On the Gaussian Product Inequality

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Abstract

On the Gaussian Product Inequality

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The long-standing Gaussian product inequality (GPI) conjecture states that $E[\prod_{j=1}^{n} |X_j|^{y_j}] \geq \prod_{j=1}^{n} E[|X_j|^{y_j}]$ for any centered Gaussian random vector $(X_1, \ldots, X_n)$ and any non-negative real numbers y_j, $j = 1, \ldots, n$. First, we complete the picture of bivariate Gaussian product relations by proving a novel "opposite GPI" when $-1 < y_1 < 0$ and $y_2 > 0$: $E[|X_1|^{y_1}|X_2|^{y_2}] \leq E[|X_1|^{y_1}]E[|X_2|^{y_2}]$. Next, we investigate the three-dimensional inequality $E[X_1^2 X_2^{2m_2} X_3^{2m_3}] \geq E[X_1^2]E[X_2^{2m_2}]E[X_3^{2m_3}]$ for any $m_2, m_3 \in \mathbb{N}$. We show that this inequality is implied by a combinatorial inequality which we verify directly for small values of m_2 and arbitrary m_3. Then, we complete the proof through the discovery of a novel inequality for the moment ratio $\frac{|E[X_2^{2m_2+1} X_3^{2m_3+1}]|}{E[X_2^{2m_2} X_3^{2m_3}]}$, which implies this three-dimensional GPI. We then extend these three-dimensional results to the case where the exponents in the GPI can be real numbers rather than simply even integers. Finally, we describe two computational algorithms involving sums-of-squares representations of polynomials that can be used to resolve the GPI conjecture. To exhibit the power of these novel methods, we apply them to prove new four- and five-dimensional GPIs: $E[X_1^{2m} X_2^2 X_3^2 X_4^2] \geq E[X_1^{2m}]E[X_2^2]E[X_3^2]E[X_4^2]$ for any $m \in \mathbb{N}$, and $E[|X_1|^y X_2^2 X_3^2 X_4^2 X_5^2] \geq E[|X_1|^y]E[X_2^2]E[X_3^2]E[X_4^2]E[X_5^2]$ for any $y \geq \frac{1}{10}$.

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Contribution of Authors

This thesis is based on the four papers [35–38]. All of the work was an equal collaboration between myself and Dr. Wei Sun. We both discussed questions, provided ideas, wrote proofs and proofread.

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Chapter 1

Introduction

Correlation is one of the basic themes of science, but also one of the most mysterious. Bell's inequality, quantum entanglement and the other foundations of quantum information theory all depend on our grasp of the concept of correlation. Even before the discovery of the quantum world, correlation was a fundamental notion of science. Similarly, the Gaussian distribution, which is a building block of probability and statistics, has a nearly 300-year history dating back to Abraham de Moivre. When combined, these two concepts have proven essential to many branches of mathematics and physics. However, our understanding of them is far from complete.

Inequalities involving correlations between Gaussian random variables have important connections to a number of active research areas, such as small-ball probabilities (cf. [24] and [40]) and the zeros of random polynomials (cf. [25]). In 2014, Royen's proof of the elusive Gaussian correlation inequality (GCI) conjecture [22, 34] was a significant breakthrough that inspired renewed hope in solving another long-standing and challenging problem, the Gaussian product inequality (GPI) conjecture.

Most generally, the GPI conjecture (see Li and Wei [26]) states that for any non-negative real numbers y_j, $j = 1, \dots, n$, and any n-dimensional real-valued centered Gaussian random vector $(X_1, \dots, X_n)$,

$$E\left[\prod_{j=1}^{n} |X_j|^{y_j}\right] \geq \prod_{j=1}^{n} E[|X_j|^{y_j}]. \tag{1.1}$$

The verification of the GPI would have immediate and significant impact on multiple problems in other fields. For example (see Malicet et al. [28]), if (1.1) holds when the y_j's are any equal positive even integers, then the 'real linear polarization constant' conjecture raised by Benítez et al. [6] is true, and (1.1) is deeply linked to the celebrated U-conjecture by Kagan, Linnik and Rao [17]. Furthermore, there is a strong connection between the GCI and GPI. In [9], Edelmann et al. showed that the GPI with negative exponents follows directly from the GCI.

Despite the misleadingly simple appearance of the statement of the GPI conjecture (1.1), the difficulty of proving the GPI lies in the subtlety of the inequality, and thus, so far, only partial results have been obtained.

This thesis is based on our four papers [35–38] and is structured as follows. In Chapter 2, we summarize the findings published prior to the start of our studies on the GPI. We briefly introduce the Gaussian hypergeometric function, an invaluable tool which we use frequently in our proofs. We then present some of our preliminary results, which set the stage for the body of our work.

In Chapter 3, we complete the picture of bivariate Gaussian product relations by proving a novel "opposite GPI" when $-1 < y_1 < 0$ and $y_2 > 0$:

$$E[|X_1|^{y_1}|X_2|^{y_2}] \leq E[|X_1|^{y_1}]E[|X_2|^{y_2}].$$

In Chapter 4, we investigate the three-dimensional inequality

$$E[X_1^2 X_2^{2m_2} X_3^{2m_3}] \geq E[X_1^2]E[X_2^{2m_2}]E[X_3^{2m_3}]$$

for any $m_2, m_3 \in \mathbb{N}$. We show that this inequality is implied by a combinatorial inequality which can be verified directly for small values of m_2 and arbitrary m_3. Hence the corresponding cases of the three-dimensional inequality are proved. In Chapter 5, we complete the proof through the discovery of a novel inequality for the moment ratio $\frac{|E[X_2^{2m_2+1}X_3^{2m_3+1}]|}{E[X_2^{2m_2}X_3^{2m_3}]}$, which implies this three-dimensional GPI. The interplay between computing and hard analysis plays a crucial role in the proofs. In Chapter 6, we extend these three-dimensional results to the case where the exponents in the GPI can be real numbers rather than simply even integers.

In Chapter 7, we describe two computational algorithms involving sums-of-squares representations of polynomials that can be used to resolve the GPI conjecture. To exhibit the power of these novel methods, we apply them to prove new four- and five-dimensional GPIs:

$$E[X_1^{2m} X_2^2 X_3^2 X_4^2] \geq E[X_1^{2m}]E[X_2^2]E[X_3^2]E[X_4^2]$$

for any $m \in \mathbb{N}$, and

$$E[|X_1|^y X_2^2 X_3^2 X_4^2 X_5^2] \geq E[|X_1|^y]E[X_2^2]E[X_3^2]E[X_4^2]E[X_5^2]$$

for any $y \geq \frac{1}{10}$.

Finally, in Chapter 8, we propose the next steps for those interested in continuing to study the GPI.

Chapter 2

Background

2.1 Past results

Prior to the discoveries contained in this thesis, no universal method had been proposed for proving the GPI conjecture (1.1); however, several special cases had been solved. The remainder of this section will serve to outline the state of the art with regards to the progress on the GPI conjecture when we began our studies.

Frenkel [10, Theorem 2.1] proved (1.1) when all exponents equal 2 using algebraic methods. In [44], Wei used integral representations to prove a stronger version of (1.1) for $y_j \in (-1, 0)$ as follows:

$$E\left[\prod_{j=1}^{n} |X_j|^{y_j}\right] \geq E\left[\prod_{j=1}^{k} |X_j|^{y_j}\right] E\left[\prod_{j=k+1}^{n} |X_j|^{y_j}\right], \quad \forall 1 \leq k \leq n-1.$$

Malicet et al. [28] gave a GPI involving Hermite polynomials, which provides a substantial generalization as well as a new analytical proof of Frenkel [10, Theorem 2.1].

By Karlin and Rinott [19, Corollary 1.1, Theorem 3.1 and Remark 1.4], (1.1) holds for $n = 2$ by virtue of $(|X_1|, |X_2|)$ possessing the MTP_2 property. Thus, we know the following two-dimensional GPI holds:

Theorem 2.1.1 *Let (X_1, X_2) be centered bivariate Gaussian random variables and $y_1, y_2 > 0$ or $-1 < y_1, y_2 < 0$. Then,*

$$E[|X_1|^{y_1}|X_2|^{y_2}] \geq E[|X_1|^{y_1}]E[|X_2|^{y_2}], \tag{2.1}$$

and the equality sign holds if and only if X_1, X_2 are independent.

Liu et al. [27] also used integral representations to provide an alternate proof of (2.1) when $y_1, y_2 \in (0, 2)$.

By using the Gaussian hypergeometric function as a powerful tool, Lan et al. [21] were successful in proving the following three-dimensional GPI: for any $m_1, m_2 \in \mathbb{N}$ and any centered Gaussian random vector (X_1, X_2, X_3),

$$E[X_1^{2m_1} X_2^{2m_2} X_3^{2m_2}] \geq E[X_1^{2m_1}]E[X_2^{2m_2}]E[X_3^{2m_2}].$$

The analogue of the GPI for complex-valued Gaussian random variables was solved over 20 years ago [3]. However, that problem is quite different from the GPI treated here. For example, in a recent preprint, Barthe and Cordero-Erausquin prove the *strong* version of the GPI for complex-valued random variables [4, Theorem 3], whereas it is well known that the strong GPI does not hold in general for the real case (see, for example, [21] or [9]). Thus, it may be argued that the real GPI (1.1) is a much more subtle inequality.

In the interest of maintaining a chronological structure within this thesis, we choose to discuss the further findings of our contemporaries alongside our own in the appropriate chapters below.

Throughout this thesis, any Gaussian random variable is assumed to be real-valued and non-degenerate, i.e., has positive variance.

2.2 The Gaussian hypergeometric function

This section is devoted to one of the main tools used in this thesis, which we felt deserved its own brief introduction.

The Gaussian hypergeometric function (cf. [33]) is defined as:

$$F(a, b; c; z) := \sum_{j=0}^{\infty} \frac{(a)_{(j)}(b)_{(j)}}{(c)_{(j)}} \cdot \frac{z^j}{j!},$$

where z can be any complex number that satisfies $|z| < 1$, and for $\alpha \neq 0$, the factorial function is defined by

$$(\alpha)_{(j)} = \begin{cases} 1, & j = 0, \\ \alpha(\alpha+1)\cdots(\alpha+j-1), & j \geq 1. \end{cases}$$

Consider the following functions contiguous to $F(a, b; c; z)$:

$$F(a \pm 1, b; c; z), \quad F(a, b \pm 1; c; z), \quad F(a, b; c \pm 1; z).$$

To simplify notation, we denote $F(a, b; c; z)$ and these six functions respectively by

$$F, \quad F(a \pm 1), \quad F(b \pm 1), \quad F(c \pm 1).$$

We have the following relations of Gauss between contiguous functions (cf. [5, 2.8-(21), (31), (37), (38), pages 102 and 103]):

$$\begin{aligned}
&F' = \frac{a[F(a+1) - F]}{z}, \\
&[c - 2a - (b-a)z]F + a(1-z)F(a+1) - (c-a)F(a-1) = 0, \\
&(b-a)(1-z)F - (c-a)F(a-1) + (c-b)F(b-1) = 0, \\
&c(1-z)F - cF(a-1) + (c-b)zF(c+1) = 0.
\end{aligned}$$

Gaussian hypergeometric functions have been studied extensively and thus, these nice relations are just a few of many, making them easy to work with. Additionally, what makes the hypergeometric function so useful to us is its intrinsic link to bivariate absolute Gaussian moments (for example, see (3.3) and (5.39) below). This connection will be investigated and manipulated thoroughly throughout this thesis.

2.3 Preliminaries

2.3.1 Non-triviality of GPI

Note that the product inequality (1.1) may not hold for non-Gaussian random vectors. Let V_1 and V_2 be independent Rademacher random variables, i.e., $P(V_i = 1) = P(V_i = -1) = \frac{1}{2}$, $i = 1, 2$. Define $Y_1 = V_1 + 0.9V_2$ and $Y_2 = V_1 - 0.9V_2$. Then, (Y_1, Y_2) is a centered random vector with

$$\mathrm{Cov}(Y_1, Y_2) = E[V_1^2 - 0.81V_2^2] = 0.19.$$

We have

$$E[Y_1^2 Y_2^2] = E[(V_1^2 - 0.81V_2^2)^2] = (0.19)^2,$$

and

$$E[Y_1^2]E[Y_2^2] = (1.81)^2.$$

Hence

$$E[Y_1^2 Y_2^2] < E[Y_1^2]E[Y_2^2].$$

2.3.2 Positive covariances

The following is taken from our paper [36].

Lemma 2.3.1 *Let $(X_1, \ldots, X_n)$ be a centered Gaussian random vector such that $E[X_i X_j] \geq 0$ for any $i \neq j$. Then,*

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq E\left[\prod_{j=1}^{k} X_j^{2m_j}\right] E\left[\prod_{j=k+1}^{n} X_j^{2m_j}\right], \qquad \forall 1 \leq k \leq n-1, \tag{2.2}$$

and

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq \prod_{j=1}^{n} E[X_j^{2m_j}]. \tag{2.3}$$

Proof. By induction, (2.3) is a direct consequence of (2.2). In the following, we prove (2.2).

We assume without loss of generality that $\mathrm{Var}(X_i) = 1$ for $1 \leq i \leq n$. Denote by Λ the covariance matrix of $(X_1, \dots, X_n)$. Let $1 \leq k \leq n-1$. Define

$$\tilde{\Lambda}_{pq} = \begin{cases} \Lambda_{pq}, & \text{if } 1 \leq p, q \leq k \text{ or } k+1 \leq p, q \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\tilde{\Lambda} := (\tilde{\Lambda}_{pq})_{1 \leq p,q \leq n}$ is a covariance matrix. Let $(Y_1, \dots, Y_n)$ be a Gaussian random vector with covariance matrix $\tilde{\Lambda}$. Then, $(Y_1, \dots, Y_k)$ and $(X_1, \dots, X_k)$ have the same distribution, $(Y_{k+1}, \dots, Y_n)$ and $(X_{k+1}, \dots, X_n)$ have the same distribution, and $(Y_1, \dots, Y_k)$ and $(Y_{k+1}, \dots, Y_n)$ are independent.

Denote by T the set of all symmetric matrices $s = (s_{kl})_{1 \leq k,l \leq n}$ satisfying $s_{kl} \in \mathbb{N} \cup \{0\}$ and

$$s_{kk} + \sum_{l=1}^{n} s_{kl} = 2m_k, \quad 1 \leq k \leq n.$$

By [42, Formula (44)] (see [41] for the detailed proof), we get

$$\begin{aligned} E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] &= \left(\prod_{j=1}^{n} (2m_j)!\right) \sum_{s \in T} \frac{\prod_{k<l} \Lambda_{kl}^{s_{kl}}}{2^{\sum_{k=1}^{n} s_{kk}} \left(\prod_{k=1}^{n} s_{kk}!\right) \left(\prod_{k<l} s_{kl}!\right)} \\ &\geq \left(\prod_{j=1}^{n} (2m_j)!\right) \sum_{s \in T} \frac{\prod_{k<l} \tilde{\Lambda}_{kl}^{s_{kl}}}{2^{\sum_{k=1}^{n} s_{kk}} \left(\prod_{k=1}^{n} s_{kk}!\right) \left(\prod_{k<l} s_{kl}!\right)} \\ &= E\left[\prod_{j=1}^{n} Y_j^{2m_j}\right] \\ &= E\left[\prod_{j=1}^{k} Y_j^{2m_j}\right] E\left[\prod_{j=k+1}^{n} Y_j^{2m_j}\right] \\ &= E\left[\prod_{j=1}^{k} X_j^{2m_j}\right] E\left[\prod_{j=k+1}^{n} X_j^{2m_j}\right]. \end{aligned}$$

The proof is complete. □

Remark 2.3.2 *(i) The GPI conjecture is especially difficult when some components of the Gaussian random vector are negatively correlated. It is known that if assuming positive correlations then one can verify special cases of the conjecture (cf. e.g., [44, page 1060]). Recently, Genest and Ouimet [11] showed that (2.3) holds if there exists a matrix* $C \in [0, \infty)^{n \times n}$ *such that* $(X_1, \dots, X_n) = (Z_1, \dots, Z_n)C$ *in law, where* $(Z_1, \dots, Z_n)$ *is an n-dimensional standard*

Gaussian random vector. Furthermore, Edelmann et al. [9] used a different method to extend (2.2) to the multivariate gamma distribution. Then, Genest and Ouimet also provided another distributional generalization (see [12]).

(ii) By replacing X_2 with $-X_2$ if necessary, Lemma 2.3.1 implies the well-known two-dimensional GPI: for any centered bivariate Gaussian random variables (X_1, X_2),

$$E[X_1^{2m_1} X_2^{2m_2}] \geq E[X_1^{2m_1}]E[X_2^{2m_2}].$$

2.3.3 Rank-reducing technique

The following is taken from our paper [36].

Lemma 2.3.3 *Let $n \geq 3$ and $m_1, \ldots, m_n \in \mathbb{N}$. If for any centered Gaussian random vector $(Y_1, \ldots, Y_n)$ with $Y_n = \alpha_1 Y_1 + \cdots + \alpha_{n-1} Y_{n-1}$ for some constants $\alpha_1, \ldots, \alpha_{n-1}$,*

$$E\left[\left\{\prod_{j=1}^{n-1} Y_j^{2m_j}\right\} Y_n^{2k}\right] \geq \left\{\prod_{j=1}^{n-1} E[Y_j^{2m_j}]\right\} E[Y_n^{2k}], \qquad 0 \leq k \leq m_n, \tag{2.4}$$

then for any centered Gaussian random vector $(X_1, \ldots, X_n)$,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq \prod_{j=1}^{n} E[X_j^{2m_j}]. \tag{2.5}$$

Additionally, if inequality (2.4) is strict when $k = m_n$, then the equality sign of (2.5) holds only if X_n is independent of $X_1, \ldots, X_{n-1}$.

Proof. Let $(X_1, \ldots, X_n)$ be a centered Gaussian random vector. Define

$$Z_0 = E[X_n | X_1, \ldots, X_{n-1}], \quad Z_1 = X_n - Z_0.$$

Then,

$$X_n^{2m_n} = (Z_0 + Z_1)^{2m_n} = \sum_{i=0}^{2m_n} \binom{2m_n}{i} Z_0^{2m_n - i} Z_1^i. \tag{2.6}$$

Note that Z_1 is independent of $X_1, \ldots, X_{n-1}$. Hence

$$E[Z_0^{2m_n - i} Z_1^i | X_1, \ldots, X_{n-1}] = Z_0^{2m_n - i} E[Z_1^i], \tag{2.7}$$

which is equal to zero if i is an odd number.

By (2.6) and (2.7), we get

$$E[X_n^{2m_n} | X_1, \ldots, X_{n-1}] = \sum_{i=0}^{m_n} \binom{2m_n}{2i} Z_0^{2m_n - 2i} E[Z_1^{2i}]. \tag{2.8}$$

Note that $Z_0 = \alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1}$ for some constants $\alpha_1, \ldots, \alpha_{n-1}$. Then, it follows from (2.4) that

$$E\left[\left(\prod_{j=1}^{n-1} X_j^{2m_j}\right) Z_0^{2m_n-2i}\right] \geq \left(\prod_{j=1}^{n-1} E[X_j^{2m_j}]\right) E[Z_0^{2m_n-2i}]. \tag{2.9}$$

Thus, by (2.6), (2.8) and (2.9), we obtain that

$$\begin{aligned} E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] &= E\left[\left(\prod_{j=1}^{n-1} X_j^{2m_j}\right) [E[X_n^{2m_n}|X_1, \ldots, X_{n-1}]\right] \\ &= \sum_{i=0}^{m_n} \binom{2m_n}{2i} E\left[\left(\prod_{j=1}^{n-1} X_j^{2m_j}\right) Z_0^{2m_n-2i}\right] E[Z_1^{2i}] \\ &\geq \sum_{i=0}^{m_n} \binom{2m_n}{2i} \left(\prod_{j=1}^{n-1} E[X_j^{2m_j}]\right) E[Z_0^{2m_n-2i}] E[Z_1^{2i}] \\ &= \left(\prod_{j=1}^{n-1} E[X_j^{2m_j}]\right) \sum_{i=0}^{m_n} \binom{2m_n}{2i} E[Z_0^{2m_n-2i} Z_1^{2i}] \\ &= \prod_{j=1}^{n} E[X_j^{2m_j}]. \end{aligned} \tag{2.10}$$

Now, suppose inequality (2.4) is strict when $k = m_n$ and X_n is not independent of $X_1, \ldots, X_{n-1}$. Then, $Z_0 = E[X_n|X_1, \ldots, X_{n-1}] \neq 0$ and

$$E\left[\left(\prod_{j=1}^{n-1} X_j^{2m_j}\right) Z_0^{2m_n}\right] > \left(\prod_{j=1}^{n-1} E[X_j^{2m_j}]\right) E[Z_0^{2m_n}].$$

Thus, by (2.10), we get

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] > \prod_{j=1}^{n} E[X_j^{2m_j}].$$

Therefore, the proof is complete. □

Corollary 2.3.4 *The following two claims are equivalent.*

Claim I: For any $n \geq 3$, $m_1, \ldots, m_n \in \mathbb{N}$, and centered Gaussian random vector $(X_1, \ldots, X_n)$ with $X_n = \alpha_1 X_1 + \cdots + \alpha_{n-1} X_{n-1}$ for some constants $\alpha_1, \ldots, \alpha_{n-1}$,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] > \prod_{j=1}^{n} E[X_j^{2m_j}].$$

Claim II: For any $n \geq 3$, $m_1, \ldots, m_n \in \mathbb{N}$, and centered Gaussian random vector $(X_1, \ldots, X_n)$,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq \prod_{j=1}^{n} E[X_j^{2m_j}],$$

and the equality holds if and only if $X_1, \ldots, X_n$ *are independent.*

Proof. Obviously, Claim II implies Claim I. The assertion that Claim I implies Claim II is a direct consequence of the proof of Lemma 2.3.3, where symmetry ensures the equality is equivalent to the independence of $X_1, \ldots, X_n$. □

Chapter 3

"An opposite Gaussian product inequality"

This chapter is based on our paper "An opposite Gaussian product inequality," published in Statistics and Probability Letters in 2022 [35].

From Theorem 2.1.1 above, we know the two-dimensional GPI holds when the exponents have the same sign. In this chapter, we complete the picture of two-dimensional Gaussian product relations by proving what we call an "opposite GPI" when y_1 and y_2 are of opposite sign. In Section 3.1, using the hypergeometric function, we state and prove this main result (see Theorem 3.1.1) and then give a more explicit expression for the two-dimensional GPI and opposite GPI when one of y_1, y_2 is a positive even integer. In section 3.2, we prove a one-dimensional GPI and opposite GPI. Finally, in section 3.3, we demonstrate that no such opposite GPI exists in higher dimensions. The results contained in this chapter extend and conclude the study of the two-dimensional GPI in its full generality.

3.1 Opposite GPI

Throughout this chapter, any Gaussian random variable is assumed to be real-valued and non-degenerate, i.e., has positive variance. Our main result is the following novel opposite GPI:

Theorem 3.1.1 *Let (X_1, X_2) be centered bivariate Gaussian random variables, $-1 < y_1 < 0$ and $y_2 > 0$. Then,*

$$E[|X_1|^{y_1}|X_2|^{y_2}] \leq E[|X_1|^{y_1}]E[|X_2|^{y_2}],$$

and the equality sign holds if and only if X_1, X_2 are independent.

Proof. Let U be a standard Gaussian random variable and $\nu > -1$. We have

$$\begin{aligned} E[|U|^\nu] &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^\nu e^{-\frac{x^2}{2}} dx \\ &= \frac{2^{\nu/2}}{\sqrt{\pi}} \int_0^\infty z^{(\nu-1)/2} e^{-z} dz \\ &= \frac{2^{\nu/2}\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}}. \end{aligned} \tag{3.1}$$

We assume without loss of generality that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 1$. Denote by ρ the correlation of X_1 and X_2. By (3.1), we get

$$E[|X_1|^{y_1}]E[|X_2|^{y_2}] = \frac{2^{(y_1+y_2)/2}\Gamma(\frac{y_1+1}{2})\Gamma(\frac{y_2+1}{2})}{\pi}. \tag{3.2}$$

If $|\rho| < 1$, by Nabeya [30], we have that

$$E[|X_1|^{y_1}|X_2|^{y_2}] = \frac{2^{(y_1+y_2)/2}\Gamma(\frac{y_1+1}{2})\Gamma(\frac{y_2+1}{2})}{\pi} \cdot F\left(-\frac{y_1}{2}, -\frac{y_2}{2}; \frac{1}{2}; \rho^2\right), \tag{3.3}$$

where we recall that F is a Gaussian hypergeometric function given by

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha\beta}{1!\,\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)} z^2 + \cdots, \qquad |z| < 1.$$

Define

$$G(z) := F\left(-\frac{y_1}{2}, -\frac{y_2}{2}; \frac{1}{2}; z\right), \qquad |z| < 1.$$

By Bateman [5, 2.8-(20), page 102] and the Euler transformation (cf. Rainville [33, Chapter 4, Theorem 21, page 60]), we get

$$\begin{aligned} G'(z) &= \frac{y_1 y_2}{2} \cdot F\left(1 - \frac{y_1}{2}, 1 - \frac{y_2}{2}; \frac{3}{2}; z\right) \\ &= \frac{y_1 y_2}{2} \cdot (1-z)^{(y_1+y_2-1)/2} F\left(\frac{y_1+1}{2}, \frac{y_2+1}{2}; \frac{3}{2}; z\right) \\ &< 0, \qquad |z| < 1. \end{aligned} \tag{3.4}$$

Therefore, the proof is complete by (3.2) and (3.3). □

Remark 3.1.2 *Following the proof of Theorem 3.1.1, we can give a new proof of Theorem 2.1.1 using the hypergeometric function. The main difference is that now $y_1 y_2 > 0$ and hence (3.4) is replaced by the inequality $G'(z) > 0$, $|z| < 1$.*

If the exponent y_2 is a positive even integer, then we can get a more explicit comparison of the joint moment and the product of marginal moments. Obviously, this comparison result gives a different proof for the GPI (when $y_1 > 0$) and the opposite GPI (when $-1 < y_1 < 0$).

Proposition 3.1.3 *Let (X_1, X_2) be centered bivariate Gaussian random variables with correlation ρ, $y_1 > -1$ and $m \in \mathbb{N}$. Then,*

$$\begin{aligned}
&\frac{E[|X_1|^{y_1}|X_2|^{2m}]}{E[|X_1|^{y_1}]E[|X_2|^{2m}]} \\
&= (1-\rho^2)^m + \sum_{j=1}^{m}\binom{m}{j}\rho^{2j}(1-\rho^2)^{m-j}\frac{[y_1+(2j-1)][y_1+(2j-3)]\cdots(y_1+1)}{(2j-1)!!}.
\end{aligned}$$

Proof. We assume without loss of generality that $X_1 = U_1$ and $X_2 = aU_1 + U_2$, where U_1, U_2 are independent standard Gaussian random variables and $a \in \mathbb{R}$.

By (3.1), we get

$$E[|X_1|^{y_1}]E[|X_2|^{2m}] = E[|U_1|^{y_1}]E[|aU_1+U_2|^{2m}] = \frac{2^{y_1/2}\Gamma\left(\frac{y_1+1}{2}\right)(1+a^2)^m(2m-1)!!}{\sqrt{\pi}}, \quad (3.5)$$

and

$$\begin{aligned}
&E[|X_1|^{y_1}|X_2|^{2m}] \\
&= E[|U_1|^{y_1}|aU_1+U_2|^{2m}] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1|^{y_1}|ax_1+x_2|^{2m}e^{-\frac{x_1^2+x_2^2}{2}}dx_1dx_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|x_1|^{y_1}e^{-\frac{x_1^2}{2}}\int_{-\infty}^{\infty}\left[\sum_{i=0}^{2m}\binom{2m}{i}(ax_1)^i x_2^{2m-i}\right]e^{-\frac{x_2^2}{2}}dx_2dx_1 \\
&= \sum_{j=0}^{m-1}\binom{2m}{2j}(2m-2j-1)!!\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a^{2j}|x_1|^{y_1+2j}e^{-\frac{x_1^2}{2}}dx_1 \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a^{2m}|x_1|^{y_1+2m}e^{-\frac{x_1^2}{2}}dx_1 \\
&= (2m-1)!!\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x_1|^{y_1}e^{-\frac{x_1^2}{2}}dx_1 + \sum_{j=1}^{m}\frac{\binom{m}{j}a^{2j}}{(2j-1)!!}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x_1|^{y_1+2j}e^{-\frac{x_1^2}{2}}dx_1\right\} \\
&= \frac{(2m-1)!!}{\sqrt{\pi}}\left\{2^{y_1/2}\Gamma\left(\frac{y_1+1}{2}\right) + \sum_{j=1}^{m}\frac{\binom{m}{j}a^{2j}2^{(y_1+2j)/2}\Gamma(\frac{y_1+2j+1}{2})}{(2j-1)!!}\right\} \\
&= \frac{2^{y_1/2}\Gamma\left(\frac{y_1+1}{2}\right)(2m-1)!!}{\sqrt{\pi}}\left\{1\right. \\
&\quad \left. + \sum_{j=1}^{m}\binom{m}{j}a^{2j}\frac{[y_1+(2j-1)][y_1+(2j-3)]\cdots(y_1+1)}{(2j-1)!!}\right\}. \qquad (3.6)
\end{aligned}$$

Note that

$$a^2 = \frac{\rho^2}{1-\rho^2}, \qquad 1 + a^2 = \frac{1}{1-\rho^2}.$$

Therefore, the proof is complete by (3.5) and (3.6). □

3.2 One-dimensional inequalities

In this section, we present a one-dimensional GPI and opposite GPI.

Proposition 3.2.1 *Let X be a centered Gaussian random variable.*

(i) If $-1 < y_1 < 0$ and $y_2 > 0$, then

$$E[|X|^{y_1+y_2}] < \frac{(y_1+1)(y_2+1)}{y_1+y_2+1} E[|X|^{y_1}]E[|X|^{y_2}].$$

(ii) If $y_1, y_2 > 0$ or $-1 < y_1, y_2 < 0$ with $y_1 + y_2 > -1$, then

$$E[|X|^{y_1+y_2}] > \frac{(y_1+1)(y_2+1)}{y_1+y_2+1} E[|X|^{y_1}]E[|X|^{y_2}].$$

Proof. We assume without loss of generality that $\mathrm{Var}(X) = 1$. By (3.1), we get

$$E[|X|^{y_1+y_2}] = \frac{2^{(y_1+y_2)/2}\Gamma(\frac{y_1+y_2+1}{2})}{\sqrt{\pi}},$$

and

$$E[|X|^{y_1}]E[|X|^{y_2}] = \frac{2^{(y_1+y_2)/2}\Gamma(\frac{y_1+1}{2})\Gamma(\frac{y_2+1}{2})}{\pi}.$$

Hence,

$$\frac{E[|X|^{y_1+y_2}]}{E[|X|^{y_1}]E[|X|^{y_2}]} = \frac{\Gamma(\frac{y_1+y_2+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{y_1+1}{2})\Gamma(\frac{y_2+1}{2})} = \frac{B(\frac{y_1+y_2+1}{2}, \frac{1}{2})}{B(\frac{y_1+1}{2}, \frac{y_2+1}{2})}. \tag{3.7}$$

We have (cf. [2, (1.1.26), page 8])

$$B(x, y) = \frac{x+y}{xy} \prod_{n=1}^{\infty} \left(1 + \frac{xy}{n(x+y+n)}\right)^{-1}, \qquad x, y > 0.$$

Note that

$$\frac{y_1+y_2+1}{2} + \frac{1}{2} = \frac{y_1+1}{2} + \frac{y_2+1}{2},$$

and

$$\begin{aligned} &\frac{y_1+y_2+1}{2} \cdot \frac{1}{2} > \frac{y_1+1}{2} \cdot \frac{y_2+1}{2}, &&\text{if } -1 < y_1 < 0 \text{ and } y_2 > 0, \\ &\frac{y_1+y_2+1}{2} \cdot \frac{1}{2} < \frac{y_1+1}{2} \cdot \frac{y_2+1}{2}, &&\text{if } y_1, y_2 > 0 \text{ or } -1 < y_1, y_2 < 0. \end{aligned}$$

Then,

$$\frac{B(\frac{y_1+y_2+1}{2},\frac{1}{2})}{B(\frac{y_1+1}{2},\frac{y_2+1}{2})} < \frac{(y_1+1)(y_2+1)}{y_1+y_2+1}, \quad \text{if } -1<y_1<0 \text{ and } y_2>0,$$
$$\frac{B(\frac{y_1+y_2+1}{2},\frac{1}{2})}{B(\frac{y_1+1}{2},\frac{y_2+1}{2})} > \frac{(y_1+1)(y_2+1)}{y_1+y_2+1}, \quad \text{if } y_1,y_2>0 \text{ or } -1<y_1,y_2<0 \text{ with } y_1+y_2>-1.$$

Therefore, the proof is complete by (3.7). □

Remark 3.2.2 *We would like to thank an anonymous referee for pointing us towards an alternate proof of Proposition 3.2.1. The proof given below reveals even more explicitly the role of monotonicity properties of ratios of gamma functions (cf. [1, 23, 31]).*

Note that $\Gamma(x)=\frac{\Gamma(x+1)}{x}$ *for any* $x>0$*. Then, (3.7) equals*

$$\frac{(y_1+1)(y_2+1)}{y_1+y_2+1}\cdot\frac{\Gamma(\frac{y_1+y_2+1}{2}+1)\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{y_1+1}{2}+1)\Gamma(\frac{y_2+1}{2}+1)}.$$

Hence, the proof of Proposition 3.2.1 is complete if we can show that

$$\frac{\Gamma(\frac{y_1+y_2+1}{2}+1)\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{y_1+1}{2}+1)\Gamma(\frac{y_2+1}{2}+1)}<1, \text{ if } -1<y_1<0 \text{ and } y_2>0,$$
$$\frac{\Gamma(\frac{y_1+y_2+1}{2}+1)\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{y_1+1}{2}+1)\Gamma(\frac{y_2+1}{2}+1)}>1, \text{ if } y_1,y_2>0 \text{ or } -1<y_1,y_2<0 \text{ with } y_1+y_2>-1. \quad (3.8)$$

It is interesting to notice that (3.8) is implied by the following fact, which can be deduced by the log-convexity of the gamma function (cf. [2, Corollary 1.2.6]) and an adaptation of the proof of [1, Corollary 3]:

Suppose that $a,b,c>0$ *and* $a<c$*. Then,*

$$\Gamma(a+b+1)\Gamma(c+1)<\Gamma(a+1)\Gamma(b+c+1). \quad (3.9)$$

In fact, (3.8) is a direct consequence of (3.9) if we suitably choose a,b,c *for different cases:*

Case 1: Suppose $-1<y_1<0$ *and* $y_2>0$*. Set*

$$a=\frac{y_1+1}{2},\quad b=-\frac{y_1}{2},\quad c=\frac{y_1+y_2+1}{2}.$$

Case 2: Suppose $y_1,y_2>0$*. Set*

$$a=\frac{1}{2},\quad b=\frac{y_1}{2},\quad c=\frac{y_2+1}{2}.$$

Case 3: Suppose $-1<y_1,y_2<0$ *with* $y_1+y_2>-1$*. Set*

$$a=\frac{y_1+y_2+1}{2},\quad b=-\frac{y_1}{2},\quad c=\frac{y_1+1}{2}.$$

Remark 3.2.3 *By making a connection between (3.7) and hypergeometric functions, we can get an even stronger statement of Proposition 3.2.1.*

Note that $\Gamma(x) = \frac{\Gamma(x+1)}{x}$ *for any* $x > 0$. *Then, (3.7) equals*

$$\begin{aligned}
&\frac{(y_1+1)(y_2+1)}{y_1+y_2+1}\cdot\frac{\Gamma(\frac{y_1+y_2+3}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{y_1+3}{2})\Gamma(\frac{y_2+3}{2})}\\
=\;&\frac{(y_1+1)(y_1+3)(y_2+1)(y_2+3)}{(y_1+y_2+1)(y_1+y_2+3)\cdot 1\cdot 3}\cdot\frac{\Gamma(\frac{y_1+y_2+5}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{y_1+5}{2})\Gamma(\frac{y_2+5}{2})}\\
=\;&\dots\\
=\;&\frac{(y_1+1)(y_1+3)\cdots(y_1+(2n-1))(y_2+1)(y_2+3)\cdots(y_2+(2n-1))}{(y_1+y_2+1)(y_1+y_2+3)\cdots(y_1+y_2+(2n-1))\cdot 1\cdot 3\cdots(2n-1)}\\
&\cdot\frac{\Gamma(\frac{y_1+y_2+(2n+1)}{2})\Gamma(\frac{(2n+1)}{2})}{\Gamma(\frac{y_1+(2n+1)}{2})\Gamma(\frac{y_2+(2n+1)}{2})}\\
=\;&\frac{\left(\frac{y_1+1}{2}\right)_{(n)}\left(\frac{y_1+1}{2}\right)_{(n)}}{\left(\frac{y_1+y_2+1}{2}\right)_{(n)}\left(\frac{1}{2}\right)_{(n)}}\cdot F\left(-\frac{y_1}{2},-\frac{y_2}{2};n+\frac{1}{2};1\right) \qquad (3.10)
\end{aligned}$$

for any $n \in \mathbb{N}$, *where the last equality follows from [5, 2.8-(46), page 104],*

By a similar argument to (3.4), Remark 3.1.2 and the continuity of $G(z)$, *we also have*

$$\begin{aligned}
&F\left(-\frac{y_1}{2},-\frac{y_2}{2};n+\frac{1}{2};1\right) < 1, \quad \text{if } -1<y_1<0 \text{ and } y_2>0,\\
&F\left(-\frac{y_1}{2},-\frac{y_2}{2};n+\frac{1}{2};1\right) > 1, \quad \text{if } y_1,y_2>0 \text{ or } -1<y_1,y_2<0 \qquad (3.11)
\end{aligned}$$

for any $n \in \mathbb{N}$.

Therefore, by (3.10) and (3.11), we may strengthen Proposition 3.2.1 as follows:

Let X *be a centered Gaussian random variable.*

(i) If $-1 < y_1 < 0$ *and* $y_2 > 0$, *then*

$$E[|X|^{y_1+y_2}] < \frac{\left(\frac{y_1+1}{2}\right)_{(n)}\left(\frac{y_1+1}{2}\right)_{(n)}}{\left(\frac{y_1+y_2+1}{2}\right)_{(n)}\left(\frac{1}{2}\right)_{(n)}}E[|X|^{y_1}]E[|X|^{y_2}].$$

for any $n \in \mathbb{N}$.

(ii) If $y_1, y_2 > 0$ *or* $-1 < y_1, y_2 < 0$ *with* $y_1 + y_2 > -1$, *then*

$$E[|X|^{y_1+y_2}] > \frac{\left(\frac{y_1+1}{2}\right)_{(n)}\left(\frac{y_1+1}{2}\right)_{(n)}}{\left(\frac{y_1+y_2+1}{2}\right)_{(n)}\left(\frac{1}{2}\right)_{(n)}}E[|X|^{y_1}]E[|X|^{y_2}].$$

for any $n \in \mathbb{N}$.

Note that Proposition 3.2.1 is a special case ($n = 1$) *of the above.*

3.3 Higher dimension

As already mentioned, the bivariate GPI had been proved for exponents with the same signs, but we could find no prior study of cases with opposite signs. In this chapter, we used the Gaussian hypergeometric function to prove an opposite GPI, which we believe to be the first of its kind. Furthermore, our method provides an extremely simple alternative proof to Theorem 2.1.1. Therefore, not only do our new results fill in the last remaining piece of the two-dimensional puzzle, but this chapter also serves as a self-contained complete picture.

Now that the two-dimensional case is covered, it is natural to consider (1.1) when all y_j's belong to $(-1, 0) \cup (0, \infty)$. We thank another anonymous referee for suggesting we address the general case. Suppose $n \geq 3$ and the signs of the y_j's are not all the same. Then, neither the GPI nor the opposite GPI holds. To prove this assertion, without loss of generality, we only consider the case that $n = 3$. Suppose $-1 < y_1 < 0$ and $y_2 > 0$. By Theorems 2.1.1 and 3.1.1, we get

Case 1: if X_1 and X_2 are dependent, and X_3 is independent of $\{X_1, X_2\}$, then

$$E[|X_1|^{y_1}|X_2|^{y_2}|X_3|^{y_3}] < E[|X_1|^{y_1}]E[|X_2|^{y_2}]E[|X_3|^{y_3}].$$

Case 2: if $-1 < y_3 < 0$, X_1 and X_3 are dependent, and X_2 is independent of $\{X_1, X_3\}$, then

$$E[|X_1|^{y_1}|X_2|^{y_2}|X_3|^{y_3}] > E[|X_1|^{y_1}]E[|X_2|^{y_2}]E[|X_3|^{y_3}].$$

Case 3: if $y_3 > 0$, X_2 and X_3 are dependent, and X_1 is independent of $\{X_2, X_3\}$, then

$$E[|X_1|^{y_1}|X_2|^{y_2}|X_3|^{y_3}] > E[|X_1|^{y_1}]E[|X_2|^{y_2}]E[|X_3|^{y_3}].$$

Hence, neither the GPI nor the opposite GPI holds. Therefore, when we consider the validity of the GPI in higher dimensions, we need only focus on the case that all y_j's are positive.

Chapter 4

“Some new Gaussian product inequalities”

This chapter is based on our paper “Some new Gaussian product inequalities,” published in Journal of Mathematical Analysis and Applications in 2022 [36].

In [21, Theorem 3.2], Lan et al. used the Gaussian hypergeometric functions to prove the following inequality: for any $m_1, m_2 \in \mathbb{N}$ and any centered Gaussian random vector (X_1, X_2, X_3),

$$E[X_1^{2m_1} X_2^{2m_2} X_3^{2m_2}] \geq E[X_1^{2m_1}]E[X_2^{2m_2}]E[X_3^{2m_2}]. \tag{4.1}$$

Note that the assumption X_2 and X_3 have the same exponent m_2 plays an essential role in the proof of [21, Theorem 3.2]. A natural question is whether the three-dimensional GPI still holds when the exponents of X_1, X_2, X_3 are all different. Surprisingly, this question is very difficult, which motivated us to consider the following special case of the GPI conjecture.

Conjecture 4.0.1 *Let $m_2, m_3 \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2 X_2^{2m_2} X_3^{2m_3}] \geq E[X_1^2]E[X_2^{2m_2}]E[X_3^{2m_3}]. \tag{4.2}$$

The equality holds if and only if X_1, X_2, X_3 are independent.

In Section 4.1, we will show that Conjecture 4.0.1 is implied by a combinatorial inequality conjecture. The combinatorial inequality can be verified directly for small values of m_2 and arbitrary m_3. Hence the corresponding cases of Conjecture 4.0.1 are proved (see Theorem 4.1.6 below). In Section 4.2, we will show that Conjecture 4.0.1 is equivalent to an improved Cauchy-Schwarz inequality. This observation leads us to derive some novel moment inequalities for bivariate Gaussian random variables (see Theorem 4.2.3 below).

4.1 Combinatorial inequality and proof of GPI for small m_2

In this section, we propose a combinatorial inequality conjecture (see Conjecture 4.1.3 below) and show that it implies Conjecture 4.0.1. Although we do not completely resolve this combinatorial inequality conjecture here, it can be verified directly for small values of m_2 and arbitrary m_3. Thereby, the corresponding cases of Conjecture 4.0.1 are proved (see Theorem 4.1.6 below). First, an important preliminary lemma.

Lemma 4.1.1 *Let $m_2, m_3 \in \mathbb{N}$ and U_1, U_2 be independent standard Gaussian random variables. Then, Conjecture 4.0.1 is equivalent to the following claim:*

Claim A: For any $x_2 \in \mathbb{R}$,

$$\begin{aligned} & (E[U_2(x_2U_1+U_2)^{2m_2}U_1^{2m_3+1}])^2 \\ < \; & \{E[(x_2U_1+U_2)^{2m_2}U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2}\} \\ & \cdot\{E[U_2^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2}\}. \end{aligned} \tag{4.3}$$

Proof. By Lemma 2.3.3, Conjecture 4.0.1 is equivalent to the following inequality:

$$\begin{aligned} & E[(x_1U_1+U_2)^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] \\ > \; & E[(x_1U_1+U_2)^2]E[(x_2U_1+U_2)^{2m_2}]E[U_1^{2m_3}], \quad \forall x_1, x_2 \in \mathbb{R}. \end{aligned} \tag{4.4}$$

We have

$$\begin{aligned} & (4.4) \text{ holds} \\ \Leftrightarrow \quad & x_1^2E[(x_2U_1+U_2)^{2m_2}U_1^{2m_3+2}] + E[U_2^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] \\ & +2x_1E[U_2(x_2U_1+U_2)^{2m_2}U_1^{2m_3+1}] \\ & > (2m_2-1)!!\,(2m_3-1)!!\,(1+x_1^2)(1+x_2^2)^{m_2}, \quad \forall x_1, x_2 \in \mathbb{R} \\ \Leftrightarrow \quad & x_1^2\{E[(x_2U_1+U_2)^{2m_2}U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2}\} \\ & +2x_1E[U_2(x_2U_1+U_2)^{2m_2}U_1^{2m_3+1}] \\ & +\{E[U_2^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2}\} \\ & > 0, \quad \forall x_1, x_2 \in \mathbb{R}. \end{aligned}$$

Note that

$$\begin{aligned} & E[(x_2U_1+U_2)^{2m_2}U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\ \geq \; & E[(x_2U_1+U_2)^{2m_2}]E[U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\ = \; & (2m_2-1)!!\,(2m_3+1)!!\,(1+x_2^2)^{m_2} - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\ = \; & 2m_3(2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\ > \; & 0, \end{aligned}$$

and Lemma 2.3.1 implies that

$$\begin{aligned}
& E[U_2^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\
= \;& E[U_2^2(|x_2|U_1+U_2)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\
\geq \;& (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\
= \;& 0.
\end{aligned}$$

Hence, (4.4) holds if and only if (4.3) holds for any $x_2 \in \mathbb{R}$. The proof is complete. □

Let $m_2, m_3 \in \mathbb{N}$. For $0 \leq p \leq 2m_2$, define

$$\begin{aligned}
c_p^{m_2,m_3} \;=\; & \sum_{i=\max(0,p-m_2)}^{\min(m_2,p)} \left[\frac{2^{2i}\left(m_3+i+\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i)!\,(m_2-i)!} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(i!\,)(m_2-i)!}\right] \\
& \cdot \left[\frac{2^{2p-2i-1}(2m_2+2i-2p+1)\left(m_3+p-i-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2p-2i)!\,(m_2+i-p)!} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2[(p-i)!\,](m_2+i-p)!}\right] \\
& - \sum_{i=\max(0,p-m_2)}^{\min(m_2,p)-1} \frac{2^{2p}\left(m_3+i+\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3+p-i-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i+1)!\,(2p-2i-1)!\,(m_2-i-1)!\,(m_2+i-p)!}.
\end{aligned}$$

Lemma 4.1.2 *Let $m_2, m_3 \in \mathbb{N}$. Then, we have*

(i) $c_0^{m_2,m_3}, c_2^{m_2,m_3} > 0$; $c_1^{m_2,m_3} \geq 0$ *if* $m_3 \leq m_2 - 1$.

(ii) $(c_1^{m_2,m_3})^2 < 4c_0^{m_2,m_3}c_2^{m_2,m_3}$.

(iii) $c_{2m_2}^{m_2,m_3} > 0$.

Proof. To simplify notation, we denote $c_p^{m_2,m_3}$ by c_p for $0 \leq p \leq 2m_2$. Define $r := \left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}$.

We have

$$c_0 \;=\; \frac{r^2 m_3}{m_2!\,(m_2-1)!},$$

$$\begin{aligned}
c_1 \;=\; & r^2 \cdot \frac{m_3}{m_2!} \cdot \frac{2m_2m_3+m_2-m_3-1}{(m_2-1)!} + r^2 \cdot \frac{2m_3^2+4m_3+1}{(m_2-1)!} \cdot \frac{m_2}{m_2!} - r^2 \cdot \frac{4\left(m_3+\frac{1}{2}\right)^2 m_2}{(m_2-1)!\,m_2!} \\
=\; & -\frac{r^2 m_3}{m_2!\,(m_2-1)!} \cdot (m_3+1-m_2),
\end{aligned}$$

$$\begin{aligned}
c_2 &= r^2 \cdot \frac{m_3}{m_2!} \cdot \frac{8(2m_2-3)(m_3+\frac{3}{2})(m_3+\frac{1}{2})-6}{4!\,(m_2-1)!} \cdot (m_2-1) \\
&\quad + r^2 \cdot \frac{12(2m_3^2+4m_3+1)}{(m_2-1)!} \cdot \frac{2(2m_2-1)\left(m_3+\frac{1}{2}\right)-1}{4!\,m_2!} \cdot m_2 \\
&\quad + r^2 \cdot \frac{16\left(m_3+\frac{5}{2}\right)\left(m_3+\frac{3}{2}\right)\left(m_3+\frac{1}{2}\right)-6}{4!\,(m_2-1)!} \cdot \frac{m_2(m_2-1)}{m_2!} \\
&\quad - r^2 \cdot \frac{128\left(m_3+\frac{1}{2}\right)^2\left(m_3+\frac{3}{2}\right)}{4!\,m_2!\,(m_2-1)!} \cdot m_2(m_2-1) \\
&= \frac{r^2 m_3}{m_2!\,(m_2-1)!} \cdot (m_2m_3^2+m_3^2+m_2^2m_3+m_2m_3+2m_3+m_2^2+1-m_2) \\
&> 0,
\end{aligned}$$

$$\begin{aligned}
c_{2m_2} &= \left[\frac{2^{2m_2}\left(m_3+m_2+\frac{1}{2}\right)\cdots\frac{1}{2}}{(2m_2)!} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(m_2!\,)}\right] \\
&\quad \cdot \left[\frac{2^{2m_2-1}\left(m_3+m_2-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2m_2)!} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(m_2!\,)}\right] \\
&= \left[\frac{\left(m_3+m_2+\frac{1}{2}\right)\cdots\left(m_2+\frac{1}{2}\right)}{m_2!} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(m_2!\,)}\right] \\
&\quad \cdot \left[\frac{\left(m_3+m_2-\frac{1}{2}\right)\cdots\left(m_2+\frac{1}{2}\right)}{2(m_2!\,)} - \frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(m_2!\,)}\right] \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
c_1^2 < 4c_0c_2 &\Leftrightarrow (m_3+1-m_2)^2 < 4(m_2m_3^2+m_3^2+m_2^2m_3+m_2m_3+2m_3+m_2^2+1-m_2) \\
&\Leftrightarrow 4m_2m_3^2+3m_3^2+4m_2^2m_3+6m_2m_3+6m_3+3m_2^2+3-2m_2>0,
\end{aligned}$$

which is clearly true. □

Now we state the combinatorial inequality conjecture.

Conjecture 4.1.3 *Let $m_2 \geq 2$ and $m_3 \in \mathbb{N}$. Then, we have*

$$c_p^{m_2,m_3} > 0 \text{ for } 3 \leq p \leq 2m_2-1. \tag{4.5}$$

Proposition 4.1.4 *If Conjecture 4.1.3 is true, then Conjecture 4.0.1 is also true.*

Proof. Assume that Conjecture 4.1.3 is true. We we will apply Lemma 4.1.1 to show that Conjecture 4.0.1 is also true.

Let U_1, U_2 be independent standard Gaussian random variables. For $x_2 \in \mathbb{R}$, we have

$$\begin{aligned}
& (E[U_2(x_2U_1+U_2)^{2m_2}U_1^{2m_3+1}])^2 \\
= & \left(\sum_{k \text{ odd}} \binom{2m_2}{k} x_2^k (2m_3+k)!!\,(2m_2-k)!!\right)^2 \\
= & \sum_{k \text{ odd}} \sum_{l \text{ odd}} x_2^{k+l} \binom{2m_2}{k}\binom{2m_2}{l} (2m_3+k)!!\,(2m_2-k)!!\,(2m_3+l)!!\,(2m_2-l)!!,
\end{aligned}$$

$$\begin{aligned}
& E[(x_2U_1+U_2)^{2m_2}U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\
= & \sum_{k \text{ even}} x_2^k \left[\binom{2m_2}{k}(2m_3+k+1)!!\,(2m_2-k-1)!! - \binom{m_2}{\frac{k}{2}}(2m_2-1)!!\,(2m_3-1)!!\right],
\end{aligned}$$

and

$$\begin{aligned}
& E[U_2^2(x_2U_1+U_2)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,(1+x_2^2)^{m_2} \\
= & \sum_{l \text{ even}} x_2^l \left[\binom{2m_2}{l}(2m_3+l-1)!!\,(2m_2-l+1)!! - \binom{m_2}{\frac{l}{2}}(2m_2-1)!!\,(2m_3-1)!!\right].
\end{aligned}$$

Hence, inequality (4.3) becomes

$$\begin{aligned}
& \sum_{k \text{ odd}} \sum_{l \text{ odd}} x_2^{k+l} \binom{2m_2}{k}\binom{2m_2}{l} (2m_3+k)!!\,(2m_2-k)!!\,(2m_3+l)!!\,(2m_2-l)!! \\
< & \sum_{k \text{ even}} \sum_{l \text{ even}} x_2^{k+l} \left[\binom{2m_2}{k}(2m_3+k+1)!!\,(2m_2-k-1)!!\right. \\
& \left. - \binom{m_2}{\frac{k}{2}}(2m_2-1)!!\,(2m_3-1)!!\right] \\
& \cdot \left[\binom{2m_2}{l}(2m_3+l-1)!!\,(2m_2-l+1)!! - \binom{m_2}{\frac{l}{2}}(2m_2-1)!!\,(2m_3-1)!!\right]. \quad (4.6)
\end{aligned}$$

We have

$$\begin{aligned}
& \binom{2m_2}{k}\binom{2m_2}{l}(2m_3+k)!!\,(2m_2-k)!!\,(2m_3+l)!!\,(2m_2-l)!! \\
= & \frac{(2m_2)!}{k!\,(2m_2-k)!} \cdot \frac{(2m_2)!}{l!\,(2m_2-l)!} \cdot (2m_2-k)!!\,(2m_2-l)!!\,(2m_3+k)!!\,(2m_3+l)!! \\
= & \frac{(2m_2)!}{k!\,2^{m_2-\frac{k+1}{2}}(m_2-\frac{k+1}{2})!} \cdot \frac{(2m_2)!}{l!\,2^{m_2-\frac{l+1}{2}}(m_2-\frac{l+1}{2})!} \cdot (2m_3+k)!!\,(2m_3+l)!! \\
= & \frac{(2m_2)!\,(2m_2)!}{2^{2m_2-\frac{k+l}{2}-1}k!\,l!\,(m_2-\frac{k+1}{2})!\,(m_2-\frac{l+1}{2})!} \\
& \cdot 2^{m_3+\frac{k+1}{2}} \cdot \left(m_3+\frac{k}{2}\right)\cdots\frac{1}{2} \cdot 2^{m_3+\frac{l+1}{2}} \cdot \left(m_3+\frac{l}{2}\right)\cdots\frac{1}{2} \\
= & \frac{2^{2m_3-2m_2+k+l+2}(2m_2)!\,(2m_2)!\,\left(m_3+\frac{k}{2}\right)\cdots\frac{1}{2}\,\left(m_3+\frac{l}{2}\right)\cdots\frac{1}{2}}{k!\,l!\,(m_2-\frac{k+1}{2})!\,(m_2-\frac{l+1}{2})!},
\end{aligned}$$

$$
\begin{aligned}
& \binom{2m_2}{k}(2m_3+k+1)!!\,(2m_2-k-1)!!-\binom{m_2}{\frac{k}{2}}(2m_2-1)!!\,(2m_3-1)!! \\
= \; & \frac{(2m_2)!}{k!\,(2m_2-k)!}\cdot(2m_2-k-1)!!\cdot 2^{m_3+\frac{k+2}{2}}\cdot\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2} \\
& -\frac{m_2!}{(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\cdot 2^{m_2}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\cdot 2^{m_3}\cdot\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2} \\
= \; & \frac{(2m_2)!}{k!\,(m_2-\frac{k}{2})!}\cdot 2^{m_3-m_2+k+1}\cdot\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2} \\
& -\frac{m_2!}{(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
& \binom{2m_2}{l}(2m_3+l-1)!!\,(2m_2-l+1)!!-\binom{m_2}{\frac{l}{2}}(2m_2-1)!!\,(2m_3-1)!! \\
= \; & \frac{(2m_2)!}{l!\,(2m_2-l)!}\cdot(2m_2-l+1)!!\cdot 2^{m_3+\frac{l}{2}}\cdot\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2} \\
& -\frac{m_2!}{(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\cdot 2^{m_2}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\cdot 2^{m_3}\cdot\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2} \\
= \; & \frac{(2m_2)!\,(2m_2-l+1)}{l!\,(m_2-\frac{l}{2})!}\cdot 2^{m_3-m_2+l}\cdot\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2} \\
& -\frac{m_2!}{(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& \left[\binom{2m_2}{k}(2m_3+k+1)!!\,(2m_2-k-1)!!-\binom{m_2}{\frac{k}{2}}(2m_2-1)!!\,(2m_3-1)!!\right] \\
& \cdot\left[\binom{2m_2}{l}(2m_3+l-1)!!\,(2m_2-l+1)!!-\binom{m_2}{\frac{l}{2}}(2m_2-1)!!\,(2m_3-1)!!\right] \\
= \; & \left[\frac{(2m_2)!}{k!\,(m_2-\frac{k}{2})!}\cdot 2^{m_3-m_2+k+1}\cdot\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2}\right. \\
& \left.-\frac{m_2!}{(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}\right] \\
& \cdot\left[\frac{(2m_2)!\,(2m_2-l+1)}{l!\,(m_2-\frac{l}{2})!}\cdot 2^{m_3-m_2+l}\cdot\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2}\right. \\
& \left.-\frac{m_2!}{(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}\right].
\end{aligned}
$$

Hence, (4.6) becomes

$$\begin{aligned}
&\sum_{k\text{ odd}}\sum_{l\text{ odd}} x_2^{k+l}\cdot\frac{2^{2m_3-2m_2+k+l+2}(2m_2)!\,(2m_2)!\,\left(m_3+\frac{k}{2}\right)\cdots\frac{1}{2}\left(m_3+\frac{l}{2}\right)\cdots\frac{1}{2}}{k!\,l!\,(m_2-\frac{k+1}{2})!\,(m_2-\frac{l+1}{2})!}\\
<\ &\sum_{k\text{ even}}\sum_{l\text{ even}} x_2^{k+l}\left[\frac{(2m_2)!}{k!\,(m_2-\frac{k}{2})!}\cdot 2^{m_3-m_2+k+1}\cdot\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2}\right.\\
&\left.-\frac{m_2!}{(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}\right]\\
&\cdot\left[\frac{(2m_2)!\,(2m_2-l+1)}{l!\,(m_2-\frac{l}{2})!}\cdot 2^{m_3-m_2+l}\cdot\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2}\right.\\
&\left.-\frac{m_2!}{(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\cdot 2^{m_2+m_3}\cdot\left(m_2-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}\right].
\end{aligned}\tag{4.7}$$

Diving both sides of inequality (4.7) by $2^{m_3-m_2+1}(2m_2)!\,(2m_2)!$, we get

$$\begin{aligned}
&\sum_{k\text{ odd}}\sum_{l\text{ odd}} x_2^{k+l}\cdot\frac{2^{k+l}\left(m_3+\frac{k}{2}\right)\cdots\frac{1}{2}\left(m_3+\frac{l}{2}\right)\cdots\frac{1}{2}}{k!\,l!\,(m_2-\frac{k+1}{2})!\,(m_2-\frac{l+1}{2})!}\\
<\ &\sum_{k\text{ even}}\sum_{l\text{ even}} x_2^{k+l}\left[\frac{2^k\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2}}{k!\,(m_2-\frac{k}{2})!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\right]\\
&\cdot\left[\frac{2^{l-1}(2m_2-l+1)\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2}}{l!\,(m_2-\frac{l}{2})!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\right].
\end{aligned}\tag{4.8}$$

Define

$$x=x_2^2.$$

We have

$$\begin{aligned}
&\sum_{k\text{ odd}}\sum_{l\text{ odd}} x_2^{k+l}\cdot\frac{2^{k+l}\left(m_3+\frac{k}{2}\right)\cdots\frac{1}{2}\left(m_3+\frac{l}{2}\right)\cdots\frac{1}{2}}{k!\,l!\,(m_2-\frac{k+1}{2})!\,(m_2-\frac{l+1}{2})!}\\
=\ &\sum_{i=1}^{m_2}\sum_{j=1}^{m_2} x_2^{2(i+j-1)}\cdot\frac{2^{2(i+j-1)}\left(m_3+i-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3+j-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i-1)!\,(2j-1)!\,(m_2-i)!\,(m_2-j)!}\\
=\ &\sum_{p=1}^{2m_2-1} x^p\sum_{i=\max(1,p+1-m_2)}^{\min(m_2,p)}\frac{2^{2p}\left(m_3+i-\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3+p-i+\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i-1)!\,(2p-2i+1)!\,(m_2-i)!\,(m_2+i-p-1)!}\\
=\ &\sum_{p=1}^{2m_2-1} x^p\sum_{i=\max(0,p-m_2)}^{\min(m_2,p)-1}\frac{2^{2p}\left(m_3+i+\frac{1}{2}\right)\cdots\frac{1}{2}\left(m_3+p-i-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i+1)!\,(2p-2i-1)!\,(m_2-i-1)!\,(m_2+i-p)!},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k\text{ even}}\sum_{l\text{ even}} x_2^{k+l}\left[\frac{2^k\left(m_3+\frac{k+1}{2}\right)\cdots\frac{1}{2}}{k!\,(m_2-\frac{k}{2})!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(\frac{k}{2})!\,(m_2-\frac{k}{2})!}\right]\\
&\quad\cdot\left[\frac{2^{l-1}(2m_2-l+1)\left(m_3+\frac{l-1}{2}\right)\cdots\frac{1}{2}}{l!\,(m_2-\frac{l}{2})!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(\frac{l}{2})!\,(m_2-\frac{l}{2})!}\right]\\
=&\sum_{i=0}^{m_2}\sum_{j=0}^{m_2} x_2^{2(i+j)}\left[\frac{2^{2i}\left(m_3+i+\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i)!\,(m_2-i)!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(i!\,)(m_2-i)!}\right]\\
&\quad\cdot\left[\frac{2^{2j-1}(2m_2-2j+1)\left(m_3+j-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2j)!\,(m_2-j)!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(j!\,)(m_2-j)!}\right]\\
=&\sum_{p=0}^{2m_2} x^p\sum_{i=\max(0,p-m_2)}^{\min(m_2,p)}\left[\frac{2^{2i}\left(m_3+i+\frac{1}{2}\right)\cdots\frac{1}{2}}{(2i)!\,(m_2-i)!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2(i!\,)(m_2-i)!}\right]\\
&\quad\cdot\left[\frac{2^{2p-2i-1}(2m_2+2i-2p+1)\left(m_3+p-i-\frac{1}{2}\right)\cdots\frac{1}{2}}{(2p-2i)!\,(m_2+i-p)!}-\frac{\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}}{2[(p-i)!\,](m_2+i-p)!}\right].
\end{aligned}$$

Define $F(x)$ to be the polynomial given by (RHS - LHS) of inequality (4.8). Then, we have that

$$F(x)=\sum_{p=0}^{2m_2} c_p^{m_2,m_3}x^p.$$

By (4.5) and Lemma 4.1.2, we get

$$F(x)\geq c_0^{m_2,m_3}+c_1^{m_2,m_3}x+c_2^{m_2,m_3}x^2>0,\qquad \forall x\geq 0.$$

Then, inequality (4.8) and hence inequality (4.3) hold. Therefore, the proof is complete by Lemma 4.1.1. □

Lemma 4.1.5 *Let $m_2\in\{2,3\}$ and $m_3\in\mathbb{N}$. Then, we have*

$$c_p^{m_2,m_3}>0 \text{ for } 3\leq p\leq 2m_2-1.$$

Proof. To simplify notation, we denote $c_p^{m_2,m_3}$ by c_p for $0\leq p\leq 2m_2$. Define $r:=\left(m_3-\frac{1}{2}\right)\cdots\frac{1}{2}$.

First, we consider the case $m_2=2$. We only need to show that $c_3>0$. We have

$$\begin{aligned}
c_3 &= r^2\cdot\frac{1}{3}m_3\,(m_3+2)\left(2m_3^2+4m_3+1\right)\\
&\quad +r^2\cdot\frac{1}{6}\,(3m_3+1)\left(4m_3^3+18m_3^2+23m_3+6\right)\\
&\quad -r^2\cdot\frac{16}{9}\left(m_3+\frac{3}{2}\right)^2\left(m_3+\frac{1}{2}\right)^2\\
&= \frac{r^2m_3}{18}\cdot\left(16m_3^3+94m_3^2+139m_3+39\right)>0,
\end{aligned}$$

which completes the proof for this case.

Next, consider the case $m_2 = 3$. We need to show that

$$c_3 > 0,\ c_4 > 0 \ \text{ and } \ c_5 > 0.$$

We have

$$\begin{aligned}
c_3 &= r^2 \cdot \frac{1}{540} m_3^2 \left(4m_3^2 + 18m_3 + 23\right) \\
&\quad + r^2 \cdot \left(m_3^2 + 2m_3 + \frac{1}{2}\right)^2 \\
&\quad + r^2 \cdot \frac{1}{12} \left(4m_3^3 + 18m_3^2 + 23m_3 + 6\right)(5m_3 + 2) \\
&\quad + r^2 \cdot \frac{1}{180} \left(8m_3^4 + 64m_3^3 + 172m_3^2 + 176m_3 + 45\right) \\
&\quad - r^2 \cdot \frac{8}{15} \left(m_3 + \frac{5}{2}\right)\left(m_3 + \frac{3}{2}\right)\left(m_3 + \frac{1}{2}\right)^2 \\
&\quad - r^2 \cdot \frac{16}{9} \left(m_3 + \frac{3}{2}\right)^2 \left(m_3 + \frac{1}{2}\right)^2 \\
&= \frac{r^2 m_3}{54} \cdot \left(22m_3^3 + 150m_3^2 + 245m_3 + 78\right) > 0,
\end{aligned}$$

$$\begin{aligned}
c_4 &= r^2 \cdot \frac{1}{90} m_3 \left(m_3^2 + 2m_3 + \frac{1}{2}\right)\left(4m_3^2 + 18m_3 + 23\right) \\
&\quad + r^2 \cdot \frac{1}{6} \left(4m_3^3 + 18m_3^2 + 23m_3 + 6\right)\left(m_3^2 + 2m_3 + \frac{1}{2}\right) \\
&\quad + r^2 \cdot \frac{1}{180} \left(8m_3^4 + 64m_3^3 + 172m_3^2 + 176m_3 + 45\right)(5m_3 + 2) \\
&\quad - r^2 \cdot \frac{32}{45} \left(m_3 + \frac{5}{2}\right)\left(m_3 + \frac{3}{2}\right)^2 \left(m_3 + \frac{1}{2}\right)^2 \\
&= \frac{r^2 m_3}{180} \cdot \left(40m_3^4 + 336m_3^3 + 956m_3^2 + 1020m_3 + 273\right) > 0,
\end{aligned}$$

and

$$\begin{aligned}
c_5 &= r^2 \cdot \frac{1}{540} m_3 \left(4m_3^3 + 18m_3^2 + 23m_3 + 6\right)\left(4m_3^2 + 18m_3 + 23\right) \\
&\quad + r^2 \cdot \frac{1}{90} \left(8m_3^4 + 64m_3^3 + 172m_3^2 + 176m_3 + 45\right)\left(m_3^2 + 2m_3 + \frac{1}{2}\right) \\
&\quad - r^2 \cdot \frac{16}{225} \left(m_3 + \frac{5}{2}\right)^2 \left(m_3 + \frac{3}{2}\right)^2 \left(m_3 + \frac{1}{2}\right)^2 \\
&= \frac{r^2 m_3}{2700} \cdot \left(128m_3^5 + 1392m_3^4 + 5564m_3^3 + 10164m_3^2 + 8087m_3 + 1890\right) > 0,
\end{aligned}$$

which completes the proof for this case. □

Theorem 4.1.6 *Let $m_3 \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2 X_2^4 X_3^{2m_3}] \geq E[X_1^2]E[X_2^4]E[X_3^{2m_3}],$$

and

$$E[X_1^2 X_2^6 X_3^{2m_3}] \geq E[X_1^2]E[X_2^6]E[X_3^{2m_3}].$$

The equalities hold if and only if X_1, X_2, X_3 are independent.

Proof. This is a direct consequence of Proposition 4.1.4 and Lemma 4.1.5. □

4.2 Improved Cauchy-Schwarz inequality for bivariate Gaussian random variables

In this section, we continue studying Conjecture 4.0.1. First, we show that Conjecture 4.0.1 is equivalent to an improved Cauchy-Schwarz inequality.

Lemma 4.2.1 *Let $m_2, m_3 \in \mathbb{N}$. Then, Conjecture 4.0.1 is equivalent to the following claim:*

Claim B: Let (U, V) be bivariate Gaussian random variables with $U \sim N(0,1)$ and $V \sim N(0,1)$. Then,

$$\begin{aligned}
&(E[U^{2m_2+1}V^{2m_3+1}])^2 + (2m_2-1)!!\,(2m_3-1)!!\,\{E[U^{2m_2}V^{2m_3+2}] + E[V^{2m_2+2}U^{2m_3}]\} \\
\leq\ & E[U^{2m_2}V^{2m_3+2}]E[V^{2m_2+2}U^{2m_3}] + (2m_2-1)!!\,(2m_3-1)!! \\
&\cdot\{2E[U^{2m_2+1}V^{2m_3+1}]E[UV] + (2m_2-1)!!\,(2m_3-1)!!\,(1-(E[UV])^2)\}. \qquad (4.9)
\end{aligned}$$

The equality holds if and only if $|E[UV]|= 1$.

Proof. By Lemma 4.1.1, we only need to show that inequality (4.3) is equivalent to inequality (4.9).

Obviously, (4.3) holds if $x = 0$. To prove (4.3), we assume without loss of generality that $x_2 > 0$. Define

$$\theta = \arccos\left(\frac{x_2}{\sqrt{1+x_2^2}}\right).$$

Then, $\theta \in (0, \pi/2)$ and (4.3) is equivalent to

$$\begin{aligned}
&(E[U_2(U_1\cos\theta + U_2\sin\theta)^{2m_2}U_1^{2m_3+1}])^2 \\
<\ & \{E[(U_1\cos\theta + U_2\sin\theta)^{2m_2}U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\} \\
&\cdot\{E[U_2^2(U_1\cos\theta + U_2\sin\theta)^{2m_2}U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\}, \quad \forall\theta \in (0, \pi/2). \quad (4.10)
\end{aligned}$$

Define

$$V = U_1\cos\theta + U_2\sin\theta.$$

Then, we have that

$$
\begin{aligned}
& \quad (4.10) \text{ holds} \\
\Leftrightarrow & \quad (E[U_2 V^{2m_2} U_1^{2m_3+1}])^2 \\
& < \{E[V^{2m_2} U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,\} \\
& \quad \cdot\{E[U_2^2 V^{2m_2} U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\,\} \\
\Leftrightarrow & \quad (E[(U_2 \sin\theta) V^{2m_2} U_1^{2m_3+1}])^2 \\
& < \{E[V^{2m_2} U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,\} \\
& \quad \cdot\{E[(U_2 \sin\theta)^2 V^{2m_2} U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\sin^2\theta\} \\
\Leftrightarrow & \quad (E[(V - U_1\cos\theta) V^{2m_2} U_1^{2m_3+1}])^2 \\
& < \{E[V^{2m_2} U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,\} \\
& \quad \cdot\{E[(V - U_1\cos\theta)^2 V^{2m_2} U_1^{2m_3}] - (2m_2-1)!!\,(2m_3-1)!!\sin^2\theta\} \\
\Leftrightarrow & \quad (E[V^{2m_2+1} U_1^{2m_3+1}] - E[V^{2m_2} U_1^{2m_3+2}]E[VU_1])^2 \\
& < \{E[V^{2m_2} U_1^{2m_3+2}] - (2m_2-1)!!\,(2m_3-1)!!\,\} \\
& \quad \cdot\{E[V^{2m_2+2} U_1^{2m_3}] + E[V^{2m_2} U_1^{2m_3+2}](E[VU_1])^2 - 2E[V^{2m_2+1} U_1^{2m_3+1}]E[VU_1] \\
& \qquad -(2m_2-1)!!\,(2m_3-1)!!\,(1-(E[VU_1])^2)\} \\
\Leftrightarrow & \quad (E[V^{2m_2+1} U_1^{2m_3+1}])^2 \\
& < E[V^{2m_2} U_1^{2m_3+2}]E[V^{2m_2+2} U_1^{2m_3}] \\
& \quad -(2m_2-1)!!\,(2m_3-1)!!\,\{E[V^{2m_2+2} U_1^{2m_3}] + E[V^{2m_2} U_1^{2m_3+2}] \\
& \qquad -2E[V^{2m_2+1} U_1^{2m_3+1}]E[VU_1] - (2m_2-1)!!\,(2m_3-1)!!\,(1-(E[VU_1])^2)\} \\
\Leftrightarrow & \quad (E[V^{2m_2+1} U_1^{2m_3+1}])^2 + (2m_2-1)!!\,(2m_3-1)!!\,\{E[V^{2m_2} U_1^{2m_3+2}] + E[V^{2m_2+2} U_1^{2m_3}]\} \\
& < E[V^{2m_2} U_1^{2m_3+2}]E[V^{2m_2+2} U_1^{2m_3}] + (2m_2-1)!!\,(2m_3-1)!! \\
& \cdot\{2E[V^{2m_2+1} U_1^{2m_3+1}]E[VU_1] + (2m_2-1)!!\,(2m_3-1)!!\,(1-(E[VU_1])^2)\}.
\end{aligned}
$$

The proof is complete. □

The GPI (4.1) was established in [21] based on the intrinsic connection between moments of Gaussian distributions and the Gaussian hypergeometric functions. Following along these lines, by virtue of Lemma 4.2.1, we can show that the GPI (4.2) is equivalent to an inequality in terms of the Gaussian hypergeometric functions (see Lemma 4.2.2 below).

Recall that we denote by $F(a, b; c; z)$ the hypergeometric function (cf. [33]):

$$
F(a, b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_{(j)}(b)_{(j)}}{(c)_{(j)}} \cdot \frac{z^j}{j!}, \quad |z| < 1,
$$

where, for $\alpha \neq 0$, the factorial function is defined by

$$
(\alpha)_{(j)} = \begin{cases} 1, & j = 0, \\ \alpha(\alpha+1)\cdots(\alpha+j-1), & j \geq 1. \end{cases}
$$

Lemma 4.2.2 *Let $m_2, m_3 \in \mathbb{N}$. Then, Conjecture 4.0.1 is equivalent to the following claim:*

Claim C: For $0 < |x| < 1$,

$$\begin{aligned}
&(2m_2+1)^2\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]^2 \\
< \;& 2(m_3-m_2)x^2\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right]F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\
&+x^2\left[(2m_2+1)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right]^2. \qquad (4.11)
\end{aligned}$$

Proof. By Lemma 4.2.1, we only need to show that Claim B is equivalent to Claim C. Let (U,V) be bivariate Gaussian random variables with $U \sim N(0,1)$ and $V \sim N(0,1)$. Define

$$x = E[UV].$$

By the moment formula (cf. [29] and [42]), we get

$$\begin{aligned}
&E[U^{2m_2+1}V^{2m_3+1}] \\
=\;& \frac{(2m_2+1)!\,(2m_3+1)!\,x}{2^{m_2+m_3}}\sum_{j=0}^{\min(m_2,m_3)}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-j)!\,(2j+1)!} \\
=\;& \frac{(2m_2+1)!\,(2m_3+1)!\,x}{m_2!\,m_3!\,2^{m_2+m_3}}\sum_{j=0}^{\infty}\frac{(-m_2)_{(j)}(-m_3)_{(j)}}{(\frac{3}{2})_{(j)}}\cdot\frac{(x^2)^j}{j!} \\
=\;& \frac{(2m_2+1)!\,(2m_3+1)!\,x}{m_2!\,m_3!\,2^{m_2+m_3}}F\left(-m_2,-m_3;\frac{3}{2};x^2\right) \\
=\;& (2m_2+1)!!\,(2m_3+1)!!\,xF\left(-m_2,-m_3;\frac{3}{2};x^2\right), \qquad (4.12)
\end{aligned}$$

$$\begin{aligned}
&E[U^{2m_2}V^{2m_3+2}] \\
=\;& \frac{(2m_2)!\,(2m_3+2)!}{2^{m_2+m_3+1}}\sum_{j=0}^{\min(m_2,m_3+1)}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3+1-j)!\,(2j)!} \\
=\;& \frac{(2m_2)!\,(2m_3+2)!}{m_2!\,(m_3+1)!\,2^{m_2+m_3+1}}\sum_{j=0}^{\infty}\frac{(-m_2)_{(j)}(-m_3-1)_{(j)}}{(\frac{1}{2})_{(j)}}\cdot\frac{(x^2)^j}{j!} \\
=\;& \frac{(2m_2)!\,(2m_3+2)!}{m_2!\,(m_3+1)!\,2^{m_2+m_3+1}}F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right) \\
=\;& (2m_2-1)!!\,(2m_3+1)!!\,F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right),
\end{aligned}$$

and

$$
\begin{aligned}
& E[U^{2m_2+2}V^{2m_3}] \\
= \;& \frac{(2m_2+2)!\,(2m_3)!}{2^{m_2+m_3+1}} \sum_{j=0}^{\min(m_2+1,m_3)} \frac{(2x)^{2j}}{(m_2+1-j)!\,(m_3-j)!\,(2j)!} \\
= \;& \frac{(2m_2+2)!\,(2m_3)!}{(m_2+1)!\,m_3!\,2^{m_2+m_3+1}} \sum_{j=0}^{\infty} \frac{(-m_2-1)_{(j)}(-m_3)_{(j)}}{(\frac{1}{2})_{(j)}} \cdot \frac{(x^2)^j}{j!} \\
= \;& \frac{(2m_2+2)!\,(2m_3)!}{(m_2+1)!\,m_3!\,2^{m_2+m_3+1}} F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
= \;& (2m_2+1)!!\,(2m_3-1)!!\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right). \qquad (4.13)
\end{aligned}
$$

Then, (4.9) is equivalent to

$$
\begin{aligned}
& \left[(2m_2+1)(2m_3+1)xF\left(-m_2,-m_3;\frac{3}{2};x^2\right)\right]^2 \\
& +(2m_3+1)F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right)+(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
< \;& (2m_2+1)(2m_3+1)F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
& +2(2m_2+1)(2m_3+1)x^2F\left(-m_2,-m_3;\frac{3}{2};x^2\right)+(1-x^2), \qquad 0<|x|<1. \qquad (4.14)
\end{aligned}
$$

Consider the following functions contiguous to $F(a,b;c;z)$:

$$
F(a\pm 1,b;c;z), \quad F(a,b\pm 1;c;z), \quad F(a,b;c\pm 1;z). \qquad (4.15)
$$

To simplify notation, we denote $F(a,b;c;z)$ and the six functions in (4.15) respectively by

$$
F, \quad F(a\pm 1), \quad F(b\pm 1), \quad F(c\pm 1).
$$

By the following relations of Gauss between contiguous functions (cf. [5, 2.8-(37), (38), page 103]):

$$
\begin{aligned}
& c(1-z)F-cF(a-1)+(c-b)zF(c+1)=0, \\
& (b-a)(1-z)F-(c-a)F(a-1)+(c-b)F(b-1)=0,
\end{aligned}
$$

we get

$$
\begin{aligned}
& x^2F\left(-m_2,-m_3;\frac{3}{2};x^2\right) \\
= \;& \frac{1}{2m_3+1}\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right) \\
= \;& \frac{1}{2m_3+1}\Bigg[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
& +2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\Bigg].
\end{aligned}
$$

Hence, we have that

$$
\begin{aligned}
& (4.14) \text{ holds} \\
\Leftrightarrow \quad & \frac{(2m_2+1)^2}{x^2}\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]^2 \\
& +2\Bigg[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
& +(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\Bigg] \\
& < (2m_2+1)\Bigg[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
& +2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\Bigg]F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) \\
& +2(2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right] \\
& +(1-x^2) \\
\Leftrightarrow \quad & (2m_2+1)^2\left\{\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)\right]^2+(1-x^2)\left[F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]^2\right\} \\
& +2(m_2+m_3+1)x^2F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\
& < 2(2m_2+1)[2m_2+1+(m_3-m_2)x^2] \\
& \cdot F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)+x^2 \\
\Leftrightarrow \quad & (2m_2+1)^2\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]^2 \\
& < 2(m_3-m_2)x^2\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right]F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\
& +x^2\left[(2m_2+1)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right]^2, \qquad 0<|x|<1.
\end{aligned}
$$

Therefore, the proof is complete. □

By Lemmas 4.2.1 and 4.2.2, we find that (4.2) ⇔ (4.9) ⇔ (4.11). Although we do not prove inequality (4.11) in this chapter, we prove a slightly weaker inequality (see (4.23) below). This weaker inequality leads us to derive the following novel moment inequalities for bivariate Gaussian random variables.

Theorem 4.2.3 *Let $m_2, m_3 \in \mathbb{N}$ and (U, V) be bivariate Gaussian random variables with $U \sim N(0,1)$ and $V \sim N(0,1)$.*

(i) We have

$$E[U^{2m_2}V^{2m_3}] \leq \frac{E[U^{2m_2+2}V^{2m_3}]}{2m_2+1}. \tag{4.16}$$

The equality holds if and only if U and V are independent.

(ii) Suppose that $m_3 \leq m_2$. Then, we have

$$\left|E[U^{2m_2+1}V^{2m_3-1}]\right| < \frac{2m_2+1}{2m_3}E[U^{2m_2}V^{2m_3}]. \tag{4.17}$$

Proof. First, we prove assertion (i). By [5, 2.8-(20), (21), page 102], we get

$$F(a+1) = F + \frac{z}{a}F' = F + \frac{bz}{c}F(a+1, b+1; c+1; z). \tag{4.18}$$

Hence,

$$F\left(-m_2-1, -m_3; \frac{1}{2}; z\right) = F\left(-m_2, -m_3; \frac{1}{2}; z\right) + 2m_3 z F\left(-m_2, -m_3+1; \frac{3}{2}; z\right). \tag{4.19}$$

By (4.13) and (4.19), we obtain that

$$\begin{aligned}
& E[U^{2m_2+2}V^{2m_3}] - (2m_2+1)E[U^{2m_2}V^{2m_3}] \\
= & \frac{(2m_2+1)!\,(2m_3)!}{m_2!\,m_3!\,2^{m_2+m_3}}F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right) \\
& - \frac{(2m_2+1)!\,(2m_3)!}{m_2!\,m_3!\,2^{m_2+m_3}}F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\
= & \frac{(2m_2+1)!\,(2m_3)!}{m_2!\,(m_3-1)!\,2^{m_2+m_3-1}}x^2 F\left(-m_2, -m_3+1; \frac{3}{2}; x^2\right).
\end{aligned}$$

Then, (4.16) holds.

In the following, we prove assertion (ii). Suppose that

$$m_3 \leq m_2. \tag{4.20}$$

Step 1. For $0 \leq x \leq 1$, define

$$\begin{aligned}
G(x) \; := \; & (2m_2+1)(1+x)F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\
& - (2m_2+1)F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right) - x.
\end{aligned} \tag{4.21}$$

We have

$$G(0) = 0,$$

and

$$\begin{aligned}
& G(1) \\
= \; & 2(2m_2+1)\frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{1}{2})}{\Gamma(m_2+\frac{1}{2})\Gamma(m_3+\frac{1}{2})} - (2m_2+1)\frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{3}{2})}{\Gamma(m_2+\frac{3}{2})\Gamma(m_3+\frac{1}{2})} - 1 \\
= \; & (2m_2-2m_3+1)\frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{1}{2})}{\Gamma(m_2+\frac{1}{2})\Gamma(m_3+\frac{1}{2})} - 1 \\
\geq \; & \frac{(m_2+m_3-\frac{1}{2})\cdots(m_3+\frac{1}{2})}{(m_2-\frac{1}{2})\cdots\frac{1}{2}} - 1 \\
> \; & 0.
\end{aligned}$$

We will show that

$$G(x) > 0, \quad 0 < x < 1. \tag{4.22}$$

Once (4.22) is proved, by (4.21), we get

$$\begin{aligned}
& (2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) - F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right] \\
< \; & |x|\left[(2m_2+1)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right], \quad 0<|x|<1.
\end{aligned} \tag{4.23}$$

Step 2. We now prove (4.22). By (4.18), we get

$$\begin{aligned}
& G'(x) \\
= \; & (2m_2+1)F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\
& + \frac{2m_2(2m_2+1)(1+x)}{x}\left[F\left(-m_2,-m_3;\frac{1}{2};x^2\right) - F\left(-m_2+1,-m_3;\frac{1}{2};x^2\right)\right] \\
& - \frac{2(m_2+1)(2m_2+1)}{x}\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) - F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right] - 1 \\
= \; & \frac{(2m_2+1)^2(2+x)}{x}F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\
& - \frac{2m_2(2m_2+1)(1+x)}{x}F\left(-m_2+1,-m_3;\frac{1}{2};x^2\right) \\
& - \frac{2(m_2+1)(2m_2+1)}{x}F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right) - 1.
\end{aligned}$$

Hence

$$\begin{aligned} & G'(x) = 0 \\ \Leftrightarrow \quad & (2m_2+1)^2(2+x)F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\ & = 2m_2(2m_2+1)(1+x)F\left(-m_2+1, -m_3; \frac{1}{2}; x^2\right) \\ & \quad +2(m_2+1)(2m_2+1)F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right) + x. \end{aligned} \tag{4.24}$$

By the following relation of Gauss between contiguous functions (cf. [5, 2.8-(31), page 103]):

$$[2a - c + (b-a)z]F = a(1-z)F(a+1) - (c-a)F(a-1),$$

we get

$$\begin{aligned} & \left[-2m_2 - \frac{1}{2} + (m_2 - m_3)x^2\right] F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\ = \quad & -m_2(1-x^2)F\left(-m_2+1, -m_3; \frac{1}{2}; x^2\right) \\ & -\left(m_2 + \frac{1}{2}\right) F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right), \end{aligned}$$

which implies that

$$\begin{aligned} & [(4m_2+1) + 2(m_3 - m_2)x^2]F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\ = \quad & 2m_2(1-x^2)F\left(-m_2+1, -m_3; \frac{1}{2}; x^2\right) \\ & +(2m_2+1)F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right). \end{aligned} \tag{4.25}$$

By (4.24) and (4.25), we get

$$\begin{aligned} & G'(x) = 0 \\ \Leftrightarrow \quad & (2m_2+1)[1 - (2m_2+1)x - (2m_3+1)x^2]F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\ & = (2m_2+1)[1 - 2(m_2+1)x]F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right) + x(1-x), \end{aligned}$$

which implies that

$$\begin{aligned} & G'(x) = 0 \\ \Leftrightarrow \quad & F\left(-m_2-1, -m_3; \frac{1}{2}; x^2\right) \\ & = \frac{(2m_2+1)[1 - (2m_2+1)x - (2m_3+1)x^2]F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) - x(1-x)}{(2m_2+1)[1 - 2(m_2+1)x]}. \end{aligned}$$

Thus, for $x \in (\frac{1}{2(m_2+1)}, 1)$ with $G'(x) = 0$, we have

$$\begin{aligned}
& G(x) \\
= \;& (2m_2+1)(1+x)F\left(-m_2, -m_3; \frac{1}{2}; x^2\right) \\
& -\frac{(2m_2+1)[1-(2m_2+1)x-(2m_3+1)x^2]F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-x(1-x)}{1-2(m_2+1)x} - x \\
= \;& \frac{(2m_2+1)x^2\{[1+2(m_2-m_3)]F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\}}{2(m_2+1)x-1} \\
> \;& 0.
\end{aligned}$$

By (4.19), for $0 < x < 1$, we have that

$$\begin{aligned}
& \frac{G(x)}{x} \\
= \;& (2m_2+1)F\left(-m_2,-m_3;\frac{1}{2};x^2\right) - 1 - 2m_3(2m_2+1)xF\left(-m_2,-m_3+1;\frac{3}{2};x^2\right) \\
= \;& (2m_2+1)m_2!\,m_3!\sum_{j=0}^{m_3}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-j)!\,(2j)!} - 1 \\
& -2(2m_2+1)m_2!\,m_3!\,x\sum_{j=0}^{m_3-1}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-1-j)!\,(2j+1)!} \\
= \;& (2m_2+1) - 1 - 2(2m_2+1)m_3x \\
& +(2m_2+1)m_2!\,m_3!\sum_{j=1}^{m_3}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-j)!\,(2j)!} \\
& -2(2m_2+1)m_2!\,m_3!\,x\sum_{j=1}^{m_3-1}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-1-j)!\,(2j+1)!}.
\end{aligned} \tag{4.26}$$

If $m_2 = 1$, then $G(x) > 0$ for $x \in (0,1)$ by (4.26) and the following elementary inequality:

$$3|x| < 3x^2 + 1, \qquad x \in \mathbb{R}.$$

If $m_2 \geq 2$, then $G(x) > 0$ for $x \in (0, \frac{1}{2(m_2+1)}]$ by (4.20), (4.26) and the following inequality:

$$\begin{aligned}
& (2m_2+1)\cdot\frac{1}{m_2+1} - 1 \\
& +(2m_2+1)m_2!\,m_3!\sum_{j=1}^{m_2}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-j)!\,(2j)!} \\
& -(2m_2+1)m_2!\,m_3!\sum_{j=1}^{m_2-1}\frac{(2x)^{2j}}{(m_2-j)!\,(m_3-j)!\,(2j)!} \\
> \;& 0, \qquad 0 < x < 1.
\end{aligned}$$

Thus far we have shown that $G(0) = 0$, $G(x) > 0$ for $x \in (0, \frac{1}{2(m_2+1)}]$, $G(x) > 0$ for $x \in (\frac{1}{2(m_2+1)}, 1)$ with $G'(x) = 0$, and $G(1) > 0$. Then, (4.22) and hence (4.23) hold.

Step 3. By (4.19) and (4.23), we get

$$2m_3|x|F\left(-m_2, -m_3+1; \frac{3}{2}; x^2\right) < F\left(-m_2, -m_3; \frac{1}{2}; x^2\right), \qquad 0 < |x| < 1.$$

Thus, by (4.12) and (4.13), we obtain that

$$\frac{m_2!\, m_3!\, 2^{m_2+m_3}}{(2m_2+1)!\,(2m_3-1)!}\left|E[U^{2m_2+1}V^{2m_3-1}]\right| < \frac{m_2!\, m_3!\, 2^{m_2+m_3}}{(2m_2)!\,(2m_3)!}E[U^{2m_2}V^{2m_3}].$$

Therefore, (4.17) holds. □

Remark 4.2.4 *To the best of our knowledge, the moment comparison inequality (4.17) is not given in the literature. From the proof of Theorem 4.2.3, we can see that (4.17) is implied by (4.23). By Lemma 4.2.2, we find that inequality (4.23) is equivalent to the GPI (4.2) if $m_3 = m_2$ and is weaker than (4.2) if $m_3 < m_2$.*

Chapter 5

“Moment ratio inequality of bivariate Gaussian distribution and three-dimensional Gaussian product inequality”

This chapter is based on our paper “Moment ratio inequality of bivariate Gaussian distribution and three-dimensional Gaussian product inequality,” published in Journal of Mathematical Analysis and Applications in 2023 [38].

As described in the previous chapter, in [36] we used combinatorial methods to prove the following GPIs: for any $m_3 \in \mathbb{N}$ and any centered Gaussian random vector (X_1, X_2, X_3),

$$E[X_1^2 X_2^4 X_3^{2m_3}] \geq E[X_1^2]E[X_2^4]E[X_3^{2m_3}],$$

and

$$E[X_1^2 X_2^6 X_3^{2m_3}] \geq E[X_1^2]E[X_2^6]E[X_3^{2m_3}],$$

and left the remainder of Theorem 5.0.1 below as a conjecture. Here, we complete the proof.

Theorem 5.0.1 *Let $m_2, m_3 \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2 X_2^{2m_2} X_3^{2m_3}] \geq E[X_1^2]E[X_2^{2m_2}]E[X_3^{2m_3}]. \tag{5.1}$$

The equality sign holds if and only if X_1, X_2, X_3 are independent.

In [21], all 3 exponents were unbounded, but 2 were equal to each other, allowing for some symmetry and convexity to be exploited. The main contribution of this chapter lies in the fact that we consider 3 *distinct* exponents, 2 of which are unbounded. Even this special case is extremely challenging since the subtle nature of the GPI resists even small modifications. For this reason, we used the computer (**Mathematica**) as a time-saving guide at key junctures to help us guess appropriate simplifications under which the GPI still holds. We emphasize that despite this guidance, the computer was not a crutch but a shot in the arm, and all proofs contained in this chapter remain rigorous and easily verifiable. We strongly believe that computing will be an essential tool used in the eventual complete proof of the GPI. For further explanation regarding the crucial role played by the computer, we refer the reader to Remark 1 of Section 5.4 below.

As mentioned before, there is still much to be discovered about the multivariate Gaussian distribution. In particular, we found comparisons of moments to be scarce in the literature even for the bivariate case. In an attempt to address this gap, we discovered a very close relationship between Gaussian moment ratio inequalities (MRIs) and the GPI. On one hand, in Theorem 4.2.3 above, we gave a novel bivariate MRI that was weaker than the GPI. On the other hand, in this chapter, we prove a new MRI (see Theorem 5.0.2 below) that is stronger than Theorem 5.0.1.

Define

$$\mathcal{S} := \{(1, m_3) : m_3 \geq 5\} \bigcup \{(2, m_3) : m_3 \geq 3\} \bigcup \{(m_2, m_3) : m_3 \geq m_2 \geq 3\},$$

$$r_{m_2,m_3} = (2m_2 + 1)(2m_3 + 1) + 1, \qquad t_{m_2,m_3} = \frac{1}{r_{m_2,m_3} + \left(1 + \frac{1}{2m_2}\right)\left(1 + \frac{1}{2m_3}\right)},$$

and for $\frac{1}{r^2_{m_2,m_3}} < z \leq 1$,

$$H_{m_2,m_3}(z) = \frac{(m_2 + m_3 + 1)(r_{m_2,m_3}z - 1) + \sqrt{[(m_3 - m_2)(r_{m_2,m_3}z - 1)]^2 + (2m_2 + 1)^3(2m_3 + 1)^3 z}}{r^2_{m_2,m_3}z - 1}.$$

For random variables X and Y, denote by $\mathrm{Cov}(X, Y)$ and $\mathrm{Corr}(X, Y)$ their covariance and correlation, respectively. We will show that the GPI (5.1) is implied by the following MRI.

Theorem 5.0.2 *Let (X_2, X_3) be a centered Gaussian random vector. If $(m_2, m_3) \in \mathcal{S}$, then*

$$\frac{\left|E[X_2^{2m_2+1} X_3^{2m_3+1}]\right|}{(2m_2 + 1)(2m_3 + 1)E[X_2^{2m_2} X_3^{2m_3}]} \leq \begin{cases} |\mathrm{Cov}(X_2, X_3)|, & \text{if } |\mathrm{Corr}(X_2, X_3)| \leq \sqrt{t_{m_2,m_3}}, \\ H_{m_2,m_3}([\mathrm{Corr}(X_2, X_3)]^2) \cdot |\mathrm{Cov}(X_2, X_3)|, & \text{if } \sqrt{t_{m_2,m_3}} < |\mathrm{Corr}(X_2, X_3)|. \end{cases} \tag{5.2}$$

The equality sign holds if and only if X_2 and X_3 are independent.

Note that the MRI (5.2) does not hold if $m_2 = 1$ and $1 \leq m_3 \leq 4$, or $m_2 = m_3 = 2$. However, for these particular cases, the GPI (5.1) has been proved by [21]. These cases can also be easily handled by the SOS method described in Chapter 7 below.

In addition to the intrinsic value of solving the GPI and the implications a proof will have on related problems and fields, we can see that the study of the GPI itself has already led to some interesting new results, including the MRIs discussed above, and highly non-trivial combinatorial identities such as [21, Lemma 2.5 and Corollary 2.8]. Therefore, the GPI certainly deserves careful consideration. Finally, we hope that the interplay between computing and hard analysis that we describe in this chapter will inspire confidence in the research community to extend our results.

The remainder of this chapter is structured as follows. In Section 5.1, we explain the relation between the GPI (5.1) and the MRI (5.2). First, we derive an inequality (see (5.15) below) which is slightly stronger than the GPI. Despite being stronger, this step provided a much needed simplification in the subsequent calculations. Then, we show that this inequality is equivalent to a hypergeometric function ratio inequality (HFRI) (see (5.18) below) and hence is equivalent to the MRI (5.2). This HFRI has a very simple structure, but has much more subtle behaviour for small m_2. Thus, in Sections 5.2 and 5.3, we use different methods to prove the HFRI (5.18) for cases $1 \leq m_2 \leq 7$ and $m_2 \geq 8$ separately. In Section 5.4, we make concluding remarks and extend some of our results to the case when m_2 and m_3 can be real numbers rather than simply even integers.

To simplify notation, we will use r, t and H to denote r_{m_2,m_3}, t_{m_2,m_3} and H_{m_2,m_3}, respectively, whenever no confusion is caused. Throughout this chapter, we assume without loss of generality that $m_3 \geq m_2$.

5.1 Relation between GPI and MRI

In this section, we derive the MRI (5.2) and show that its validity implies the validity of the GPI (5.1).

Recall that we denote by $F(a,b;c;z)$ the hypergeometric function (cf. [33]):

$$F(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \cdot \frac{z^j}{j!}, \qquad |z| < 1,$$

where $(\alpha)_j := \alpha(\alpha+1)\cdots(\alpha+j-1)$ for $j \geq 1$, and $(\alpha)_0 = 1$ for $\alpha \neq 0$. Let (X_2, X_3) be a centered Gaussian random vector and $x = \mathrm{Corr}(X_2, X_3)$. By the moment formula (cf. [20, Page 261]), we get

$$\begin{aligned} & E[X_2^{2m_2} X_3^{2m_3}] \\ = \; & (2m_2-1)!!\,(2m_3-1)!!\,[\mathrm{Var}(X_2)]^{m_2}[\mathrm{Var}(X_3)]^{m_3} F\left(-m_2, -m_3; \frac{1}{2}; x^2\right), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} & E[X_2^{2m_2+1} X_3^{2m_3+1}] \\ = \; & (2m_2+1)!!\,(2m_3+1)!!\,[\mathrm{Var}(X_2)]^{\frac{2m_2+1}{2}}[\mathrm{Var}(X_3)]^{\frac{2m_3+1}{2}} xF\left(-m_2, -m_3; \frac{3}{2}; x^2\right). \end{aligned} \tag{5.4}$$

Consider the following functions contiguous to $F(a, b; c; z)$:

$$F(a \pm 1, b; c; z), \quad F(a, b \pm 1; c; z), \quad F(a, b; c \pm 1; z). \tag{5.5}$$

To simplify notation, we denote $F(a, b; c; z)$ and the six functions in (5.5) respectively by

$$F, \quad F(a \pm 1), \quad F(b \pm 1), \quad F(c \pm 1).$$

We have the following relations of Gauss between contiguous functions (cf. [5, 2.8-(21), (31), (37), (38), pages 102 and 103]):

$$\begin{aligned}
&F' = \frac{a[F(a+1) - F]}{z}, \\
&[c - 2a - (b-a)z]F + a(1-z)F(a+1) - (c-a)F(a-1) = 0, \\
&(b-a)(1-z)F - (c-a)F(a-1) + (c-b)F(b-1) = 0, \\
&c(1-z)F - cF(a-1) + (c-b)zF(c+1) = 0.
\end{aligned} \tag{5.6}$$

5.1.1 A stronger inequality

By Lemma 2.3.3, to prove the GPI (5.1), we may assume without loss of generality that $X_1 = X_2 + aX_3$ for some $a \in \mathbb{R}$ and $E[X_2^2] = E[X_3^2] = 1$. This rank-reducing technique greatly simplifies calculations by allowing us to apply nice formulas for bivariate Gaussian moments involving Gaussian hypergeometric functions to this trivariate problem. Define

$$x = E[X_2 X_3].$$

Then,

$$E[X_1^2] = a^2 + 1 + 2ax.$$

Consider the moment ratio:

$$\begin{aligned}
&\frac{E[(X_2 + aX_3)^2 X_2^{2m_2} X_3^{2m_3}]}{E[X_1^2]E[X_2^{2m_2}]E[X_3^{m_3}]} \\
=\ &\frac{E[a^2 X_2^{2m_2} X_3^{2m_3+2} + X_2^{2m_2+2} X_3^{2m_3} + 2aX_2^{2m_2+1} X_3^{2m_2+1}]}{(a^2 + 1 + 2ax)(2m_2 - 1)!!\,(2m_3 - 1)!!} \\
=\ &\left[a^2\,(2m_3+1)\,F\left(-m_3 - 1, -m_2; \frac{1}{2}; x^2\right) + (2m_2+1)\,F\left(-m_3, -m_2 - 1; \frac{1}{2}; x^2\right)\right. \\
&\left. +2ax\,(2m_3+1)\,(2m_2+1)\,F\left(-m_3, -m_2; \frac{3}{2}; x^2\right)\right] \cdot \frac{1}{a^2 + 1 + 2ax}.
\end{aligned} \tag{5.7}$$

Then, the proof of Theorem 5.0.1 is complete if we can show that for any $a \in \mathbb{R}$ and $x \in [-1, 1]$,

$$\begin{aligned}
&a^2\,(2m_3+1)\,F\left(-m_3 - 1, -m_2; \frac{1}{2}; x^2\right) + (2m_2+1)\,F\left(-m_3, -m_2 - 1; \frac{1}{2}; x^2\right) \\
&+2ax\,(2m_3+1)\,(2m_2+1)\,F\left(-m_3, -m_2; \frac{3}{2}; x^2\right) \\
>\ &a^2 + 1 + 2ax.
\end{aligned} \tag{5.8}$$

Equivalently, we need to show that for any $a \in (-\infty, 0]$ and $x \in [0, 1]$,

$$\begin{aligned} F(a,x) \quad := \quad & a^2 \left[(2m_3+1)\, F\left(-m_2, -m_3-1; \frac{1}{2}; x^2 \right) - 1 \right] \\ & +2ax \left[(2m_3+1)\,(2m_2+1)\, F\left(-m_2, -m_3; \frac{3}{2}; x^2 \right) - 1 \right] \\ & + \left[(2m_2+1)\, F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) - 1 \right] \\ > \quad & 0. \end{aligned} \tag{5.9}$$

Note that (5.9) holds when either $x \in \{0, 1\}$ or $a = 0$, and

$$\lim_{a \to -\infty} \min_{x \in [0,1]} F(a, x) = \infty.$$

Thus, we need only show that (5.9) holds under the following condition:

$$\frac{\partial F(a,x)}{\partial a} = 0, \qquad a \in (-\infty, 0),\ x \in (0, 1).$$

By (5.6), we get

$$\begin{aligned} & F\left(-m_2, -m_3-1; \frac{1}{2}; x^2 \right) \\ = \quad & \frac{1}{2m_3+1} \left[(2m_2+1) F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) \right. \\ & \left. +2(m_3-m_2)(1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) \right], \end{aligned}$$

and

$$\begin{aligned} & x^2 F\left(-m_2, -m_3; \frac{3}{2}; x^2 \right) \\ = \quad & \frac{1}{2m_3+1} \left[F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) - (1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) \right]. \end{aligned} \tag{5.10}$$

Then,

$$\begin{aligned} & \frac{\partial F(a,x)}{\partial a} = 0 \\ \Leftrightarrow \quad & 0 = a \left[(2m_3+1)\, F\left(-m_2, -m_3-1; \frac{1}{2}; x^2 \right) - 1 \right] \\ & \quad +x \left[(2m_3+1)\,(2m_2+1)\, F\left(-m_2, -m_3; \frac{3}{2}; x^2 \right) - 1 \right] \\ \Leftrightarrow \quad & 0 = ax \left[(2m_2+1) F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) + 2(m_3-m_2)(1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) - 1 \right] \\ & \quad + \left\{ (2m_2+1) \left[F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) - (1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) \right] - x^2 \right\} \\ \Leftrightarrow \quad & a = -\frac{(2m_2+1)\left[F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) - (1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) \right] - x^2}{x \left[(2m_2+1) F\left(-m_2-1, -m_3; \frac{1}{2}; x^2 \right) + 2(m_3-m_2)(1-x^2) F\left(-m_2, -m_3; \frac{1}{2}; x^2 \right) - 1 \right]}. \end{aligned}$$

Hence, we have that

$$\begin{aligned} F(a,x) &= a^2\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)+2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right] \\ &\quad +\frac{2a}{x}\left\{(2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]-x^2\right\} \\ &\quad +\left[(2m_2+1)\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right] \\ &= \frac{\{(2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]-x^2\}^2}{x^2\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)+2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right]} \\ &\quad -\frac{2\{(2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]-x^2\}^2}{x^2\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)+2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right]} \\ &\quad +\left[(2m_2+1)\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right]. \end{aligned}$$

Thus, to prove the GPI (5.1), we need to show that

$$\begin{aligned} &\left\{(2m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)\right]-x^2\right\}^2 \\ <\ & x^2\left[(2m_2+1)\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right] \\ &\cdot\left[(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)+2(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right)-1\right]. \end{aligned} \tag{5.11}$$

It can be shown that (5.11) is equivalent to Claim C (4.11).

By (5.6), we get

$$\begin{aligned} &(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\ =\ & F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-(2m_3+1)x^2F\left(-m_2,-m_3;\frac{3}{2};x^2\right), \end{aligned}$$

and

$$\begin{aligned} 0 &= -(m_3-m_2)(1-x^2)F\left(-m_2,-m_3;\frac{1}{2};x^2\right) \\ &\quad -\left(m_2+\frac{1}{2}\right)F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)+\left(m_3+\frac{1}{2}\right)F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right). \end{aligned}$$

Then, (5.11) becomes

$$\begin{aligned} & x^2\left\{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};x^2\right)-1\right\}^2 \\ <\ & \left[(2m_2+1)\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right] \\ &\cdot\left[(2m_3+1)\,F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right)-1\right]. \end{aligned} \tag{5.12}$$

Note that, different from (5.11), the roles played by m_2 and m_3 in (5.12) are symmetric. This key fact will lead us to derive the stronger inequality (5.15) given below.

Remark 5.1.1 *Let (X_2, X_3) be a centered Gaussian random vector with $\mathrm{Var}(X_2) = \mathrm{Var}(X_3) = 1$. By (5.3) and (5.4), we can rewrite (5.12) as the following interesting and delicate moment inequality, which is equivalent to the GPI (5.1).*

$$\begin{aligned} & \left\{E[X_2^{2m_2+1}X_3^{2m_3+1}] - E[X_2^{2m_2}]E[X_3^{2m_3}]E[X_2X_3]\right\}^2 \\ < \; & \left\{E[X_2^{2m_2+2}X_3^{2m_3}] - E[X_2^{2m_2}]E[X_3^{2m_3}]\right\} \\ & \cdot \left\{E[X_2^{2m_2}X_3^{2m_3+2}] - E[X_2^{2m_2}]E[X_3^{2m_3}]\right\}. \end{aligned}$$

Let $z = x^2$. Then, (5.12) becomes

$$\begin{aligned} & z\left\{(2m_2+1)\,(2m_3+1)F\left(-m_2, -m_3; \frac{3}{2}; z\right) - 1\right\}^2 \\ < \; & \left[(2m_2+1)\,F\left(-m_2-1, -m_3; \frac{1}{2}; z\right) - 1\right] \\ & \cdot\left[(2m_3+1)\,F\left(-m_2, -m_3-1; \frac{1}{2}; z\right) - 1\right]. \end{aligned} \tag{5.13}$$

Further, by (5.10), we find that (5.13) is equivalent to

$$\begin{aligned} & z\left\{(2m_2+1)\,(2m_3+1)F\left(-m_2, -m_3; \frac{3}{2}; z\right) - 1\right\}^2 \\ < \; & \Bigg[(2m_2+1)\,(1-z)F\left(-m_2, -m_3; \frac{1}{2}; z\right) \\ & + (2m_2+1)\,(2m_3+1)zF\left(-m_2, -m_3; \frac{3}{2}; z\right) - 1\Bigg] \\ & \cdot\Bigg[(2m_3+1)\,(1-z)F\left(-m_2, -m_3; \frac{1}{2}; z\right) \\ & + (2m_2+1)\,(2m_3+1)zF\left(-m_2, -m_3; \frac{3}{2}; z\right) - 1\Bigg], \end{aligned}$$

i.e.,

$$\begin{aligned} & z\left\{(2m_2+1)\,(2m_3+1)F\left(-m_2, -m_3; \frac{3}{2}; z\right) - 1\right\}^2 \\ < \; & \Bigg[\left\{(2m_2+1)\,F\left(-m_2, -m_3; \frac{1}{2}; z\right) - 1\right\}(1-z) \\ & + (2m_2+1)\,(2m_3+1)zF\left(-m_2, -m_3; \frac{3}{2}; z\right) - z\Bigg] \\ & \cdot\Bigg[\left\{(2m_3+1)\,F\left(-m_2, -m_3; \frac{1}{2}; z\right) - 1\right\}(1-z) \\ & + (2m_2+1)\,(2m_3+1)zF\left(-m_2, -m_3; \frac{3}{2}; z\right) - z\Bigg]. \end{aligned}$$

Hence, we need only show that for any $z \in (0,1)$,

$$\begin{aligned} z \quad < \quad & \left[\frac{\left\{(2m_2+1)\, F\left(-m_2,-m_3;\frac{1}{2};z\right)-1\right\}(1-z)}{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)-1} + z \right] \\ & \cdot \left[\frac{\left\{(2m_3+1)\, F\left(-m_2,-m_3;\frac{1}{2};z\right)-1\right\}(1-z)}{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)-1} + z \right]. \end{aligned} \tag{5.14}$$

Note that

$$\begin{aligned} & \frac{\left\{(2m_2+1)\, F\left(-m_2,-m_3;\frac{1}{2};z\right)-1\right\}(1-z)}{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)-1} \\ \geq \quad & \frac{(2m_2+1)\, F\left(-m_2,-m_3;\frac{1}{2};z\right)(1-z)}{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)} + \frac{-(1-z)}{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)} \\ \geq \quad & \frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)(1-z)}{(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)} + \frac{-(1-z)}{(2m_2+1)\,(2m_3+1)}. \end{aligned}$$

Therefore, to prove (5.14), it suffices to show that the following stronger inequality holds for $0 < z < 1$:

$$\begin{aligned} z \quad < \quad & \left[\frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)(1-z)}{(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)} + \frac{[(2m_2+1)(2m_3+1)+1]z-1}{(2m_2+1)\,(2m_3+1)} \right] \\ & \cdot \left[\frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)(1-z)}{(2m_2+1)F\left(-m_2,-m_3;\frac{3}{2};z\right)} + \frac{[(2m_2+1)(2m_3+1)+1]z-1}{(2m_2+1)\,(2m_3+1)} \right]. \end{aligned} \tag{5.15}$$

Using **Mathematica**, it appears that inequality (5.15) holds if $(m_2, m_3) \in \mathcal{S}$. In Sections 5.2 and 5.3 below, we will give a rigorous proof for (5.15) by combining computing and hard analysis.

5.1.2 MRI $\Leftrightarrow$ HFRI $\Rightarrow$ GPI

In this subsection, we define a hypergeometric function ratio inequality (HFRI) and show that it is equivalent to the MRI by proving its equivalence to (5.15). Once proven, the GPI follows.

For $0 < z < 1$, define

$$y = \frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)}{F\left(-m_2,-m_3;\frac{3}{2};z\right)}.$$

Then, $y > 1$ and

$$\begin{aligned}
& (5.15) \text{ holds} \\
\Leftrightarrow\ & 0 < \frac{(1-z)^2}{(2m_2+1)(2m_3+1)} \cdot y^2 \\
& + \frac{2(m_2+m_3+1)\{[(2m_2+1)(2m_3+1)+1]z-1\}(1-z)}{(2m_2+1)^2(2m_3+1)^2} \cdot y \\
& + \frac{[(2m_2+1)(2m_3+1)+1]^2 z^2 - \{[(2m_2+1)(2m_3+1)+1]^2+1\}z+1}{(2m_2+1)^2(2m_3+1)^2} \\
\Leftrightarrow\ & 0 < (1-z)y^2 + 2\beta y + \gamma,
\end{aligned}$$

where

$$\begin{aligned}
\beta &:= -\frac{(m_2+m_3+1)\{1-[(2m_2+1)(2m_3+1)+1]z\}}{(2m_2+1)(2m_3+1)}, \\
\gamma &:= \frac{1-[(2m_2+1)(2m_3+1)+1]^2 z}{(2m_2+1)(2m_3+1)}.
\end{aligned}$$

By the identity

$$(m_2+m_3+1)^2 = (m_3-m_2)^2 + (2m_2+1)(2m_3+1),$$

we get

$$\begin{aligned}
& \beta^2 - (1-z)\gamma \\
=\ & \left(\frac{(m_3-m_2)\{1-[(2m_2+1)(2m_3+1)+1]z\}}{(2m_2+1)(2m_3+1)} \right)^2 + (2m_2+1)(2m_3+1)z. \qquad (5.16)
\end{aligned}$$

Then,

$$\begin{aligned}
& 0 < (1-z)y^2 + 2\beta y + \gamma \\
\Leftrightarrow\ & \left| y + \frac{\beta}{1-z} \right| > \frac{\sqrt{\beta^2-(1-z)\gamma}}{1-z}.
\end{aligned}$$

For $0 < z \leq \frac{1}{r^2}$, we have

$$\begin{aligned}
& y + \frac{\beta}{1-z} - \frac{\sqrt{\beta^2-(1-z)\gamma}}{1-z} \\
>\ & 1 - \frac{m_2+m_3+1}{(2m_2+1)(2m_3+1)\left[1 - \frac{1}{[(2m_2+1)(2m_3+1)+1]^2}\right]} \\
& - \frac{\sqrt{\left[\frac{m_3-m_2}{(2m_2+1)(2m_3+1)}\right]^2 + \frac{(2m_2+1)(2m_3+1)}{[(2m_2+1)(2m_3+1)+1]^2}}}{1 - \frac{1}{[(2m_2+1)(2m_3+1)+1]^2}} \\
>\ & 1 - \frac{1}{(2m_2+1)\left(1-\frac{1}{10^2}\right)} - \frac{\sqrt{\left[\frac{1}{2(2m_2+1)}\right]^2 + \frac{1}{(2m_2+1)^2}}}{1-\frac{1}{10^2}} \\
>\ & 0.
\end{aligned}$$

Hence, to prove (5.15), it suffices to show that for $\frac{1}{r^2} < z < 1$,

$$y > \frac{-\beta + \sqrt{\beta^2 - (1-z)\gamma}}{1-z}.$$

Note that

$$\begin{aligned} \frac{1}{H(z)} &= \frac{\gamma}{-\beta - \sqrt{\beta^2 - (1-z)\gamma}} \\ &= \frac{-\beta + \sqrt{\beta^2 - (1-z)\gamma}}{1-z}, \qquad 0 < z < 1. \end{aligned} \tag{5.17}$$

Hence we need show that the following HFRI holds:

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)}, \qquad \frac{1}{r^2} < z < 1.$$

For $\frac{1}{r^2} < z < \frac{1}{r}$, we have

$$\begin{aligned} & H(z) \geq 1 \\ \Leftrightarrow \quad & [(m_3 - m_2)(rz-1)]^2 + (r-1)^3 z \geq [(r^2 z - 1) + (m_2 + m_3 + 1)(1 - rz)]^2 \\ \Leftrightarrow \quad & (r-1)^3 z - (r^2 z - 1)^2 - (r-1)(rz-1)^2 \geq 2(m_2 + m_3 + 1)(1 - rz)(r^2 z - 1) \\ \Leftrightarrow \quad & (r-1)(1-z)(r^2 z - 1) - (r^2 z - 1)^2 \geq 2(m_2 + m_3 + 1)(1 - rz)(r^2 z - 1) \\ \Leftrightarrow \quad & (r-1)(1-z) - (r^2 z - 1) \geq 2(m_2 + m_3 + 1)(1 - rz) \\ \Leftrightarrow \quad & r(1 - rz) - (r-1)z \geq 2(m_2 + m_3 + 1)(1 - rz) \\ \Leftrightarrow \quad & (2m_2)(2m_3)(1 - rz) \geq (r-1)z \\ \Leftrightarrow \quad & z \leq t. \end{aligned}$$

Then,

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)}, \qquad \frac{1}{r^2} < z \leq t.$$

Thus, to prove the GPI (5.1), we need only prove

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)}, \qquad t < z < 1. \tag{5.18}$$

Finally, we show that the HFRI (5.18) is equivalent to the MRI (5.2) with the equality sign holding if and only if X_2 and X_3 are independent. We assume without loss of generality that $\mathrm{Var}(X_2) = \mathrm{Var}(X_3) = 1$. By (5.3) and (5.4), we get

$$\frac{|E[X_2^{2m_2+1} X_3^{2m_3+1}]|}{E[X_2^{2m_2} X_3^{2m_3}]} = (2m_2+1)(2m_3+1)|x| \frac{F\left(-m_2, -m_3; \frac{3}{2}; x^2\right)}{F\left(-m_2, -m_3; \frac{1}{2}; x^2\right)}. \tag{5.19}$$

Note that

$$\frac{F\left(-m_2,-m_3;\frac{1}{2};0\right)}{F\left(-m_2,-m_3;\frac{3}{2};0\right)}=1,$$

and

$$\frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)}{F\left(-m_2,-m_3;\frac{3}{2};z\right)}>1,\qquad 0<z<1.$$

Therefore, by (5.19) and (5.32) (see below), we conclude that (5.18) is equivalent to (5.2) with the equality sign holding if and only if X_2 and X_3 are independent.

Remark 5.1.2 *On one hand, the MRI (5.2) implies the GPI (5.1). On the other hand, Theorem 4.2.3 and Remark 4.2.4 show that the following MRI holds and is implied by the GPI (5.1):*

$$\frac{\left|E[X_2^{2m_2-1}X_3^{2m_3+1}]\right|}{E[X_2^{2m_2}X_3^{2m_3}]}<\frac{2m_3+1}{2m_2}, \tag{5.20}$$

where (X_2,X_3) *is a centered Gaussian random vector with* $\mathrm{Var}(X_2)=\mathrm{Var}(X_3)=1$. *Hence the MRI (5.2) is stronger than the MRI (5.20). However, these two MRIs have independent interests.*

5.2 Proof of HFRI (5.18) for the case $1\le m_2\le 7$

In this section, we show that (5.18) holds for $1\le m_2\le 7$. We achieve this by considering each of the seven possible values of m_2 separately and by reducing each problem into proving the positivity of bivariate polynomials. By converting these polynomials into sums-of-squares polynomials, their positivity is validated.

5.2.1 Positiveness of bivariate polynomials

Note that

$$\begin{aligned}
F\left(-m_2,-m_3;\frac{1}{2};z\right) &= m_2!\,m_3!\sum_{j=0}^{m_2}\frac{2^{2j}z^j}{(m_2-j)!\,(m_3-j)!\,(2j)!},\\
F\left(-m_2,-m_3;\frac{3}{2};z\right) &= m_2!\,m_3!\sum_{j=0}^{m_2}\frac{2^{2j}z^j}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}.
\end{aligned} \tag{5.21}$$

To simplify notation, we denote

$$f_1(z):=F\left(-m_2,-m_3;\frac{1}{2};z\right),\qquad f_2(z):=F\left(-m_2,-m_3;\frac{3}{2};z\right).$$

Then, we have

$$
\begin{aligned}
&\quad (5.18) \text{ holds} \\
\Leftarrow &\quad \left[(m_2+m_3+1)(rz-1)+\sqrt{[(m_3-m_2)(rz-1)]^2+(2m_2+1)^3(2m_3+1)^3z}\right] f_1(z) \\
&> (r^2z-1)f_2(z) \\
\Leftarrow &\quad \left\{[(m_3-m_2)(rz-1)]^2+(r-1)^3z\right\} f_1^2(z) \\
&> \left\{(r^2z-1)f_2(z)-(m_2+m_3+1)(rz-1)f_1(z)\right\}^2 \\
\Leftrightarrow &\quad (r-1)^3zf_1^2(z) \\
&> (r^2z-1)^2f_2^2(z)+(r-1)(rz-1)^2f_1^2(z)-2(m_2+m_3+1)(rz-1)(r^2z-1)f_1(z)f_2(z) \\
\Leftrightarrow &\quad (r-1)(1-z)(r^2z-1)f_1^2(z) \\
&> (r^2z-1)^2f_2^2(z)-2(m_2+m_3+1)(rz-1)(r^2z-1)f_1(z)f_2(z) \\
\Leftrightarrow &\quad (r-1)(1-z)f_1^2(z)+2(m_2+m_3+1)(rz-1)f_1(z)f_2(z) \\
&> (r^2z-1)f_2^2(z) \\
\Leftrightarrow &\quad S_{m_2,m_3}(z) := (2m_2+1)(2m_3+1)\,(1-z)\left[\sum_{j=0}^{m_2}\frac{2^{2j}z^jm_2!\,m_3!}{(m_2-j)!\,(m_3-j)!\,(2j)!}\right]^2 \\
&\qquad +2(m_2+m_3+1)\left\{[(2m_2+1)(2m_3+1)+1]z-1\right\} \\
&\qquad \cdot\left[\sum_{j=0}^{m_2}\frac{2^{2j}z^jm_2!\,m_3!}{(m_2-j)!\,(m_3-j)!\,(2j)!}\right]\left[\sum_{j=0}^{m_2}\frac{2^{2j}z^jm_2!\,m_3!}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}\right] \\
&\qquad -\left\{[(2m_2+1)(2m_3+1)+1]^2z-1\right\}\left[\sum_{j=0}^{m_2}\frac{2^{2j}z^jm_2!\,m_3!}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}\right]^2 \\
&> 0, \qquad \frac{1}{r^2}<z<1.
\end{aligned}
\tag{5.22}
$$

By using the transformations

$$z=\frac{c^2}{1+c^2},$$

and

$$m_3=\begin{cases} b^2+5, & \text{if } m_2=1, \\ b^2+3, & \text{if } m_2=2, \\ b^2+m_2, & \text{if } m_2\geq 3, \end{cases}$$

we define

$$h_{m_2}(b,c) := (1+c^2)^{2m_2+1}\cdot S_{m_2,m_3}(z), \qquad b,c\in\mathbb{R}.$$

Then, by (5.22), we find that the HFRI (5.18) holds for $1\leq m_2\leq 7$ if h_{m_2} is a positive bivariate polynomial for each $1\leq m_2\leq 7$.

By virtue of **Mathematica**, we obtain the expansion of h_{m_2}, a polynomial of b and c. See the next subsection. Note that all exponents of b and c are even. We sum all terms with negative coefficients with some of the terms with positive coefficients and then apply the package 'SumsOfSquares' in **Macaulay2** [8, 32] to obtain exact SOS decompositions. Hence, since h_{m_2} is strictly positive, we conclude that

$$(5.18) \text{ holds for } 1 \leq m_2 \leq 7.$$

5.2.2 Expansion of functions h_{m_2}

Let $1 \leq m_2 \leq 7$. We have the following expansions of the functions h_{m_2} for any $b, c \in \mathbb{R}$ and the SOS decompositions corresponding to the sums of the 10 terms in parentheses:

$$h_1(b,c) = \frac{8b^6c^6}{3} + \frac{124b^4c^6}{3} + \frac{622b^2c^6}{3} + \frac{1}{9}\left(48b^6c^4 + 664b^4c^4 - 48b^4c^2 + 2884b^2c^4 - 546b^2c^2 + 36b^2 + 3003c^6 + 3766c^4 - 1557c^2 + 180\right).$$

Using 'SumsOfSquares' in **Macaulay2**, we get

$$\begin{aligned}
&48b^6c^4 + 664b^4c^4 - 48b^4c^2 + 2884b^2c^4 - 546b^2c^2 + 36b^2 + 3003c^6 + 3766c^4 - 1557c^2 + 180 \\
&= 4924\left(\frac{3377}{39392}\,b^2c^2 + c^2 - \frac{35173}{196960}\right)^2 + 3003\left(\frac{151}{8008}\,b^2c + c^3 - \frac{193}{1001}\,c\right)^2 + \frac{3853}{2}\left(\frac{557}{77060}\,b^3c^2 \right. \\
&\quad \left. + b\,c^2 - \frac{16027}{200356}\,b\right)^2 + \frac{945348187}{1575680}\left(b^2c^2 - \frac{1722648687}{12289526431}\right)^2 + \frac{1591}{13}\left(b\,c\right)^2 \\
&\quad + \frac{1802093}{20020}\left(-\frac{193955}{1802093}\,b^2c + c\right)^2 + \frac{147644951}{3082400}\left(b^3c^2 - \frac{1109415365}{1919384363}\,b\right)^2 \\
&\quad + \frac{178655579224727}{15976384360300}\left(1\right)^2 + \frac{191380511436}{24951996719}\left(b\right)^2 + \frac{56147093713}{7496706880}\left(b^2c\right)^2.
\end{aligned}$$

$$\begin{aligned}
h_2(b,c) = {} & \frac{32b^{10}c^{10}}{15} + \frac{128b^{10}c^8}{45} + \frac{1936b^8c^{10}}{45} + \frac{13376b^8c^8}{225} + \frac{128b^8c^6}{15} + \frac{5104b^6c^{10}}{15} + \frac{7424b^6c^8}{15} \\
& + \frac{400b^6c^6}{3} + \frac{59192b^4c^{10}}{45} + \frac{454432b^4c^8}{225} + \frac{32872b^4c^6}{45} + \frac{12374b^2c^{10}}{5} + \frac{19896b^2c^8}{5} \\
& + \frac{4892b^2c^6}{3} + \frac{9009c^{10}}{5} + \frac{74484c^8}{25} + \frac{1}{45}\left(576b^6c^4 + 3920b^4c^4 - 480b^4c^2 + 5376b^2c^4 \right. \\
& \left. - 3210b^2c^2 + 360b^2 + 53838c^6 - 4500c^4 - 5535c^2 + 1080\right).
\end{aligned}$$

Using 'SumsOfSquares' in **Macaulay2**, we get

$$
\begin{aligned}
&576b^6c^4 + 3920b^4c^4 - 480b^4c^2 + 5376b^2c^4 - 3210b^2c^2 + 360b^2 + 53838c^6 \\
&\quad - 4500c^4 - 5535c^2 + 1080 = 53838\left(\frac{24865}{3876336}\,b^2c + c^3 - \frac{26099}{107676}\,c\right)^2 \\
&\qquad + 21599\left(-\frac{5209}{194391}\,b^2c^2 + c^2 - \frac{18043}{86396}\right)^2 + \frac{210343}{36}\left(\frac{5751}{60098}\,b^3c^2 + b\,c^2\right. \\
&\qquad \left. - \frac{538218}{2734459}\,b\right)^2 + \frac{19498324709}{6998076}\left(b^2c^2 - \frac{88115791563}{506956442434}\right)^2 + \frac{251207361}{480784}\left(b^3c^2\right. \\
&\qquad \left. - \frac{717899012}{1814275385}\,b\right)^2 + \frac{69666947}{215352}\left(-\frac{15907018825}{32604131196}\,b^2c + c\right)^2 + \frac{15599}{52}\left(b\,c\right)^2 \\
&\qquad + \frac{11349643727532583}{152587333997280}\left(b^2c\right)^2 + \frac{1418170065555879}{26361735006568}\left(1\right)^2 + \frac{42786226455192}{825495300175}\left(b\right)^2.
\end{aligned}
$$

We proceed in the exact same way for $h_3(b, c), \ldots, h_7(b, c)$, but leave the full details for the careful reader at `https://arxiv.org/abs/2208.13957`.

5.3 Proof of HFRI (5.18) for the case $m_2 \geq 8$

We will show that (5.18) holds for $m_2 \geq 8$. To this end, we split the unit interval into two subintervals according to the bound

$$B = \frac{2.75}{m_2 m_3}. \tag{5.23}$$

We use different techniques to prove that

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)} \tag{5.24}$$

for the cases $z \leq B$ and $z > B$ separately. In §5.3.1, we apply a truncation method to reduce the validity of (5.24) for $z \leq B$ to the positiveness of a trivariate polynomial, which will be established through the SOS expansion obtained by **Mathematica**. In §5.3.2, we show that (5.24) holds for $z > B$ by delicate analysis. The classical relations (5.6) play an important role in the proof.

Here we would like to point out that although we choose to use the bound (5.23) corresponding to the truncation number "4" (see (5.27) below), we may also use the following bound corresponding to the truncation number "5":

$$B = \frac{3.8}{m_2 m_3}. \tag{5.25}$$

The advantage of using the bound (5.25) is that we can apply the same method of this section to establish the inequality (5.24) for all $m_2 \geq 6$. The price is that the polynomial in

§5.3.1 would become much longer. With this in mind, to shorten the length of our original arXiv preprint of this chapter, we decided to use the truncation number "4" and the bound (5.23).

5.3.1 The case that $z \le \frac{2.75}{m_2 m_3}$

First, we present a useful lemma.

Lemma 5.3.1 *For $m_2, m_3 \in \mathbb{N}$, we have*

$$H(z) > \begin{cases} \frac{1}{2}, & \text{if } \frac{1}{r^2} < z \le \frac{1}{r}, \\ \frac{1}{7}, & \text{if } \frac{1}{r} < z \le \frac{2.75}{m_2 m_3}. \end{cases} \tag{5.26}$$

Proof. For $\frac{1}{r^2} < z \le \frac{1}{r}$, we have

$$\begin{aligned} & H(z) > \frac{1}{2} \\ \Leftrightarrow\ & 4[(m_3 - m_2)(rz-1)]^2 + 4(r-1)^3 z > [(r^2 z - 1) + 2(m_2 + m_3 + 1)(1 - rz)]^2 \\ \Leftrightarrow\ & 4(r-1)^3 z - (r^2 z - 1)^2 - 4(r-1)(rz-1)^2 > 4(m_2 + m_3 + 1)(1 - rz)(r^2 z - 1) \\ \Leftrightarrow\ & 4(r-1)(1-z)(r^2 z - 1) - (r^2 z - 1)^2 > 4(m_2 + m_3 + 1)(1 - rz)(r^2 z - 1) \\ \Leftrightarrow\ & 4(r-1)(1-z) - (r^2 z - 1) > 4(m_2 + m_3 + 1)(1 - rz) \\ \Leftarrow\ & 3r - 7 > 4(m_2 + m_3 + 1)(1 - rz) \\ \Leftarrow\ & 3r - 7 > 4(m_2 + m_3 + 1). \end{aligned}$$

Then,

$$H(z) > \frac{1}{2}, \quad \frac{1}{r^2} < z \le \frac{1}{r}.$$

For $\frac{1}{r} < z \le \frac{2.75}{m_2 m_3}$, we have

$$\begin{aligned} & H(z) > \frac{1}{7} \\ \Leftrightarrow\ & 49[(m_3 - m_2)(rz-1)]^2 + 49(r-1)^3 z > [(r^2 z - 1) - 7(m_2 + m_3 + 1)(rz - 1)]^2 \\ \Leftrightarrow\ & 49(r-1)^3 z - (r^2 z - 1)^2 - 49(r-1)(rz-1)^2 > -14(m_2 + m_3 + 1)(rz - 1)(r^2 z - 1) \\ \Leftrightarrow\ & 49(r-1)(1-z)(r^2 z - 1) - (r^2 z - 1)^2 > -14(m_2 + m_3 + 1)(rz - 1)(r^2 z - 1) \\ \Leftrightarrow\ & 49(r-1)(1-z) - (r^2 z - 1) > -14(m_2 + m_3 + 1)(rz - 1) \\ \Leftrightarrow\ & 49r - r^2 z + 49z + [14(m_2 + m_3 + 1) - 49]rz > 14(m_2 + m_3 + 1) + 48 \\ \Leftarrow\ & 14.5r > 14(m_2 + m_3 + 1) + 48. \end{aligned}$$

Then,

$$H(z) > \frac{1}{7}, \quad \frac{1}{r} < z \le \frac{2.75}{m_2 m_3}.$$

□

By (5.21), we obtain that for $\frac{1}{r^2} < z < 1$,

$$
\begin{aligned}
&\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} - \frac{1}{H(z)} > 0 \\
\Leftrightarrow \quad & \left[(m_2+m_3+1)(rz-1) + \sqrt{[(m_3-m_2)(rz-1)]^2 + (2m_2+1)^3(2m_3+1)^3 z}\right] \\
&\cdot \sum_{j=0}^{m_2} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j)!} \\
&> (r^2 z - 1) \sum_{j=0}^{m_2} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j+1)!} \\
\Leftarrow \quad & \left[(m_2+m_3+1)(rz-1) + \sqrt{[(m_3-m_2)(rz-1)]^2 + (2m_2+1)^3(2m_3+1)^3 z}\right] \\
&\cdot \sum_{j=0}^{4} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j)!} \\
&> (r^2 z - 1) \sum_{j=0}^{4} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}, \\
&\text{and} \quad H(z) \sum_{j=5}^{m_2} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j)!} > \sum_{j=5}^{m_2} \frac{2^{2j} z^j}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}.
\end{aligned}
\tag{5.27}
$$

Define

$$u = m_2 m_3 z.$$

Similar to (5.22), we can show that

$$
\begin{aligned}
&\left[(m_2+m_3+1)(rz-1)+\sqrt{[(m_3-m_2)(rz-1)]^2+(2m_2+1)^3(2m_3+1)^3 z}\right]\\
&\cdot\left[\sum_{j=0}^{4}\frac{2^{2j}z^j}{(m_2-j)!\,(m_3-j)!\,(2j)!}\right]\\
&>(r^2z-1)\left[\sum_{j=0}^{4}\frac{2^{2j}z^j}{(m_2-j)!\,(m_3-j)!\,(2j+1)!}\right]\\
\Leftarrow\quad &(2m_2+1)(2m_3+1)\left(1-\frac{u}{m_2m_3}\right)\left[1+\sum_{j=1}^{4}\frac{2^{2j}u^j(m_2-1)!\,(m_3-1)!}{(m_2m_3)^{j-1}(m_2-j)!\,(m_3-j)!\,(2j)!}\right]^2\\
&+2(m_2+m_3+1)\left\{\frac{[(2m_2+1)(2m_3+1)+1]u}{m_2m_3}-1\right\}\\
&\cdot\left[1+\sum_{j=1}^{4}\frac{2^{2j}u^j(m_2-1)!\,(m_3-1)!}{(m_2m_3)^{j-1}(m_2-j)!\,(m_3-j)!\,(2j)!}\right]\\
&\cdot\left[1+\sum_{j=1}^{4}\frac{2^{2j}u^j(m_2-1)!\,(m_3-1)!}{(m_2m_3)^{j-1}(m_2-j)!\,(m_3-j)!\,(2j+1)!}\right]\\
&>\left\{\frac{[(2m_2+1)(2m_3+1)+1]^2u}{m_2m_3}-1\right\}\\
&\cdot\left[1+\sum_{j=1}^{4}\frac{2^{2j}u^j(m_2-1)!\,(m_3-1)!}{(m_2m_3)^{j-1}(m_2-j)!\,(m_3-j)!\,(2j+1)!}\right]^2.
\end{aligned}
\tag{5.28}
$$

Obviously, the second inequality of (5.28) is a direct consequence of the following inequality: for $u\in(0,2.75)$ and $x_2\geq 8$, $x_3\geq x_2$,

$$
\begin{aligned}
&f(x_2,x_3,u)\\
:=\quad &(2x_2+1)(2x_3+1)\,(x_2x_3-u)\left[17!\,(x_2x_3)^3+\sum_{j=1}^{4}\frac{2^{2j}u^j(x_2x_3)^{4-j}(x_2-1)!\,(x_3-1)!\,17!}{(x_2-j)!\,(x_3-j)!\,(2j)!}\right]^2\\
&+2(x_2+x_3+1)\left\{[(2x_2+1)(2x_3+1)+1]u-x_2x_3\right\}\\
&\cdot\left[17!\,(x_2x_3)^3+\sum_{j=1}^{4}\frac{2^{2j}u^j(x_2x_3)^{4-j}(x_2-1)!\,(x_3-1)!\,17!}{(x_2-j)!\,(x_3-j)!\,(2j)!}\right]\\
&\cdot\left[17!\,(x_2x_3)^3+\sum_{j=1}^{4}\frac{2^{2j}u^j(x_2x_3)^{4-j}(x_2-1)!\,(x_3-1)!\,17!}{(x_2-j)!\,(x_3-j)!\,(2j+1)!}\right]\\
&-\left\{[(2x_2+1)(2x_3+1)+1]^2u-x_2x_3\right\}\\
&\cdot\left[17!\,(x_2x_3)^3+\sum_{j=1}^{4}\frac{2^{2j}u^j(x_2x_3)^{4-j}(x_2-1)!\,(x_3-1)!\,17!}{(x_2-j)!\,(x_3-j)!\,(2j+1)!}\right]^2\\
>\quad &0.
\end{aligned}
\tag{5.29}
$$

By using the transformations

$$u = \frac{2.75c^2}{1+c^2}, \quad x_2 = a^2 + 8, \quad x_3 = b^2 + 8,$$

we define

$$g(a,b,c) := (1+c^2)^9 \cdot f(x_2, x_3, u), \qquad a,b,c \in \mathbb{R}.$$

Then, (5.29) holds for any $u \in (0, 2.75)$ and $x_2 \geq 8$, $x_3 \geq x_2$ if g is positive on $\mathbb{R}^3$. By virtue of **Mathematica**, we obtain the expansion of g for any $a,b,c \in \mathbb{R}$ (see the Appendix of https://arxiv.org/abs/2208.13957 for the full polynomial):

```
g(a,b,c)=148260632637820250986905600 + 148260632637820250986905600 a^2 +
64864026779046359806771200 a^4 + 16216006694761589951692800 a^6 +
2533751046056498429952000 a^8 + 253375104605649842995200 a^10 +
15835944037853115187200 a^12 + 565569429923325542400 a^14 +
8837022342551961600 a^16 + 148260632637820250986905600 b^2 +
148260632637820250986905600 a^2 b^2 +
64864026779046359806771200 a^4 b^2 +
...
+ 139790538825606949/2 a^12 b^14 c^18 +
1773890417556559 a^14 b^14 c^18 + 15738263341034 a^16 b^14 c^18 +
23549293329040239264 b^16 c^18 +
21275411352230094024 a^2 b^16 c^18 +
8248665160417831936 a^4 b^16 c^18 +
1779973287150248186 a^6 b^16 c^18 +
231114800310247544 a^8 b^16 c^18 +
18098762270078228 a^10 b^16 c^18 + 797031849137168 a^12 b^16 c^18 +
15738263341034 a^14 b^16 c^18 + 33866423320 a^16 b^16 c^18
```

Note that this is an SOS and thus all terms in the expansion are positive. Hence (5.29) holds. Therefore, the first inequality of (5.28) holds, which together with (5.26) and (5.27) implies that

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)}, \qquad t < z \leq \frac{2.75}{m_2 m_3}.$$

5.3.2 The case that $z > \frac{2.75}{m_2 m_3}$

In this subsection, we show that for $m_2 \geq 8$,

$$\frac{F\left(-m_2, -m_3; \frac{1}{2}; z\right)}{F\left(-m_2, -m_3; \frac{3}{2}; z\right)} > \frac{1}{H(z)}, \qquad \frac{2.75}{m_2 m_3} < z < 1.$$

Note that

$$
\begin{aligned}
& \frac{F\left(-m_2,-m_3;\frac{1}{2};z\right)}{F\left(-m_2,-m_3;\frac{3}{2};z\right)} > \frac{1}{H(z)} \\
\Leftrightarrow\ & F\left(-m_2,-m_3;\frac{3}{2};z\right) - H(z)F\left(-m_2,-m_3;\frac{1}{2};z\right) < 0 \\
\Leftrightarrow\ & \frac{F\left(-m_2-1,-m_3;\frac{1}{2};z\right)-(1-z)F\left(-m_2,-m_3;\frac{1}{2};z\right)}{(2m_3+1)z} - H(z)F\left(-m_2,-m_3;\frac{1}{2};z\right) < 0 \\
\Leftrightarrow\ & F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - [(1-z)+(2m_3+1)zH(z)]F\left(-m_2,-m_3;\frac{1}{2};z\right) < 0.
\end{aligned}
$$

For $0 < z \leq 1$, define

$$
G(z) := F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - [(1-z)+(2m_3+1)zH(z)]F\left(-m_2,-m_3;\frac{1}{2};z\right). \tag{5.30}
$$

We will show that

$$
G(z) < 0, \qquad \frac{2.75}{m_2 m_3} < z < 1. \tag{5.31}
$$

By (5.16) and (5.17), we get

$$
\begin{aligned}
& H(1) \\
= & \lim_{z\to 1}\frac{1-z}{-\beta+\sqrt{\beta^2-(1-z)\gamma}} \\
= & \lim_{z\to 1}\frac{1-z}{(m_2+m_3+1)\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]+\sqrt{(m_3-m_2)^2\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]^2+(2m_2+1)(2m_3+1)z}} \\
= & \lim_{z\to 1}\Bigg\{(1-z) \\
& \cdot\frac{(m_2+m_3+1)\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]-\sqrt{(m_3-m_2)^2\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]^2+(2m_2+1)(2m_3+1)z}}{(2m_2+1)(2m_3+1)\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]^2-(2m_2+1)(2m_3+1)z}\Bigg\} \\
= & \lim_{z\to 1}\frac{(m_2+m_3+1)\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]-\sqrt{(m_3-m_2)^2\left[\frac{1-z}{(2m_2+1)(2m_3+1)}-z\right]^2+(2m_2+1)(2m_3+1)z}}{\frac{1-z}{(2m_2+1)(2m_3+1)}-2z-(2m_2+1)(2m_3+1)z} \\
= & \frac{2(m_2+m_3+1)}{2+(2m_2+1)(2m_3+1)}.
\end{aligned} \tag{5.32}
$$

Then,

$$
\begin{aligned}
G(1) &= F\left(-m_2-1,-m_3;\frac{1}{2};1\right)-(2m_3+1)\cdot H(1)\cdot F\left(-m_2,-m_3;\frac{1}{2};1\right) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{3}{2})}{\Gamma(m_2+\frac{3}{2})\Gamma(m_3+\frac{1}{2})} - \frac{2(2m_3+1)(m_2+m_3+1)}{2+(2m_2+1)(2m_3+1)}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{1}{2})}{\Gamma(m_2+\frac{1}{2})\Gamma(m_3+\frac{1}{2})} \\
&= -\frac{(2m_2-1)(2m_3-1)-2}{(2m_2+1)[2+(2m_2+1)(2m_3+1)]}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(m_2+m_3+\frac{1}{2})}{\Gamma(m_2+\frac{1}{2})\Gamma(m_3+\frac{1}{2})} \\
&< 0.
\end{aligned}
$$

Hence, to prove (5.31), we need only show that for $\frac{2.75}{m_2m_3} < z < 1$,

$$G'(z) = 0 \Rightarrow G(z) < 0. \tag{5.33}$$

By (5.6), we get

$$\begin{aligned}
& G'(z) = 0 \\
\Leftrightarrow\ & \left\{F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - [(1-z)+(2m_3+1)zH(z)]F\left(-m_2,-m_3;\frac{1}{2};z\right)\right\}' = 0 \\
\Leftrightarrow\ & 0 = \frac{(m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - F\left(-m_2,-m_3;\frac{1}{2};z\right)\right]}{z} \\
& \quad -[(1-z)+(2m_3+1)zH(z)]'F\left(-m_2,-m_3;\frac{1}{2};z\right) \\
& \quad -[(1-z)+(2m_3+1)zH(z)]\frac{m_2\left[F\left(-m_2,-m_3;\frac{1}{2};z\right) - F\left(-m_2+1,-m_3;\frac{1}{2};z\right)\right]}{z} \\
\Leftrightarrow\ & 0 = \frac{(m_2+1)\left[F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - F\left(-m_2,-m_3;\frac{1}{2};z\right)\right]}{z} \\
& \quad -[(1-z)+(2m_3+1)zH(z)]'F\left(-m_2,-m_3;\frac{1}{2};z\right) \\
& \quad -[(1-z)+(2m_3+1)zH(z)] \\
& \quad \cdot\frac{(2m_2+1)F\left(-m_2-1,-m_3;\frac{1}{2};z\right) - [(2m_2+1)+2m_3z]F\left(-m_2,-m_3;\frac{1}{2};z\right)}{2z(1-z)} \\
\Leftrightarrow\ & \quad \{2(m_2+1)(1-z) - (2m_2+1)[(1-z)+(2m_3+1)zH(z)]\}F\left(-m_2-1,-m_3;\frac{1}{2};z\right) \\
& = \{2(m_2+1)(1-z) + 2z(1-z)[(1-z)+(2m_3+1)zH(z)]' \\
& \quad -[(2m_2+1)+2m_3z][(1-z)+(2m_3+1)zH(z)]\}F\left(-m_2,-m_3;\frac{1}{2};z\right) \\
\Leftrightarrow\ & \quad \{(1-z) - (2m_2+1)(2m_3+1)zH(z)\}F\left(-m_2-1,-m_3;\frac{1}{2};z\right) \\
& = \{[1-2(m_3+1)z](1-z) + 2(2m_3+1)z^2(1-z)H'(z) \\
& \quad +(2m_3+1)z[1-2m_2-2(m_3+1)z]H(z)\}F\left(-m_2,-m_3;\frac{1}{2};z\right).
\end{aligned} \tag{5.34}$$

Note that for $\frac{2.75}{m_2m_3} < z < 1$,

$$\begin{aligned}
& (1-z) - (2m_2+1)(2m_3+1)zH(z) \\
< \ & 1 - \frac{[(2m_2+1)(2m_3+1)]^{5/2}z^{3/2}}{r^2z-1} \\
< \ & 1 - \frac{[(2m_2+1)(2m_3+1)]^{5/2}z^{1/2}}{[(2m_2+1)(2m_3+1)+1]^2} \\
< \ & 0.
\end{aligned}$$

Thus, by (5.30) and (5.34), to complete the proof of (5.31) it suffices to show that for $\frac{2.75}{m_2 m_3} < z < 1$,

$$\begin{aligned}
& [1-2(m_3+1)z](1-z)+2(2m_3+1)z^2(1-z)H'(z) \\
& +(2m_3+1)z[1-2m_2-2(m_3+1)z]H(z) \\
> \; & [(1-z)-(2m_2+1)(2m_3+1)zH(z)]\cdot[(1-z)+(2m_3+1)zH(z)],
\end{aligned}$$

i.e.,

$$(1-z)+[2(m_2+m_3+1)z-1]H(z) < (2m_2+1)(2m_3+1)zH^2(z)+2z(1-z)H'(z). \quad (5.35)$$

We have

$$\begin{aligned}
H'(z) \; = \; & \frac{1}{2(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(2m_2+1)^3(2m_3+1)^3z}} \\
& \cdot\Big\{2r(r-1)(m_2+m_3+1)\sqrt{[(m_3-m_2)(rz-1)]^2+(2m_2+1)^3(2m_3+1)^3z} \\
& -(1+r^2z)(2m_2+1)^3(2m_3+1)^3+2(m_3-m_2)^2r(r-1)(rz-1)\Big\}.
\end{aligned}$$

Then, for $\frac{2.75}{m_2m_3} < z < 1$, we get

(5.35) holds

$$\begin{aligned}
\Leftrightarrow\quad & (r-1)z\left\{\frac{(m_2+m_3+1)(rz-1)+\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}{r^2z-1}\right\}^2 \\
& +\frac{z(1-z)}{(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}} \\
& \cdot\Big\{2r(r-1)(m_2+m_3+1)\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z} \\
& +2(m_3-m_2)^2r(r-1)(rz-1)\Big\} \\
& > \frac{(r-1)^3z(1-z)(1+r^2z)}{(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}+(1-z) \\
& +[2(m_2+m_3+1)z-1]\cdot\frac{(m_2+m_3+1)(rz-1)+\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}{r^2z-1} \\
\Leftrightarrow\quad & \frac{(r-1)z\{[(r-1)+2(m_3-m_2)^2](rz-1)^2+(r-1)^3z\}}{(r^2z-1)^2} \\
& +\frac{2(m_2+m_3+1)(r-1)z(rz-1)\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}{(r^2z-1)^2} \\
& +\frac{2(m_2+m_3+1)r(r-1)z(1-z)}{(r^2z-1)^2} \\
& +\frac{2(m_3-m_2)^2r(r-1)(rz-1)z(1-z)}{(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}} \\
& > \frac{(r-1)^3z(1-z)(1+r^2z)}{(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}+(1-z) \\
& +\frac{[2(m_2+m_3+1)z-1]\left[(m_2+m_3+1)(rz-1)+\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}\right]}{r^2z-1} \\
\Leftrightarrow\quad & \frac{(r-1)^4z^2+(m_2+m_3+1)[2r(r-1)z(1-z)+(rz-1)(r^2z-1)]}{(r^2z-1)^2} \\
& > \frac{(r-1)z(1-z)\left[(r-1)^2(1+r^2z)-2(m_3-m_2)^2r(rz-1)\right]}{(r^2z-1)^2\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}+(1-z) \\
& +\frac{[2(m_2+m_3+1)z(r+rz-2)-(r^2z-1)]\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}{(r^2z-1)^2} \\
& +\frac{(r-1)z(rz-1)(r^2z+rz+r-3)+2(m_3-m_2)^2z(rz-1)(rz+r-2)}{(r^2z-1)^2} \\
\Leftrightarrow\quad & (r-1)^4z^2+(m_2+m_3+1)[2r(r-1)z(1-z)+(rz-1)(r^2z-1)] \\
& > \frac{(r-1)z(1-z)\left[(r-1)^2(1+r^2z)-2(m_3-m_2)^2r(rz-1)\right]}{\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}}+(r^2z-1)^2(1-z) \\
& +[2(m_2+m_3+1)z(r+rz-2)-(r^2z-1)]\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z} \\
& +(r-1)z(rz-1)(r^2z+rz+r-3)+2(m_3-m_2)^2z(rz-1)(rz+r-2)
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow \quad & -1+4z-4rz+3r^2z+z^2-8rz^2+8r^2z^2-4r^3z^2+r^2z^3 \\
& +(m_2+m_3+1)(1-3rz+r^2z+2rz^2-2r^2z^2+r^3z^2) \\
& -2(m_3-m_2)^2(2z-rz-3rz^2+r^2z^2+r^2z^3) \\
> & \frac{1}{\sqrt{[(m_3-m_2)(rz-1)]^2+(r-1)^3z}} \\
& \cdot \left\{(r-1)z(1-z)\left[(r-1)^2(1+r^2z)-2(m_3-m_2)^2r(rz-1)\right]\right. \\
& \left. +[2(m_2+m_3+1)z(r+rz-2)-(r^2z-1)]\{[(m_3-m_2)(rz-1)]^2+(r-1)^3z\}\right\}.
\end{aligned} \tag{5.36}
$$

For $\frac{2.75}{m_2m_3} < z < 1$, we have

$$
\begin{aligned}
& -1+4z-4rz+3r^2z+z^2-8rz^2+8r^2z^2-4r^3z^2+r^2z^3 \\
& +(m_2+m_3+1)(1-3rz+r^2z+2rz^2-2r^2z^2+r^3z^2) \\
& -2(m_3-m_2)^2(2z-rz-3rz^2+r^2z^2+r^2z^3) \\
> \quad & -4r^3z^2+\sqrt{(m_3-m_2)^2+(r-1)}(-3r^2z^2+r^3z^2)-4(m_3-m_2)^2r^2z^2 \\
\geq \quad & -4r^3z^2+\sqrt{(m_3-m_2)^2+(r-1)}\cdot\frac{287r^3z^2}{290}-4(m_3-m_2)^2r^2z^2 \\
> \quad & \left(\frac{1}{2}\sqrt{(m_3-m_2)^2+(r-1)}-4\right)r^3z^2+\frac{142(m_3-m_2)r^3z^2}{290}-4(m_3-m_2)^2r^2z^2 \\
> \quad & 0,
\end{aligned}
$$

and, for $z \geq \frac{2.1}{2m_2+1}$, we have

$$
\begin{aligned}
& (r-1)z(1-z)\left[(r-1)^2(1+r^2z)-2(m_3-m_2)^2r(rz-1)\right] \\
& +[2(m_2+m_3+1)z(r+rz-2)-(r^2z-1)]\{[(m_3-m_2)(rz-1)]^2+(r-1)^3z\} \\
< \quad & \left(1-\frac{2.1}{2m_2+1}\right)(r-1)^3z(r^2z+1)+\{4(m_2+m_3+1)rz-(r^2z-1)\}(r-1)^3z \\
= \quad & \left\{2\left(1-\frac{2.1}{2m_2+1}\right)+4(m_2+m_3+1)rz-\frac{2.1(r^2z-1)}{2m_2+1}\right\}(r-1)^3z \\
= \quad & \left\{\frac{2-\frac{2.1}{2m_2+1}}{\frac{2.1r^2z}{2m_2+1}}+\frac{4(m_2+m_3+1)rz}{\frac{2.1r^2z}{2m_2+1}}-1\right\}\frac{2.1r^2(r-1)^3z^2}{2m_2+1} \\
< \quad & \left\{\frac{2}{(2.1)^2(2m_3+1)^2}+\frac{2}{2.1}-1\right\}\frac{2.1r^2(r-1)^3z^2}{2m_2+1} \\
< \quad & 0.
\end{aligned}
$$

Then (5.36) holds and hence (5.35) holds for $\frac{2.1}{2m_2+1} \leq z < 1$.

Finally, we will show that (5.36) holds for $\frac{2.75}{m_2m_3} < z < \frac{2.1}{2m_2+1}$. To this end, we assume without loss of generality that the right hand side of (5.36) is positive. Then, for $\frac{2.75}{m_2m_3} <$

$z < \frac{2.1}{2m_2+1}$, we have

$$
\begin{aligned}
& (5.36) \text{ holds} \\
\Leftarrow \quad & \{-1 + 4z - 4rz + 3r^2z + z^2 - 8rz^2 + 8r^2z^2 - 4r^3z^2 + r^2z^3 \\
& +\sqrt{(m_3-m_2)^2+(r-1)}(1 - 3rz + r^2z + 2rz^2 - 2r^2z^2 + r^3z^2) \\
& -2(m_3-m_2)^2(2z - rz - 3rz^2 + r^2z^2 + r^2z^3)\}^2\{[(m_3-m_2)(rz-1)]^2 + (r-1)^3z\} \\
& > \Big\{(r-1)z(1-z)\left[(r-1)^2(1+r^2z) - 2(m_3-m_2)^2r(rz-1)\right] \\
& +[2\sqrt{(m_3-m_2)^2+(r-1)}z(r+rz-2) - (r^2z-1)] \\
& \cdot\{[(m_3-m_2)(rz-1)]^2 + (r-1)^3z\}\Big\}^2 \\
\Leftrightarrow \quad & [(m_3-m_2)^2(rz-1)^2 + (r-1)^3z] \\
& \cdot\big\{-1 + 4z - 4rz + 3r^2z + z^2 - 8rz^2 + 8r^2z^2 - 4r^3z^2 + r^2z^3 \\
& \quad +\sqrt{(m_3-m_2)^2+(r-1)}[1 + r^3z^2 + rz(-3+2z) + r^2(z-2z^2)] \\
& \quad -2(m_3-m_2)^2z[2 + r^2z(1+z) - r(1+3z)]\big\}^2 \\
& -\Big\{[(m_3-m_2)^2(rz-1)^2 + (r-1)^3z][1 + 2\sqrt{(m_3-m_2)^2+(r-1)}z(r+rz-2)] \\
& \quad -(m_3-m_2)^2rz(rz-1)[2(r-1)(1-z) + r(rz-1)] \\
& \quad -(r-1)^3r^2z^3 + (r-1)^3z(1-z)\Big\}^2 \\
& > 0.
\end{aligned}
$$

Therefore, the proof of (5.36) for $\frac{2.75}{m_2 m_3} < z < \frac{2.1}{2m_2+1}$ is complete by

$$\begin{aligned}
& [(m_3-m_2)^2(rz-1)^2+(r-1)^3z] \\
& \cdot \big\{-1+4z-4rz+3r^2z+z^2-8rz^2+8r^2z^2-4r^3z^2+r^2z^3 \\
& \quad +\sqrt{(m_3-m_2)^2+(r-1)}[1+r^3z^2+rz(-3+2z)+r^2(z-2z^2)] \\
& \quad -2(m_3-m_2)^2z[2+r^2z(1+z)-r(1+3z)]\big\}^2 \\
& -\Big\{[(m_3-m_2)^2(rz-1)^2+(r-1)^3z][1+2\sqrt{(m_3-m_2)^2+(r-1)}z(r+rz-2)] \\
& \quad -(m_3-m_2)^2rz(rz-1)[2(r-1)(1-z)+r(rz-1)] \\
& \quad -(r-1)^3r^2z^3+(r-1)^3z(1-z)\big\}^2 \\
> \; & (r-1)^3z\left\{-4r^3z^2+\sqrt{(m_3-m_2)^2+(r-1)}(r^3z^2-3rz)-2(m_3-m_2)^2r^2z^2(1+z)\right\}^2 \\
& -\left\{(r-1)^3z(2-z-r^2z^2)+2\sqrt{(m_3-m_2)^2+(r-1)}(r-1)^3rz^2(1+z)\right\}^2 \\
> \; & (r-1)^3r^6z^5[(m_3-m_2)^2+(r-1)]\left\{1-\frac{4}{(r-1)^{1/2}}-\frac{3}{r^2z}-\frac{2(m_3-m_2)(1+z)}{r}\right\}^2 \\
& -4[(m_3-m_2)^2+(r-1)](r-1)^6r^2z^4(1+z)^2 \\
> \; & [(m_3-m_2)^2+(r-1)](r-1)^6r^2z^4 \\
& \cdot\left[\left\{1-\frac{4}{(r-1)^{1/2}}-\frac{3}{r^2z}-\frac{1+\frac{2.1}{2m_2+1}}{2m_2+1}\right\}^2rz-4(1+z)^2\right] \\
> \; & [(m_3-m_2)^2+(r-1)](r-1)^6r^2z^4 \\
& \cdot\left[\left\{1-\frac{4}{17}-\frac{3}{(17^2)(4)(2.75)}-\frac{1+\frac{2.1}{17}}{17}\right\}^2(4)(2.75)-4\left(1+\frac{2.1}{17}\right)^2\right] \\
> \; & 0.3[(m_3-m_2)^2+(r-1)](r-1)^6r^2z^4 \\
> \; & 0.
\end{aligned}$$

5.4 Remarks

Remark 1 Computer-assisted proof methods have been applied in this chapter to establish the three-dimensional GPI (5.1). The idea of using non-traditional methods to solve challenging math problems is not new. For example, Hales gave a computer-assisted proof of the Kepler conjecture [13]. The novelty of our work is that the computer not only verifies some finite statements, which can be completed by pencil and paper in principle, but also helps us make use of traditional methods to complete the main part of the proofs and provides us with deeper insight into why the results are correct.

An equivalent form of the inequality (5.12), which is equivalent to (5.1), has been given in Chapter 4. However, the proof of its validity is extremely challenging. In §5.1.1, we derive the inequality (5.15), which is slightly stronger than (5.12). Considering that the GPI usually resists even small modifications, we first used **Mathematica** to check the validity of (5.15). **Mathematica** suggests that (5.15) holds if $(m_2, m_3) \in \mathcal{S}$, which improved our confidence to give a rigorous proof.

Similar to [21], we use the Gaussian hypergeometric function as the main tool to consider the three-dimensional GPI. It is natural to use the relations of Gauss to establish the key claim (5.33), which is implied by the elementary inequality (5.36) in terms of three variables m_2, m_3, z. Unfortunately, **Mathematica** shows that (5.36) only holds when z is not small. This discovery led us to use traditional methods to prove that (5.15) holds when m_2 is not small, e.g. ≥ 8, and z is bigger than a bound B, whose suitable value was also discovered with the help of **Mathematica**. For the case $z \leq B$, we had to use a completely different method to prove (5.15). Again, with the help of **Mathematica**, we determined a suitable truncation number and then used the (traditional) truncation method to investigate the case that $z \leq B$. In this step, we also used the SOS expansion obtained by **Mathematica** to quickly prove the positiveness of the trivariate polynomial g, although a manual proof might exist.

As for the case that m_2 is small, e.g. $1 \leq m_2 \leq 7$, we needed to show that the bivariate polynomials h_{m_2} given in §5.2.2 are positive on $\mathbb{R}^2$. Although it is possible to figure out clever, traditional methods to establish the positiveness of these polynomials individually, it seems more natural to use the computer to quickly present SOS expansions so as to solve the problem. We believe that mathematicians should feel free to use computing to verify special cases.

Remark 2 In this chapter, the exponents m_2 and m_3 are assumed to be natural numbers. However, Theorems 5.0.1 and 5.0.2 can be extended to the case that m_2 and m_3 are positive real numbers.

Let $y_2, y_3 \in (0, \infty)$. We consider a centered Gaussian random vector (X_1, X_2, X_3). Let $X_1 = X_2 + aX_3$ for some $a \in \mathbb{R}$. Define $x = E[X_2X_3]$. Assume without loss of generality that $E[X_2^2] = E[X_3^2] = 1$. We have

$$E[|X_2|^{y_2}] = \frac{2^{y_2/2}\Gamma(\frac{y_2+1}{2})}{\sqrt{\pi}}, \qquad E[|X_3|^{y_3}] = \frac{2^{y_3/2}\Gamma(\frac{y_3+1}{2})}{\sqrt{\pi}}. \tag{5.37}$$

By Nabeya [30], we get

$$E[|X_2|^{y_2}|X_3|^{y_3}] = \frac{2^{(y_2+y_3)/2}\Gamma(\frac{y_2+1}{2})\Gamma(\frac{y_3+1}{2})}{\pi} F\left(-\frac{y_2}{2}, -\frac{y_3}{2}; \frac{1}{2}; x^2\right). \tag{5.38}$$

Denote by $p(x_2, x_3)$ the probability density function of (X_2, X_3). Let k and l be positive odd integers. By Kamat [18, display (4)], we have

$$\begin{aligned}
& E[|X_2|^{y_2}X_2^k|X_3|^{y_3}X_3^l] \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_2|^{y_2}x_2^k|x_3|^{y_3}x_3^l p(x_2,x_3)dx_2dx_3 \\
= & \left[\int_0^{\infty}\int_0^{\infty}+\int_{-\infty}^{0}\int_{-\infty}^{0}-\int_0^{\infty}\int_{-\infty}^{0}-\int_{-\infty}^{0}\int_0^{\infty}\right]|x_2|^{y_2+k}|x_3|^{y_3+l}p(x_2,x_3)dx_2dx_3 \\
= & 2\cdot\frac{2^{(y_2+k+y_3+l-4)/2}}{\pi}\left[\Gamma\left(\frac{y_2+k+1}{2}\right)\Gamma\left(\frac{y_3+l+1}{2}\right)F\left(-\frac{y_2+k}{2},-\frac{y_3+l}{2};\frac{1}{2};x^2\right)\right. \\
& \left.+2x\Gamma\left(\frac{y_2+k+2}{2}\right)\Gamma\left(\frac{y_3+l+2}{2}\right)F\left(-\frac{y_2+k-1}{2},-\frac{y_3+l-1}{2};\frac{3}{2};x^2\right)\right] \\
& -2\cdot\frac{2^{(y_2+k+y_3+l-4)/2}}{\pi}\left[\Gamma\left(\frac{y_2+k+1}{2}\right)\Gamma\left(\frac{y_3+l+1}{2}\right)F\left(-\frac{y_2+k}{2},-\frac{y_3+l}{2};\frac{1}{2};x^2\right)\right. \\
& \left.-2x\Gamma\left(\frac{y_2+k+2}{2}\right)\Gamma\left(\frac{y_3+l+2}{2}\right)F\left(-\frac{y_2+k-1}{2},-\frac{y_3+l-1}{2};\frac{3}{2};x^2\right)\right] \\
= & x(y_2+k)\cdot(y_2+k-2)\cdots(y_2+1)\cdot(y_3+l)\cdot(y_3+l-2)\cdots(y_3+1) \\
& \cdot\frac{2^{(y_2+y_3)/2}\Gamma\left(\frac{y_2+1}{2}\right)\Gamma\left(\frac{y_3+1}{2}\right)}{\pi}F\left(-\frac{y_2+k-1}{2},-\frac{y_3+l-1}{2};\frac{3}{2};x^2\right). \qquad (5.39)
\end{aligned}$$

Then,

$$\begin{aligned}
& \frac{E[(X_2+aX_3)^2|X_2|^{y_2}|X_3|^{y_3}]}{E[X_1^2]E[|X_2|^{y_2}]E[|X_3|^{y_3}]} \\
= & \frac{E[a^2|X_2|^{y_2}|X_3|^{y_3+2}]+|X_2|^{y_2+2}|X_3|^{y_3}+2a|X_2|^{y_2}X_2|X_3|^{y_3}X_3]}{(a^2+1+2ax)E[|X_2|^{y_2}]E[|X_3|^{y_3}]} \\
= & \left[a^2(y_3+1)F\left(-\frac{y_3}{2}-1,-\frac{y_2}{2};\frac{1}{2};x^2\right)+(y_2+1)F\left(-\frac{y_3}{2},-\frac{y_2}{2}-1;\frac{1}{2};x^2\right)\right. \\
& \left.+2ax(y_3+1)(y_2+1)F\left(-\frac{y_3}{2},-\frac{y_2}{2};\frac{3}{2};x^2\right)\right]\cdot\frac{1}{a^2+1+2ax}
\end{aligned}$$

by (5.37)–(5.39). Hence, by a modified version of Lemma 2.3.3, verifying the corresponding GPIs is equivalent to showing that for any $a\in\mathbb{R}$ and $x\in[-1,1]$,

$$\begin{aligned}
& a^2(y_3+1)F\left(-\frac{y_3}{2}-1,-\frac{y_2}{2};\frac{1}{2};x^2\right)+(y_2+1)F\left(-\frac{y_3}{2},-\frac{y_2}{2}-1;\frac{1}{2};x^2\right) \\
& +2ax(y_3+1)(y_2+1)F\left(-\frac{y_3}{2},-\frac{y_2}{2};\frac{3}{2};x^2\right) \\
> & a^2+1+2ax. \qquad (5.40)
\end{aligned}$$

Note that the inequality (5.40) is exactly the same as the inequality (5.8) except that integer-valued exponents m_2, m_3 are replaced with real-valued exponents $\frac{y_2}{2}, \frac{y_3}{2}$. The arguments used in Sections 5.1 and 5.3 can be applied without any change. Additionally, as explained in the beginning of Section 5.3, we can cover the case that $\frac{y_2}{2}, \frac{y_3}{2} \geq 6$ by using the truncation number "5" and the bound $B = \frac{3.8}{\frac{y_2}{2} \cdot \frac{y_3}{2}}$. Therefore, we have the following propositions.

Proposition 5.4.1 *Let $y_2, y_3 \in [12, \infty)$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2 |X_2|^{y_2} |X_3|^{y_3}] \geq E[X_1^2] E[|X_2|^{y_2}] E[|X_3|^{y_3}]. \tag{5.41}$$

The equality sign holds if and only if X_1, X_2, X_3 are independent.

Define

$$r_{y_2,y_3} = (y_2+1)(y_3+1)+1, \qquad t_{y_2,y_3} = \frac{1}{r_{y_2,y_3} + \left(1+\frac{1}{y_2}\right)\left(1+\frac{1}{y_3}\right)},$$

and for $\frac{1}{r^2_{y_2,y_3}} < z \leq 1$,

$$H_{y_2,y_3}(z) = \frac{\frac{(y_2+y_3+2)(r_{y_2,y_3}z-1)}{2} + \sqrt{\frac{[(y_3-y_2)(r_{y_2,y_3}z-1)]^2}{4} + (y_2+1)^3(y_3+1)^3 z}}{r^2_{y_2,y_3}z - 1}.$$

Proposition 5.4.2 *Let (X_2, X_3) be a centered Gaussian random vector. If $y_2, y_3 \in [12, \infty)$, then*

$$\begin{aligned} &\frac{|E[|X_2|^{y_2} X_2 |X_3|^{y_3} X_3]|}{(y_2+1)(y_3+1)E[|X_2|^{y_2}|X_3|^{y_3}]} \\ \leq &\begin{cases} |\mathrm{Cov}(X_2, X_3)|, & \text{if } |\mathrm{Corr}(X_2,X_3)| \leq \sqrt{t_{y_2,y_3}}, \\ H_{y_2,y_3}([\mathrm{Corr}(X_2,X_3)]^2) \cdot |\mathrm{Cov}(X_2,X_3)|, & \text{if } \sqrt{t_{y_2,y_3}} < |\mathrm{Corr}(X_2,X_3)|. \end{cases} \end{aligned} \tag{5.42}$$

The equality sign holds if and only if X_2 and X_3 are independent.

Through more delicate analysis, the interval $[12, \infty)$ used in Propositions 5.4.1 and 5.4.2 can be enlarged. However, Proposition 5.4.2 does not hold for all $y_2, y_3 \in (0, \infty)$. For example, **Mathematica** has shown that the inequality (5.42) does not hold when $y_2 = 4$ and $y_3 \in [4, 4.58]$, or when $y_2 = 2$ and $y_3 \in [2, 8.15]$. Therefore, the validity of the GPI (5.41) for the most general case that $y_2, y_3 \in (0, \infty)$ still remains open, although using **Mathematica** to check the inequality (5.40), it appears to be true.

Remark 3 It is natural to ask if the MRI method developed in this chapter can be adapted to solve the following three-dimensional GPI.

Conjecture 5.4.3 *Let $m_1 \in \mathbb{N}$ and $y_2, y_3 \in (0, \infty)$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^{2m_1} |X_2|^{y_2} |X_3|^{y_3}] \geq E[X_1^{2m_1}] E[|X_2|^{y_2}] E[|X_3|^{y_3}].$$

The equality sign holds if and only if X_1, X_2, X_3 are independent.

By a modified version of Lemma 2.3.3, we may assume without loss of generality that $X_1 = X_2 + aX_3$ for some $a \in \mathbb{R}$ and $E[X_2^2] = E[X_3^2] = 1$. Define $x = E[X_2X_3]$. By (5.37)–(5.39), we can compute the moment ratio:

$$\begin{aligned}
& \frac{E[(X_2+aX_3)^{2m_1}|X_2|^{y_2}|X_3|^{y_3}]}{E[X_1^{2m_1}]E[|X_2|^{y_2}]E[|X_3|^{y_3}]} \\
= \quad & \frac{\sum_{k=0}^{2m_1}\binom{2m_1}{k}a^k E[|X_2|^{y_2}X_2^{2m_1-k}|X_3|^{y_3}X_3^k]}{(a^2+1+2ax)^{m_1}(2m_1-1)!!\left(\frac{2^{y_2/2}\Gamma(\frac{y_2+1}{2})}{\sqrt{\pi}}\right)\left(\frac{2^{y_3/2}\Gamma(\frac{y_3+1}{2})}{\sqrt{\pi}}\right)} \\
= \quad & \left[\sum_{i=0}^{m_1}\binom{2m_1}{2i}a^{2i}(y_3+2i-1)\cdot(y_3+2i-3)\cdots(y_3+1)\right. \\
& \cdot(y_2+2m_1-2i-1)\cdot(y_2+2m_1-2i-3)\cdots(y_2+1)\cdot F\left(-\frac{y_3}{2}-i,-\frac{y_2}{2}-m_1+i;\frac{1}{2};x^2\right) \\
& +x\sum_{j=1}^{m_1}\binom{2m_1}{2j-1}a^{2j-1}(y_3+2j-1)\cdot(y_3+2j-3)\cdots(y_3+1) \\
& \cdot(y_2+2m_1-2j+1)\cdot(y_2+2m_1-2j-1)\cdots(y_2+1) \\
& \left.\cdot F\left(-\frac{y_3}{2}-j+1,-\frac{y_2}{2}-m_1+j;\frac{3}{2};x^2\right)\right]\cdot\frac{1}{(a^2+1+2ax)^{m_1}(2m_1-1)!!}. \qquad (5.43)
\end{aligned}$$

The proof of Conjecture 5.4.3 is complete if we can show that this moment ratio > 1.

Obviously, this is an extension of the inequality (5.40). Further, by using the relations (5.6), we may use only two hypergeometric functions, $F\left(-\frac{y_2}{2},-\frac{y_3}{2};\frac{1}{2};x^2\right)$ and $F\left(-\frac{y_2}{2},-\frac{y_3}{2};\frac{3}{2};x^2\right)$, to simplify (5.43). We hope the MRI method introduced in this chapter can be improved so as to prove (5.43) with all three exponents unbounded.

Chapter 6

Further three-dimensional GPI results

In an impressive and surprising turn of events, several months after the preprint of our publication [38] (see Chapter 5) was initially posted online, Herry et al. [15] appear to have proved the three-dimensional GPI for any even-integer exponents. Despite the fact that this proof covers our Theorem 5.0.1 above, we feel that our work on the three-dimensional GPI is still unique and valuable for some important reasons. First, we prove the stronger MRI (Theorem 5.0.2 above) which will surely have independent interest for those studying the bivariate Gaussian distribution. Furthermore, as described in Section 5.4 above, our method of proof applies also to the case where the exponents in Theorems 5.0.1 and 5.0.2 are *real*-valued. Since Herry et al.'s proof only allows for these exponents to be even integers, it may be argued that our paper [38] is still the only existing avenue towards a *complete* proof of the general GPI conjecture (1.1).

In this chapter, we build on our work from Chapters 4 and 5 as we reconsider the three-dimensional GPI (5.1) without restricting the exponents to being even integers. We succeed in extending our results further by proving two new and non-trivial extensions to Proposition 5.4.1 as we continue to study the following GPI conjecture:

Conjecture 6.0.1 *Let $y_2, y_3 \in (0, \infty)$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2|X_2|^{y_2}|X_3|^{y_3}] \geq E[X_1^2]E[|X_2|^{y_2}]E[|X_3|^{y_3}].$$

The equality sign holds if and only if X_1, X_2, X_3 are independent.

6.1 Three-dimensional GPI with two equal real exponents

In this section, we prove a new three-dimensional GPI in which 2 exponents are positive real numbers by first extending Theorem 4.2.3 to the case where the exponents can be real numbers rather than simply even integers.

Theorem 6.1.1 *Let $y_2, y_3 \in (0, \infty)$ and (U, V) be bivariate Gaussian random variables with $U \sim N(0,1)$ and $V \sim N(0,1)$.*

(i) We have

$$E[|U|^{y_2}|V|^{y_3}] \leq \frac{E[|U|^{y_2+2}|V|^{y_3}]}{y_2+1}. \tag{6.1}$$

The equality holds if and only if U and V are independent.

(ii) Suppose that $y_3 \leq y_2$. Then, we have

$$\left|E[|U|^{y_2}U|V|^{y_3}V^{-1}]\right| < \frac{y_2+1}{y_3}E[|U|^{y_2}|V|^{y_3}]. \tag{6.2}$$

Proof. After substituting $2m_2$ with y_2, the proof of this theorem remains mostly the same as that of Theorem 4.2.3 except for a few steps which we outline below.

First, we prove assertion (i). The steps here are the same except we now use the bivariate absolute moment formulas (5.38) instead of the even moment formulas (4.13) to confirm (6.1) holds.

Next, we prove assertion (ii). "*Step 1*" can be copied. "*Step 2*" is also identical up until we must prove $G(x) > 0$ for $x \in (0, \frac{1}{y_2+2}]$. We will prove the latter separately for real y_2, y_3 (although it must be noted that this new proof would have been slightly simpler than what we have in the corresponding part of Theorem 4.2.3). First, some preliminary results concerning hypergeometric functions:

By the Euler transformation (cf. Rainville [33, Chapter 4, Theorem 21, page 60]), we get

$$F\left(-y_2, -y_3; \frac{1}{2}; x^2\right) = (1-x^2)^{\frac{1}{2}+y_2+y_3} F\left(\frac{1}{2}+y_2, \frac{1}{2}+y_3; \frac{1}{2}; x^2\right),$$

and

$$F\left(-y_2, -y_3+1; \frac{3}{2}; x^2\right) = (1-x^2)^{\frac{1}{2}+y_2+y_3} F\left(\frac{3}{2}+y_2, \frac{1}{2}+y_3; \frac{3}{2}; x^2\right).$$

We have

$$\begin{aligned} & F\left(\frac{1}{2}+y_2, \frac{1}{2}+y_3; \frac{1}{2}; x^2\right) \\ = \;& 1 + \frac{(\frac{1}{2}+y_2)(\frac{1}{2}+y_3)}{\frac{1}{2}}\frac{x^2}{1!} + \frac{(\frac{1}{2}+y_2)(\frac{1}{2}+y_2+1)(\frac{1}{2}+y_3)(\frac{1}{2}+y_3+1)}{\frac{1}{2}(\frac{1}{2}+1)}\frac{x^4}{2!} + \cdots, \end{aligned}$$

and

$$\begin{aligned} & F\left(\frac{3}{2}+y_2, \frac{1}{2}+y_3; \frac{3}{2}; x^2\right) \\ = \;& 1 + \frac{(\frac{3}{2}+y_2)(\frac{1}{2}+y_3)}{\frac{3}{2}}\frac{x^2}{1!} + \frac{(\frac{3}{2}+y_2)(\frac{3}{2}+y_2+1)(\frac{1}{2}+y_3)(\frac{1}{2}+y_3+1)}{\frac{3}{2}(\frac{3}{2}+1)}\frac{x^4}{2!} + \cdots. \end{aligned}$$

Note that for $a > b > 0$,

$$\frac{a}{b} > \frac{a+1}{b+1}.$$

Then, for $0 < x < 1$,

$$F\left(\frac{1}{2}+y_2, \frac{1}{2}+y_3; \frac{1}{2}; x^2\right) > F\left(\frac{3}{2}+y_2, \frac{1}{2}+y_3; \frac{3}{2}; x^2\right),$$

which implies that

$$F\left(-y_2, -y_3; \frac{1}{2}; x^2\right) > F\left(-y_2, -y_3+1; \frac{3}{2}; x^2\right). \tag{6.3}$$

Furthermore, from Remark 3.1.2, we know

$$F\left(-y_2, -y_3; \frac{1}{2}; x^2\right) \geq 1. \tag{6.4}$$

Then, by (6.3) and (6.4), for $x \in (0, \frac{1}{y_2+2}]$, we have that

$$\begin{aligned} & \frac{G(x)}{x} \\ = \; & (y_2+1)F\left(-\frac{y_2}{2}, -\frac{y_3}{2}; \frac{1}{2}; x^2\right) - 1 - y_3(y_2+1)xF\left(-\frac{y_2}{2}, -\frac{y_3}{2}+1; \frac{3}{2}; x^2\right) \\ > \; & 0. \end{aligned}$$

Thus, the analogue of (4.23) holds:

$$\begin{aligned} & (y_2+1)\left[F\left(-\frac{y_2}{2}-1, -\frac{y_3}{2}; \frac{1}{2}; x^2\right) - F\left(-\frac{y_2}{2}, -\frac{y_3}{2}; \frac{1}{2}; x^2\right)\right] \\ < \; & |x|\left[(y_2+1)F\left(-\frac{y_2}{2}, -\frac{y_3}{2}; \frac{1}{2}; x^2\right) - 1\right], \quad 0 < |x| < 1. \end{aligned} \tag{6.5}$$

For "*Step 3*", the steps are the same except we make use of (5.39) instead of (4.12) to confirm (6.2) holds. □

Since (6.5) is equivalent to (5.40) when $y_2 = y_3$, the following theorem is proved.

Theorem 6.1.2 *Let $y \in (0, \infty)$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2|X_2|^y|X_3|^y] \geq E[X_1^2]E[|X_2|^y]E[|X_3|^y].$$

The equality holds if and only if X_1, X_2, X_3 are independent.

6.2 Three-dimensional GPI with two distinct integer exponents

In this section, we prove most of the following three-dimensional GPI conjecture in which 2 exponents are distinct positive integers (even or odd).

Conjecture 6.2.1 *Let $y_2, y_3 \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^2|X_2|^{y_2}|X_3|^{y_3}] \geq E[X_1^2]E[|X_2|^{y_2}]E[|X_3|^{y_3}].$$

The equality holds if and only if X_1, X_2, X_3 are independent.

We will split the proof into two cases: y_2 even and y_2 odd.

Case 1: First, let y_2 be even. By Proposition 5.4.1 it suffices to prove the following 5 GPIs, where we let $y_3 = y \in (0, \infty)$:

$$E[X_1^2X_2^2|X_3|^y] \geq E[X_1^2]E[X_2^2]E[|X_3|^y], \tag{6.6}$$
$$E[X_1^2X_2^4|X_3|^y] \geq E[X_1^2]E[X_2^4]E[|X_3|^y], \tag{6.7}$$
$$E[X_1^2X_2^6|X_3|^y] \geq E[X_1^2]E[X_2^6]E[|X_3|^y], \tag{6.8}$$
$$E[X_1^2X_2^8|X_3|^y] \geq E[X_1^2]E[X_2^8]E[|X_3|^y], \tag{6.9}$$
$$E[X_1^2X_2^{10}|X_3|^y] \geq E[X_1^2]E[X_2^{10}]E[|X_3|^y]. \tag{6.10}$$

Here, we only go through the proof of (6.7) as an example, since the proofs of the other 4 were completed by following the exact same steps.

From (5.12), we need only prove the following inequality:

$$\begin{aligned} &x^2\left\{(2m_2+1)\,(2m_3+1)F\left(-m_2,-m_3;\frac{3}{2};x^2\right)-1\right\}^2 \\ < \;&\left[(2m_2+1)\,F\left(-m_2-1,-m_3;\frac{1}{2};x^2\right)-1\right] \\ &\cdot\left[(2m_3+1)\,F\left(-m_2,-m_3-1;\frac{1}{2};x^2\right)-1\right], \end{aligned}$$

where we let $2m_2 = y_2 (= 4$ in this case) and $2m_3 = y$. By direct calculation this yields:

$$\begin{aligned} &-\frac{y^5z^5}{9}+\frac{y^5z^4}{9}-\frac{5y^4z^5}{9}-\frac{y^4z^4}{3}+\frac{8y^4z^3}{9}+\frac{8y^3z^5}{9}-\frac{16y^3z^4}{3}+\frac{13y^3z^3}{9}+3y^3z^2+\frac{4y^2z^5}{3} \\ &+\frac{52y^2z^4}{9}-\frac{262y^2z^3}{9}+24y^2z^2-2y^2z+\frac{32yz^4}{3}-\frac{104yz^3}{3}+36yz^2-16yz+4y>0. \end{aligned}$$

Now, since $y > 0$ and $0 < x^2 < 1$, we may set $y = p^2$ and $x^2 = \frac{1}{1+q^2}$ so that we are working strictly with the free variables $p, q \in \mathbb{R}$. Then, we multiply both sides of the resulting inequality by $(1+q^2)^{y_2+1}$ to remove the denominator:

$$\frac{16p^2q^2}{3} + \frac{104p^4q^2}{9} + \frac{59p^6q^2}{9} + \frac{13p^8q^2}{9} + \frac{p^{10}q^2}{9} + \frac{52p^2q^4}{3} + \frac{278p^4q^4}{9} + \frac{94p^6q^4}{9} + \frac{8p^8q^4}{9} + 12p^2q^6 + 16p^4q^6 + 3p^6q^6 + 4p^2q^8 - 2p^4q^8 + 4p^2q^{10} > 0.$$

Only one of the terms of the above polynomial can be negative. By applying the **Mathematica** function `PolynomialSumOfSquaresList[]` to the last 4 terms of the polynomial, we quickly get an exact (and easily verifiable) sums-of-squares (SOS) representation:

$$3p^6q^6 + 4p^2q^8 - 2p^4q^8 + 4p^2q^{10} = \left(2pq^4\right)^2 + \left(pq^3\left(2q^2 - \frac{19p^2}{16}\right)\right)^2 + \left(\frac{1}{2}\sqrt{11}p^2q^4\right)^2 + \left(\frac{1}{16}\sqrt{407}p^3q^3\right)^2.$$

Thus, the proof of the GPI (6.7) is complete. As mentioned before, we have also verified (6.6) and (6.8)-(6.10). Therefore we have completed *Case 1*.

Case 2: Now, let y_2 be odd. By Theorem 6.1.2, we may assume without loss of generality that $y_2 < y_3$. Furthermore, by Proposition 5.4.1, we need not consider $12 \leq y_2 < y_3$. Thus, we will split *Case 2* into the 2 subcases: $0 < y_2 < y_3 \leq 11$ and $0 < y_2 < 12 \leq y_3$.

Subcase 1: Let $0 < y_2 < y_3 \leq 11$. By *Case 1* and Proposition 5.4.1 we need only consider the cases where y_3 is also odd. Namely, we have proved the following 15 GPIs:

$$E[X_1^2|X_2|^1|X_3|^3] \geq E[X_1^2]E[|X_2|^1]E[|X_3|^3], \tag{6.11}$$
$$E[X_1^2|X_2|^1|X_3|^5] \geq E[X_1^2]E[|X_2|^1]E[|X_3|^5], \tag{6.12}$$
$$E[X_1^2|X_2|^1|X_3|^7] \geq E[X_1^2]E[|X_2|^1]E[|X_3|^7], \tag{6.13}$$
$$E[X_1^2|X_2|^1|X_3|^9] \geq E[X_1^2]E[|X_2|^1]E[|X_3|^9], \tag{6.14}$$
$$E[X_1^2|X_2|^1|X_3|^{11}] \geq E[X_1^2]E[|X_2|^1]E[|X_3|^{11}], \tag{6.15}$$
$$E[X_1^2|X_2|^3|X_3|^5] \geq E[X_1^2]E[|X_2|^3]E[|X_3|^5], \tag{6.16}$$
$$E[X_1^2|X_2|^3|X_3|^7] \geq E[X_1^2]E[|X_2|^3]E[|X_3|^7], \tag{6.17}$$
$$E[X_1^2|X_2|^3|X_3|^9] \geq E[X_1^2]E[|X_2|^3]E[|X_3|^9], \tag{6.18}$$
$$E[X_1^2|X_2|^3|X_3|^{11}] \geq E[X_1^2]E[|X_2|^3]E[|X_3|^{11}], \tag{6.19}$$
$$E[X_1^2|X_2|^5|X_3|^7] \geq E[X_1^2]E[|X_2|^5]E[|X_3|^7], \tag{6.20}$$
$$E[X_1^2|X_2|^5|X_3|^9] \geq E[X_1^2]E[|X_2|^5]E[|X_3|^9], \tag{6.21}$$
$$E[X_1^2|X_2|^5|X_3|^{11}] \geq E[X_1^2]E[|X_2|^5]E[|X_3|^{11}], \tag{6.22}$$

$$E[X_1^2|X_2|^7|X_3|^9] \geq E[X_1^2]E[|X_2|^7]E[|X_3|^9], \tag{6.23}$$

$$E[X_1^2|X_2|^7|X_3|^{11}] \geq E[X_1^2]E[|X_2|^7]E[|X_3|^{11}], \tag{6.24}$$

$$E[X_1^2|X_2|^9|X_3|^{11}] \geq E[X_1^2]E[|X_2|^9]E[|X_3|^{11}]. \tag{6.25}$$

Before giving one proof as an example, let us describe the general idea behind the proofs.

Let y_2, y_3 be positive odd integers, $a < 0$ and $x \in (0, 1)$. Proving Conjecture 6.2.1 is equivalent to proving the following inequality given in Chapter 5 as display (5.40):

$$\begin{aligned} & a^2\left(y_3+1\right) F\left(-\frac{y_3}{2}-1,-\frac{y_2}{2};\frac{1}{2};x^2\right)+\left(y_2+1\right) F\left(-\frac{y_3}{2},-\frac{y_2}{2}-1;\frac{1}{2};x^2\right) \\ & +2ax\left(y_3+1\right)\left(y_2+1\right) F\left(-\frac{y_3}{2},-\frac{y_2}{2};\frac{3}{2};x^2\right) \\ > \;& a^2+1+2ax. \end{aligned} \tag{6.26}$$

By the Euler transformation (cf. Rainville [33, Chapter 4, Theorem 21, page 60]), we get

$$\begin{aligned} F\left(-\frac{y_3}{2}-1,-\frac{y_2}{2};\frac{1}{2};x^2\right) &= (1-x^2)^{\frac{y_2+y_3+3}{2}} F\left(\frac{y_3+3}{2},\frac{y_2+1}{2};\frac{1}{2};x^2\right), \\ F\left(-\frac{y_3}{2},-\frac{y_2}{2}-1;\frac{1}{2};x^2\right) &= (1-x^2)^{\frac{y_2+y_3+3}{2}} F\left(\frac{y_3+1}{2},\frac{y_2+3}{2};\frac{1}{2};x^2\right), \\ F\left(-\frac{y_3}{2},-\frac{y_2}{2};\frac{3}{2};x^2\right) &= (1-x^2)^{\frac{y_2+y_3+3}{2}} F\left(\frac{y_3+3}{2},\frac{y_2+3}{2};\frac{3}{2};x^2\right). \end{aligned} \tag{6.27}$$

By repeated use of the contiguous function relation (cf. Rainville [33, Chapter 4, Exercise 21, Page 71])

$$[2a-c+(b-a)z]F = a(1-z)F(a+) - (c-a)F(a-),$$

we can use $F\left(1,1;\frac{3}{2};x^2\right)$ to express $F\left(\frac{y_3+3}{2},\frac{y_2+3}{2};\frac{3}{2};x^2\right)$ and use $F\left(1,1;\frac{1}{2};x^2\right)$ to express $F\left(\frac{y_3+3}{2},\frac{y_2+1}{2};\frac{1}{2};x^2\right)$ and $F\left(\frac{y_3+1}{2},\frac{y_2+3}{2};\frac{1}{2};x^2\right)$. Fortunately, the function `ReplaceRepeated (//.)` in **Mathematica** allows us to do this easily for fixed y_2 and y_3. By the contiguous function relation

$$(1-z)F = F(a-) - c^{-1}(c-b)zF(c+),$$

we get

$$(1-x^2)F\left(1,1;\frac{1}{2};x^2\right) = 1 + x^2 F\left(1,1;\frac{3}{2};x^2\right).$$

Then, we can use $F\left(1,1;\frac{3}{2};x^2\right)$ to express $F\left(\frac{y_3+3}{2},\frac{y_2+3}{2};\frac{3}{2};x^2\right)$, $F\left(\frac{y_3+3}{2},\frac{y_2+1}{2};\frac{1}{2};x^2\right)$ and $F\left(\frac{y_3+1}{2},\frac{y_2+3}{2};\frac{1}{2};x^2\right)$.

By the Euler transformation, we get

$$F\left(1,1;\frac{3}{2};x^2\right) = (1-x^2)^{-\frac{1}{2}} F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) = (1-x^2)^{-\frac{1}{2}} \cdot \frac{\arcsin x}{x}.$$

Then, we can use $\arcsin x$ to express $F\left(\frac{y_3+3}{2}, \frac{y_2+3}{2}; \frac{3}{2}; x^2\right)$, $F\left(\frac{y_3+3}{2}, \frac{y_2+1}{2}; \frac{1}{2}; x^2\right)$ and $F\left(\frac{y_3+1}{2}, \frac{y_2+3}{2}; \frac{1}{2}; x^2\right)$. Thus, by (6.26) and (6.27), we obtain an inequality which is equivalent to (6.26) and contains the function $\arcsin x$ but does not contain any hypergeometric function.

Now, let us use this strategy to begin the proof of the GPI (6.18). Here, we only go through this particular proof as an example, since the proofs of the other 14 inequalities were completed by following the exact same steps. Fix $y_2 = 3$ and $y_3 = 9$. Following the above steps (the use of **Mathematica** makes the calculations quick), (6.26) becomes:

$$\begin{aligned}
&315\left(11a^2x\left(10x^2+3\right)+2a\left(80x^4+60x^2+3\right)+x\left(48x^4+80x^2+15\right)\right)\arcsin x\\
&\quad+a^2\Big(33125\sqrt{1-x^2}x^2+2560\sqrt{1-x^2}-96\sqrt{1-x^2}x^{10}\\
&\quad+832\sqrt{1-x^2}x^8-3556\sqrt{1-x^2}x^6+12180\sqrt{1-x^2}x^4-256\Big)\\
&\quad+2a\Big(31670\sqrt{1-x^2}x^2+9295\sqrt{1-x^2}+64\sqrt{1-x^2}x^8-688\sqrt{1-x^2}x^6\\
&\quad+4704\sqrt{1-x^2}x^4-256\Big)x+18827\sqrt{1-x^2}x^2+1024\sqrt{1-x^2}\\
&\quad-80\sqrt{1-x^2}x^8+1400\sqrt{1-x^2}x^6+23874\sqrt{1-x^2}x^4-256>0.
\end{aligned} \tag{6.28}$$

The remainder of the proof relies on the removal of $\arcsin x$ from (6.28) by approximating it from both above and below as follows. Note that for $0 < x < 1$,

$$x < \arcsin x < \frac{x}{\sqrt{1-x^2}}.$$

If we can show that (6.28) still holds when $\arcsin x$ is replaced by both x and $\frac{x}{\sqrt{1-x^2}}$, the proof is complete.

First, we replace $\arcsin x$ with x in (6.28). Then, proving this inequality is equivalent to showing that the discriminant with respect to a of the LHS (note that the polynomial on the LHS is quadratic in a) is negative. Computing this discriminant, we obtain:

$$\Big[4\left(x^2-1\right)\Big(-512\left(114555733\sqrt{1-x^2}-17365845\right)x^2-16777216\left(424\sqrt{1-x^2}-181\right)$$
$$+344064\sqrt{1-x^2}x^{28}-57344\left(241\sqrt{1-x^2}+2430\right)x^{26}-2048\left(37823\sqrt{1-x^2}-914760\right)x^{24}$$
$$+3072\left(828397\sqrt{1-x^2}-3716685\right)x^{22}-1792\left(11704615\sqrt{1-x^2}-24058992\right)x^{20}$$
$$+896\left(74304221\sqrt{1-x^2}+65202948\right)x^{18}-288\left(505311237\sqrt{1-x^2}+2646568904\right)x^{16}$$
$$+48\left(9348788063\sqrt{1-x^2}+27508005495\right)x^{14}-336\left(7349756819\sqrt{1-x^2}-4197860610\right)x^{12}$$
$$+\left(3310457519376-9979274753304\sqrt{1-x^2}\right)x^{10}$$
$$-2\left(2017029202343\sqrt{1-x^2}+4503182520564\right)x^8$$
$$-259\left(14493444155\sqrt{1-x^2}+10958917527\right)x^6+1260\left(41052415\sqrt{1-x^2}+19368769\right)x^4\Big)\Big]$$
$$\cdot\Big[-5\left(6625\sqrt{1-x^2}+2079\right)x^2-2560\sqrt{1-x^2}+96\sqrt{1-x^2}x^{10}$$
$$-832\sqrt{1-x^2}x^8+3556\sqrt{1-x^2}x^6-210\left(58\sqrt{1-x^2}+165\right)x^4+256\Big]^{-1}<0.$$

Then, performing the substitution $x=\sqrt{1-z}$ (note that $0<z<1$) and simplifying (again, **Mathematica** helps), we get

$$\begin{aligned}&916037850z^{3/2}-965546568z^{5/2}+380775528z^{7/2}-23247616z^{9/2}\\&-4132800z^{11/2}-4677120z^{13/2}+1451520z^{15/2}+3584z^9+80640z^8\\&-2692224z^7+1764672z^6+6313608z^5-66735900z^4+220910130z^3\\&-409730895z^2+391127310z-299743290\sqrt{z}-143727901<0.\end{aligned}$$

Substituting $z=\left(\frac{1}{1+q^2}\right)^2$ where $q\in\mathbb{R}$ and expanding, we arrive at

$$\begin{aligned}&143727901q^{36}+2886845508q^{34}+26694877473q^{32}+150872979846q^{30}\\&+583366700205q^{28}+1636336197216q^{26}+3443944013433q^{24}\\&+5546206904130q^{22}+6903228914703q^{20}+6657338832756q^{18}\\&+4951600692075q^{16}+2805460675434q^{14}+1183949030439q^{12}+358500910824q^{10}\\&+73099063035q^8+8912051406q^6+493682688q^4+7077888q^2+1769472>0,\end{aligned}$$

which is clearly true.

Now, we replace $\arcsin x$ with $\frac{x}{\sqrt{1-x^2}}$ in (6.28). The exact same steps as above can be followed to arrive at

$$\begin{aligned}&156080925q^{40} + 3109199940q^{38} + 29162992021q^{36} + 171201952818q^{34} \\ &+ 705104263953q^{32} + 2163827962416q^{30} + 5127688826325q^{28} \\ &+ 9592542436806q^{26} + 14356071174123q^{24} + 17307025090380q^{22} \\ &+ 16833762813303q^{20} + 13164801289566q^{18} + 8204727229155q^{16} \\ &+ 4012177350024q^{14} + 1501808920479q^{12} + 413799923754q^{10} \\ &+ 78663065220q^{8} + 9148537656q^{6} + 493682688q^{4} + 7077888q^{2} + 1769472 > 0,\end{aligned}$$

which is clearly true.

Thus, the proof of the GPI (6.18) is complete. As mentioned before, we have also verified (6.11)-(6.17) and (6.19)-(6.25). Therefore we have completed *Subcase 1*.

Subcase 2: Let $0 < y_2 < 12 \leq y_3$. Once this can be verified, Conjecture 6.2.1 will be fully proved. We present this as an open problem to the community and leave the following theorem here as a summary of the findings of this section.

Theorem 6.2.2 *Let* $(y_2, y_3) \in (\mathbb{N} \times \mathbb{N}) \setminus \{(y_2, y_3) \,|\, 1 \leq y_2 < 12 \leq y_3$ *and* y_2 *odd*$\}$. *For any centered Gaussian random vector* (X_1, X_2, X_3),

$$E[X_1^2 |X_2|^{y_2} |X_3|^{y_3}] \geq E[X_1^2] E[|X_2|^{y_2}] E[|X_3|^{y_3}].$$

The equality holds if and only if X_1, X_2, X_3 *are independent.*

Chapter 7

"Using sums-of-squares to prove Gaussian product inequalities"

This chapter is based on our paper "Using sums-of-squares to prove Gaussian product inequalities," first posted on arXiv in May 2022 [37].

In this chapter, we study the following conjecture.

Conjecture 7.0.1 *Let $n \geq 3$ and $m_1, \ldots, m_n \in \mathbb{N}$. For any centered Gaussian random vector $(X_1, \ldots, X_n)$,*

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq \prod_{j=1}^{n} E[X_j^{2m_j}]. \tag{7.1}$$

The equality holds if and only if $X_1, \ldots, X_n$ are independent.

We develop an efficient computational algorithm that produces *exact* sums-of-squares (SOS) polynomials to tackle the GPI. We describe this method in Section 7.1 and use it to rigorously prove two special cases of the GPI conjecture with fixed exponents in Section 7.2. Then, in Section 7.3, we reveal the true power of the SOS method by extending these special cases to the stronger result where one exponent is unbounded. Finally, in Section 7.4, we prove a five-dimensional GPI as a template for a new and improved SOS method that handles the case where the unbounded exponent can be real rather than simply an even integer. In theory, our algorithm is applicable to *any* GPI of the form (1.1) with dimension and all but one exponent fixed even integers, and is therefore the first universal method for solving the GPI.

7.1 The SOS method of solving the GPI

An SOS representation of a polynomial is of the form $\sum_{i=1}^{p} f_i^2$, where the f_i's are real-coefficient polynomials. It is clear that any polynomial with an SOS representation is necessarily non-negative.

Lemma 7.1.1 *Let $(X_1, \ldots, X_n)$ be a centered Gaussian random vector. Denote by Λ the covariance matrix of $(X_1, \ldots, X_n)$ and $c_{m_1,\ldots,m_n}$ the coefficient corresponding to the term $t_1^{2m_1} \cdots t_n^{2m_n}$ of the polynomial*

$$G(t_1, \ldots, t_n) = \left(\sum_{k,l=1}^{n} \Lambda_{kl} t_k t_l \right)^{\sum_{j=1}^n m_j}, \qquad t_1, \ldots, t_n \in \mathbb{R}.$$

Then,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j} \right] = \frac{\prod_{j=1}^n (2m_j)!}{2^{\sum_{j=1}^n m_j} (\sum_{j=1}^n m_j)!} \cdot c_{m_1,\ldots,m_n}. \tag{7.2}$$

Proof. We use $'$ to denote the transpose of a matrix or a vector. Define

$$f_\Lambda(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Lambda)}} e^{-\frac{1}{2} x' \Lambda^{-1} x}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

and

$$\Psi_\Lambda(t) = \int_{\mathbb{R}^n} e^{it'x} f_\Lambda(x) dx, \qquad t = (t_1, \ldots, t_n) \in \mathbb{R}^n.$$

Then,

$$\Psi_\Lambda(t) = \exp\left(-\frac{1}{2} t' \Lambda t \right).$$

For $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta_j \in \mathbb{N} \cup \{0\}$, we have

$$\partial_t^\beta \Psi_\Lambda(0) = i^{\sum_{j=1}^n \beta_j} \int_{\mathbb{R}^n} \left\{ \prod_{j=1}^n x_j^{\beta_j} \right\} f_\Lambda(x) dx.$$

Hence,

$$\begin{aligned} E\left[\prod_{j=1}^{n} X_j^{2m_j} \right] &= (-1)^{\sum_{j=1}^n m_j} \partial_t^{(2m_1,\ldots,2m_n)} \exp\left(-\frac{1}{2} t' \Lambda t \right)(0) \\ &= \frac{1}{2^{\sum_{j=1}^n m_j} (\sum_{j=1}^n m_j)!} \cdot \partial_t^{(2m_1,\ldots,2m_n)} \left(\sum_{k,l=1}^n \Lambda_{kl} t_k t_l \right)^{\sum_{j=1}^n m_j} (0). \end{aligned}$$

Therefore, the proof is complete. □

Let U_j, $1 \le j \le n$, be independent standard Gaussian random variables. Define

$$X_k = \sum_{j=1}^n x_{kj} U_j, \quad 1 \le k \le n,$$

where each $x_{kj} \in \mathbb{R}$, $1 \le k, j \le n$. Then, we have

$$\Lambda_{kk} = \sum_{j=1}^{n} x_{kj}^2, \quad 1 \le k \le n,$$
$$\Lambda_{kl} = \sum_{j=1}^{n} x_{kj} x_{lj}, \quad 1 \le k < l \le n. \tag{7.3}$$

Define

$$\begin{aligned} F_{m_1,\ldots,m_n}(\Lambda) &= E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] - \prod_{j=1}^{n} E[X_j^{2m_j}] \\ &= E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] - \prod_{k=1}^{n} [(2m_k - 1)!!\,\Lambda_{kk}^{m_k}]. \end{aligned}$$

By (7.2) and (7.3), it is easy to see that $F_{m_1,\ldots,m_n}(\Lambda)$ can be expressed as a polynomial of the x_{ij}'s, say $F_{m_1,\ldots,m_n}(x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn})$.

By (7.2) and using the **Mathematica** functions `Expand[]` and `Coefficient[]`, we get

$$\begin{aligned} \text{Poly} &:= \texttt{Expand}\left[\left(\sum_{k,l=1}^{n} \Lambda_{kl} t_k t_l\right)^{\sum_{j=1}^{n} m_j}\right], \\ F_{m_1,\ldots,m_n}(\Lambda) &= \texttt{Expand}\left[\frac{\prod_{j=1}^{n}(2m_j)!}{2^{\sum_{j=1}^{n} m_j}(\sum_{j=1}^{n} m_j)!} \cdot \texttt{Coefficient}\left[\text{Poly}, \prod_{j=1}^{n} t_j^{2m_j}\right]\right. \\ &\qquad \left. - \prod_{k=1}^{n} [(2m_k - 1)!!\,\Lambda_{kk}^{m_k}]\right]. \end{aligned} \tag{7.4}$$

Combining (7.3) and (7.4), we have an algorithm to get the expression of $F_{m_1,\ldots,m_n}(x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn})$. Suppose we consider (7.4) for fixed $n, m_1, \ldots, m_n$. Applying the **Macaulay2** package `'SumsOfSquares'` [8] to this polynomial, we may attempt to obtain an SOS decomposition $\sum_{i=1}^{p} c_i f_i^2$ of $F_{m_1,\ldots,m_n}$, where the c_i's are positive rational numbers and the f_i's are rational-coefficient polynomials. Although not every rational-coefficient SOS polynomial necessarily has such a decomposition (see Scheiderer [39, Theorem 2.1]), this software aims to produce one regardless (see Peyrl and Parrilo [32]). If it is obtained, then (7.1) is verified for this case.

This package is very user-friendly and, along with the semi-definite programming package it uses, comes pre-installed in the newest versions of **Macaulay2**. By using a "mixed symbolic-numerical approach" [32], `'SumsOfSquares'` takes advantage of the speed of approximate numerical calculations, yet still produces a final SOS decomposition that is exact (*not* an approximation). This SOS polynomial can then be expanded and checked to match the original $F_{m_1,\ldots,m_n}$ using the `value()` function.

To further increase the efficiency of our method, we will use the rank-reducing trick, Lemma 2.3.3, to reduce the degree of the polynomial $F_{m_1,\ldots,m_n}$. In this way, calculation of the SOS decomposition becomes significantly less computationally intensive and much faster. This rank-reducing technique was proved in §2.3.3, but we restate it here for convenience.

Lemma 7.1.2 *([36, Lemma 2.1]) Let $n \geq 3$ and $m_1, \ldots, m_n \in \mathbb{N}$. If for any centered Gaussian random vector $(Y_1, \ldots, Y_n)$ with $Y_n = \alpha_1 Y_1 + \cdots + \alpha_{n-1} Y_{n-1}$ for some constants $\alpha_1, \ldots, \alpha_{n-1}$,*

$$E\left[\left\{\prod_{j=1}^{n-1} Y_j^{2m_j}\right\} Y_n^{2k}\right] \geq \left\{\prod_{j=1}^{n-1} E[Y_j^{2m_j}]\right\} E[Y_n^{2k}], \qquad 0 \leq k \leq m_n, \tag{7.5}$$

then for any centered Gaussian random vector $(X_1, \ldots, X_n)$,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] \geq \prod_{j=1}^{n} E[X_j^{2m_j}]. \tag{7.6}$$

Additionally, if inequality (7.5) is strict when $k = m_n$, then the equality sign of (7.6) holds only if X_n is independent of $X_1, \ldots, X_{n-1}$.

7.2 Applications of the SOS method

First, we will verify (7.1) for the case $n = 3, m_1 = 4, m_2 = 3, m_3 = 2$.

Theorem 7.2.1 *For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^8 X_2^6 X_3^4] \geq E[X_1^8] E[X_2^6] E[X_3^4].$$

The equality holds if and only if X_1, X_2, X_3 are independent.

Proof. Let X_3 be a linear combination of X_1 and X_2, and let U_1 and U_2 be independent standard Gaussian random variables. Then, without loss of generality, we may write

$$X_1 = U_1,\ \ X_2 = aU_1 + U_2,\ \ X_3 = bU_1 + U_2,\ \ a, b \in \mathbb{R}.$$

By Lemma 7.1.2, we need only show that $F_{4,3,1}$ and $F_{4,3,2}$ are strictly positive. By Theorem 4.1.6, $F_{4,3,1} > 0$. We will complete the proof by giving an SOS decomposition of $F_{4,3,2}$. By (7.4), we get

$$\begin{aligned}F_{4,3,2}(a,b) = {} & 94500 + 1474200\,a^2 + 2324700\,a^4 + 400680\,a^6 + 2381400\,a\,b + 12474000\,a^3\,b \\ & + 9729720\,a^5\,b + 585900\,b^2 + 14004900\,a^2\,b^2 + 36458100\,a^4\,b^2 \\ & + 12152700\,a^6\,b^2 + 3742200\,a\,b^3 + 32432400\,a^3\,b^3 + 48648600\,a^5\,b^3 \\ & + 151200\,b^4 + 6066900\,a^2\,b^4 + 30391200\,a^4\,b^4 + 34454700\,a^6\,b^4\end{aligned}$$

$$
\begin{aligned}
&= \frac{145216503}{4}\left(\frac{60146041}{290433006}\,a^3b + a^2b^2 + \frac{2953715}{145216503}\,a^2 + \frac{88269419}{290433006}\,a\,b - \frac{244387}{12627522}\,b^2 + \frac{54414145}{2468680551}\right)^2 + 34454700\Big(a^3b^2 \\
&\quad - \frac{8427157}{275637600}\,a^3 + \frac{134448359}{275637600}\,a^2b - \frac{875989}{10208800}\,a\,b^2 + \frac{639317}{9844200}\,a - \frac{97603}{5512752}\,b\Big)^2 + \frac{43421589679843919}{2205100800}\Big(\frac{2207851366454963}{43421589679843919}\,a^3 \\
&\quad + a^2b + \frac{16239747161830977}{43421589679843919}\,a\,b^2 + \frac{9538031563407116}{43421589679843919}\,a + \frac{55963631836960250}{738167024557346623}\,b\Big)^2 + \frac{29513864367243803}{2323464048}\Big(a^3b \\
&\quad + \frac{9815490958053974}{29513864367243803}\,a^2 - \frac{2663681330163887}{29513864367243803}\,a\,b + \frac{36010094112641}{29513864367243803}\,b^2 - \frac{15346278961059626}{501735694243144651}\Big)^2 \\
&\quad + \frac{775345651901116499475901}{173686358719375676}\Big(-\frac{512147584469856960982333}{775345651901116499475901}\,a^3 + a\,b^2 - \frac{124746121201440746832731}{131808760823189804910903 17}\,a \\
&\quad + \frac{641351442519001011532 6451}{2636175216463796098218063 4}\,b\Big)^2 + \frac{355765177630182709268 3371}{1003471388486289302}\Big(\frac{509726612981214583208031 1}{142306071052073083707334 84}\,a^2 + a\,b \\
&\quad + \frac{1562593143932213174576929}{14230607105207308370733484}\,b^2 + \frac{1355975034048165669823823}{14230607105207308370733484}\Big)^2 + \frac{69744372474088890758457139816341}{113844856841658466965867872}\Big(a^2 \\
&\quad - \frac{171575936011195533752480675 1893}{13173937022883457143264126409753 3}\,b^2 - \frac{186306503137515128525276484100099}{118565433205951114289377137687779 7}\Big)^2 \\
&\quad + \frac{4972165669876574823877172353515 91}{896299573597690673394141556}\Big(\frac{2658217817396466488457326602434 40}{4972165669876574823877172353515 91}\,b \\
&\quad - \frac{2112568700279405295506800353629 88}{4972165669876574823877172353515 91}\,a^3 + a\Big)^2 + \frac{98420585988442784935014259601772304508}{497216566987657482387717235351591}\Big(a^3 \\
&\quad + \frac{223480058219178036298172060362961185 3225}{133851996944282187511619393058410334130 88}\,b\Big)^2 + \frac{212159700340573881370359421737581680593}{22395692938901877143549014896580 61}\Big(b^2 \\
&\quad - \frac{40248036588293144056679077001926201625}{121234114480327932211633955278618103196}\Big)^2 + \frac{9419358853851710805077475715972917781209 1894697}{18203871584422377501580237455943805441799 68}\Big(b\Big)^2 \\
&\quad + \frac{10120785250613810234800774009900874989594890 65}{14014663633925908963664885230208252729457 6}\Big(1\Big)^2 \\
&> 0,
\end{aligned}
$$

where the second expression (i.e. the SOS decomposition) is obtained by an application of `'SumsOfSquares'` to the first expression. □

Next, we will verify (7.1) for the case $n = 4, m_1 = 2, m_2 = 1, m_3 = 1, m_4 = 1$.

Theorem 7.2.2 *For any centered Gaussian random vector* (X_1, X_2, X_3, X_4),

$$E[X_1^4X_2^2X_3^2X_4^2] \geq E[X_1^4]E[X_2^2]E[X_3^2]E[X_4^2]. \tag{7.7}$$

The equality holds if and only if X_1, X_2, X_3, X_4 *are independent.*

Proof. Let X_4 be a linear combination of X_1, X_2 and X_3, and let U_1, U_2 and U_3 be independent standard Gaussian random variables. Then, without loss of generality, we may consider 3 constructions of $X_1, \ldots, X_4$ in terms of U_1, U_2, U_3:

Case 1:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + cU_2 + U_3,\ X_4 = dU_1 + eU_2 + U_3,\quad a, b, c, d, e \in \mathbb{R}.$$

Case 2:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + U_2,\ X_4 = cU_1 + dU_2 + U_3,\quad a, b, c, d \in \mathbb{R}.$$

Case 3:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + U_2,\ X_4 = cU_1 + U_2,\ \ a, b, c \in \mathbb{R}.$$

By Lemma 7.1.2, we need only show that, in each of these 3 cases, $F_{2,1,1,1} > 0$.

Case 1:

$$\begin{aligned} F_{2,1,1,1}(a,b,c,d,e) = {} & 6 + 42\,a^2 + 12\,b^2 + 102\,a^2\,b^2 + 60\,a\,b\,c + 6\,c^2 + 12\,a^2\,c^2 + 60\,b\,d \\ & + 420\,a^2\,b\,d + 120\,a\,c\,d + 12\,d^2 + 102\,a^2\,d^2 + 102\,b^2\,d^2 + 942\,a^2\,b^2\,d^2 \\ & + 420\,a\,b\,c\,d^2 + 42\,c^2\,d^2 + 102\,a^2\,c^2\,d^2 + 120\,a\,b\,e + 36\,c\,e + 60\,a^2\,c\,e \\ & + 60\,a\,d\,e + 420\,a\,b^2\,d\,e + 180\,b\,c\,d\,e + 420\,a^2\,b\,c\,d\,e + 180\,a\,c^2\,d\,e + 6\,e^2 \\ & + 12\,a^2\,e^2 + 42\,b^2\,e^2 + 102\,a^2\,b^2\,e^2 + 180\,a\,b\,c\,e^2 + 42\,c^2\,e^2 + 42\,a^2\,c^2\,e^2 \end{aligned}$$

$$\begin{aligned} = {} & 942\left(a\,b\,d + \frac{1169}{7536}\,a\,c\,e + \frac{1159}{7536}\,c\,d + \frac{1159}{7536}\,b\,e + \frac{1217}{7536}\,a\right)^2 + 102\left(a\,b + \frac{463}{816}\,a\,d + \frac{29}{136}\,c + \frac{7}{51}\,e\right)^2 \\ & + 102\left(\frac{511}{816}\,a\,c\,d + a\,b\,e + \frac{521}{816}\,b\,d + \frac{71}{136}\,c\,e + \frac{11}{68}\right)^2 + \frac{451487}{6528}\left(a\,d + \frac{10830}{451487}\,c + \frac{90128}{451487}\,e\right)^2 \\ & + \frac{404735}{6528}\left(a\,c\,d + \frac{521}{1327}\,b\,d + \frac{426}{1327}\,c\,e + \frac{132}{1327}\right)^2 + \frac{269975}{5308}\left(b\,d + \frac{25609}{107990}\,c\,e + \frac{10397}{53995}\right)^2 \\ & + \frac{1188815}{60288}\left(\frac{860713}{1188815}\,a\,c\,e + \frac{834623}{1188815}\,c\,d + b\,e + \frac{134377}{1188815}\,a\right)^2 + \frac{81700883}{4755260}\left(\frac{17007629}{163401766}\,a\,c\,e\right. \\ & \left. + \frac{6315719}{163401766}\,c\,d + a\right)^2 + 12\left(\frac{1}{4}\,a\,c + a\,e + \frac{31}{96}\,b + \frac{11}{16}\,d\right)^2 + \frac{45}{4}\left(a\,c + \frac{233}{360}\,b + \frac{29}{180}\,d\right)^2 \\ & + \frac{13038091541}{1307214128}\left(\frac{116381935}{277406203}\,a\,c\,e + c\,d\right)^2 + \frac{7828043977}{1109624812}\left(a\,c\,e\right)^2 + \frac{2173}{360}\left(b + \frac{9829}{17384}\,d\right)^2 + \frac{855377}{172784}\left(c\,e\right. \\ & \left. + \frac{230438}{855377}\right)^2 + \frac{13706281}{3337728}\left(d\right)^2 + \frac{597018}{451487}\left(c + \frac{782927}{2388072}\,e\right)^2 + \frac{11273665}{9552288}\left(e\right)^2 + \frac{403986}{855377}\left(1\right)^2 \\ > {} & 0, \end{aligned}$$

Case 2:

$$\begin{aligned} F_{2,1,1,1}(a,b,c,d) = {} & 6 + 12\,a^2 + 60\,a\,b + 12\,b^2 + 102\,a^2\,b^2 + 42\,c^2 + 102\,a^2\,c^2 + 420\,a\,b\,c^2 \\ & + 102\,b^2\,c^2 + 942\,a^2\,b^2\,c^2 + 180\,a\,c\,d + 180\,b\,c\,d + 420\,a^2\,b\,c\,d \\ & + 420\,a\,b^2\,c\,d + 42\,d^2 + 42\,a^2\,d^2 + 180\,a\,b\,d^2 + 42\,b^2\,d^2 + 102\,a^2\,b^2\,d^2 \end{aligned}$$

$$\begin{aligned} = {} & 942\left(a\,b\,c + \frac{583}{3768}\,a\,d + \frac{583}{3768}\,b\,d + \frac{583}{3768}\,c\right)^2 + 102\left(a\,b + \frac{43}{204}\right)^2 + 102\left(a\,b\,d + \frac{257}{408}\,a\,c\right. \\ & \left. + \frac{257}{408}\,b\,c + \frac{25}{48}\,d\right)^2 + \frac{100415}{1632}\left(a\,c + \frac{257}{665}\,b\,c + \frac{85}{266}\,d\right)^2 + \frac{69611}{1330}\left(b\,c + \frac{425}{1844}\,d\right)^2 \\ & + \frac{293135}{15072}\left(a\,d + \frac{215891}{293135}\,b\,d + \frac{215891}{293135}\,c\right)^2 + 12\left(\frac{17}{24}\,a + b\right)^2 + \frac{10435033}{1172540}\left(b\,d\right. \\ & \left. + \frac{215891}{509026}\,c\right)^2 + \frac{29721597}{4072208}\left(c\right)^2 + \frac{287}{48}\left(a\right)^2 + \frac{77709}{14752}\left(d\right)^2 + \frac{599}{408}\left(1\right)^2 \\ > {} & 0, \end{aligned}$$

Case 3:

$$F_{2,1,1,1}(a,b,c) = 42 + 42\,a^2 + 180\,a\,b + 42\,b^2 + 102\,a^2\,b^2 + 180\,a\,c + 180\,b\,c + 420\,a^2\,b\,c + 420\,a\,b^2\,c + 42\,c^2 + 102\,a^2\,c^2 + 420\,a\,b\,c^2 + 102\,b^2\,c^2 + 942\,a^2\,b^2\,c^2$$

$$= 942\left(a\,b\,c + \frac{593}{3768}\,a + \frac{593}{3768}\,b + \frac{593}{3768}\,c\right)^2 + 102\left(a\,b + \frac{247}{408}\,a\,c + \frac{247}{408}\,b\,c + \frac{109}{204}\right)^2 + \frac{105455}{1632}\left(a\,c + \frac{247}{655}\,b\,c + \frac{218}{655}\right)^2 + \frac{72611}{1310}\left(b\,c + \frac{109}{451}\right)^2 + \frac{281375}{15072}\left(a + \frac{183407}{281375}\,b + \frac{183407}{281375}\,c\right)^2 + \frac{3021083}{281375}\left(b + \frac{183407}{464782}\,c\right)^2 + \frac{8426457}{929564}\left(c\right)^2 + \frac{2241}{902}\left(1\right)^2$$
$$> 0,$$

where in each case, the first expression is obtained by (7.4) and the second expression (i.e. the SOS decomposition) is obtained by an application of `'SumsOfSquares'` to the first expression. □

Remark 7.2.3 *To establish the inequality (7.7), we need only consider Case 1. However, to show that the equality sign holds if and only if X_1, X_2, X_3, X_4 are independent, we must check all three cases.*

7.3 The SOS method with one exponent unbounded

In this section, we demonstrate an even more powerful application of the SOS method to be used when one exponent is unknown. In particular, we extend Theorems 7.2.1 and 7.2.2 to the case where m_1 is unbounded by exploiting the fact that verifying these GPIs can be reduced to proving the non-negativity of a multivariate polynomial with variables $a, b, ..., m_1$.

Theorem 7.3.1 *Let $m \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3),*

$$E[X_1^{2m} X_2^6 X_3^4] \geq E[X_1^{2m}]E[X_2^6]E[X_3^4].$$

The equality holds if and only if X_1, X_2, X_3 are independent.

Proof. Let X_3 be a linear combination of X_1 and X_2, and let U_1 and U_2 be independent standard Gaussian random variables. Then, without loss of generality, we may write

$$X_1 = U_1,\ \ X_2 = aU_1 + U_2,\ \ X_3 = bU_1 + U_2,\ \ a, b \in \mathbb{R}.$$

By Lemma 7.1.2, we need only show that $F_{m,3,1}$ and $F_{m,3,2}$ are strictly positive. By Theorem 4.1.6, $F_{m,3,1} > 0$. We will complete the proof by giving an SOS decomposition of $F_{m,3,2}$. By direct calculation (expanding and taking expectations), then letting $m = p^2 + 1$, we have that for any $p \in \mathbb{R}$,

$$\begin{aligned}\frac{F_{m,3,2}(a,b,p)}{2(2m-1)!!} = {} & 450 + 2295\,a^2 + 1620\,a^4 + 135\,a^6 + 3780\,a\,b + 9000\,a^3\,b + 3780\,a^5\,b \\ & + 900\,b^2 + 9990\,a^2\,b^2 + 14040\,a^4\,b^2 + 2790\,a^6\,b^2 + 2700\,a\,b^3 + 12600\,a^3\,b^3 \\ & + 11340\,a^5\,b^3 + 90\,b^4 + 2295\,a^2\,b^4 + 7020\,a^4\,b^4 + 5175\,a^6\,b^4 + 1575\,a^2\,p^2 \\ & + 1800\,a^4\,p^2 + 213\,a^6\,p^2 + 2520\,a\,b\,p^2 + 9600\,a^3\,b\,p^2 + 5112\,a^5\,b\,p^2 \\ & + 630\,b^2\,p^2 + 10800\,a^2\,b^2\,p^2 + 19170\,a^4\,b^2\,p^2 + 4464\,a^6\,b^2\,p^2 + 2880\,a\,b^3\,p^2 \\ & + 17040\,a^3\,b^3\,p^2 + 17856\,a^5\,b^3\,p^2 + 120\,b^4\,p^2 + 3195\,a^2\,b^4\,p^2 \\ & + 11160\,a^4\,b^4\,p^2 + 9129\,a^6\,b^4\,p^2 + 450\,a^4\,p^4 + 90\,a^6\,p^4 + 2400\,a^3\,b\,p^4 \\ & + 2160\,a^5\,b\,p^4 + 2700\,a^2\,b^2\,p^4 + 8100\,a^4\,b^2\,p^4 + 2472\,a^6\,b^2\,p^4 + 720\,a\,b^3\,p^4 \\ & + 7200\,a^3\,b^3\,p^4 + 9888\,a^5\,b^3\,p^4 + 30\,b^4\,p^4 + 1350\,a^2\,b^4\,p^4 + 6180\,a^4\,b^4\,p^4 \\ & + 6020\,a^6\,b^4\,p^4 + 12\,a^6\,p^6 + 288\,a^5\,b\,p^6 + 1080\,a^4\,b^2\,p^6 + 576\,a^6\,b^2\,p^6 \\ & + 960\,a^3\,b^3\,p^6 + 2304\,a^5\,b^3\,p^6 + 180\,a^2\,b^4\,p^6 + 1440\,a^4\,b^4\,p^6 + 1880\,a^6\,b^4\,p^6 \\ & + 48\,a^6\,b^2\,p^8 + 192\,a^5\,b^3\,p^8 + 120\,a^4\,b^4\,p^8 + 280\,a^6\,b^4\,p^8 + 16\,a^6\,b^4\,p^{10}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2461933}{196}\Big(-\frac{46403}{29543196}\,a^3b^2p^5 + \frac{381808}{2461933}\,a^3b^2p^3 + \frac{12180469}{29543196}\,a^3b^2p - \frac{8183}{4923866}\,a^3p^3 + \frac{188501}{2461933}\,a^2b\,p^3 + \frac{57820}{2461933}\,a\,b^2p^3 + \frac{1233281}{29543196}\,a^3p + a^2b\,p \\
&\quad + \frac{837655}{2461933}\,a\,b^2p + \frac{8357783}{39390928}\,a\,p + \frac{8216565}{91091521}\,b\,p\Big)^2 + 10899\Big(-\frac{2665}{87192}\,a^3b\,p^3 + \frac{29641}{726600}\,a^2b^2p^3 + \frac{233017}{523152}\,a^3b\,p + a^2b^2p + \frac{297779}{4272408}\,a^2p + \frac{1193}{4844}\,a\,b\,p \\
&\quad - \frac{193}{7266}\,b^2p\Big)^2 + \frac{41695}{4}\Big(\frac{107}{83390}\,a^3b^2p^4 + \frac{37177}{500340}\,a^3b^2p^2 + \frac{29011}{83390}\,a^3b^2 - \frac{3259}{500340}\,a^3p^2 + \frac{293859}{4086110}\,a^2b\,p^2 - \frac{1574}{41695}\,a\,b^2p^2 + \frac{118}{8339}\,a^3 + a^2b \\
&\quad + \frac{7844}{41695}\,a\,b^2 + \frac{15677}{41695}\,a + \frac{2921}{41695}\,b\Big)^2 + \frac{70071801709219}{7090367040}\Big(-\frac{245635624885}{70071801709219}\,a^3b^2p^5 - \frac{11198510281680}{70071801709219}\,a^3b^2p^3 + a^3b^2p + \frac{1830613217106}{350359008546095}\,a^3p^3 \\
&\quad - \frac{2274370721580}{70071801709219}\,a^2b\,p^3 - \frac{1799591724552}{350359008546095}\,a\,b^2p^3 - \frac{86820255707345}{1191220629056723}\,a^3p - \frac{13009331255460}{70071801709219}\,a\,b^2p - \frac{697867593895245}{13734073135006924}\,a\,p \\
&\quad - \frac{81360974520540}{2592656663241103}\,b\,p\Big)^2 + \frac{172103}{20}\Big(-\frac{420}{172103}\,a^3b\,p^4 - \frac{3227}{1721030}\,a^2b^2p^4 - \frac{161515}{2065236}\,a^3b\,p^2 + \frac{16035}{172103}\,a^2b^2p^2 + \frac{81745}{344206}\,a^3b + a^2b^2 \\
&\quad - \frac{208955}{16866094}\,a^2p^2 + \frac{4780}{172103}\,a\,b\,p^2 + \frac{1305}{344206}\,b^2p^2 + \frac{10055}{172103}\,a^2 + \frac{191845}{344206}\,a\,b - \frac{9953}{344206}\,b^2 + \frac{10375}{172103}\Big)^2 + \frac{182859069551}{24016320}\Big(-\frac{6418879954}{182859069551}\,a^3b^2p^4 \\
&\quad + a^3b^2p^2 - \frac{41161024698}{182859069551}\,a^3b^2 - \frac{20615021137}{914295347755}\,a^3p^2 + \frac{154694373066}{1280013486857}\,a^2b\,p^2 - \frac{40319988936}{914295347755}\,a\,b^2p^2 + \frac{22432660800}{3108604182367}\,a^3 - \frac{12775700256}{182859069551}\,a\,b^2 \\
&\quad + \frac{11681404428}{8960094407999}\,a - \frac{8345914044}{182859069551}\,b\Big)^2 + \frac{98097336461}{17210300}\Big(-\frac{14512945525}{1177168037532}\,a^3b\,p^4 + \frac{39929834765}{3335309439674}\,a^2b^2p^4 + \frac{191874337775}{392389345844}\,a^3b\,p^2 + a^2b^2p^2 \\
&\quad - \frac{221498760325}{1177168037532}\,a^3b + \frac{181601825675}{9613538973178}\,a^2p^2 + \frac{24812335375}{196194672922}\,a\,b\,p^2 - \frac{7224882875}{1667654719837}\,b^2p^2 - \frac{111802291925}{4806769486589}\,a^2 + \frac{57444211575}{196194672922}\,a\,b \\
&\quad - \frac{403439350}{98097336461}\,b^2 + \frac{267399874525}{3629601449057}\Big)^2 + \frac{1881745972159}{426892032}\Big(-\frac{102378079626}{47043649303975}\,a^3b\,p^3 - \frac{6247680732882}{47043649303975}\,a^2b^2p^3 + a^3b\,p + \frac{40590877543374}{92205552635791}\,a^2p \\
&\quad + \frac{371809874700}{13172221805113}\,a\,b\,p + \frac{35462399976}{1881745972159}\,b^2p\Big)^2 + \frac{1290111735374379}{365718139102}\Big(-\frac{1254213486987157}{38703352061231370}\,a^3b^2p^4 + a^3b^2 + \frac{55648064894758831}{219318995013644300}\,a^3p^2 \\
&\quad - \frac{8137138834833332681}{75858570040013485200}\,a^2b\,p^2 - \frac{919990297705031}{21501862256239650}\,a\,b^2p^2 - \frac{16733411228329253}{219318995013644430}\,a^3 - \frac{2332609290373289}{5160446941497516}\,a\,b^2 - \frac{72218787593927536}{316077375166722855}\,a \\
&\quad - \frac{5342404109573243}{25802234707487580}\,b\Big)^2 + \frac{281259138332925112160414689}{89209678367055858595200}\Big(\frac{448365562781576358871926668}{281259138332925112160414689}\,a^3b^2p^4 + \frac{237991835996651005758467525 4}{239070267582986345336352485 65}\,a^3p^2 \\
&\quad + a^2b\,p^2 + \frac{512872285198992142407483276}{1406295691664625560802073445}\,a\,b^2p^2 - \frac{246842735840239738068445884}{478140535165972690672704971 3}\,a^3 - \frac{419647754541046390803597 30}{281259138332925112160414689}\,a\,b^2 \\
&\quad + \frac{823265119693667706118614 66}{281259138332925112160414689}\,a + \frac{534358664600268384534896826}{1040658811831822914993534349 3}\,b\Big)^2 + \frac{724171982368916897}{282520329007680}\Big(-\frac{78866109211794023}{3620859911844584485}\,a^3b\,p^4 \\
&\quad - \frac{169675800886557258}{12310923700271587249}\,a^2b^2p^4 - \frac{169055740849158255}{12310923700271587249}\,a^3b\,p^2 + a^3b - \frac{2958248375541023850}{35484427136076927953}\,a^2p^2 - \frac{3538484271647279130}{35484427136076927953}\,a\,b\,p^2 \\
&\quad - \frac{367156853667452 40}{12310923700271587249}\,b^2p^2 + \frac{22124076308603404500}{35484427136076927953}\,a^2 - \frac{49371151709178630}{724171982368916897}\,a\,b - \frac{1219853157005100}{724171982368916897}\,b^2 \\
&\quad - \frac{2201018396447410620}{26794363347649925189}\Big)^2 + \frac{26212746613691772978097}{11586751717902670352}\Big(-\frac{11755791135165677987753 7}{64221229203544843796337 65}\,a^3b\,p^4 + \frac{1091778806240355865724 49}{2228083462163800703138245}\,a^2b^2p^4 \\
&\quad - \frac{28860678489162367020455027}{13101130757523148134452880 6}\,a^3b\,p^2 - \frac{576983703856223757853069}{5137698336283587503707012}\,a^2p^2 + \frac{77738012351906108282953 10}{47523709610623184409289861}\,a\,b\,p^2 \\
&\quad - \frac{43531084034378371097507}{2228083462163800703138245}\,b^2p^2 + \frac{518731965484307539879 38}{18348922629584241084667 9}\,a^2 + a\,b + \frac{251870594346332635453 1}{26212746613691772978097}\,b^2 \\
&\quad + \frac{27504586918885906005756 2}{96987162470659560018958 9}\Big)^2 + \frac{53631965997981898754}{23881238132669869}\Big(\frac{10635346068515502628 7}{42905572798385519003 2}\,a^3b\,p^3 + \frac{47411312002258617407 61}{10726393199596379750800}\,a^2b^2p^3 \\
&\quad + \frac{1420797638906740449692 5}{2402712076709589064179 2}\,a^2p + a\,b\,p + \frac{369985339153504062 03}{268159829989909493770}\,b^2p\Big)^2 + \frac{1225720890883722229}{630646215382971}\Big(a^3b^2p^3 - \frac{227202054695158443}{12257208908837222290}\,a^3b^2p^5 \\
&\quad + \frac{1430298266691753873}{83349020580093111572 0}\,a^3p^3 + \frac{249545321744126943}{19611534254139555664}\,a^2b\,p^3 - \frac{1412535516833558454 3}{41674510290046555786 0}\,a\,b^2p^3 - \frac{13824776582094829637 7}{10418627572511638946 50}\,a^3p \\
&\quad - \frac{1063149061912372040 67}{2451441781767444458 00}\,a\,b^2p - \frac{4787605614525387063 3}{9609651784528382275 36}\,a\,p - \frac{8296396344452300805}{9070334592539544494 6}\,b\,p\Big)^2 + \frac{9408995393797413663945824457325061}{5456620960508391197999624769900}\Big(\\
&\quad - \frac{350828535653402740933920328198325}{376359815751896546557832978293002 44}\,a^3b\,p^4 - \frac{264206950012992941237276145096437 5}{376359815751896546557832978293002 44}\,a^2b^2p^4 \\
&\quad + a^3b\,p^2 + \frac{333261827005163898518330473543885 25}{301087852601517237246266382634401952}\,a^2p^2 - \frac{343825436303795505990899637151088 25}{348132829570504305565995504921027257}\,a\,b\,p^2 \\
&\quad + \frac{309060196300930354417419284937655}{376359815751896546557832978293002 44}\,b^2p^2 + \frac{101118483817605990989094637303180925}{376359815751896546557832978293002 44}\,a^2 + \frac{239385063650020948927997733088 75}{101718869122134201772387291430541 2}\,b^2 \\
&\quad - \frac{454164830667551963943552068421383175}{557012527312806888905592807873643611 2}\Big)^2 + \frac{897565122952924936950449636848 7}{562518276665850224320829378 0}\Big(-\frac{719450408374770344006189110565 9}{269269536885877481085134891054610}\,a^3b^2p^4 \\
&\quad + \frac{698906413627722036900956744291156}{373835873709893236239862273747483 55}\,a^3p^2 + \frac{221365135598121496382892455517 87}{897565122952924936950449636848 70}\,a\,b^2p^2 - \frac{669195416391111346459662135871 5}{152586070901997239281576438264279}\,a^3 \\
&\quad + a\,b^2 + \frac{154044808862422276693276218417 95}{351845528197546575284576257644690 4}\,a + \frac{337263748895723350866231041032825}{664198190985164453343332731268038}\,b\Big)^2 + \frac{569931772878771325112087 41}{490288356353488891600 00}\Big(a\,b^2p \\
&\quad - \frac{865364627242242832841612 20}{290665204168173375807164579 1}\,a^3b^2p^5 + \frac{919138351456343807483206 45}{474753166808016513818368812 53}\,a^3p^3 - \frac{194262980186734439069317 5}{162837649393934664317739 26}\,a^2b\,p^3 \\
&\quad + \frac{946885743036362774926648 90}{968884013893911252690548597}\,a\,b^2p^3 - \frac{584907206707934744989138728 4}{474753166808016513818368812 53}\,a^3p - \frac{724494446808429120253409792 5}{206657260845842482485642894866}\,a\,p \\
&\quad + \frac{577737201220275968873509500}{210874755965145390291472341 7}\,b\,p\Big)^2 + \frac{636404863726500276579335103458310528 13}{599471382261619873194352909427292 80}\Big(\frac{618969801136233110624963188166810006 0}{190921459117950082973800531037493158439}\,a^3b^2p^5 \\
&\quad + \frac{161395857438882863823635771555608693 2}{636404863726500276579335103458310528 13}\,a^3p^3 + \frac{418738997799009453539775739657958987 70}{636404863726500276579335103458310528 13}\,a^2b\,p^3 \\
&\quad + \frac{318259569856788654338942826117002255662}{108188826833505047018486967587912789782 1}\,a\,b^2p^3 - \frac{275581975118119595227601337446431540897 0}{140645474883556561124033057864286626716 73}\,a^3p + a\,p \\
&\quad + \frac{337294616383143401018744158202571089 20}{636404863726500276579335103458310528 13}\,b\,p\Big)^2 + \frac{219605904998627431295491364061}{255288158150393838069040000}\Big(\frac{315464086652016426885241091137}{658817714995882293886474092183}\,a^3b\,p^3 \\
&\quad + a^2b^2p^3 - \frac{220649831012248101513653499047 5}{286951715864873176892775382373 04}\,a^2p - \frac{460845432252523937695968761 20}{219605904998627431295491364061}\,b^2p\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{751298069739776111324182015106815207080556113}{1051520590138866293781255650802756146623706}\Bigg(\frac{1259576947571548888826894805716902024509294379}{10017307596530348150989093534757536094407414840}\,a^3b\,p^4\\
&+\frac{166918854479147820517146353370037156333739946 53}{6010384557918208890593456120854521656644448904 0}\,a^2b^2p^4+\frac{6868670651718458722983859238852927535461291677}{1602769215444855704158254965561205775105186374 4}\,a^2p^2\\
&+a\,b\,p^2+\frac{5380260843731654887855542070217310749754641 39}{6010384557918208890593456120854521656644448904}\,b^2p^2-\frac{7046972944780067201238165098772217493260539121}{2404153823167283556237382448341808662657779561 6}\,a^2\\
&-\frac{2521771873821077005927768728351947837906418779}{3005192278959104445296728060427260828322224452 0}\,b^2+\frac{7946375997650306751578466826267626390558497827}{2404153823167283556237382448341808662657779561 6}\Bigg)^2\\
&+\frac{132662645389010932413304395114633357161}{194974125872985827926077779809734912}\Bigg(\frac{172157153865929309787097560237174342664 34}{298490952125274597929934889007925053612 25}\,a^3b\,p^3+a^2p\\
&-\frac{231023089838336117344966735522934217196}{1989939680835163986199565926719500357415}\,b^2p\Bigg)^2+\frac{237215090816801266864395381046926131941 7}{394066991581252164318725408562053248 0}\Bigg(a\\
&-\frac{7224668581148161324290528330833614546 28}{7116452724504038005931861431407783958251}\,a^3b^2p^4-\frac{1788269034078999459801323438819254803189 68}{3669717454935915598392196544795947261138099}\,a^3p^2\\
&+\frac{4010373078726940161739654321501343546996}{8776958360221646873982629098736266881842 9}\,a\,b^2p^2-\frac{763049967021614686210405363839675392463 2}{4032656543885621536694721477797744243008 9}\,a^3\\
&+\frac{4812523797773549441155268401777899391380}{8776958360221646873982629098736266881842 9}\,b\Bigg)^2+\frac{446320302804558864290705474002225207256098724366 73}{966123691501397779653637898043394577018857217 50}\Bigg(a\,b^2p^2\\
&-\frac{769234113654606139865326123252153059330050200917 13}{178528121121823545716282189600890082902439489746692}\,a^3b^2p^4+\frac{805039349079600695791544916353094327301134808218347}{2843060328865039965531793869394174570221348874216070 1}\,a^3p^2\\
&+\frac{3395782447168338760733203573364905836866229486755}{104141403987730401667831277267185881693089702352237}\,a^3+\frac{1333938366007804057089027197196527174315962138880 0}{446320302804558864290705474002225207256098724366 73}\,b\Bigg)^2\\
&+\frac{5798854105401307440575336297554933781480328659771088743}{1847159465415887301928305682709978431693225192768512 0}\Bigg(\frac{3027712794814245608843724682535620428804385607420 7028}{5798854105401307440575336297554933781480328659771088743}\,a^3b\,p^4\\
&+\frac{5782049733604745109787099157079690598436073253290462402}{985805197918222264897807170584338742851655872161085086 31}\,a^2b^2p^4\\
&-\frac{525063018878492683857706609932970458212390107047712088 5}{150770206740433993454958743736428278318488545154048307318}\,a^2p^2\\
&+\frac{16821799130622556325370384188403513951111124434919485295 6}{36474792322974223801218865311620533485511267269960148193 47}\,b^2p^2+a^2\\
&-\frac{2863732754223278884840005573314180044783970413145596 12}{5798854105401307440575336297554933781480328659771088743}\,b^2-\frac{554025412012552882233933470527110694752783861012465623 53}{21455760189984837530128744300953254991477216041153028349 1}\Bigg)^2\\
&+\frac{6023346958102712039190062093270040252063816812126410557433925809861}{28913605404605437364487949936098114669496215305584495511450013696}\Bigg(a^2p^2\\
&+\frac{193280308201686379299841666534655221642273097735852449932155458773 6}{6023346958102712039190062093270040252063816812126410557433925809861}\,a^3b\,p^4\\
&-\frac{196690878624138143109711574034444250359950552362800932488019174160 8}{301167347905135601959503104663502012603190840606320527871696290493 05}\,a^2b^2p^4\\
&+\frac{163133385022351151733609322371234731365888488183905033480594310165 6}{187558675081515142210423220627072045472680236377599615872571254178345}\,b^2p^2\\
&+\frac{164250264269349073981479303374507844208541859793506504207049023090168}{111431918724900172725016148725495744663180611024338595312527627482428 5}\,b^2\\
&-\frac{4140160673746606165742599183199372844824760745830814670694335652293 98}{111431918724900172725016148725495744663180611024338595312527627482428 5}\Bigg)^2\\
&+\frac{204173885286736969155087340782345215909798682252 63}{119407108045100265632312872315209014648092782928}\Bigg(\frac{278739558055840991519654788478814962652349348860 2}{183756496758063272239578606704110694318818814027367}\,a^3b^2p^5\\
&-\frac{772960345519102603146661898113933611086632403522}{6805796176224565638502911359411507196993289408421}\,a^3p^3-\frac{1950870272617726351443232491042466090559368814603}{204173885286736969155087340782345215909798682252 63}\,a^2b\,p^3\\
&+\frac{1153578369605926458091217357612762660459762317997}{340289808811228281925145567970575359849664470421 05}\,a\,b^2p^3+a^3p+\frac{2019305919977093630271343170846884619431723059891579}{1510886751121853571747646321789354597732510248669462 0}\,b\,p\Bigg)^2\\
&+\frac{975840908493215452011616978230731711441435479919041136 21}{6069956118142000554353594446430262818682942651387528 00}\Bigg(a^3b^2p^4\\
&-\frac{318104825423653875812296734091051913949016886942961959180 80}{621610658710178242931400015132976100188194400708429203765 77}\,a^3p^2\\
&+\frac{1006018235753527927148487466661960944067037160967221844 64}{683088635945250816408131884761512198009004835943328795347}\,a^3+\frac{587538523063443535832416944491098895488438015055366215340}{361061136142489717244298281945370733233331127570045220397 7}\,b\Bigg)^2\\
&+\frac{2547979952487198785212709307211844770932994888524125459}{16007232606480178381758847517335864927328216688606192}\Bigg(\frac{6292438296998353428279666999940232914543435522598175947 6}{389840932730541414137544524003412249952748217944191195227}\,a^3b^2p^5\\
&+\frac{2201488955207890396175570299050169861913938160204508176 7}{2165782959614118967430802911130068055293045655245506640 15}\,a^3p^3+a^2b\,p^3\\
&+\frac{8727525970574264925933028607383516156041413338036393263 3}{4331565919228237934861605822260136110586091310491013280 30}\,a\,b^2p^3\\
&-\frac{9403912318199261971375257692393709136891230429950032823483}{3205358780228896071797588308472500721833707569763349827422 0}\,b\,p\Bigg)^2\\
&+\frac{1099497882555389016616929857482383397985239609783}{730705850802672215732480608291400531242788000 0}\Bigg(a^3b\,p^3+\frac{8590683514112276150746077669097583411718303463050}{4068142165454939361482640472684818572545386556197 1}\,b^2p\Bigg)^2\\
&+\frac{870934070277681879777500987129841029272919024547310237028790711 83}{806468269105590251664273218411681181613360824552458816579375200}\Bigg(b\,p\\
&-\frac{831257129467483701216157930059731328868285473355929242085516178 40}{783840663249913691799750888416856926345627122092579213325911640647}\,a^3b^2p^5\\
&+\frac{708091356514018127524294496942354079001077521241902749004131948 1}{174186814055536375955500197425968205854583804909462047405758142366}\,a^3p^3\\
&+\frac{270622143972119826134314027023228971339653128390218279989585760815}{348373628111072751911000394851936411709167609818924094811516284732}\,a\,b^2p^3\Bigg)^2
\end{aligned}
$$

$$+\frac{3626429004572319085753795422317372679543247056461954508217084358 7}{48366090868103484055777778100538871180027587408967395354544335 0}\Bigg(a^3$$

$$-\frac{7875000105462815664557044530875488162146430465438705584698594227565}{1980030236496486220821572300585285483030612892828227161486528059850 2}a^3p^2$$

$$+\frac{1026777627345518238625370370074903889261048605992655266995572575265}{8050672390150548370373425837544567348586008465345539008241927276314}b\Bigg)^2$$

$$+\frac{4612456516990299011626286929285025079287525560142295386215021494741165825137 3}{824352720463386196367795989248493895120777369755011430114035281352837256680}\Bigg(a^3p^2$$

$$+\frac{909430248736402465331554525718246874903474929005477255592135773072634604828 83}{368996521359223920930102954342802006343002044811383630897201719579293266010984}b\Bigg)^2$$

$$+\frac{116546146207272521398437649456424314040744052124686424683325885779277934 23}{21759340886146047241574099311938020410580538233806658138730056988122862 5}\Bigg(a^2b^2p^4$$

$$+\frac{3247955943684499047856151546250849813470530300608484812572503274440601115 5}{699276877243635128390625896738545884244464312748118548099955314675667605 38}a^3b\,p^4$$

$$-\frac{40668347617625912109693480264518557280996056768455837008104257741174086062 5}{86244148193381665834843860597753992390150598572267954265661155476665671330 2}b^2p^2$$

$$+\frac{9492160981712982744111424250920078932583306744948235015932050827742767665 85}{3449765927735266633393754423910159695606023942890718170626446219066626853208}b^2$$

$$-\frac{2932964600106421094570667692811376749159303863835736746419697511042555299635}{1379906371094106653357501769564063878242409577156287268250578487626650741283 2}\Bigg)^2$$

$$+\frac{98646021534217902642439074825525876002685390632749029556209866761808199298924218932521 9}{18913049541014711387883961761903880243832853287543600099743197129655724894968748874240}\Bigg(b\Bigg)^2$$

$$+\frac{12188036789943262576845007782131009030329310053570090 11}{25588614220711568583725808573187508821310481438479759}\Bigg(b^2p\Bigg)^2$$

$$+\frac{1916626494093696300812313131704954609924467023454863298760201687830882752085814460 7}{4084522858438555693938205237909629079597532348382610314021712323374886194198272 00}\Bigg(1$$

$$+\frac{584311941674477796520267738474306974837386448845579226424762316625748544019304120 0}{574987948228108890243693939511486382977340107036458989628060506349264825625744338 21}a^3b\,p^4$$

$$-\frac{929256451648865700992701939228920543140348117589146307366672463711807200768703774 24}{325826503995928371138093232389842283687159393987326760789234286931250067854588458319}b^2p^2$$

$$-\frac{908203852527895715321838424232358926942421712285691445460394597329762572656226261 2}{1916626494093696300812313131704954609924467023454863298760201687830882752085814460 7}b^2\Bigg)^2$$

$$+\frac{235785679163682336185210833470212615176350437974854938356037405465351540137197886919547 86}{65596541760356755895301416932602071524664883877742696400067902766011962190136999917457 5}\Bigg(b^2$$

$$-\frac{346679262621556250751885708185364173302659574615670367990160664199558166306587057116409 77}{565885629992837606844506000328510276423241051139651852054489773116843696329274928606914864}a^3b\,p^4$$

$$-\frac{379790205139115770223371942380753605549056103972285505555244454989200424844212580689344 5}{246668095125083059393759025784222428184448968895830978166477759340990622660506855862352699 2}b^2p^2\Bigg)^2$$

$$+\frac{2598555810676178916779140148947017846855205084513645624441686884500531149 7}{1610879656385600404836465825795353967743191027802705014408451300600768000}\Bigg(a\,b^2p^3$$

$$-\frac{1014301109629513360454932873827289111748771704034229991926466049222310352 00}{2338700229608561025101226134052316062169684576062281061997518196050478034 73}a^3b^2p^5$$

$$+\frac{762977954223280987438738328446777740832088159677043338195949920309412055 66}{1273292347231327669221778672984038744959050491411686355976426573405260263353}a^3p^3\Bigg)^2$$

$$+\frac{1247026159056318920889118972193317004988891724553734109809904660526782706729276279}{1803109292914283112384960778812697266736511400888089048698217670599189058934183 30}\Bigg(a^3p^3$$

$$-\frac{1307348510000282560934993074207227911352222060016099574684350921618463121254000815 9}{2244647086301374057600414149947970608980005104196721397657828388948208872112697302 2}a^3b^2p^5\Bigg)^2$$

$$+\frac{66353643877134205004971671183473765682665660329591607114997757767589484541872517863443777595410 93}{11481413155692116082541907614152417142275257062253487097368737862857495223559521129690687369632 00}\Bigg(b^2p^2$$

$$+\frac{11033434426734849202372000295956059607726548933570936180092623498588194212727967314509876582015657}{19906093163140261501491501355042129704799698098877482134499327330276845362561755359033133278623279}a^3b\,p^4\Bigg)^2$$

$$+\frac{388681935647012442251946102542793082043360840320090931730502746670863314264566630906737289084953671697}{172519474080548933012926345077031790774930716856938178498994170195732659808868546444953821748068418000}\Bigg(a^3b\,p^4\Bigg)^2$$

$$+\frac{2629929480562269930314071400782243373877386514044892496969174184208223685867430351966437}{1167665414293974784763735440802934310791398655203134471061602327930858255273025136604440}\Bigg(a^3b^2p^5\Bigg)^2$$

$$> 0,$$

where the second expression (i.e. the SOS decomposition) is obtained by an application of `'SumsOfSquares'` to the first expression. □

Theorem 7.3.2 *Let $m \in \mathbb{N}$. For any centered Gaussian random vector (X_1, X_2, X_3, X_4),*

$$E[X_1^{2m}X_2^2X_3^2X_4^2] \geq E[X_1^{2m}]E[X_2^2]E[X_3^2]E[X_4^2]. \tag{7.8}$$

The equality holds if and only if X_1, X_2, X_3, X_4 are independent.

Proof. Let X_4 be a linear combination of X_1, X_2 and X_3, and let U_1, U_2 and U_3 be independent standard Gaussian random variables. Then, without loss of generality, we may consider 3 constructions of $X_1, \ldots, X_4$ in terms of U_1, U_2, U_3:

Case 1:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + cU_2 + U_3,\ X_4 = dU_1 + eU_2 + U_3,\ \ a, b, c, d, e \in \mathbb{R}.$$

Case 2:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + U_2,\ X_4 = cU_1 + dU_2 + U_3,\ \ a, b, c, d \in \mathbb{R}.$$

Case 3:

$$X_1 = U_1,\ X_2 = aU_1 + U_2,\ X_3 = bU_1 + U_2,\ X_4 = cU_1 + U_2,\ \ a, b, c \in \mathbb{R}.$$

By Lemma 7.1.2, we need only show that, in each of these 3 cases, $F_{m,1,1,1} > 0$. By direct calculation (expanding and taking expectations), then letting $m = p^2 + 1$, we have that for any $p \in \mathbb{R}$,

Case 1:

$$\begin{aligned}\frac{F_{m,1,1,1}(a,b,c,d,e,p)}{2(2m-1)!!} = {} & 1+4\,a^2+b^2+7\,a^2\,b^2+6\,a\,b\,c+c^2+a^2\,c^2+6\,b\,d+30\,a^2\,b\,d+12\,a\,c\,d+d^2 \\ & +7\,a^2\,d^2+7\,b^2\,d^2+52\,a^2\,b^2\,d^2+30\,a\,b\,c\,d^2+4\,c^2\,d^2+7\,a^2\,c^2\,d^2+12\,a\,b\,e \\ & +6\,c\,e+6\,a^2\,c\,e+6\,a\,d\,e+30\,a\,b^2\,d\,e+18\,b\,c\,d\,e+30\,a^2\,b\,c\,d\,e+18\,a\,c^2\,d\,e \\ & +e^2+a^2\,e^2+4\,b^2\,e^2+7\,a^2\,b^2\,e^2+18\,a\,b\,c\,e^2+7\,c^2\,e^2+4\,a^2\,c^2\,e^2 \\ & +3\,a^2\,p^2+b^2\,p^2+8\,a^2\,b^2\,p^2+4\,a\,b\,c\,p^2+a^2\,c^2\,p^2+4\,b\,d\,p^2+32\,a^2\,b\,d\,p^2 \\ & +8\,a\,c\,d\,p^2+d^2\,p^2+8\,a^2\,d^2\,p^2+8\,b^2\,d^2\,p^2+71\,a^2\,b^2\,d^2\,p^2+32\,a\,b\,c\,d^2\,p^2 \\ & +3\,c^2\,d^2\,p^2+8\,a^2\,c^2\,d^2\,p^2+8\,a\,b\,e\,p^2+4\,a^2\,c\,e\,p^2+4\,a\,d\,e\,p^2 \\ & +32\,a\,b^2\,d\,e\,p^2+12\,b\,c\,d\,e\,p^2+32\,a^2\,b\,c\,d\,e\,p^2+12\,a\,c^2\,d\,e\,p^2+a^2\,e^2\,p^2 \\ & +3\,b^2\,e^2\,p^2+8\,a^2\,b^2\,e^2\,p^2+12\,a\,b\,c\,e^2\,p^2+3\,a^2\,c^2\,e^2\,p^2+2\,a^2\,b^2\,p^4 \\ & +8\,a^2\,b\,d\,p^4+2\,a^2\,d^2\,p^4+2\,b^2\,d^2\,p^4+30\,a^2\,b^2\,d^2\,p^4+8\,a\,b\,c\,d^2\,p^4 \\ & +2\,a^2\,c^2\,d^2\,p^4+8\,a\,b^2\,d\,e\,p^4+8\,a^2\,b\,c\,d\,e\,p^4+2\,a^2\,b^2\,e^2\,p^4+4\,a^2\,b^2\,d^2\,p^6\end{aligned}$$

$$
\begin{aligned}
&= \frac{123}{2}\Big(\frac{27}{410}\,a\,b\,d\,p^3 + a\,b\,d\,p + \frac{137}{1230}\,a\,c\,e\,p + \frac{71}{615}\,c\,d\,p + \frac{71}{615}\,b\,e\,p + \frac{9}{82}\,a\,p\Big)^2 + 52\Big(\frac{19}{208}\,a\,b\,d\,p^2 + a\,b\,d + \frac{83}{416}\,a\,c\,e + \frac{81}{416}\,c\,d + \frac{81}{416}\,b\,e + \frac{85}{416}\,a\Big)^2 \\
&\quad + \frac{89299}{4160}\Big(a\,b\,d\,p^2 + \frac{3301}{25514}\,a\,c\,e + \frac{20177}{178598}\,c\,d + \frac{20177}{178598}\,b\,e + \frac{3155}{25514}\,a\Big)^2 + 7\Big(\frac{5}{14}\,a\,b\,p^2 + \frac{5}{14}\,a\,d\,p^2 + a\,b + \frac{5}{8}\,a\,d + \frac{9}{28}\,c + \frac{55}{224}\,e\Big)^2 \\
&\quad + 7\Big(\frac{13}{35}\,a\,c\,d\,p^2 + \frac{9}{28}\,a\,b\,e\,p^2 + \frac{13}{35}\,b\,d\,p^2 + \frac{37}{56}\,a\,c\,d + a\,b\,e + \frac{39}{56}\,b\,d + \frac{45}{56}\,c\,e + \frac{59}{224}\Big)^2 + \frac{273}{64}\Big(\frac{20}{91}\,a\,b\,p^2 + \frac{20}{91}\,a\,d\,p^2 + a\,d + \frac{20}{273}\,c + \frac{43}{156}\,e\Big)^2 \\
&\quad + \frac{1767}{448}\Big(\frac{1192}{8835}\,a\,c\,d\,p^2 + \frac{2494}{8835}\,a\,b\,e\,p^2 + \frac{104}{465}\,b\,d\,p^2 + a\,c\,d + \frac{13}{31}\,b\,d + \frac{15}{31}\,c\,e + \frac{59}{372}\Big)^2 + \frac{30613}{8200}\Big(a\,b\,d\,p^3 + \frac{15776}{30613}\,a\,c\,e\,p + \frac{15641}{30613}\,c\,d\,p \\
&\quad + \frac{15641}{30613}\,b\,e\,p + \frac{16855}{30613}\,a\,p\Big)^2 + \frac{7}{2}\Big(\frac{9}{140}\,a\,c\,d\,p + a\,b\,e\,p + \frac{1}{10}\,b\,d\,p\Big)^2 + \frac{19519}{5600}\Big(a\,c\,d\,p + \frac{14}{149}\,b\,d\,p\Big)^2 + 3\Big(a\,b\,p + \frac{5}{24}\,a\,d\,p\Big)^2 + \frac{361}{124}\Big(\frac{351}{1805}\,a\,c\,d\,p^2 \\
&\quad + \frac{351}{1805}\,a\,b\,e\,p^2 + \frac{353}{1805}\,b\,d\,p^2 + b\,d + \frac{287}{722}\,c\,e + \frac{411}{1444}\Big)^2 + \frac{551}{192}\Big(a\,d\,p\Big)^2 + \frac{3627}{1490}\Big(b\,d\,p\Big)^2 + \frac{6267251}{3571960}\Big(\frac{7437577}{12534502}\,a\,c\,e + \frac{3141786}{6267251}\,c\,d + b\,e \\
&\quad - \frac{3928365}{25069004}\,a\Big)^2 + \frac{1169378377}{802208128}\Big(\frac{260572758}{1169378377}\,a\,c\,e - \frac{109994220}{1169378377}\,c\,d + a\Big)^2 + \frac{6084238559}{4677513508}\Big(\frac{367352216}{869176937}\,a\,c\,e + c\,d\Big)^2 + \frac{915139}{734712}\Big(a\,c\,e\,p \\
&\quad + \frac{717236}{915139}\,c\,d\,p + \frac{717236}{915139}\,b\,e\,p - \frac{136728}{915139}\,a\,p\Big)^2 + \frac{1006310}{915139}\Big(a\,p - \frac{8122597}{32201920}\,c\,d\,p - \frac{8122597}{32201920}\,b\,e\,p\Big)^2 + \frac{6333}{5776}\Big(\frac{19586}{31665}\,a\,c\,d\,p^2 + \frac{19586}{31665}\,a\,b\,e\,p^2 \\
&\quad + \frac{4886}{10555}\,b\,d\,p^2 + c\,e + \frac{3019}{12666}\Big)^2 + 1\Big(a\,c + \frac{3}{8}\,a\,e + \frac{3}{4}\,b + \frac{11}{32}\,d\Big)^2 + 1\Big(\frac{3}{8}\,a\,c\,p + a\,e\,p + \frac{17}{32}\,b\,p + \frac{7}{8}\,d\,p\Big)^2 + \frac{82}{91}\Big(\frac{73}{164}\,a\,b\,p^2 + a\,d\,p^2 + \frac{463}{2624}\,c \\
&\quad + \frac{23}{82}\,e\Big)^2 + \frac{903}{1024}\Big(\frac{212}{301}\,a\,e + \frac{376}{903}\,b + d\Big)^2 + \frac{55}{64}\Big(a\,c\,p + \frac{173}{220}\,b\,p + \frac{13}{55}\,d\,p\Big)^2 + \frac{237}{328}\Big(a\,b\,p^2 + \frac{955}{3792}\,c + \frac{61}{948}\,e\Big)^2 + \frac{2253517443}{3476707748}\Big(a\,c\,e\Big)^2 \\
&\quad + \frac{77689}{158325}\Big(\frac{146437}{621512}\,a\,c\,d\,p^2 + \frac{146437}{621512}\,a\,b\,e\,p^2 + b\,d\,p^2 + \frac{22520}{77689}\Big)^2 + \frac{127}{301}\Big(a\,e - \frac{1889}{4064}\,b\Big)^2 + \frac{114972691}{283409472}\Big(\frac{582802531}{2874317275}\,a\,c\,d\,p^2 + a\,b\,e\,p^2 \\
&\quad - \frac{133581818}{574863455}\Big)^2 + \frac{111810532673}{287431727500}\Big(a\,c\,d\,p^2 - \frac{17576555}{90976837}\Big)^2 + \frac{381989069}{1030461440}\Big(c\,d\,p + \frac{124373709}{381989069}\,b\,e\,p\Big)^2 + \frac{253181389}{763978138}\Big(b\,e\,p\Big)^2 \\
&\quad + \frac{75473}{390144}\Big(b\Big)^2 + \frac{41}{220}\Big(b\,p + \frac{883}{1312}\,d\,p\Big)^2 + \frac{38225}{212352}\Big(c - \frac{59373}{152900}\,e\Big)^2 + \frac{35901007}{234854400}\Big(e\Big)^2 + \frac{17121}{167936}\Big(d\,p\Big)^2 + \frac{229159727}{5822517568}\Big(1\Big)^2 \\
&> 0,
\end{aligned}
$$

Case 2:

$$
\begin{aligned}
\frac{F_{m,1,1,1}(a,b,c,d,p)}{2(2m-1)!!} &= 1 + a^2 + 6\,a\,b + b^2 + 7\,a^2\,b^2 + 4\,c^2 + 7\,a^2\,c^2 + 30\,a\,b\,c^2 + 7\,b^2\,c^2 + 52\,a^2\,b^2\,c^2 \\
&\quad + 18\,a\,c\,d + 18\,b\,c\,d + 30\,a^2\,b\,c\,d + 30\,a\,b^2\,c\,d + 7\,d^2 + 4\,a^2\,d^2 + 18\,a\,b\,d^2 \\
&\quad + 4\,b^2\,d^2 + 7\,a^2\,b^2\,d^2 + a^2\,p^2 + 4\,a\,b\,p^2 + b^2\,p^2 + 8\,a^2\,b^2\,p^2 + 3\,c^2\,p^2 \\
&\quad + 8\,a^2\,c^2\,p^2 + 32\,a\,b\,c^2\,p^2 + 8\,b^2\,c^2\,p^2 + 71\,a^2\,b^2\,c^2\,p^2 + 12\,a\,c\,d\,p^2 \\
&\quad + 12\,b\,c\,d\,p^2 + 32\,a^2\,b\,c\,d\,p^2 + 32\,a\,b^2\,c\,d\,p^2 + 3\,a^2\,d^2\,p^2 + 12\,a\,b\,d^2\,p^2 \\
&\quad + 3\,b^2\,d^2\,p^2 + 8\,a^2\,b^2\,d^2\,p^2 + 2\,a^2\,b^2\,p^4 + 2\,a^2\,c^2\,p^4 + 8\,a\,b\,c^2\,p^4 + 2\,b^2\,c^2\,p^4 \\
&\quad + 30\,a^2\,b^2\,c^2\,p^4 + 8\,a^2\,b\,c\,d\,p^4 + 8\,a\,b^2\,c\,d\,p^4 + 2\,a^2\,b^2\,d^2\,p^4 + 4\,a^2\,b^2\,c^2\,p^6
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{368}{5}\Big(\frac{25}{2944}\,a\,b\,c\,p^3 + a\,b\,c\,p + \frac{281}{2944}\,a\,d\,p + \frac{281}{2944}\,b\,d\,p + \frac{281}{2944}\,c\,p\Big)^2 + 52\Big(\frac{41}{208}\,c - \frac{1}{40}\,a\,b\,c\,p^2 + a\,b\,c \\
&\quad + \frac{41}{208}\,a\,d + \frac{41}{208}\,b\,d\Big)^2 + \frac{11487}{400}\Big(a\,b\,c\,p^2 + \frac{2825}{22974}\,a\,d + \frac{2825}{22974}\,b\,d + \frac{2825}{22974}\,c\Big)^2 + 7\Big(\frac{73}{280}\,a\,b\,p^2 + a\,b \\
&\quad + \frac{9}{28}\Big)^2 + 7\Big(\frac{19}{56}\,a\,b\,d\,p^2 + \frac{53}{140}\,a\,c\,p^2 + \frac{53}{140}\,b\,c\,p^2 + a\,b\,d + \frac{19}{28}\,a\,c + \frac{19}{28}\,b\,c + \frac{23}{28}\,d\Big)^2 + \frac{87}{20}\Big(a\,b\,p\Big)^2 \\
&\quad + \frac{94083}{23552}\Big(a\,b\,c\,p^3 + \frac{18177}{31361}\,a\,d\,p + \frac{18177}{31361}\,b\,d\,p + \frac{18177}{31361}\,c\,p\Big)^2 + \frac{423}{112}\Big(\frac{1163}{4230}\,a\,b\,d\,p^2 + \frac{323}{2115}\,a\,c\,p^2 \\
&\quad + \frac{53}{235}\,b\,c\,p^2 + a\,c + \frac{19}{47}\,b\,c + \frac{23}{47}\,d\Big)^2 + \frac{13}{4}\Big(a\,b\,d\,p + \frac{8}{65}\,a\,c\,p + \frac{8}{65}\,b\,c\,p\Big)^2 + \frac{4161}{1300}\Big(a\,c\,p + \frac{8}{73}\,b\,c\,p\Big)^2 \\
&\quad + \frac{4617}{1460}\Big(b\,c\,p\Big)^2 + \frac{297}{94}\Big(\frac{1163}{5940}\,a\,b\,d\,p^2 + \frac{1163}{5940}\,a\,c\,p^2 + \frac{437}{5940}\,b\,c\,p^2 + b\,c + \frac{23}{66}\,d\Big)^2 + \frac{3692291}{2389296}\Big(a\,d \\
&\quad + \frac{1900319}{3692291}\,b\,d + \frac{1900319}{3692291}\,c\Big)^2 + \frac{17071}{11200}\Big(a\,b\,p^2 + \frac{6030}{17071}\Big)^2 + \frac{8388915}{7384582}\Big(b\,d + \frac{1900319}{5592610}\,c\Big)^2 \\
&\quad + \frac{22478787}{22370440}\Big(c\Big)^2 + 1\Big(a\,p + \frac{7}{8}\,b\,p\Big)^2 + 1\Big(\frac{3}{4}\,a + b\Big)^2 + \frac{87}{88}\Big(\frac{1519}{2610}\,a\,b\,d\,p^2 + \frac{1519}{2610}\,a\,c\,p^2 + \frac{1519}{2610}\,b\,c\,p^2 \\
&\quad + d\Big)^2 + \frac{619373}{627220}\Big(\frac{34867}{141790}\,a\,b\,d\,p^2 + \frac{34867}{141790}\,a\,c\,p^2 + b\,c\,p^2\Big)^2 + \frac{14179}{31320}\Big(a\Big)^2 + \frac{7}{16}\Big(\frac{462568}{619373}\,a\,d\,p \\
&\quad + b\,d\,p + \frac{462568}{619373}\,c\,p\Big)^2 + \frac{1081941}{2477492}\Big(\frac{462568}{1081941}\,a\,d\,p + c\,p\Big)^2 + \frac{217111453}{510444000}\Big(a\,b\,d\,p^2 \\
&\quad + \frac{34867}{176657}\,a\,c\,p^2\Big)^2 + \frac{21663583}{52997100}\Big(a\,c\,p^2\Big)^2 + \frac{1544509}{4327764}\Big(a\,d\,p\Big)^2 + \frac{15}{64}\Big(b\,p\Big)^2 + \frac{2957}{34142}\Big(1\Big)^2 \\
&> 0,
\end{aligned}
$$

Case 3:

$$
\begin{aligned}
\frac{F_{m,1,1,1}(a,b,c,p)}{2(2m-1)!!} &= 7 + 4\,a^2 + 18\,a\,b + 4\,b^2 + 7\,a^2\,b^2 + 18\,a\,c + 18\,b\,c + 30\,a^2\,b\,c + 30\,a\,b^2\,c + 4\,c^2 \\
&\quad + 7\,a^2\,c^2 + 30\,a\,b\,c^2 + 7\,b^2\,c^2 + 52\,a^2\,b^2\,c^2 + 3\,a^2\,p^2 + 12\,a\,b\,p^2 + 3\,b^2\,p^2 \\
&\quad + 8\,a^2\,b^2\,p^2 + 12\,a\,c\,p^2 + 12\,b\,c\,p^2 + 32\,a^2\,b\,c\,p^2 + 32\,a\,b^2\,c\,p^2 + 3\,c^2\,p^2 \\
&\quad + 8\,a^2\,c^2\,p^2 + 32\,a\,b\,c^2\,p^2 + 8\,b^2\,c^2\,p^2 + 71\,a^2\,b^2\,c^2\,p^2 + 2\,a^2\,b^2\,p^4 + 8\,a^2\,b\,c\,p^4 \\
&\quad + 8\,a\,b^2\,c\,p^4 + 2\,a^2\,c^2\,p^4 + 8\,a\,b\,c^2\,p^4 + 2\,b^2\,c^2\,p^4 + 30\,a^2\,b^2\,c^2\,p^4 + 4\,a^2\,b^2\,c^2\,p^6
\end{aligned}
$$

$$= \frac{372}{5}\left(\frac{5}{2976}\,a\,b\,c\,p^3 + a\,b\,c\,p + \frac{139}{1488}\,a\,p + \frac{139}{1488}\,b\,p + \frac{139}{1488}\,c\,p\right)^2 + 52\left(\frac{81}{416}\,c - \frac{17}{520}\,a\,b\,c\,p^2 + a\,b\,c\right.$$
$$\left. + \frac{81}{416}\,a + \frac{81}{416}\,b\right)^2 + \frac{154411}{5200}\left(a\,b\,c\,p^2 + \frac{76045}{617644}\,a + \frac{76045}{617644}\,b + \frac{76045}{617644}\,c\right)^2 + 7\left(\frac{13}{28}\,a\,b\,p^2 + \frac{13}{28}\,a\,c\,p^2\right.$$
$$\left. + \frac{13}{28}\,b\,c\,p^2 + \frac{23}{28}\,a\,b + \frac{23}{28}\,a\,c + \frac{23}{28}\,b\,c + 1\right)^2 + \frac{95227}{23808}\left(a\,b\,c\,p^3 + \frac{56266}{95227}\,a\,p + \frac{56266}{95227}\,b\,p + \frac{56266}{95227}\,c\,p\right)^2$$
$$+ \frac{63}{20}\left(a\,b\,p + \frac{13}{126}\,a\,c\,p + \frac{13}{126}\,b\,c\,p\right)^2 + \frac{15707}{5040}\left(a\,c\,p + \frac{13}{139}\,b\,c\,p\right)^2 + \frac{2147}{695}\left(b\,c\,p\right)^2 + \frac{255}{112}\left(\frac{1}{75}\,a\,b\,p^2\right.$$
$$\left. - \frac{137}{1275}\,a\,c\,p^2 + \frac{1}{75}\,b\,c\,p^2 + \frac{1}{15}\,a\,b + a\,c + \frac{1}{15}\,b\,c\right)^2 + \frac{34}{15}\left(\frac{1}{80}\,a\,b\,p^2 + \frac{7}{340}\,a\,c\,p^2 - \frac{37}{340}\,b\,c\,p^2\right.$$
$$\left. + \frac{1}{16}\,a\,b + b\,c\right)^2 + \frac{289}{128}\left(a\,b - \frac{159}{1445}\,a\,b\,p^2 + \frac{28}{1445}\,a\,c\,p^2 + \frac{28}{1445}\,b\,c\,p^2\right)^2 + \frac{15598365}{9882304}\left(\frac{2728879}{5199455}\,a\right.$$
$$\left. + b + \frac{2728879}{5199455}\,c\right)^2 + \frac{11892501}{10398910}\left(a + \frac{2728879}{7928334}\,c\right)^2 + \frac{10657213}{10571112}\left(c\right)^2 + \frac{908823}{952270}\left(a\,p\right.$$
$$\left. + \frac{1341511}{1817646}\,b\,p + \frac{1341511}{1817646}\,c\,p\right)^2 + \frac{669}{1445}\left(a\,b\,p^2 + \frac{3571}{13380}\,a\,c\,p^2 + \frac{3571}{13380}\,b\,c\,p^2\right)^2 + \frac{3159157}{7270584}\left(b\,p\right.$$
$$\left. + \frac{1341511}{3159157}\,c\,p\right)^2 + \frac{9780727}{22746000}\left(\frac{3571}{16951}\,a\,c\,p^2 + b\,c\,p^2\right)^2 + \frac{5920597}{14408350}\left(a\,c\,p^2\right)^2 + \frac{1125167}{3159157}\left(c\,p\right)^2$$
$$> 0,$$

where in each case, the second expression (i.e. the SOS decomposition) is obtained by an application of '`SumsOfSquares`' to the first expression. □

Remark 7.3.3 *To establish the inequality (7.8), we need only consider Case 1. However, to show that the equality sign holds if and only if X_1, X_2, X_3, X_4 are independent, we must check all three cases.*

7.4 Alternative SOS method in Mathematica

Since **Mathematica** is much more user-friendly than **Macaulay2**, we were pleased to see that, in the latest versions of **Mathematica**, a new function `PolynomialSumOfSquaresList[]` has been added. Although this function does not give SOS decompositions with strictly rational components, it still produces an exact and verifiable SOS decomposition when it works.

This motivated us to develop a more efficient SOS method that only necessitates the use of **Mathematica**. The remaining pages of this section are an excerpt from a **Mathematica** notebook in which we prove a new five-dimensional GPI in only a few steps.

- Here, we use Mathematica only to demonstrate a new SOS method by proving the following 5-D GPI:
- E[|X1|^y X2^2 X3^2 X4^2 X5^2] >= E[|X1|^y] E[X2^2] E[X3^2] E[X4^2] E[X5^2] for any y >= 1/10.
- Note that both Macaulay2 and Mathematica fail to find an SOS decomposition for the polynomial F (below) when y can equal 0. Thus, we are choosing to set the lower bound of y as 1/10, although even lower values would likely work as well.
- Again, we will use our rank-reducing technique. The definitions of X1, X2, X3, X4, X5 are set accordingly in terms of the independent standard normal random variables U1, U2, U3, U4. The handy Expectation[] function is used below to facilitate calculations.

In[3]:=
```
Clear[y, a, b, c, d, e, f, g, h, k, p, U1, U2, U3, U4];
m2 = 1;
m3 = 1;
m4 = 1;
m5 = 1;
X1 = U1;
X2 = a * U1 + U2;
X3 = b * U1 + c * U2 + U3;
X4 = d * U1 + e * U2 + f * U3 + U4;
X5 = g * U1 + h * U2 + k * U3 + U4;
LHS = Expectation[Abs[X1] ^ y * X2 ^ (2 * m2) * X3 ^ (2 * m3) * X4 ^ (2 * m4) * X5 ^ (2 * m5),
   {U1 ≈ NormalDistribution[], U2 ≈ NormalDistribution[],
    U3 ≈ NormalDistribution[], U4 ≈ NormalDistribution[]}];
E1 = Expectation[Abs[X1] ^ y, {U1 ≈ NormalDistribution[]}];
E2 =
  Expectation[X2 ^ (2 * m2), {U1 ≈ NormalDistribution[], U2 ≈ NormalDistribution[]}];
E3 = Expectation[X3 ^ (2 * m3), {U1 ≈ NormalDistribution[],
    U2 ≈ NormalDistribution[], U3 ≈ NormalDistribution[]}];
E4 = Expectation[X4 ^ (2 * m4), {U1 ≈ NormalDistribution[], U2 ≈ NormalDistribution[],
    U3 ≈ NormalDistribution[], U4 ≈ NormalDistribution[]}];
E5 = Expectation[X5 ^ (2 * m5), {U1 ≈ NormalDistribution[], U2 ≈ NormalDistribution[],
    U3 ≈ NormalDistribution[], U4 ≈ NormalDistribution[]}];
RHS = E1 * E2 * E3 * E4 * E5;
```

$$F = \text{Expand}\left[\text{Expand}\left[(\text{LHS} - \text{RHS}) \Big/ \frac{2^{y/2}\,\text{Gamma}\left[\frac{1+y}{2}\right]}{\sqrt{\pi}}\right] \text{ /. } y \to (p\text{^}2 + 1/10)\right]$$

Out[20]=

$$2 + \frac{23 a^2}{10} + \frac{23 b^2}{10} + \frac{923 a^2 b^2}{100} + \frac{66 a b c}{5} + 8 c^2 + \frac{23 a^2 c^2}{10} + \frac{d^2}{10} + \frac{241 a^2 d^2}{100} + \frac{241 b^2 d^2}{100} +$$
$$\frac{16391 a^2 b^2 d^2}{1000} + \frac{341}{25} a b c d^2 + \frac{23 c^2 d^2}{10} + \frac{241}{100} a^2 c^2 d^2 + \frac{22 a d e}{5} + \frac{341}{25} a b^2 d e + \frac{66}{5} b c d e +$$
$$\frac{341}{25} a^2 b c d e + \frac{66}{5} a c^2 d e + 2 e^2 + \frac{a^2 e^2}{10} + \frac{23 b^2 e^2}{10} + \frac{241}{100} a^2 b^2 e^2 + \frac{66}{5} a b c e^2 + 14 c^2 e^2 +$$

$$\frac{23}{10} a^2 c^2 e^2 + \frac{22 b d f}{5} + \frac{341}{25} a^2 b d f + \frac{44}{5} a c d f + \frac{44}{5} a b e f + 12 c e f + \frac{22}{5} a^2 c e f + 2 f^2 +$$

$$\frac{23 a^2 f^2}{10} + \frac{b^2 f^2}{10} + \frac{241}{100} a^2 b^2 f^2 + \frac{22}{5} a b c f^2 + 2 c^2 f^2 + \frac{1}{10} a^2 c^2 f^2 + \frac{22 d g}{5} + \frac{341}{25} a^2 d g +$$

$$\frac{341}{25} b^2 d g + \frac{17391}{250} a^2 b^2 d g + \frac{1364}{25} a b c d g + \frac{66}{5} c^2 d g + \frac{341}{25} a^2 c^2 d g + \frac{44 a e g}{5} +$$

$$\frac{682}{25} a b^2 e g + \frac{132}{5} b c e g + \frac{682}{25} a^2 b c e g + \frac{132}{5} a c^2 e g + \frac{44 b f g}{5} + \frac{682}{25} a^2 b f g + \frac{88}{5} a c f g +$$

$$\frac{g^2}{10} + \frac{241 a^2 g^2}{100} + \frac{241 b^2 g^2}{100} + \frac{16391 a^2 b^2 g^2}{1000} + \frac{341}{25} a b c g^2 + \frac{23 c^2 g^2}{10} + \frac{241}{100} a^2 c^2 g^2 +$$

$$\frac{241 d^2 g^2}{100} + \frac{16391 a^2 d^2 g^2}{1000} + \frac{16391 b^2 d^2 g^2}{1000} + \frac{1224761 a^2 b^2 d^2 g^2}{10000} + \frac{17391}{250} a b c d^2 g^2 +$$

$$\frac{923}{100} c^2 d^2 g^2 + \frac{16391 a^2 c^2 d^2 g^2}{1000} + \frac{341}{25} a d e g^2 + \frac{17391}{250} a b^2 d e g^2 + \frac{1023}{25} b c d e g^2 +$$

$$\frac{17391}{250} a^2 b c d e g^2 + \frac{1023}{25} a c^2 d e g^2 + \frac{23 e^2 g^2}{10} + \frac{241}{100} a^2 e^2 g^2 + \frac{923}{100} b^2 e^2 g^2 +$$

$$\frac{16391 a^2 b^2 e^2 g^2}{1000} + \frac{1023}{25} a b c e^2 g^2 + \frac{31}{2} c^2 e^2 g^2 + \frac{923}{100} a^2 c^2 e^2 g^2 + \frac{341}{25} b d f g^2 +$$

$$\frac{17391}{250} a^2 b d f g^2 + \frac{682}{25} a c d f g^2 + \frac{682}{25} a b e f g^2 + \frac{66}{5} c e f g^2 + \frac{341}{25} a^2 c e f g^2 +$$

$$\frac{23 f^2 g^2}{10} + \frac{923}{100} a^2 f^2 g^2 + \frac{241}{100} b^2 f^2 g^2 + \frac{16391 a^2 b^2 f^2 g^2}{1000} + \frac{341}{25} a b c f^2 g^2 + \frac{23}{10} c^2 f^2 g^2 +$$

$$\frac{241}{100} a^2 c^2 f^2 g^2 + \frac{44 a d h}{5} + \frac{682}{25} a b^2 d h + \frac{132}{5} b c d h + \frac{682}{25} a^2 b c d h + \frac{132}{5} a c^2 d h +$$

$$12 e h + \frac{22}{5} a^2 e h + \frac{66}{5} b^2 e h + \frac{341}{25} a^2 b^2 e h + \frac{264}{5} a b c e h + 60 c^2 e h + \frac{66}{5} a^2 c^2 e h +$$

$$\frac{88}{5} a b f h + 24 c f h + \frac{44}{5} a^2 c f h + \frac{22 a g h}{5} + \frac{341}{25} a b^2 g h + \frac{66}{5} b c g h + \frac{341}{25} a^2 b c g h +$$

$$\frac{66}{5} a c^2 g h + \frac{341}{25} a d^2 g h + \frac{17391}{250} a b^2 d^2 g h + \frac{1023}{25} b c d^2 g h + \frac{17391}{250} a^2 b c d^2 g h +$$

$$\frac{1023}{25} a c^2 d^2 g h + \frac{66}{5} d e g h + \frac{341}{25} a^2 d e g h + \frac{1023}{25} b^2 d e g h + \frac{17391}{250} a^2 b^2 d e g h +$$

$$\frac{4092}{25} a b c d e g h + 66 c^2 d e g h + \frac{1023}{25} a^2 c^2 d e g h + \frac{66}{5} a e^2 g h + \frac{1023}{25} a b^2 e^2 g h +$$

$$66 b c e^2 g h + \frac{1023}{25} a^2 b c e^2 g h + 66 a c^2 e^2 g h + \frac{1364}{25} a b d f g h + \frac{132}{5} c d f g h +$$

$$\frac{682}{25} a^2 c d f g h + \frac{132}{5} b e f g h + \frac{682}{25} a^2 b e f g h + \frac{264}{5} a c e f g h + \frac{66}{5} a f^2 g h +$$

$$\frac{341}{25} a b^2 f^2 g h + \frac{66}{5} b c f^2 g h + \frac{341}{25} a^2 b c f^2 g h + \frac{66}{5} a c^2 f^2 g h + 2 h^2 + \frac{a^2 h^2}{10} + \frac{23 b^2 h^2}{10} +$$

$$\frac{241}{100} a^2 b^2 h^2 + \frac{66}{5} a b c h^2 + 14 c^2 h^2 + \frac{23}{10} a^2 c^2 h^2 + \frac{23 d^2 h^2}{10} + \frac{241}{100} a^2 d^2 h^2 + \frac{923}{100} b^2 d^2 h^2 +$$

$$\frac{16391 a^2 b^2 d^2 h^2}{1000} + \frac{1023}{25} a b c d^2 h^2 + \frac{31}{2} c^2 d^2 h^2 + \frac{923}{100} a^2 c^2 d^2 h^2 + \frac{66}{5} a d e h^2 +$$
$$\frac{1023}{25} a b^2 d e h^2 + 66 b c d e h^2 + \frac{1023}{25} a^2 b c d e h^2 + 66 a c^2 d e h^2 + 14 e^2 h^2 + \frac{23}{10} a^2 e^2 h^2 +$$
$$\frac{31}{2} b^2 e^2 h^2 + \frac{923}{100} a^2 b^2 e^2 h^2 + 66 a b c e^2 h^2 + 104 c^2 e^2 h^2 + \frac{31}{2} a^2 c^2 e^2 h^2 + \frac{66}{5} b d f h^2 +$$
$$\frac{341}{25} a^2 b d f h^2 + \frac{132}{5} a c d f h^2 + \frac{132}{5} a b e f h^2 + 60 c e f h^2 + \frac{66}{5} a^2 c e f h^2 + 8 f^2 h^2 +$$
$$\frac{23}{10} a^2 f^2 h^2 + \frac{23}{10} b^2 f^2 h^2 + \frac{241}{100} a^2 b^2 f^2 h^2 + \frac{66}{5} a b c f^2 h^2 + 14 c^2 f^2 h^2 + \frac{23}{10} a^2 c^2 f^2 h^2 +$$
$$\frac{44 b d k}{5} + \frac{682}{25} a^2 b d k + \frac{88}{5} a c d k + \frac{88}{5} a b e k + 24 c e k + \frac{44}{5} a^2 c e k + 12 f k + \frac{66}{5} a^2 f k +$$
$$\frac{22}{5} b^2 f k + \frac{341}{25} a^2 b^2 f k + \frac{88}{5} a b c f k + 12 c^2 f k + \frac{22}{5} a^2 c^2 f k + \frac{22 b g k}{5} + \frac{341}{25} a^2 b g k +$$
$$\frac{44}{5} a c g k + \frac{341}{25} b d^2 g k + \frac{17391}{250} a^2 b d^2 g k + \frac{682}{25} a c d^2 g k + \frac{1364}{25} a b d e g k +$$
$$\frac{132}{5} c d e g k + \frac{682}{25} a^2 c d e g k + \frac{66}{5} b e^2 g k + \frac{341}{25} a^2 b e^2 g k + \frac{132}{5} a c e^2 g k + \frac{66}{5} d f g k +$$
$$\frac{1023}{25} a^2 d f g k + \frac{341}{25} b^2 d f g k + \frac{17391}{250} a^2 b^2 d f g k + \frac{1364}{25} a b c d f g k + \frac{66}{5} c^2 d f g k +$$
$$\frac{341}{25} a^2 c^2 d f g k + \frac{132}{5} a e f g k + \frac{682}{25} a b^2 e f g k + \frac{132}{5} b c e f g k + \frac{682}{25} a^2 b c e f g k +$$
$$\frac{132}{5} a c^2 e f g k + \frac{66}{5} b f^2 g k + \frac{1023}{25} a^2 b f^2 g k + \frac{132}{5} a c f^2 g k + \frac{44}{5} a b h k + 12 c h k +$$
$$\frac{22}{5} a^2 c h k + \frac{682}{25} a b d^2 h k + \frac{66}{5} c d^2 h k + \frac{341}{25} a^2 c d^2 h k + \frac{132}{5} b d e h k + \frac{682}{25} a^2 b d e h k +$$
$$\frac{264}{5} a c d e h k + \frac{132}{5} a b e^2 h k + 60 c e^2 h k + \frac{66}{5} a^2 c e^2 h k + \frac{132}{5} a d f h k + \frac{682}{25} a b^2 d f h k +$$
$$\frac{132}{5} b c d f h k + \frac{682}{25} a^2 b c d f h k + \frac{132}{5} a c^2 d f h k + 36 e f h k + \frac{66}{5} a^2 e f h k +$$
$$\frac{66}{5} b^2 e f h k + \frac{341}{25} a^2 b^2 e f h k + \frac{264}{5} a b c e f h k + 60 c^2 e f h k + \frac{66}{5} a^2 c^2 e f h k +$$
$$\frac{132}{5} a b f^2 h k + 36 c f^2 h k + \frac{66}{5} a^2 c f^2 h k + 2 k^2 + \frac{23 a^2 k^2}{10} + \frac{b^2 k^2}{10} + \frac{241}{100} a^2 b^2 k^2 +$$
$$\frac{22}{5} a b c k^2 + 2 c^2 k^2 + \frac{1}{10} a^2 c^2 k^2 + \frac{23 d^2 k^2}{10} + \frac{923}{100} a^2 d^2 k^2 + \frac{241}{100} b^2 d^2 k^2 + \frac{16391 a^2 b^2 d^2 k^2}{1000} +$$
$$\frac{341}{25} a b c d^2 k^2 + \frac{23}{10} c^2 d^2 k^2 + \frac{241}{100} a^2 c^2 d^2 k^2 + \frac{66}{5} a d e k^2 + \frac{341}{25} a b^2 d e k^2 + \frac{66}{5} b c d e k^2 +$$
$$\frac{341}{25} a^2 b c d e k^2 + \frac{66}{5} a c^2 d e k^2 + 8 e^2 k^2 + \frac{23}{10} a^2 e^2 k^2 + \frac{23}{10} b^2 e^2 k^2 + \frac{241}{100} a^2 b^2 e^2 k^2 +$$
$$\frac{66}{5} a b c e^2 k^2 + 14 c^2 e^2 k^2 + \frac{23}{10} a^2 c^2 e^2 k^2 + \frac{66}{5} b d f k^2 + \frac{1023}{25} a^2 b d f k^2 + \frac{132}{5} a c d f k^2 +$$
$$\frac{132}{5} a b e f k^2 + 36 c e f k^2 + \frac{66}{5} a^2 c e f k^2 + 14 f^2 k^2 + \frac{31}{2} a^2 f^2 k^2 + \frac{23}{10} b^2 f^2 k^2 +$$

$$\frac{923}{100} a^2 b^2 f^2 k^2 + \frac{66}{5} a b c f^2 k^2 + 8 c^2 f^2 k^2 + \frac{23}{10} a^2 c^2 f^2 k^2 + 3 a^2 p^2 + 3 b^2 p^2 + \frac{63}{5} a^2 b^2 p^2 +$$

$$12 a b c p^2 + 3 a^2 c^2 p^2 + d^2 p^2 + \frac{21}{5} a^2 d^2 p^2 + \frac{21}{5} b^2 d^2 p^2 + \frac{2483}{100} a^2 b^2 d^2 p^2 + \frac{84}{5} a b c d^2 p^2 +$$

$$3 c^2 d^2 p^2 + \frac{21}{5} a^2 c^2 d^2 p^2 + 4 a d e p^2 + \frac{84}{5} a b^2 d e p^2 + 12 b c d e p^2 + \frac{84}{5} a^2 b c d e p^2 +$$

$$12 a c^2 d e p^2 + a^2 e^2 p^2 + 3 b^2 e^2 p^2 + \frac{21}{5} a^2 b^2 e^2 p^2 + 12 a b c e^2 p^2 + 3 a^2 c^2 e^2 p^2 + 4 b d f p^2 +$$

$$\frac{84}{5} a^2 b d f p^2 + 8 a c d f p^2 + 8 a b e f p^2 + 4 a^2 c e f p^2 + 3 a^2 f^2 p^2 + b^2 f^2 p^2 + \frac{21}{5} a^2 b^2 f^2 p^2 +$$

$$4 a b c f^2 p^2 + a^2 c^2 f^2 p^2 + 4 d g p^2 + \frac{84}{5} a^2 d g p^2 + \frac{84}{5} b^2 d g p^2 + \frac{2483}{25} a^2 b^2 d g p^2 +$$

$$\frac{336}{5} a b c d g p^2 + 12 c^2 d g p^2 + \frac{84}{5} a^2 c^2 d g p^2 + 8 a e g p^2 + \frac{168}{5} a b^2 e g p^2 + 24 b c e g p^2 +$$

$$\frac{168}{5} a^2 b c e g p^2 + 24 a c^2 e g p^2 + 8 b f g p^2 + \frac{168}{5} a^2 b f g p^2 + 16 a c f g p^2 + g^2 p^2 +$$

$$\frac{21}{5} a^2 g^2 p^2 + \frac{21}{5} b^2 g^2 p^2 + \frac{2483}{100} a^2 b^2 g^2 p^2 + \frac{84}{5} a b c g^2 p^2 + 3 c^2 g^2 p^2 + \frac{21}{5} a^2 c^2 g^2 p^2 +$$

$$\frac{21}{5} d^2 g^2 p^2 + \frac{2483}{100} a^2 d^2 g^2 p^2 + \frac{2483}{100} b^2 d^2 g^2 p^2 + \frac{48421}{250} a^2 b^2 d^2 g^2 p^2 + \frac{2483}{25} a b c d^2 g^2 p^2 +$$

$$\frac{63}{5} c^2 d^2 g^2 p^2 + \frac{2483}{100} a^2 c^2 d^2 g^2 p^2 + \frac{84}{5} a d e g^2 p^2 + \frac{2483}{25} a b^2 d e g^2 p^2 + \frac{252}{5} b c d e g^2 p^2 +$$

$$\frac{2483}{25} a^2 b c d e g^2 p^2 + \frac{252}{5} a c^2 d e g^2 p^2 + 3 e^2 g^2 p^2 + \frac{21}{5} a^2 e^2 g^2 p^2 + \frac{63}{5} b^2 e^2 g^2 p^2 +$$

$$\frac{2483}{100} a^2 b^2 e^2 g^2 p^2 + \frac{252}{5} a b c e^2 g^2 p^2 + 15 c^2 e^2 g^2 p^2 + \frac{63}{5} a^2 c^2 e^2 g^2 p^2 + \frac{84}{5} b d f g^2 p^2 +$$

$$\frac{2483}{25} a^2 b d f g^2 p^2 + \frac{168}{5} a c d f g^2 p^2 + \frac{168}{5} a b e f g^2 p^2 + 12 c e f g^2 p^2 + \frac{84}{5} a^2 c e f g^2 p^2 +$$

$$3 f^2 g^2 p^2 + \frac{63}{5} a^2 f^2 g^2 p^2 + \frac{21}{5} b^2 f^2 g^2 p^2 + \frac{2483}{100} a^2 b^2 f^2 g^2 p^2 + \frac{84}{5} a b c f^2 g^2 p^2 +$$

$$3 c^2 f^2 g^2 p^2 + \frac{21}{5} a^2 c^2 f^2 g^2 p^2 + 8 a d h p^2 + \frac{168}{5} a b^2 d h p^2 + 24 b c d h p^2 + \frac{168}{5} a^2 b c d h p^2 +$$

$$24 a c^2 d h p^2 + 4 a^2 e h p^2 + 12 b^2 e h p^2 + \frac{84}{5} a^2 b^2 e h p^2 + 48 a b c e h p^2 + 12 a^2 c^2 e h p^2 +$$

$$16 a b f h p^2 + 8 a^2 c f h p^2 + 4 a g h p^2 + \frac{84}{5} a b^2 g h p^2 + 12 b c g h p^2 + \frac{84}{5} a^2 b c g h p^2 +$$

$$12 a c^2 g h p^2 + \frac{84}{5} a d^2 g h p^2 + \frac{2483}{25} a b^2 d^2 g h p^2 + \frac{252}{5} b c d^2 g h p^2 + \frac{2483}{25} a^2 b c d^2 g h p^2 +$$

$$\frac{252}{5} a c^2 d^2 g h p^2 + 12 d e g h p^2 + \frac{84}{5} a^2 d e g h p^2 + \frac{252}{5} b^2 d e g h p^2 + \frac{2483}{25} a^2 b^2 d e g h p^2 +$$

$$\frac{1008}{5} a b c d e g h p^2 + 60 c^2 d e g h p^2 + \frac{252}{5} a^2 c^2 d e g h p^2 + 12 a e^2 g h p^2 + \frac{252}{5} a b^2 e^2 g h p^2 +$$

$$60 b c e^2 g h p^2 + \frac{252}{5} a^2 b c e^2 g h p^2 + 60 a c^2 e^2 g h p^2 + \frac{336}{5} a b d f g h p^2 + 24 c d f g h p^2 +$$

$$\frac{168}{5} a^2 c d f g h p^2 + 24 b e f g h p^2 + \frac{168}{5} a^2 b e f g h p^2 + 48 a c e f g h p^2 + 12 a f^2 g h p^2 +$$
$$\frac{84}{5} a b^2 f^2 g h p^2 + 12 b c f^2 g h p^2 + \frac{84}{5} a^2 b c f^2 g h p^2 + 12 a c^2 f^2 g h p^2 + a^2 h^2 p^2 + 3 b^2 h^2 p^2 +$$
$$\frac{21}{5} a^2 b^2 h^2 p^2 + 12 a b c h^2 p^2 + 3 a^2 c^2 h^2 p^2 + 3 d^2 h^2 p^2 + \frac{21}{5} a^2 d^2 h^2 p^2 + \frac{63}{5} b^2 d^2 h^2 p^2 +$$
$$\frac{2483}{100} a^2 b^2 d^2 h^2 p^2 + \frac{252}{5} a b c d^2 h^2 p^2 + 15 c^2 d^2 h^2 p^2 + \frac{63}{5} a^2 c^2 d^2 h^2 p^2 + 12 a d e h^2 p^2 +$$
$$\frac{252}{5} a b^2 d e h^2 p^2 + 60 b c d e h^2 p^2 + \frac{252}{5} a^2 b c d e h^2 p^2 + 60 a c^2 d e h^2 p^2 + 3 a^2 e^2 h^2 p^2 +$$
$$15 b^2 e^2 h^2 p^2 + \frac{63}{5} a^2 b^2 e^2 h^2 p^2 + 60 a b c e^2 h^2 p^2 + 15 a^2 c^2 e^2 h^2 p^2 + 12 b d f h^2 p^2 +$$
$$\frac{84}{5} a^2 b d f h^2 p^2 + 24 a c d f h^2 p^2 + 24 a b e f h^2 p^2 + 12 a^2 c e f h^2 p^2 + 3 a^2 f^2 h^2 p^2 + 3 b^2 f^2 h^2 p^2 +$$
$$\frac{21}{5} a^2 b^2 f^2 h^2 p^2 + 12 a b c f^2 h^2 p^2 + 3 a^2 c^2 f^2 h^2 p^2 + 8 b d k p^2 + \frac{168}{5} a^2 b d k p^2 + 16 a c d k p^2 +$$
$$16 a b e k p^2 + 8 a^2 c e k p^2 + 12 a^2 f k p^2 + 4 b^2 f k p^2 + \frac{84}{5} a^2 b^2 f k p^2 + 16 a b c f k p^2 +$$
$$4 a^2 c^2 f k p^2 + 4 b g k p^2 + \frac{84}{5} a^2 b g k p^2 + 8 a c g k p^2 + \frac{84}{5} b d^2 g k p^2 + \frac{2483}{25} a^2 b d^2 g k p^2 +$$
$$\frac{168}{5} a c d^2 g k p^2 + \frac{336}{5} a b d e g k p^2 + 24 c d e g k p^2 + \frac{168}{5} a^2 c d e g k p^2 + 12 b e^2 g k p^2 +$$
$$\frac{84}{5} a^2 b e^2 g k p^2 + 24 a c e^2 g k p^2 + 12 d f g k p^2 + \frac{252}{5} a^2 d f g k p^2 + \frac{84}{5} b^2 d f g k p^2 +$$
$$\frac{2483}{25} a^2 b^2 d f g k p^2 + \frac{336}{5} a b c d f g k p^2 + 12 c^2 d f g k p^2 + \frac{84}{5} a^2 c^2 d f g k p^2 + 24 a e f g k p^2 +$$
$$\frac{168}{5} a b^2 e f g k p^2 + 24 b c e f g k p^2 + \frac{168}{5} a^2 b c e f g k p^2 + 24 a c^2 e f g k p^2 + 12 b f^2 g k p^2 +$$
$$\frac{252}{5} a^2 b f^2 g k p^2 + 24 a c f^2 g k p^2 + 8 a b h k p^2 + 4 a^2 c h k p^2 + \frac{168}{5} a b d^2 h k p^2 +$$
$$12 c d^2 h k p^2 + \frac{84}{5} a^2 c d^2 h k p^2 + 24 b d e h k p^2 + \frac{168}{5} a^2 b d e h k p^2 + 48 a c d e h k p^2 +$$
$$24 a b e^2 h k p^2 + 12 a^2 c e^2 h k p^2 + 24 a d f h k p^2 + \frac{168}{5} a b^2 d f h k p^2 + 24 b c d f h k p^2 +$$
$$\frac{168}{5} a^2 b c d f h k p^2 + 24 a c^2 d f h k p^2 + 12 a^2 e f h k p^2 + 12 b^2 e f h k p^2 + \frac{84}{5} a^2 b^2 e f h k p^2 +$$
$$48 a b c e f h k p^2 + 12 a^2 c^2 e f h k p^2 + 24 a b f^2 h k p^2 + 12 a^2 c f^2 h k p^2 + 3 a^2 k^2 p^2 + b^2 k^2 p^2 +$$
$$\frac{21}{5} a^2 b^2 k^2 p^2 + 4 a b c k^2 p^2 + a^2 c^2 k^2 p^2 + 3 d^2 k^2 p^2 + \frac{63}{5} a^2 d^2 k^2 p^2 + \frac{21}{5} b^2 d^2 k^2 p^2 +$$
$$\frac{2483}{100} a^2 b^2 d^2 k^2 p^2 + \frac{84}{5} a b c d^2 k^2 p^2 + 3 c^2 d^2 k^2 p^2 + \frac{21}{5} a^2 c^2 d^2 k^2 p^2 + 12 a d e k^2 p^2 +$$
$$\frac{84}{5} a b^2 d e k^2 p^2 + 12 b c d e k^2 p^2 + \frac{84}{5} a^2 b c d e k^2 p^2 + 12 a c^2 d e k^2 p^2 + 3 a^2 e^2 k^2 p^2 +$$

$3 b^2 e^2 k^2 p^2 + \frac{21}{5} a^2 b^2 e^2 k^2 p^2 + 12 a b c e^2 k^2 p^2 + 3 a^2 c^2 e^2 k^2 p^2 + 12 b d f k^2 p^2 +$

$\frac{252}{5} a^2 b d f k^2 p^2 + 24 a c d f k^2 p^2 + 24 a b e f k^2 p^2 + 12 a^2 c e f k^2 p^2 + 15 a^2 f^2 k^2 p^2 +$

$3 b^2 f^2 k^2 p^2 + \frac{63}{5} a^2 b^2 f^2 k^2 p^2 + 12 a b c f^2 k^2 p^2 + 3 a^2 c^2 f^2 k^2 p^2 + 3 a^2 b^2 p^4 + a^2 d^2 p^4 +$

$b^2 d^2 p^4 + \frac{93}{10} a^2 b^2 d^2 p^4 + 4 a b c d^2 p^4 + a^2 c^2 d^2 p^4 + 4 a b^2 d e p^4 + 4 a^2 b c d e p^4 + a^2 b^2 e^2 p^4 +$

$4 a^2 b d f p^4 + a^2 b^2 f^2 p^4 + 4 a^2 d g p^4 + 4 b^2 d g p^4 + \frac{186}{5} a^2 b^2 d g p^4 + 16 a b c d g p^4 +$

$4 a^2 c^2 d g p^4 + 8 a b^2 e g p^4 + 8 a^2 b c e g p^4 + 8 a^2 b f g p^4 + a^2 g^2 p^4 + b^2 g^2 p^4 + \frac{93}{10} a^2 b^2 g^2 p^4 +$

$4 a b c g^2 p^4 + a^2 c^2 g^2 p^4 + d^2 g^2 p^4 + \frac{93}{10} a^2 d^2 g^2 p^4 + \frac{93}{10} b^2 d^2 g^2 p^4 + \frac{4543}{50} a^2 b^2 d^2 g^2 p^4 +$

$\frac{186}{5} a b c d^2 g^2 p^4 + 3 c^2 d^2 g^2 p^4 + \frac{93}{10} a^2 c^2 d^2 g^2 p^4 + 4 a d e g^2 p^4 + \frac{186}{5} a b^2 d e g^2 p^4 +$

$12 b c d e g^2 p^4 + \frac{186}{5} a^2 b c d e g^2 p^4 + 12 a c^2 d e g^2 p^4 + a^2 e^2 g^2 p^4 + 3 b^2 e^2 g^2 p^4 +$

$\frac{93}{10} a^2 b^2 e^2 g^2 p^4 + 12 a b c e^2 g^2 p^4 + 3 a^2 c^2 e^2 g^2 p^4 + 4 b d f g^2 p^4 + \frac{186}{5} a^2 b d f g^2 p^4 +$

$8 a c d f g^2 p^4 + 8 a b e f g^2 p^4 + 4 a^2 c e f g^2 p^4 + 3 a^2 f^2 g^2 p^4 + b^2 f^2 g^2 p^4 + \frac{93}{10} a^2 b^2 f^2 g^2 p^4 +$

$4 a b c f^2 g^2 p^4 + a^2 c^2 f^2 g^2 p^4 + 8 a b^2 d h p^4 + 8 a^2 b c d h p^4 + 4 a^2 b^2 e h p^4 + 4 a b^2 g h p^4 +$

$4 a^2 b c g h p^4 + 4 a d^2 g h p^4 + \frac{186}{5} a b^2 d^2 g h p^4 + 12 b c d^2 g h p^4 + \frac{186}{5} a^2 b c d^2 g h p^4 +$

$12 a c^2 d^2 g h p^4 + 4 a^2 d e g h p^4 + 12 b^2 d e g h p^4 + \frac{186}{5} a^2 b^2 d e g h p^4 + 48 a b c d e g h p^4 +$

$12 a^2 c^2 d e g h p^4 + 12 a b^2 e^2 g h p^4 + 12 a^2 b c e^2 g h p^4 + 16 a b d f g h p^4 + 8 a^2 c d f g h p^4 +$

$8 a^2 b e f g h p^4 + 4 a b^2 f^2 g h p^4 + 4 a^2 b c f^2 g h p^4 + a^2 b^2 h^2 p^4 + a^2 d^2 h^2 p^4 + 3 b^2 d^2 h^2 p^4 +$

$\frac{93}{10} a^2 b^2 d^2 h^2 p^4 + 12 a b c d^2 h^2 p^4 + 3 a^2 c^2 d^2 h^2 p^4 + 12 a b^2 d e h^2 p^4 + 12 a^2 b c d e h^2 p^4 +$

$3 a^2 b^2 e^2 h^2 p^4 + 4 a^2 b d f h^2 p^4 + a^2 b^2 f^2 h^2 p^4 + 8 a^2 b d k p^4 + 4 a^2 b^2 f k p^4 + 4 a^2 b g k p^4 +$

$4 b d^2 g k p^4 + \frac{186}{5} a^2 b d^2 g k p^4 + 8 a c d^2 g k p^4 + 16 a b d e g k p^4 + 8 a^2 c d e g k p^4 +$

$4 a^2 b e^2 g k p^4 + 12 a^2 d f g k p^4 + 4 b^2 d f g k p^4 + \frac{186}{5} a^2 b^2 d f g k p^4 + 16 a b c d f g k p^4 +$

$4 a^2 c^2 d f g k p^4 + 8 a b^2 e f g k p^4 + 8 a^2 b c e f g k p^4 + 12 a^2 b f^2 g k p^4 + 8 a b d^2 h k p^4 +$

$4 a^2 c d^2 h k p^4 + 8 a^2 b d e h k p^4 + 8 a b^2 d f h k p^4 + 8 a^2 b c d f h k p^4 + 4 a^2 b^2 e f h k p^4 +$

$a^2 b^2 k^2 p^4 + 3 a^2 d^2 k^2 p^4 + b^2 d^2 k^2 p^4 + \frac{93}{10} a^2 b^2 d^2 k^2 p^4 + 4 a b c d^2 k^2 p^4 + a^2 c^2 d^2 k^2 p^4 +$

$4 a b^2 d e k^2 p^4 + 4 a^2 b c d e k^2 p^4 + a^2 b^2 e^2 k^2 p^4 + 12 a^2 b d f k^2 p^4 + 3 a^2 b^2 f^2 k^2 p^4 +$

$a^2 b^2 d^2 p^6 + 4 a^2 b^2 d g p^6 + a^2 b^2 g^2 p^6 + a^2 d^2 g^2 p^6 + b^2 d^2 g^2 p^6 + \frac{82}{5} a^2 b^2 d^2 g^2 p^6 +$

$4 a b c d^2 g^2 p^6 + a^2 c^2 d^2 g^2 p^6 + 4 a b^2 d e g^2 p^6 + 4 a^2 b c d e g^2 p^6 + a^2 b^2 e^2 g^2 p^6 +$

$4 a^2 b d f g^2 p^6 + a^2 b^2 f^2 g^2 p^6 + 4 a b^2 d^2 g h p^6 + 4 a^2 b c d^2 g h p^6 + 4 a^2 b^2 d e g h p^6 + a^2 b^2 d^2 h^2 p^6 + 4 a^2 b d^2 g k p^6 + 4 a^2 b^2 d f g k p^6 + a^2 b^2 d^2 k^2 p^6 + a^2 b^2 d^2 g^2 p^8$

- In the previous step, we have subtracted the right-hand side of the GPI from the left-hand side and then divided out the common factor E[|U1|^y] = $\frac{2^{y/2}\,\text{Gamma}\left[\frac{1+y}{2}\right]}{\sqrt{\pi}}$. This leaves us with the polynomial above, which we will show is non-negative by asking Mathematica to find an (exact) SOS decomposition for it. Also, we have replaced y with p^2 + 1/10 so that we are working only with the free variables {a,b,c,d,e,f,g,h,k,p} that can take any value on the reals.
- From the Mathematica documentation:
  PolynomialSumOfSquaresList[f, $vars$]
  attempts to find polynomials with real coefficients $\{f_1, \ldots, f_n\}$ such that $f == f_1^2 + \ldots + f_n^2$.

In[21]:=
```
Fsos = PolynomialSumOfSquaresList[F, {a, b, c, d, e, f, g, h, k, p}]
```

Out[21]=

$$\Big\{\sqrt{2} + \frac{5\,854\,679\,373\, d\, g}{2\,750\,058\,580\,\sqrt{2}} + \frac{9\,190\,281\,809\,417\, a\, e\, g}{5\,790\,187\,167\,380\,\sqrt{2}} + \frac{307\,791\,432\,241\,133\,623\, b\, f\, g}{207\,238\,439\,470\,989\,080\,\sqrt{2}} + \frac{98\,206\,714\,699\,376\,480\,445\,958\,058\,561\, a\, c\, f\, g}{81\,984\,699\,384\,664\,350\,563\,993\,624\,128\,\sqrt{2}} + \frac{2\,871\,741\,304\,886\,781\, a\, d\, h}{1\,809\,293\,744\,250\,080\,\sqrt{2}} + \frac{27\,969\, e\, h}{5785\,\sqrt{2}} + \ldots 9 \ldots + \frac{37\,805\,185\,925\,964\,067\, b\, f\, g\, p^2}{35\,278\,014\,265\,328\,560\,\sqrt{2}} + \frac{160\,434\,733\,512\,065\,401\,997\,652\,981\,187\, a\, c\, f\, g\, p^2}{184\,169\,669\,767\,022\,495\,609\,582\,723\,040\,\sqrt{2}} + \frac{1\,860\,238\,623\,975\,053\, a\, d\, h\, p^2}{1\,606\,615\,443\,841\,200\,\sqrt{2}} + \frac{35\,118\,985\,844\,680\,204\,246\,967\,500\,066\,207\, a\, b\, f\, h\, p^2}{38\,270\,036\,244\,993\,879\,903\,661\,565\,601\,120\,\sqrt{2}} + \frac{59\,675\,165\,948\,284\,951\, b\, d\, k\, p^2}{55\,686\,048\,649\,070\,112\,\sqrt{2}} + \frac{197\,612\,343\,504\,552\,784\,532\,856\,552\,481\, a\, c\, d\, k\, p^2}{226\,847\,398\,636\,801\,313\,708\,576\,229\,120\,\sqrt{2}} + \frac{7\,125\,642\,256\,642\,552\,070\,871\,547\,234\,286\,509\, a\, b\, e\, k\, p^2}{7\,764\,991\,629\,911\,448\,449\,941\,363\,895\,957\,760\,\sqrt{2}},\ \ldots 222 \ldots,\ \frac{\sqrt{\ldots 1 \ldots}\ \ldots 4 \ldots\ p^4}{69\,454\,403\ \ldots 210 \ldots\ 01\,309\,000}\Big\}$$

Size in memory: 2.7 MB | + Show more | Show all | Iconize | Store full expression in notebook

In[22]:=
```
Fsos.Fsos - F // Expand
```

Out[22]=

```
0
```

- This last step confirms that the SOS decomposition, i.e. the dot product of Fsos with itself, is exactly equal to the original polynomial, F.
- Thus, by approximation on {a,b,c,d,e,f,g,h,k}, **E[|X1|^y X2^2 X3^2 X4^2 X5^2] >= E[|X1|^y] E[X2^2] E[X3^2] E[X4^2] E[X5^2] for y >= 1/10 is proved**.
- Note that to prove that the equality sign holds if and only if X1, X2, X3, X4, X5 are independent, we would need to consider multiple cases with different definitions of X3, X4 and X5, similar to what we did in our proofs of the 4-D GPIs in previous sections. Since the goal of this section is simply to demonstrate a new technique, we omit those extra cases here in the interest of space, but encourage the interested reader to verify and expand upon our results.

7.5 Discussion

The GPI conjecture is an extremely difficult problem to solve, particularly when some of the correlations are negative. The SOS method described in this chapter can be used to rigorously verify *any* specific case of the GPI (7.1), constrained only by computing power. Furthermore, as demonstrated in Section 7.3, this method is even powerful enough to prove GPIs with one exponent unbounded, a feat that is extremely difficult by purely theoretical methods. Moreover, in Section 7.4 we proved a new five-dimensional GPI, showing that we can even use SOS methods to verify more general GPIs with at least one unbounded *real* exponent. On the other hand, should the GPI conjecture not hold in its full generality, our methods may prove quite useful in the search for a counterexample. Our algorithms are efficient, straightforward and produce exact results. Furthermore, whereas calculations of multivariate Gaussian moments are often burdened by the constraints imposed by the covariance matrix, our methods have the advantage of using free variables (with domain over the reals).

Despite the fact that the GPI is widely believed to be true, as of yet there has not been much to strongly support this presumption. At the time the original preprint of this work was posted online, all theorems in the paper [37] constituted never-before obtained results. Still today, to the best of our knowledge, Theorem 7.3.2 and the five-dimensional GPI of Section 7.4 have not been proved by any other methods. In fact, using the methods outlined above, we were able to verify many more GPIs with one exponent unbounded and higher fixed exponents, the proofs of which we omit seeing as the SOS decompositions, although exact, can be quite long. Thus, with the help of software, our work provides some of the first legitimate and substantial support to the correctness of the GPI. Therefore, we propose a stronger version of the GPI:

Conjecture 7.5.1 *Let $n \geq 3$, $m_1, \ldots, m_n \in \mathbb{N}$, and $U_1, \ldots, U_n$ be independent standard Gaussian random variables. Define the polynomial H on $\mathbb{R}^{n(n-1)/2}$ by*

$$\begin{aligned} & H(x_{21}, x_{31}, x_{32}, \ldots, x_{n1}, \ldots, x_{n,n-1}) \\ = \; & E\left[U_1^{2m_1} \prod_{i=2}^{n} \left(\sum_{j=1}^{i-1} x_{ij} U_j + U_i\right)^{2m_i}\right] - (2m_1 - 1)!! \prod_{i=2}^{n} E\left[\left(\sum_{j=1}^{i-1} x_{ij} U_j + U_i\right)^{2m_i}\right]. \end{aligned}$$

Then, H has an SOS representation.

By approximation, we find that if Conjecture 7.5.1 is true then the GPI (7.1) holds. By virtue of this conjecture, we have connected the probability inequality to analysis, algebra, geometry, combinatorics, mathematical programming and computer science. In particular, research on SOS is abundant and ongoing (see, for example, [14]), but so far, no theoretical results in that field have been used to prove a GPI. Although software is used, our rigorous proofs establish the first meaningful link between the GPI and SOS, and should stimulate those working on SOS.

Chapter 8

Next steps

In this chapter, we propose future avenues of research which stem naturally from our work. We hope that our contributions towards solving the GPI conjecture can lead to further progress and, hopefully, an eventual full proof.

8.1 Three-dimensional GPI for real exponents

The global minimum of $E\left[\prod_{j=1}^{n} X_j^{2m_j}\right]$ occurs when the covariance matrix of $(X_1, \ldots, X_n)$ is either in the interior or on the boundary of the set of real symmetric positive matrices. To prove the three-dimensional GPI when all exponents are even integers, Herry et al. [15] consider these two cases separately. When the minimum is in the interior, they use induction starting with the already proven two-dimensional GPI by applying formulas that dictate the partial derivatives of multivariate Gaussian moments (see their Lemma 2.1).

The reliance on these formulas, however, is what might prevent their method from being applied to the general three-dimensional GPI where the exponents, y_j, are real numbers:

$$E\left[\prod_{j=1}^{3}|X_j|^{y_j}\right] \geq \prod_{j=1}^{3} E[|X_j|^{y_j}].$$

In this case, the absolute values of the Gaussian random variables are present, which results in an extra layer of complication.

On the other hand, using the methods of Chapters 5 and 6, we may prove three-dimensional GPIs that involve absolute values and exponents which are not restricted to being even integers (see Proposition 5.4.1, Theorem 6.1.2 and Theorem 6.2.2 above).

Thus, we propose the continued use of hypergeometric functions to attempt to fully solve the general three-dimensional GPI.

8.2 Four-dimensional GPI

When Herry et al. [15] consider the case where the minimum is on the boundary, they use a result that relies on the fact that the rank of the covariance matrix corresponding to the minimum point is exactly 1 less than n (see their Lemma 2.3). For the three-dimensional case, this is true. However, for dimension four and beyond, this strategy might not work.

On the other hand, using our SOS methods in Chapter 7, we were already able to produce partial results for the four-dimensional and five-dimensional GPI (see Theorem 7.3.2 and Section 7.4 above). In theory, limited only by computer processing, we may prove GPIs in any dimension, even when we let one exponent be unbounded.

Furthermore, the standard multivariate Gaussian distribution is the invariant distribution of the Ornstein-Uhlenbeck semigroup, so it is natural to consider it when approaching the GPI. This is the strategy employed in the paper by Malicet et al. [28] when they prove an alternate form of the GPI for Hermite polynomials.

Thus, as a future avenue of research, we propose adapting Herry et al.'s and Malicet et al.'s methods in tandem with our own to prove the four-dimensional GPI.

8.3 SOS representation of GPI

The contents of this section are taken from our paper [37].

It is well-known that a non-negative multivariate polynomial may not have an SOS representation. Denote by $P_{n,2d}$ the set of all non-negative polynomials in n variables of degree at most $2d$ and $\sum_{n,2d}$ the set of all polynomials in $P_{n,2d}$ that are SOS. In 1888, Hilbert proved that $\sum_{n,2d} = P_{n,2d}$ if and only if $n = 1$ or $d = 1$ or $(n, d) = (2, 2)$ (see [16]). However, we would like to point out that $E\left[\prod_{j=1}^{n} X_j^{2m_j}\right]$ itself is an SOS. To see this, let $U_1, \dots, U_n$ be independent standard Gaussian random variables. For $x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn} \in \mathbb{R}$, we can write:

$$X_i = \sum_{j=1}^{n} x_{ij} U_j, \qquad 1 \le i \le n.$$

Then, we have

$$\prod_{j=1}^{n} X_j^{2m_j} = a'b,$$

where $a' = (1, x_{11}, \dots, x_{nn}, x_{11}^2, x_{11}x_{12}, \dots, x_{1n}^{m_1} \cdots x_{nn}^{m_n})$ is a vector of monomials of length $l \le n^{\sum_{j=1}^{n} m_j}$ and $b' = (P_1(U_1, \dots, U_n), P_2(U_1, \dots, U_n), \dots, P_l(U_1, \dots, U_n))$ is a vector of polynomials also of length l. Thus,

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] = E\left[\left(\prod_{j=1}^{n} X_j^{m_j}\right)^2\right] = E[a'bb'a] = a'Qa,$$

where $Q = bb'$ is a non-negative definite matrix. Therefore, $E\left[\prod_{j=1}^{n} X_j^{2m_j}\right]$ has an SOS representation.

Thus, another way to attempt to solve the GPI would be to show that

$$E\left[\prod_{j=1}^{n} X_j^{2m_j}\right] - \prod_{j=1}^{n} E[X_j^{2m_j}]$$

has an SOS representation.

8.4 IRGA

Our papers have received attention from some prominent figures in the research community. In addition to our results being cited in important papers like the three-dimensional proof by Herry et al. [15], Thomas Royen, Donald Richards and Victor Magron have all shown interest and support for our work. More recently, as a result of Jeffrey Uhlmann discovering our SOS paper [37], we are collaborating with him to help finish proving that the inverse of the relative gain array (IRGA) of a d-dimensional positive definite matrix is doubly stochastic for $d \leq 6$ using SOS methods.

The relative gain array (RGA) is defined as

$$RGA(G) = G \circ (G^{-1})^T,$$

where $\circ$ represents the Hadamard product, and the IRGA is its inverse. This concept is important in the design of process control systems [7].

In a 2021 paper by Uhlmann and Wang [43], Uhlmann provides the following conjecture:

Conjecture 8.4.1 *For an $n \times n$, $n \leq 6$, positive definite matrix G, IRGA(G) is positive definite, has unit row and column sums, and is non-negative and hence doubly stochastic.*

The critical part of solving this conjecture is proving non-negativity. In order to do this for, say dimension k, it is sufficient to consider a general $k \times k$ positive definite matrix G, compute its IRGA and then show that a typical off-diagonal element is non-negative [43].

As it turns out, the off-diagonal elements of these matrices are polynomials. Thus, one way of proving non-negativity is by demonstrating that such a polynomial has an SOS decomposition. As in the case of our proofs in Chapter 7, the authors realized that such a decomposition must be exact, or rational, for the proof to be considered rigorous.

In their paper, they are able to finish the proof up to and including $n = 4$ (see [43, Theorem 3]). For the 4×4 case, using software it was not difficult for them to obtain an approximate SOS decomposition. On the other hand, it required much work on their part to convert this into a rational SOS decomposition.

After discussing the problem with Dr. Uhlmann, we used our usual SOS techniques, which involve a combination of **Mathematica** and **Macaulay2**, to directly obtain a rational SOS decomposition of a typical off-diagonal element of the 4×4 IRGA, thereby bypassing the step that gave them the most trouble, and providing a quicker alternate proof of their main result.

Thus, as a future avenue of research, we propose developing SOS techniques in an attempt to prove the $n = 5$ and $n = 6$ cases of Dr. Uhlmann's IRGA conjecture.

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