P}_{s_i}, \mathcal{P}_{s_j}$ are corresponding homology coordinates, then formulas (3.1.42) imply

$$\frac{\partial^2 \log \tau_B}{\partial \mathcal{P}_{s_i} \partial \mathcal{P}_{s_j}} = -\frac{1}{16\pi^2}\oint_{s_i^*}\oint_{s_j^*} \frac{B^+(x,t)^2}{v(x)v(t)}, \tag{3.2.23}$$

which is symmetric under the exchange of derivatives.

3.3 Spaces $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ of generalized SL(2) spectral covers

We introduce a Riemann surface $\mathcal{C}$ of genus g, with m marked points $z_1, ..., z_m$ and associated positive multiplicities $k_1, ..., k_m$. The Higgs bundle on $\mathcal{C}$ is a pair (E, Φ), where E is a holomorphic vector bundle and Φ (the *Higgs field*) is holomorphic (or more generally meromorphic) Ad_E-valued 1-form on $\mathcal{C}$.

Consider a meromorphic $SL(2)$ Higgs field Φ with poles at z_j's of the corresponding orders k_j, $j = 1, ..., m$. We also assume a generic form of the singular parts of Φ near these poles. The *generalized spectral cover* $\hat{\mathcal{C}}$ defined as a locus in $T^*\mathcal{C}$ by the equation $\det(\Phi - v\mathrm{Id}) = 0$, which can be written as

$$v^2 = Q, \tag{3.3.1}$$

with Q being a quadratic differential with simple zeroes and poles at z_j of order $2k_j$ due to the genericity assumption.

For a fixed base $\mathcal{C}$ and positions of marked points we introduce the moduli space $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ of quadratic differentials with simple zeroes and poles at marked points of associate orders $\mathbf{k} = (2k_1, ..., 2k_m)$. The definition of covering surface $\hat{\mathcal{C}}$ of genus $\hat{g} = 4g - 3 + \sum_{j=1}^{m} k_j$, projection π and involution μ are in accordance with the Section 3.1. Differential v has $r = 4g - 4 + 2\sum_{j=1}^{m} k_j$ double zeroes at branch points x_i and $2m$ poles at the preimages $\pi^{-1}(z_j) = \{z_j^{(1)}, z_j^{(2)}\}$ of orders k_j. Denote by χ_j a local coordinate on $\mathcal{C}$ near z_j. We can also use χ_j as local coordinate near both $z_j^{(1)}$ and $z_j^{(2)}$. Consider the singular parts of v near $z_j^{(1)}$:

$$v(\chi_j(x))\big|_{x \to z_j^{(1)}} = \left(\frac{C_j^{k_j}}{\chi_j^{k_j}} + \frac{C_j^{k_j - 1}}{\chi_j^{k_j - 1}} + ... + \frac{C_j^1}{\chi_j} + O(1) \right) d\chi_j. \tag{3.3.2}$$

As v is skew-symmetric under the involution and $\mu(z_j^{(1)}) = z_j^{(2)}$, we have that near a point $z_j^{(2)}$ using the same coordinate χ_j we have the following expansion:

$$v(\chi_j(x))\Big|_{x \to z_j^{(2)}} = \left(-\frac{C_j^{k_j}}{\chi_j^{k_j}} - \frac{C_j^{k_j - 1}}{\chi_j^{k_j - 1}} - ... - \frac{C_j^1}{\chi_j} + O(1) \right) d\chi_j. \tag{3.3.3}$$

The dimension of $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ consists of the sums of dimensions of meromorphic and holomorphic parts of a quadratic differential Q which equals

$$\dim \mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}] = 2\sum_{j=1}^{m} k_j + (3g - 3). \tag{3.3.4}$$

We introduce the following set of local coordinates on the moduli space $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$:

$$\Big\{ \{A_\alpha\}_{\alpha=1}^{3g-3+\sum_{j=1}^{m} k_j}, \{C_j^l\},\ j = 1, ..., m,\ l = 1, ..., k_j \Big\}. \tag{3.3.5}$$

while C_j^l are coefficients of singular parts of v near $z_j^{(1)}$ (or $z_j^{(2)}$), A_α are integrals over skew-symmetric part of the a-cycles on $\hat{\mathcal{C}}$ (defined explicitly in (3.1.11-3.1.12))

$$A_\alpha = \oint_{a_\alpha^-} v. \tag{3.3.6}$$

Remark 3.3.1. Moduli C_j^1 are coordinate-independent residues which coincide with the coordinates r_j on the space $\mathcal{Q}_{g,m}[\mathbf{k}]$ defined by (3.1.15). Moduli $C_j^l,\ l \geq 2$ clearly depend on the choice of local coordinates χ_j. Integrals A_α depend on the choices of Torelli marking on the base surface $\mathcal{C}$ and symplectic basis in $H_-(\hat{\mathcal{C}})$.

3.4 Variational formulas on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$

The assumption that moduli of the base curve $\mathcal{C}$ are kept fixed allows us to well-define the variations on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ for any Abelian differential associated with spectral curve $\hat{\mathcal{C}}$. For any fixed local chart D and corresponding local coordinate ξ on $\mathcal{C}$, we can lift D to $\hat{\mathcal{C}}$ via π^{-1} and use ξ as a local coordinate on each connected component of $\pi^{-1}(D)$ outside the branch points. Then if p_i is any coordinate on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ from the list (3.3.5), the variation of an Abelian differential $w = f(\xi)d\xi$ is defined by

$$\frac{dw}{dp_i} = \frac{df(\xi)}{dp_i} d\xi, \tag{3.4.1}$$

assuming that the coordinate ξ does not depend on p_i. Such definition is clearly independent of the choice ξ on D.

In order to compute variations of the differential v with respect to moduli (3.3.5), we introduce additional (meromorphic) Abelian differentials attributed to the double cover. Let $\left\{\{\hat{w}_j^l\}_{l=2}^{k_j}\right\}_{j=1}^m$ denote the second-kind differentials on $\hat{\mathcal{C}}$ with prescribed singular part at $z_j^{(1)}$ and normalized over a-cycles of the homology basis (3.1.6). That is,

$$\hat{w}_j^l(x) = \Big(\frac{1}{\chi_j^l} + O(1)\Big)d\chi_j, \qquad x \to z_j^{(1)}. \tag{3.4.2}$$

By $\{\hat{\eta}_j\}_{j=1}^m$ we denote third-kind differentials on $\hat{\mathcal{C}}$, normalized over a-cycles, with simple poles at $z_j^{(1)}$ and x_1 with residues $+1$ and -1, respectively. Now put

$$w_j^{l-} := \hat{w}_j^l - \mu^*\hat{w}_j^l, \tag{3.4.3}$$

$$\eta_j^- := \hat{\eta}_j - \mu^*\hat{\eta}_j. \tag{3.4.4}$$

We will call w_j^{l-} the normalized *Prym second-kind differential* and η_j^- the normalized *Prym third-kind differential.* The following lemma outlines the properties of defined objects

Lemma 3.4.1. *The differentials w_j^{l-} and η_j^- have the following properties:*

(i)

$$\mu^* w_j^{l-} = -w_j^{l-}, \tag{3.4.5}$$

(ii)

$$\mu^*\eta_j^- = -\eta_j^-, \tag{3.4.6}$$

(iii)

$$\oint_{a_k^+} w_j^{l-} = \oint_{a_k^-} w_j^{l-} = \oint_{b_k^+} w_j^{l-} = 0, \tag{3.4.7}$$

(iv)

$$\oint_{a_k^+} \eta_j^- = \oint_{a_k^-} \eta_j^- = \oint_{b_k^+} \eta_j^- = 0, \tag{3.4.8}$$

(v) the differential w_j^{l-} has the following singular parts:

$$w_j^{l-}(x) = \left(\frac{1}{\chi_j^l} + O(1)\right)d\chi_j, \qquad x \to z_j^{(1)}, \tag{3.4.9}$$

$$w_j^{l-}(x) = -\left(\frac{1}{\chi_j^l} + O(1)\right)d\chi_j, \qquad x \to z_j^{(2)} \tag{3.4.10}$$

and it is holomorphic elsewhere,

(vi) the differential η_j^- is of third kind with simple poles at $z_j^{(1)}$ and $z_j^{(2)}$ with residues $+1$ and -1, respectively,

(vii)

$$\oint_{b_k^-} w_j^{l-} = 2\pi i \frac{u_k^{(l-2)-}(z_j^{(1)})}{(l-1)!}, \qquad \oint_{b_k^-} \eta_j^- = \pi i \int_{z_j^{(2)}}^{z_j^{(1)}} u_k^-, \tag{3.4.11}$$

where $u_k^{(l-2)-}(z_j^{(1)})$ stands for $\frac{d^{l-2}}{d\chi_j^{l-2}}\left(\frac{u_k^-}{d\chi_j}\right)\Big|_{\chi_j=0}$.

Proof. (i)-(vi) follows from the definitions and the facts that $\mu(z_j^{(1)}) = z_j^{(2)},\ \mu(x_1) = x_1$.

To prove (vii) we apply the Riemann Bilinear Identity (later RBI) to w_j^{l-} with u_k^- (η_j^- with u_k^-) by integrating them over the cycles in $H_-(\hat{\mathcal{C}})$ in the following way: write

$$\oint_{b_k^-} w_j^{l-} = \sum_{j=1}^{g^-} \Big[\oint_{b_j^-} w_j^{l-} \oint_{a_j^-} u_k^- - \oint_{a_j^-} w_j^{l-} \oint_{b_j^-} u_k^- \Big]. \tag{3.4.12}$$

We can assume that the boundary of the universal cover $\hat{\mathcal{C}}_0$ of $\hat{\mathcal{C}}$ is invariant under the involution μ. Then we can extend this sum by adding integrals over $H_+(\hat{\mathcal{C}})$. The integrands are skew-symmetric with respect to involution, so their integrals over the cycles in $H_+(\hat{\mathcal{C}})$ give zero contribution. Then by the Stokes' theorem (3.4.12) could be represented as the sum over residues near poles inside $\hat{\mathcal{C}}_0$:

$$\frac{1}{2}(2\pi i) \underset{(z_j^{(1)}, z_j^{(2)})}{res} \left(w_j^{l-} \int_{p_0}^{x} u_k^- \right), \tag{3.4.13}$$

where the factor $1/2$ is due to the intersection index $a_i^- \circ b_k^- = \frac{1}{2}\delta_{ik}$; p_0 is a reference point. This expression does not depend on the choice of the point p_0 since the difference between two choices is the sum of the residues of w_j^{l-} which is clearly zero. For convenience, we put $p_0 = x_1$. Then skew-symmetry of both differentials in (3.4.13) implies that the residues at $z_j^{(1)}$ and $z_j^{(2)}$ are equal. Their computation leads to the result. The second formula is proven by analogy, noticing that

$$2 \int_{x_1}^{z_j^{(1)}} u_k^- = \int_{z_j^{(2)}}^{z_j^{(1)}} u_k^-. \tag{3.4.14}$$

□

Proposition 3.4.1. *The following variational formulas with respect to the coordinates (3.3.5) on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ hold:*

$$\frac{\partial v}{\partial A_\alpha} = u_\alpha^-, \tag{3.4.15}$$

$$\frac{\partial v}{\partial C_j^1} = \eta_j^-, \tag{3.4.16}$$

$$\frac{\partial v}{\partial C_j^l} = w_j^{l-},\ l \geq 2, \tag{3.4.17}$$

here u_α^- is a normalized holomorphic Prym differential (3.1.20), η_j^- and w_j^{l-} are normalized Prym differentials of second kind (3.4.3) and third kind (3.4.4).

Proof. The proof follows the idea in [4]. Consider an expansion of the Abelian differential v near a branch point x_i on $\hat{\mathcal{C}}$. The coordinate could be taken as $(\xi - \xi_i)^{1/2}$. While ξ is moduli-independent, $\xi_i = \xi(\pi(x_i))$ changes when $\hat{\mathcal{C}}$ varies. Thus, the dependence of ξ_i on moduli should be taken into account. v has a double zero at each x_i, then locally it can be written as follows:

$$v(\xi) = (\xi - \xi_j)(a_0 + a_1(\xi - \xi_j)^{1/2} + ...)d(\xi - \xi_j)^{1/2} = \frac{1}{2}(a_0(\xi - \xi_j)^{1/2} + a_1(\xi - \xi_j) + ...)d\xi. \tag{3.4.18}$$

Performing the differentiation by the rule (3.4.1) with respect to any coordinate p from the list (3.3.5) we obtain

$$\frac{\partial v}{\partial p} = \frac{1}{4}\Bigg(-\frac{a_0(\xi_j)'_p}{(\xi-\xi_j)^{1/2}} + O(1)\Bigg)d\xi = -\frac{1}{2}\Bigg(a_0(\xi_j)'_p + O(1)\Bigg)d(\xi-\xi_j)^{1/2}. \tag{3.4.19}$$

From this formula it is clear that $\frac{\partial v}{\partial p}$ is holomorphic at the branch points.

The differential $\frac{\partial v}{\partial A_\alpha}$ also holomorphic at all poles z_j since the singular parts of v do not depend on the moduli A_α. Moreover, all periods of $\frac{\partial v}{\partial A_\alpha}$ vanish except for the period over a_α^-, which is 1. Take the difference $w := \frac{\partial v}{\partial A_\alpha} - u_\alpha^-$. The differential w is holomorphic, its a^- periods vanish by construction and a^+ vanish due to skew-symmetry of v and u_α^-. Thus, we have that $w \equiv 0$, and so (3.4.15) holds.

Consider $\frac{\partial v}{\partial C_j^l}$. Its singular part coincides with the one of w_j^{l-}. Also its a^--periods vanish since C_j^k are independent of $\{A_\alpha\}_{\alpha=1}^{g^-}$. Similarly to the previous argument we obtain (3.4.17).

Finally, $\frac{\partial v}{\partial C_j^1}$ equals to the Prym third-kind differential η_j^- again due to the coincidence of singular parts. □

3.4.1 Variations of Prym matrix

In this section we discuss variations of the Prym matrix Ω^- on the spaces of $SL(2)$ spectral covers. While the derivatives with respect to the a^-- periods of v reproduce the Prym version of Donagi-Markman cubic [13], variations with respect to coordinates spanning the singular part of v extend this result to meromorphic case and involve Prym meromorphic differentials.

Theorem 3.4.1. *The variations of Prym matrix Ω^- with respect to the coordinates (3.3.5) on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ take the following form:*

$$\frac{\partial \Omega_{\alpha\beta}^-}{\partial A_\gamma} = -\pi i \sum_{i=1}^{r} \underset{x_i}{res}\Bigg(\frac{u_\alpha^- u_\beta^- u_\gamma^-}{d\xi\, d(v/d\xi)}\Bigg), \tag{3.4.20}$$

$$\frac{\partial \Omega_{\alpha\beta}^-}{\partial C_j^1} = -\pi i \sum_{i=1}^{r} \underset{x_i}{res}\Bigg(\frac{u_\alpha^- u_\beta^- \eta_j^-}{d\xi\, d(v/d\xi)}\Bigg), \tag{3.4.21}$$

$$\frac{\partial \Omega_{\alpha\beta}^-}{\partial C_j^l} = -\pi i \sum_{i=1}^{r} \underset{x_i}{res}\Bigg(\frac{u_\alpha^- u_\beta^- w_j^{l-}}{d\xi\, d(v/d\xi)}\Bigg), \tag{3.4.22}$$

where $r = 4g-4+2\sum_{j=1}^m k_j$ is a number of branch points (zeroes of v). ξ denotes a local coordinate on $\mathcal{C}$ near a branch point x_i. The above formulas do not depend on the choice of ξ.

Proof. Let us proof (3.4.20). On the subspace $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}] \subset \mathcal{Q}_{g,m}[\mathbf{k}]$ the coordinates $\{B_i\}_{i=1}^{g^-}$ become dependent functions of $\{A_i\}_{i=1}^{g^-}$. Thus, we compute the derivative of Prym matrix applying the chain rule as follows:

$$\frac{d\Omega_{\alpha\beta}^-}{dA_\gamma} = \frac{\partial \Omega_{\alpha\beta}^-}{\partial A_\gamma}\Big|_{\{B_i\}_{i=1}^{g^-}=const} + \sum_{i=1}^{g^-} \frac{\partial \Omega_{\alpha\beta}^-}{\partial B_i}\frac{\partial B_i}{\partial A_\gamma}. \tag{3.4.23}$$

Using the variational formulas (3.1.39) and (3.4.15) we further rewrite this expression as

$$\frac{1}{2}\sum_{i=1}^{g^-}\Bigg(-\oint_{2b_i^-}\frac{u_\alpha^- u_\beta^-}{v}\oint_{a_i^-}u_\gamma^- + \oint_{2a_i^-}\frac{u_\alpha^- u_\beta^-}{v}\oint_{b_i^-}u_\gamma^-\Bigg). \tag{3.4.24}$$

Similarly to Lemma 3.4.1, this sum could be represented as a sum of residues inside the fundamental polygon $\hat{\mathcal{C}}_0$ of $\hat{\mathcal{C}}$:

$$- \pi i \sum_{i=1}^{r} \underset{x_i}{res} \bigg(\frac{u_\alpha^- u_\beta^-}{v} \int_{p_0}^{x} u_\gamma^- \bigg), \tag{3.4.25}$$

where p_0 is a reference point. To simplify the residues at first notice that

$$\underset{x_i}{res} \bigg(\frac{u_\alpha^- u_\beta^-}{v} \int_{p_0}^{x} u_\gamma^- \bigg) = \underset{x_i}{res} \bigg(\frac{u_\alpha^- u_\beta^-}{v} \int_{x_i}^{x} u_\gamma^- \bigg), \tag{3.4.26}$$

(taking the difference of the above expressions it equals $\underset{x_i}{res}\big(\frac{u_\alpha^- u_\beta^-}{v}\big)$ up to an explicit constant C_i, then the skew-symmetry of the differential $\frac{u_\alpha^- u_\beta^-}{v}$ implies it has vanishing residue). We introduce a local coordinate $\hat{\xi}$ near a branch point on $\hat{\mathcal{C}}$. $\hat{\xi}$ can be chosen such that $\hat{\xi}(x^\mu) = -\hat{\xi}(x)$. Then for u^-, v being skew-symmetric it implies that they have expansion by even powers of $\hat{\xi}$. Moreover, v has double zeroes at branch points. Then one has:

$$u_\alpha^- = ((u_0)_\alpha + O(\hat{\xi}^2))d\hat{\xi}, \tag{3.4.27}$$

$$v = \hat{\xi}^2 (v_0 + O(\hat{\xi}^2))d\hat{\xi}. \tag{3.4.28}$$

And

$$\frac{u_\alpha^- u_\beta^-}{v} = \Big(\frac{(u_0)_\alpha (u_0)_\beta}{v_0} \frac{1}{\hat{\xi}^2} + O(1) \Big) d\hat{\xi}, \tag{3.4.29}$$

$$\int_{x_i}^{x} u_\gamma^- = \Big((u_0)_\gamma \hat{\xi} + O(\hat{\xi}^3) \Big). \tag{3.4.30}$$

Therefore, the sum (3.4.25) becomes

$$- \pi i \sum_{i=1}^{r} \frac{(u_0)_\alpha (u_0)_\beta (u_0)_\gamma}{v_0}, \tag{3.4.31}$$

which could be rewritten in invariant form (3.4.20) (there $\xi - \xi_i = \hat{\xi}^2$, and the expression $d\xi d(v/d\xi)$ is proportional exactly to $v_0 \hat{\xi}(d\hat{\xi})^2$).

The formula (3.4.21) could be proven in a similar way. Applying the chain rule, we write:

$$\frac{d\Omega_{\alpha\beta}^-}{dC_j^1} = \frac{\partial \Omega_{\alpha\beta}^-}{\partial C_j^1}\Big|_{\{A_i, B_i\}_{i=1}^{g^-} = const} + \sum_{i=1}^{g^-} \bigg(\frac{\partial \Omega_{\alpha\beta}^-}{\partial A_i} \frac{\partial A_i}{\partial C_j^1} + \frac{\partial \Omega_{\alpha\beta}^-}{\partial B_i} \frac{\partial B_i}{\partial C_j^1} \bigg). \tag{3.4.32}$$

According to (3.3.2) and (3.1.39)

$$\frac{\partial \Omega_{\alpha\beta}^-}{\partial C_j^1} = (2\pi i) \, \frac{1}{2} \int_{2\kappa_j^-} \frac{u_\alpha^- u_\beta^-}{v} = \pi i \int_{z_j^{(1)}}^{z_j^{(2)}} \frac{u_\alpha^- u_\beta^-}{v}. \tag{3.4.33}$$

Then using variation formulae (3.1.39) and (3.4.16)

$$\frac{d\Omega_{\alpha\beta}^-}{dC_j^1} = \pi i \int_{z_j^{(1)}}^{z_j^{(2)}} \frac{u_\alpha^- u_\beta^-}{v} + \frac{1}{2} \sum_{i=1}^{g^-} \bigg(- \oint_{2b_i^-} \frac{u_\alpha^- u_\beta^-}{v} \oint_{a_i^-} \eta_j^- + \oint_{2a_i^-} \frac{u_\alpha^- u_\beta^-}{v} \oint_{b_i^-} \eta_j^- \bigg). \tag{3.4.34}$$

With the help of the RBI, we obtain the sum over residues at the branch points plus the residues at $\{z_j^{(1)}, z_j^{(2)}\}$ which conveniently cancel with the first integral.

Finally, to prove (3.4.22), we use variations (3.1.39) with (3.4.17) to write

$$\frac{d\Omega^-_{\alpha\beta}}{dC^l_j} = \sum_{i=1}^{g^-}\left(\frac{\partial\Omega^-_{\alpha\beta}}{\partial A_i}\frac{\partial A_i}{\partial C^l_j} + \frac{\partial\Omega^-_{\alpha\beta}}{\partial B_i}\frac{\partial B_i}{\partial C^l_j}\right) = \tag{3.4.35}$$

$$= \frac{1}{2}\sum_{i=1}^{g^-}\left(-\oint_{2b_i^-}\frac{u^-_\alpha u^-_\beta}{v}\oint_{a_i^-} w_j^{l-} + \oint_{2a_i^-}\frac{u^-_\alpha u^-_\beta}{v}\oint_{b_i^-} w_j^{l-}\right) \tag{3.4.36}$$

which due to the RBI is equal to the required expression (notice that the residues at the poles $\{z_j^{(1)}, z_j^{(2)}\}$ of w_j^{l-} vanish). □

Remark 3.4.1. By assumption the base curve $\mathcal{C}$ is kept fixed, so variations on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ of its complex structure coded by Period matrix Ω^+ must be zero. Consider, for example $\frac{\partial\Omega^+_{\alpha\beta}}{\partial A_\gamma}$. Then, similarly to the derivation of (3.4.20), applying variational formulas (3.1.39) it equals to

$$\frac{\partial\Omega^+_{\alpha\beta}}{\partial A_\gamma} = -\pi i\sum_{i=1}^{n}\underset{x_i}{res}\left(\frac{u^+_\alpha u^+_\beta}{v}\int_{p_0}^{x} u^-_\gamma\right). \tag{3.4.37}$$

The local analysis shows that each differential u^+ gains simple zeros at branch points x_i when being lifted from the base curve $\mathcal{C}$. Thus, the residues over x_i vanish.

The difference between the dimensions of $\mathcal{Q}_{g,m}[\mathbf{k}]$ and moduli space of curves $\mathcal{M}_{g,m}(\mathcal{C})$

$$dim(\mathcal{Q}_{g,m}[\mathbf{k}]) - dim(\mathcal{M}_{g,m}(\mathcal{C})) = (6g-6+m+2\sum_{j=1}^{m} k_j) - (3g-3+m) = 3g-3+2\sum_{j=1}^{m} k_j \tag{3.4.38}$$

implies the existence of $3g-3+2\sum_{j=1}^m k_j$ linearly independent fields on $\mathcal{Q}_{g,m}[\mathbf{k}]$ that preserve a complex structure of $\mathcal{C}$ and positions of poles. It is instructive to find these vector fields.

Proposition 3.4.2. *The following linearly independent vector fields defined on* $\mathcal{Q}_{g,m}[\mathbf{k}]$

$$V_{A_\gamma} = \frac{\partial}{\partial A_\gamma} + \sum_{i=1}^{g^-}\Omega^-_{\gamma i}\frac{\partial}{\partial B_i}, \qquad \gamma = 1, ..., g^-, \tag{3.4.39}$$

$$V_{C^1_j} = \frac{\partial}{\partial(2\pi r_j)} + \pi i\sum_{i=1}^{g^-}\left(\int_{z_j^{(2)}}^{z_j^{(1)}} u^-_i\right)\frac{\partial}{\partial B_i}, \qquad j = 1, ..., m, \tag{3.4.40}$$

$$V_{C^l_j} = 2\pi i\sum_{i=1}^{g^-}\left(\frac{u_i^{(l-2)-}(z_j^{(1)})}{(l-1)!}\right)\frac{\partial}{\partial B_i}, \qquad j = 1, ..., m, \quad l = 2, ..., k_j \tag{3.4.41}$$

preserve the moduli of $\mathcal{C}$ *and positions of poles.*

Proof. Consider a perturbation of the original quadratic differential $Q^\epsilon = Q + \epsilon\tilde{Q}$, where $\tilde{Q}$ is an arbitrary quadratic differential on $\mathcal{C}$ with simple zeroes and poles at z_j of even order no greater than $2k_j$. The differential Q^ϵ also has simple zeroes for ϵ small enough, is defined on the same Riemann surface and has the same set of poles as Q. Thus, the vector field $\frac{d}{d\epsilon}$ does not change the complex structure of $\mathcal{C}$ and positions of poles. Having $(v^\epsilon)^2 = Q^\epsilon$ we observe that

$$v^\epsilon = v + \epsilon\frac{\tilde{Q}}{2v} + O(\epsilon^2), \tag{3.4.42}$$

leading to

$$\frac{\partial}{\partial\epsilon}\left(\oint_{s^\epsilon} v^\epsilon\right)\Bigg|_{\epsilon=0} = \oint_s \frac{\tilde{Q}}{2v}, \quad s^\epsilon \in H_-, \tag{3.4.43}$$

since the derivative commutes with the integral over a closed contour. Expressing vector field $\frac{d}{d\epsilon}$ via the coordinates (3.1.15) one has:

$$\frac{d}{d\epsilon} = \sum_{j=1}^{m}\left(\oint_{t_j^-}\frac{\tilde{Q}}{2v}\right)\frac{\partial}{\partial(2\pi i r_j)} + \sum_{i=1}^{g^-}\left[\left(\oint_{a_i^-}\frac{\tilde{Q}}{2v}\right)\frac{\partial}{\partial A_i} + \left(\oint_{b_i^-}\frac{\tilde{Q}}{2v}\right)\frac{\partial}{\partial B_i}\right]. \tag{3.4.44}$$

Then taking $\tilde{Q}$ to be equal either to $2vu_\gamma, 2v\eta_j^-$ or $2vw_j^{l-}$ and applying formulas (3.4.11) we obtain the result. Notice that (3.4.39-3.4.41) are exactly the expressions that appear in the proof of Theorem 3.4.1 when performing the chain rule. The fields are independent since on the submanifold $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ they are equal to the derivatives over independent coordinates.

□

Remark 3.4.2. In the holomorphic case, when $m = 0$, one has the isomorphism between holomorphic quadratic differentials $\mathcal{Q}_g$ and the cotangent bundle of the moduli space of curves $T^*\mathcal{M}_g$. It follows from the proposition that $V_{A_\gamma}, \gamma = 1, ..., 3g-3$ act trivially on the coordinates $\{q_i\}_{i=1}^{3g-3}$, represented by the entries of Ω^+, of the canonical Darboux coordinate set $\{p_i, q_i\}_{i=1}^{3g-3}$ on $T^*\mathcal{M}_g$. Thus, V_{A_γ} span the vertical bundle $V_*(T^*\mathcal{M})$.

3.4.2 Variations of Prym differentials

Similarly to (3.4.1) we define on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ derivatives for any coordinate p_i in the list (3.3.5) as follows

$$\frac{\partial B^-(x,y)}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{B^-(x,y)}{d\xi(x)d\xi(y)}\right)d\xi(x)d\xi(y), \tag{3.4.45}$$

where ξ is any fixed coordinate on $\mathcal{C}$ lifted to the covering surface via π^{-1}. The derivatives of differentials depending on the point (points) on $\hat{\mathcal{C}}$ should be treated in a greater accuracy, since the latter is deforming. Notice that on the space $\mathcal{Q}_{g,m}[\mathbf{k}]$ the differentiation is performed according to the rule (3.1.41), when the flat coordinates $z(x)$ and $z(y)$ are kept constant, while on the subspace $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ they become moduli-dependent. Restricting the variational formulas from $\mathcal{Q}_{g,m}[\mathbf{k}]$ to $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ we obtain the following:

Theorem 3.4.2. *The variations of Prym bidifferential $B^-(x,y)$ with respect to the coordinates (3.3.5) on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ take the following form:*

$$\frac{\partial B^-(x,y)}{\partial A_\gamma} = -\frac{1}{2}\sum_{i=1}^{r}\underset{x_i}{res}\left(\frac{u_\gamma^-(t)B^-(x,t)B^-(t,y)}{d\xi\, d(v/d\xi)}\right), \tag{3.4.46}$$

$$\frac{\partial B^-(x,y)}{\partial C_j^1} = -\frac{1}{2}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{\eta_j^-(t)B^-(x,t)B^-(t,y)}{d\xi\, d(v/d\xi)}\right), \tag{3.4.47}$$

$$\frac{\partial B^-(x,y)}{\partial C_j^l} = -\frac{1}{2}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{w_j^{l-}(t)B^-(x,t)B^-(t,y)}{d\xi\, d(v/d\xi)}\right), \tag{3.4.48}$$

ξ denotes a local coordinate on $\mathcal{C}$ near a branch point x_i. The above formulas do not depend on the choice of ξ.

Proof. Let us prove (3.4.46). Denote by $b^-(x,y) := \frac{B^-(x,y)}{v(x)v(y)}$. Then one has

$$\frac{\partial B^-(x,y)}{\partial A_\gamma}\Big|_{\xi(x),\xi(y)} = \frac{\partial[b^-(x,y)v(x)v(y)]}{\partial A_\gamma}\Big|_{\xi(x),\xi(y)} = \tag{3.4.49}$$

$$= \frac{\partial b^-(x,y)}{\partial A_\gamma}\Big|_{\xi(x),\xi(y)} v(x)v(y) + b^-(x,y)v(y)\frac{\partial v(x)}{\partial A_\gamma}\Big|_{\xi(x)} + b^-(x,y)v(x)\frac{\partial v(y)}{\partial A_\gamma}\Big|_{\xi(y)} = \tag{3.4.50}$$

$$\overset{(3.4.15)}{=} \frac{\partial b^-(x,y)}{\partial A_\gamma}\Big|_{z(x),z(y)} v(x)v(y) + \frac{\partial b^-(x,y)}{\partial z(x)}\frac{\partial z(x)}{\partial A_\gamma}\Big|_{\xi(x),\xi(y)} v(x)v(y) + \frac{\partial b^-(x,y)}{\partial z(y)}\frac{\partial z(y)}{\partial A_\gamma}\Big|_{\xi(x),\xi(y)} v(x)v(y) + \tag{3.4.51}$$

$$+ b^-(x,y)v(y)u_\gamma^-(x) + b^-(x,y)v(x)u_\gamma^-(y) =$$

$$\overset{(3.4.15)}{=} \frac{\partial b^-(x,y)}{\partial A_\gamma}\Big|_{z(x),z(y)} dz(x)dz(y) + \left[\Big(b^-(x,y)\Big)'_{z(x)}\int_{x_1}^x u_\gamma^- + \Big(b^-(x,y)\Big)'_{y(x)}\int_{x_1}^y u_\gamma^-\right] dz(x)dz(y) + \tag{3.4.52}$$

$$+ b^-(x,y)u_\gamma^-(x)dz(y) + b^-(x,y)u_\gamma^-(y)dz(x).$$

To compute the term $\frac{\partial b^-(x,y)}{\partial A_\gamma}\Big|_{z(x),z(y)}$ we, similarly to (3.4.23), apply the chain rule and then variational formulas (3.1.41), (3.4.15) along with the RBI to obtain

$$\frac{\partial b^-(x,y)}{\partial A_\gamma}\Big|_{z(x),z(y)} = -\sum_{t\in int(\hat{C})} \frac{1}{2}\underset{t}{res}\left(b^-(x,t)b^-(t,y)v(t)\int_{p_0}^t u_\gamma^-\right). \tag{3.4.53}$$

To evaluate the residues, introduce the differentials $V(t) = b^-(x,t)b^-(t,y)v(t)$ and $W(t) = u_\gamma^-(t)$. $W(t)$ is holomorphic, while $V(t)$, in addition to second order poles at $\{x_i\}_{i=1}^r$ has four second order poles at $t = x,\ x^\mu$ and at $t = y,\ y^\mu$. As in the proof of Lemma 3.4.1, we can put $p_0 = x_1$. Then residues at the pairs (x, x^μ) and (y, y^μ) are the same due to $V(t), W(t)$ being skew-symmetric under the involution and, thus, their contribution to the sum double. Using the expansion of the Prym differential (3.1.33) the residue at $t = x$ equals

$$\left(b^-(x,y)\int_{x_1}^x u_\gamma^-\right)'_{z(x)}. \tag{3.4.54}$$

The residue at $t = y$ is

$$\left(b^-(x,y)\int_{x_1}^y u_\gamma^-\right)'_{z(y)}. \tag{3.4.55}$$

Then it it easy to see that these terms, multiplied by $dz(x)dz(y)$, cancel the last four terms of the sum (3.4.52). The evaluation of residues at x_i, similarly to the proof of (3.4.20), leads to the result. (3.4.47) and (3.4.48) are obtained by analogy. □

Integrating above formulas over the cycles b^- an using the simple fact that

$$\oint_{b^-_\alpha} B^-(x,t) = 2\pi i \; u^-_\alpha(x) \tag{3.4.56}$$

we derive variational formulas for Prym normalized differentials:

Theorem 3.4.3. *The variations of normalized Prym differentials u^-_α with respect to the coordinates (3.3.5) on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ take the following form:*

$$\frac{\partial u^-_\alpha(x)}{\partial A_\gamma} = -\frac{1}{2}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{u^-_\gamma(t)B^-(x,t)u^-_\alpha(t)}{d\xi \; d(v/d\xi)}\right), \tag{3.4.57}$$

$$\frac{\partial u^-_\alpha(x)}{\partial C^1_j} = -\frac{1}{2}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{\eta^-_j(t)B^-(x,t)u^-_\alpha(t)}{d\xi \; d(v/d\xi)}\right), \tag{3.4.58}$$

$$\frac{\partial u^-_\alpha(x)}{\partial C^l_j} = -\frac{1}{2}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{w^{l-}_j(t)B^-(x,t)u^-_\alpha(t)}{d\xi \; d(v/d\xi)}\right), \tag{3.4.59}$$

ξ denotes a local coordinate on $\mathcal{C}$ near a branch point x_i. The above formulas do not depend on the choice of ξ.

Remark 3.4.3. Analogously one can derive variations of Prym differentials of second and third kinds by noticing that

$$\int_{z^{(2)}_j}^{z^{(1)}_j} B^-(x,t) = \eta^-_j(x), \tag{3.4.60}$$

$$-\frac{1}{l}\underset{t=z^{(1)}_j}{res}\left(\chi_j(t)^{-l}B^-(x,t)\right) = w^{l-}_j(x) \tag{3.4.61}$$

and applying these formulas on both sides of (3.4.46-3.4.48).

Remark 3.4.4. Similarly to Remark 3.4.1, one can show that the variations of $B^+(x,y)$ and u^+_α on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ are zero. There is no surprise since these objects are the pullbacks from the base curve $\mathcal{C}$ which is assumed not to depend on moduli.

3.4.3 Variations of Bergman tau-function

Initially, Bergman tau-function defined by (3.2.12) solves the system of differential equations on the space $\mathcal{Q}_{g,m}[\mathbf{k}]$ with a variable base. Having the base curve $\mathcal{C}$ and positions of poles (z_j) fixed on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$. we define tau-functions on this subspace by the restriction of variational formulas from Theorem 3.2.1.

Let $\{\tilde{C}^1_j\}^n_{j=1}$ denote the local coordinates on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ from the list (3.3.5) corresponding to the residues near simple poles $\{\tilde{z}^{(1)}_j, \tilde{z}^{(2)}_j\}^n_{j=1}$ of the differential v, whereas coordinates $\{C^1_j\}^m_{j=n+1}$ are residues near the higher order poles.

Theorem 3.4.4. *Bergman tau-function τ_B satisfies the following system of differential equations on the space $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$*

$$\frac{\partial \log \tau_B}{\partial A_\gamma} = \frac{5}{432}\sum_{i=1}^r \underset{x_i}{res}\left(\frac{u^-_\gamma}{\int_{x_i}^x v}\right) - \sum_{k=1}^n \frac{1}{48\tilde{r}_k}\int_{\tilde{z}^{(2)}_k}^{\tilde{z}^{(1)}_k} u^-_\gamma, \tag{3.4.62}$$

$$\frac{\partial \log \tau_B}{\partial \tilde{C}_j^1} = \frac{5}{432}\sum_{i=1}^{r} \underset{x_i}{res}\left(\frac{\eta_j^-}{\int_{x_i}^x v}\right) - \sum_{k=1,k\neq j}^{n} \frac{1}{48\tilde{r}_k}\int_{\tilde{z}_k^{(2)}}^{\tilde{z}_k^{(1)}} \eta_j^- - \frac{1}{48\tilde{r}_j}\int_{\tilde{z}_j^{(2)}}^{\tilde{z}_j^{(1)}}\left(\frac{v}{\tilde{r}_j} - \eta_j^-\right), \tag{3.4.63}$$

$j = 1, ..., n,$

$$\frac{\partial \log \tau_B}{\partial C_j^1} = \frac{5}{432}\sum_{i=1}^{r} \underset{x_i}{res}\left(\frac{\eta_j^-}{\int_{x_i}^x v}\right) - \sum_{k=1}^{n} \frac{1}{48\tilde{r}_k}\int_{\tilde{z}_k^{(2)}}^{\tilde{z}_k^{(1)}} \eta_j^-, \tag{3.4.64}$$

$j = n+1, ..., m,$

$$\frac{\partial \log \tau_B}{\partial C_j^l} = \frac{5}{432}\sum_{i=1}^{r} \underset{x_i}{res}\left(\frac{w_j^{l-}}{\int_{x_i}^x v}\right) - \sum_{k=1}^{n} \frac{1}{48\tilde{r}_k}\int_{\tilde{z}_k^{(2)}}^{\tilde{z}_k^{(1)}} w_j^{l-}, \tag{3.4.65}$$

$j = n+1, ..., m, \quad l = 2, ..., k_j - 1.$

$$\frac{\partial \log \tau_B}{\partial C_j^{k_j}} = \frac{5}{432}\sum_{i=1}^{r} \underset{x_i}{res}\left(\frac{w_j^{k_j-}}{\int_{x_i}^x v}\right) - \sum_{k=1}^{n} \frac{1}{48\tilde{r}_k}\int_{\tilde{z}_k^{(2)}}^{\tilde{z}_k^{(1)}} w_j^{k_j-} + \frac{1}{(k_j-1)}\frac{2k_j - k_j^2}{24C_j^{k_j}}, \tag{3.4.66}$$

$j = n+1, ..., m.$

Proof. In parallel to (3.4.20), we apply the chain rule to the equations (3.2.18) and use (3.4.15) with the RBI to have

$$\frac{\partial \log \tau_B}{\partial A_\gamma} = \frac{1}{4}\sum_{t=\{x_i, \tilde{z}_k^{(1)}, \tilde{z}_k^{(2)}\}} \underset{t}{res}\left(\frac{B_{reg}^+}{v}\int_{x_1}^x u_\gamma^-\right). \tag{3.4.67}$$

Notice that in addition to the residues at branch points $(x_i)_{i=1}^r$ we have residues at simple poles of v at $(\tilde{z}_k^{(1)}, \tilde{z}_k^{(2)})_{k=1}^n$. To compute the residue near x_i we represent B_{reg}^+ as the difference of projective connections:

$$B_{reg}^+ = \frac{S_B^+ - S_v}{6}. \tag{3.4.68}$$

While S_B^+ is the Bergman projective connection of (3.1.33), S_v is Schwarzian projective connection defined by

$$S_v = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2, \tag{3.4.69}$$

where $v' = (v/d\hat{\xi})'$ for any local coordinate $\hat{\xi}$ on $\hat{\mathcal{C}}$. Recall that by (3.1.34) we have $S_B^+(x) = \hat{S}_B(x) + 6\mu_y^*\hat{B}(x,y)|_{x=y}$. In the neighborhood of x_i we chose a local coordinate $\hat{\xi}$ such that $v = \hat{\xi}^2 d\hat{\xi}$. Then near x_i we have

$$\frac{S_v}{6v} = -\frac{2}{3\hat{\xi}^4}d\hat{\xi}. \tag{3.4.70}$$

Moreover, we can choose $\hat{\xi}$ in such a way that $\hat{\xi}(\mu(x)) = -\hat{\xi}(x)$. Therefore, near x_i we also have

$$\hat{B}(x,\mu(x)) = \left[\frac{1}{(\hat{\xi}(x) - \hat{\xi}(\mu(x)))^2} + \frac{1}{6}\hat{S}_B(\hat{\xi}(x)) + O\Big((\hat{\xi}(x) - \hat{\xi}(\mu(x)))^2\Big)\right] d\hat{\xi}(x)d\hat{\xi}(\mu(x)) = \tag{3.4.71}$$

$$= \left[-\frac{1}{4\hat{\xi}^2} - \frac{1}{6}\hat{S}_B(\hat{\xi}) + O(\hat{\xi}^2)\right](d\hat{\xi})^2, \tag{3.4.72}$$

so that

$$\frac{S_B^+}{6v} = \left[-\frac{1}{4\hat{\xi}^4} + O(1) \right] d\hat{\xi} \tag{3.4.73}$$

and

$$\frac{B_{reg}^+}{v} = \left[\frac{5}{12\xi^4} + O(1) \right] d\hat{\xi}, \tag{3.4.74}$$

leading to

$$\operatorname*{res}_{x_i}\left(\frac{B_{reg}^+}{v}\int_{x_1}^{x} u_\gamma^-\right) = \operatorname*{res}_{x_i}\left(\frac{5d\hat{\xi}}{12\hat{\xi}^4}\int_{x_1}^{x} u_\gamma^-\right) = \frac{5}{12}\frac{1}{3!}\left(\frac{u_\gamma^-}{d\hat{\xi}}\right)''(x_i) = \frac{5}{108}\operatorname*{res}_{x_i}\left(\frac{u_\gamma^-}{\int_{x_i}^{x} v}\right). \tag{3.4.75}$$

To compute residues near simple poles $\tilde{z}_k$ we use the local coordinate ζ (3.2.5) to write near $\tilde{z}_k^{(1)}$:

$$\frac{1}{6}\frac{S_B^+ - S_v}{v} = \frac{1}{6}\frac{S_B^+(\zeta) - \frac{1}{2\zeta^2}}{\frac{\tilde{r}_k}{\zeta}} d\zeta = \left(-\frac{1}{12\tilde{r}_k\zeta} + O(1) \right) d\zeta. \tag{3.4.76}$$

Thus,

$$(\operatorname*{res}_{z_k^{(1)}} + \operatorname*{res}_{z_k^{(2)}})\left(\frac{B_{reg}^+}{v}\int_{x_1}^{x} u_\gamma^-\right) = -\frac{1}{12\tilde{r}_k}\int_{\tilde{z}_k^{(2)}}^{\tilde{z}_k^{(1)}} u_\gamma^- \tag{3.4.77}$$

and the formula (3.4.62) results.

To obtain (3.4.63) we apply the chain rule with (3.2.18), (3.2.19) and (3.4.16) to write

$$\frac{d\log\tau_B}{d\tilde{C}_j^1} = -\frac{1}{4}\int_{z_j^{(1)}}^{z_j^{(2)}}\left(\frac{B_{reg}^+}{v} + \frac{1}{12\tilde{r}_j^2}v\right) + \frac{1}{8\pi i}\sum_{i=1}^{g^-}\left(\oint_{2b_i^-}\frac{B_{reg}^+}{v}\oint_{a_i^-}\eta_j^- - \oint_{2a_i^-}\frac{B_{reg}^+}{v}\oint_{b_i^-}\eta_j^-\right). \tag{3.4.78}$$

Notice that in this case both differentials (B_{reg}^+/v) and η_j^- have simple poles at $(\tilde{z}_j^{(1)}, \tilde{z}_j^{(2)})$. From (3.4.76) it follows that in order to regularize (B_{reg}^+/v) near these points we need to add $\frac{1}{12\tilde{r}_j}\eta_j^-$. Then the sum could be rewritten as

$$\frac{1}{8\pi i}\sum_{i=1}^{g^-}\left[\oint_{2b_i^-}\left(\frac{B_{reg}^+}{v} + \frac{1}{12\tilde{r}_j}\eta_j^-\right)\oint_{a_i^-}\eta_j^- - \oint_{2a_i^-}\left(\frac{B_{reg}^+}{v} + \frac{1}{12\tilde{r}_j}\eta_j^-\right)\oint_{b_i^-}\eta_j^-\right], \tag{3.4.79}$$

which is due to the RBI equals

$$-\frac{1}{4}\sum_{t=\{x_i,\tilde{z}_k^{(1)},\tilde{z}_k^{(2)}\}}\operatorname*{res}_{t}\left[\eta_j^-\int_{x_1}^{x}\left(\frac{B_{reg}^+}{v} + \frac{1}{12\tilde{r}_j}\eta_j^-\right)\right] \tag{3.4.80}$$

and the evaluation of residues provides the formula (3.4.63).

For $\frac{\partial\log\tau_B}{\partial C_j^1}$, $j = n+1, ..., m$ we have, using (3.2.18), (3.2.20) and (3.4.16),

$$\frac{d\log\tau_B}{dC_j^1} = -\frac{1}{4}\int_{z_j^{(1)}}^{z_j^{(2)}}\left(\frac{B_{reg}^+}{v}\right) + \frac{1}{8\pi i}\sum_{i=1}^{g^-}\left(\oint_{2b_i^-}\frac{B_{reg}^+}{v}\oint_{a_i^-}\eta_j^- - \oint_{2a_i^-}\frac{B_{reg}^+}{v}\oint_{b_i^-}\eta_j^-\right). \tag{3.4.81}$$

Here differential $\frac{B^+_{reg}}{v}$ is holomorphic at $z_j^{(1)}$ and $z_j^{(2)}$ where it gains a zero of order $k_j - 2$. Then no regularization needed and applying the RBI we obtain (3.4.64).

Similarly to (3.4.64) one derives (3.4.65) and (3.4.66) for $\frac{\partial \log \tau_B}{\partial C_j^l}$, $l \geq 2$. For $l = k_j$ extra term appears due to nontrivial coinciding residues near poles $z_j^{(1)}$ and $z_j^{(1)}$ of $w_j^{k_j-}$:

$$\frac{1}{(k_j-1)!}\left(\frac{S_v}{6v}\right)^{(k_j-2)}\left(z_j^{(1)}\right), \tag{3.4.82}$$

where the derivative is taken in a local coordinate χ_j. Using the expansion (3.3.2) of v and the expression (3.4.69) for S_v, one derives

$$\frac{1}{(k_j-1)!}\left(\frac{S_v}{6v}\right)^{(k_j-2)}\left(z_j^{(1)}\right) = \frac{1}{(k_j-1)}\frac{2k_j - k_j^2}{12C_j^{k_j}}, \tag{3.4.83}$$

which finalize the computation. □

3.5 Higher order variations on $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ and topological recursion

Topological recursion is a recursive procedure, which takes a Riemann surface C with underlying tensor objects ("spectral data") to produce a collection of meromorphic multidifferentials $W_k^{(s)} = W_k^{(s)}(p_1, ..., p_k)$ (called Eynard-Orantin invariants), which are defined on C^k and symmetric with respect to permutation of variables. Introduced originally in [16] for moduli spaces of algebraic curves, it was later generalized to spaces of holomorphic $SL(2)$ [15] and $GL(n)$ [3] Hitchin's covers. In this section, we will adapt the framework of topological recursion of [16] to meromorphic $SL(2)$ Hitchin's covers, addressing their specifics governed by the global holomorphic involution of the canonical double cover. The recursion that will be used here is different from the one studied in [15] and only applies to multidifferentials $W_k^{(0)}$.

• **Spectral data** (C, y, x, B). It consists of a compact Riemann surface C; differential ydx with meromorphic functions $y, x : \mathcal{C} \to \mathbb{C}P^1$, such that all branchpoints a (zeroes of dx) are simple with $y'(a) \neq 0$; symmetric bidifferential B defined on $C \times C$. Functions y, x are only needed to be locally defined in the neighborhoods of zeroes of dx.

• **Bidifferential**. Choosing a canonical basis of cycles (a_α, b_α) on C, one defines a unique bilinear form $B(p_1, p_2)$ on $C \times C$, which is normalized over a-cycles and has a residueless double pole on the diagonal $p_1 = p_2$, such that in some local coordinate ξ we have the following expansion of $B(p_1, p_2)$:

$$B(p_1, p_2) = \frac{d\xi(p_1)d\xi(p_2)}{(\xi(p_1) - \xi(p_2))^2} + O(1). \tag{3.5.1}$$

• **Recursion Kernel**. Let p any point on C, while q lies within a small neighborhood of point a. By assumption the map $x : C \to \mathbb{C}P^1$ is simply branched. Thus, for any ramification point $a \in C$, we can find a neighborhood $a \in U \subset C$ and a local non-trivial involution $\sigma_a : U \to U$ such that $x \circ \sigma_a = x$. The recursion kernel is defined by the formula

$$K_q(p) = \frac{1}{2}\frac{\int_q^{\sigma_a(q)} B(\,\cdot\,, p)}{(y(q) - y(\sigma_a(q))dx(q)}, \tag{3.5.2}$$

where integration path lies within the neighborhood of the branch point a.

• **Recursion**. Base of recursion:

$$W_k^{(s)}(p) = 0 \quad \text{if } s < 0, \tag{3.5.3}$$

$$W_1^{(0)}(p) = 0, \tag{3.5.4}$$

$$W_2^{(0)}(p_1, p_2) = B(p_1, p_2). \tag{3.5.5}$$

Given a set of points $p_1, \dots, p_n$ on C, if $K = \{i_1, \dots, i_k\}$ is any subset of $\{1, 2, \dots, n\}$, we let p_K be the k-tuple $p_K = (p_{i_1}, p_{i_2}, \dots, p_{i_k})$. Then for $2s - 2 + k \geq 0$ we define

$$W_{k+1}^{(s)}(p, p_K) = \sum_a \underset{q=a}{res} \left(K_q(p) \sum_{m=0}^{s} \sum_{J \subseteq K} W_{|J|+1}^{(m)}(q, p_J) W_{k-|J|+1}^{(s-m)}(\sigma_a(q), p_{K \setminus J}) + W_{k+2}^{(s-1)}(q, \sigma_a(q), p_K) \right) \tag{3.5.6}$$

where the sum $\sum_{J \subseteq K}$ is over all subsets $J \subseteq K$.

In case of spaces $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ of $SL(2)$ covers: $C = \hat{\mathcal{C}}$ and the map $x : C \to \mathbb{C}P^1$ is replaced by the map $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, defined by the equation $v^2 = Q$ in $T^*\mathcal{C}$. Having that zeros of quadratic differential Q are simple, the map π is simply branched and the differential v, written as $v = \frac{v}{d\xi} d\xi$ in terms of a local coordinate ξ on $\mathcal{C}$, plays the role of 1-form ydx. The base bidifferential B that we will use in this setting is Prym bidifferential (3.1.26). Namely, we set

$$B := B^-(p_1, p_2). \tag{3.5.7}$$

Additionally, the surface $\hat{\mathcal{C}}$ is equipped with global holomorphic involution μ, which, being restricted to the neighborhoods of the branch points x_i (zeroes of Q), provides local involutions σ_{x_i}. Then

$$(y(q) - y(\sigma_{x_i}(q))dx(q) = v(q) - v(\mu(q)) = 2v(q), \tag{3.5.8}$$

since differential v is skew-symmetric under μ. The recursion kernel (3.5.2) written in terms of v and $B^-(p_1, p_2)$ and denoted by $K_q^-(p)$ takes the form

$$K_q^-(p) = -\frac{1}{2} \frac{\int_{x_i}^q B^-(\,\cdot\,, p)}{v(q)}, \tag{3.5.9}$$

where we used

$$\int_q^{\mu(q)} B^-(\,\cdot\,, p) = -2 \int_{x_i}^q B^-(\,\cdot\,, p), \tag{3.5.10}$$

due to skew-symmetry of $B^-(p_1, p_2)$ under μ. Skew-symmetry of $B^-(p_1, p_2)$ also implies that all the subsequent multidifferentials of the recursion (3.5.6) for $s = 0$ are skew-symmetric under involution μ in all their variables. We denote the corresponding multidifferentials by $W_k^{(0)-}(p, p_K)$ and have from (3.5.6) that

$$W_1^{(0)-}(p) = 0, \tag{3.5.11}$$

$$W_2^{(0)-}(p_1, p_2) = B^-(p_1, p_2), \tag{3.5.12}$$

$$W_{k+1}^{(0)-}(p, p_K) = \frac{1}{2} \sum_{x_i} \underset{q=x_i}{res} \left(\frac{\int_{x_i}^q B^-(\,\cdot\,, p)}{v(q)} \sum_{J \subseteq K} W_{|J|+1}^{(0)-}(q, p_J) W_{k-|J|+1}^{(0)-}(q, p_{K \setminus J}) \right), \quad k \geq 2. \tag{3.5.13}$$

Remark 3.5.1. We do not consider recursion with $s \geq 1$ for technical reason: writing down the first $s = 1$ differential

$$W_1^{(1)}(p) = -\frac{1}{2}\sum_{x_i} \underset{q=x_i}{res}\left(\frac{\int_{x_i}^q B^-(\,\cdot\,,p)}{v(q)} B^-(q,\mu(q))\right) \tag{3.5.14}$$

we observe that $B^-(q,\mu(q)) = -B^-(q,q)$ is not correctly defined. This issue could be circumvented by instead considering the recursion with the base bidifferential

$$B := \hat{B}(p_1,p_2), \tag{3.5.15}$$

which is a standard Bergman bidifferential associated with the canonical cover $\hat{\mathcal{C}}$. Such approach was taken in [15].

Multidifferentials $W_k^{(0)-} = W_k^{(0)-}(p_1,...,p_k)$ share the following properties: skew-symmetry under involution μ in all variables implies that their only poles, located at the branch points x_i, have vanishing residues. Additionally, $W_k^{(0)-}$ are symmetric with respect to permutation of their variables (Theorem 4.6, [16]) and have vanishing periods over a^+ and a^--cycles.

We are ready to formulate a theorem that relates variations of Eynard-Orantin invariants $W_k^{(0)-}$ with their recursive definition.

Theorem 3.5.1. *The variations of Eynard-Orantin invariants $W_k^{(0)-}$ on the space $\mathcal{M}_{g,m}^{\mathfrak{sl}_2}[\mathbf{k}]$ with respect to the coordinates (3.3.5) take the following form:*

$$\frac{\partial}{\partial A_\gamma} W_k^{(0)-}(p_1,\ldots,p_k) = -\frac{1}{4\pi i}\int_{p\in b_\gamma^-} W_{k+1}^{(0)-}(p,p_1,\ldots,p_k). \tag{3.5.16}$$

$$\frac{\partial}{\partial C_j^1} W_k^{(0)-}(p_1,\ldots,p_k) = -\frac{1}{2}\int_{p\in z_j^{(2)}}^{z_j^{(1)}} W_{k+1}^{(0)-}(p,p_1,\ldots,p_k). \tag{3.5.17}$$

$$\frac{\partial}{\partial C_j^l} W_k^{(0)-}(p_1,\ldots,p_k) = -\frac{1}{2}\underset{p=z_j^{(1)}}{res}\left(\chi_j(p)^{-l} W_{k+1}^{(0)-}(p,p_1,\ldots,p_k)\right). \tag{3.5.18}$$

Proof. This is an immediate adaptation of Theorem 5.1 of [16], whose proof is essentially a local statement only involving variational formulas (3.4.46-3.4.48) and combinatorial representation of differentials $W_k^{(0)-}$. □

Let us consider the case $k = 2$. Formula (3.5.16) implies that

$$\frac{\partial}{\partial A_\gamma} B^-(p_1,p_2) = -\frac{1}{4\pi i}\int_{p\in b_\gamma^-} W_3^{(0)-}(p,p_1,p_2). \tag{3.5.19}$$

From (3.5.13) we get

$$W_3^{(0)-}(p,p_1,p_2) = \sum_{x_i} \underset{q=x_i}{res}\left(\frac{\int_{x_i}^q B^-(\,\cdot\,,p)}{v(q)} B^-(q,p_1)B^-(q,p_2)\right). \tag{3.5.20}$$

Calculation similar to the proof of the formula (3.4.20) in Theorem 3.4.1 allows us to rewrite it as

$$W_3^{(0)-}(p,p_1,p_2) = \sum_{i=1}^{r} \underset{x_i}{res}\left(\frac{B^-(q,p)B^-(q,p_1)B^-(q,p_2)}{d\xi\, d(v/d\xi)}\right). \tag{3.5.21}$$

Plugging this formula into (3.5.19) and additionally integrating two times with respect to b_α^- and b_β^-, using identities (3.4.56) and (3.1.22), we obtain the formula (3.4.20) for the Donagi-Markman cubic

$$\frac{\partial \Omega^-_{\alpha\beta}}{\partial A_\gamma} = -\pi i \sum_{i=1}^{r} \underset{x_i}{res} \left(\frac{u^-_\alpha u^-_\beta u^-_\gamma}{d\xi \, d(v/d\xi)} \right) \tag{3.5.22}$$

that measures a variation of the Prym matrix Ω^- with respect to periods A_γ. The formulas (3.5.17-3.5.18) could be similarly applied to obtain (3.4.21-3.4.22) with the help of identities (3.4.60-3.4.61).

Repeatedly applying variational formulas from Theorem 3.5.1 one may obtain a Taylor expansion of the Prym matrix Ω^-, and, consequently, of the period matrix $\hat{\Omega}$ (3.1.24) of the double cover $\hat{\mathcal{C}}$ on the space $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$. On the submanifold $\mathcal{M}^{\mathfrak{sl}_2}_{g} \subset \mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ for $m = 0$, consisting of holomorphic $SL(2)$ covers this formula has a more compact and familiar form.

Proposition 3.5.1. *On spaces $\mathcal{M}^{\mathfrak{sl}_2}_{g}$ of holomorphic $SL(2)$ covers, variations of the Prym matrix Ω^- take the form:*

$$\frac{\partial}{\partial A_{i_1}} \frac{\partial}{\partial A_{i_2}} \cdots \frac{\partial}{\partial A_{i_{k-2}}} \Omega^-_{i_{k-1} i_k} = -\left(\frac{i}{4\pi} \right)^{k-1} \int_{p_1 \in b^-_{i_1}} \cdots \int_{p_k \in b^-_{i_k}} W^{(0)-}_k (p_1, \ldots, p_k). \tag{3.5.23}$$

This formula provides an $SL(2)$ specialization of a more general result for $GL(n)$ spectral covers previously obtained in [3]. The moral of this proposition is that a single spectral curve (its invariants are computed on the right-hand side) knows about the geometry of families of spectral curves (on the left-hand side). While the right-hand side is clearly symmetric in indices i_k due to symmetry of $W^{(0)-}_k$ with respect to permuting its variables, the symmetry of the left-hand side follows from an observation, that the Prym matrix $\Omega^-_{\alpha\beta}$ itself is a second partial derivative of a single function $\mathcal{F}$ called the prepotential:

$$F = \frac{1}{2} \sum_{\gamma=1}^{g^-} A_\gamma B_\gamma, \tag{3.5.24}$$

such that

$$\Omega^-_{\alpha\beta} = \frac{\partial^2 F}{\partial A_\alpha \partial A_\beta}. \tag{3.5.25}$$

Remark 3.5.2. Prepotential $\mathcal{F}$ is known to be a generating function between homological and canonical coordinates on symplectic space $\mathcal{Q}_g$ of holomorphic quadratic differentials with variable base (see [8]).

We conclude this section with the formula for the second variation of the Prym matrix $\frac{\partial^2 \Omega^-_{\alpha\beta}}{\partial A_\delta \partial A_\gamma}$. Due to (3.5.25) the resulting expression must be symmetric with respect to all 4 indices. It could be computed either by topological recursion, or by direct differentiation of the formula (3.4.20) and using variations of Prym differentials (3.4.57). The latter computation could be performed following the Proposition 5.1 in [4] with a small alteration and results in

Proposition 3.5.2. *Second derivative of the Prym matrix $\Omega^-_{\alpha\beta}$ on $\mathcal{M}^{\mathfrak{sl}_2}_{g,m}[\mathbf{k}]$ is given by the following expression:*

$$\frac{1}{2\pi i}\frac{\partial^2\Omega^-_{\alpha\beta}}{\partial A_\delta \partial A_\gamma} = \frac{1}{16}\sum_{x_a\neq x_b}\left\{B^-(x_a,x_b)\frac{u^-_\delta(x_a)u^-_\gamma(x_a)u^-_\alpha(x_b)u^-_\beta(x_b) + cycl\ of\ (\alpha,\beta,\gamma)}{y'(x_a)y'(x_b)}\right\} +$$
$$+\frac{1}{16}\sum_{x_a}\left\{\left(\frac{\hat{S}_B}{y'^2}-\frac{y'''}{y'^3}\right)u^-_\alpha u^-_\beta u^-_\gamma u^-_\delta(x_a) + \frac{1}{y'^2}((u^-_\alpha)'' u^-_\beta u^-_\gamma u^-_\delta(x_a) + cycl\ of\ (\alpha,\beta,\gamma,\delta))\right\}, \tag{3.5.26}$$

where $\hat{S}$ is Bergman projective connection (3.1.32); $y = \frac{v}{d\xi}$. Values and derivatives at brach points x_a are computed in a local coordinate $\hat{\xi}$. This formula does not depend on the choice of local coordinates $\xi, \hat{\xi}$ on $\mathcal{C}$ and $\hat{\mathcal{C}}$, respectively, provided $\hat{\xi}^2 = \xi$.

Remark 3.5.3. Corresponding second-order variations with respect to moduli C^k_j are obtained by replacing Prym holomorphic differentials u^-_δ, u^-_γ with corresponding Prym differentials of second (3.4.3) and third (3.4.4) kinds.

Chapter 4

Symplectic geometry of spaces of quadratic differentials

This chapter is organized as follows: in **Section 4.1** we introduce a linear second order equation on a Riemann surface and associated monodromy map between the moduli space of quadratic differentials with second order poles and $PSL(2)$ character variety. We define local coordinates on the moduli space of curves $\mathcal{M}_{g,n}$ and state the theorem which gives a condition for a monodromy map to become a symplectomorphism in terms of closeness of the 1-form on $\mathcal{M}_{g,n}$. In **Section 4.2** we describe the geometry, main objects and variational formulas linked to the moduli space of quadratic differentials with second order poles, which is a special case of general moduli spaces of meromorphic quadratic differentials discussed in Chapter 3. **Section 4.3** is devoted to the symplectic properties of the monodromy map. In particular, we prove the theorem stated in Section 4.1. We also introduce the generating function for the monodromy symplectomorphism (Yang-Yang function). In **Section 4.4** we perform generalized WKB expansion of the monodromy generating function and compute its asymptotics.

4.1 Definition of the monodromy map

Consider the linear second order equation on a Riemann surface $\mathcal{C}$ of genus g with n punctures (the stationary Schrödinger equation) in the form

$$\partial^2\phi + \left(\frac{1}{2}S - Q\right)\phi = 0, \tag{4.1.1}$$

where S is a fixed meromorphic projective connection on $\mathcal{C}$ with at most simple poles at the punctures $(z_j)_{j=1}^n$ and depending holomorphically on the moduli of $\mathcal{M}_{g,n}$, Q is a meromorphic quadratic differential with simple zeroes and double poles at (z_j) with the asymptotics:

$$Q(x) \sim \Big(\frac{r_j^2}{\xi_j^2} + O(\xi_j^{-1})\Big)(d\xi_j)^2. \tag{4.1.2}$$

Here and below, we will assume that Q is Gaiotto-Moore-Nietzke (GMN) differential (i.e., none of horizontal trajectories of Q connect two of its zeros [20]). In particular, this implies that $r_j^2 \notin \mathbb{R}_-$. Denote by $\mathcal{Q}_{g,n}$ the moduli space of pairs $(\mathcal{C}, Q)$ and by $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ its corresponding stratum for fixed values of r_j's. The solution to (4.1.1) is locally a $-\frac{1}{2}$ differential [22] which could be written as

$$\phi = \phi(\xi)(d\xi)^{-\frac{1}{2}}. \tag{4.1.3}$$

Take two linearly independent solutions ϕ_1, ϕ_2 and consider their ratio $f = \phi_1/\phi_2$. Analytic continuation of f along the cycles of $\pi(\mathcal{C}\backslash\{z_i\}_{j=1}^n, x_0)$ determines a $PSL(2,\mathbb{C})$ monodromy representation of the fundamental group with the chosen basepoint x_0. The choice of standard generators $(\{\kappa\}_{j=1}^n, \{\alpha,\beta\}_{j=1}^g)$ of the fundamental group with single relation

$$\kappa_1 ... \kappa_n \prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = id \tag{4.1.4}$$

yield the same relation on the monodromy matrices

$$M_{\kappa_1} ... M_{\kappa_n} \prod_{i=1}^{g} M_{\alpha_i} M_{\beta_i} M_{\alpha_i}^{-1} M_{\beta_i}^{-1} = I. \tag{4.1.5}$$

The matrix M_{κ_j} corresponding to the monodromy around the pole z_j has the following diagonal form:

$$D_j = \begin{pmatrix} m_j & 0 \\ 0 & m_j^{-1} \end{pmatrix}, \tag{4.1.6}$$

where

$$m_j^2 = e^{4\pi i \lambda_j}. \tag{4.1.7}$$

Local analysis of the solutions for (4.1.1) implies the following relation between the biresidues (r_j) and eigenvalues (m_j)

$$r_j^2 = \lambda_j(\lambda_j - 1). \tag{4.1.8}$$

We denote by $CV_{g,n}$ the $PSL(2)$ character variety corresponding to the representation (4.1.5). It is well known that the stratum $CV_{g,n}\{\mathbf{m}\}$ for fixed values m_j is a symplectic leaf with a Poisson structure given by the Goldman bracket [21].

The space $\mathcal{Q}_{g,n}$ is a special case of spaces $\mathcal{Q}_{g,m}[\mathbf{k}]$ of meromorphic quadratic differentials discussed in Chapter 3. It admits a system of local coordinates defined in the following way: for every pair $(\mathcal{C}, Q) \in \mathcal{Q}_{g,n}$ consider equation $v^2 = Q$ in $T^*\mathcal{C}$. This equation induces a double covering $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, where v is a globally defined Abelian differential. The map π is branched at (simple) zeroes of Q denoted by $(x_j)_{j=1}^{4g-4+2n}$. Thus, each double pole z_j has two preimages that we call $(z_j^{(1)}, z_j^{(2)})$. The enumeration of these points is chosen such that the residue of v at $z_j^{(1)}$ equals r_j and the residue of v at $z_j^{(2)}$ equals $-r_j$ .The genus of $\hat{\mathcal{C}}$ is $\hat{g} = g + g^-$, with $g^- = 3g - 3 + n$. The surface $\hat{\mathcal{C}}$ is equipped with a natural holomorphic involution $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ which interchanges the sheets of the double cover. The involution induces the splitting of the homology group $H_1(\hat{\mathcal{C}}\backslash(z_j^{(1)}, z_j^{(2)})_{j=1}^n, \mathbb{Z})$ into even H_+ and odd H_- parts. Local coordinates on $\mathcal{Q}_{g,n}$ are defined by integrating the differential v over a basis of H_-. We choose appropriate subset of cycles $(a_i^-, b_i^-)_{i=1}^{g^-} \in H_-$ with intersection index $a_i^- \circ b_j^- = \frac{1}{2}\delta_{ij}$ so that the integrals

$$A_j = \oint_{a_j^-} v, \quad B_j = \oint_{b_j^-} v \tag{4.1.9}$$

become local period (or homological) coordinates on the stratum $\mathcal{Q}_{g,n}\{\mathbf{r}\}$. The intersection pairing defines the natural symplectic form on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$

$$\Omega_{\text{hom}} = \sum_{j=1}^{g^-} 2dB_j \wedge dA_j. \tag{4.1.10}$$

The character variety $CV_{g,n}\{\mathbf{m}\}$ is equipped with a symplectic form that inverts a natural Poisson structure given by the Goldman bracket. It could be written in terms of homological shear coordinates which are appropriate linear combinations of the logarithms of Thurston's shear coordinates [43]:

$$\Omega_G = \sum_{j=1}^{g^-} 2d\rho_{a_j^-} \wedge d\rho_{b_j^-}. \tag{4.1.11}$$

Introduce the Bergman projective connection S_B defined in terms of the canonical bidifferential $B(x,y)$ on $\mathcal{C}$, which is normalized with respect to chosen Torelli marking in $H_1(\mathcal{C},\mathbb{Z})$:

$$S_B(x) = \Big(B(x,y) - \frac{d\xi(x)d\xi(y)}{(\xi(x)-\xi(y))^2}\Big)\Big|_{y=x}, \tag{4.1.12}$$

where ξ is any local coordinate near point x. As long as S_B depends holomorphically on the conformal structure of $\mathcal{C}$, the difference $S - S_B$ becomes a family of quadratic differentials with at most simple poles at the punctures (z_j), depending holomorphically on moduli of $\mathcal{M}_{g,n}$. Using the identification of the moduli space of quadratic differentials with simple poles and the cotangent bundle $T^*\mathcal{M}_{g,n}$, we can associate $S - S_B$ with the 1-form $\Theta_{(S-S_B)}$, locally defined on $\mathcal{M}_{g,n}$, in the following way.

At first, introduce the set of holomorphic local coordinates (Ω_{jk}, q_l) on $\mathcal{M}_{g,n}$, $g \geq 2$. To determine locally the conformal structure of $\mathcal{C}$ we pick at generic point of $\mathcal{M}_{g,n}$ (outside of hyperelliptic locus for $g \geq 3$) a set D of $3g-3$ entries of the period matrix Ω of $\mathcal{C}$. The quadratic differentials corresponding to cotangent vectors $d\Omega_{jk}$ are products $u_j u_k$ of normalized holomorphic differentials. An additional set of n coordinates which determine the positions of punctures $(z_l)_{l=1}^n$ on $\mathcal{C}$ we choose to be $q_l = (u_i/u_j)(z_l)$ where u_i and u_j form a pair of normalized holomorphic 1-forms on $\mathcal{C}$, such that $u_j(z_l) \neq 0$. The quadratic differential corresponding to cotangent vector dq_l is the meromorphic quadratic differential Q^{z_l} (given by the formula (4.3.59) below) whose only simple pole is at z_l. These coordinates are local: in different coordinate charts on $\mathcal{M}_{g,n}$ one might need to choose other pairs of normalized holomorphic differentials and/or different Torelli markings. The momenta p_l are then defined to be coefficients of decomposition of the quadratic differential $S - S_B$ in the basis described above. Writing down the quadratic differential $S - S_B$ as

$$S - S_B = \sum_{(jk)\in D} p_{jk}\, u_j u_k + \sum_{l=1}^n p_l Q^{z_l}, \tag{4.1.13}$$

where p_{jk} and p_l are holomorphic functions of (Ω_{jk}, q_l), the corresponding 1-form $\Theta_{(S-S_B)}$ on $\mathcal{M}_{g,n}$ reads as

$$\Theta_{(S-S_B)} = \sum_{(jk)\in D} p_{jk}\, d\Omega_{jk} + \sum_{l=1}^n p_l dq_l. \tag{4.1.14}$$

Local coordinates on $\mathcal{M}_{g,n}$ for $g = 0, 1$ have a special description and were covered in [32].

First main result of this chapter imposes a condition on projective connection S of (4.1.1) for the monodromy map to become a symplectomorphism.

Theorem 4.1.1 (Theorem 4.3.2). *The monodromy map*

$$\mathcal{F}_{(S)} : \mathcal{Q}_{g,n}\{\mathbf{r}\} \to CV_{g,n}\{\mathbf{m}\} \tag{4.1.15}$$

is a symplectomorphism with $\mathcal{F}_{(S)}^*\Omega_G = -\Omega_{\rm hom}$ iff the 1-form $\Theta_{(S-S_B)}$, corresponding to family of quadratic differentials $S - S_B$ (which is locally defined on the moduli space $\mathcal{M}_{g,n}$), is closed, $d\Theta_{(S-S_B)} = 0$.

Statement of the theorem generalizes the results proven in [8], [32], where the differential Q is assumed to be holomorphic or with simple poles, respectively. The proofs were based on the identification of the homological symplectic form with the canonical form on $T^*\mathcal{M}_{g,n}$. which does not hold in presence of second order poles.

4.2 Spaces $\mathcal{Q}_{g,n}$ of quadratic differentials with second order poles

4.2.1 Canonical double cover

Denote by $\mathcal{Q}_{g,n}$ the moduli space of meromorphic quadratic differentials on Riemann surface $\mathcal{C}$ of genus g with n double poles $(z_1, ..., z_n)$ and $4g-4+2n$ simple zeroes $(x_1, ..., x_{4g-4+2n})$. We assume that any quadratic differential $Q \in \mathcal{Q}_{g,n}$ has the following asymptotics near poles:

$$Q(x) \sim \Big(\frac{r_j^2}{\xi_j^2} + O(\xi_j^{-1})\Big)(d\xi_j)^2, \tag{4.2.1}$$

as $x \to z_i$, here ξ_j is any local coordinate near pole z_j. For all such Q the equation $v^2 = Q$ in the cotangent bundle $T^*\mathcal{C}$ defines double covering $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, branched at zeroes of Q. The covering surface $\hat{\mathcal{C}}$ possesses a natural holomorphic involution $\mu : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$. The differential v is single-valued on $\hat{\mathcal{C}}$ and skew-symmetric under the involution: $v(x^\mu) = -v(x)$. It has double zeroes at branch points $(x_j)_{j=1}^{4g-4+2n}$ and simple poles at $2n$ preimages of $(z_j)_{j=1}^n$ denoted by $z_j^{(1)}$ and $z_j^{(2)}$ with residues r_j and $-r_j$, respectively. The Riemann-Hurwitz formula implies the genus of the covering surface $\hat{\mathcal{C}}$ equals

$$\hat{g} = 4g - 3 + n. \tag{4.2.2}$$

We decompose the first homology group of $H_1(\hat{\mathcal{C}}\backslash\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n, \mathbb{Z})$ into

$$H_1(\hat{\mathcal{C}}\backslash\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n, \mathbb{Z}) = H_+ \oplus H_-, \tag{4.2.3}$$

which are the $+1$ and -1 eigenspaces of the map, induced by the involution μ. $\dim(H_+) = 2g+n-1$ and $\dim(H_-) = 6g - 6 + 3n := 2g^- + n$. The canonical basis of $H_1(\hat{\mathcal{C}}\backslash\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n, \mathbb{Z})$ and corresponding bases in $H_\pm$ are chosen in accordance with the previous chapter. Recall, that the classes

$$\{a_k^+, b_k^+, t_j^+\}, \quad k = 1, ..., g, \quad j = 1, ..., n, \tag{4.2.4}$$

with the single relation

$$\sum_{j=1}^n t_j^+ = 0. \tag{4.2.5}$$

generate the group H^+ with the intersection index

$$a_i^+ \circ b_k^+ = \frac{1}{2}\delta_{ik}, \tag{4.2.6}$$

while t_j^+'s have zero intersection with all cycles. The following cycles

$$\{a_k^-, b_k^-, t_j^-\}, \quad k = 1, ..., 3g-3+n, \quad j = 1, ..., n. \tag{4.2.7}$$

are the generators of the group H_-. Similarly, their intersection index is

$$a_i^- \circ b_k^- = \frac{1}{2}\delta_{ik} \tag{4.2.8}$$

and all other intersections are zero.

The differential v is used to introduce a system of local coordinates on both $\mathcal{C}$ and $\hat{\mathcal{C}}$. If x is a point of $\hat{\mathcal{C}}$ which does not coincide with branch points $\{x_i\}$ and poles $\{z_j^{(1)}, z_j^{(2)}\}$ then the local coordinate (also called "flat" coordinate) near x is given by

$$z(x) = \int_{x_1}^{x} v, \tag{4.2.9}$$

x_1 is a chosen "first" zero of v. Coordinate $z(x)$ could also be used on $\mathcal{C}$ outside branch points and poles. Notice that in this case $v = dz$. Near a branch point x_i on $\hat{\mathcal{C}}$ the *distinguished* local coordinate is given by

$$\hat{\xi}_i(x) = \left(\int_{x_i}^{x} v \right)^{\frac{1}{3}}. \tag{4.2.10}$$

On the curve $\mathcal{C}$ the local coordinate near x_i is

$$\xi_i(x) = \hat{\xi}_i^2(x) = \left(\int_{x_i}^{x} v \right)^{\frac{2}{3}}. \tag{4.2.11}$$

Near a double pole z_j on $\mathcal{C}$ and simple poles $(z_j^{(1)}, z_j^{(2)})$ on $\hat{\mathcal{C}}$ the local coordinate is

$$\zeta_j(x) = \exp\left(\frac{1}{r_j} \int_{x_1}^{x} v \right). \tag{4.2.12}$$

To define local coordinates near the poles uniquely we on $\mathcal{C}$ connect first zero x_1 with a chosen first double pole z_1 by a branch cut, then connect z_1 with the remaining poles $\{z_i\}_{j=2}^n$ forming a tree. Then we lift this tree to $\hat{\mathcal{C}}$ via π^{-1}.

4.2.2 Period coordinates. Homological symplectic form

The dimension of H_- coincides with the dimension of $\mathcal{Q}_{g,n}$. We introduce the following set of *period* (homological) local coordinates on $\mathcal{Q}_{g,n}$:

$$A_j = \oint_{a_j^-} v, \quad B_j = \oint_{b_j^-} v, \quad 2\pi r_k = \oint_{t_k^-} v. \tag{4.2.13}$$

We fix the values (r_k) and denote the corresponding stratum of moduli space by $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ (for the definition of the homological shear coordinates and their WKB expansion it is necessary for Q to be free from saddle trajectories, so we assume that $r_k^2 \notin \mathbb{R}_-$ [7]). Then $(A_j, B_j)_{j=1}^{g^-}$ become local coordinates on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$. There is a natural Poisson structure on $\mathcal{Q}_{g,n}$ between periods of v induced by the intersection index of the corresponding cycles $s_1, s_2 \in H_-$:

$$\left\{ \int_{s_1} v, \int_{s_2} v \right\} = s_1 \circ s_2. \tag{4.2.14}$$

This Poisson structure is degenerate, with Casimir functions $r_1, ..., r_n$. The stratum $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ becomes a symplectic leaf for given bracket. It allows us to introduce the following symplectic form on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$:

$$\Omega_{\text{hom}} = \sum_{j=1}^{g^-} 2dB_j \wedge dA_j. \tag{4.2.15}$$

4.2.3 Variational formulas on $\mathcal{Q}_{g,n}$

In this section we introduce several meromorphic functions associated with surfaces $\hat{\mathcal{C}}$ and $\mathcal{C}$. Then we recall their variational formulas with respect to period coordinates on the moduli space $\mathcal{Q}_{g,n}$. Let us denote by $\{a_k, b_k\}_{k=1}^g$ the canonical basis of cycles on $\mathcal{C}$.

• The matrix $\Omega_{ij} = \oint_{b_j} u_i$ represents the $g \times g$ period matrix of the base surface $\mathcal{C}$.
• Meromorphic functions $f_j : \hat{\mathcal{C}} \to \mathbb{C}P^1$ given by

$$f_j(x) = \frac{u_j(x)}{v(x)}, \quad j = 1, ..., g. \tag{4.2.16}$$

These functions are skew-symmetric under the involution and generically (when zeroes of $u_j(x)$ and $v(x)$ differ) have simple poles at the branch points (x_j).
• The meromorphic function $q : \mathcal{C} \to \mathbb{C}P^1$

$$q(x) = \frac{S_B - S_v}{2v^2}, \tag{4.2.17}$$

where S_v is the Schwarzian projective connection defined by

$$S_v(\xi(x)) = \Big\{ \int_{p_0}^x v, \xi(x) \Big\} (d\xi(x))^2 \tag{4.2.18}$$

in any local coordinate ξ. Here $\{f, \xi\} = \left(\frac{f''(\xi)}{f'(\xi)}\right)' - \frac{1}{2}\left(\frac{f''(\xi)}{f'(\xi)}\right)^2$ – Schwarzian derivative. The pullback of q to $\hat{\mathcal{C}}$ has residueless 6-order poles at (x_j).
• The meromorphic function $b : \hat{\mathcal{C}} \times \hat{\mathcal{C}} \to \mathbb{C}P^1$

$$b(x, y) = \frac{B(x, y)}{v(x)v(y)}, \tag{4.2.19}$$

skew-symmetric in both arguments, with simple poles at (x_j) on $\hat{\mathcal{C}}$ with respect to each argument. In addition, it has a double pole on the diagonal outside the branch points.
• The meromorphic function $h : \mathcal{C} \times \mathcal{C} \to \mathbb{C}P^1$,

$$h(x, y) = \frac{B^2(x, y)}{Q(x)Q(y)} = b^2(x, y). \tag{4.2.20}$$

Its pullback to $\hat{\mathcal{C}} \times \hat{\mathcal{C}} \to \mathbb{C}P^1$ is symmetric, has residueless double poles at (x_j) in both arguments, and a fourth order pole on the diagonal away from branch points.

We introduce the periods $\mathcal{P}_{s_i} = \oint_{s_i} v$ for s_i being an element from the canonical basis of H_- :

$$\{s_i\}_{i=1}^{6g-6+3n} = \Big\{ \{a_j^-, b_j^-\}_{j=1}^{g^-}, \{t_k^-\}_{k=1}^n \Big\}. \tag{4.2.21}$$

The dual basis $\{s_i^*\}$ is defined by the condition

$$s_i^* \circ s_j = \delta_{ij} \tag{4.2.22}$$

and is given by

$$\{s_i^*\}_{i=1}^{6g-6+3n} = \Big\{ \{-2b_j^-, 2a_j^-\}_{j=1}^{g^-}, \{2\kappa_k^-\}_{k=1}^n \Big\}, \tag{4.2.23}$$

here κ_j^- is a 1/2 of the contour connecting poles $z_j^{(1)}$ with $z_j^{(2)}$ and skew-symmetric under the involution, not intersecting other contours.

Choose a fundamental polygon $\hat{\mathcal{C}}_0$ with vertex at x_1 and dissected along paths represented by the graph $\hat{G}$ (see Section 3.1.3). On the simply connected domain $\hat{\mathcal{C}}_0 \backslash \hat{G}$ we define the "flat" coordinate

$$z(x) = \int_{x_1}^{x} v. \tag{4.2.24}$$

The following variational formulas were derived in Proposition 3.1.1 in Chapter 3. Note that the period matrix Ω of the base curve $\mathcal{C}$ is just Ω^+, while the pullbacks of $u_j(x), B(x,y)$ and $S_B(x)$ to $\hat{\mathcal{C}}$ are naturally identified with the corresponding elements $u_j^+(x), B^+(x,y)$ and $S_B^+(x)$.

Proposition 4.2.1. *For arbitrary basis $\{s_i\}_{i=1}^{6g-6+3n}$ of H_- and its dual basis $\{s_i^*\}_{i=1}^{6g-6+3n}$ the following formulas hold on $\mathcal{Q}_{g,n}$:*

$$\frac{\partial \Omega_{ij}}{\partial \mathcal{P}_{s_i}} = \frac{1}{2} \oint_{s_i{}^*} f_i f_j v, \tag{4.2.25}$$

$$\frac{\partial f_j(x)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x)=const} = \frac{1}{4\pi i} \oint_{s_i{}^*} f_j(t) b(x,t) v(t), \tag{4.2.26}$$

$$\frac{\partial b(x,y)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x),z(y)=const} = \frac{1}{4\pi i} \oint_{s_i{}^*} b(x,t) b(t,y) v(t), \tag{4.2.27}$$

$$\frac{\partial q(x)}{\partial \mathcal{P}_{s_i}}\Big|_{z(x)=const} = \frac{3}{4\pi i} \oint_{s_i{}^*} h(x,t) v(t). \tag{4.2.28}$$

4.2.4 Bergman tau-function

The explicit formula and main properties of Bergman tau-function on moduli spaces of meromorphic quadratic differentials with even order poles were outlined in previous chapter. In our framework we consider a special case of Bergman tau-function τ_B associated with moduli spaces $\mathcal{Q}_{g,n}$ of quadratic meromorphic differentials with second order poles. In the present context we only need its defining differential equations and transformation under rescaling of the differential Q by a constant.

Proposition 4.2.2 (Theorem 3.2.1)**.** *Bergman tau-function τ_B satisfies the following system of differential equations on $\mathcal{Q}_{g,n}$:*

$$\frac{\partial \log \tau_B}{\partial A_j} = \frac{1}{12\pi i} \oint_{b_j^-} qv, \quad \frac{\partial \log \tau_B}{\partial B_j} = -\frac{1}{12\pi i} \oint_{a_j^-} qv, \tag{4.2.29}$$

for $j = 1, ..., 3g-3+n$ and

$$\frac{\partial \log \tau_B}{\partial (2\pi i r_k)} = -\frac{1}{12\pi i} \int_{\kappa_k^-} \left(qv + \frac{1}{4 r_k^2} v \right), \tag{4.2.30}$$

for $k = 1, ..., n$.

The function τ_B satisfies the following homogeneity property (see (3.2.15)):

$$\tau_B(\mathcal{C}, \kappa Q) = \kappa^{\frac{5(2g-2+n)}{72}} \tau_B(\mathcal{C}, Q). \tag{4.2.31}$$

Equivalently, defining the Euler vector field via

$$E = \sum_{j=1}^{g^-} \left(A_j \frac{\partial}{\partial A_j} + B_j \frac{\partial}{\partial B_j} \right) + \sum_{j=1}^{n} r_j \frac{\partial}{\partial r_j} \tag{4.2.32}$$

we have that

$$E \log \tau_B = \frac{5(2g-2+n)}{72}. \tag{4.2.33}$$

We also notice that on the stratum $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ the differential $d\log\tau_B|_r$ is given by

$$d\log\tau_B|_r = -\frac{1}{12\pi i} \sum_{j=1}^{g_-} \left[\left(\oint_{a_j^-} qv \right) dB_j - \left(\oint_{b_j^-} qv \right) dA_j \right]. \tag{4.2.34}$$

4.3 Symplectic properties of the monodromy map

4.3.1 Monodromy symplectomorphism and Goldman bracket

Consider the monodromy map

$$\mathcal{F}_{(S)} : \mathcal{Q}_{g,n} \to CV_{g,n} \tag{4.3.1}$$

for the equation (4.1.1) defined by (4.1.5). The Goldman bracket on $CV_{g,n}\{\mathbf{m}\}$ is defined as follows [21]. For two arbitrary loops γ and $\tilde{\gamma}$

$$\{\mathrm{tr} M_\gamma, \mathrm{tr} M_{\tilde{\gamma}}\}_G = \frac{1}{2} \sum_{p\in\gamma\circ\tilde{\gamma}} \nu(p)(\mathrm{tr} M_{\gamma_p\tilde{\gamma}} - \mathrm{tr} M_{\gamma_p\tilde{\gamma}^{-1}}), \tag{4.3.2}$$

where the monodromy matrices $M_\gamma, M_{\tilde{\gamma}} \in PSL(2,\mathbb{C})$; $\gamma_p\tilde{\gamma}$ and $\gamma_p\tilde{\gamma}^{-1}$ are paths obtained by resolving the intersection point p in two different ways (see [21]); $\nu(p) = \pm 1$ is the contribution of the point p to the intersection index of γ and $\tilde{\gamma}$.

The following theorem was stated in [7] and it is a natural extension of the results proven in [8] for holomorphic potentials and in [32] for potentials with simple poles.

Theorem 4.3.1 ([7]). *For the Bergman projective connection S_B (4.1.12) chosen to be the base projective connection S the monodromy map $\mathcal{F}_{(S_B)}$ (4.3.1) of equation (4.1.1) is Poisson. Namely, the homological bracket implies minus the Goldman bracket between traces of monodromy matrices.*

The homological shear coordinates that invert the Goldman bracket (4.3.2) are constructed in the following way (based on appendix of [7], also [6], [11]). Assume that quadratic differential Q is generic i.e it does not have any saddle connections (as in the definition of the "Gaiotto-Moore-Nietzke differential" [20]). Then each horizontal trajectory given by $\mathfrak{Im} \int_{x_1}^{x} v = 0$, where x_1 is an arbitrary "first" zero, starting at a zero x_j of Q ends at one of the poles z_k, defining critical graph Γ_Q. Additionally, three horizontal trajectories meet at each zero, determining three vertices of the triangle at the poles, therefore, defining the triangulation Σ_Q of $\mathcal{C}$. The dual graph with vertices at x_j is denoted by Σ_Q^* (see Figure 3). Notice that the number of edges of Σ_Q equals to $6g-6+3n = \dim(CV_{g,n})$. The Thurston shear coordinate is a value $\zeta_e \in \mathbb{C}$ attached to each edge e of the graph Σ_Q.

Using the graph Σ_Q^* one defines a two-sheeted branch covering $\hat{\mathcal{C}}_{\Sigma_B}$ by assuming that all edges of Σ_Q^* are branch cuts. To each coordinate ζ_e we assign skew-symmetric cycle l_e on $\hat{\mathcal{C}}_{\Sigma_B}$, which is a double cover of the corresponding dual edge $e^* \in \Sigma_Q^*$. Lemma A.1 of [7] states that the Goldman

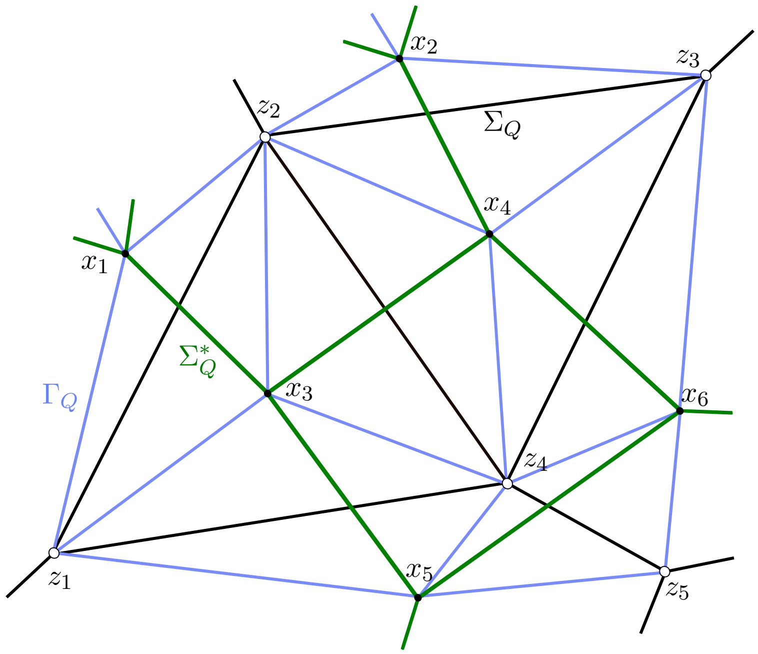

Figure 3: Critical graph Γ_Q and corresponding triangulation Σ_Q and dual Σ_Q^* graphs

Poisson brackets (4.3.2) between the coordinates ζ_e can be expressed via the intersection indices of the cycles l_e as

$$\{\zeta_e, \zeta_{e'}\}_G = \frac{1}{4} l_e \circ l_{e'}. \tag{4.3.3}$$

An observation (Proposition A.6, [7]) that the double cover $\hat{\mathcal{C}}_{\Sigma_B}$ is holomorphically equivalent to the canonical double cover $\hat{\mathcal{C}}$, defined analytically by $v^2 = Q$, allows us to view these brackets in terms of corresponding cycles on $\hat{\mathcal{C}}$. Then one can consider linear combinations with half-integer coefficients of cycles l_e, generating the elements $\{a_j^-, b_j^-, t_k^-\}$ of the homology group H_-. Taking the same linear combinations of the elements $2\zeta_e$ one defines the homological shear coordinates $\{\rho_{a_j^-}, \rho_{b_j^-}, \rho_{t_k^-}\}$ with the following Poisson brackets:

$$\{\rho_{a_j^-}, \rho_{b_k^-}\}_G = \frac{\delta_{jk}}{2}, \quad \{\rho_{a_j^-}, \rho_{a_k^-}\}_G = \{\rho_{b_j^-}, \rho_{b_k^-}\}_G = 0, \tag{4.3.4}$$

while $\rho_{t_k^-}$ lie in the center of Poisson algebra. The coordinates $\rho_{t_k^-}$ are related to the monodromy eigenvalues as follows:

$$\rho_{t_k^-} = \log m_k. \tag{4.3.5}$$

Thus, on the symplectic leaf $CV_{g,n}\{\mathbf{m}\}$ for fixed values of m_k the Goldman symplectic form is written as

$$\Omega_G = \sum_{j=1}^{g^-} 2d\rho_{a_j^-} \wedge d\rho_{b_j^-}. \tag{4.3.6}$$

Corollary 4.3.1. *The homological symplectic form Ω_{hom} on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ is minus pullback of the Goldman symplectic form Ω_G by the map $\mathcal{F}_{(S_B)}$*

$$\mathcal{F}^*_{(S_B)} \Omega_G = -\Omega_{hom}. \tag{4.3.7}$$

4.3.2 Admissible meromorphic projective connections

The map between the space of quadratic differentials $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ and the character variety $CV_{g,n}\{\mathbf{m}\}$ essentially depends on the choice of the base projective connection on a Riemann surface $\mathcal{C}$. To parametrize space of such connections we consider the holomorphic affine bundle $\mathbb{S}_{g,n}$ of meromorphic projective connections with at most simple poles at the punctures over the moduli space of closed curves $\mathcal{M}_{g,n}$. The Theorem 4.3.1 states that for the choice $S = S_B$ the monodromy map is symplectic. The naturally arising question is when the monodromy map with a fixed projective connection S other than S_B is also a symplectomorphism.

Definition 4.3.1. A holomorphic section S of the affine bundle $\mathbb{S}_{g,n}$ is called *admissible* if the homological symplectic structure on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ implies Goldman bracket on the character variety $CV_{g,n}\{\mathbf{m}\}$.

For any two choices of $S_0, S_1 \in \mathbb{S}_{g,n}$ we write the same equation in two ways

$$\partial^2\phi + \left(\frac{1}{2}S_0 - Q_0\right)\phi = 0 \tag{4.3.8}$$

and

$$\partial^2\phi + \left(\frac{1}{2}S_1 - Q_1\right)\phi = 0, \tag{4.3.9}$$

where both Q_0 and Q_1 belong to $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ and are related by

$$Q_1 - Q_0 = \frac{1}{2}(S_0 - S_1). \tag{4.3.10}$$

We have the following diagram of maps

$$\begin{tikzcd} \mathcal{Q}_{g,n}\{\mathbf{r}\} \arrow[r, "Q_0 \xrightarrow{\mathcal{H}} Q_1"] \arrow[dr, "\mathcal{F}_{(S_0)}"'] & \mathcal{Q}_{g,n}\{\mathbf{r}\} \arrow[d, "\mathcal{F}_{(S_1)}"] \\ & CV_{g,n}\{\mathbf{m}\} \end{tikzcd}$$

Assuming that S_0 is admissible, the condition for S_1 to be also admissible (or *equivalent* to S_0) is that the map

$$\mathcal{H} : Q_0 \to Q_0 + \frac{1}{2}(S_0 - S_1) \tag{4.3.11}$$

is a symplectomorphism implying the coincidence of homological 2-forms calculated via the periods of v_0 and v_1, where $v_0^2 = Q_0,\ v_1^2 = Q_1$ define canonical coverings with different conformal structures. The following proposition gives a condition for the map (4.3.11) to be a symplectomorphism.

Proposition 4.3.1. *1) Two meromorphic differentials Q_0 and Q_1 induce the same homological 2-form on $\mathcal{Q}_{n,g}\{r\}$ iff the 1-form $\Theta_{(S_0-S_1)}$, corresponding to family of quadratic differentials $S_0 - S_1$ and locally defined on $\mathcal{M}_{g,n}$, is closed, $d\Theta_{(S_0-S_1)} = 0$.*

2) The generating function of the symplectomorphism between the periods $(A_k^{(0)}, B_k^{(0)})$ of v_0 and $(A_k^{(1)}, B_k^{(1)})$ of v_1 for the chosen potentials

$$\theta_0 = \sum_{k=1}^{g^-}(B_k^{(0)}dA_k^{(0)} - A_k^{(0)}dB_k^{(0)}), \quad \theta_1 = \sum_{k=1}^{g^-}(B_k^{(1)}dA_k^{(1)} - A_k^{(1)}dB_k^{(1)}), \tag{4.3.12}$$

defined by

$$d\mathcal{G}_{hom} = \mathcal{H}^*\theta_1 - \theta_0, \tag{4.3.13}$$

has the following form:

$$\mathcal{G}_{hom} = \sum_{i=1}^{n} \pi i r_i \left(reg \int_{z_i^{(2)}}^{z_i^{(1)}} v_1 - reg \int_{z_i^{(2)}}^{z_i^{(1)}} v_0 \right) + \frac{1}{2} G_{(S_0 - S_1)}, \tag{4.3.14}$$

where there exists a local holomorphic function $G_{(S_0-S_1)}$ on $\mathcal{M}_{g,n}$, such that

$$dG_{(S_0-S_1)} = \Theta_{(S_0-S_1)}. \tag{4.3.15}$$

Remark 4.3.1. For any $Q \in \mathcal{Q}_{g,n}\{\mathbf{r}\}$ the integral $\int_{z_i^{(2)}}^{z_i^{(1)}} v$ is singular at the endpoints. We define its regularization by removing the divergent part as follows: fix a coordinate ξ_j near z_j, such that

$$Q(x) \sim \Big(\frac{r_j^2}{\xi_j^2} + O(\xi_j^{-1})\Big)(d\xi_j)^2. \tag{4.3.16}$$

ξ_j can also serve as a local coordinate on $\hat{\mathcal{C}}$ near the lifts $\{z_j^{(1)}, z_j^{(2)}\}$ with

$$v(x) \sim \pm\Big(\frac{r_j}{\xi_j} + O(1)\Big)d\xi_j. \tag{4.3.17}$$

Let z_j^t be an arbitrary sequence of points on $\mathcal{C}$ converging to z_j, such that in the local coordinate ξ_j

$$\mathfrak{Re}(\xi_j(z_j^t)) \sim \frac{1}{t},\ t \to \infty; \quad \mathfrak{Im}(\xi_j(z_j^t)) = 0. \tag{4.3.18}$$

Then the regularization is defined by

$$\operatorname{reg} \int_{z_j^{(2)}}^{z_j^{(1)}} v := \lim_{t\to\infty} \left(\int_{z_j^{t\,(2)}}^{z_j^{t\,(1)}} v - 2r_j \log t \right). \tag{4.3.19}$$

Before proceeding to the proof of Proposition 4.3.1 we will prove the following technical lemma, which could be viewed as a 1-form version of Riemann Bilinear Identity (the RBI). Introduce the pairing between any two meromorphic differentials w_1, w_2 on $\hat{\mathcal{C}}$.

$$\left\langle \oint w_1, \oint w_2 \right\rangle := \sum_{j=1}^{g^-} \left[\oint_{b_j^-} w_1 \oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \oint_{b_j^-} w_2 \right] \tag{4.3.20}$$

Lemma 4.3.1. *Let w_1 and w_2 be two meromorphic differentials on $\hat{\mathcal{C}}$, skew-symmetric under involution. Assuming both w_1 and w_2 holomorphically depend on moduli $(A_i, B_i)_{i=1}^{g^-}$, the following identity of 1-forms on $\mathcal{Q}_{g,n}\{\boldsymbol{r}\}$ holds:*

$$\left\langle \oint w_1, d\oint w_2 \right\rangle = -\frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0} \left(\delta w_2 \int_{p_0}^{x} w_1 \right) + \left\langle \oint \frac{w_1 w_2}{v}, d\oint v \right\rangle \tag{4.3.21}$$

or

$$\left\langle \oint w_1, d\oint w_2 \right\rangle = \frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0} \left(w_1 \int_{p_0}^{x} \delta w_2 \right) + \left\langle \oint \frac{w_1 w_2}{v}, d\oint v \right\rangle. \tag{4.3.22}$$

Here $\hat{\mathcal{C}}_0$ is the fundamental polygon of the covering surface $\hat{\mathcal{C}}$, p_0 is a generic point.

Remark 4.3.2. Note that by $d\oint w$ appearing in the pairing we mean differential applied to the periods of w, while δw is defined by

$$\delta w := \sum_{i=1}^{g^-}\left(\frac{\partial w}{\partial A_i}\Big|_{z(x)} dA_i + \frac{\partial w}{\partial B_i}\Big|_{z(x)} dB_i\right), \tag{4.3.23}$$

where the differentiation is performed assuming the coordinate $z(x)$ (4.2.9) is independent of the moduli.

Proof. Expressing the differential in coordinates $(A_i, B_i)_{i=1}^{g^-}$ we write the pairing on the left-hand side of (4.3.21) as follows

$$\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \; d\oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \; d\oint_{b_j^-} w_2\right) = \tag{4.3.24}$$

$$= \sum_{i=1}^{g^-}\left[\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \; \frac{\partial}{\partial A_i}\oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \; \frac{\partial}{\partial A_i}\oint_{b_j^-} w_2\right) dA_i + \right. \tag{4.3.25}$$

$$\left. + \sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \; \frac{\partial}{\partial B_i}\oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \; \frac{\partial}{\partial B_i}\oint_{b_j^-} w_2\right)\right] dB_i. \tag{4.3.26}$$

Take a reference point p_0 and consider a canonical dissection of the covering surface along the cycles in $H_1(\hat{\mathcal{C}})$ to obtain the fundamental polygon $\hat{\mathcal{C}}_0$. The coordinate $z(x) = \int_{p_0}^{x} v$ serves as a local coordinate on $\hat{\mathcal{C}}_0$ outside branch points and poles and is kept fixed while differentiating with respect to the moduli. Consider the expression (4.3.25) near dA_i. When we differentiate the integral over a_i^- with respect to the variable $A_i = \oint_{a_i^-} v$ an additional term appears:

$$\frac{\partial}{\partial A_i}\oint_{a_i^-} w_2 = \frac{w_2}{v}(R_i) + \oint_{a_i^-}\frac{\partial w_2}{\partial A_i}, \tag{4.3.27}$$

where R_i is an intersection point of the cycles a_i^- and $2b_i^-$ (recall the intersection index $a_i^- \circ b_j^- = \frac{\delta_{ij}}{2}$), whereas all other integrals commute with the differentiation with respect to coordinate A_i. That is due to the following fact: in terms of "flat" coordinate $z(x)$, cycle a_i^- becomes a part of the boundary of the fundamental polygon $\hat{\mathcal{C}}_0$. Let us write $w_2(x) = f(z)dz$, for $x \in \hat{\mathcal{C}}_0$. Then the integral $\oint_{a_i^-} w_2 = \oint_{a_i^-} f(z)dz$ is an integral with variable upper limit: when the coordinate $A_i = \oint_{a_i^-} dz$ gets an increment this upper limit gets the same increment. Thus, after differentiation of the integral $\oint_{a_i^-} w_2$ an extra term appears: the value of the integrand at the end point of the contour a_i^- (that is the point R_i). In our case all cycles deformed to have a common intersection point p_0, so $R_i = p_0$. Therefore, we can rewrite the term near dA_i as

$$\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \; \oint_{a_j^-}\frac{\partial w_2}{\partial A_i} - \oint_{a_j^-} w_1 \; \oint_{b_j^-}\frac{\partial w_2}{\partial A_i}\right) + \frac{w_2}{v}(p_0)\oint_{b_j^-} w_1. \tag{4.3.28}$$

Since w_2 is globally defined on $\hat{\mathcal{C}}$, it is invariant under analytic continuation along the cycles in $H_-(\hat{\mathcal{C}})$. Writing $w_2(x + a_i^-) = w_2(x)$, we have that in coordinate $z(x)$:

$$f(z + A_i) = f(z). \tag{4.3.29}$$

Differentiating this equality with respect to z, we get

$$\frac{\partial f(z+A_i)}{\partial z}=\frac{\partial f(z)}{\partial z}. \tag{4.3.30}$$

Differentiating (4.3.29) again with respect to A_i, while z is kept constant, we also mind that f implicitly depends on A_i:

$$\frac{\partial f(z+A_i)}{\partial z}+\frac{\partial f(z+A_i)}{\partial A_i}=\frac{\partial f(z)}{\partial A_i}. \tag{4.3.31}$$

Combining these formulas, we write

$$\frac{\partial f(z+A_i)}{\partial A_i}dz-\frac{\partial f(z)}{\partial A_i}dz=-\frac{\partial f(z)}{\partial z}dz, \tag{4.3.32}$$

or in invariant form

$$\frac{\partial w_2}{\partial A_i}(x+a_i^-)-\frac{\partial w_2}{\partial A_i}(x)=-d\Big(\frac{w_2}{v}\Big). \tag{4.3.33}$$

Hence, the differential $\frac{\partial}{\partial A_i}w_2$ could be seen as meromorphic on $\hat{\mathcal{C}}$ with a jump discontinuity $-d(\frac{w_2}{v})$ on the cycle $2b_i^-$. Denote by $F:=\int_{p_0}^x w_1$. We apply a modification of the RBI for differentials having discontinuities along the homology cycles. Splitting the integral over the boundary of $\hat{\mathcal{C}}_0$ into even and odd parts of $H_1(\hat{\mathcal{C}},\mathbb{Z})$ and recalling that the intersection index is $a_i^+\circ b_j^+=a_i^-\circ b_j^-=\frac{\delta_{ij}}{2}$ one has

$$\int_{\partial\hat{\mathcal{C}}_0}F\frac{\partial w_2}{\partial A_i}=\sum_{j=1}^{g^-}\Bigg[\Bigg(\oint_{2b_j^-}F\frac{\partial w_2}{\partial A_i}+\oint_{(2b_j^-)^{-1}}F\frac{\partial w_2}{\partial A_i}\Bigg)+\Bigg(\oint_{a_j^-}F\frac{\partial w_2}{\partial A_i}+\oint_{(a_j^-)^{-1}}F\frac{\partial w_2}{\partial A_i}\Bigg)+ \tag{4.3.34}$$

$$+\Bigg(\oint_{2b_j^+}F\frac{\partial w_2}{\partial A_i}+\oint_{(2b_j^+)^{-1}}F\frac{\partial w_2}{\partial A_i}\Bigg)+\Bigg(\oint_{a_j^+}F\frac{\partial w_2}{\partial A_i}+\oint_{(a_j^+)^{-1}}F\frac{\partial w_2}{\partial A_i}\Bigg)\Bigg]. \tag{4.3.35}$$

Consider the following term of the above sum:

$$\oint_{2b_i^-}F\frac{\partial w_2}{\partial A_i}+\oint_{(2b_i^-)^{-1}}F\frac{\partial w_2}{\partial A_i}. \tag{4.3.36}$$

It could be rewritten as

$$\oint_{2b_i^-}\frac{\partial w_2}{\partial A_i}(P)\int_{p_0}^{P}w_1-\oint_{2b_i^-}\frac{\partial w_2}{\partial A_i}(P')\int_{p_0}^{P'}w_1, \tag{4.3.37}$$

where P,P' are identified points on $2b_i^-$ and $(2b_i^-)^{-1}$ cycles, respectively. $P'=P-a_i^-$. That means that $P,\ P'$ lie on the different sides of a cycle $2b_i^-$, where $\frac{\partial w_2}{\partial A_i}$ gains a jump. Then

$$\frac{\partial w_2}{\partial A_i}(P')=\frac{\partial w_2}{\partial A_i}(P)-\text{"}jump\text{"}=\frac{\partial w_2}{\partial A_i}(P)+d\Big(\frac{w_2}{v}\Big)(P). \tag{4.3.38}$$

Plugging it into (4.3.37) we rewrite this expression as

$$\oint_{2b_i^-}\frac{\partial w_2}{\partial A_i}\oint_{a_i^-}w_1-\oint_{2b_i^-}d\Big(\frac{w_2}{v}\Big)(P)\int_{p_0}^{P}w_1. \tag{4.3.39}$$

Integrating the second term by parts we get:

$$\oint_{2b_i^-} F\frac{\partial w_2}{\partial A_i} + \oint_{(2b_i^-)^{-1}} F\frac{\partial w_2}{\partial A_i} = \oint_{2b_i^-} \frac{\partial w_2}{\partial A_i} \oint_{a_i^-} w_1 - \frac{w_2}{v}(p_0) \oint_{2b_i^-} w_1 + \oint_{2b_i^-} \frac{w_1 w_2}{v}. \tag{4.3.40}$$

In all the the remaining terms of (4.3.34) the differential $\frac{\partial w_2}{\partial A_i}$ does not gain jump discontinuities and they could be commonly expressed:

$$\oint_{2b_j^-} F\frac{\partial w_2}{\partial A_i} + \oint_{(2b_j^-)^{-1}} F\frac{\partial w_2}{\partial A_i} = \oint_{2b_j^-} \frac{\partial w_2}{\partial A_i} \oint_{a_j^-} w_1, \quad j \neq i, \tag{4.3.41}$$

$$\oint_{a_j^-} F\frac{\partial w_2}{\partial A_i} + \oint_{(a_j^-)^{-1}} F\frac{\partial w_2}{\partial A_i} = -\oint_{a_j^-} \frac{\partial w_2}{\partial A_i} \oint_{2b_j^-} w_1, \quad \forall j, \tag{4.3.42}$$

$$\oint_{2b_j^+} F\frac{\partial w_2}{\partial A_i} + \oint_{(2b_j^+)^{-1}} F\frac{\partial w_2}{\partial A_i} = \oint_{2b_j^+} \frac{\partial w_2}{\partial A_i} \oint_{a_j^+} w_1, \quad \forall j, \tag{4.3.43}$$

$$\oint_{a_j^+} F\frac{\partial w_2}{\partial A_i} + \oint_{(a_j^+)^{-1}} F\frac{\partial w_2}{\partial A_i} = -\oint_{a_j^+} \frac{\partial w_2}{\partial A_i} \oint_{2b_j^+} w_1, \quad \forall j. \tag{4.3.44}$$

The integrals in (4.3.43-4.3.44) over a^+, b^+ cycles vanish due to skew symmetry of w_1. Thus, (4.3.34) could be rewritten as

$$\frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0} F\frac{\partial w_2}{\partial A_i} = -\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \oint_{a_j^-} \frac{\partial w_2}{\partial A_i} - \oint_{a_j^-} w_1 \oint_{b_j^-} \frac{\partial w_2}{\partial A_i}\right) - \frac{w_2}{v}(p_0)\oint_{b_i^-} w_1 + \oint_{b_i^-} \frac{w_1 w_2}{v}. \tag{4.3.45}$$

Comparing the expressions (4.3.28) and (4.3.45) we see that

$$\left[\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \frac{\partial}{\partial A_i}\oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \frac{\partial}{\partial A_i}\oint_{b_j^-} w_2\right)\right] dA_i = \left[-\frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0} F\frac{\partial w_2}{\partial A_i} + \oint_{b_i^-} \frac{w_1 w_2}{v}\right] dA_i \tag{4.3.46}$$

Similarly, one can show that

$$\left[\sum_{j=1}^{g^-}\left(\oint_{b_j^-} w_1 \frac{\partial}{\partial B_i}\oint_{a_j^-} w_2 - \oint_{a_j^-} w_1 \frac{\partial}{\partial B_i}\oint_{b_j^-} w_2\right)\right] dB_i = \left[-\frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0} F\frac{\partial w_2}{\partial B_i} - \oint_{a_i^-} \frac{w_1 w_2}{v}\right] dB_i. \tag{4.3.47}$$

Plugging these expressions into (4.3.25, 4.3.26) one obtains the formula (4.3.21). (4.3.22) follows from (4.3.21) by applying the Stokes' theorem and the fact that in the interior of $\hat{\mathcal{C}}_0$ away from poles, when differentiating with respect to any local coordinate ξ, one has:

$$d_\xi\left(\frac{\partial w_2}{\partial \mathcal{P}}\int_{p_0}^x w_1\right) = \frac{\partial w_2}{\partial \mathcal{P}} \wedge w_1 = -w_1 \wedge \frac{\partial w_2}{\partial \mathcal{P}} = -d_\xi\left(w_1 \int_{p_0}^x \frac{\partial w_2}{\partial \mathcal{P}}\right) = 0, \tag{4.3.48}$$

where $\mathcal{P} \in (A_i, B_i)_{i=1}^{g^-}$ □

Proof of Proposition 4.3.1. **Case of two nearby differentials** $Q_0, Q_1 \in \mathcal{Q}_{g,n}\{\mathbf{r}\}$: assume that a differential Q_0 defines double cover $\hat{\mathcal{C}}$ by equation $v_0^2 = Q_0$. Let U be a simply-connected neighborhood of Q_0 and take $Q_1 \in U$. For sufficiently small $\hbar$ this differential may be expressed as

$Q_1 = Q_0 + \hbar\tilde{Q}$, where $\tilde{Q}$ is a quadratic differential with at most simple poles at $(z_j)_{j=1}^n$. The canonical cover $\hat{\mathcal{C}}_\hbar$, defined by $(v_1^\hbar)^2 = Q_1$, becomes $\hbar$-dependent. Consider $\mathcal{P}_\hbar$ to be one of the periods $(A_i^{(1)}, B_i^{(1)})$ of $v_1^\hbar$. Then its k'th derivative with respect to $\hbar$ is given by:

$$\frac{\partial^k}{\partial\hbar^k}\mathcal{P}_\hbar\Big|_{\hbar=0} = (-1)^{k+1}\frac{(2k-3)!!}{2^k}\oint_s \frac{\tilde{Q}^k}{v_0^{2k-1}}, \tag{4.3.49}$$

where s is an element of $H_-(\hat{\mathcal{C}}, \mathbb{Z})$.

To justify these formulas consider $\mathcal{P}_\hbar = \oint_{s(\hbar)} v_1^\hbar$ as the integral on the base curve $\mathcal{C}$ via the projection $\pi : \hat{\mathcal{C}}_\hbar \to \mathcal{C}$. If cycle $s(\hbar)$ belongs to the subset of $H_-(\hat{\mathcal{C}}_\hbar)$, that could be represented in the form $(c - c^\mu)(\hbar)$, where $c, c^\mu(\hbar)$ are two lifts of the corresponding cycle $c \in H_1(\mathcal{C})$, using skew-symmetry of $v_1^\hbar$ it projects onto this cycle, which is independent of $\hbar$. If $s(\hbar)$ is a cycle on a handle of $\hat{\mathcal{C}}_\hbar$ obtained by gluing along branch cuts on two copies of $\mathcal{C}$, it projects onto the contour encircling or passing through the branch cut arranged between pairs of zeroes of $Q_1^\hbar$. Despite the positions of the branch points vary along with $\hbar$, as the integral depends only on the homotopy class of the cycle, we may assume that the projection $\pi(s(\hbar))$ is kept fixed on $\mathcal{C}$. In either case, differentiation commutes with the integral and one has

$$\frac{\partial}{\partial\hbar}\oint_{s(\hbar)} v_1^\hbar = \frac{\partial}{\partial\hbar}\oint_{\pi(s(\hbar))}\sqrt{Q_0 + \hbar\tilde{Q}} = \oint_{\pi(s(\hbar))}\frac{\partial}{\partial\hbar}\sqrt{Q_0 + \hbar\tilde{Q}}, \tag{4.3.50}$$

and the differentiation is followed by pullback to $\hat{\mathcal{C}}_\hbar$. Higher derivatives are obtained the same way.

Applying this argument, we can expand period coordinates by powers of $\hbar$. Write

$$\mathcal{P}_\hbar = \mathcal{P}_\hbar\Big|_{\hbar=0} + \frac{\partial}{\partial\hbar}\mathcal{P}_\hbar\Big|_{\hbar=0}\hbar + \dots + \frac{\partial^k}{\partial\hbar^k}\mathcal{P}_\hbar\Big|_{\hbar=0}\frac{\hbar^k}{k!} + \dots = \tag{4.3.51}$$

$$= \oint_s v_0 + \frac{\hbar}{2}\oint_s \frac{\tilde{Q}}{v_0} + \dots + \hbar^k\binom{\frac{1}{2}}{k}\oint_s \frac{\tilde{Q}^k}{v_0^{2k-1}} + \dots \tag{4.3.52}$$

Plugging $\hbar$-expansions of the periods $(A_i^{(1)}, B_i^{(1)})$ into the potential θ_1 (4.3.12) and arranging terms by powers of $\hbar$, we write with the help of the pairing notation (4.3.20):

$$\theta_1 = \left\langle \oint v_1, d\oint v_1 \right\rangle = \left\langle \oint v_0, d\oint v_0 \right\rangle + \hbar\left[\frac{1}{2}\left\langle \oint v_0, d\oint \frac{\tilde{Q}}{v_0}\right\rangle + \frac{1}{2}\left\langle \oint \frac{\tilde{Q}}{v_0}, d\oint v_0 \right\rangle\right] + \tag{4.3.53}$$

$$+\sum_{k=2}^{\infty}\hbar^k\left[\binom{\frac{1}{2}}{k}\left(\left\langle \oint v_0, d\oint \frac{\tilde{Q}^k}{v_0^{2k-1}}\right\rangle + \left\langle \oint \frac{\tilde{Q}^k}{v_0^{2k-1}}, d\oint v_0\right\rangle\right) + \right. \tag{4.3.54}$$

$$\left. + \sum_{l=1}^{k-1}\binom{\frac{1}{2}}{l}\binom{\frac{1}{2}}{k-l}\left\langle \oint \frac{\tilde{Q}^l}{v_0^{2l-1}}, d\oint \frac{\tilde{Q}^{k-l}}{v_0^{2(k-l)-1}}\right\rangle\right]. \tag{4.3.55}$$

We will treat separately the expressions near $\hbar^1$ and $\hbar^k,\ k \geq 2$.

Coefficient near $\hbar^1$: noticing that

$$d\left\langle \oint v_0, \oint \frac{\tilde{Q}}{v_0}\right\rangle = \left\langle d\oint v_0, \oint \frac{\tilde{Q}}{v_0}\right\rangle + \left\langle \oint v_0, d\oint \frac{\tilde{Q}}{v_0}\right\rangle, \tag{4.3.56}$$

the expression near $\hbar^1$ could be rewritten as

$$\frac{1}{2}d\left\langle \oint v_0, \oint \frac{\tilde{Q}}{v_0}\right\rangle + \left\langle \oint \frac{\tilde{Q}}{v_0}, d\oint v_0\right\rangle. \tag{4.3.57}$$

Applying the RBI, the pairing $\left\langle \oint v_0, \oint \frac{\tilde{Q}}{v_0}\right\rangle$ could be written as the sum over residues inside the fundamental domain. Differential $\tilde{Q}$, being lifted to $\hat{\mathcal{C}}$, gains double zeroes at branch points x_i, which are the zeroes of Q_0, and simple poles at preimages of z_j. That makes $\frac{\tilde{Q}}{v_0}$ holomorphic, while v_0 has poles at $\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n$. Therefore using that near $z_j^{(1)}$ ($z_j^{(2)}$) in the local coordinate (4.2.12) $v_0 = \pm\frac{r_j}{\zeta_i}d\zeta_i$ we write

$$d\left[\sum_{j=1}^n \pi i \underset{\{z_j^{(1)}, z_j^{(2)}\}}{res}\left(v_0\int_{p_0}^x \frac{\tilde{Q}}{v_0}\right)\right] = d\left[\sum_{j=1}^n \pi i\, r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{\tilde{Q}}{v_0}\right]. \tag{4.3.58}$$

We will also express the pairing $\left\langle \oint \frac{\tilde{Q}}{v_0}, d\oint v_0\right\rangle$ in a different form, introducing the system of local coordinates on $\mathcal{M}_{g,n}$. For simplicity here we restrict us to the case $g \geq 2$ (low genus cases $g = 0, 1$ could be covered by analogy following [32]). At generic point of the moduli space $\mathcal{M}_{g,n}$ the quadratic differential $\tilde{Q}$ could be represented as a linear combination of $3g-3$ products of normalized holomorphic differentials $u_j u_k$, where $(jk) \in D$ for some subset D of entries of matrix Ω, and additional n quadratic differentials encoding the meromorphic part could be represented by the following generically meromorphic differentials Q_k whose only pole of order one located at z_k :

$$Q^{z_k}(t) = \frac{1}{4\pi i}\frac{u_i(t)u_j(z_k) - u_i(z_k)u_j(t)}{u_j^2(z_k)}B(t, z_k), \tag{4.3.59}$$

here u_i and u_j are two arbitrary normalized holomorphic differentials such that $u_j(z_k) \neq 0$.

Using the variational formulas (4.2.26) after lifting the function $\frac{u_i}{u_j}(x)$ to $\hat{\mathcal{C}}$ we get

$$\frac{\partial}{\partial \mathcal{P}_s}\left[\frac{u_i}{u_j}(z_k)\right] = \frac{\partial}{\partial \mathcal{P}_s}\left[\frac{u_i}{u_j}(z_k^{(1)})\right] = \oint_{s^*}\frac{Q^{z_k^{(1)}}}{v_0}, \tag{4.3.60}$$

where $Q^{z_k^{(1)}}$ is a lift of Q^{z_k} to $\hat{\mathcal{C}}$. The entries Ω_{jk}, $(jk) \in D$ of the period matrix can serve as the moduli of the base curve $\mathcal{C}$, while $\frac{u_i}{u_j}(z_k) := q_k$ code the positions of poles, providing in total $3g - 3 + n$ local coordinates on $\mathcal{M}_{g,n}$.

At generic point of $\mathcal{M}_{g,n}$ quadratic differential $\tilde{Q}$ can be expressed as

$$\tilde{Q} = \sum_{(jk)\in D} p_{jk}\, u_j u_k + \sum_{l=1}^n p_l Q^{z_l}, \quad p_{jk}, p_l \in \mathbb{C}. \tag{4.3.61}$$

Then applying variational formulas (4.2.25) and (4.3.60) one has

$$\left\langle \oint \frac{\tilde{Q}}{v_0}, d\oint v_0\right\rangle = \sum_{j=1}^{g_-}\left[\Big(\oint_{b_j^-}\frac{\tilde{Q}}{v_0}\Big)dA_j^{(0)} - \Big(\oint_{a_j^-}\frac{\tilde{Q}}{v_0}\Big)dB_j^{(0)}\right] = \tag{4.3.62}$$

$$= \sum_{(jk)\in D} p_{jk} \sum_{j=1}^{g-} \left[\left(\oint_{b_j^-} \frac{u_j u_k}{v_0} \right) dA_j^{(0)} - \left(\oint_{a_j^-} \frac{u_j u_k}{v_0} \right) dB_j^{(0)} \right] + \tag{4.3.63}$$

$$+ \sum_{l=1}^{n} p_l \sum_{j=1}^{g-} \left[\left(\oint_{b_j^-} \frac{Q^{z_l^{(1)}}}{v_0} \right) dA_j^{(0)} - \left(\oint_{a_j^-} \frac{Q^{z_l^{(1)}}}{v_0} \right) dB_j^{(0)} \right] = \tag{4.3.64}$$

$$= \sum_{(jk)\in D} p_{jk} \sum_{j=1}^{g-} \left[\frac{\partial \Omega_{ij}}{\partial A_j^{(0)}} dA_j^{(0)} + \frac{\partial \Omega_{ij}}{\partial B_j^{(0)}} dB_j^{(0)} \right] + \sum_{l=1}^{n} p_l \sum_{j=1}^{g-} \left[\frac{\partial q_l}{\partial A_j^{(0)}} dA_j^{(0)} + \frac{\partial q_l}{\partial B_j^{(0)}} dB_j^{(0)} \right] = \tag{4.3.65}$$

$$= \sum_{(jk)\in D} p_{jk} d\Omega_{jk} + \sum_{l=1}^{n} p_l dq_l. \tag{4.3.66}$$

Therefore, the term near $\hbar^1$ becomes

$$d\left[\sum_{j=1}^{n} \frac{\pi i\, r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{\tilde{Q}}{v_0} \right] + \sum_{(jk)\in D} p_{jk} d\Omega_{jk} + \sum_{l=1}^{n} p_l dq_l. \tag{4.3.67}$$

Coefficients near $\hbar^k,\ k \geq 2$: by Lemma 4.3.1 we can rewrite the pairings appearing in (4.3.54) as

$$\left\langle \oint \frac{\tilde{Q}^l}{v_0^{2l-1}}, d \oint \frac{\tilde{Q}^{k-l}}{v_0^{2(k-l)-1}} \right\rangle = \frac{1}{2} \int_{\partial \hat{\mathcal{C}}_0} \left(\frac{\tilde{Q}^l}{v_0^{2l-1}} \int_{p_0}^{x} \delta \frac{\tilde{Q}^{k-l}}{v_0^{2(k-l)-1}} \right) + \left\langle \oint \frac{\tilde{Q}^k}{v_0^{2k-1}}, d \oint v_0 \right\rangle. \tag{4.3.68}$$

Also

$$\left\langle \oint v_0, d \oint \frac{\tilde{Q}^k}{v_0^{2k-1}} \right\rangle = \frac{1}{2} \int_{\partial \hat{\mathcal{C}}_0} \left(v_0 \int_{p_0}^{x} \delta \frac{\tilde{Q}^k}{v_0^{2k-1}} \right) + \left\langle \oint \frac{\tilde{Q}^k}{v_0^{2k-1}}, d \oint v_0 \right\rangle. \tag{4.3.69}$$

Thus, the expression (4.3.54) near $\hbar^k,\ k \geq 2$ becomes

$$\binom{\frac{1}{2}}{k} \frac{1}{2} \int_{\partial \hat{\mathcal{C}}_0} \left(v_0 \int_{p_0}^{x} \delta \frac{\tilde{Q}^k}{v_0^{2k-1}} \right) - \sum_{l=1}^{k-1} \binom{\frac{1}{2}}{l} \binom{\frac{1}{2}}{k-l} \frac{1}{2} \int_{\partial \hat{\mathcal{C}}_0} \left(\delta \frac{\tilde{Q}^{k-l}}{v_0^{2(k-l)-1}} \int_{p_0}^{x} \frac{\tilde{Q}^l}{v_0^{2l-1}} \right) + \tag{4.3.70}$$

$$+ \left[\sum_{l=0}^{k} \binom{\frac{1}{2}}{l} \binom{\frac{1}{2}}{k-l} \right] \left\langle \oint \frac{\tilde{Q}^k}{v_0^{2k-1}}, d \oint v_0 \right\rangle. \tag{4.3.71}$$

Using the identity

$$1 + t = (\sqrt{1+t})^2 = \left(\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} t^k \right)^2 = \sum_{k=0}^{\infty} t^k \left[\sum_{l=0}^{k} \binom{\frac{1}{2}}{l} \binom{\frac{1}{2}}{k-l} \right],\ |t| \leq 1. \tag{4.3.72}$$

and comparing the expressions near same powers of t we conclude that the piece (4.3.71) is zero. Further, we can represent the expression in (4.3.70) as the sum over residues at the branch points x_i. Notice that the first term also has additional residues near $\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^{n}$ due to simple poles of v_0:

$$\sum_{j=1}^{n} \binom{\frac{1}{2}}{k} \pi i \underset{\{z_j^{(1)}, z_j^{(2)}\}}{res} \left(v_0 \int_{p_0}^{x} \delta \frac{\tilde{Q}^k}{v_0^{2k-1}} \right) + \tag{4.3.73}$$

$$+\sum_{i=1}^{4g-4+2n} \pi i \operatorname*{res}_{x_i}\left[\binom{\frac{1}{2}}{k} v_0 \int_{p_0}^{x} \delta \frac{\tilde{Q}^k}{v_0^{2k-1}} + \sum_{l=1}^{k-1} \binom{\frac{1}{2}}{l}\binom{\frac{1}{2}}{k-l}\left(\frac{\tilde{Q}^l}{v_0^{2l-1}} \int_{p_0}^{x} \delta \frac{\tilde{Q}^{k-l}}{v_0^{2(k-l)-1}}\right)\right]. \tag{4.3.74}$$

Similarly to (4.3.58), the sum (4.3.73) could be rewritten as

$$d\left[\sum_{j=1}^{n} \binom{\frac{1}{2}}{k} \pi i\, r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{\tilde{Q}^k}{v_0^{2k-1}}\right], \tag{4.3.75}$$

here we used that the derivatives with respect to the coordinates $(A_i^{(0)}, B_i^{(0)})_{i=1}^{g^-}$ commute with the integral.

Consider the expression defined on the double cover $\pi : \hat{\mathcal{C}} \to \mathcal{C}$, given by $v_0^2 = Q_0$ in $T^*\mathcal{C}$:

$$\operatorname*{res}_{x_i}\left(\sqrt{Q_0 + \hbar\tilde{Q}} \int_{p_0}^{x} \delta\sqrt{Q_0 + \hbar\tilde{Q}}\right), \tag{4.3.76}$$

where x_i is a zero of Q_0 on $\mathcal{C}$.

Formally, the Abelian differential $\hat{v}_\hbar = \sqrt{Q_0 + \hbar\tilde{Q}}$ is globally defined on the h-dependent double cover $\hat{\pi} : \hat{\hat{\mathcal{C}}}_\hbar \to \hat{\mathcal{C}}$, given by $(\hat{v}_\hbar)^2 = Q_0 + \hbar\tilde{Q}$ in $T^*\hat{\mathcal{C}}$ (note that $\hat{\mathcal{C}}$ itself is a double cover of $\mathcal{C}$). Lifted from $\mathcal{C}$ to $\hat{\mathcal{C}}$, Q_0 has a 4th-order zeros at x_i and double poles at $(z_j^{(1)}, z_j^{(2)})_{j=1}^{n}$, while $\tilde{Q}$ gains a 2nd-order zero at x_i and simple poles at $(z_j^{(1)}, z_j^{(2)})$. Thus, the map $\hat{\pi}$ is brached at $8g-8+4n$ simple zeroes $\tilde{x}_j^\hbar$ of $Q_0 + \hbar\tilde{Q}$. The double cover $\hat{\hat{\mathcal{C}}}_\hbar$ is smooth everywhere except for preimages of double zeroes $(x_i)_{i=1}^{4g-4+2n}$ of $Q_0 + \hbar\tilde{Q}$, where $\hat{\hat{\mathcal{C}}}_\hbar$ gains nodes. The genus $\hat{\hat{g}}$ of $\hat{\hat{\mathcal{C}}}_\hbar$ equals $12g - 11 + 4n$. Letting $\hbar \to 0$, the nodes smoothen out and the covering surface $\hat{\hat{\mathcal{C}}}_\hbar$ degenerates to the pair of smooth surfaces $\hat{\mathcal{C}}^{(1,2)}$. On the base curve $\hat{\mathcal{C}}$ that corresponds to the merging of triplets of points: two simple zeroes $\tilde{x}_{i_1}^\hbar, x_{i_2}^\hbar$ of $Q_0 + \hbar\tilde{Q}$ converge to a double zero at x_i, increasing its multiplicity to 4.

In the local coordinate $z(x)$ on $\hat{\mathcal{C}}$: $Q_0 = v_0^2 = dz^2$, $\tilde{Q} = \tilde{Q}(z)dz^2$. Differentiating with respect to the coordinates $(A_i^{(0)}, B_i^{(0)})_{i=1}^{g^-}$ according to the rule (4.3.23), when the coordinate $z(x)$ is kept fixed, one has that the residue could be written as

$$\operatorname*{res}_{x_i}\left(\sqrt{Q_0 + \hbar\tilde{Q}} \int_{p_0}^{x} \frac{\hbar\,\delta\tilde{Q}}{2\sqrt{Q_0 + \hbar\tilde{Q}}}\right) = 0. \tag{4.3.77}$$

The residue vanishes since the expression inside is holomorphic at x_i. Then we can expand the left-hand side by powers of $\hbar$ and observe that the coefficients near the powers of $\hbar$ in the series are exactly the terms appearing in the sum (4.3.74). It follows that these coefficients must vanish too. Thus, the coefficient near $\hbar^n$, $n \geq 2$ reduces to the expression (4.3.75).

Full expansion: combining (4.3.67) and (4.3.75) we have that

$$\theta_1 - \theta_0 = d\left[\sum_{j=1}^{n} \pi i\, r_j \left[\sum_{k=1}^{\infty} \hbar^k \binom{\frac{1}{2}}{k} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{\tilde{Q}^k}{v_0^{2k-1}}\right]\right] + \hbar\Big[\sum_{(jk)\in D} p_{jk} d\Omega_{jk} + \sum_{l=1}^{n} p_l dq_l\Big]. \tag{4.3.78}$$

We further notice, similarly to the argument in (4.3.49), that the infinite series is formally the Taylor expansion by powers of $\hbar$ of the expression

$$\int_{z_j^{(2)}(\hbar)}^{z_j^{(1)}(\hbar)} v_1^\hbar - \int_{z_j^{(2)}}^{z_j^{(1)}} v_0. \tag{4.3.79}$$

The issue is that the integrands are singular at the endpoints of the integration path, and one requires a regularization of the integral to have a proper identity. Considering the regularization proposed in (4.3.19) one can see that

$$\operatorname{reg}\int_{z_j^{(2)}(\hbar)}^{z_j^{(1)}(\hbar)} v_1^\hbar = \operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_0 + \sum_{k=1}^{\infty} \hbar^k \binom{\frac{1}{2}}{k} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{\tilde{Q}^k}{v_0^{2k-1}}, \tag{4.3.80}$$

where the integrals near $\hbar^k$ are already regular. Using that we rewrite the difference of potentials as

$$\theta_1 - \theta_0 = d\left[\sum_{j=1}^{n} \pi i\, r_j \left(\operatorname{reg}\int_{z_j^{(2)}(\hbar)}^{z_j^{(1)}(\hbar)} v_1^\hbar - \operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_0\right)\right] + \hbar\Big[\sum_{(jk)\in D} p_{jk} d\Omega_{jk} + \sum_{l=1}^{n} p_l dq_l\Big]. \tag{4.3.81}$$

Case of two arbitrary differentials $Q_0, Q_1 \in \mathcal{Q}_{g,n}\{\mathbf{r}\}$. Theorem 1.3 of [12] asserts that generically (outside of hyperelliptic locus for $g \geq 3$) space $\mathcal{Q}_{g,n}$, and thus $\mathcal{Q}_{g,n}\{\mathbf{r}\}$, is connected. Let $\gamma_t = Q_t : [0,1] \to \mathcal{Q}_{g,n}\{\mathbf{r}\}$ be a path such that $\gamma(0) = Q_0, \gamma(1) = Q_1$. For each t the function $\oint_{s_i} v_t$ is holomorphic on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ and could be expanded by the Taylor series in some simply-connected open neighborhood U_t of Q_t. Then $\bigcup_t U_t$ provides and open cover for γ_t. Due to compactness of γ_t we can choose some finite subcover $\bigcup_{t_i} U_{t_i},\ i \in \{0, ..., N\}$. We can assume $Q_0 = Q_{\hat{t}_0} \in U_{t_0}$, $Q_1 = Q_{\hat{t}_{N+1}} \in U_{t_N}$ and take $Q_{\hat{t}_i} \in \gamma \cap U_{t_i} \cap U_{t_{i-1}},\ i \in \{1, ..., N\}$ (see Figure 4).

Due to (4.3.81)

$$\theta_{\hat{t}_{i+1}} - \theta_{\hat{t}_i} = d\left[\sum_{j=1}^{n} \pi i r_j \Big(\operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_{\hat{t}_{i+1}} - \operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_{\hat{t}_i}\Big)\right] + \Big[\sum_{(jk)\in D} p_{jk}^{(\hat{t}_{i+1},\hat{t}_i)} d\Omega_{jk} + \sum_{j=1}^{n} p_l^{(\hat{t}_{i+1},\hat{t}_i)} dq_l\Big], \tag{4.3.82}$$

where $(p_{jk}^{(\hat{t}_{i+1},\hat{t}_i)}, p_l^{(\hat{t}_{i+1},\hat{t}_i)})$ are coefficients of the linear representation of $(Q_{\hat{t}_{i+1}} - Q_{\hat{t}_i})$ in the basis $(u_j u_k, Q^{z_l})$.

Then applying the telescoping series, one has

$$\begin{aligned} \theta_1 - \theta_0 &= \sum_{i=0}^{N} (\theta_{\hat{t}_{i+1}} - \theta_{\hat{t}_i}) = \\ &= d\left[\sum_{j=1}^{n} \pi i r_j \Big(\operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_1 - \operatorname{reg}\int_{z_j^{(2)}}^{z_j^{(1)}} v_0\Big)\right] + \Big[\sum_{(jk)\in D} p_{jk}^{(1,0)} d\Omega_{jk} + \sum_{l=1}^{n} p_l^{(1,0)} dq_l\Big], \end{aligned} \tag{4.3.83}$$

where the latter expression, using (4.3.10), is exactly the 1-form $\frac{1}{2}\Theta_{(S_0 - S_1)}$. Applying differential to both sides of (4.3.83) one obtains the first statement of the proposition. If $\Theta_{(S_0-S_1)}$ is assumed closed, then by the Poincare lemma it could be locally integrated, leading to the second statement. □

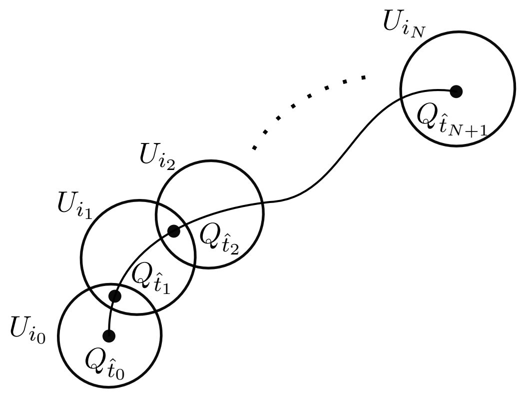

Figure 4: Sequence of differentials $Q_{\hat{t}_i}$

Remark 4.3.3. In cases when Q_i, $i = 0, 1$ are both holomorphic or with first order poles, following the lines (4.3.62-4.3.66) we can identify the potentials

$$\sum_{k=1}^{g^-}(B_k^{(i)} dA_k^{(i)} - A_k^{(i)} dB_k^{(i)}) = \sum_{(jk)\in D} p_{jk}^{(i)} d\Omega_{jk} + \sum_{l=1}^{n} p_l^{(i)} dq_l \tag{4.3.84}$$

and Proposition 4.3.1 immediately follows. Such approach was taken in [8] and [32].

Combining the results of Theorem 4.3.1 and Proposition 4.3.1 we can formulate a condition for projective connection S to become admissible.

Theorem 4.3.2. *The monodromy map*

$$\mathcal{F}_{(S)} : \mathcal{Q}_{g,n}\{\mathbf{r}\} \to CV_{g,n}\{\mathbf{m}\} \tag{4.3.85}$$

is a symplectomorphism with $\mathcal{F}_{(S)}^* \Omega_G = -\Omega_{\rm hom}$ iff the 1-form $\Theta_{(S-S_B)}$, corresponding to family of quadratic differentials $S - S_B$ and locally defined on $\mathcal{M}_{g,n}$, is closed, $d\Theta_{(S-S_B)} = 0$. Equivalently, iff there exists a local holomorphic function $G_{(S-S_B)}$ on $\mathcal{M}_{g,n}$, such that

$$dG_{(S-S_B)} = \Theta_{(S-S_B)}. \tag{4.3.86}$$

The computation similar to (4.3.62-4.3.66) (performed backwards) allows us to characterize the admissible projective connection in terms of the 1-form defined on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ in period coordinates.

Corollary 4.3.2. *The projective connection $S \in \mathbb{S}_{g,n}$ is admissible iff the following locally defined 1-form on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$*

$$\Theta_{(S-S_B)} = \sum_{j=1}^{g^-}\left[\left(\oint_{b_j^-}\frac{S-S_B}{v}\right)dA_j - \left(\oint_{a_j^-}\frac{S-S_B}{v}\right)dB_j\right] \tag{4.3.87}$$

is closed, $d\Theta_{(S-S_B)} = 0$.

The following corollary gives an alternative characterization of admissible projective connections which does not refer to the Bergman projective connection:

Corollary 4.3.3. *The projective connection $S \in \mathbb{S}_{g,n}$ is admissible iff the following locally defined 1-form on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$*

$$\Theta_{(S-S_v)} = \sum_{j=1}^{g^-} \left[\left(\oint_{b_j^-} \frac{S - S_v}{v} \right) dA_j - \left(\oint_{a_j^-} \frac{S - S_v}{v} \right) dB_j \right] \tag{4.3.88}$$

is closed, $d\Theta_{(S-S_v)} = 0$.

Proof. Notice, that from (4.2.34) it follows that 1-forms $\Theta_{(S-S_v)}$ and $\Theta_{(S-S_B)}$ differ by the closed form $(24\pi i) d \log \tau_B|_r$ so, their conditions of closeness are equivalent. □

In [8] authors discussed alternative ways of fixing the reference projective connection. It was showed that if S is chosen to be either Schottky, Wirtinger or Bers projective connection, it is equivalent to the Bergman projective connection S_B in the sense (4.3.86). While explicit formulas $G_{(S-S_B)}$ for Schottky and Wirtinger connections were derived in [8], for Bers connection it was only conjectured, and recently proven in [17]. Moreover, the definition of Bergman projective connection itself depends on the choice of Torelli marking on $\mathcal{C}$. Let two Torelli markings α^σ and α be related by $Sp(2g, \mathbb{Z})$ matrix

$$\sigma = \begin{pmatrix} D & C \\ B & A \end{pmatrix} : \quad \begin{pmatrix} b \\ a \end{pmatrix}^\sigma = \sigma \begin{pmatrix} b \\ a \end{pmatrix}. \tag{4.3.89}$$

Then two corresponding Bergman projective connections S_B^σ and S_B are related by

$$S_B^\sigma = S_B - 12\pi i \sum_{1 \le j \le k \le g} u_j u_k \frac{\partial}{\partial \Omega_{jk}} \log \det(C\Omega + D). \tag{4.3.90}$$

and also equivalent due (4.3.86) with the generating function $G_{(S_B^\sigma - S_B)}$ given by

$$G_{(S_B^\sigma - S_B)} = -12\pi i \log \det(C\Omega + D). \tag{4.3.91}$$

That allows us to formulate the following corollary of Theorem 4.3.2.

Corollary 4.3.4. *If $S \in \mathbb{S}_{g,n}$ is chosen to be either Bergman (corresponding to any Torelli marking), Schottky, Wirtinger or Bers (defined with respect their own data) then the monodromy map*

$$\mathcal{F}_{(S)} : \mathcal{Q}_{g,n}\{\mathbf{r}\} \to CV_{g,n}\{\mathbf{m}\} \tag{4.3.92}$$

is a symplectomorphism with $\mathcal{F}_{(S)}^* \Omega_G = -\Omega_{\text{hom}}$.

4.3.3 Definition of the monodromy generating function

Fixing the Bergman projective connection as the base connection $S = S_B$ we may choose a symplectic potential on the moduli space $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ in period coordinates

$$\theta_{hom} = \sum_{j=1}^{g^-} \Big(B_j dA_j - A_j dB_j \Big) \tag{4.3.93}$$

with another symplectic potential on the character variety $CV_{g,n}\{\mathbf{m}\}$ in homological shear coordinates

$$\theta_G = \sum_{j=1}^{g^-} \left(\rho_{b_j^-} d\rho_{a_j^-} - \rho_{a_j^-} d\rho_{b_j^-} \right) \tag{4.3.94}$$

and consider the generating function of symplectomorphism $\mathcal{F}_{(S_B)}$ (the Yang-Yang function introduced in [41]) given by

$$d\mathcal{G}_B = \mathcal{F}^*_{(S_B)}\theta_G - \theta_{hom}. \tag{4.3.95}$$

Assuming that the triangulation of the surface $\mathcal{C}$ used to define homological shear coordinates is specified by the horizontal trajectories of the GMN-differential Q, the remaining parameters that define the function $\mathcal{G}_B$ include the choice of the Torelli marking on $\mathcal{C}$ and the choice of generators (a_j^-, b_j^-) in H_-. It is easy to see, that the symplectic potentials are invariant under symplectic transformations of the generators in H_-. Namely, under the transformation $\sigma \in Sp(2g, \mathbb{Z})$

$$\sigma = \begin{pmatrix} C_- & A_- \\ D_- & B_- \end{pmatrix} : \quad \begin{pmatrix} b_- \\ a_- \end{pmatrix}^\sigma = \sigma \begin{pmatrix} b_- \\ a_- \end{pmatrix} \tag{4.3.96}$$

the potentials θ_{hom} and θ_G remain the same, leaving the function $\mathcal{G}_B$ also invariant. The question how the change of Torelli marking affects the monodromy generating function was posed in [7] and Proposition 4.3.1 allows us to provide the answer. Under the change (4.3.89) of the canonical basis of $\mathcal{C}$ the Goldman potential θ_G remains invariant, while the homological potentials θ^σ_{hom}, θ_{hom} for new and old Torelli markings are related by the term (4.3.14)

$$\theta^\sigma_{hom} = \theta_{hom} + d\mathcal{G}_{hom} \tag{4.3.97}$$

In our setting, with the help of (4.3.90) and (4.3.91) one has

$$Q_0 = Q, \quad Q_1 = Q + 6\pi i \sum_{1\le j\le k\le g} u_j u_k \frac{\partial}{\partial \Omega_{jk}} \log\det(C\Omega + D) \tag{4.3.98}$$

and

$$d\mathcal{G}_{hom} = d\left[\sum_{i=1}^n \pi i r_i \left(\operatorname{reg}\int_{z_i^{(2)}}^{z_i^{(1)}} v_1 - \operatorname{reg}\int_{z_i^{(2)}}^{z_i^{(1)}} v_0\right)\right] + 6\pi i d \log\det(C\Omega + D), \tag{4.3.99}$$

where $v_0^2 = Q_0$, $v_1^2 = Q_1$ define two different canonical coverings. Combining that with the definition of the generating function (4.3.95) we have

Proposition 4.3.2. *Under the change (4.3.89) of the Torelli marking the monodromy generating function transforms as*

$$\mathcal{G}^\sigma_B = \mathcal{G}_B + \sum_{i=1}^n \pi i r_i \left(reg\int_{z_i^{(2)}}^{z_i^{(1)}} v_1 - reg\int_{z_i^{(2)}}^{z_i^{(1)}} v_0\right) + 6\pi i \log\det(C\Omega + D), \tag{4.3.100}$$

where $v_0^2 = Q$ *and* $v_1^2 = Q + 6\pi i \sum_{1\le j\le k\le g} u_j u_k \frac{\partial}{\partial\Omega_{jk}} \log\det(C\Omega + D)$.

4.4 Generalized WKB expansion of the Monodromy generating function

The WKB method was originally introduced by Wentzel, Kramers, and Brillouin as a way of finding approximate solutions of the Schrödinger equation

$$\hbar^2\partial^2\phi(x,\hbar) + U(x,\hbar)\phi(x,\hbar) = 0 \tag{4.4.1}$$

in the semiclassical limit $\hbar \ll 1$. The asymptotic series that solve (4.4.1) are generically divergent and one needs to apply Borel resummation to obtain genuine analytic solutions [44]. Here we employ exact WKB method to study the asymptotic expansion of the Monodromy generating function. We assume that the potential $U(x, \hbar)$ has the form

$$U(x, \hbar) = Q(x) + \hbar Q_1(x) + \hbar^2 Q_2(x). \tag{4.4.2}$$

While this potential appears in a problem of characterization of Stokes graphs [25] and genus zero explicit computations of the isomonodromy deformations [10], in present context it serves as a generalization of the equation previously studied in [7] with regards to the symplectic geometry of the monodromy map.

In our assumption Q is a quadratic differential varying within $\mathcal{Q}_{g,n}\{\mathbf{r}\}$, Q_1 is a holomorphic section of $T^*\mathcal{M}_{g,n}$ (i.e., a quadratic differential with at most simple poles at the marked points), and $Q_2 = \frac{1}{2} S_B$ is the Bergman projective connection.

4.4.1 WKB approximation of the Schrödinger equation

To study the asymptotic expansion of the monodromy generating function $\mathcal{G}_B$ we consider the second order equation in the form

$$\partial^2 \phi + \Big(\frac{1}{2} S_B - \frac{Q_1}{\hbar} - \frac{Q}{\hbar^2}\Big)\phi = 0, \tag{4.4.3}$$

where $Q \in \mathcal{Q}_{g,n}\{\mathbf{r}\}$, while Q_1 is a fixed meromorphic quadratic differential assumed to depend holomorphically on moduli of $\mathcal{M}_{g,n}$, with at most simple poles at the punctures $(z_j)_{j=1}^n$. This is a generalized version of the equation originally studied in [7], where $Q_1 \equiv 0$.

The WKB approximation for this equation is performed in the following way: consider the canonical double cover $\hat{\mathcal{C}}_\hbar$ given by the equation

$$v_\hbar^2 = \frac{Q}{\hbar^2}. \tag{4.4.4}$$

Rescaling the differential $v = \hbar v_\hbar$ pass to the cover $v^2 = Q$ which is now independent of $\hbar$. Choose some base point x_0. In terms of local coordinate $z(x) = \int_{x_0}^x v$ and the function $\varphi(x) = \phi\sqrt{v(x)}$ equation (4.4.3) takes the form

$$\varphi_{zz} + (q(z) - \hbar^{-2} - p(z)\hbar^{-1})\varphi = 0, \tag{4.4.5}$$

where

$$q = \frac{S_B - S_v}{2v^2} \quad p = \frac{Q_1}{v^2}, \tag{4.4.6}$$

(notice that in local coordinate $z(x)$ the Schwarzian projective connection (4.2.18) vanishes). Introducing the asymptotic series $s = \sum_{k=-1}^{\infty} \hbar^k s_k$ write the solution for (4.4.5) in the form

$$f_{x_0} = v^{-\frac{1}{2}} \exp \int_{x_0}^x (\hbar^{-1} s_{-1} + s_0 + \hbar s_1 + ...)v, \tag{4.4.7}$$

where s_k are meromorphic functions on $\hat{\mathcal{C}}$. The asymptotic series s satisfies the Ricatti equation:

$$ds + s^2 v = -qv + \hbar^{-1} pv + \hbar^{-2} v. \tag{4.4.8}$$

Then plugging its expansion into (4.4.8) and comparing terms near the same powers of $\hbar$ one gets the following first terms s_k :

$$s_{-1} = \pm 1, \quad s_0 = \frac{p}{2s_{-1}}, \quad s_1 = -\frac{dp}{4v} - \frac{1}{2s_{-1}}\left(\frac{p^2}{4} + q\right), \tag{4.4.9}$$

while the consecutive terms satisfy the recurrence relation

$$s_{k+1} = -\frac{1}{2s_{-1}}\left(\frac{ds_k}{v} + \sum_{\substack{j+l=k \\ j,l\geq 0}} s_j s_l\right), \ k \geq 1. \tag{4.4.10}$$

In particular, when $k = 2$, we get:

$$s_2 = \frac{1}{8v} d\left(\frac{dp}{s_{-1}v} + \frac{p^2}{2} + 2q\right) - \frac{pdp}{8v} + \frac{1}{4s_{-1}}\left(qp - \frac{p^3}{4}\right). \tag{4.4.11}$$

There is an ambiguity in choosing the value of s_{-1}, which corresponds to the choice of the sign for the square root $\sqrt{Q}$. Further below we shall assume that $s_{-1} = +1$. To obtain another asymptotic series corresponding to $s_{-1} = -1$ it is sufficient to apply involution μ to get $\mu^* v = -v$. We define *even* and *odd* part of the asymptotic series s by

$$s_{odd} = \frac{1}{2}(s + \mu^* s), \quad s_{even} = \frac{1}{2}(s - \mu^* s). \tag{4.4.12}$$

Notice that $\mu^* s_{odd} = s_{odd}$ and $\mu^* s_{even} = -s_{even}$, while

$$\mu^*(s_{odd} v) = -s_{odd} v, \quad \mu^*(s_{even} v) = s_{even} v. \tag{4.4.13}$$

Lemma 4.4.1. *The following equation holds:*

$$ds_{odd} = -2 s_{even} s_{odd} v. \tag{4.4.14}$$

Proof. Expressing $s = s_{odd} + s_{even}$ and plugging it into (4.4.8) we have

$$d(s_{odd} + s_{even}) + (s_{odd}^2 + s_{even}^2 + 2 s_{odd} s_{even}) v = -qv - \hbar^{-1} pv - \hbar^{-2} v. \tag{4.4.15}$$

This equality contains terms both symmetric and skew-symmetric under involution. Comparing only symmetric terms one gets

$$ds_{odd} + 2 s_{even} s_{odd} v = 0. \tag{4.4.16}$$

□

Using this relation, it is easy to obtain two local WKB-solutions for the equation (4.4.3):

$$f^{\pm}_{x_0} = \frac{1}{(s_{odd} v)^{\frac{1}{2}}} \exp\left[\pm \int_{x_0}^{x} s_{odd} v\right]. \tag{4.4.17}$$

Solutions are unique in each triangle face of the graph Σ_Q from Figure 3. In the i'th face of Σ_Q the initial point of integration x_0 is chosen to coincide with zero x_i contained in this face and called a turning point (see f.e. [28]). The differential $s_{odd} v$ is multi-valued on the base curve $\mathcal{C}$ and generically singular at x_i. To define the integral correctly we pass to the double cover $\hat{\mathcal{C}}$ where $s_{odd} v$

is well-defined. Skew-symmetry of $s_{odd}v$ implies it has a vanishing residue at x_i. Therefore, we can define the integral by

$$\int_{x_i}^{x} s_{odd}v = \frac{1}{2}\int_{x^{(2)}}^{x^{(1)}} s_{odd}v, \tag{4.4.18}$$

where we join preimages $x^{(1)}$ and $x^{(2)}$ of x by an arc passing through the branch cut, which connects x_i with some other branch point.

Introduce the meromorphic differentials

$$v_k = (s_{odd})_k v. \tag{4.4.19}$$

Analytic continuation of the WKB-solutions (4.4.17) through the edges of graph Σ_Q (from Figure 3) gives rise to the relation between the homological shear coordinates and the Voros symbols – integrals of $s_{odd}v$ over the elements of H_-. The following proposition generalizes the one stated in [7] to the case when $Q_1 \neq 0$ and is proven in complete analogy.

Proposition 4.4.1 ([7])**.** *For each $l \in H_-$ the homological shear coordinate ρ_l admits the following asymptotic expansion (in Poincaré sense)*

$$\rho_l(\hbar) \sim \int_l s_{odd}v = \frac{1}{\hbar}\int_l v_{-1} + \int_l v_0 + \hbar\int_l v_1 + \ldots \quad \hbar \to 0^+, \tag{4.4.20}$$

where the relation is understood modulo an addition of $\pi i k,\ k \in \mathbb{Z}$.

Remark 4.4.1. The similar result was present in [2] where the meromorphic potential in (4.1.1) was arranged in a different way, such that the double poles with fixed biresidues were attached to the reference meromorphic projective connection, while in our case these double poles belong to the quadratic differential Q.

4.4.2 WKB expansion of the Yang-Yang function

The requirement for the monodromy map of the equation (4.4.3) to be a symplectomorphism imposes a restriction on the differential Q_1. We can regard

$$S = S_B - \frac{2Q_1}{\hbar} \tag{4.4.21}$$

as a chosen base projective connection, then the condition for it to be admissible is ruled by Theorem 4.3.2. Namely, the form $\Theta_{(Q_1)}$, defined locally on $\mathcal{M}_{g,n}$, must be closed, $d\Theta_{(Q_1)} = 0$. The generating function $\mathcal{G}_B(\hbar)$ of this symplectomorphism is defined by

$$d\mathcal{G}_B(\hbar) = \mathcal{F}^*_{(S_B - 2Q_1/\hbar)}\theta_G(\hbar) - \theta_{hom}(\hbar), \tag{4.4.22}$$

where the symplectic potentials θ_G and θ_{hom} are defined by (4.3.93) and (4.3.94):

$$\theta_G(\hbar) = \sum_{j=1}^{g^-}\Big(\rho_{b_j}d\rho_{a_j} - \rho_{a_j}d\rho_{b_j}\Big)(\hbar), \tag{4.4.23}$$

here $\rho_l(\hbar)$ is the homological shear coordinate corresponding to a loop $l \in H_-$ and

$$\theta_{hom}(\hbar) = \frac{1}{\hbar^2}\sum_{j=1}^{g^-}\Big(B_j dA_j - A_j dB_j\Big), \tag{4.4.24}$$

here $(A_j = \oint_{a_j^-} v, B_j = \oint_{b_j^-} v)$ are period coordinates on $\mathcal{Q}_{g,n}\{\mathbf{r}\}$. Using the pairing notation (4.3.20) the symplectic potential θ_{hom} in period coordinates reads as

$$\theta_{hom}(\hbar) = \frac{1}{\hbar^2}\left\langle \oint v, d\oint v \right\rangle. \tag{4.4.25}$$

The potential θ_G in homological shear coordinates $\rho_l(\hbar)$ by means of the expansion (4.4.20) of Proposition 4.4.1 has the following expression

$$\theta_G(\hbar) = \sum_{i=-2}^{\infty} \hbar^i \sum_{\substack{l+k=i \\ l,k\geq -1}} \left\langle \oint v_l, d\oint v_k \right\rangle. \tag{4.4.26}$$

Meromorphic differentials v_k could be obtained by antisymmetrizing the differentials $s_k v$, where functions s_k are given by (4.4.9-4.4.10). First four differentials v_k take the following form:

$$v_{-1} = v, \qquad v_0 = \frac{Q_1}{2v}, \tag{4.4.27}$$

$$v_1 = -\frac{Q_1^2}{8v^3} - \frac{qv}{2}, \qquad v_2 = \frac{1}{4}\left(q\frac{Q_1}{v} - \frac{Q_1^3}{v^5}\right). \tag{4.4.28}$$

By plugging (4.4.26) in (4.4.22) we see that the coefficient in front of $\hbar^{-2}$ in the expansion of (4.4.22) vanishes and

$$d\mathcal{G}_B(\hbar) = \sum_{i=-1}^{\infty} \hbar^i \sum_{\substack{l+k=i \\ l,k\geq -1}} \left\langle \oint v_l, d\oint v_k \right\rangle := \sum_{i=-1}^{\infty} \hbar^i dG_i, \quad \hbar \to 0^+. \tag{4.4.29}$$

Equation for dG_{-1}:

$$dG_{-1} = \left\langle \oint v_{-1}, d\oint v_0 \right\rangle + \left\langle \oint v_0, d\oint v_{-1} \right\rangle \tag{4.4.30}$$

using the expressions (4.4.27-4.4.28) for v_k could be written as follows

$$dG_{-1} = \frac{1}{2}\left\langle \oint v, \oint \frac{Q_1}{v} \right\rangle + \frac{1}{2}\left\langle \oint \frac{Q_1}{v}, d\oint v \right\rangle. \tag{4.4.31}$$

Notice that (after relabeling $\tilde{Q} \to Q_1, v_0 \to v$) this is exactly the term (4.3.53) near $\hbar^1$ appearing in the expansion of the potential θ_1 in the proof of Proposition 4.3.1. Thus, we immediately get

$$dG_{-1} = \Theta_{(Q_1)} + d\left[\sum_{j=1}^{n} \frac{\pi i\, r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v}\right], \tag{4.4.32}$$

where before we assumed that the form $\Theta_{(Q_1)}$ is closed on $\mathcal{M}_{g,n}$. Then the integration leads to

Proposition 4.4.2. *The term G_{-1} of the generalized WKB expansion of the monodromy generating function has the following expression*

$$G_{-1} = G_{(Q_1)} + \sum_{j=1}^{n} \frac{\pi i\, r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v}, \tag{4.4.33}$$

where there exists local holomorphic function $G_{(Q_1)}$ on the moduli space $\mathcal{M}_{g,n}$, such that

$$dG_{(Q_1)} = \Theta_{(Q_1)}. \tag{4.4.34}$$

The geometrical meaning of the term G_{-1} is that the condition of closeness of dG_{-1} (or equivalently the existence of $G_{(Q_1)}$) is an obstruction for the monodromy map $\mathcal{F}_{(S_B-Q_1/\hbar)}$ to be a symplectomorphism.

Equation for dG_0:

$$dG_0 = \left\langle \oint v_{-1}, d\oint v_1 \right\rangle + \left\langle \oint v_0, d\oint v_0 \right\rangle + \left\langle \oint v_1, d\oint v_{-1} \right\rangle \tag{4.4.35}$$

using the expressions (4.4.27-4.4.28) for v_k could be written as follows

$$dG_0 = \left[\frac{1}{4}\left\langle \oint \frac{Q_1}{v}, d\oint \frac{Q_1}{v} \right\rangle - \frac{1}{8}\left\langle \oint v, d\oint \frac{Q_1^2}{v^3} \right\rangle - \frac{1}{8}\left\langle \oint \frac{Q_1^2}{v^3}, d\oint v \right\rangle \right] - \tag{4.4.36}$$

$$- \left[\frac{1}{2}\left\langle \oint v, d\oint qv \right\rangle + \frac{1}{2}\left\langle \oint qv, d\oint v \right\rangle \right]. \tag{4.4.37}$$

The term in the first bracket is the coefficient (4.3.54) near $\hbar^2$ in the expansion of the potential θ_1 in the proof in Proposition 4.3.1 and it equals

$$d\left[\sum_{j=1}^{n} \pi i\, r_j \binom{\frac{1}{2}}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^2}{v^3} \right]. \tag{4.4.38}$$

To compute the term in the second bracket we notice that

$$\frac{1}{2}\left\langle \oint v, d\oint qv \right\rangle + \frac{1}{2}\left\langle \oint qv, d\oint v \right\rangle = \frac{1}{2} d\left\langle \oint v, \oint qv \right\rangle + \left\langle \oint qv, d\oint v \right\rangle. \tag{4.4.39}$$

It follows from (4.2.34) that $\langle \int qv, d\int v \rangle$ is a differential of the Bergman tau-function, namely

$$\left\langle \oint qv, d\oint v \right\rangle = 12\pi i d \log \tau_B|_r. \tag{4.4.40}$$

Applying the variational formulas (4.2.29, 4.2.30) and the homogeneity property (4.2.32) of the function τ_B on the full space $\mathcal{Q}_{g,n}$ the term $\langle \oint v, \oint qv \rangle$ could be written as

$$\begin{aligned} \left\langle \oint v, \oint qv \right\rangle &= -12\pi i \sum_{j=1}^{g^-} \Big(A_j \frac{\partial}{\partial A_j} + B_j \frac{\partial}{\partial B_j} \Big) \log \tau_B = \\ &= -12\pi i \left[\frac{5(2g-2+n)}{72} - \sum_{j=1}^{n} r_j \frac{\partial}{\partial r_j} \log \tau_B \right] = \end{aligned} \tag{4.4.41}$$

$$= -12\pi i \left[\frac{5(2g-2+n)}{72} - \sum_{j=1}^{n} \frac{r_j}{12} \int_{z_j^{(2)}}^{z_j^{(1)}} \Big(qv + \frac{1}{4r_k^2} v \Big) \right]. \tag{4.4.42}$$

Restricting this formula to $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ we get

$$d\left\langle \oint v, \oint qv \right\rangle = d\left[\sum_{j=1}^{n} \pi i r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \Big(qv + \frac{1}{4r_k^2} v \Big) \right], \tag{4.4.43}$$

(alternatively, this term could be computed via the resides after applying the RBI to differentials v and qv). Putting all terms together in (4.4.36) and integrating we obtain

Proposition 4.4.3. *The term G_0 of the generalized WKB expansion of the monodromy generating function has the following expression*

$$G_0 = -12\pi i \log \tau_B|_r - \sum_{j=1}^{n} \frac{\pi i r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \Big(qv + \frac{1}{4r_k^2} v\Big) + \sum_{j=1}^{n} \pi i \, r_j \binom{\frac{1}{2}}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^2}{v^3}. \tag{4.4.44}$$

Equation for dG_1:

$$dG_1 = \Big\langle \oint v_{-1}, d\oint v_2 \Big\rangle + \Big\langle \oint v_0, d\oint v_1 \Big\rangle + \Big\langle \oint v_1, d\oint v_0 \Big\rangle + \Big\langle \oint v_2, d\oint v_{-1} \Big\rangle \tag{4.4.45}$$

using the expressions (4.4.27-4.4.28) for v_k could be written as follows

$$dG_1 = \Bigg[-\frac{1}{16} \Big\langle \oint \frac{Q_1^2}{v^3}, d\oint \frac{Q_1}{v} \Big\rangle - \frac{1}{16} \Big\langle \oint \frac{Q_1}{v}, d\oint \frac{Q_1^2}{v^3} \Big\rangle + \frac{1}{16} \Big\langle \oint v, d\oint \frac{Q_1^3}{v^5} \Big\rangle + \frac{1}{16} \Big\langle \oint \frac{Q_1^3}{v^5}, d\oint v \Big\rangle \Bigg] +$$
$$+ \Bigg[-\frac{1}{4} \Big\langle \oint \frac{Q_1}{v}, d\oint qv \Big\rangle - \frac{1}{4} \Big\langle \oint qv, d\oint \frac{Q_1}{v} \Big\rangle + \frac{1}{4} \Big\langle \oint v, d\oint q\frac{Q_1}{v} \Big\rangle + \frac{1}{4} \Big\langle \oint q\frac{Q_1}{v}, d\oint v \Big\rangle \Bigg]. \tag{4.4.46}$$

The term in the first brackets is the coefficient (4.3.54) near $\hbar^3$ in the expansion of the potential θ_1 in the proof of Proposition 4.3.1 and it equals

$$d\Bigg[\sum_{j=1}^{n} \pi i \, r_j \binom{\frac{1}{2}}{3} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^3}{v^5} \Bigg]. \tag{4.4.47}$$

To treat the term in the second bracket first notice that it could be rewritten as

$$-\frac{1}{4} d \Big\langle \oint \frac{Q_1}{v}, \oint qv \Big\rangle - \frac{1}{2} \Big\langle \oint qv, d\oint \frac{Q_1}{v} \Big\rangle + \frac{1}{4} d \Big\langle \oint v, \oint q\frac{Q_1}{v} \Big\rangle + \frac{1}{2} \Big\langle \oint q\frac{Q_1}{v}, d\oint v \Big\rangle. \tag{4.4.48}$$

Lemma 4.3.1 in the form (4.3.22) implies that

$$\Big\langle \oint qv, d\oint \frac{Q_1}{v} \Big\rangle = \frac{1}{2} \int_{\partial \hat{\mathcal{C}}_0} \Big(qv \int_{p_0}^{x} \delta \, \frac{Q_1}{v} \Big) + \Big\langle \oint q\frac{Q_1}{v}, d\oint v \Big\rangle, \tag{4.4.49}$$

so (4.4.48) becomes

$$-\frac{1}{4} d \Big\langle \oint \frac{Q_1}{v}, \oint qv \Big\rangle + \frac{1}{4} d \Big\langle \oint v, \oint q\frac{Q_1}{v} \Big\rangle - \frac{1}{4} \int_{\partial \hat{\mathcal{C}}_0} \Big(qv \int_{p_0}^{x} \delta \, \frac{Q_1}{v} \Big). \tag{4.4.50}$$

Let us consider the last integral. While Abelian differential $\delta \frac{Q_1}{v}$ is holomorphic, qv has residueless 4-order poles at the branch points (x_i) and simple poles at the punctures $\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n$. So, the integral over the boundary reduces to the computation of residues:

$$\int_{\partial \hat{\mathcal{C}}_0} \Big(qv \int_{p_0}^{x} \delta \, \frac{Q_1}{v} \Big) = \sum_{i=1}^{4g-4+2n} 2\pi i \, \underset{x_i}{res} \Big[qv \int_{p_0}^{x} \delta \frac{Q_1}{v} \Big] + \sum_{j=1}^{n} 2\pi i \underset{\{z_j^{(1)}, z_j^{(2)}\}}{res} \Big[qv \int_{p_0}^{x} \delta \frac{Q_1}{v} \Big]. \tag{4.4.51}$$

To compute residues near simple poles we recall the formulas (4.2.17) for $q(x)$ and S_v and use the local coordinate ζ (4.2.12) to write near $z_j^{(1)}$:

$$qv = \frac{S_B - S_v}{2v} = \frac{S_B(\zeta) - \frac{1}{2\zeta^2}}{2\frac{r_j}{\zeta}} d\zeta = \left(-\frac{1}{4r_j\zeta} + O(1) \right) d\zeta. \tag{4.4.52}$$

Due to skew-symmetry of qv the expansion near $z_j^{(2)}$ is the negation of the above formula. Thus,

$$(\underset{z_j^{(1)}}{res} + \underset{z_j^{(2)}}{res}) \left(qv \int_{p_0}^{x} \delta \frac{Q_1}{v} \right) = d\left[-\frac{1}{4r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v} \right], \tag{4.4.53}$$

since the differential d commutes with the line integral. To simplify the residue near a branch point x_i at first notice that the variational formula (4.2.28) implies that the differential $\delta(qv)$ is holomorphic at x_i, so

$$\underset{x_i}{res} \left[qv \int_{p_0}^{x} \delta \frac{Q_1}{v} \right] = d\left(\underset{x_i}{res} \left[qv \int_{p_0}^{x} \frac{Q_1}{v} \right] \right). \tag{4.4.54}$$

Let's assume that the Bergman projective connection admits the following expansion near x_i on the base curve $\mathcal{C}$ in local coordinate (4.2.11):

$$S_B(\xi) = S_B(x_i) + {S_B}'(x_i)\xi + ... \tag{4.4.55}$$

Being lifted to $\hat{\mathcal{C}}$, it transforms like

$$S_B(\hat{\xi})(d\hat{\xi})^2 = S_B(\xi)(d\xi)^2 + S(\xi, \hat{\xi}), \tag{4.4.56}$$

where $\hat{\xi}$ is local coordinate (4.2.10) near x_i on $\hat{\mathcal{C}}$, $S(\xi, \hat{\xi})$ is the Schwarzian derivative

$$S(\xi, \hat{\xi}) = \left(\frac{\xi''}{\xi'} \right)' - \frac{1}{2} \left(\frac{\xi''}{\xi'} \right)^2 \tag{4.4.57}$$

where derivatives are taken with respect to $\hat{\xi}$. Having that $\xi = \hat{\xi}^2$ we write

$$S_B(\hat{\xi})(d\hat{\xi})^2 = 4(S_B(x_i) + {S_B}'(x_i)\hat{\xi}^2)\hat{\xi}^2(d\hat{\xi})^2 - \frac{3}{2\hat{\xi}^2}(d\hat{\xi})^2. \tag{4.4.58}$$

Also

$$S_v = -\frac{4}{\hat{\xi}^2}(d\hat{\xi})^2, \quad v = 3\hat{\xi}^2 d\hat{\xi}, \tag{4.4.59}$$

leading to

$$qv = \left[\frac{5}{12\hat{\xi}^4} + O(1) \right] d\hat{\xi}. \tag{4.4.60}$$

Then the residue near x_i could be expressed as

$$\underset{x_i}{res} \left[qv \int_{p_0}^{x} \frac{Q_1}{v} \right] = \underset{x_i}{res} \left(\frac{5d\hat{\xi}}{12\hat{\xi}^4} \int_{p_0}^{x} \frac{Q_1}{v} \right) = \frac{5}{12} \frac{1}{3!} \left(\frac{(Q_1/v)}{d\hat{\xi}} \right)''(x_i) = \frac{5}{36} \underset{x_i}{res} \left(\frac{Q_1/v}{\int_{x_i}^{x} v} \right). \tag{4.4.61}$$

Therefore, integral (4.4.51) takes the following form

$$\int_{\partial\hat{\mathcal{C}}_0}\left(qv\int_{p_0}^{x}\delta\,\frac{Q_1}{v}\right)=d\left[\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{18}\underset{x_i}{res}\left(\frac{Q_1/v}{\int_{x_i}^{x}v}\right)-\sum_{j=1}^{n}\frac{\pi i}{2r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1}{v}\right]. \tag{4.4.62}$$

The differential of the first pairing $d\left\langle\oint\frac{Q_1}{v},\oint qv\right\rangle$ in (4.4.50) is computed by analogy. Applying the RBI, one has

$$\left\langle\oint\frac{Q_1}{v},\oint qv\right\rangle=-\frac{1}{2}\int_{\partial\hat{\mathcal{C}}_0}\left(qv\int_{p_0}^{x}\frac{Q_1}{v}\right), \tag{4.4.63}$$

resulting in

$$d\left\langle\oint\frac{Q_1}{v},\oint qv\right\rangle=d\left[-\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{36}\underset{x_i}{res}\left(\frac{Q_1/v}{\int_{x_i}^{x}v}\right)+\sum_{j=1}^{n}\frac{\pi i}{4r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1}{v}\right]. \tag{4.4.64}$$

Finally, applying the RBI to the pairing $\left\langle\oint v,\oint q\frac{Q_1}{v}\right\rangle$ we have

$$\left\langle\oint v,\oint q\frac{Q_1}{v}\right\rangle=-\sum_{i=1}^{4g-4+2n}\pi i\,\underset{x_i}{res}\left[q\frac{Q_1}{v}\int_{p_0}^{x}v\right]+\sum_{j=1}^{n}\pi i\underset{\{z_j^{(1)},z_j^{(2)}\}}{res}\left[v\int_{p_0}^{x}q\frac{Q_1}{v}\right]. \tag{4.4.65}$$

While

$$\underset{\{z_j^{(1)},z_j^{(2)}\}}{res}\left[v\int_{p_0}^{x}q\frac{Q_1}{v}\right]=r_j\int_{z_j^{(2)}}^{z_j^{(1)}}q\frac{Q_1}{v}, \tag{4.4.66}$$

the residue near x_i is computed noticing that

$$q\frac{Q_1}{v}=\left(\frac{5}{36\hat{\xi}^6}+O\left(\frac{1}{\hat{\xi}^2}\right)\right)\frac{Q_1}{v},\qquad\int_{p_0}^{x}v=C(p_0)+\hat{\xi}^3. \tag{4.4.67}$$

Differential $\frac{Q_1}{v}$ is antisymmetric, and, thus, expands by even powers of $\hat{\xi}$ near a branch point. That leads to

$$\underset{x_i}{res}\left[q\frac{Q_1}{v}\int_{p_0}^{x}v\right]=\frac{5}{36}\underset{x_i}{res}\left[\frac{Q_1/v}{\hat{\xi}^3}\right]=\frac{5}{36}\underset{x_i}{res}\left(\frac{Q_1/v}{\int_{x_i}^{x}v}\right). \tag{4.4.68}$$

Overall,

$$d\left\langle\oint v,\oint q\frac{Q_1}{v}\right\rangle=d\left[-\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{36}\underset{x_i}{res}\left(\frac{Q_1/v}{\int_{x_i}^{x}v}\right)+\sum_{j=1}^{n}\pi i r_j\int_{z_j^{(2)}}^{z_j^{(1)}}q\frac{Q_1}{v}\right]. \tag{4.4.69}$$

Plugging derived expressions (4.4.62), (4.4.64) and (4.4.69) into (4.4.50) and integrating we obtain

Proposition 4.4.4. *The term G_1 of the generalized WKB expansion of the monodromy generating function has the following expression*

$$\begin{aligned}G_1=-\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{72}\underset{x_i}{res}\left(\frac{Q_1/v}{\int_{x_i}^{x}v}\right)+\sum_{j=1}^{n}\frac{\pi i r_j}{4}\int_{z_j^{(2)}}^{z_j^{(1)}}q\frac{Q_1}{v}+\\+\sum_{j=1}^{n}\frac{\pi i}{16r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1}{v}+\sum_{j=1}^{n}\pi i\,r_j\binom{\frac{1}{2}}{3}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1^3}{v^5}.\end{aligned} \tag{4.4.70}$$

To sum up, we can formulate the following theorem.

Theorem 4.4.1. *Consider the differential equation*

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q_1}{\hbar} - \frac{Q}{\hbar^2}\Big)\phi = 0, \tag{4.4.71}$$

on a Riemann surface $\mathcal{C}$. Let Q be a quadratic differential on $\mathcal{C}$ with simple zeroes and n second order poles at $z_1, ..., z_n$ and biresidues $r_1^2, ..., r_n^2$. Q_1 is a meromorphic quadratic differential which depends holomorphically on moduli of $\mathcal{M}_{g,n}$, with at most simple poles at z_j. S_B is the Bergman projective connection (4.1.12) defined with respect to some Torelli marking on $\mathcal{C}$. For the chosen base projective connection $S_B - \frac{2Q_1}{\hbar}$ denote by $\mathcal{F}_{(S_B - 2Q_1/\hbar)}$ the monodromy map between the moduli space $\mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\}$ of pairs $(\mathcal{C}, Q/\hbar^2)$ and the symplectic leaf $CV_{g,n}\{\mathbf{m}(\hbar)\}$ of the $PSL(2,\mathbb{C})$ character variety, where each $m_j(\hbar)$ is related to r_j via (4.1.7 – 4.1.8) as

$$\frac{r_j}{\hbar} = \pm\left[\frac{\log m_j}{2\pi i}\left(\frac{\log m_j}{2\pi i} - 1\right)\right]^{1/2}. \tag{4.4.72}$$

- *The map $\mathcal{F}_{(S_B - 2Q_1/\hbar)}$ is a symplectomorphism, provided that the 1-form $\Theta_{(Q_1)}$, locally defined on $\mathcal{M}_{g,n}$, is closed, $d\Theta_{(Q_1)} = 0$.*
- *Introduce the symplectic potential θ_{hom} (4.4.24) of the homological symplectic form on $\mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\}$ and symplecic potential θ_G (4.4.23) for the Goldman form on $CV_{g,n}\{\mathbf{m}(\hbar)\}$ The generating function $\mathcal{G}_B$ of the monodromy symplectomorphism between $\mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\}$ and $CV_{g,n}\{\mathbf{m}(\hbar)\}$ is defined by*

$$d\mathcal{G}_B(\hbar) = \mathcal{F}^*_{(S_B - 2Q_1/\hbar)}\theta_G(\hbar) - \theta_{hom}(\hbar) \tag{4.4.73}$$

and has the following asymptotics as $\hbar \to 0^+$:

$$\mathcal{G}_B(\hbar) = \frac{G_{-1}}{\hbar} + G_0 + G_1\hbar + O(\hbar^2). \tag{4.4.74}$$

Here

$$G_{-1} = G_{(Q_1)} + \sum_{j=1}^{n} \frac{\pi i\, r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v}, \tag{4.4.75}$$

where the function $G_{(Q_1)}$ is defined by (4.4.34). Its explicit form depends on the concrete choice of Q_1;

$$G_0 = -12\pi i \log \tau_B|_r - \sum_{j=1}^{n} \frac{\pi i r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \Big(qv + \frac{1}{4r_j^2}v\Big) + \sum_{j=1}^{n} \pi i\, r_j \begin{pmatrix}\frac{1}{2}\\ 2\end{pmatrix} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^2}{v^3}, \tag{4.4.76}$$

here $\log \tau_B|_r$ is the Bergman tau-function defined by (3.2.12) on stratum of the moduli space of quadratic differentials with second order poles, $q(x)$ is a meromorphic function on $\mathcal{C}$ given by

$$q(x) = \frac{S_B - S_v}{2v^2}, \tag{4.4.77}$$

where S_v is the Schwarzian projective connection (4.2.17);

$$\begin{aligned} G_1 = -\sum_{i=1}^{4g-4+2n} \frac{5\pi i}{72} \mathop{res}_{x_i}\left(\frac{Q_1/v}{\int_{x_i}^{x} v}\right) + \sum_{j=1}^{n} \frac{\pi i r_j}{4} \int_{z_j^{(2)}}^{z_j^{(1)}} q\frac{Q_1}{v} + \\ + \sum_{j=1}^{n} \frac{\pi i}{16 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v} + \sum_{j=1}^{n} \pi i\, r_j \begin{pmatrix}\frac{1}{2}\\ 3\end{pmatrix} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^3}{v^5}, \end{aligned} \tag{4.4.78}$$

here the first sum runs over the branch points of the double cover.

Chapter 5

Higher WKB asymptotics and deformation of Hitchin's covers

We begin this chapter by formulating a proposition that allows us to derive generalized WKB expansion of the Yang-Yang function from the regular expansion by deforming the spectral cover. With that we adapt variational techniques from Chapters 3 and 4 to propose an alternative derivation of the term G_1. We also take one step further and derive the term G_2 of the expansion.

5.1 Regular and generalized WKB expansions

We can propose an alternative way of computing the expansion of Yang-Yang function $\mathcal{G}_B(\hbar)$, assuming we obtained the "regular" expansion of the related generating function $\mathcal{G}_B^{\circ}(\hbar)$ for $Q_1 \equiv 0$. Rewrite the equation (4.4.71) in the form

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q+\hbar Q_1}{\hbar^2}\Big)\phi = 0, \tag{5.1.1}$$

corresponding to detachment of differential $\frac{Q_1}{\hbar}$ from projective connection S_B and joining it with $\frac{Q_1}{\hbar^2}$, so that $\frac{Q+\hbar Q_1}{\hbar^2} \in \mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\}$. This operation induces the following diagram of maps, where symplectomorphism $\mathcal{F}_{(S_B-2Q_1/\hbar)}$ factors through the composition $\mathcal{F}_{(S_B)} \circ \mathcal{H}$ of symplectomorphisms:

$$\begin{tikzcd}
\mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\} \arrow[r, "Q \xrightarrow{\mathcal{H}} Q+\hbar Q_1"] \arrow[dr, "\mathcal{F}_{(S_B-2Q_1/\hbar)}"'] & \mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\} \arrow[d, "\mathcal{F}_{(S_B)}"] \\
 & CV_{g,n}\{\mathbf{m}(\hbar)\}
\end{tikzcd}$$

Denote by $\theta^1_{hom}(\hbar)$ the symplectic potential computed via the periods of the Abelian differential $v^\hbar$, which defines canonical double cover $\hat{\mathcal{C}}_\hbar$ by $(v^\hbar)^2 = Q+\hbar Q_1$. Then the monodromy generating function (4.4.73) admits the following representation

$$d\mathcal{G}_B(\hbar) = \mathcal{H}^*\big(\mathcal{F}^*_{(S_B)}\theta_G(\hbar) - \theta^1_{hom}(\hbar)\big) + \big(\mathcal{H}^*\theta^1_{hom}(\hbar) - \theta_{hom}(\hbar)\big). \tag{5.1.2}$$

The $\hbar$-expansion of the generating function for map $\mathcal{H}$ was computed in (4.3.78) along with the proof of Proposition 4.3.1:

$$\mathcal{H}^*\theta^1_{hom}(\hbar) - \theta_{hom}(\hbar) = d\left[\sum_{j=1}^{n}\pi i\, r_j\left[\sum_{k=1}^{\infty}\hbar^{k-2}\binom{\frac{1}{2}}{k}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1^k}{v^{2k-1}}\right]\right] + \hbar^{-1}dG_{(Q_1)}. \tag{5.1.3}$$

Homological coordinates on $\mathcal{Q}_{g,n}\{\mathbf{r}/\hbar\}$, defined by the periods of $v^\hbar$, holomorphically depend on $\hbar$. That allows us to perform the $\hbar$-expansion of the pullback

$$\mathcal{H}^*\big(\mathcal{F}^*_{(S_B)}\theta_G(\hbar) - \theta^1_{hom}(\hbar)\big) \tag{5.1.4}$$

in two steps. At first, obtain the WKB-expansion of the generating function

$$\mathcal{F}^*_{(S_B)}\theta_G(\hbar) - \theta^1_{hom}(\hbar), \tag{5.1.5}$$

assuming that $\hbar$ of $v^\hbar$ is fixed. This is equivalent to setting $Q_1 \equiv 0$ and performing the computations as in Section 4.4.1. One can notice that in this case only the differentials with odd indices defined by (4.4.19) are non-zero. We denote these differentials by v°_{2k+1}. Corresponding regular $\hbar$-expansion of $\mathcal{G}^\circ_B(\hbar)$ contains only even non-vanishing terms:

$$d\mathcal{G}^\circ_B(\hbar) = \sum_{i=-1}^{\infty} \hbar^{2i+2} \sum_{\substack{l+k=i \\ l,k\geq -1}} \left\langle \oint v^\circ_{2l+1}, d\oint v^\circ_{2k+1} \right\rangle := \sum_{i=0}^{\infty} \hbar^{2i} dG^\circ_{2i}, \quad \hbar \to 0^+, \tag{5.1.6}$$

and after integration reads as

$$\mathcal{G}^\circ_B(\hbar) = \sum_{i=0}^{\infty} \hbar^{2i} G^\circ_{2i}, \quad \hbar \to 0^+, \tag{5.1.7}$$

where each term G°_{2i} depends on the differential Q via period coordinates. On the second step, vary Q by the differential $\hbar Q_1$ and expand the terms G°_{2i} by Taylor series to obtain the full $\hbar$-expansion for (5.1.4). The following proposition holds

Proposition 5.1.1. *Let*

$$\mathcal{G}^\circ_B(\hbar) = \sum_{i=0}^{\infty} \hbar^{2i} G^\circ_{2i}, \quad \hbar \to 0^+ \tag{5.1.8}$$

be a WKB expansion of the monodromy generating function $d\mathcal{G}^\circ_B(\hbar) = \mathcal{F}^*_{(S_B)}\theta_G(\hbar) - \theta_{hom}(\hbar)$ of the equation

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q}{\hbar^2}\Big)\phi = 0. \tag{5.1.9}$$

Then the generalized WKB expansion

$$\mathcal{G}_B(\hbar) = \sum_{i=-1}^{\infty} \hbar^i G_i, \quad \hbar \to 0^+ \tag{5.1.10}$$

of the monodromy generating function $d\mathcal{G}_B(\hbar) = \mathcal{F}^*_{(S_B - 2Q_1/\hbar)}\theta_G(\hbar) - \theta_{hom}(\hbar)$ of the equation

$$\partial^2\phi + \Big(\frac{1}{2}S_B - \frac{Q_1}{\hbar} - \frac{Q}{\hbar^2}\Big)\phi = 0 \tag{5.1.11}$$

is related to (5.1.8) by

$$G_{-1} = G_{(Q_1)} + \sum_{j=1}^{n} \frac{\pi i\, r_j}{2} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v}, \tag{5.1.12}$$

$$G_{2k} = \sum_{i=0}^{k} \delta^{(2i)}_{Q_1} \frac{G^\circ_{2k-2i}}{(2i)!} + \sum_{j=1}^{n} \binom{\frac{1}{2}}{2k+2} \pi i\, r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^{2k+2}}{v^{4k+3}}, \quad k = 0, \dots \infty \tag{5.1.13}$$

$$G_{2k+1}=\sum_{i=0}^{k}\delta_{Q_1}^{(2i+1)}\frac{G^\circ_{2k-2i}}{(2i+1)!}+\sum_{j=1}^{n}\binom{\frac{1}{2}}{2k+3}\pi i\, r_j\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1^{2k+3}}{v^{4k+5}},\quad k=0,...\infty, \tag{5.1.14}$$

where $\delta_{Q_1}^{(k)}f$ denotes a k'th variation of a moduli-dependent function f by differential Q_1: $\delta_{Q_1}^{(k)}f=\frac{\partial^k}{\partial\hbar^k}f[Q+\hbar Q_1]\big|_{\hbar=0}$.

Let us provide an alternative computation of the term G_1 of the generalized expansion by applying Proposition 5.1.1. Formula (5.1.14) implies

$$G_1=\delta_{Q_1}G^\circ_0++\sum_{j=1}^{n}\binom{\frac{1}{2}}{3}\pi i\, r_j\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1^3}{v^5}. \tag{5.1.15}$$

The variation δ_{Q_1} on the space $\mathcal{Q}_{g,n}\{\mathbf{r}\}$ of pairs $(\mathcal{C},Q)$ only affects quadratic differential Q, while the Riemann surface $\mathcal{C}$ and positions of poles are kept fixed. Thus, δ_{Q_1} could be instead viewed on the subspace $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}\subset\mathcal{Q}_{g,n}\{\mathbf{r}\}$ of SL(2) spectral covers, defined by the quadratic differential Q having simple zeroes and n double poles with fixed biresidues,

$$\dim\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}=g^-=3g-3+n. \tag{5.1.16}$$

The local coordinates on $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ are given by the subset of elements (3.3.5), consisting of the periods of v over a^--cycles:

$$A_\gamma=\oint_{a^-_\gamma}v. \tag{5.1.17}$$

Having that

$$\delta_{Q_1}v=\frac{\partial}{\partial\hbar}[v]\Big|_{\hbar=0}=\frac{Q_1}{2v} \tag{5.1.18}$$

we represent the variation δ_{Q_1} on $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ in local coordinates A_γ as

$$\delta_{Q_1}=\sum_{\gamma=1}^{g^-}\left(\oint_{a^-_\gamma}\frac{Q_1}{2v}\right)\frac{\partial}{\partial A_\gamma}, \tag{5.1.19}$$

implying the computation of the variation δ_{Q_1} basically boils down to varying with respect to A_γ.

It follows from Proposition 4.4.3 by letting $Q_1=0$, that

$$G^\circ_0=-12\pi i\log\tau_B|_r-\sum_{j=1}^{n}\frac{\pi i r_j}{2}\int_{z_j^{(2)}}^{z_j^{(1)}}\Big(qv+\frac{1}{4r_k^2}v\Big). \tag{5.1.20}$$

The variation of the tau-function τ_B is essentially the formula (3.4.62) of Theorem 3.4.4. Namely,

$$\frac{\partial\log\tau_B|_r}{\partial A_\gamma}=\frac{5}{432}\sum_{i=1}^{r}\underset{x_i}{res}\left(\frac{u^-_\gamma}{\int_{x_i}^{x}v}\right)-\sum_{j=1}^{n}\frac{1}{48r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}u^-_\gamma, \tag{5.1.21}$$

where u^-_γ is a Prym holomorphic differential.

To compute variation of the integral involving function $q(x)$, we first prove the following lemma:

Lemma 5.1.1. *The variations of function $q(x)$ (4.4.6) on $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ with respect to the coordinates A_γ take the following form:*

$$\frac{\partial q(x)}{\partial A_\gamma} = -\frac{1}{2}\left(\frac{u_\gamma^-}{v}\right)''_{zz} - 2\frac{qu_\gamma^-}{v}, \tag{5.1.22}$$

where the second derivative of the right-hand side is computed in local coordinate $z(x) = \int_{x_1}^x v$.

Proof. The computation follows the logic of the proof of formula (3.4.46) in Theorem 3.4.2. Recall that on the space $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$ with fixed base curve the derivative with respect to the coordinate A_γ is defined by taking any local coordinate ξ on $\mathcal{C}$ away from branch points and lifting it to the spectral curve $\hat{\mathcal{C}}$. In this case the deformation of the coordinate $z(x)$ should be kept in mind. One has

$$\frac{\partial q(x)}{\partial A_\gamma}\Big|_{\xi(x)} = \frac{\partial q(x)}{\partial A_\gamma}\Big|_{z(x)} + \frac{\partial q(x)}{\partial z(x)}\frac{\partial z(x)}{\partial A_\gamma} = \frac{\partial q(x)}{\partial A_\gamma}\Big|_{z(x)} + \left[q_z\int_{x_1}^x u_\gamma^-\right], \tag{5.1.23}$$

where we used that

$$\frac{\partial v}{\partial A_\gamma} = u_\gamma^-, \tag{5.1.24}$$

(see Proposition 3.4.1).

To compute the term $\frac{\partial q(x)}{\partial A_\gamma}\Big|_{z(x)}$ we, similarly to (3.4.46), apply the chain rule with variational formulas (4.2.28), (5.1.24) along with Riemann Bilinear Identity to obtain

$$\frac{\partial q(x)}{\partial A_\gamma}\Big|_{z(x)} = -\sum_{t\in int(\hat{\mathcal{C}}_0)}\frac{3}{4\pi i}\operatorname*{res}_t\left(h(x,t)v(t)\int_{x_1}^t u_\gamma^-\right), \tag{5.1.25}$$

where

$$h(x,t) = \frac{B^2(x,t)}{Q(x)Q(t)}. \tag{5.1.26}$$

The integrand has poles of order four on $\hat{\mathcal{C}}$ at $t = x$ and $t = x^\mu$. Residues at x and x^μ coincide due to μ-symmetry of the expression under the residue. The residue at $t = x$ is computed using the asymptotics of $h(x,t)$ as $t \to x$.

Recall that if the points x, y belong to the same chart of any local coordinate ξ, then $B(x,y)$ has the following expansion near the diagonal

$$B(x,t) = \frac{d\xi(x)d\xi(t)}{(\xi(x)-\xi(t))^2} + \frac{1}{6}S_B\left(\frac{\xi(x)+\xi(t)}{2}\right) + O((\xi(x)-\xi(t))^2). \tag{5.1.27}$$

Therefore, asymptotics of the function $b(x,t) = \frac{B(x,t)}{v(x)v(t)}$ in local coordinate $z(x)$ take the form:

$$b(x,t) = \frac{1}{(z(x)-z(t))^2} + \frac{1}{3}q(x) + \frac{1}{6}q_z(x)(z(t)-z(x)) + O((z(x)-z(t))^2) \qquad t \to x\ . \tag{5.1.28}$$

Hence, the function $h(x,t) = b^2(x,t)$ expands as follows near the diagonal:

$$h(x,t) = \frac{1}{(z(t)-z(x))^4} + \frac{2}{3}\frac{q(x)}{(z(t)-z(x))^2} + \frac{q_z(x)}{3(z(t)-z(x))} + O(1) \qquad t \to x\ . \tag{5.1.29}$$

The result of the residue computation, multiplied by 2 to take into account contributions of both x and x^μ, reads as

$$\frac{\partial q(x)}{\partial A_\gamma}\Big|_{z(x)} = -\frac{1}{2}\Big(\frac{u_\gamma^-}{v}\Big)''_{zz} - 2\frac{qu_\gamma^-}{v} - q_z\int_{x_1}^{x} u_\gamma^-. \tag{5.1.30}$$

Plugging it into (5.1.23) we finish the computation. □

Applying Lemma 5.1.1 with the product rule we have

$$\frac{\partial}{\partial A_\gamma}\int_{z_j^{(2)}}^{z_j^{(1)}}\left(qv + \frac{1}{4r_k^2}v\right) = \frac{1}{2}\Big(\frac{u_\gamma^-}{v}\Big)'_z(z_j^{(2)}) - \frac{1}{2}\Big(\frac{u_\gamma^-}{v}\Big)'_z(z_j^{(1)}) - \int_{z_j^{(2)}}^{z_j^{(1)}} qu_\gamma^- + \frac{1}{4r_k^2}\int_{z_j^{(2)}}^{z_j^{(1)}} u_\gamma^-. \tag{5.1.31}$$

To evaluate derivatives at $z_j^{(1)}$ we use distinguished local coordinate $\zeta(x)$ (4.2.12):

$$\zeta(x) = \exp\Big(\frac{z(x)}{r_j}\Big). \tag{5.1.32}$$

Since v has first order pole at $z_j^{(1)}$, while Prym differential u_γ^- is holomorphic, the function $\frac{u_\gamma^-}{v}$ has a simple zero at $z_j^{(1)}$. Writing the expansion of $\frac{u_\gamma^-}{v}$ in local coordinate ζ

$$\frac{u_\gamma^-}{v}(\zeta) = \zeta(a_0 + a_1\zeta + \dots), \tag{5.1.33}$$

and having that

$$\zeta_z(x) = \frac{\zeta(x)}{r_j}, \tag{5.1.34}$$

we obtain

$$\Big(\frac{u_\gamma^-}{v}\Big)'_z(\zeta) = \frac{\zeta}{r_j}(a_0 + O(\zeta)). \tag{5.1.35}$$

Thus, the derivative vanishes at $z_j^{(1)}$ and, similarly, at $z_j^{(2)}$.

Combining the variations (5.1.21) and (5.1.31) together with (5.1.20) we have

$$\frac{\partial G_0^\circ}{\partial A_\gamma} = -\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{36}\operatorname*{res}_{x_i}\left(\frac{u_\gamma^-}{\int_{x_i}^x v}\right) + \sum_{j=1}^{n}\frac{\pi i r_j}{2}\int_{z_j^{(2)}}^{z_j^{(1)}} qu_\gamma^- + \sum_{j=1}^{n}\frac{\pi i}{8r_j}\int_{z_j^{(2)}}^{z_j^{(1)}} u_\gamma^-. \tag{5.1.36}$$

We apply differential operator (5.1.19) to G_0° using the above result and employ a simple fact

$$\frac{Q_1}{2v} = \sum_{\gamma=1}^{g^-}\left(\oint_{a_\gamma^-}\frac{Q_1}{2v}\right)u_\gamma^-, \tag{5.1.37}$$

to obtain

$$\delta_{Q_1}G_0^\circ = -\sum_{i=1}^{4g-4+2n}\frac{5\pi i}{72}\operatorname*{res}_{x_i}\left(\frac{Q_1/v}{\int_{x_i}^x v}\right) + \sum_{j=1}^{n}\frac{\pi i r_j}{4}\int_{z_j^{(2)}}^{z_j^{(1)}} q\frac{Q_1}{v} + \sum_{j=1}^{n}\frac{\pi i}{16r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\frac{Q_1}{v}. \tag{5.1.38}$$

Plugging the result into (5.1.15) we finally get

$$
\begin{aligned}
G_1 = -\sum_{i=1}^{4g-4+2n} \frac{5\pi i}{72} \operatorname*{res}_{x_i}\left(\frac{Q_1/v}{\int_{x_i}^{x} v}\right) + \sum_{j=1}^{n} \frac{\pi i r_j}{4} \int_{z_j^{(2)}}^{z_j^{(1)}} q\frac{Q_1}{v} + \\
+ \sum_{j=1}^{n} \frac{\pi i}{16 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1}{v} + \sum_{j=1}^{n} \pi i\, r_j \binom{\frac{1}{2}}{3} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^3}{v^5},
\end{aligned}
\tag{5.1.39}
$$

which coincides with the formula of Proposition 4.4.4.

Let us use the Proposition 5.1.1 to compute the next term of the generalized expansion. By the formula (5.1.13) one has:

$$
G_2 = \delta_{Q_1}^{(2)} \frac{G_0^\circ}{2} + G_2^\circ + \sum_{j=1}^{n} \binom{\frac{1}{2}}{4} \pi i\, r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^4}{v^7}. \tag{5.1.40}
$$

While the second derivative with respect to $\hbar$ might be obtained by differentiation (5.1.38), the term G_2° should be computed independently.

Proposition 5.1.2. *The term G_2° of the regular WKB expansion of the monodromy generating function has the following expression*

$$
G_2^\circ = \frac{5\pi i}{144} \sum_i \operatorname*{res}_{x_i}\left(\frac{qv}{\int_{x_i}^{x} v}\right) - \frac{\pi i}{16} \sum_{j=1}^{n} \frac{1}{r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \left(qv + \frac{1}{4r_j^2} v\right) + \left\langle \oint v_{-1}^\circ, \oint v_3^\circ \right\rangle. \tag{5.1.41}
$$

Proof. Equation for dG_2° has the following form:

$$
dG_2^\circ = \left\langle \oint v_{-1}^\circ, d\oint v_3^\circ \right\rangle + \left\langle \oint v_1^\circ, d\oint v_1^\circ \right\rangle + \left\langle \oint v_3^\circ, d\oint v_{-1}^\circ \right\rangle. \tag{5.1.42}
$$

Using the formulas from Section 4.4.1 we compute first three non-zero differentials

$$
v_{-1}^\circ = v,\ v_1^\circ = -\frac{qv}{2},\ v_3^\circ = -\frac{q^2 v}{8}, \tag{5.1.43}
$$

so that the term dG_2° reads as

$$
dG_2^\circ = -\frac{1}{8}\left\langle \oint v, d\oint q^2 v \right\rangle + \frac{1}{4}\left\langle \oint qv, d\oint qv \right\rangle - \frac{1}{8}\left\langle \oint q^2 v, d\oint v \right\rangle. \tag{5.1.44}
$$

Notice that this sum could be represented as

$$
-\frac{1}{8} d\left\langle \oint v, \oint q^2 v \right\rangle + \frac{1}{4}\left\langle \oint qv, d\oint qv \right\rangle - \frac{1}{4}\left\langle \oint q^2 v, d\oint v \right\rangle. \tag{5.1.45}
$$

Lemma 4.3.1 in the form (4.3.22) implies that

$$
\left\langle \oint qv, d\oint qv \right\rangle = \frac{1}{2} \int_{\partial \hat{C}_0} \left(qv \int_{p_0}^{x} \delta\, qv \right) + \left\langle \oint q^2 v, d\oint v \right\rangle, \tag{5.1.46}
$$

so (5.1.44) becomes

$$dG_2^\circ = -\frac{1}{8} d \left\langle \oint v, \oint q^2 v \right\rangle + \frac{1}{8} \int_{\partial \hat{\mathcal{C}}_0} \left(qv \int_{p_0}^x \delta\, qv \right). \tag{5.1.47}$$

Let us analyze the last integral. While qv has residueless 4-order poles at the branch points (x_i) and simple poles at the punctures $\{z_j^{(1)}, z_j^{(2)}\}_{j=1}^n$, the variational formula (4.2.28) implies that the differential δqv is holomorphic, So, the integral over the boundary reduces to the computation of residues:

$$\int_{\partial \hat{\mathcal{C}}_0} \left(qv \int_{p_0}^x \delta\, qv \right) = \sum_{i=1}^{4g-4+2n} 2\pi i \operatorname*{res}_{x_i} \left[qv \int_{p_0}^x \delta qv \right] + \sum_{i=1}^{n} 2\pi i \operatorname*{res}_{\{z_j^{(1)}, z_j^{(2)}\}} \left[qv \int_{p_0}^x \delta qv \right]. \tag{5.1.48}$$

Consider

$$\operatorname*{res}_{x_i} \left[qv \int_{p_0}^x \delta qv \right]. \tag{5.1.49}$$

To integrate this term with respect to moduli we use expansions (4.4.60) for qv and v near a branch point x_i computed in the distinguished coordinate $\hat{\xi}$:

$$qv = \left[\frac{5}{12 \hat{\xi}^4} + O(1) \right] d\hat{\xi}, \tag{5.1.50}$$

$$v = d\hat{\xi}^3. \tag{5.1.51}$$

Then the residue at the branch point x_i could be rewritten as

$$\operatorname*{res}_{x_i} \Big(qv \int_{p_0}^x \delta qv \Big) = \frac{5}{12} \frac{1}{3!} \Big(\delta q \frac{v}{d\hat{\eta}} \Big)'' = \frac{5}{36} \operatorname*{res}_{x_i} \left(\frac{\delta qv}{\int_{x_i}^x v} \right). \tag{5.1.52}$$

Write down the following identity

$$d \left[\operatorname*{res}_{x_i} \left(\frac{qv}{\int_{x_i}^x v} \right) \right] = \operatorname*{res}_{x_i} \left(\frac{\delta qv}{\int_{x_i}^x v} \right) + \operatorname*{res}_{x_i} \left(qv\ \delta \left[\frac{1}{\int_{x_i}^x v} \right] \right) \tag{5.1.53}$$

and consider differential

$$d \left[\int_{x_i}^x v \right] = d \left[z(x) - z(x_i) \right]. \tag{5.1.54}$$

While $z(x)$ is kept fixed under the differentiation, $z(x_i)$, which is an integral of v between branch points, according to Lemma 3.1.1 could be represented by linear combination of the homological coordinates A_k, B_k with half-integer coefficients. If $\mathcal{P}_k \in (A_k, B_k)$, then

$$\frac{\partial}{\partial \mathcal{P}_k}\Big|_{z(x)} \left[\int_{x_i}^x v \right] = a_k,\ a_k = const, \tag{5.1.55}$$

and

$$\operatorname*{res}_{x_i} \left(qv\ \delta \left[\frac{1}{\int_{x_i}^x v} \right] \right) = - \sum_k \operatorname*{res}_{x_i} \left(a_k \frac{qv}{\left(\int_{x_i}^x v \right)^2} \right) d\mathcal{P}_k = 0, \tag{5.1.56}$$

since the term in the residue is skew-symmetric under involution μ and, therefore, has a vanishing residue at the branch point. Therefore, we have

$$\operatorname*{res}_{x_i}\left[qv\int_{p_0}^{x}\delta qv\right]=d\left[\frac{5}{36}\operatorname*{res}_{x_i}\left(\frac{qv}{\int_{x_i}^{x}v}\right)\right]. \tag{5.1.57}$$

Now consider

$$\operatorname*{res}_{\{z_j^{(1)},z_j^{(2)}\}}\left[qv\int_{p_0}^{x}\delta qv\right]. \tag{5.1.58}$$

To compute residue near the pole $z_j^{(1)}$ we recall the expansion (4.4.52) of qv in distinguished local coordinate ζ:

$$qv=\frac{S_B-S_v}{2v}=\frac{S_B(\zeta)-\frac{1}{2\zeta^2}}{2\frac{r_j}{\zeta}}d\zeta=\left(-\frac{1}{4r_j\zeta}+O(1)\right)d\zeta. \tag{5.1.59}$$

Expansion near $z_j^{(2)}$ is just the negation of this formula. Having that δqv is holomorphic at poles of v, we write

$$(\operatorname*{res}_{z_j^{(1)}}+\operatorname*{res}_{z_j^{(2)}})\left(qv\int_{p_0}^{x}\delta qv\right)=-\frac{1}{4r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\delta qv. \tag{5.1.60}$$

We cannot simply pullout the differential δ, since qv is singular at the endpoints of the integral. To circumvent this issue notice that in local coordinate $z(x)$ one has:

$$\delta[qv]=\delta\left[qv+\frac{1}{4r_j^2}v\right], \tag{5.1.61}$$

where the expression on right-hand side is regular. Hence,

$$(\operatorname*{res}_{z_j^{(1)}}+\operatorname*{res}_{z_j^{(2)}})\left(qv\int_{p_0}^{x}\delta qv\right)=d\left[-\frac{1}{4r_j}\int_{z_j^{(2)}}^{z_j^{(1)}}\left(qv+\frac{1}{4r_j^2}v\right)\right]. \tag{5.1.62}$$

Plugging (5.1.57) and (5.1.62) into (5.1.47) and integrating we obtain the desired formula. □

Let us compute $\delta^{(2)}_{Q_1}G_0^\circ$. As before, first we vary the formula (5.1.38) for $\delta_{Q_1}G_0^\circ$ with respect to A_γ on the space $\mathcal{M}^{\mathfrak{sl}_2}_{g,n}\{\mathbf{r}\}$. While the variations of differential v and function q are known and given by (5.1.24) and (5.1.22), we also need to vary quadratic differential Q_1 and the integral $\int_{x_i}^{x}v$ which involves a moving branch point.

Since the differential Q_1 is assumed to depend only on moduli of $\mathcal{M}_{g,n}$, which are kept fixed while varying with respect to A_γ, we immediately get

$$\frac{\partial Q_1}{\partial A_\gamma}=0. \tag{5.1.63}$$

The variation of $\int_{x_i}^{x}v$ is slightly more involved. Let $x_i\in\hat{\mathcal{C}}$ be a ramification point of $\hat{\mathcal{C}}$ and $\xi_i=\xi(\pi(x_i))\in\mathbb{C}$ be the corresponding critical value in some local coordinate ξ on $\mathcal{C}$ which remains fixed under deformation of $\hat{\mathcal{C}}$; let $\xi-\xi_i$ be a coordinate on $\mathbb{C}$ vanishing at $\pi(x_i)$ (this coordinate deforms when $\hat{\mathcal{C}}$ varies). A suitable local coordinate on $\hat{\mathcal{C}}$ near x_i can then be chosen to be $\sqrt{\xi-\xi_i}$.

Then by the Leibniz integral rule the differentiation with respect to A_α of the lower endpoint also gives a contribution to the derivative and we get

$$\frac{\partial(\int_{x_i}^{x}v)}{\partial A_\gamma}=\int_{x_i}^{x}u_\gamma^-+\frac{\partial\xi_i}{\partial A_\gamma}\frac{v}{d\xi}(x_i). \tag{5.1.64}$$

Since on $\hat{\mathcal{C}}$ differential v vanishes at x_i of order 2, the function $\frac{v}{d\xi}$ has simple zero at x_i and, thus,

$$\frac{\partial(\int_{x_i}^x v)}{\partial A_\gamma} = \int_{x_i}^x u_\gamma^-. \tag{5.1.65}$$

We apply $\frac{\partial}{\partial A_\gamma}$ to (5.1.38) along with variational formulas (5.1.22), (5.1.24), (5.1.63) and (5.1.65) to obtain

$$\frac{\partial}{\partial A_\gamma}[\delta_{Q_1} G_0^\circ] = \sum_{i=1}^{4g-4+2n} \frac{5\pi i}{72} \underset{x_i}{res} \left(\frac{\frac{Q_1 u_\gamma^-}{v^2}\left(\int_{x_i}^x v\right) + \frac{Q_1}{v}\left(\int_{x_i}^x u_\gamma^-\right)}{\left(\int_{x_i}^x v\right)^2} \right) - \tag{5.1.66}$$

$$-\sum_{j=1}^n \frac{\pi i r_j}{4} \int_{z_j^{(2)}}^{z_j^{(1)}} \left[\frac{1}{2}\left(\frac{u_\gamma^-}{v}\right)''_{zz} \frac{Q_1}{v} + 3q\frac{Q_1 u_\gamma^-}{v^2} \right] - \sum_{j=1}^n \frac{\pi i}{16 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1 u_\gamma^-}{v^2}. \tag{5.1.67}$$

Applying differential operator (5.1.19) to $\delta_{Q_1} G_0^\circ$ along with the formula (5.1.37), we get

$$\delta_{Q_1}^{(2)} G_0^\circ = \sum_{i=1}^{4g-4+2n} \frac{5\pi i}{144} \underset{x_i}{res} \left(\frac{\frac{Q_1^2}{v^3}\left(\int_{x_i}^x v\right) + \frac{Q_1}{v}\left(\int_{x_i}^x \frac{Q_1}{v}\right)}{\left(\int_{x_i}^x v\right)^2} \right) - \tag{5.1.68}$$

$$-\sum_{j=1}^n \frac{\pi i r_j}{8} \int_{z_j^{(2)}}^{z_j^{(1)}} \left[\frac{1}{2}\left(\frac{Q_1}{v^2}\right)''_{zz} \frac{Q_1}{v} + 3q\frac{Q_1^2}{v^3} \right] - \sum_{j=1}^n \frac{\pi i}{32 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^2}{v^3}. \tag{5.1.69}$$

On the final step, we plug this expression into (5.1.40) to obtain

Proposition 5.1.3. *The term G_2 of the generalized WKB expansion of the monodromy generating function has the following expression*

$$\begin{aligned} G_2 = &\sum_{i=1}^{4g-4+2n} \frac{5\pi i}{288} \underset{x_i}{res} \left(\frac{\left(2qv + \frac{Q_1^2}{v^3}\right)\left(\int_{x_i}^x v\right) + \frac{Q_1}{v}\left(\int_{x_i}^x \frac{Q_1}{v}\right)}{\left(\int_{x_i}^x v\right)^2} \right) - \sum_{j=1}^n \frac{\pi i}{16 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \left(qv + \frac{1}{4r_j^2} v\right) - \\ &-\sum_{j=1}^n \frac{\pi i r_j}{16} \int_{z_j^{(2)}}^{z_j^{(1)}} \left[\frac{1}{2}\left(\frac{Q_1}{v^2}\right)''_{zz} \frac{Q_1}{v} + 3q\frac{Q_1^2}{v^3} \right] - \sum_{j=1}^n \frac{\pi i}{64 r_j} \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^2}{v^3} + \\ &+ \sum_{j=1}^n \binom{\frac{1}{2}}{4} \pi i\, r_j \int_{z_j^{(2)}}^{z_j^{(1)}} \frac{Q_1^4}{v^7} + \left\langle \oint v_{-1}^\circ, \oint v_3^\circ \right\rangle. \end{aligned} \tag{5.1.70}$$

Bibliography

[1] Abikoff, W., *The real analytic theory of Teichmüller space. Lecture Notes in Math.*, **820** Springer, Berlin, 144 p. (1980)

[2] D. Allegretti. Voros symbols as cluster coordinates. *Journal of Topology*, 12(4):1031–1068, May 2019.

[3] D. Baraglia and Z. Huang. Special Kähler geometry of the Hitchin system and topological recursion. *Advances in Theoretical and Mathematical Physics*, 23(8):1981–2024, 2019.

[4] M. Bertola and D. Korotkin. Spaces of Abelian Differentials and Hitchin's Spectral Covers. *International Mathematics Research Notices*, 2019.

[5] M. Bertola and D. Korotkin. Hodge and Prym Tau Functions, Strebel Differentials and Combinatorial Model of $\mathcal{W}_{g,n}$. *Communications in Mathematical Physics*, 378(2):1279–1341, 2020.

[6] M. Bertola and D. Korotkin. Tau-Functions and Monodromy Symplectomorphisms. *Communications in Mathematical Physics*, 388(1):245–290, Oct 2021.

[7] M. Bertola and D. A. Korotkin. WKB expansion for a Yang–Yang generating function and the Bergman tau function. *Theor. Math. Phys.*, 206(3):258–295, 2021.

[8] M. Bertola, D. Korotkin, and C. Norton. Symplectic geometry of the moduli space of projective structures in homological coordinates. *Inventiones mathematicae*, 210(3):759–814, 2017.

[9] C. Boissy. Connected components of the strata of the moduli space of meromorphic differentials. *Commentarii Mathematici Helvetici*, 90(2):255–286, 2015.

[10] T. Bridgeland and D. Masoero, On the monodromy of the deformed cubic oscillator, arXiv:2006.10648, 2020.

[11] L. Chekhov. Symplectic Structures on Teichmüller Spaces $\mathfrak{T}_{g,s,n}$ and Cluster Algebras. *Proceedings of the Steklov Institute of Mathematics*, 309(1):87–96, May 2020.

[12] D. Chen and Q. Gendron. Towards a classification of connected components of the strata of k-differentials, arXiv:2101.01650, 2021.

[13] R. Donagi and E. Markman. Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles. *Integrable Systems and Quantum Groups*, page 1–119, 1996.

[14] B. Dubrovin. Geometry of 2D topological field theories, in: *Integrable systems and quantum groups. Proceedings,* Montecatini Terme, 1993, pp. 120-348, Lecture Notes in Math., v.1620, Berlin: Springer, 1996

[15] O. Dumitrescu and M. Mulase. Quantum curves for Hitchin fibrations and the Eynard–Orantin theory. *Letters in Mathematical Physics*, 104(6):635–671, feb 2014.

[16] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Communications in Number Theory and Physics*, 1(2):347–452, 2007.

[17] D. Eriksson, G. Montplet, and R. Wentworth. Complex Chern–Simons bundles in the relative setting, arXiv:2109.02033, 2021.

[18] J. Fay. Kernel functions, analytic torsion, and moduli spaces. *Memoirs of the American Mathematical Society*, 96(464), 1992.

[19] J. Fay. Theta Functions on Riemann Surfaces. *Lecture Notes in Mathematics*, 1973.

[20] D. Gaiotto, G. Moore, and A. Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Advances in Mathematics*, 234:239–403, Feb 2013.

[21] W. Goldman. The symplectic nature of fundamental groups of surfaces. *Advances in Mathematics*, 54(2):200–225, Nov 1984.

[22] N. Hawley and M. Schiffer. Half-order differentials on Riemann surfaces. *Acta Mathematica*, 115:199 – 236, 1966.

[23] N. Hitchin. The Self-Duality Equations on a Riemann Surface. *Proceedings of the London Mathematical Society*, s3-55(1):59–126, 1987.

[24] J. Hurtubise. The geometry of generalised Hitchin systems. *Integrable Systems: From Classical to Quantum*, page 55–76, 2000.

[25] K. Iwaki and T. Nakanishi. Exact WKB analysis and cluster algebras. *Journal of Physics A: Mathematical and Theoretical*, 47(47):474009, 2014.

[26] C. Kalla and D. Korotkin. Baker–Akhiezer Spinor Kernel and Tau-functions on Moduli Spaces of Meromorphic Differentials. *Communications in Mathematical Physics*, 331(3):1191–1235, 2014.

[27] S. Kawai. The symplectic nature of the space of projective connections on Riemann surfaces, Math Ann **305** 161-182 (1996)

[28] T. Kawai and Y. Takei. *Algebraic analysis of singular perturbation theory*, volume 227. AMS, 2005.

[29] R. Klimov. On generalized WKB expansion of monodromy generating function, arXiv:2206.10578, 2022.

[30] R. Klimov. Variational formulas on spaces of SL(2) Hitchin's spectral covers, arXiv:2108.12911, 2021.

[31] D. Korotkin. Bergman Tau-Function: From Einstein Equations and Dubrovin-Frobenius Manifolds to Geometry of Moduli Spaces. *Integrable Systems and Algebraic Geometry*, page 215–287, 2020.

[32] D. Korotkin. Periods of meromorphic quadratic differentials and Goldman bracket. *Topological Recursion and its Influence in Analysis, Geometry, and Topology*, page 491–515, 2018.

[33] D. Korotkin, A. Sauvaget, and P. Zograf. Tau functions, Prym-Tyurin classes and loci of degenerate differentials. *Mathematische Annalen*, 375(1-2):213–246, 2019.

[34] D. Korotkin and P. Zograf. Tau function and moduli of differentials. *Mathematical Research Letters*, 18(3):447–458, 2011.

[35] D. Korotkin and P. Zograf. Tau function and the Prym class. *Contemporary Mathematics*, page 241–261, 2013.

[36] D. Korotkin and P. Zograf. Tau functions, Hodge classes, and discriminant loci on moduli spaces of Hitchin's spectral covers. *Journal of Mathematical Physics*, 59(9):091412, Sep 2018.

[37] A. Kokotov, D. Korotkin, Tau-functions on Hurwitz spaces. *Mathematical Physics, Analysis and Geometry*, **7** N.1 47-96 (2004)

[38] A. Kokotov and D. Korotkin. Tau-functions on spaces of Abelian differentials and higher genus generalizations of Ray-Singer formula. *Journal of Differential Geometry*, 82(1):35–100, 2009.

[39] M. Kontsevitch, A. Zorich, Lyapunov exponents and Hodge theory, hep-th/9701164

[40] M. Kontsevich and A. Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Inventiones Mathematicae*, 153(3):631–678, 2003.

[41] N. Nekrasov, A. Rosly, and S. Shatashvili. Darboux coordinates, Yang-Yang functional, and gauge theory. *Nuclear Physics B - Proceedings Supplements*, 216(1):69–93, Jul 2011.

[42] H. Rauch. Weierstrass points, branch points, and moduli of Riemann surfaces. *Communications on Pure and Applied Mathematics*, 12:543–560, 1959.

[43] W. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.

[44] A. Voros. The return of the quartic oscillator. The complex WKB method. *Annales de l'Institut Henri Poincaré. Section A, Physique théorique*, **39**(3), 211–338, 1983