THE COBORDISM RING: THE PERSPECTIVE OF CHARACTERISTIC CLASSES

Benedetta Andina

A Thesis
in the Department
of
Mathematics and statistics

Presented in Partial Fulfillment of the
Requirements
for the Degree of
Master of Science, Faculty of Arts & Science
(Mathematics) at
Concordia University
Montréal, Québec, Canada

July 2023

CONCORDIA UNIVERSITY
School of Graduate Studies

This is to certify that the thesis
By: **Benedetta Andina**
Entitled: **The cobordism ring: the perspective of characteristic classes**
and submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE, FACULTY OF ARTS & SCIENCE (MATHEMATICS)

complies with the regulations of this University and meets the accepted standards with respect to originality and quality. Signed by the final examining commitee:

________________________________ Chair
Dr. Bruno Chiareleotto

________________________________ External Examiner
Dr. Denis Benois

________________________________ Examiner
Dr. Adrian Iovita

________________________________ Supervisor
Dr. Adrian Iovita

Approved by: ________________________________
Dr Lea Popovic
Graduate Program Director

______________ ________________________________
Date Dr. Pascale Sicotte
Dean of Faculty of Arts and Science

Abstract

The cobordism ring: the prospective of characteristic classes.
Benedetta Andina

After providing sufficient preliminaries to make this thesis accessible to any graduate student, it clearly highlights how both the oriented and unoriented cobordism rings can be studied using characteristic classes, which are usually easily computable. To demonstrate this relationship in the unoriented case (as presented by Pontrjagin and Thom), the author unveils an explicit structure of the unoriented cobordism ring. This structure provides a clear classification of smooth closed manifolds.

Acknowledgments

First of all, I want to thank Duncan McCoy for supervising my master thesis. Every answer to my questions, every advice on the math itself, the structure and the depth of my dissertation was helpful and necessary for me in order to complete this written work. I am really grateful for his patience and his constant help throughout this journey.

I wish to thank also Adrian Iovita and Giovanni Rosso, for their tips ad their guidance during this year at Concordia. I would like to acknowledge the importance of Misja Steinmetz, who really counseled me during my first year at Leiden University.

A special mention goes to Carmen Buffone and Laura van Kempen, the best assistants I could have ever asked for. I probably would not have the right bureaucratic form to graduate without them.

Moreover, I would like to express my gratitude to all the people that carry on the ALGANT program: it was a really crucial professional experience. Thanks to all the professors that gave me the opportunity to deepen my mathematical knowledge.

Lastly, I would like to show my appreciation to my family, my friends and colleagues, who believed in me and supported me, no matter the distance.

To my family,

and to the friends who became my family.

Contents

Introduction

In the contest of smooth closed manifolds, it is typical to wonder whether two manifolds are diffeormorphic. However, this problem can be unsolvable in general. One might try to classify manifolds with a less strict equivalence relation. This is where the concept of cobordism arises.

Two manifolds are considered cobordant if their union is the boundary of some manifold with boundary. Using this equivalence, it is possible to create a group of equivalence classes, and moreover it is possible to give an internal product in order to create the cobordism ring.

In this thesis we try to understand the structure of this latter, using as a tool some characteristic classes. In particular, we focus on Stiefel–Whitney and Pontrjagin classes. The first ones are fundamental in the study of the unoriented cobordism ring whereas the second are used to study the oriented cobordism ring (this considers oriented manifolds).

We see how computing Stiefel–Whitney and Pontrjagin classes on the tangent bundle of a manifold tells us if this manifold is a boundary or not and also the converse implication. Indeed, by two theorems by Pontrjagin, if a manifold is a boundary then all its Stiefel–Whitney and Pontrjagin numbers will be zero. The converse statement is true by a theorem by Thom (in the unoriented case) and a theorem by Wall (in the oriented case).

We finally prove Thom's theorem. In order to do that, we study in more depth the structure of the unoriented cobordism ring. After defining the Thom spectrum, we see how its homology groups allow a product operation between them, and thus theirs direct sum gives the structure of a graded ring. We show, using the Thom–Pontrjagin construction, that this graded ring is isomorphic as an algebra to the cobordism ring. Using the Hurewicz morphism we relate this ring to the homology ring of the Thom prespectrum, that has a known structure thanks to its relation to the Grassmannian manifold. Through this process, we are able to even see the explicit generators of the unoriented cobordism ring as a $\mathbb{Z}/2\mathbb{Z}$-algebra.

Contents

1. In the first chapter we focus on the prerequisites concerning the characteristic classes we use in the thesis. First of all, we define vector bundles and some structures on them. We then pass to the study of the unoriented characteristic classes, the Stiefel–Whitney classes, and we define the Stiefel–Whitney numbers of a smooth closed manifold. We also give an example of computation of these for the projective space. We then center our attention to oriented vector bundles and characteristic classes constructed on them. We study the Euler class and, in the contest of complex vector bundles, the Chern classes. These are necessary in order to construct the Pontrjagin class.

2. In the second chapter, we finally give the definition of (unoriented and oriented) cobordism equivalence, n-cobordism group and (unoriented and oriented) cobordism ring. We highlight its relation with characteristic classes by proving the Pontrjagin theorems (for the unoriented and the oriented scenario) and by stating Thom and Wall's theorem.

3. In the last chapter, we concentrate on the unoriented cobordism ring, and we prove Thom's theorem. In order to do so, we define the Grassmann manifold and the universal vector bundle on it. We then define the Thom space of a vector bundle, and we study some properties, both in general and in the case of the Thom space of the universal bundle, which in particular defines the Thom spectrum. We construct the homotopy ring of the Thom spectrum, and we prove its isomorphism with the cobordism ring. Moreover, knowing the cohomology ring of the Grassmannian, using the Thom isomorphism and some duality, we give the structure of the homology ring of the Thom spectrum. The last thing we use to prove Thom's theorem is the Hurewicz homomorphism and its injectivity. This also gives us the exact structure of the cobordism ring, and so we also discuss its explicit generators.

I

Prerequisites

In this chapter we are going to give some definitions and first results about vector bundles and characteristic classes in order to use these notions later in the thesis. Indeed, the goal of the dissertation is to show the connection between characteristic classes and numbers (which are numbers defined out of characteristic classes and associated to some manifold) and the unoriented and oriented cobordism theory.

I.1 Vector bundles

The aim of this section is to define the structure of vector bundles, together with maps between them and ways to construct new vector bundles from old ones. This will give us the opportunity to define various characteristic classes of vector bundles.

Definition I.1.1. Let B be a topological space. A *real vector bundle* ξ over B is a map $\pi : E \to B$ where $E = E(\xi)$ is a topological space and for each $b \in B$, $\pi^{-1}(b)$ is a real vector space. Moreover, these objects must satisfy the *local triviality* condition: for every $b \in B$, there exist a neighborhood of b $U \subset B$, a non-negative integer n and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$. For each $b \in U$,

$$\begin{aligned} \mathbb{R}^n &\to \pi^{-1}(b) \\ x &\mapsto h(b, x) \end{aligned}$$

defines an isomorphism $\mathbb{R}^n \cong \pi^{-1}(b)$. The space B is called the *base space*, E the *total space* and π the *projection map* of the vector bundle. Lastly, (U, h) will be called a *local coordinate system* for ξ. If we can choose $U = B$, then we will call ξ a *trivial bundle*.

For $b \in B$ we may denote the vector space $\pi^{-1}(b)$ (which is the *fiber* over b) by F_b or $F_b(\xi)$. If the dimension of every fiber is n, we will call the vector bundle an *n-dimensional vector bundle* (or *n-plane bundle* or even *$\mathbb{R}^n$-bundle*). In any case, this dimension will be locally constant.

We will define in a later section a *complex vector bundle* similarly, using a complex vector space instead of a real vector space, and taking homeomorphisms $h : U \times \mathbb{C}^n \to \pi^{-1}(U)$ for the local triviality condition.

Furthermore, we can also define the concept of a *smooth vector bundle*, taking B and E as smooth manifolds, π as a smooth map and each h as a diffeomorphism.

Note that even though we do not require explicitly the projection π to be surjective, the fact that for each $b \in B$ its fiber is a vector space implies that this fibre must be non empty (indeed, if it is isomorphic to a 0-dimensional vector space it consists of a single point).

Let now see some examples:

Example I.1.1.

1. We will denote by ε^n_B the trivial $\mathbb{R}^n$-bundle with base space B, total space $B \times \mathbb{R}^n$ and projection maps $\pi(b, x) = b$;

2. The *tangent bundle* τ_M of a smooth manifold M will have as the total space the tangent manifold

$$DM = \{(x, v)|\ x \in M,\ v \text{ tangent to } M \text{ at } x\} = \cup_{x \in M} DM_x,$$

base space M and projection maps $\pi(x, v) = x$. We denoted by DM_x the tangent space of M at $x \in M$: it is the fibre of the point x. By saying that a vector is tangent to the manifold at a point we mean that the vector may be expressed as the velocity vector $\frac{dp}{dt}|_{t=0}$ of some smooth path $p : (-\delta, \delta) \to M$, $\delta \in \mathbb{R}_{>0}$, such that $p(0) = x$. The bundle τ_M is an example of a smooth vector bundle. If M has constant dimension n, then also each DM_x has dimension n, so τ_M is an n-dimensional vector bundle;

3. The *normal bundle* ν of a smooth manifold M embedded in a manifold N has as total space

$$\{(x, v)|\ x \in M,\ v \text{ orthogonal to } DM_x\},$$

base space M and projection maps $\pi(x, v) = x$, where by $v \in N$ orthogonal we mean that the standard real inner product of v with any vector in DM_x is zero. If M has constant dimension n and it is embedded in a manifold of dimension $n+q$, the space of normal vectors to a point will have dimension q, and thus ν will be an q-dimensional vector bundle;

4. Taking the real projective space $\mathbb{P}^n$ as base space (seen as the quotient $\mathbb{S}^n$ via the equivalent relation $x \sim -x$ for $x \in \mathbb{S}^n$), the *canonical line bundle* is denoted by γ_n^1. It consists of total space $E \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$, with $([x], v) \in E$ where $x \in \mathbb{S}^n$, $[x] = \{x, -x\}$ and v a scalar multiple of x. The projection maps are just $\pi([x], v) = [x]$ and $\pi^{-1}([x]) = \{[x]\} \times l$ where $l \subset \mathbb{R}^{n+1}$ is the line containing x and $-x$.

We defer to [MS05] for the proofs of local triviality and the structures of vector spaces of the fibres. In particular, the structure of the fibres is always quite natural to describe.

Now, let η and ξ be two vector bundles over the same base space B.

Definition I.1.2. η is *isomorphic* to ξ ($\eta \cong \xi$) if there exists a homeomorphism $f : E(\eta) \to E(\xi)$ such that for every $b \in B$, $F_b(\eta)$ is sent via f isomorphically (as vector spaces, so in particular f is linear) to $F_b(\xi)$.

We now need to define the concept of sections and nowhere dependency:

Definition I.1.3. Let ξ be a vector bundle with base space B. A *section* of ξ is a continuous function

$$s : B \to E(\xi)$$

that takes each $b \in B$ to a point in $F_b(\xi)$. A section is *nowhere zero* if $s(b)$ is a non-zero vector of $F_b(\xi)$ for every $b \in B$.

Remark I.1.4. Asking that $s(b)$ is a vector of $F_b(\xi)$ is equivalent to asking that $\pi \circ s = \mathrm{id}_B$.

Moreover, notice that each vector bundle has as a canonical section which just maps each point of the base space to the zero vector in each fibre.

Definition I.1.5. This section is called the *zero section*, and we will denote it by s_0.

Note that for any smooth manifold M, the zero section of its tangent bundle and of its normal bundle are diffeomorphic to M itself.

Since we are working with vectors in vector spaces, we need to define some condition regarding the linear dependence/independence.

Definition I.1.6. Let $\{s_1, \dots, s_n\}$ be sections of a vector bundle ξ with base space B. These sections are *nowhere dependent* if the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in $F_b(\xi)$ for every $b \in B$.

Obviously, if the minimal dimension of the fibres of ξ is some natural number m, we can have at most m nowhere linear sections.

Remark I.1.7. The existence of nowhere dependent sections gives us a nice characterization of trivial n-plane bundles. Indeed, given an n-dimensional vector bundle, this is isomorphic (as a vector bundle) to the trivial bundle if and only if there exist n nowhere dependent sections $s_1, \dots, s_n$. The first direction is obvious since given a trivial vector bundle, we easily can find n nowhere dependent sections, and the isomorphism will take independent vectors to independent vectors, which will be the nowhere dependent sections of our n-plane bundle. On the other hand, if we have nowhere dependent sections $s_1, \dots, s_n$ of $\xi : E \to B$, we can define the map

$$\begin{aligned} f : B \times \mathbb{R}^n &\to E \\ (b, t_1, \dots, t_n) &\mapsto \sum_{i=1}^{n} t_i s_i(b) \end{aligned}$$

which is linear in each fibre and will be an isomorphism thanks to [Hat17, Lemma 1.1].

Now that we have constructed the basic concepts to work with vector bundles, we would like to construct and study new bundles from pre-existing ones.

Let ξ be a vector bundle with projection $\pi : E \to B$. It is possible to construct new vector bundles from ξ.

(i) Restricting a bundle to a subset of the base space: let $\overline{B} \subset B$. If we set $\overline{E} = \pi^{-1}(\overline{B})$ and $\overline{\pi} = \pi|_{\overline{E}} : \overline{E} \to \overline{B}$, we get a vector bundle that we will denote by $\xi|_{\overline{B}}$ and call the *restriction* of ξ to $\overline{B}$. The fibres will be the same of the original bundle ($F_b(\xi) = F_b(\xi|_{\overline{B}})$ for $b \in \overline{B}$) and the same vector space structure of the corresponding ones in ξ.

(ii) Induced bundles: let B' be a topological space and $f : B' \to B$ any map. We can construct the *induced bundle* $f^*\xi$ over B' as the vector bundle with total space $E' = \{(b, e) \in B' \times E | \ f(b) = \pi(e)\}$ and projection map

$$\begin{aligned} \pi_1 : E' &\to B' \\ (b, e) &\mapsto b. \end{aligned}$$

If we define a map $\hat{f} : E' \to E$ as $\hat{f}(b, e) = e$, we thus get a commuting diagram

$$\begin{array}{ccc} E' & \xrightarrow{\hat{f}} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array} .$$

Via $\hat{f}$, $F_b(f^*\xi) \cong F_{f(b)}(\xi)$.

Observe that the bundle induced by the inclusion $\iota : \overline{B} \hookrightarrow B$ is isomorphic to the restriction bundle.

We can generalize the construction of $\hat{f}$ through the following definition:

Definition I.1.8. Let η and ξ be vector bundles, with total spaces $E(\eta), E(\xi)$ and base spaces $B(\eta), B(\xi)$. A *bundle map* from η to ξ is a continuous linear (i.e. linear on the fibers) function

$$g : E(\eta) \to E(\xi)$$

such that for each $b \in B(\eta)$, g sends isomorphically $F_b(\eta)$ into $F_{b'}(\xi)$ (as vector spaces) for some $b' \in B(\xi)$.

If we set $\overline{g}(b) = b'$, we get a continuous function $\overline{g} : B(\eta) \to B(\xi)$. We say that g *covers* $\overline{g}$ (or *lies over* $\overline{g}$).

Remark I.1.9. Any isomorphism of vector bundles is a bundle map between vector bundles with the same base space, $b = b'$ (i.e. $\overline{g}$ is the identity map) and $f : E(\eta) \to E(\xi)$ a homeomorphism (and not just continuous and linear).

Example I.1.2. If ε is a trivial vector bundle, then there exists a bundle map from ε to a vector bundle with base space a point. In particular, this vector bundle is the vector bundle with base space $\{p\}$, total space $\mathbb{R}^n$ and projection map $\pi(x) = p$: obviously $\pi^{-1}(p) = \mathbb{R}^n$ is a vector space and moreover, it obviously satisfies the local triviality condition. Then, the bundle map is a continuous function

$$g : B(\varepsilon) \times \mathbb{R}^n \to \mathbb{R}^n$$

with $g(F_b(\varepsilon)) = g(\{b\} \times \mathbb{R}^n) \cong \mathbb{R}^n = \pi^{-1}(p)$.

Moreover, let ξ_1, ξ_2 be a vector bundle with projection maps $\pi_i : E_i \to B_i$, $i = 1, 2$. It is possible to construct new vector bundles from ξ_1, ξ_2.

(iii) Cartesian products: The *Cartesian product* $\xi_1 \times \xi_2$ is the bundle with total space $E_1 \times E_2$, base space $B_1 \times B_2$ and projection map $\pi_1 \times \pi_2$. The fibers are given by $F_{b_1}(\xi_1) \times F_{b_2}(\xi_2)$ for each $(b_1, b_2) \in B_1 \times B_2$.

(iv) Whitney sums: Let $B_1 = B_2$. Consider the diagonal embedding

$$\begin{aligned} d : B &\to B \times B \\ b &\mapsto (b, b) \end{aligned}.$$

The induced bundle $d^*(\xi_1 \times \xi_2) := \xi_1 \oplus \xi_2$ is called the *Whitney sum* of ξ_1 and ξ_2. Each fiber $F_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum of vector spaces $F_b(\xi_1) \oplus F_b(\xi_2)$.

Lastly, we want to give other two definitions in order to present another construction:

Definition I.1.10. Let ξ and η be two vector bundles over the same base space B such that the total space of ξ is a subset of the total space of η. Then, we say that ξ is a *sub-bundle* of η ($\xi \subset \eta$) if for each $b \in B$, $F_b(\xi) \subset F_b(\eta)$ as vector spaces.

Recall that an *Euclidean vector space* is a real vector space V with a positive definite quadratic function $\mu : V \to \mathbb{R}$ (which determines an *inner product* $v \cdot w = \frac{1}{2}(\mu(v+w) - \mu(v) - \mu(w))$, for each $v, w \in V$). Then:

Definition I.1.11. A real vector bundle ξ, together with a continuous function $\mu : E(\xi) \to \mathbb{R}$, is an *Euclidean vector bundle* if the restriction of μ to each fiber of ξ is positive definite and quadratic.

(v) Orthogonal complements: If η is an Euclidean bundle with a sub-bundle ξ, we can define the *orthogonal complement* $\xi^\perp$ of ξ in η as the sub-bundle with total space $E(\xi^\perp)$, where this is the union of all the $F_b(\xi^\perp) = \{v \in F_b(\eta) |\ v \cdot w = 0\ \forall w \in F_b(\xi)\}$. By [MS05, Theorem 3.3], $\xi^\perp$ is actually a sub-bundle of η and $\eta \cong \xi \oplus \xi^\perp$.

If a smooth manifold M is embedded in some real space $\mathbb{R}^d$ for d big enough, then the tangent bundle of M, τ_M is a sub-bundle of the restriction $\tau_{\mathbb{R}^d}|_M$, which is a trivial bundle. Moreover, the orthogonal complement of τ_M in $\tau_{\mathbb{R}^d}|_M$ is by definition the normal bundle, so we get that

$$\tau_M \oplus \nu_M \cong \varepsilon^d.$$

I.2 Stiefel–Whitney Classes

In this section we will introduce a first kind of characteristic classes, the Stiefel–Whitney classes. We will define them without proving their existence or their uniqueness, and we will defer the proof of that to [MS05, Chapter 8]. Moreover, we will present the Stiefel–Whitney numbers, important cohomology invariants. These last objects are Stiefel–Whitney classes of tangent bundles of manifolds, and can be crucial in the study of the unoriented cobordism ring. We will also define the normal Stiefel–Whitney numbers and see their connection with (tangential) Stiefel–Whitney numbers.

Our definition of Stiefel–Whitney classes will be the following axiomatic one.

Definition I.2.1. Let ξ be a vector bundle. We can define the *Stiefel–Whitney classes* of ξ as the cohomology classes $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$, for $i \in \mathbb{Z}_{\geq 0}$, that satisfy the following axioms:

1. The 0th-class is $w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{Z}/2\mathbb{Z})$ and $w_i(\xi) = 0$ for $i > n$ if ξ is an n-plane bundle;

2. Naturality: If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then
$$w_i(\xi) = f^* w_i(\eta);$$

3. The Whitney product theorem: If ξ and η are vector bundles over the same base space, then
$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta);$$

4. Non-triviality: The first Stiefel–Whitney class $w_1(\gamma_1^1)$, where γ_1^1 is the line bundle over $\mathbb{P}^1$, is non zero.

Remark I.2.2. Notice that the fourth condition excludes the case of having every Stiefel–Whitney class w_i with $i > 0$ equal to zero. Moreover, recalling that $H^1(\mathbb{P}^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, due to the non-triviality condition we get that we must have $w_1(\gamma_1^1) \cong 1$ (as expected since the Stiefel–Whitney classes are unique).

Indeed, we have the following theorem, that we will not prove, which states the existence and uniqueness of the Stiefel–Whitney classes.

Theorem I.2.3. *There exists one and only one correspondence $\xi \mapsto w_i(\xi)$ for every $i \geq 0$, which associates to each vector bundle ξ over a paracompact base space B a cohomology class $w_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$ which satisfy the four axioms of the definition above.*

For the proof of the uniqueness, we defer to [MS05, Theorem 7.3], whereas for the existence we cite the proof in [MS05, Chapter 8].

We have some straightforward consequences given by these four axioms. In particular, thanks to the second and third axioms:

Proposition I.2.4. *If ξ is isomorphic to η, then $w_i(\xi) = w_i(\eta)$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $g : E(\xi) \to E(\eta)$ be the isomorphism. By Remark I.1.9, we get the map $\mathrm{id} : B(\xi) \to B(\eta)$. We conclude by axiom 2, $w_i(\xi) = \mathrm{id}^* w_i(\eta) = w_i(\eta)$. ☐

Proposition I.2.5. *If ε is a trivial bundle, then $w_i(\varepsilon) = 0$ for every $i > 0$.*

Proof. It is a direct consequence of Example I.1.2 and axiom 2. ☐

Proposition I.2.6. *If ε is a trivial vector bundle and η is any vector bundle, then $w_i(\eta \oplus \varepsilon) = w_i(\eta)$ for $i > 0$.*

Proof. For $i = 0$, $w_i(\eta \oplus \varepsilon) = 1 = w_i(\eta)$ by definition. For $i > 0$, by the previous proposition and axiom 3

$$w_i(\eta \oplus \varepsilon) = \sum_{i=0}^{k} w_i(\eta) \smile w_{k-i}(\varepsilon) = w_k(\eta) \smile w_0(\varepsilon) + \sum_{i=0}^{k-1} w_i(\eta) \smile 0 =$$
$$= w_k(\eta) \smile w_0(\varepsilon) = w_k(\eta) \smile 1 = w_k(\eta),$$

where we also used axiom 1. ☐

Proposition I.2.7. *Let ξ be an Euclidean n-plane bundle with a nowhere zero section. Then $w_n(\xi) = 0$. More generally, if ξ has k nowhere linearly dependent sections, then*

$$w_{n-k+1}(\xi) = \cdots = w_n(\xi) = 0.$$

Proof. If $k = 1$, being nowhere zero is equivalent to being nowhere dependent, so it is enough to prove the second part of the statement. By Remark I.1.7, there is a sub-bundle of ξ given by the k nowhere dependent sections. This bundle, which we will denote ε, is a trivial k-plane bundle. Thanks to I.1, ξ splits as a Whitney sum $\varepsilon \oplus \varepsilon^\perp$ (where $\varepsilon^\perp$ is an $(n-k)$-plane bundle). By Proposition I.2.6,

$$w_i(\xi) = w_i(\varepsilon \oplus \varepsilon^\perp) = w_i(\varepsilon^\perp)$$

and by axiom 1, $w_i(\varepsilon^\perp) = 0$ for $i > n - k$. This proves our statement. ☐

We can now see how these classes can be seen as elements of the following graded ring.

Definition I.2.8. We will denote by $H^{\Pi}(B;\mathbb{Z}/2\mathbb{Z})$ the ring with elements the formal series

$$a = a_0 + a_1 + a_2 + \dots$$

with $a_i \in H^i(B;\mathbb{Z}/2\mathbb{Z})$.

This is indeed a commutative ring, with the obvious sum and product operations, where we will just need to be careful about the degrees. For example, taking $a = a_0 + a_1 + a_2 + \dots, b = b_0 + b_1 + b_2 + \dots \in H^{\Pi}(B;\mathbb{Z}/2\mathbb{Z})$, $(a+b)_i = a_i + b_i \in H^i(B;\mathbb{Z}/2\mathbb{Z})$ and $(a \cdot b)_i = \sum_{k=0}^{i} a_k b_{i-k} \in H^i(B;\mathbb{Z}/2\mathbb{Z})$. The product is associative and commutative (since we are working modulo 2).

This ring contains a very special element:

Definition I.2.9. Let ξ be an n-dimensional bundle over B. We will call *total Stiefel–Whitney class* of ξ the element

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \dots + w_n(\xi) + 0 + \dots \in H^{\Pi}(B;\mathbb{Z}/2\mathbb{Z}).$$

Thanks to this formulation, we can now restate the *Whitney product theorem* (axiom 3) as

$$w(\xi \oplus \eta) = w(\xi) \smile w(\eta). \tag{I.1}$$

Note that we usually work only with finite dimensional vector bundles (i.e. vector bundles whose fibers are all finite dimensional vector spaces), and the only example of infinite dimensional vector bundle that we will encounter (and thus the only case when we will need this generality about the formal group) will be the universal bundle in Chapter 3.

Let's now introduce the concept of Stiefel–Whitney numbers. These will take an important role in the study of the cobordism ring: in particular, we will see that these will be cobordism invariants.

Consider a smooth closed (i.e. compact without boundary) n-dimensional manifold M. Since every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable and there exists a unique $\mathbb{Z}/2\mathbb{Z}$ orientation, there exists a unique fundamental class $\mu_M \in$

$H_n(M;\mathbb{Z}/2\mathbb{Z})$ (for more details, check Appendix A). Moreover, the *Kronecker pairing*

$$\langle v, \mu_M \rangle := v[M] \in \mathbb{Z}/2\mathbb{Z}$$

is well defined for every $v \in H^n(M;\mathbb{Z}/2\mathbb{Z})$.

Moreover, if we consider integers $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$ such that

$$r_1 + 2r_2 + \cdots + nr_n = n,$$

we get that for any vector bundle ξ,

$$w_1(\xi)^{r_1} \smile \cdots \smile w_n(\xi)^{r_n}$$

is an element of $H^n(B(\xi);\mathbb{Z}/2\mathbb{Z})$.

Combining these two things allows us to define the Stiefel–Whitney numbers:

Definition I.2.10. Given a manifold M and integers $r_1, \ldots, r_n$ as above, we can define the *(tangential) Stiefel–Whitney number* associated to $r_1, \ldots, r_n$ as

$$\langle w_1(\tau_M)^{r_1} \smile \cdots \smile w_n(\tau_M)^{r_n}, \mu_M \rangle := w_1^{r_1} \cdots w_n^{r_n}[M] \in \mathbb{Z}/2\mathbb{Z}.$$

We will say that two different n-manifolds M, M' have the same Stiefel–Whitney numbers if $w_1^{r_1} \cdots w_n^{r_n}[M] = w_1^{r_1} \cdots w_n^{r_n}[M']$ for every combination of $r_1, \ldots, r_n$ of dimension n. Note that Stiefel–Whitney numbers are invariant under diffeomorphism. However two manifolds with the same Stiefel–Whitney numbers are not diffeomorphic in general.

We can also define the normal Stiefel–Whitney numbers, but to that we first need some differential topology notions.

First of all, we need to see a theorem about embeddings.

Theorem I.2.11 (Strong embedding theorem)**.** *Every compact boundaryless smooth n-dimensional manifold M can be embedded in $\mathbb{R}^{2n}$.*

This theorem is also called the Whitney embedding theorem, and can be found in [Ben21, Theorem 7.17].

Whilst discussing embeddings, we want to recall what an isotopy is.

Definition I.2.12. Let f, g be two embeddings $X \to Y$. An *isotopy* between f and g is an homotopy $H : X \times [0;1] \to Y$ between them such that for each $s \in [0,1]$, $H_s = H(\cdot, s) : X \to Y$ gives an embedding. The maps f and g will be called *isotopic*.

Note that for q large enough, any two embedding of an n-dimensional manifold M in $\mathbb{R}^{n+q}$ are in fact isotopic. Moreover, if we consider the normal bundle of M in $\mathbb{R}^{n+q}$, any two embedding of an n-dimensional manifold M in $\mathbb{R}^{n+q}$ have equivalent normal bundles (for a reference of this statement and the relative proof, check [Lan02]). Thus the isomorphism class of the normal bundle is uniquely defined and therefore the (isomorphism class of the) stable normal bundle of M is defined abstractly, independently of an embedding.

Definition I.2.13. Given a manifold M and integers $r_1, \ldots, r_n$ as above, we can define the *normal Stiefel–Whitney number* associated to $r_1, \ldots, r_n$ as

$$\langle w_1(\nu_M)^{r_1} \smile \cdots \smile w_n(\nu_M)^{r_n}, \mu_M \rangle := w_1^{r_1} \cdots w_n^{r_n}[\nu(M)] \in \mathbb{Z}/2\mathbb{Z},$$

where ν_M is the normal bundle of M in $\mathbb{R}^{n+q}$ for q sufficiently large.

These are well-defined because, as we stated before, any two embedding of an n-dimensional manifold M in $\mathbb{R}^{n+q}$ have equivalent normal bundles.

We can relate tangential and normal Stiefel–Whitney numbers as a consequence of the following formula.

Lemma I.2.14 (Whitney duality formula)**.** *Let M be a smooth closed n-dimensional manifold, embedded in $\mathbb{R}^{n+q}$ for q large enough. Then*

$$w(\tau_M) \smile w(\nu_M) = 1.$$

Proof. Since the tangent bundle of any real space is trivial, for q large enough $\tau_M \oplus \nu_M \cong \varepsilon^{n+q}$. By the product formula I.1, this implies that

$$w(\tau_M) \smile w(\nu_M) = w(\tau_M \oplus \nu_M) = w(\varepsilon^{n+q}),$$

which is 1 by Proposition I.2.5. □

As a corollary, we get the following lemma.

Lemma I.2.15. *Let M be a smooth closed n-dimensional manifold. Then all its tangential Stiefel–Whitney numbers are zero if and only if all its normal Stiefel–Whitney numbers are zero.*

Proof. The Whitney duality formula implies that every tangential Stiefel–Whitney class $w_i(\tau_M)$ is a polynomial with variables $w_1(\nu_M), \dots, w_q(\nu_M)$ and that every normal Stiefel–Whitney class $w_i(\nu_M)$ is a polynomial with variables $w_1(\tau_M), \dots, w_n(\tau_M)$. Indeed, the formula tells us that for every $i > 0$, the i-th degree part of $w(\tau_M) \smile w(\nu_M)$ (which is $\sum_{k=0}^{i} w_k(\tau_M) \smile w_{i-k}(\nu_M)$) must be zero. Through an induction process over i, we get these polynomials. This tells us that if every tangential Stiefel–Whitney number of M is zero, then every normal Stiefel–Whitney number is zero, and conversely if every normal Stiefel–Whitney number of M is zero, then every tangential Stiefel–Whitney number is zero. □

I.3 Stiefel–Whitney numbers of the projective space

In this section we are going to give an actual example of computation of Stiefel–Whitney classes and numbers. In particular, we are going to study the tangent bundle of the projective space, in order to compute its Stiefel–Whitney numbers. This will be an important example in later chapters, since it gives us the description of some elements of the unoriented cobordism ring.

Consider a projective space $\mathbb{P}^n$. Firstly, we need to understand better the structure of the tangent bundle $\tau_{\mathbb{P}^n}$. In order to do so, we need the following new construction, the Hom vector bundle. Let ξ_1 and ξ_2 be two vector bundles with the same base space B. Given two vector spaces V_1, V_2, we get the vector space $\mathrm{Hom}(V_1, V_2)$ of linear maps with domain V_1 and codomain V_2. We can extend this concept also to vector bundles: $\mathrm{Hom}(\xi_1, \xi_2)$ will be the vector bundle over B with fibres $F_b(\mathrm{Hom}(\xi_1, \xi_2)) = \mathrm{Hom}(F_b(\xi_1), F_b(\xi_2))$, total space the disjoint union of these and projection map $\pi(F_b) = b$. There exists a unique canonical topology on the total space so that this is the total space of a vector bundle with projection π and with fibres F_b by [MS05, Theorem 3.6]. The idea behind this is that we can construct local coordinates using the local coordinates of the vector bundles we are starting with: given an open $U \subset B$ and maps $h_1, h_2 : U \times \mathbb{R}^{n_i} \to \pi_i^{-1}(U)$, for every $b \in B$ we can define the isomorphism $\mathrm{Hom}(h_1(b, \cdot), h_2(b, \cdot)) : \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to \mathrm{Hom}(F_b(\xi_1), F_b(\xi_2))$. It can be proved that there exists a unique topology on the total space such that

$$h : U \times \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \to \mathrm{Hom}(F_b(\xi_1), F_b(\xi_2))$$
$$(b, \psi) \mapsto \mathrm{Hom}(h_1(b, \psi), h_2(b, \psi))$$

is an homeomorphism (and thus we can choose it as local coordinate system for this Hom bundle).

We have the following property about the Hom construction when we work with Euclidean vector bundles.

Remark I.3.1. If ξ is an Euclidean vector bundle of finite dimension (i.e. all the fibres are finite dimensional vector spaces), then ξ is isomorphic to the bundle $\mathrm{Hom}(\xi, \varepsilon^1)$ where ε^1 is a 1-dimensional trivial bundle over the same base space as ξ. This vector bundle is usually called the *dual bundle* of ξ. The isomorphism is just the linear map that takes each fibre $\mathrm{Hom}(F_b(\xi), F_b(\varepsilon^1)) = \mathrm{Hom}(F_b(\xi), \{b\} \times \mathbb{R}) \cong \mathrm{Hom}(F_b(\xi), \mathbb{R}) = F_b(\xi)^\vee$ isomorphically to $F_b(\xi)$ (using the symmetric form given by the Euclidean metric on $F_b(\xi)$).

We also have the following lemma that carries with it a really important consequence on bundles over the real projective space $\mathbb{P}^n$.

Lemma I.3.2. *The tangent bundle of the real projective space $\mathbb{P}^n$ is isomorphic to the bundle $\mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp)$.*

Here, we considered γ_n^1 as a sub-bundle of the trivial bundle ε^{n+1} over $\mathbb{P}^n$ (for each $x \in \mathbb{P}^n$, its fibre $F_x(\gamma_n^1)$ is the line $l \in \mathbb{R}^{n+1}$ through x and $-x$, which is a linear subspace of $\mathbb{R}^{n+1}$, which is the fibre $F_x(\varepsilon^{n+1})$), so by [MS05, Theorem 3.3], $\gamma_n^1 \oplus (\gamma_n^1)^\perp \cong \varepsilon^{n+1}$.

For the proof of this lemma, check [MS05, Lemma 4.4]. We use this lemma to prove the following theorem.

Theorem I.3.3. *Let $\tau_{\mathbb{P}^n}$ be the tangent bundle of $\mathbb{P}^n$, and let ε^1 be a trivial line bundle over $\mathbb{P}^n$. Then, the Whitney sum $\tau_{\mathbb{P}^n} \oplus \varepsilon^1$ is isomorphic to the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \cdots \oplus \gamma_n^1$.*

Proof. By I.3.2, we know that $\tau_{\mathbb{P}^n} \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp) \oplus \varepsilon^1$.

Since $\varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ (for each $x \in \mathbb{P}^n$, its fibre $F_x(\mathrm{Hom}(\gamma_n^1, \gamma_n^1))$ is $\mathrm{Hom}(l, l) \cong \mathbb{R}^1$, where l is the line containing x and $-x$, whereas its fibre

$F_x(\varepsilon^1)$ is $\{x\} \times \mathbb{R}^1$, so these vector bundles are isomorphic) we get that $\tau_{\mathbb{P}^n} \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$. This is isomorphic to $\mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp \oplus \gamma_n^1)$. Knowing that $(\gamma_n^1)^\perp \oplus \gamma_n^1 \cong \varepsilon^{n+1}$ and that $\varepsilon^{n+1} \cong \varepsilon^1 \oplus \cdots \oplus \varepsilon^1$, we get the following chain of isomorphisms

$$\begin{aligned}\tau_{\mathbb{P}^n} \oplus \varepsilon^1 &\cong \mathrm{Hom}(\gamma_n^1, (\gamma_n^1)^\perp \oplus \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \cong \\ &\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \cong \\ &\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \mathrm{Hom}(\gamma_n^1, \varepsilon^1).\end{aligned}$$

By the Remark I.3.1 $\mathrm{Hom}(\gamma_n^1, \varepsilon^1) \cong \gamma_n^1$, and we get the desired result

$$\tau_{\mathbb{P}^n} \oplus \varepsilon^1 \cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1.$$

□

We can finally introduce an example of computation of Stiefel–Whitney classes: the Stiefel–Whitney classes of the tangent bundle of real projective space $\mathbb{P}^n$.

Example I.3.1. We want to compute the total Stiefel–Whitney class $w(\tau_{\mathbb{P}^n})$. By Proposition I.2.6, we know that $w(\tau_{\mathbb{P}^n}) = w(\tau_{\mathbb{P}^n} \oplus \varepsilon^1)$. Using Theorem I.3.3, we get that $\tau_{\mathbb{P}^n} \oplus \varepsilon^1 \cong \gamma_n^1 \oplus \cdots \oplus \gamma_n^1$, and thus by Proposition I.2.4, $w(\tau_{\mathbb{P}^n} \oplus \varepsilon^1) = w(\gamma_n^1 \oplus \cdots \oplus \gamma_n^1)$. By the product formula (axiom 3) $w(\gamma_n^1 \oplus \cdots \oplus \gamma_n^1) = w(\gamma_n^1) \cup \cdots \cup w(\gamma_n^1)$. We claim that $w(\gamma_n^1) = 1 + a$ where $a \in H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$ is the non-zero element. Recall that $H^*(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a]\langle a^{n+1}\rangle$, thus

$$w(\tau_{\mathbb{P}^n}) = (1+a)^{n+1} \in \frac{\mathbb{Z}/2\mathbb{Z}[a]}{\langle a^{n+1}\rangle}.$$

Now, we need to prove the claim. Since γ_n^1 is a line bundle, by axiom 1 $w_i(\gamma_n^1) = 0$ for $i > 1$. Thus, $w(\gamma_n^1) = 1 + w_1(\gamma_n^1)$. To compute the first Stiefel–Whitney class, note that the inclusion map $\iota : \mathbb{P}^1 \to \mathbb{P}^n$ is covered by a bundle map $\gamma_1^1 \to \gamma_n^1$. Thus, by naturality 2 and the non-triviality 4 we have that

$$\iota^*(w_1(\gamma_n^1)) = w_1(\gamma_1^1) \neq 0$$

and thus $w_1(\gamma_n^1) \neq 0$. Indeed, $\iota^* : H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{P}^1; \mathbb{Z}/2\mathbb{Z})$ is a linear map, so the zero element cannot be mapped in a non zero element. Since $H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z}) = \{0, a\} \cong \mathbb{Z}/2\mathbb{Z}$, $w_1(\gamma_n^1)$ must be the element a.

Now that we have computed the Stiefel–Whitney classes of $\tau_{\mathbb{P}^n}$, computing the Stiefel–Whitney numbers of $\mathbb{P}^n$ will be straightforward.

Example I.3.2. Using Example I.3.1, we see that $w(\tau_{\mathbb{P}^n}) = (1+a)^{n+1} = \sum_{j=0}^{n} \binom{n+1}{j} a^j$. We need to divide in two cases now. If n is odd, $n+1 = 2k$ for some $k > 0$. Since we are working in characteristic 2,

$$w(\tau_{\mathbb{P}^n}) = (1+a)^{2k} = (1+a^2)^k = 1 + \binom{n+1}{1} a^2 + \binom{n+1}{2} a^4 + \dots.$$

It is clear that there will be no classes in odd degrees, so $w_j(\tau_{\mathbb{P}^n}) = 0$ for j odd. Now, let's consider integers $r_1, \dots, r_n \in \mathbb{Z}_{\geq 0}$ such that $r_1 + 2r_2 + \dots + nr_n = n$. Since n is odd, there will exists an odd integer $j \leq n$ such that $r_j \neq 0$. Therefore, $w_1(\tau_{\mathbb{P}^n})^{r_1} \smile \dots \smile w_j(\tau_{\mathbb{P}^n})^{r_j} \smile \dots \smile w_n(\tau_{\mathbb{P}^n})^{r_n} = w_1(\tau_{\mathbb{P}^n})^{r_1} \smile \dots \smile 0 \smile \dots \smile w_n(\tau_{\mathbb{P}^n})^{r_n} = 0$, so $w_1^{r_1} \cdots w_n^{r_n}[\mathbb{P}^n] = 0$. Therefore, every Stiefel–Whitney number of an odd dimensional real projective space is zero.

On the other hand, if n is even, we have that some Stiefel–Whitney are always non zero. For example, $w_n(\tau_{\mathbb{P}^n}) = \binom{n+1}{n} a^n = (n+1)a^n$ and $w_1(\tau_{\mathbb{P}^n}) = \binom{n+1}{1} a = (n+1)a$, which are non zero since $n+1 \equiv 1 \mod 2$ and a is the non zero element of the first cohomology group of $\mathbb{P}^n$. Therefore, if we choose $r_n = 1$ we get the Stiefel–Whitney number $w_n^1[\mathbb{P}^n] = \langle (n+1)a^n, \mu_{\mathbb{P}^n} \rangle = (n+1) \equiv 1 \mod 2$ and for $r_1 = n$, $w_1^n[\mathbb{P}^n] = \langle (n+1)^n a^n, \mu_{\mathbb{P}^n} \rangle = (n+1)^n \equiv 1 \mod 2$.

I.4 Oriented vector bundles and Euler class

In this section, we want to present the Euler class of an oriented vector bundle. In order to do so, we first need to define the concept of orientation of a vector bundle. The Euler class will be essential in order to define the Chern classes, that we will see afterwards.

Moreover, we will focus several times on orientation of the tangent bundles and manifolds (which we will see are equivalent), so it might be important to see the following results.

Firstly, recall the following definition:

Definition I.4.1. An *orientation* of a real n-dimensional vector space V is an equivalence class of ordered bases, where the equivalence relation that defines the class is the following:

$$v_1, \dots, v_n \sim v'_1, \dots, v'_n \iff \det(A_{vv'}) > 0$$

where $A_{vv'}$ is the transition matrix from $v_1, \dots, v_n$ to $v'_1, \dots, v'_n$ (i.e. $v'_i = \sum_{j=1}^n a_{ij} v_j$ and $A = (a_{ij})_{i,j}$).

We can thus define what an oriented vector bundle is.

Definition I.4.2. An *orientation on an n-plane bundle* $\xi : E \xrightarrow{\pi} B$ is a choice of *compatible* orientations for each fibre $F_b(\xi)$, where compatible means that for each $b_0 \in B$, there exists a trivializing chart (U, h) for ξ such that $b_0 \in U$ and for every $b \in U$, $h|_{\{b_0\}\times\mathbb{R}^n} : \{b_0\} \times \mathbb{R}^n \xrightarrow{\cong} F_b(\xi)$ is orientation preserving ($\mathbb{R}^n$ has its standard orientation).

Equivalently, there exist nowhere dependent sections $s_1, \dots, s_n$ of $\xi|_U$ such that for every $b \in U$, $s_1(b), \dots, s_n(b)$ is an orientation basis for $F_b(\xi)$.

In the appendix, we give the definition of orientation for a manifold. Sometimes, we will use the following equivalent definition of orientable manifold:

Definition I.4.3. A smooth n-manifold M is orientable if its tangent bundle τ_M is orientable. An *orientation* on M is an orientation on τ_M.

The two definitions are equivalent. Indeed, we have the following lemma.

Lemma I.4.4. *Let M be a manifold and τ_M be its tangent bundle. Any orientation for τ_M gives rise to an orientation for the underlying manifold M and conversely any orientation for M induces an orientation on τ_M.*

This is proved in [MS05, Lemma 11.6].

In the interest of stating and proving the following theorem, we want to recall that, given an oriented n-plane bundle $\xi : E \xrightarrow{\pi} B$, for each fibre F, the cohomology group $H^n(F, F_0; \mathbb{Z})$ has a canonical generator $\tilde{u}_F$. We denote by F_0 the set

$$F_0 = F \cap E_0 = F \cap \{e \in E |\ e \in F_{\pi(e)}(\xi) \setminus s_0(\pi(e))\} = F \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\},$$

where $E_0 = E \setminus s_0(B)$ is called the *deleted total space* and s_0 is the zero section Thus, $H^n(F, F_0; \mathbb{Z}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$. This isomorphism is specified by the orientation preserving map h that we find in the definition of oriented bundle. A choice of generator of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ gives us a generator for every fiber and therefore $H^n(F, F_0; \mathbb{Z})$ possesses a canonical generator corresponding to 1 or -1, (depending on the orientation of F, which is determined by the orientation of ξ) which we will denote $\tilde{u}_F$. Analogously, for any commutative ring with unity R, $H^n(F, F_0; R) \cong R = \langle 1_R \rangle$, and we will denote the preimage of 1_R by u_F. This corresponds to an R-orientation of the bundle. Analogously to the case of oriented manifolds, if a bundle is oriented (i.e. $\mathbb{Z}$-oriented) it is also R-oriented for all commutative rings with unity R.

If we take any commutative ring with unity R, the ring homomorphism $\mathbb{Z} \to R$ that sends 1 to the unity of R, determines a coefficient homomorphism

$$\begin{aligned} H^n(F, F_0; \mathbb{Z}) &\to H^n(F, F_0; R) \\ \tilde{u}_F &\mapsto u_F \end{aligned}.$$

We will consider only the case $R = \mathbb{Z}$ or $R = \mathbb{Z}/2\mathbb{Z}$, so in the first case this map will be the identity and in the latter, u_F will be the only non-trivial element of $H^n(F, F_0; \mathbb{Z}/2\mathbb{Z})$, and the map will send $k\tilde{u}_F$ to 0 for k even and to u_F for k odd.

Now that we noted the importance of the class u_F, we are going to see a theorem that highlights it. Observe that if we are working with $R = \mathbb{Z}/2\mathbb{Z}$, for each fiber F there is just one generator u_F, so there is just one choice of orientation for each fibre. Therefore, there always exists a $\mathbb{Z}/2\mathbb{Z}$-orientation and it is unique.

Theorem I.4.5 (Thom isomorphism)**.** *Let R be a commutative ring with unity and $\xi : E \xrightarrow{\pi} B$ an R-oriented n-plane bundle. Then there is one and* *only class $u_R \in H^n(E, E_0; R)$ whose restriction to each fibre F is u_F, i.e.*

$$\begin{aligned} H^n(E, E_0; R) &\to H^n(F, F_0; R) \\ u_R &\mapsto u_F \end{aligned}.$$

Moreover, for every $q \geq 0$, $H^q(E; R) = H^q(E, \emptyset; R) \xrightarrow{\smile u_R} H^{q+n}(E, E_0; R)$ is an isomorphism.

For the proof of this theorem, we recommend [MS05, Chapter 10].

Observe that the last map is defined using the relative cup product $\smile$: $H^q(E, \emptyset; R) \times H^n(E, E_0; R) \to H^{q+n}(E, E_0 \cup \emptyset; R) = H^{q+n}(E, E_0; R)$, that one might study in depth on [Mun84, Chapter 48, pages 290-291]. This relative cup product can be seen intuitively in the following way: if $\alpha \in H^q(E; R)$ and $\beta \in H^n(E, E_0; R)$, then $\alpha \smile \beta$ is an element of $H^{n+q}(E; R)$ and since β is zero on the simplices of E_0, also the cup product will be the zero homomorphism on the simplices of E_0, and thus $\alpha \smile \beta \in H^{n+q}(E, E_0; R)$.

The theorem lets us define important objects, needed to define the Euler class.

Definition I.4.6. We will call u_R the $(R$-$)$ *Thom class* and

$$\begin{aligned} \phi = (\smile u_R) \circ \pi^* : H^q(B; R) &\to H^{q+n}(E, E_0; R) \\ x &\mapsto \pi^*(x) \smile u_R \end{aligned}$$

the $(R$-$)$ *Thom isomorphism.*

Before continuing, please note that the map $\pi^* : H^q(B; R) \to H^q(E; R)$ induced by the projection $\pi : E \to B$ is an isomorphism, since B is isomorphic to $s_0(B) \subset E$, and $s_0(B)$ is a deformation retract of E, where the retraction is π itself. Therefore, the Thom isomorphism is an actual isomorphism, since it is the composition of two isomorphisms.

Finally we can define the Euler class. Let $\xi : E \xrightarrow{\pi} B$ be an oriented n-plane bundle, $u \in H^n(E, E_0; \mathbb{Z})$ the associated Thom class and

$$\begin{aligned} \phi : H^i(B; \mathbb{Z}) &\to H^{i+n}(E, E_0; \mathbb{Z}) \\ x &\mapsto \pi^*(x) \smile u \end{aligned}$$

the Thom isomorphism.

Definition I.4.7. The *Euler class* of ξ, $e(\xi) \in H^n(B;\mathbb{Z})$ is given by

$$e(\xi) = (\pi^*)^{-1}(i^*(u)) = (\pi^*)^{-1}(u|_E)$$

where $i : (E, \emptyset) \to (E, E_0)$ is the inclusion.

Remark I.4.8. We have the following commuting diagram

$$\begin{array}{ccccc} e(\xi) & \longmapsto & i^*(u) & \longleftarrow\!\shortmid & u \\ & & & & \\ H^n(B;\mathbb{Z}) & \xrightarrow{\pi^*} & H^n(E;\mathbb{Z}) & \xleftarrow{i^*} & H^n(E,E_0;\mathbb{Z}) \\ & \searrow^{\phi} & \downarrow{\smile u} & \swarrow_{\smile u} & \\ & & H^{2n}(E,E_0;\mathbb{Z}) & & \end{array}$$

so

$$\phi(e(\xi)) = \pi^*(e(\xi)) \smile u = i^*(u) \smile u = u \smile u$$

and thus an equivalent definition of the Euler class is

$$e(\xi) = \phi^{-1}(u \smile u).$$

The Euler class possesses some nice and important properties that we will use frequently. The first of these is its naturality with respect to bundle maps:

Theorem I.4.9. *Let $f : B \to B'$ be covered by a preserving bundle map of oriented n-plane bundles $\xi \to \xi'$. Then the Euler class of ξ is the pullback of the Euler class of ξ'*

$$e(\xi) = f^*(e(\xi')).$$

Remark I.4.10. This theorem implies that for a trivial n-plane bundle ξ (for $n > 0$), $e(\xi) = 0$. Indeed, by Example I.1.2, there exists a vector bundle map such that the following diagram commutes

$$\begin{array}{cccc} \xi : & E(\xi) & \longrightarrow & B(\xi) \\ & \downarrow & & \downarrow f \\ \xi' : & \{p\} \times \mathbb{R}^n & \longrightarrow & \{p\} \end{array} .$$

In this case, since $H^n(\{p\};\mathbb{Z}) = 0$, and thus $e(\xi') = 0$, we get

$$e(\xi) = f^*(e(\xi')) = f^*(0) = 0.$$

Since we are working with oriented bundles, it shouldn't be a surprise that the Euler class is dependent on the orientation. Indeed, we have the following result:

Proposition I.4.11. *Let ξ be an oriented n-plane bundle. Let $-\xi$ denote ξ with the opposite orientation. Then*

$$e(-\xi) = -e(\xi).$$

The last property that we want to highlight is the product formula.

Theorem I.4.12. *Let ξ, ξ' be two oriented vector bundles. Then,*

$$e(\xi \times \xi') = e(\xi) \times e(\xi').$$

If the base spaces of the two vector bundles coincide, we get that

$$e(\xi \oplus \xi') = e(\xi) \smile e(\xi').$$

Note that we have defined Euler classes only on oriented bundles. Both for the product and the Whitney sum of oriented vector bundles, their orientation is induced by the orientations of ξ, ξ' in the following way: for each fibre, the compatible orientation is just given by concatenating one orientation basis for ξ' to an orientation basis for ξ.

I.5 Complex vector bundles

In this section we will give the definition of complex vector bundles, and we will see which objects a complex vector bundle allows us to define. The main purpose of doing this is to be able to define the Chern classes, which are characteristic classes on complex vector bundles, and some of these properties on complexified vector bundles will be essential in the definition of Pontrjagin classes, which play an important role in the oriented cobordism ring.

The definition of a complex vector bundle is actually identical to the one of a real vector bundle, having the foresight to ask the fibres to be complex vector spaces and to substitute $\mathbb{R}^n$ with $\mathbb{C}^n$ in the local triviality condition:

Definition I.5.1. A *complex vector bundle* of complex dimension n over a base space B, which is a topological space, (or a *complex n-plane bundle*) is a topological space E (which is called the total space) together with a projection map $\pi : E \to B$ such that for each $b \in B$, $\pi^{-1}(b)$ is a complex vector space of complex dimension n. Furthermore, we require the complex vector bundle to satisfy the local triviality condition: for every point $b \in B$, there must exists a neighborhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times \mathbb{C}^n$, and this homeomorphism takes each fibre $\pi^{-1}(b)$ linearly to $b \times \mathbb{C}^n$.

We can construct, just as in the case of real vector bundles, the restriction of a vector bundle, induced bundles, Cartesian products and Whitney sums.

Just as with vector spaces, we can associate to a real vector bundle of real dimension $2n$ a complex vector bundle of complex dimension n. To do so, we need to add to the real $2n$-plane bundle a complex structure.

Definition I.5.2. Let ξ be a real $2n$-plane bundle. A *complex structure* is a continuous map $J : E(\xi) \to E(\xi)$ such that J sends each fibre into itself, $\mathbb{R}$-linearly and $J(J(v)) = -v$ for every vector $v \in E(\xi)$.

In particular, once given the complex structure to the real vector bundle, we can give to each fibre $F_b(\xi)$ the structure of a complex vector space by defining the multiplication for a complex number as

$$(x + iy)v = xv + J(yv)$$

for all $x + iy \in \mathbb{C}$ and every $v \in F_b(\xi)$.

Conversely, once given a complex vector bundle, we can think of it as a real vector bundle (with double dimension) just forgetting its complex structure. This will be called the *underlying real vector bundle* of a complex bundle, and it will be denoted by adding $\mathbb{R}$ to the subscript of the complex vector bundle.

In both these constructions, projection maps, base spaces and total spaces remain unchanged. The following definition will be about the orientation of a complex vector bundle. Note that also giving an orientation to a complex vector bundle coincide to give an orientation on the underlying real vector bundle.

Definition I.5.3. Let ω be a complex vector bundle. Then, the *canonical preferred orientation* of its underlying real vector bundle $\omega_{\mathbb{R}}$ is given in the following way: once chosen a complex basis $v_1, \ldots, v_n$ of each fibre F of ω, the ordered basis of $F_{\mathbb{R}}$ (the underlying real vector space of F) $v_1, iv_1, \ldots, v_n, iv_n$, will be one of the compatible orientations.

Lemma I.5.4. *The preferred orientation is well defined, i.e. it does not depend on the choice of basis of each fibre.*

Proof. First of all, consider a complex vector space V and choose a basis $v_1, \ldots, v_n$. Then, $v_1, iv_1, \ldots, v_n, iv_n$ will be a basis for the underlying real vector space $V_{\mathbb{R}}$. This ordered basis gives the desired orientation on the vector space. Indeed, it does not depend on the initial choice of the basis: if we chose another basis, these two will be connected by a transition matrix $A \in \mathrm{Gl}_n(\mathbb{C})$. Since $\mathrm{Gl}_n(\mathbb{C})$ is path-connected, the matrix A will give us a continuous deformation, so the two basis will give us the same orientation.

If we apply this to each fibre of ω, this will give us the desired orientation for $\omega_{\mathbb{R}}$. □

This lemma has an important consequence, that allows us to work with characteristic classes that require oriented vector bundles, for example the Euler class.

Corollary I.5.5. *Any complex manifold has a canonical preferred orientation.*

Proof. By Definition I.4.3 a complex manifold M has an orientation if its tangent vector bundle τ_M has an orientation. Since τ_M is a complex vector bundle, by the previous lemma $(\tau_M)_{\mathbb{R}}$ has a preferred orientation, and therefore so does τ_M and M. □

Remark I.5.6. As we noted before, the Euler class of the underlying real vector bundle of a complex n-plane bundle is always well defined. Thus, by theorem I.4.12 if we have two complex bundles ω, ω' over the same base space the following formula holds:

$$e(\omega_{\mathbb{R}} \oplus \omega'_{\mathbb{R}}) = e(\omega_{\mathbb{R}}) \smile e(\omega'_{\mathbb{R}}).$$

Thanks to the fact that $\omega_{\mathbb{R}} \oplus \omega'_{\mathbb{R}}$ and $(\omega \oplus \omega)'_{\mathbb{R}}$ are isomorphic as oriented vector bundles (since the orientation basis of a Whitney sum is just the concatenation of the orientations basis of the summands), we get that

$$e((\omega \oplus \omega)'_{\mathbb{R}}) = e(\omega_{\mathbb{R}}) \smile e(\omega'_{\mathbb{R}}).$$

In the same way one can just look at a complex bundle as a real bundle, given any real bundle we can actually complexify it. To do so, we have to complexify each fibre. Given a real vector space V, its *complexification* is the complex vector bundle $V \otimes_{\mathbb{R}} \mathbb{C}$.

Definition I.5.7. Let ξ be a real n-plane bundle. Then, we call its *complexification* $\xi \otimes \mathbb{C}$ the complex n-plane bundle constructed taking for each fibre F of ξ the complex vector space $F \otimes \mathbb{C}$ over the same base space.

This procedure duplicates the real dimension of the vector bundle. Indeed, when we take the underlying real vector bundle of $\xi \otimes \mathbb{C}$, it will become a real vector bundle of dimension $2n$. The structure will remain unchanged, but if we start with fibres like F, we will end up with fibres $F \otimes \mathbb{C}$; this is canonically isomorphic to $F \oplus F$. Indeed, if $z = x + iy \in F \otimes \mathbb{C}$ with $x, y \in F$, it is easy to see that $F \otimes \mathbb{C} = F \oplus iF$. Therefore, canonically,

$$(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi.$$

The complex structure on $\xi \otimes \mathbb{C}$ corresponds to

$$J(x, y) = (-y, x)$$

on $\xi \oplus \xi$.

The last thing we want to define in this section is the notion of conjugate bundle. We will see how some characteristic classes over this vector bundle are in strict relationship with the ones over the vector bundle we started with.

Definition I.5.8. Let ω be a complex n-plane bundle. Its *conjugate vector bundle* $\overline{\omega}$ is the complex n-plane bundle with the same underlying real vector bundle ($\omega_{\mathbb{R}} = \overline{\omega}_{\mathbb{R}}$), but with opposite complex structure: if $f : E(\omega) \to E(\overline{\omega})$ is the identity map, then for every $\lambda \in \mathbb{C}$ and for every $v \in E(\omega)$ we have $f(\lambda v) = \overline{\lambda} v$.

Two results we can already see using the conjugate bundle are the following.

Lemma I.5.9. *Let ξ be a real n-plane bundle. Then*

$$\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$$

as complex vector bundles.

Proof. We want to show that this isomorphism is not just an isomorphism as real vector bundles (we already have that $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong (\overline{\xi \otimes \mathbb{C}})_{\mathbb{R}}$ by definition of conjugation), but an isomorphism of complex vector bundles, and therefore, we want to show that there is a isomorphism that sends the complex structure of the first to the second. Let f be the map

$$\begin{aligned} f : E(\xi \otimes \mathbb{C}) \to & E(\xi \otimes \mathbb{C}) = E(\overline{\xi \otimes \mathbb{C}}) \\ x + iy \mapsto & x - iy \end{aligned}.$$

This is obviously invertible (its inverse is itself). Moreover, we can prove that f is $\mathbb{R}$-linear and it satisfy $f(i(x+iy)) = -if(x+iy)$:

- $f(x_1 + iy_1 + x_2 + iy_2) = x_1 + x_2 - i(y_1 + y_2) = f(x_1 + iy_1) + f(x_2 + iy_2)$;
- $f(k(x+iy)) = kx - iky = k(x - iy) = kf(x+iy)$ for every $k \in \mathbb{R}$;
- $f(i(x+iy)) = f(-y+ix) = -y - ix = -i(x-iy) = -if(x+iy)$.

This is enough to show that $f(\lambda(x+iy)) = \overline{\lambda}(x+iy)$ for every $\lambda \in \mathbb{C}$. □

Lemma I.5.10. *Let ω be a complex vector bundle. Then we have a canonical isomorphism between the complexification of the underlying real vector bundle of ω and the Whitney sum between ω and $\overline{\omega}$:*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

This lemma is proven in [MS05, Lemma 15.4].

I.6 Chern Classes

The goal of this section is to define the Chern classes. These classes will be defined for complex vector bundles. However, as we saw in the last chapter, we can create complex vector bundles from real ones, and since this is fundamental for the definition of Pontrjagin classes, we will focus on those at the end of this section.

Let $\omega : E \xrightarrow{\pi} B$ be a complex n-plane bundle. Recall that the deleted total space is $E_0 = \{e \in E |\ e \neq s_0(\pi(e)),\ e \in F_{\pi(e)}(\omega)\}$. We can construct an $(n-1)$-plane bundle ω_0 over E_0 in the following way: each point v in E_0 is given by a fibre F of ω (the fibre $F_{\pi(v)}(\omega)$) together with a non zero vector $v \in F$. We can associate to each of this point the $(n-1)$-vector space $F/\langle v\rangle_{\mathbb{C}}$, where we indicate by $\langle v\rangle_{\mathbb{C}}$ the 1-dimensional subspace spanned by v (which is non zero so it spans a line). This will give us a fibre of ω_0. Doing this for each $v \in E_0$, we will define each fibre of ω_0.

Before defining the Chern class, we must present the Gysin sequence and show that it is exact. let $\xi : E \xrightarrow{\pi} B$ be a real oriented n-plane bundle. Then we have an exact Gysin sequence

$$\cdots \to H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \xrightarrow{\psi} H^{i-n+1}(B) \to \dots$$

where we are omitting integer coefficients.

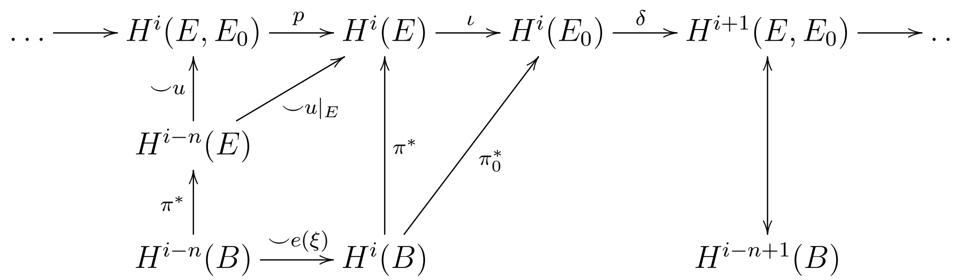
Proof. By the long exact sequence of the pair (E, E_0) we get

$$\cdots \to H^i(E, E_0) \xrightarrow{p} H^i(E) \xrightarrow{\iota} H^i(E_0) \xrightarrow{\delta} H^{i+1}(E, E_0) \to \dots .$$

Recall from Definition I.4.6, that we have the Thom isomorphism

$$\begin{aligned} \smile u \circ \pi^* : H^{i-n}(B;\mathbb{Z}) &\to H^i(E, E_0;\mathbb{Z}) \\ x &\mapsto \pi^*(x) \smile u \end{aligned}.$$

Moreover, by definition of π_0, $\pi_0^* = (\pi \circ \iota)^* = \iota^* \circ \pi^*$. Lastly, by definition of the Euler class, $e(\xi) = (\pi^*)^{-1}(i^*(u)) = (\pi^*)^{-1}(u|_E)$, so by naturality of the cup product $\smile u|_E \circ \pi^* = \pi^* \smile e(\xi)$ (indeed, for any $x \in H^{i-n}(B;\mathbb{Z}), \pi^*(x \smile e(\xi)) = \pi^*(x) \smile \pi^*(e(\xi)) = \pi^*(x) \smile u|_E$). Hence, we have the following commuting diagram



which gives us the desired sequence fixing $\psi = (\smile u \circ \pi^*)^{-1} \circ \delta$. $\square$

Now, for the purpose of defining Chern classes, we need to take a real oriented $2n$-plane bundle $E \xrightarrow{\pi} B$ and we get the exact Gysin sequence

$$\dots \to H^{i-2n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \xrightarrow{\psi} H^{i-2n+1}(B) \to \dots$$

where we are omitting the integer coefficients.

For $i < 2n-1$, $H^{i-2n}(B;\mathbb{Z}) = H^{i-2n+1}(B;\mathbb{Z}) = 0$ (since $i-2n, i-2n+1 < 0$) and thus, since the Gysin sequence is exact,

$$H^i(B;\mathbb{Z}) \cong H^i(E_0;\mathbb{Z}).$$

Thanks to all of this, we can now define a new characteristic class by induction on the dimension n of the complex plane bundle.

Definition I.6.1. Let ω be a complex n-plane bundle. Its i-th *Chern class*, denoted by $c_i(\omega) \in H^{2i}(B;\mathbb{Z})$, is defined by induction on the complex dimension of ω in the following way:

- for $i = 0$, $c_0 \equiv 1$;
- for $i < n$ (and so $i \leq 2i < 2n - 1$ for $n > 0$ since we are working with integer numbers) we define $c_i(\omega)$ as

  $$c_i(\omega) = (\pi_0^*)^{-1}(c_i(\omega_0))$$

  where since ω_0 is an $(n-1)$-plane bundle, by inductive hypothesis $c_i(\omega_0)$ is defined;

- for $i = n$ $c_i(\omega) = e(\omega_{\mathbb{R}})$ (which we recall to be well defined since ω is a complex bundle);
- for $i > n$, $c_i(\omega) = 0$.

Note that we need to work by induction just from $n = 2$. Indeed, for ω a 0-dimensional vector bundle, by the last assumption we get $i > 0$, $c_i(\omega) = 0$. For ω a 1-dimensional vector bundle, we will have $c_0(\omega) = 1$, $c_1(\omega) = e(\omega_{\mathbb{R}})$ and $c_i(\omega) = 0$ for $i > 2$. The base case of our induction will be $n = 2$, and here $c_i(\omega_0)$ will be defined as in the latter case, since ω_0 will be a 1-plane bundle. Moreover, we want to observe that ω_0 was defined as a bundle over E_0, so $c_i(\omega_0) \in H^{2i}(E_0; \mathbb{Z})$, so $c_i(\omega) = (\pi_0^*)^{-1}(c_i(\omega_0)) \in H^{2i}(B; \mathbb{Z})$ as we want.

We will define the *total Chern class* of an n-plane bundle ω as the sum

$$c(\omega) = 1 + c_1(\omega) + \cdots + c_n(\omega) \in H^{\Pi}(B; \mathbb{Z}).$$

As for other classes, also the Chern class possesses really useful properties, namely the naturality, the triviality and a product formula. Let's state them in a rigorous way.

Theorem I.6.2. *Let $f : B \to B'$ be a map covered by a bundle map $\omega \to \omega'$ between complex n-plane bundles. Then*

$$c(\omega) = f^*(c(\omega')).$$

This implies that if there exists an isomorphism of vector bundles between ω and ω', $c(\omega) = c(\omega')$.

Theorem I.6.3. *If ω is a complex vector bundle and ε^k is a trivial complex k-plane bundle, over the same base space as ω, then*

$$c(\omega \oplus \varepsilon^k) = c(\omega).$$

Theorem I.6.4. *If ω, ϕ are two complex vector bundles over the same paracompact base space, then*

$$c(\omega \oplus \phi) = c(\omega) \smile c(\phi).$$

Here by *paracompact* topological space X we mean an Hausdorff space such that for every open cover $\{U_i\}_{i\in I}$ of X there exists an open cover $\{V_j\}_{j\in J}$ such that:

- for every $j \in J$, there exists an $i(j) \in I$ such that $V_j \subset U_{i(j)}$ (i.e. $\{V_j\}_j$ is a refinement of $\{U_i\}_i$);
- If $x \in X$, then $\{j| \ x \in V_j\}$ is finite (i.e. $\{V_j\}_j$ is a locally finite open cover).

In general, we will use compact spaces, and compactness implies paracompactness.

Other than these nice properties, we have other properties due to the fact that we are working with complex vector bundles, and so conjugate bundles are defined.

Lemma I.6.5. *Let ω be an n-plane bundle. Then the k-th Chern class of its conjugate bundle is $c_k(\overline{\omega}) = (-1)^k c_k(\omega)$ and thus the total Chern class can be written as*

$$c(\overline{\omega}) = 1 - c_1(\omega) + \cdots + (-1)^n c_n(\omega).$$

Finally, we can see that combining this lemma with Lemma I.5.9 (and Theorem I.6.2), we get the following result:

$$\begin{aligned} 1 + c_1(\xi \otimes \mathbb{C}) + \cdots + c_n(\xi \otimes \mathbb{C}) =& c(\xi \otimes \mathbb{C}) = \\ =& c(\overline{\xi \otimes \mathbb{C}}) = \\ =& 1 - c_1(\xi \otimes \mathbb{C}) + \cdots + (-1)^n c_n(\xi \otimes \mathbb{C}). \end{aligned}$$

Corollary I.6.6. *Every odd Chern class $c_1(\xi \otimes \mathbb{C}), c_3(\xi \otimes \mathbb{C}), \ldots$ is of order two.*

Note that for ξ a real plane bundle, its complexification is a complex plane bundle, so its Chern class is well defined.

I.7 Pontrjagin Classes

Now that we have defined the Chern class and the complexification of a real vector bundle, it is possible to give the definition of Pontrjagin classes and some first few properties. Moreover, in this section we will define the Pontrjagin numbers, which we will see in the next chapter play an important role in cobordism theory.

As we saw in Corollary I.6.6, the odd Chern of classes of the complexification of a real plane bundle are of order 2. We will define the Pontrjagin classes as characteristic classes deriving from the even Chern classes (we forget the odd classes).

Definition I.7.1. Let ξ be a real n-plane bundle with base space B. Its i-th *Pontrjagin class* is the cohomology class

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B;\mathbb{Z}).$$

We will call the *total Pontrjagin class* of ξ the sum

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{\lfloor \frac{n}{2} \rfloor}(\xi) \in H^{\Pi}(B;\mathbb{Z}).$$

Thanks to the fact that the definition of Pontrjagin classes is so closely related to Chern classes, Pontrjagin classes inherit some of Chern classes properties.

Lemma I.7.2. *Pontrjagin classes are natural with respect to bundle maps.*

Furthermore, if ε^k is the trivial real bundle and ξ is a real vector bundle over the same base space, then

$$p(\xi \oplus \varepsilon^k) = p(\xi).$$

We would like to also have a product formula as in the previous characteristic classes we have studied. Unfortunately, we only have the following.

Theorem I.7.3. *Let ξ and η be two real plane bundles over the same base space. Then we have*

$$p(\xi \oplus \eta) \equiv p(\xi) \smile p(\eta)$$

modulo elements of order 2 (or equivalently, $2(p(\xi \oplus \eta) - p(\xi) \smile p(\eta)) = 0$).

Proof. First of all, note that we have the isomorphism $(\xi \oplus \eta) \otimes \mathbb{C} \cong (\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C})$, thus

$$c_i((\xi \oplus \eta) \otimes \mathbb{C}) = c_i((\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C})) = \sum_{k+l=i} c_k((\xi \otimes \mathbb{C}) \smile c_l((\eta \otimes \mathbb{C}))$$

by Theorems I.6.2 and I.6.4.

Since the odd Chern classes are of order 2, we can start working modulo elements of order 2 and ignore them:

$$\begin{aligned} p_i(\xi \oplus \eta) \equiv &(-1)^i c_{2i}((\xi \oplus \eta) \otimes \mathbb{C}) \equiv \sum_{k+l=i} (-1)^{k+l} c_{2k}(\xi \otimes \mathbb{C}) \smile c_{2l}(\eta \otimes \mathbb{C}) = \\ &\equiv \sum_{k+l=i} (-1)^k c_{2k}(\xi \otimes \mathbb{C}) \smile (-1)^l c_{2l}(\eta \otimes \mathbb{C}) = \\ &\equiv \sum_{k+l=i} p_k(\xi) \smile p_l(\eta). \end{aligned}$$

Hence, if we compute the total Pontrjagin class we get

$$p(\xi \oplus \eta) \equiv p(\xi) \smile p(\eta) \mod 2$$

as we wanted. □

A useful consequence of this theorem is the next corollary.

Corollary I.7.4. *Let ω be a complex n-plane bundle. Then the Chern classes $c_1(\omega), \ldots, c_n(\omega)$ determine the Pontrjagin classes $p_1(\omega_\mathbb{R}), \ldots, p_n(\omega_\mathbb{R})$ by the formula*

$$1 - p_1(\omega_\mathbb{R}) + p_2(\omega_\mathbb{R}) - \cdots + (-1)^n p_n(\omega_\mathbb{R}) = c(\omega) \smile c(\overline{\omega})$$

and dividing by grades we get

$$\begin{aligned} p_k(\omega_\mathbb{R}) = &c_k(\omega)^2 - 2c_{k-1}(\omega) \smile c_{k+1}(\omega) + \cdots + (-1)^{k+1} 2c_1(\omega) \smile c_{2k-1}(\omega) + \\ &+ (-1)^k 2c_{2k}(\omega) \end{aligned}.$$

Proof. By Lemma I.5.10, we have that $\omega_\mathbb{R} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$. Therefore,

$$p_i(\omega_\mathbb{R}) = (-1)^i c_{2i}(\omega_\mathbb{R} \otimes \mathbb{C}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega})$$

that by Theorem I.6.4 is equal to

$$p_i(\omega_\mathbb{R}) = (-1)^i \sum_{k+l=2i} c_k(\omega) \smile c_l(\overline{\omega}).$$

Thanks to Lemma I.6.5, we also have that

$$p_i(\omega_{\mathbb{R}}) = (-1)^i \sum_{k+l=2i} c_k(\omega) \smile (-1)^l c_l(\omega)$$

as we wanted.

Now, computing the product of sums in

$$c(\omega) \smile c(\overline{\omega}) = \left(\sum_{i=0}^{n} c_i(\omega) \right) \smile \left(\sum_{i=0}^{n} (-1)^i c_i(\omega) \right)$$

we see that this is equal to

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \cdot p_i(\omega_{\mathbb{R}}) &= \sum_{i=0}^{n} (-1)^i \cdot (-1)^i \sum_{k+l=2i} c_k(\omega) \smile (-1)^l c_l(\omega) = \\ &= \sum_{i=0}^{n} \sum_{k+l=2i} c_k(\omega) \smile (-1)^l c_l(\omega). \end{aligned}$$

□

In special cases, computing Pontrjagin classes can be even easier than this. One example would be the corollary to this lemma:

Lemma I.7.5. *Let ξ be a real n-plane bundle. Then $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ (which is a real $2n$-plane bundle) is isomorphic to $\xi \oplus \xi$. This isomorphism preserves the orientation if $\frac{n(n-1)}{2}$ is even, and reverses it if $\frac{n(n-1)}{2}$ is odd.*

Corollary I.7.6. *If ξ is a real oriented $2k$-plane bundle, then*

$$p_k(\xi) = e(\xi) \smile e(\xi).$$

Proof. We just need to show that

$$\begin{aligned} p_k(\xi) &= (-1)^k c_{2k}(\xi \otimes \mathbb{C}) = (-1)^k e(\xi \otimes \mathbb{C}) = (-1)^k (-1)^{\frac{2k(2k-1)}{2}} e(\xi \oplus \xi) = \\ &= (-1)^{2k^2} e(\xi) \smile e(\xi) = e(\xi) \smile e(\xi) \end{aligned}$$

where we used the definition of Pontrjagin classes and of the top Chern class, Proposition I.4.11 together with the last lemma and Theorem I.4.12. □

Lastly, as we did for Stiefel–Whitney numbers, we are going to introduce also Pontrjagin numbers. In order to do so, we firstly need to give the definition of partitions.

Definition I.7.7. Let k be a non-negative integer. A *partition* of k is an unordered sequence $I = i_1, \dots, i_r$ with $i_j > 0$ for every $j = 1, \dots, r$ and $k = i_1 + \cdots + i_r$.

We will now define the Pontrjagin numbers over compact $4n$-manifolds:

Definition I.7.8. Let M be a $4n$-dimensional smooth compact oriented manifold. Let $I = i_1, \dots, i_r$ be a partition of n. The I-th *Pontrjagin number* is

$$p_I[M] = p_{i_1 \cdots i_r}[M] = \langle p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M), \mu_{4n} \rangle,$$

where τ_M denotes the tangent bundle of M and μ_{4n} the fundamental homology class of M. As a convention $p_I[M] = 0$ for I not a partition of n.

Note that, if $p(n)$ indicates the number of partitions of n, a complex $4n$-manifold has $p(n)$ Pontrjagin numbers.

Remark I.7.9. Looking at Pontrjagin numbers we also can learn whether a manifold M^{4n} possesses reverse orientation diffeomorphisms. Indeed, since Pontrjagin classes don't depend on the orientation (they are invariant under any diffeomorphism), if we reverse the orientation of M, for every partition $I = i_1, \dots, i_r$ $p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M) = p_{i_1}(-\tau_M) \smile \cdots \smile p_{i_r}(-\tau_M)$. On the other hand, changing orientation sends μ_{4n} to $-\mu_{4n}$ (by definition of fundamental homology class). Hence, $p_I[M^{4n}]$ changes sign, and thus if there exists a partition I of n such that its Pontrjagin number is different from 0, M can't have any reverse orientation diffeomorphism.

To show this last statement, we will prove a more general statement. Let M, N be two $4n$-dimensional smooth compact oriented manifolds and let $f : M \to N$ a diffeomorphism such that $f_*(\mu_{4n}) = \nu_{4n}$, where μ_{4n}, ν_{4n} are the fundamental homology classes of M and N respectively. Now, denote by μ^*_{4n}, ν^*_{4n} be the fundamental cohomology classes. Since $f_*(\mu_{4n}) = \nu_{4n}$, we also have $f^*(\nu_{4n}^*) = \mu_{4n}^*$. Then, once chosen a partition $I = i_1, \ldots, i_r$, $p_{i_1}(\tau_N) \smile \cdots \smile p_{i_r}(\tau_N) \in H^{4n}(N^{4n}; \mathbb{Z}) = \langle \nu_{4n}^* \rangle$ so $p_{i_1}(\tau_N) \smile \cdots \smile p_{i_r}(\tau_N) = P\nu_{4n}^*$. Hence by naturality of Pontrjagin classes,

$$p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M) = f^*(p_{i_1}(\tau_N) \smile \cdots \smile p_{i_r}(\tau_N)) = f^*(P\nu_{4n}^*) = \\ = Pf^*(\nu_{4n}^*) = P\mu_{4n}^*.$$

Since $\langle \mu_{4n}^*, \mu_{4n} \rangle = \langle \nu_{4n}^*, \nu_{4n} \rangle = 1$

$$p_I[M] = p_I[N] = P.$$

Now, if we take N to be the manifold M with reverse orientation, so $N = -M$, $\nu_{4n} = -\mu_{4n}$ and $\nu_{4n}^* = -\mu_{4n}^*$. Moreover, $f^*(-\mu_{4n}^*) = \mu_{4n}^*$ and if $p_{i_1}(\tau_{-M}) \smile \cdots \smile p_{i_r}(\tau_{-M}) = P(-\mu_{4n}^*)$

$$f^*(p_{i_1}(\tau_{-M}) \smile \cdots \smile p_{i_r}(\tau_{-M})) = Pf^*(-\mu_{4n}^*) = P\mu_{4n}^*$$

but on the other hand, by naturality

$$p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M) = f^*(p_{i_1}(\tau_{-M}) \smile \cdots \smile p_{i_r}(\tau_{-M}))$$

and by invariance of Pontrjagin classes under any diffeomorphism,

$$p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M) = P(-\mu_{4n}^*),$$

so if $P \neq 0$, this leads to a contradiction.

II
The cobordism ring

In this chapter, we are going to define the unoriented and the oriented cobordism ring and see some first results that principally make use of Stiefel–Whitney and Pontrjagin numbers. These results are due to Pontrjagin, Thom and Wall. We will prove the Pontrjagin theorems in this chapter, whereas we postpone a discussion about the proof of Thom's theorem on the third chapter.

One of the main purposes in studying the cobordism relation is to classify closed manifolds, instead of dividing them up to diffeomorphism (which is a stronger relation).

II.1 Manifolds with boundary and the cobordism ring

In this section, we are going to see how to generalize the concept of a manifold to a manifold with boundary, and we are going to finally define cobordism classes and the cobordism ring.

As we can imagine any manifold as locally $\mathbb{R}^n$, we can do the same with manifolds with boundary, which will be locally identified with $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n | \; x_1 \geq 0\}$ (with the relative topology). We can see this formally in the following definition.

Definition II.1.1. Let X be a second countable Hausdorff space. It is called a *smooth n-dimensional manifold with boundary* if it has a smooth structure on it. A *smooth structure* is a family of pairs $\{(U_i, \phi_i)\}_{i \in I}$ where U_i is an open set of X and ϕ_i is a homeomorphism of U_i onto an open subset of $\mathbb{H}^n$ such that:

- $\{U_i\}_{i \in I}$ is an open cover of X;
- For every $i, j \in I$, where $U_i \cap U_j \neq \emptyset$, the homeomorphisms $\phi_i \circ \phi_j^{-1}$: $\phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are smooth maps between open sets of $\mathbb{H}^n$;
- the family $\{(U_i, \phi_i)\}_{i \in I}$ is maximal (i.e. it contains all the possible (U_i, ϕ_i) satisfying the first two conditions).

In contrast to a general manifold, the points of a manifold with boundary are of different type. As suggested by the name, there will be points on the boundary, that will correspond to the points of $\mathbb{H}^n$ with $x_1 = 0$.

Definition II.1.2. Let X be a manifold with boundary. We say that $x \in X$ in an *interior point* of X if there exists a local parameterization $\phi_i : U_i \to \mathbb{H}^n$ where $\phi_i(U_i)$ is an open of $\mathbb{R}^n$ (rather than of $\mathbb{H}^n$), i.e. $\phi_i(U_i) \cap \{(0, x_2 \ldots, x_n) \in \mathbb{H}^n\} = \emptyset$. We will denote by $\mathring{X}$ the set of all the interior points. The points that are not interior points will be called *boundary points*, and the *boundary* will be denoted by ∂X.

There are two remarks to be made here. Firstly, we need to observe that the set of all the interior points is an open set (since it is the union of all the open neighborhoods of the interior points), and a smooth n-dimensional manifold in X by definition. Secondly, the boundary is a smooth $(n-1)$-dimensional manifold, and since it is the complement of an open set, it will be closed in X.

We can also state an important theorem, that describes the boundary of a paracompact manifold with boundary. Since we will make use only of compact manifolds, this theorem will always hold for us (because compact implies paracompact).

Theorem II.1.3 (Collar neighborhood theorem)**.** *Let X be a smooth paracompact manifold with boundary. Then there exists an open neighborhood of the boundary ∂X in X which is diffeomorphic to $\partial X \times [0, 1)$.*

An important consequence of this theorem is highlighted in the following remark.

Remark II.1.4. This theorem implies that we have an homotopy equivalence between X and $\mathring{X}$. Indeed, let V be the collar neighborhood given by the theorem. We have that $V \cap \mathring{X} \cong \partial X \times (0,1)$, since $V \cap \mathring{X} = V \setminus \partial X$ and $\partial X \times (0,1) = \partial X \times [0,1) \setminus \partial(\partial X \times [0,1)) = \partial X \times [0,1) \setminus (\partial X \times \{0\})$. It is easy to prove that both $\partial X \times [0,1)$ and $\partial X \times (0,1)$ are homotopy equivalent to $\partial X \times [0.5,1)$, and thus are homotopy equivalent to each other.

Therefore, we have

$$V \cap \mathring{X} \cong \partial X \times (0,1) \sim \partial X \times [0,1) \cong V.$$

Thus, we can define the homotopy equivalence between X and $\mathring{X}$ as the homotopy $V \cap \mathring{X} \sim V$ extended by the identity outside V.

As we did in the previous chapter, without using the classical definition of orientation of a manifold we will use the following equivalent definition: a smooth n-manifold M is orientable if its tangent bundle τ_M is orientable. An orientation on M is an orientation on τ_M. It is possible to see the correspondent definition, that uses also fundamental classes when the manifolds are compact, in the appendix A.

To give the definition this way, we first need to understand how it is defined the tangent bundle of a manifold with boundary and how this restricts to the boundary. The tangent bundle of a manifold with boundary X is still an n-plane bundle defined as in Example I.1.1. We just need to pay attention at the boundary points: the tangent manifold on a boundary point $x \in \partial X$, DX_x, has an $(n-1)$-submanifold, consisting of the tangent manifold of the boundary $D(\partial X)_x$. We have that $DX_x \setminus D(\partial X)_x$ can be divided into vectors "into" and "out of" X. We say that a vector $v \in DX_x \setminus D(\partial X)_x$ points *into* X if v is the velocity vector $\frac{dp}{dt}|_{t=0}$ of a smooth path $p : [0;\varepsilon) \to X$ with $p(0) = x$, and that $v \in DX_x \setminus D(\partial X)_x$ points *out of* X if v is the velocity vector of a smooth path $p : (-\varepsilon; 0] \to X$ with $p(0) = x$.

We want to define an orientation on ∂X once given an orientation on X. To do so, we need to see how the orientation on τ_X induces an orientation on $\tau_X|_{\partial X}$.

Let x be a point on the boundary. By the convention we will be using, once given an orientation for τ_X at x with representative basis $\{v_1, \ldots, v_n\}$

where the vector v_1 points "out of" X and the other vectors $v_2, \dots, v_n$ are tangent to ∂X, $\{v_2, \dots, v_n\}$ will represent the desired induced orientation basis of $D(\partial X)_x$. Thus, we will have

$$\tau_X|_{\partial X} \cong \tau_{\partial X} \oplus \varepsilon^1$$

where ε^1 will be the trivial line bundle of "out of" X vectors.

Even though we mainly just need to use this definition of orientation, in the following section we will consider compact oriented manifolds and we will use their fundamental classes. If a manifold without boundary is compact, it is orientable if and only if it has a fundamental class. Regarding this, let's present the following proposition.

Proposition II.1.5. *Let X be a compact and R-oriented manifold with boundary. Let $\mu_{\partial X}$ be the fundamental class of ∂X given by the induced R-orientation on ∂X. Then there exists a unique generator $\mu_X \in H_n(X, \partial X; R)$ such that $\partial_n(\mu_X) = \mu_{\partial X}$.*

For the proof of this proposition, we refer to the appendix A.1.6.

We will call this element μ_X the R-fundamental class of X determined by the R-orientation of X.

Before going on with the definition of the cobordism relation, since we have defined the induced orientation on the boundary, we can finally give an easy example of manifold with boundary.

Example II.1.1. Given an smooth oriented n-dimensional manifold M, then $M \times [0, 1]$ is a smooth $(n+1)$-dimensional manifold with boundary $M \sqcup (-M)$, where we denote by $(-M)$ the manifold M with opposite orientation.

Now that we have the definition of manifolds with boundary, we can study this boundary. As we already stated, the boundary will be a manifold of one dimension less than the manifold. We can classify the possible boundaries by an equivalence relation called the cobordism equivalence. To define the classes of this relation, we need to first see if we need to work on an unoriented or an oriented environment.

Firstly, let's see what the unoriented definition is:

Definition II.1.6. Let M_1, M_2 be two n-dimensional manifolds, which are also smooth and closed (i.e. compact without boundary). Then we say that they belong to the same *unoriented cobordism class* if and only if $M_1 \sqcup M_2$ is the boundary of a smooth compact $(n+1)$-dimensional manifold.

However, if our manifold is also oriented, usually we can get more information: for example, we can define Chern and Pontrjagin numbers. In particular, the latter ones play an important role in the study of cobordism classes. Therefore, we want to define also the notion of oriented cobordism classes.

Definition II.1.7. Let M_1, M_2 two smooth compact oriented n-dimensional manifolds. We say that they are *oriented cobordant,* or equivalently that they belong to the same *oriented cobordism class*, if there exists a smooth compact oriented $(n+1)$-dimensional manifold with boundary X such that its boundary ∂X with its induced orientation is diffeomorphic to $M_1 \sqcup (-M_2)$ (via a orientation preserving diffeomorphism).

Note that we denoted by $-M_2$ the manifold M_2 with opposite orientation.

It is easy to see that if we forget the orientation on all these manifolds, we get back the definition of unoriented cobordism classes. For this reason, the majority of the definitions that will follow will be about oriented manifolds. For instance, we could give also the definition of an unoriented cobordism ring, but that would be redundant, since we can just forget the orientation.

In all these definitions, we talked about equivalence classes. Indeed, the cobordism relation is an equivalence relation:

Lemma II.1.8. *Oriented cobordism is an equivalence relation, i.e. it is reflexive, symmetric and transitive.*

Proof. Let M_1, M_2, M_3 be any three smooth compact oriented n-dimensional manifolds, and let X, Y be smooth compact oriented manifold with boundary.

- Reflexivity: We have that M_1 is cobordant to itself. Indeed, we have that $M \sqcup (-M)$ is the boundary of $M \times [0, 1]$, as we saw in Example II.1.1.

- Symmetry: Suppose M_1, M_2 are such that $M_1 \sqcup (-M_2) \cong \partial X$. Then, if we invert each orientation, it is easy to see that $(-M_1) + M_2 \cong \partial(-X)$.

- Transitivity: Suppose M_1, M_2, M_3 are such that $M_1 \sqcup (-M_2) \cong \partial X$ and $M_2 \sqcup (-M_3) \cong \partial Y$. Using the Theorem II.1.3,we get that M_2 has a neighborhood in X diffeomorphic to $M_2 \times (-1, 0]$ and a neighborhood in Y diffeomorphic to $M_2 \times [0, 1)$. We can glue them on M_2, and get that $M_1 \sqcup (-M_3) \cong \partial(X \cup Y)$. Indeed, all the points of M_2 won't be on the on the boundary (they will be internal points of $X \cup Y$, for example choosing $U \times (-1, 1)$ as a neighborhood, where U is a neighborhood of in M_2). Note that the orientation in this way agrees.

□

We can now see how we can collect all these classes and give this set the structure of a group.

Let Ω_n denote the set of all the oriented cobordism classes of smooth closed n-dimensional manifolds. If we take the disjoint union $\sqcup$ as an operation between classes, it becomes an abelian group. Indeed, if $[M], [M']$ are two cobordism classes, then $M \sqcup M'$ is an n-dimensional manifold, so $[M \sqcup M'] \in \Omega_n$. Moreover, it is well defined: let $M_1 \sim \tilde{M}_1$ and $M_2 \sim \tilde{M}_2$. If $M_1 \sqcup (-\tilde{M}_1) = \partial X$ and $M_2 \sqcup (-\tilde{M}_2) = \partial Y$ for some manifolds X, Y, then $(M_1 \sqcup M_2) \sqcup -(\tilde{M}_1 \sqcup \tilde{M}_2) = (M_1 \sqcup (-\tilde{M}_1)) \sqcup (M_2 \sqcup (-\tilde{M}_2)) = \partial X \sqcup \partial Y = \partial(X \sqcup Y)$. The zero element is the class of the empty set $[\emptyset]$ (the empty set is an n-manifold for any $n \geq 0$, since any $x \in \emptyset$ satisfies the conditions): $M \sqcup \emptyset = M = \emptyset \sqcup M$, so obviously $[M \sqcup \emptyset] = [M] = [\emptyset \sqcup M]$. It is abelian since $M \sqcup M' = M' \sqcup M$, and the inverse of $[M]$ is $[-M]$ (indeed, $M \sqcup -M$ is the boundary of $M \times [0, 1]$, so $[M \sqcup -M] = [\emptyset]$).

One might wonder if we can connect these groups varying $n \in \mathbb{Z}_{\geq 0}$. Interestingly, using the Cartesian product

$$\begin{aligned} \Omega_m \times \Omega_n &\to \Omega_{m+n} \\ (M_1^m, M_2^n) &\mapsto M_1^m \times M_2^n \end{aligned}$$

we get an associative bilinear (with the respect to the disjoint union) product operation. This is well defined since $M_1^m \times M_2^n \in \Omega_{m+n}$ and if $[\tilde{M}_1^m] = [M_1^m]$ and $[\tilde{M}_2^n] = [M_2^n]$, where $M_1^m \sqcup (-\tilde{M}_1^m)$ bounds X and $M_2^n \sqcup (-\tilde{M}_2^n)$ bounds Y, then $(M_1^m \times M_2^n) \sqcup -(\tilde{M}_1^m \times \tilde{M}_2^n)$ bounds $X \times \partial Y \sqcup \partial X \times Y$, so $[M_1^m \times M_2^n] = [\tilde{M}_1^m \times \tilde{M}_2^n]$. The unit element of this operation is the 0-dimensional manifold with a single point. Indeed, if $M \in \Omega_m$, $[M \times \{\text{pt}\}] = [M] \in \Omega_{0+m}$.

With this operation, we get that the sequence

$$\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \dots) = \bigoplus_i \Omega_i$$

gets the structure of a graded ring.

Moreover, for any two manifolds M_1^m, M_2^n, we have

$$M_1^m \times M_2^n \cong (-1)^{mn} M_2^n \times M_1^m$$

since the orientation of $M_1^m \times M_2^n$ is the concatenation of the basis of M_1^m followed by the one of M_2^n, and to pass to the orientation of $M_2^n \times M_1^m$ (which is the concatenation of the basis of M_2^n followed by the one of M_1^m), we need a permutation of the basis of sign $(-1)^{mn}$. Thus, Ω_* is also commutative in the graded sense.

Forgetting the orientation of the manifolds, we get the unoriented cobordism ring, that we will denote by

$$\mathcal{N}_* = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \dots) = \bigoplus_i \mathcal{N}_i.$$

In this case, if M in an m-dimensional manifold, $[M] = [-M]$ since $M \sqcup M$ is the boundary of $M \times [0,1]$ (where we are not considering orientations anywhere). Thus, since we also have that $[-M]$ is the inverse of $[M]$, we get that $\mathcal{N}_m$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space for every m and the ring $\mathcal{N}_*$ has the structure of a graded $\mathbb{Z}/2\mathbb{Z}$-algebra.

II.2 Pontrjagin Theorems

We have some theorems by Pontrjagin that give us necessary conditions for a manifold to be null cobordant. These two theorems involve Stiefel–Whitney and Pontrjagin numbers, that we constructed in the first chapter.

The first one uses only Stiefel–Whitney numbers, that are defined independently from the orientability of the manifold, so we won't require the manifold to be oriented.

Theorem II.2.1 (Pontrjagin). *Let M be a closed smooth n-dimensional manifold. Let X be a smooth compact $(n+1)$-dimensional manifold with boundary M. Then, the Stiefel–Whitney numbers of M are all zero.*

Proof. First of all, denote by μ_X the fundamental class of the pair (X, M) given by Proposition II.1.5 and by μ_M the fundamental class of M induced by the $\mathbb{Z}/2\mathbb{Z}$-orientation of X. They exist since X, M are compact and every compact manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable. By Proposition II.1.5, we have that $\partial(\mu_X) = \mu_M$.

Moreover, by definition of the Kronecker pairing ($\langle [f], [c] \rangle = f(c)$ for $[f]$ a singular i-coycle and $[c]$ and i-cycle) and of the connecting homomorphism in singular homology ($f(\partial c) = (\delta f)(c)$ for $[f]$ a singular i-cocycle and $[c]$ an $(i+1)$-cycle), we have that

$$\langle v, \mu_M \rangle = \langle v, \partial \mu_X \rangle = \langle \delta v, \mu_M \rangle \in \mathbb{Z}/2\mathbb{Z} \tag{II.1}$$

for every $v \in H^n(M; \mathbb{Z}/2\mathbb{Z})$.

Now, let τ_X be the tangent bundle of X and $\tau_X|_M = \iota^* \tau_X$ its restriction to M (where $\iota : M \hookrightarrow X$ is the inclusion map). Let τ_M be the tangent bundle of M (seen as a sub-bundle of τ_X). We have

$$\tau_X|_M \cong \tau_M \oplus \varepsilon^1,$$

where ε^1 is the trivial line bundle. Thus, by Proposition I.2.6, we have that the Stiefel–Whitney classes of $\tau_X|_M$ are the same as the Stiefel–Whitney classes of τ_M

$$\iota^*(w_i(\tau_X)) = w_i(\iota^* \tau_X) = w_i(\tau_X|_M) = w_i(\tau_M) \tag{II.2}$$

for every $i \geq 0$, thanks to axiom 2.

Taking the long exact sequence of the pair (X, M) in cohomology

$$H^i(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\iota^*} H^i(M, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta^i} H^{i+1}(X, M; \mathbb{Z}/2\mathbb{Z})$$

we get that $\delta^i \circ \iota^* = 0$ for every $i \geq 0$. Therefore, by the equation II.2, we get that $\delta^i(w_i(\tau_M)) = 0$ for every $i \geq 0$, and for every $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$ such that $r_1 + 2r_2 \dots nr_n = n$,

$$\begin{aligned}\delta^n\left(w_1(\tau_M)^{r_1} \smile \cdots \smile w_n(\tau_M)^{r_n}\right) =& \delta^1\left(w_1(\tau_M)^{r_1}\right) \smile w_2(\tau_M)^{r_2} \smile \cdots \smile \\ & \smile w_n(\tau_M)^{r_n} \pm \cdots \pm w_1(\tau_M)^{r_1} \smile \cdots \smile \\ & \smile w_{n-1}(\tau_M)^{r_{n-1}} \smile \delta^n\left(w_n(\tau_M)^{r_n}\right) = \\ =& 0 \pm \cdots \pm 0 = 0\end{aligned}$$

To conclude, let's compute the Stiefel–Whitney numbers of M, using this and the equation above II.1

$$\begin{aligned}w_1^{r_1} \cdots w_n^{r_n}[M] =& \langle w_1(\tau_M)^{r_1} \smile \cdots \smile w_n(\tau_M)^{r_n}, \mu_M\rangle = \\ =& \langle \delta(w_1(\tau_X)^{r_1} \smile \cdots \smile w_n(\tau_X)^{r_n}), \mu_X\rangle = \\ =& \langle 0, \mu_X\rangle = 0.\end{aligned}$$

□

Note that if M is the boundary of X, then $M \sqcup \emptyset = \partial X$, so M is cobordant to the empty set, so $[M]$ is the zero element of $\mathcal{N}_n$.

An example where we can see that this theorem holds is the following:

Example II.2.1. If M is a closed smooth n-dimensional manifold, then $M \sqcup M$ is the boundary of the cylinder $M \times [0,1]$. If we compute any Stiefel–Whitney class of $M \sqcup M$, we see that this is zero (since $\omega_I[M \sqcup M] = \omega_I[M] + \omega_I[M] = 2\omega_I[M] \equiv 0 \in \mathbb{Z}/2\mathbb{Z}$), so every Stiefel–Whitney number of $M \sqcup M$ is zero.

Moreover, in the last chapter we also computed the Stiefel–Whitney numbers of the projective space in the Example I.3.2.

Example II.2.2. Considering the projective space $\mathbb{P}^n$, we can see that if n is odd, all its Stiefel–Whitney numbers will be zero, and therefore by the theorem II.2.1 $\mathbb{P}^n$ will be an unoriented boundary (for example, $\mathbb{P}^1 \cong \mathbb{S}^1$ is the boundary of the disk D^2). On the other hand, if n is even, $\mathbb{P}^n$ will never be a boundary, because some of its Stiefel–Whitney numbers will always be different from zero.

It has been proven by Thom that the condition given by the theorem is not only a necessary condition but a sufficient condition. Indeed, we have that

Theorem II.2.2 (Thom)**.** *Let M be a closed smooth n-dimensional manifold. If all the Stiefel–Whitney numbers of M are zero, then M is the boundary of some smooth compact $(n+1)$-dimensional manifold with boundary.*

The aim of next chapter will be to study the structure of the unoriented cobordism ring, in order to prove this theorem.

These two theorems give rise to a result which is really interesting when computing the unoriented cobordism ring.

Corollary II.2.3. *Let M_1, M_2 be two closed smooth n-dimensional manifolds. They belong to the same unoriented cobordism class if and only if their Stiefel–Whitney numbers are equal.*

Proof. We have that $\omega_1^{r_1} \cdots \omega_n^{r_n}[M_1 \sqcup M_2] = \omega_1^{r_1} \cdots \omega_n^{r_n}[M_1] + \omega_1^{r_1} \cdots \omega_n^{r_n}[M_2]$. This is zero if and only if $\omega_1^{r_1} \cdots \omega_n^{r_n}[M_1] = \omega_1^{r_1} \cdots \omega_n^{r_n}[M_2]$ (since these numbers are in $\mathbb{Z}/2\mathbb{Z}$). Thus, by theorems II.2.1 and II.2.2, we get the desired result. □

We have a second theorem by Thom, now involving Pontrjagin numbers (and therefore oriented manifolds).

Theorem II.2.4 (Pontrjagin)**.** *Let M be a closed smooth oriented manifold of dimension $4n$. Let X be a smooth compact oriented $(4n+1)$-dimensional manifold with boundary M. Then, the Pontrjagin numbers of M are all zero.*

The proof is really similar to the one of Theorem II.2.1. We are going to highlight only the crucial passages.

Proof. First of all, denote by $\mu_X \in H_{4n+1}(X, M; \mathbb{Z})$ the fundamental class of the pair (X, M) given by Proposition II.1.5 and by $\mu_M \in H_{4n}(M; \mathbb{Z})$ the fundamental class of M induced by the orientation of X. They exists since X, M are compact and $\mathbb{Z}$-orientable. Using the morphism given by the long exact sequence of the pair (X, M) with coefficients in $\mathbb{Z}$, we get $\partial(\mu_X) = \mu_M$, once again by Proposition II.1.5.

We have that

$$\langle v, \mu_M \rangle = \langle v, \partial \mu_X \rangle = \langle \delta v, \mu_M \rangle \in \mathbb{Z} \tag{II.3}$$

for every $v \in H^n(M; \mathbb{Z})$.

As for the other proof, we have

$$\tau_X|_M \cong \tau_M \oplus \varepsilon^1,$$

and thus by Lemma I.7.2, we have that the Pontrjagin classes of $\tau_X|_M$ are the same as the Pontrjagin classes of τ_M

$$\iota^*(p_i(\tau_X)) = p_i(\iota^* \tau_X) = p_i(\tau_X|_M) = p_i(\tau_M) \tag{II.4}$$

for every $i \geq 0$.

Taking the long exact sequence of the pair (X, M) in cohomology we get that $\delta \circ \iota^* = 0$. Therefore, by the equation II.4, we get that $\delta^i(p_i(\tau_M)) = 0$ for every $i \geq 0$, and for every partition $I = i_1, \dots, i_r$ of n,

$$\begin{aligned}\delta\left(p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M)\right) =& \delta\left(p_{i_1}(\tau_M)\right) \smile p_{i_2}(\tau_M) \smile \cdots p_{i_r}(\tau_M) \\ & \pm \cdots \pm p_{i_1}(\tau_M) \smile \cdots p_{i_{r-1}}(\tau_M) \smile \\ & \smile \delta\left(p_{i_r}(\tau_M)\right) = 0 \pm \cdots \pm 0 = 0\end{aligned}$$

To conclude, let's compute the Pontrjagin numbers of M, using this and the equation above II.3

$$\begin{aligned}p_I[M] =& \langle \left(p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M)\right), \mu_M \rangle = \\ =& \langle \delta\left(p_{i_1}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M)\right), \mu_X \rangle = \\ =& \langle 0, \mu_X \rangle = 0.\end{aligned}$$

□

As before, we want to give an example and a corollary to the Pontrjagin theorem in the oriented case. These are due to the fact that for M_1, M_2 two closed smooth oriented $4n$-dimensional manifolds,

$$p_I[M_1 \sqcup M_2] = p_I[M_1] + p_I[M_2] \tag{II.5}$$

for any I partition of n.

Example II.2.3. If M is a closed smooth oriented $4n$-dimensional manifold, then $M \sqcup (-M)$ is the boundary of the cylinder $M \times [0,1]$. If we compute any Pontrjagin number of $M \sqcup (-M)$, we see that this is zero by the equation above. We see that this agrees with Theorem II.2.4.

Corollary II.2.5. *Let M_1, M_2 be two closed smooth oriented $4n$-dimensional manifolds. If they belong to the same oriented cobordism class then their Pontrjagin numbers are equal.*

The proof is quite easy and similar to the one of Corollary II.2.3:

Proof. Let $[M] = [\tilde{M}]$. Then there exist a manifold X such that $\partial X = M \sqcup (-\tilde{M})$. By Theorem II.2.4, $0 = p_I(\partial X) = p_I(M \sqcup (-\tilde{M})) = p_I[M] - p_I[\tilde{M}]$ for every partition I of n. Hence, $p_I[M] = p_I[\tilde{M}]$. □

Lastly, we want to show a last corollary, which once again is a consequence of equation II.5.

Corollary II.2.6. *Let M is a closed smooth oriented $4n$-dimensional manifold. For any partition I of n, the map*

$$\begin{aligned} \Omega_{4n} &\to \mathbb{Z} \\ [M] &\mapsto p_I[M] \end{aligned}$$

is a group homomorphism.

Proof. We already have equation II.5. We still need to prove that it is well defied and the zero of Ω_{4n} is mapped to the zero of $\mathbb{Z}$. It is well defined by Corollary II.2.5 and the unit is mapped to the unit since $p_I[\emptyset] = 0$ for every partition I of n. □

As for the unoriented case, we also have that the converse implication of Theorem II.2.4 is true. This really powerful result, due to Wall, is the following.

Theorem II.2.7 (Wall)**.** *Let M is a closed smooth oriented $4n$-dimensional manifold. It is an oriented boundary if and only if all its Stiefel–Whitney and all its Pontrjagin numbers are zero.*

We will not prove this theorem, and we will defer to [Wal60] for the proof.

This theorem highlights also that if a $4n$-dimensional manifold is oriented and is an unoriented manifold (and thus by Theorem II.2.2 its Stiefel–Whitney numbers are all zero), then it is also an oriented boundary if and only if all its Pontrjagin numbers are zero.

III

Thom's theorem on Stiefel–Whitney classes

The aim of this chapter is to study the structure of the unoriented cobordism ring. We can relate it to homotopy theory, in order to then prove the Theorem II.2.2. To do so, we will need to construct the universal bundle (and before that the Grassmannian manifold) and see how to construct the Thom space of a vector bundle. Once we have done that, we will construct the Thom spectrum, give its homotopy groups a bilinear product and show that this homotopy ring is isomorphic to the cobordism ring. We will then study the homotopy ring of this Thom spectrum, using the Thom isomorphism and the Steenrod algebra, in order to get more information about the unoriented cobordism ring itself. This procedure will give us the tools to prove the Thom's theorem. Lastly, since this process clearly gives us the structure of the unoriented cobordism ring, we will be able to give explicit generators of it.

III.1 The Grassmannian and the canonical bundle

In this section we will define the Grassmannian manifold and we will construct over it the universal vector bundle. Its universality will be fundamental in order to study any other vector bundle over a paracompact vector space. We will then construct some maps that will help us define an H-space structure on the colimit of the Grassmannians. This structure will allow us to define a ring structure for the homology groups of this colimit. This section is filled with prerequisites we will need in this chapter.

Definition III.1.1. The subset of the n-fold product $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$ which consists of linearly independent n-tuples of $\mathbb{R}^{n+k}$, is an open subset of this Cartesian product which is called the *Stiefel manifold* and is denoted by $\mathbb{V}_n(\mathbb{R}^{n+k})$.

The topology of this manifold is just the relative topology as subsets of $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$. This manifold was defined just in order to give the following set a natural topology.

Definition III.1.2. The (n, k)-*Grassmannian* (or *Grassmann manifold*) is the collection of all the n-dimensional vector subspaces of $\mathbb{R}^{n+k}$. It is denoted by $\mathbb{G}_n(\mathbb{R}^{n+k})$.

Usually we omit (n, k) when the context is clear.

If $n = 1$, $\mathbb{G}_1(\mathbb{R}^{1+k})$ is exactly the projective space $\mathbb{P}^k$.

We have a natural map $q : \mathbb{V}_n(\mathbb{R}^{n+k}) \to \mathbb{G}(\mathbb{R}^{n+k})$, which take an element of the domain to the n-dimensional subspace of $\mathbb{R}^{n+k}$ that the vectors span. We can give $\mathbb{G}(\mathbb{R}^{n+k})$ the quotient topology with the respect to this map.

With this topology, it can be proven that the Grassmannian has the following properties.

Lemma III.1.3. *The topological space $\mathbb{G}_n(\mathbb{R}^{n+k})$ is a compact topological manifold of dimension nk. Moreover, the correspondence*

$$\begin{aligned} \mathbb{G}_n(\mathbb{R}^{n+k}) &\to \mathbb{G}_k(\mathbb{R}^{n+k}) \\ X &\mapsto X^{\perp} \end{aligned}$$

where $X^{\perp}$ is the orthogonal k-plane of X in $\mathbb{R}^{n+k}$, is a homeomorphism.

For the proof of this lemma, please check [MS05, Lemma 5.1]. Moreover, we can give $\mathbb{G}_n(\mathbb{R}^{n+k})$ a finite CW-complex structure, as stated in [MS05, Thorem 6.4].

We can also extend the definition of the Grassmannian to the infinite Grassmannian. To do so, consider the infinite dimensional vector space

$$\mathbb{R}^{\infty} = \{(x_1, x_2, x_3, \dots) | \ \exists i \text{ such that } x_j = 0 \text{ for } j > i\}.$$

Fixing $k > 0$, we can identify the set of all the vectors in $\mathbb{R}^\infty$ of the form $(x_1, \dots, x_k, 0, 0, \dots)$ with $\mathbb{R}^k$. Therefore we have the following chain of inclusions

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \dots$$

and their union gives us $\mathbb{R}^\infty$.

Definition III.1.4. The *infinite Grassmannian* $\mathbb{G}_n = \mathbb{G}_n(\mathbb{R}^\infty)$ is the set of all the n-dimensional subspaces of $\mathbb{R}^\infty$, with weak topology given by the sequence of inclusions

$$\mathbb{G}_n(\mathbb{R}^n) \subset \mathbb{G}_n(\mathbb{R}^{n+1}) \subset \mathbb{G}_n(\mathbb{R}^{n+2}) \subset \dots$$

(i.e. U is an open set of $\mathbb{G}_n$ if and only if $U \cap \mathbb{G}_n(\mathbb{R}^{n+k})$ is open in $\mathbb{G}_n(\mathbb{R}^{n+k})$ for every $k \geq 0$). Here $\mathbb{G}_n$ is the infinite union $\cup_k \mathbb{G}_n(\mathbb{R}^{n+k})$.

As for the finite case, if $n = 1$, then $\mathbb{G}_1$ is the infinite dimensional real projective space $\mathbb{P}^\infty$ (defined as the direct limit of $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^3 \subset \dots$).

Furthermore, also $\mathbb{G}_n$ is a (infinite) CW-complex, by [MS05, Theorem 6.4].

It can be proven that the Grassmannian is a paracompact space. To do so, we need to use the following theorem, by Morita.

Theorem III.1.5 (Morita)**.** *Let X be a regular topological space (i.e. for every point $p \in X$ and every closed set $C \subset X$, $p \notin C$, there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $p \in U$ and $C \subset V$). Suppose that X is a countable union of compact subsets. Then X is paracompact.*

The result we are interested in is a corollary of this.

Corollary III.1.6. *The infinite Grassmannian $\mathbb{G}_n$ is a paracompact space.*

Now that we have defined the Grassmann manifold, we can define a vector bundle over it. Additionally, we will see the reason why this is called the universal bundle.

Let E be the topological space $E = \{(X, v) \in \mathbb{G}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} | \ v \in X\}$ (topologised as a subset of $\mathbb{G}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$). Then we have an n-plane bundle over $\mathbb{G}_n(\mathbb{R}^{n+k})$ that we will call the *canonical vector bundle* with total space E and projection map $\pi(X, v) = X$. For every $X \in \mathbb{G}_n(\mathbb{R}^{n+k})$, its fibre $\pi^{-1}(X)$ will be isomorphic to the subspace X itself, which is an n-dimensional vector space. We will denote this vector bundle by γ^n_k or $\gamma^n(\mathbb{R}^{n+k})$.

If $n = 1$, we get back the canonical line bundle over $\mathbb{P}^1 = \mathbb{G}_1(\mathbb{R}^{1+k})$ seen in Example I.1.1.

On the other hand, if $n = \infty$, as before $E = \{(X, v) \in \mathbb{G}_n \times \mathbb{R}^\infty | \ v \in X\}$ (topologised as a subset of the Cartesian product) gives the total space of an n-dimensional vector bundle γ^n over $\mathbb{G}_n$, with projection map $\pi(X, v) = v$. We will call this bundle the *universal bundle*.

The following theorem shows why this bundle is called "universal".

Theorem III.1.7. *Given any n-plane bundle ξ over a paracompact base space, we get a bundle map $\xi \to \gamma^n$.*

This map (or sometimes the map induced on the base spaces $B(\xi) \to B(\gamma^n) = \mathbb{G}_n$) will be called a *classifying map* for the bundle ξ. This map will be unique up to homotopy. Furthermore, observe that since this is a bundle map, ξ will be the pullback of γ^n under this classifying map.

The theorem is really useful since most of the typical topological spaces are paracompact. Indeed, by *Morita's theorem* every metric space is paracompact and by Corollary III.1.6 the Grassmannian is paracompact.

The last two things we want to study in this section will be the following two constructions. We will need it in order to create useful maps between Thom spaces.

Consider an isomorphism of vector spaces $\mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$. This implies that we have an homeomorphism $\mathbb{G}_n = \mathbb{G}_n(\mathbb{R}^\infty) \cong \mathbb{G}_n(\mathbb{R}^\infty \oplus \mathbb{R})$ (whose homotopy class is independent of the choice of isomorphism) that allows us to define a map

$$\begin{aligned} \iota_n : \mathbb{G}_n &\to \mathbb{G}_{n+1} \\ X &\mapsto X \oplus \mathbb{R} \end{aligned}.$$

This map is covered by a bundle map $\gamma^n \oplus \varepsilon^1 \to \gamma^{n+1}$, where ε^1 is a trivial line bundle over $\mathbb{G}_n$, by Theorem III.1.7. Moreover, we can also easily compute any characteristic class on these bundles: if c is any of the characteristic classes we defined in Chapter I, then it satisfies naturality and triviality, and so

$$c(\gamma^n) = c(\gamma^n \oplus \varepsilon^1) = \iota_n^* c(\gamma^{n+1}).$$

Another essential use that we can make of these natural inclusion maps is to define the direct limit given by the sequence of inclusions

$$\mathbb{G}_1 \xrightarrow{\iota_1} \mathbb{G}_2 \xrightarrow{\iota_2} \mathbb{G}_3 \xrightarrow{\iota_3} \dots$$

that will give us the topological space

$$BO = \cup_q \mathbb{G}_q,$$

with the weak topology. This space becomes really useful in our study since we can study its cohomology and homology groups (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) very clearly. Later in this chapter, we will give to $\oplus_i H^i(BO; \mathbb{Z}/2\mathbb{Z})$ a graded ring structure, and in the following chapters we will relate this to the homology ring of the Thom spectrum, thanks to the Thom isomorphism.

The other construction we want to introduce uses an isomorphism $\mathbb{R}^\infty \cong \mathbb{R}^\infty \oplus \mathbb{R}^\infty$. From this isomorphism of vector spaces, we get the homeomorphism $\mathbb{G}_{m+n} = \mathbb{G}_{m+n}(\mathbb{R}^\infty) \cong \mathbb{G}_{m+n}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty)$ (note that the homotopy class of this homeomorphism is independent of the choice of isomorphism $\mathbb{R}^\infty \cong \mathbb{R}^\infty \oplus \mathbb{R}^\infty$). From this we can construct a map

$$\begin{aligned} p_{m,n} : \mathbb{G}_m \times \mathbb{G}_n \to & \mathbb{G}_{m+n}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) = \mathbb{G}_{m+n} \\ (X, Y) \mapsto & X \oplus Y \end{aligned}.$$

We have that this map is covered by a bundle map $\gamma^m \times \gamma^n \to \gamma^{m+n}$, thanks to Theorem III.1.7, since $\gamma^m \times \gamma^n$ has fibres of dimension $m+n$ and it is over $\mathbb{G}_m \times \mathbb{G}_n$ which is paracompact (since it is product of paracompact spaces). As for ι_n, we can compute the characteristic classes of $\gamma^m \times \gamma^n$ as

$$p^*_{m,n}(c(\gamma^{m+n})) = c(\gamma^m \times \gamma^n)$$

by naturality of the characteristic classes, since $p^*_{m,n}(\gamma^{m+n}) = \gamma^m \times \gamma^n$.

Using these two kind of maps, we can define an H-space structure on BO. Recall that an H-space X is a space with a continuous multiplication map $M : X \times X \to X$ and an "identity" element $e \in X$ such that $M(\cdot, e)$ and $M(e, \cdot)$ are homotopic to the identity map with base point e. Let's see the H-space structure of BO : by passage to colimits over m and n, the maps $p_{m,n}$ induce a multiplication map $\oplus : BO \times BO \to BO$, that takes two linear subspaces to its direct sum. The identity element is obviously the 0-dimensional space $\{0\}$. It is clear that with this map $\oplus$, $BO, \{0\}$ is an H-space.

Following [Hat17, Section 3.C], we get that $H^*(BO;\mathbb{Z}/2\mathbb{Z})$ is a Hopf algebra (so it's a graded $\mathbb{Z}/2\mathbb{Z}$-algebra) and we can define a product operation between the homology groups (the *Pontrjagin product*) in order to get a graded $\mathbb{Z}/2\mathbb{Z}$-algebra structure on $H_*(BO;\mathbb{Z}/2\mathbb{Z})$. We are going to explain it more clearly in the following paragraphs.

In particular, the passage to colimits of the following result will be fundamental.

Theorem III.1.8. *The cohomology ring $H^*(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ freely generated by Stiefel–Whitney classes $w_1(\gamma^q),\dots,w_q(\gamma^q)$.*

For details about this theorem, check [MS05, Chapter 7]. In particular, we can consider the Stiefel–Whitney classes $w_1,\dots,w_q$ to be elements of $H^*(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z})$.

From this we get that

$$H^*(BO;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_i|i\geq 1] \tag{III.1}$$

as an algebra. The coalgebra structure given by the Hopf algebra structure of $H^*(BO;\mathbb{Z}/2\mathbb{Z})$ will be defined by the following coproduct

$$\psi : H^*(BO;\mathbb{Z}/2\mathbb{Z}) \to H^*(BO;\mathbb{Z}/2\mathbb{Z}) \otimes H^*(BO;\mathbb{Z}/2\mathbb{Z}).$$

If we consider the maps $p_{m,n}$ defined before, we have that $p_{m,n}^*(w_k(\gamma^{m+n})) = w_k(\gamma^m\times\gamma^n) \in H^*(\mathbb{G}_m\times\mathbb{G}_n)$. Since we are working with coefficients in a field and the Grassmannian is a CW-complex, *Künneth theorem* holds and thus

$$\begin{aligned} H^*(\mathbb{G}_m\times\mathbb{G}_n) &\cong H^*(\mathbb{G}_m)\otimes H^*(\mathbb{G}_n) \\ w_k(\gamma^m\times\gamma^n) &\cong \sum_{i+j=k} w_i(\gamma^m)\otimes w_j(\gamma^n)\cdot \end{aligned}$$

This clarifies that the coalgebra structure can just be defined as

$$\psi(w_k) = \sum_{i+j=k} w_i\otimes w_j.$$

This coalgebra structure, together with equation III.1, tells us that the ring $H_*(BO;\mathbb{Z}/2\mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$-algebra with one generator for each dimension by duality. In order to give explicitly these generators, take into consideration the inclusion map $i:\mathbb{P}^\infty = \mathbb{G}_1 \hookrightarrow BO$. Then $i_*: H_j(\mathbb{P}^\infty;\mathbb{Z}/2\mathbb{Z}) \to H_j(BO;\mathbb{Z}/2\mathbb{Z})$ is injective, and if we take $x_j\in H_j(\mathbb{P}^\infty;\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$ the non-zero element, $i_*(x_j) := b_j$ is a generator of $H_*(BO;\mathbb{Z}/2\mathbb{Z})$ or degree j. Thus, we have the following result.

Theorem III.1.9. *The homology ring $H_*(BO;\mathbb{Z}/2\mathbb{Z})$ is the polynomial algebra $\mathbb{Z}/2\mathbb{Z}[b_i|i \geq 1]$.*

III.2 Thom spaces

In this section we will construct the Thom space of a vector bundle (we will give multiple constructions). We will see different properties connected to this space, for example how it relates to the Thom isomorphism and its possible CW-structure. Afterwards, we will investigate on the Thom space of the universal bundle and from this, we will give the tools to construct the Thom spectrum and the homotopy ring of the Thom spectrum, which will be essential in our study of the unoriented cobordism theory. In the next chapter we will see that this latter object is isomorphic to the unoriented cobordism ring.

Let ξ be a k-plane bundle with an Euclidean metric. Denote by A (or sometimes A_ξ) the set $\{e \in E(\xi)|\ |e| \geq 1\}$, where $E(\xi)$ is the total space of ξ and $|e|$ is the inner product $e \cdot e$.

Definition III.2.1. The *Thom space* of ξ is the quotient $T(\xi) = E(\xi)/A$. The space A will be contracted to the preferred base point $t_0 \in T(\xi)$, which we will often call the *point at infinity*. Sometimes, we will denote this by $t_0(B(\xi))$.

We have that the space $T(\xi) \setminus \{t_0\}$ is isomorphic to $E(\xi) \setminus A = \{e \in E(\xi)|\ |e| < 1\}$.

In case the base space of ξ, $B = B(\xi)$ is compact, we can construct the Thom space in the following equivalent way.

Remark III.2.2. If B is compact, $T(\xi)$ is the one point compactification of $E(\xi)$. This is equivalent to compactifying each fibre (making them spheres with base points the points at infinity) and then identifying all the points at infinity.

In order to see that this is equivalent to the definition we have already given, we can consider the diffeomorphism

$$\begin{aligned}\varphi : E(\xi) \setminus A \to & E(\xi) \\ e \mapsto & \frac{e}{\sqrt{1-|e|^2}}.\end{aligned}$$

This is a well defined map since for every e, $|e| < 1$ and thus $\sqrt{1-|e|^2} \neq 0$, it is smooth and has as inverse the smooth map

$$\begin{aligned}\varphi^{-1} : E(\xi) \to & E(\xi) \setminus A \\ e \mapsto & \frac{e}{\sqrt{1+|e|^2}},\end{aligned}$$

which is smooth since it is composition of smooth functions and $\sqrt{1+|e|^2} \neq 0$ for every $e \in E(\xi)$.

Therefore, we have that the diffeomorphism φ induces an homeomorphism between $T(\xi)$ and $E(\xi) \cup \infty$: every point of $T(\xi) \setminus \{t_0\} \cong E(\xi) \setminus A$ will be taken diffeomorphically by φ to a point of $E(\xi)$, whereas t_0 will be mapped to ∞.

Note that in the case of B not compact, this $T(\xi)$ is not the one point compactification of $E(\xi)$ (consider for example the 1-dimensional trivial bundle over $\mathbb{R}^1$). However, we can still think of $T(\xi)$ as the one point identification of the spheres coming from the one point compactification of the fibers. If the bases space is discrete, this is a bouquet of sphere.

Another essential theorem by Thom is the one about Thom's isomorphism I.4.5. This can easily be used to compute the $\mathbb{Z}/2\mathbb{Z}$-cohomology groups of the Thom space. First of all, note that we are always working with positive degrees, so the reduced cohomology groups coincide with unreduced cohomology groups. We have the following.

Lemma III.2.3. *Let $\xi : E \to B$ be a k-plane bundle. Then there is a canonical isomorphism*

$$\Phi^i : H^i(B; \mathbb{Z}/2\mathbb{Z}) \to \tilde{H}^{k+i}(T(\xi); \mathbb{Z}/2\mathbb{Z})$$

for all $i \geq 0$.

Proof. We have that B can be embedded in $E \setminus A \cong T(\xi) \setminus t_0$ as the zero section of ξ. Define by T_0 the quotient E_0/A (where E_0 is the deleted total space, i.e. the total space without the zero section). This T_0 is contractible (because if B is discrete it is homeomorphic to a bouquet of spheres each without a point, if not with the same argument we get that it is contractible). Thus, $H^{k+i}(T_0, t_0; \mathbb{Z}/2\mathbb{Z}) = 0$ and using the long exact sequence of the triple $(T(\xi), T_0, t_0)$, we get that $H^{k+i}(T(\xi), T_0; \mathbb{Z}/2\mathbb{Z}) \cong H^{k+i}(T(\xi), t_0; \mathbb{Z}/2\mathbb{Z})$ for each k, i. By an excision argument, we have that $H^{k+i}(T(\xi), T_0; \mathbb{Z}/2\mathbb{Z}) \cong H^{k+i}(E, E_0; \mathbb{Z}/2\mathbb{Z})$. Since each vector bundle is $\mathbb{Z}/2\mathbb{Z}$-orientable, we have the Thom isomorphism

$$H^i(B; \mathbb{Z}/2\mathbb{Z}) \to H^{i+k}(E, E_0; \mathbb{Z}/2\mathbb{Z})$$
$$x \mapsto \pi^*(x) \smile \mu$$

where $\mu \in H^k(E, E_0; \mathbb{Z}/2\mathbb{Z})$ is the $\mathbb{Z}/2\mathbb{Z}$-Thom class (that since we are working with the ring $\mathbb{Z}/2\mathbb{Z}$, it is equivalent to the fundamental class, because the orientation is unique).

Composing all these isomorphisms, we get

$$H^i(B; \mathbb{Z}/2\mathbb{Z}) \cong H^{k+i}(T(\xi), t_0; \mathbb{Z}/2\mathbb{Z}) = \tilde{H}^{k+i}(T(\xi); \mathbb{Z}/2\mathbb{Z}).$$

□

Sometimes, with an abuse of notation, we will write this isomorphism as $x \mapsto \pi^*(x) \smile \mu$ too.

Recall that since in this case we are working with cohomology groups with coefficients in a field, these are finite dimensional vector spaces, with duals the correspondent homology groups. In particular, we have a dual map to this Thom isomorphism:

$$\Phi_i : \tilde{H}_{k+i}(T(\xi); \mathbb{Z}/2\mathbb{Z}) \to H_i(B; \mathbb{Z}/2\mathbb{Z}).$$

It has the following explicit formulation:

$$a \mapsto \pi_*(\mu \frown a).$$

Indeed, since the reduced homology of a quotient A/B is the reduced homology of the pair (A, B) and the reduced homology is equivalent to the unreduced homology for positive degrees, we can see the homology of the Thom space of a vector bundle ξ $T(\xi) = D(\xi)/S(\xi)$ as the homology of the pair $(D(\xi), S(\xi))$. Moreover, recall that we have a relative cap product

$$\frown : H^q(D(\xi), S(\xi); \mathbb{Z}/2\mathbb{Z}) \otimes H_{n+q}(D(\xi), S(\xi); \mathbb{Z}/2\mathbb{Z}) \to H_n(D(\xi); \mathbb{Z}/2\mathbb{Z})$$

coming from the classic cap product and the universal property of the quotient. With another abuse of notation consider the source of π to be $D(\xi)$.

Since proving that Φ_i is the dual of Φ^i is equivalent to proving that Φ^i is the dual of Φ_i, we want to compute the dual of Φ_i, $(\Phi_i)^*$ and show that it is Φ^i.

$$\begin{aligned}(\Phi_i)^* : H^i(B; \mathbb{Z}/2\mathbb{Z}) &\to \tilde{H}^{k+i}(T(\xi); \mathbb{Z}/2\mathbb{Z})\\ \alpha &\mapsto (\Phi_i)^*(\alpha)\end{aligned}$$

where

$$(\Phi_i)^*(\alpha)(a) = \alpha(\Phi_i(a)) = \alpha(\pi_*(\mu \frown a)) = \pi^*(\alpha)(\mu \frown a) = (\pi^*(\alpha) \smile \mu)(a).$$

Here we used the relation between cup and cap product $\gamma(\beta \frown c) = (\beta \smile \gamma)(c)$. Since Φ^i is an isomorphism, also Φ_i is an isomorphism.

Now consider the Thom space of the universal bundle. We can give it the structure of a CW-complex.

Lemma III.2.4. *Let ξ be a vector bundle over a base space B which is a CW-complex. Then its Thom space $T(\xi)$ is a CW-complex.*

The proof of this theorem can be found at [MS05, Lemma 18.1]. The lemma and its proof also prove that if B is a finite CW-complex then also $T(\xi)$ is finite, and if B is infinite dimensional, also $T(\xi)$ is an infinite dimensional CW-complex.

This implies that for every $n, k \geq 0$, the Thom space $T(\gamma_k^n)$ is a finite CW-complex. Moreover, for $n = \infty$, also the Thom space of the universal bundle has the structure of an infinite CW-complex. Thanks to the filtration $\mathbb{G}_n(\mathbb{R}^n) \subset \mathbb{G}_n(\mathbb{R}^{n+1}) \subset \mathbb{G}_n(\mathbb{R}^{n+2}) \subset \dots$, the total spaces of the canonical bundles on these base spaces will give us a fibration

$$E(\gamma_n^n) \subset E(\gamma_{n+1}^n) \subset E(\gamma_{n+2}^n) \subset \dots .$$

Quotienting in order to get the Thom spaces gives us

$$T(\gamma_n^n) \subset T(\gamma_{n+1}^n) \subset T(\gamma_{n+2}^n) \subset \dots .$$

This fibration gives us the CW-skeleton structure of

$$T(\gamma^n) = \cup_{k\geq 0} T(\gamma^n_{n+k}).$$

Another useful result, that helps us imagine the Thom space more clearly, is given by the following lemma.

Lemma III.2.5. *There is an homotopy equivalence between the infinite dimensional real projective space (which is the first Grassmannian) and the Thom space of the universal line bundle*

$$j : \mathbb{P}^\infty = \mathbb{G}_1 \to T(\gamma^1).$$

Proof. First of all, note that $T(\gamma^1) = D(\gamma^1)/S(\gamma^1)$. We can describe $S(\gamma^1)$ in an easier way: it is the subset of $E(\gamma^1) = \{(l, v) \in \mathbb{G}_1 \times \mathbb{R}^\infty |\ v \in l\}$ with elements of length 1. This length can be defined as $|(l, v)| = |v|$ (where $|v|$ is the canonical length in $\mathbb{R}^\infty$). This is isomorphic to $\mathbb{S}^\infty$, which is contractible. Thus, $D(\gamma^1)/S(\gamma^1)$ is the quotient of $D(\gamma^1)$ by a contractible subspace, and thus is equivalent to $D(\gamma^1)$. Hence, $T(\gamma^1)$ is homotopy equivalent to $D(\gamma^1)$. Since $\mathbb{P}^\infty$ is homotopic equivalent to the zero section of γ^1, there's an homotopy equivalence $\mathbb{P}^\infty \to D(\gamma^1)$. Therefore, $\mathbb{P}^\infty \overset{j}{\sim} T(\gamma^1)$. □

This following remark is fundamental in order to create maps between Thom spaces from bundle maps. Combining this with Theorem III.1.7, will give us maps between Thom spaces of vector bundles over paracompact spaces and Thom spaces of the universal vector bundle.

Remark III.2.6. Note that every bundle map between Euclidean vector bundles $f : \eta \to \xi$ induces a map between their Thom spaces $T(f) : T(\eta) \to T(\xi)$. Indeed, we can dilate the Euclidean metric on the second vector bundle ξ (this doesn't change the topology of its Thom space) in a way such that every $|f(v)| \geq 1$ if $|v| \geq 1$. Considering the diagram

$$\begin{array}{ccc} E(\eta) & \xrightarrow{f} & E(\xi) \\ {\scriptstyle q_\eta}\downarrow & & \downarrow{\scriptstyle q_\xi} \\ T(\eta) & \underset{T(f)}{\dashrightarrow} & T(\xi) \end{array}$$

and the isomorphism $q_\eta : T(\eta) \setminus t_0(\eta) \cong E(\eta) \setminus A_\eta$, we can define $T(f)$ as

$$\begin{aligned} T(f) : T(\eta) &\to T(\xi) \\ t_0(\eta) &\mapsto t_0(\xi) \\ v &\mapsto q_\xi(f(q_\eta^{-1}(v))) \end{aligned} .$$

We have added the condition $|f(v)| \geq 1$ if $|v| \geq 1$ in order to make the diagram above commutative.

Note that these maps are such that for each q-plane bundle $\pi_\xi : E \to B$ with B a paracompact space, the following diagram commutes

$$\begin{array}{ccc} \tilde{H}_{n+q}(T(\xi);\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{T(f)_*} & \tilde{H}_{n+q}(T(\gamma^q);\mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\Phi_n} & & \downarrow{\Phi_n} \\ \tilde{H}_n(B;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow[f_*]{} & \tilde{H}_n(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z}) \end{array}$$

for $f : B \to \mathbb{G}_q$ a classifying map, by definition of Φ_n and cap product properties.

We will use these induced maps multiple times in the next section, in the proof of Thom's theorem.

Now, since we want to describe the relation between the Stiefel–Whitney classes numbers and $\mathcal{N}_*$, we would like to describe this last one as something that's more familiar. In order to do so in the next section, we need construct a prespectrum (of which we will compute the homotopy group), built thanks to the association between Thom spaces and reduced suspensions.

For any Euclidean vector bundle ξ, we have an homeomorphism

$$T(\xi \oplus \varepsilon^1) \cong \Sigma T(\xi), \tag{III.2}$$

where Σ indicates the reduced suspension.

Indeed, if D is the disk $D(\xi) = E(\xi) \setminus A = \{e \in E(\xi) |\ |e| < 1\}$, then the disk of vectors of length < 1 of $\xi \oplus \varepsilon^1$ is homeomorphic to $D \times [-1;1]$. Now, their Thom spaces will be respectively homeomorphic to $D/\partial D$ and $D \times [-1;1]/(D \times \{-1;1\} \cup \partial D \times [-1;1])$ (because contracting A is equivalent to contracting the boundary of D and $D \times [-1;1]$ respectively). Now, the latter one is homeomorphic to

$$(D/\partial D \times [-1;1])/(D/\partial D \times \{-1;1\} \cup \partial D/\partial D \times [-1;1]),$$

which is $(T(\xi) \times [-1;1])/(T(\xi) \times \{-1;1\}) \cup t_0 \times [-1;1])$ that is the reduced suspension of $T(\xi)$.

We can give an easy example of this construction.

Example III.2.1. For example, if we take $\xi = \varepsilon^1$ over a point, its Thom space will be a sphere $\mathbb{S}^1$, and the suspension of an $\mathbb{S}^1$ is a 2 dimensional sphere $\mathbb{S}^2$. On the other hand, $\varepsilon^1 \oplus \varepsilon^1 \cong \varepsilon^2$, and the Thom space of ε^2 is an $\mathbb{S}^2$, as we wanted.

This gives us all the tools to construct maps $\Sigma T(\gamma^q) \to T(\gamma^{q+1})$. Indeed, we have that $\gamma^q \oplus \varepsilon^1$ is an $(n+1)$-vector bundle over the paracompact space $\mathbb{G}_n$. By Theorem III.1.7, we get a bundle map $\gamma^q \oplus \varepsilon^1 \to \gamma^{n+1}$, which gives us a map between the Thom spaces $T(\gamma^q \oplus \varepsilon^1) \to T(\gamma^{n+1})$. By the homeomorphism $T(\gamma^q \oplus \varepsilon^1) \cong \Sigma T(\gamma^q)$, we get the desired map $\sigma_q : \Sigma T(\gamma^q) \to T(\gamma^{q+1})$.

Definition III.2.7. A *prespectrum* is a sequence of based spaces $\{T_n\}_{n\geq 0}$ and based maps $\sigma_n : \Sigma T_n \to T_{n+1}$.

It is clear that the spaces $T(\gamma^q)$ and the maps σ_q constitute a prespectrum, that we will call *Thom spectrum* and will denote by TO. By convention, $T(\gamma^0)$ is isomorphic to $\mathbb{S}^0$ (we have that $E(\gamma^0) \cong \{(\underline{0}, \underline{0})\} \in \mathbb{G}_0 \times \mathbb{R}^\infty$, so its one point compactification is a set of two isolated points, as $\mathbb{S}^0$).

We can define the homotopy group of a prespectrum as:

Definition III.2.8. The *n-homotopy group of a prespectrum* $T = \{T_q\}_q$ is the direct limit

$$\pi_n(T) = \operatorname{colim} \pi_{n+q}(T_q),$$

where the colimit is taken over the maps

$$\pi_{n+q}(T_q) \xrightarrow{\Sigma} \pi_{n+q+1}(\Sigma T_q) \xrightarrow{\sigma_{q*}} \pi_{n+q+1}(T_{q+1}).$$

Now that we have defined $\pi_n(TO)$, we want to give

$$\pi_*(TO) = (\pi_0(TO), \pi_1(TO), \pi_2(TO), \dots)$$

a ring structure.

We are going to need another kind of maps between Thom spaces of the universal bundle, in this case maps that will allow us to connect Thom spaces of different degrees from TO.

First of all, we need the following fact: for any ξ, η two vector bundles, we have the following homeomorphism

$$T(\xi \times \eta) \cong T(\xi) \wedge T(\eta). \tag{III.3}$$

Indeed, we have that if X and Y are two compact topological spaces, then $X \wedge Y$ is the one point compactification of $(X \setminus \{x_0\}) \times (Y \setminus \{y_0\})$, where $x_0 \in X$ and $y_0 \in Y$ are any points ([Bre93, page 199]). In our case $T(\xi), T(\eta)$ are compact (they are the one point compactification of $E(\xi), E(\eta)$), so $T(\xi) \wedge T(\eta)$ is the one point compactification of $(T(\xi) \setminus \{t_0(\xi)\}) \times (T(\eta) \setminus \{t_0(\eta)\}) \cong E(\xi) \times E(\eta)$. This is the total space of the product $\xi \times \eta$, and its one point compactification is the Thom space $T(\xi \times \eta)$, so we get our claim.

Thus we get an homeomorphism

$$T(\gamma^m \times \gamma^n) \cong T(\gamma^m) \wedge T(\gamma^n).$$

From the bundle map $\gamma^m \times \gamma^n \to \gamma^{m+n}$ we get a map between Thom spaces $T(\gamma^m \times \gamma^n) \to T(\gamma^{m+n})$. We can define maps $\phi_{m,n}$ as the composition

$$\phi_{m,n} : T(\gamma^m) \wedge T(\gamma^n) \cong T(\gamma^m \times \gamma^n) \to T(\gamma^{m+n}).$$

Before going on, recall some other properties of the smash product. For any pointed topological spaces X, Y, Z we have the following natural (base-point preserving) homeomorphisms:

- (unity) $X \cong \mathbb{S}^0 \wedge X$.
- (commutativity) $X \wedge Y \cong Y \wedge X$.
- (associativity) $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.
- $\Sigma X \cong \mathbb{S}^1 \wedge X$.
- $\mathbb{S}^m \wedge \mathbb{S}^n \cong \mathbb{S}^{m+n}$.

We can now define the concept of a ring prespectrum, see that TO is an example of this and show how this implies that $\pi_*(TO)$ is a commutative graded ring.

Definition III.2.9. Let $T = \{T_n, \sigma_n\}_n$ be a prespectrum. Then it is a *ring prespectrum* if there are maps $\eta : \mathbb{S}^0 \to T_0$ and $\phi_{m,n} : T_m \wedge T_n \to T_{m+n}$ such that the following three diagrams are commutative up to homotopy:

$$\begin{tikzcd} T_m \wedge \Sigma T_n \arrow[r, "\mathrm{id} \wedge \sigma_n"] \arrow[d, "="'] & T_m \wedge T_{n+1} \arrow[rd, "\phi_{m,n+1}"] & \\ \Sigma(T_m \wedge T_n) \arrow[r, "\Sigma\phi_{m,n}"] \arrow[d, "(-1)^n"'] & \Sigma T_{m+n} \arrow[r, "\sigma_{m+n}"'] & T_{m+n+1} \\ (\Sigma T_m) \wedge T_n \arrow[r, "\sigma_m \wedge \mathrm{id}"'] & T_{m+1} \wedge T_n \arrow[ru, "\phi_{m+1,n}"'] & \end{tikzcd} , \tag{III.4}$$

$$\begin{tikzcd} \mathbb{S}^0 \wedge T_n \arrow[r, "\eta \wedge \mathrm{id}"] \arrow[rd, "\cong"'] & T_0 \wedge T_n \arrow[d, "\phi_{0,n}"] \\ & T_n \end{tikzcd} \tag{III.5}$$

and

$$\begin{tikzcd} T_n \wedge T_0 \arrow[d, "\phi_{n,0}"'] & T_n \wedge \mathbb{S}^0 \arrow[l, "\mathrm{id} \wedge \eta"'] \arrow[ld, "\cong"] \\ T_n & \end{tikzcd} . \tag{III.6}$$

If the following diagram is commutative up to homotopy, we say that T is a *associative* ring prespectrum:

$$\begin{tikzcd} T_m \wedge T_n \wedge T_p \arrow[r, "\phi_{m,n} \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \phi_{n+p}"'] & T_{m+n} \wedge T_p \arrow[d, "\phi_{m+n,p}"] \\ T_m \wedge T_{n+p} \arrow[r, "\phi_{m,n+p}"'] & T_{m+n+p} \end{tikzcd} . \tag{III.7}$$

If the following diagram is commutative up to homotopy for equivalences $(-1)^{mn} : T_{m+n} \to T_{m+n}$ that suspend $(-1)^{mn}$ on ΣT_{m+n}, we say that T is a *commutative* ring prespectrum:

$$\begin{tikzcd} T_m \wedge T_n \arrow[r, "\psi"] \arrow[d, "\phi_{m,n}"'] & T_n \wedge T_m \arrow[d, "\phi_{n,m}"] \\ T_{m+n} \arrow[r, "(-1)^{mn}"'] & T_{m+n} \end{tikzcd} . \tag{III.8}$$

For more details on this definition, we recommend [May99, Chapter 25.2].

We have the following essential lemma that let us define the ring structure of $\pi_*(T)$ for a ring prespectrum T.

Lemma III.2.10. *Let T be an associative ring prespectrum. Then $\pi_*(T)$ has the structure of a graded ring. If T is also commutative than also $\pi_*(T)$ is graded commutative.*

Proof. For the addition operation we just need to prove that there exists a commutative sum inside $\pi_n(T)$. Let $[f],[g] \in \pi_n(T)$. For $q > 0$ sufficiently large, $[f],[g] \in \pi_{n+q}(T_q)$. Then $[f]+[g] \in \pi_n(T)$ is just the colimit of the element $[f*g] \in \pi_{n+q}(T_q)$. This is commutative since the π_{n+q} is commutative for $n+q > 1$.

For the graded multiplication, we will use the maps $\phi_{m,n}$. Let $[f] \in \pi_m(T)$ and $[g] \in \pi_n(T)$. Then, for $q, r > 0$ sufficiently large, $[f] \in \pi_{m+q}(T_q)$ and $[g] \in \pi_{n+r}(T_r)$. Thus $f : \mathbb{S}^{m+q} \to T_q$ and $g : \mathbb{S}^{n+r} \to T_r$. We can define the multiplication

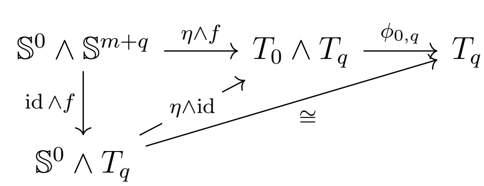
$$\begin{aligned} \cdot : \pi_m(T) \times \pi_n(T) &\to \pi_{m+n}(T) \\ ([f],[g]) &\mapsto [f]\cdot[g] \end{aligned}$$

as the colimit of the composition

$$[f]\cdot[g] : \mathbb{S}^{m+n+q+r} \cong \mathbb{S}^{m+q} \wedge \mathbb{S}^{n+r} \xrightarrow{f\wedge g} T_q \wedge T_r \xrightarrow{\phi_{q+r}} T_{q+r}.$$

We have that this map is well defined by the first diagram III.4, that is has a left and right unit by III.5 and III.6 (that is $\eta \in \pi_0(T)$, that is the limit of $\eta \wedge \mathrm{id}_{T_q}$), and it is associative by the commutativity of diagram III.7. The fact that these diagrams commute just up to homotopy does not matter since we are working with homotopy groups. For example, the right unity is proven using the diagram

$[f]\cdot[\eta]$:



which is commutative up to homotopy, and shows us that $[f]\cdot[\eta] \sim \mathrm{id}\wedge f \cong f$.

If the ring prespectrum is also commutative, by the last diagram III.8 we get that $\cdot$ is also a commutative operation. Indeed, the following diagram shows that $[g]\cdot[f] \sim (-1)^{qr}\circ([f]\cdot[g])$:

$$\begin{array}{lccccccc}
[f]\cdot[g]: & \mathbb{S}^{m+n+q+r} & \xrightarrow{\cong} & \mathbb{S}^{m+q}\wedge\mathbb{S}^{n+r} & \xrightarrow{f\wedge g} & T_q\wedge T_r & \xrightarrow{\phi_{q,r}} & T_{q+r} \\
& {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle (-1)^{qr}} \\
[g]\cdot[f]: & \mathbb{S}^{m+n+q+r} & \xrightarrow[\cong]{} & \mathbb{S}^{n+r}\wedge\mathbb{S}^{m+q} & \xrightarrow{g\wedge f} & T_r\wedge T_q & \xrightarrow{\phi_{r,q}} & T_{q+r}
\end{array}$$

□

Now that we have this result, to prove that $\pi_*(TO)$ has a commutative graded ring structure, we just need to show that TO is an associative commutative ring prespectrum. To do so, we need to show that the diagrams above are satisfied by TO, together with the maps $\phi_{m,n}$ that we have previously defined and some maps η and $(-1)^{mn}$. The latter ones, $(-1)^{mn} : T(\gamma^{m+n}) \to T(\gamma^{m+n})$ are the maps induced by the bundle maps $\gamma^{m+n} \to \gamma^{m+n}$, which cover the interchange isomorphisms $\mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$. The map η is just the isomorphism $T(\gamma^0) \cong \mathbb{S}^0$. It is easy to see that using these maps all the diagrams above commute up to homotopy so TO is a commutative associative ring prespectrum, and so $\pi_*(TO)$ is a commutative graded ring.

III.3 The structure of the unoriented cobordism ring

We finally have all the tools to study the relation between $\mathcal{N}_*$ and $\pi_*(TO)$. These two are isomorphic as $\mathbb{Z}/2\mathbb{Z}$-algebras.

In order to prove so, let's first prove the following theorem concerning just the groups $\mathcal{N}_n$ and $\pi_n(TO)$.

Theorem III.3.1 (Thom)**.** *For a sufficiently large q, we have that $\mathcal{N}_n$ is isomorphic to $\pi_{n+q}(T(\gamma^q))$, and therefore*

$$\mathcal{N}_n \cong \pi_n(TO).$$

The base point of the space $T(\gamma^q)$ is t_0, so whenever we write $\pi_{n+q}(T(\gamma^q))$ instead of $\pi_{n+q}(T(\gamma^q), t_0)$ we are implying that the base point is the point at infinity.

In the proof of this theorem we are going to use some results from differential topology. We are going to state what we need from differential topology before starting the proof of Theorem III.3.1.

First of all, we need to see two theorems about embeddings, one that we have already cited in Chapter I, I.2.11 and the other about manifolds with boundary.

Theorem III.3.2 (Strong embedding theorem)**.** *Every compact boundaryless smooth n-dimensional manifold M can be embedded in $\mathbb{R}^{2n}$.*

Theorem III.3.3 (Whitney embedding theorem with boundary)**.** *Let X be an n-dimensional manifold with boundary. Then there exists an embedding $\iota : X \to \mathbb{R}^{2n+1}$ such that $\iota(\partial X) \subset \partial\mathbb{H}^{2n+1} \cong \mathbb{R}^{2n}$ and $\iota(\mathring{X}) \subset \mathring{\mathbb{H}}^{2n+1} \cong \mathbb{R}^{2n+1}$.*

For a reference for this theorem, check [Hir76, Theorem 1.4.3].

Another useful theorem is the tubular neighborhood theorem. Let's first write what a tubular neighborhood actually is. We will give both the definition and the theorem only in the case of submanifolds of some $\mathbb{R}^m$, even though it can be extended to submanifolds of any manifold.

Definition III.3.4. A *tubular neighborhood* of a submanifold $M \subset \mathbb{R}^m$ consists of the vector bundle $\nu : E(\nu) \xrightarrow{\pi} M$ (defined in Example I.1.1) and an open embedding $\phi : E(\nu) \hookrightarrow \mathbb{R}^m$ extending the diffeomorphism of the zero section $Z = s_0(M)$ onto M (i.e. $\phi(x, 0) = \mathrm{id}(x) = x$ for every $x \in Z$).

It is possible we will refer also to $\phi(E(\nu)) \subset \mathbb{R}^m$ as *tubular neighborhood*. Note that this is an open neighborhood of M.

Theorem III.3.5 (Tubular neighborhood theorem)**.** *If M is a submanifold of $\mathbb{R}^m$ without boundary, then there exists a tubular neighborhood of M in $\mathbb{R}^m$.*

For a proof of this theorem, the reader might look at [Muk15, Theorem 7.1.5]. Note that if we take different embeddings of M in $\mathbb{R}^{n+m}$, for m large enough, the total spaces of the normal bundles will be isotopic (for a reference of this statement and the relative proof, check [Lan02]) and any two tubular neighborhoods will be isotopic (for a reference of this statement and the relative proof, check [Hir76, Chapter 4, Theorem 5.3]).

We also have a version of this theorem for manifolds with boundary.

Theorem III.3.6. *Let M, V be two manifolds with boundary, such that $M \subset V$ is a neat submanifold. Then every tubular neighborhood of ∂M in ∂V is the intersection with ∂V of a tubular neighborhood for M in V.*

For a proof of this theorem, check [Hir76, Chapter 4, Theorem 6.4].

Another classical result from differential topology is the Whitney approximation theorem for continuous maps.

Lemma III.3.7. *Let M, N be manifolds and let $f : M \to N$ be a continuous map. Then f is homotopic to smooth maps.*

A proof of this lemma can be found in [Hir76, Chapter 5, Lemma 1.5].

The last results we will use in this proof are coming from transversality theory.

The first important theorem is Sard's theorem. It can be really useful to compute dimensions.

Theorem III.3.8 (Sard)**.** *Let $f : M \to N$ be a smooth map between manifolds and let $C(f)$ be the set of all the critical points of f in M. Then, the set $f(C(f))$ has measure zero.*

A proof can be found in [Muk15, Theorem 2.3.5].

Sard's theorem is quite fundamental to differential topology because it is the base of transversality theory. We won't go deep in the study of it, but we will need the basic definition and a first important theorem.

Definition III.3.9. Let M be an n-dimensional manifold with (possibly empty) boundary ∂M and let N be an n-dimensional manifold without boundary. Let Z be a proper closed r-submanifold of N. If $f : M \to N$ is a smooth map, then we say that f is *transverse* to Z (denoted as $f \pitchfork Z$) if

- for every $x \in M$ such that $f(x) \in Z$

$$DN_y = DZ_y + d_x f(DM_x);$$

- if $\partial M \neq \emptyset$ for every $x \in \partial M$ such that $f(x) \in Z$

  $$DN_y = DZ_y + d_x \partial f(D\partial M_x),$$

  where we denoted by ∂f the restriction of f to ∂M.

Theorem III.3.10 (First basic transversality Theorem)**.** *With the same notation as above, if $f \pitchfork Z$, then $(f^{-1}(Z), \partial f^{-1}(Z))$ is a proper submanifold of $(M, \partial M)$. Moreover,*

$$\dim M - \dim f^{-1}(Z) = \dim N - \dim Z.$$

For a proof, we suggest to check [Ben21, Theorem 8.2]. We also have the following.

Theorem III.3.11 (Second basic transversality Theorem)**.** *With the same notation as above, the set of smooth maps transverse to a set Z is open and dense in the space of maps between M and N.*

For a reference to this statement, check [Ben21, Theorem 8.5].
We finally can start the proof of Thom's theorem.

Proof of Theorem III.3.1. We want to define an homomorphism $\alpha : \mathcal{N}_n \to \pi_{n+q}(T(\gamma^q))$ and an homomorphism $\beta : \pi_{n+q}(T(\gamma^q)) \to \mathcal{N}_n$ for any n, and show that β is invertible and α is the right inverse for β. From this, we get our thesis.

Let M be an n-dimensional smooth manifold (and so $[M] \in \mathcal{N}_n$). For q sufficiently large, we can embed it in $\mathbb{R}^{n+q}$ (by the Whitney Theorem I.2.11 $q = n$ is sufficient).

Let ν be the normal bundle of M in $\mathbb{R}^{n+q}$. We have that this is a q-plane bundle (since q is the codimension of M in $\mathbb{R}^{n+q}$). We have that M is diffeomorphic to the zero section of ν. As a result of Theorem III.3.5 the embedding of M in $\mathbb{R}^{n+q}$ extends to an embedding ϕ of $E(\nu)$ onto an open neighborhood $U = \phi(E(\nu))$ which we call the tubular neighborhood of M in $\mathbb{R}^{n+q}$.

This tubular neighborhood defines uniquely a map $t : \mathbb{S}^{n+q} \to T(\nu)$ in the following way. Consider $\mathbb{S}^{n+q}$ as the one point compactification of $\mathbb{R}^{n+q}$ and remember that since we are working with an embedding (and thus with an injective map) $U \cong E(\nu)$. Then the map t is defined as

$$t : \mathbb{S}^{n+q} \longrightarrow T(\nu),$$

where $t|_U \equiv \phi^{-1}$ and $t|_{\mathbb{S}^{n+q}\setminus U} \equiv t_0$. This construction is called the Thom–Pontrjagin construction. Note that sometimes we will denote this map t as t_M, where M is the manifold we are working with.

Any two tubular neighborhoods of a manifold are isotopic, as we recalled from [Hir76, Chapter 4, Theorem 5.3]. Thus, if we choose a different embedding, we still get the same homotopy class as t. The maps t and $\hat{t}$ resulting from these different tubular neighborhoods will be homotopic (via the isotopy between the tubular neighborhoods), so $[t] = [\hat{t}]$.

Furthermore, we know that since M is a compact manifold, and therefore it is paracompact. This implies, by Theorem III.1.7, that we have a bundle map $f : \nu \to \gamma^q$, since ν is a q-plane bundle over M. Since a bundle map induces a map on the Thom spaces, we thus get a map $T(f) : T(\nu) \to T(\gamma^q)$. Composing the maps t and $T(f)$, we get a map $T(f) \circ t : \mathbb{S}^{n+q} \to T(\gamma^q)$. The map α will be defined as $\alpha([M]) = [T(f) \circ t] \in \pi_{n+q}(T(\gamma^q))$. We still need to prove that this map is well defined and it is an homomorphism.

In order to prove that it is well defined, we have to show that cobordant manifolds induce homotopic maps $\mathbb{S}^{n+q} \to \pi_{n+q}(T(\gamma^q))$. Let $M \sqcup \tilde{M}$ be the boundary of an $(n+1)$-manifold X (so $[M] = [\tilde{M}]$). For a large enough q, we can embed X in $\mathbb{R}^{n+q} \times [0, 1]$, in a way such that M and $\tilde{M}$ will be embedded in $\mathbb{R}^{n+q} \times \{0, 1\}$. The normal bundle of X $\nu(X)$ in $\mathbb{R}^{n+q} \times [0, 1]$ is a q-plane bundle (because the codimension of X in $\mathbb{R}^{n+q} \times [0, 1]$ is q). Thus, by Theorem III.1.7, we have a bundle map $f : \nu(X) \to \gamma^q$. Note that $\nu(X)|_M = \nu(M)$ and $\nu(X)|_{\tilde{M}} = \nu(\tilde{M})$ and thus $A_{\nu(M)}, A_{\nu(\tilde{M})} \subset A_{\nu(X)}$. This implies that $T(\nu(M)), T(\nu(\tilde{M})) \subset T(\nu(X))$. Therefore, $f|_{\nu(M)} : \nu(M) \to \gamma^q$, $f|_{\tilde{M}} : \nu(\tilde{M}) \to \gamma^q$ and $T(f|_{\nu(M)}) = T(f)|_{T(\nu(M))} : T(\nu(M)) \to T(\gamma^q)$, $T(f|_{\tilde{M}}) =$

$T(f)|_{T(\nu(\tilde{M}))} : T(\nu(\tilde{M})) \to T(\gamma^q)$. Lastly, if U_X is the tubular neighborhood of X, observe that $U_X \cap \mathbb{R}^{n+q} \times \{0\}$ gives a tubular neighborhood of M and similarly $U_X \cap \mathbb{R}^{n+q} \times \{1\}$ gives a tubular neighborhood of $\tilde{M}$. Since the one point compactification of $\mathbb{R}^{n+q} \times [0,1]$ is $\frac{\mathbb{S}^{n+q}\times[0,1]}{s_0\times[0,1]}$, we can see $\mathbb{R}^{n+q} \times [0,1]$ (and U_X) as a subset of this, and define a relative Thom-Pontrjagin construction map $t_X : \frac{\mathbb{S}^{n+q}\times[0,1]}{s_0\times[0,1]} \to T(\nu(X))$ as we did before. Since the one point compactification of the two copies of $\mathbb{R}^{n+q}$ will be $\mathbb{S}^{n+q} \subset \frac{\mathbb{S}^{n+q}\times[0,1]}{s_0\times[0,1]}$ the restriction of $t_X : \frac{\mathbb{S}^{n+q}\times[0,1]}{s_0\times[0,1]} \to T(\nu(X))$ to these $\mathbb{S}^{n+q}$ gives two maps $t_M, t_{\tilde{M}}$. Therefore, $\alpha([M]) = [T(f)|_{T(\nu(M))} \circ t_M] \in \pi_{n+q}(T(\gamma^q))$ and $\alpha([\tilde{M}]) = [T(f)|_{T(\nu(\tilde{M}))} \circ t_{\tilde{M}}] \in \pi_{n+q}(T(\gamma^q))$. These two elements are the same, since $T(f)|_{T(\nu(M))} \circ t_M$ and $T(f)|_{T(\nu(\tilde{M}))} \circ t_{\tilde{M}}$ are homotopic, via t_X.

We can now prove that α is an homomorphism. Let M, N be two n-dimensional manifolds. We want to show that $[M] \sqcup [N] \mapsto \alpha([M] \sqcup [N]) = \alpha([M]) * \alpha([N])$. To show this, observe that we can embed M and N and their tubular neighborhoods disjointly in $\mathbb{R}^{n+q}$ for q large enough. In particular, $U_M \cap U_N = \emptyset$, so $t_M(U_N) = t_0(M)$ and $t_N(U_M) = t_0(N)$. Via the classifying maps f_M and f_N, we will get that both these points at infinity will be mapped in the point at infinity of $T(\gamma^q)$. This, which is the base point of $\pi_{n+q}(T(\gamma^q))$, is therefore the only point of intersection of the paths $\alpha([M])$ and $\alpha([N])$. We can repeat the same argument for $[M \sqcup N]$: since they are disjoint, the normal bundle of this union will be the union of the normal bundle, and similarly the classifying map and the map $t_{M\sqcup N}$ will be defined componentwise. Thus, $\alpha([M] \sqcup [N]) = \alpha([M]) * \alpha([N])$.

Now we can finally define the map β. Let $[g] \in \pi_{n+q}(T(\gamma^q))$. Since $\mathbb{S}^{n+q}$ is compact and g is continuous, $g(\mathbb{S}^{n+q}) \subset T(\gamma^q)$ will be a compact subset. Since compact subsets in CW-complexes must be contained in finite dimensional subcomplexes, and since by Proposition III.2.4 the Thom space $T(\gamma^q)$ is a CW-complex, there must exists an $r \geq 0$ such that $g(\mathbb{S}^{n+q}) \subset T(\gamma^q_r)$.

We can deform the map g to an homotopic map $\tilde{g}$ by Lemma III.3.7 and Theorem III.3.11 (so $[g] = [\tilde{g}]$) which is smooth on the restriction of g to the set

$$g^{-1}(T(\gamma_r^q) \setminus \{\infty\}) \cong g^{-1}(E(\gamma_r^q)) \subset \mathbb{S}^{n+q}$$

and transverse to the zero section of γ_r^q $Z \cong B(\gamma_r^q) = \mathbb{G}_q(\mathbb{R}^{r+q})$. We have that the set $\tilde{g}^{-1}(Z) = \tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is smooth (since it is the inverse image of a smooth manifold over a smooth map) and it is compact (since $\tilde{g}$ is a continuous map and $\mathbb{G}_q(\mathbb{R}^{r+q})$ is a closed manifold in $T(\gamma_r^q)$). Moreover, by Theorem III.3.10, it is a proper submanifold of $\mathbb{S}^{n+q}$ and it has no boundary since $\partial\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) \subset \partial\mathbb{S}^{n+q} = \emptyset$. Thanks the same theorem, we also have that

$$\dim \tilde{g}^{-1}(E(\gamma_r^q)) - \dim \tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) = \dim E(\gamma_r^q) - \dim \mathbb{G}_q(\mathbb{R}^{r+q})$$

and thus

$$\dim \tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) = \dim \tilde{g}^{-1}(E(\gamma_r^q)) - \dim E(\gamma_r^q) + \dim \mathbb{G}_q(\mathbb{R}^{r+q}).$$

We can compute these dimensions: $\dim \mathbb{G}_q(\mathbb{R}^{r+q}) = (r+q)q$ by Lemma III.1.3. The dimension of $E(\gamma_r^q) = \{(X, v) \in \mathbb{G}_q(\mathbb{R}^{r+q}) \times \mathbb{R}^{r+q} | v \in X)\}$ is the dimension of $\mathbb{G}_q(\mathbb{R}^{r+q}) \times \mathbb{R}^q$, which is $(r+q)q + q$. Lastly, by Sard's theorem III.3.8, since $C(f) = \{\infty\}$ has dimension zero, $\dim \tilde{g}^{-1}(E(\gamma_r^q)) = \dim g^{-1}(T(\gamma_r^q)) = \dim \mathbb{S}^{n+q} = n+q$. Therefore

$$\dim \tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) = n + q - (r+q)q - q + (r+q)q = n.$$

In conclusion, $\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is a smooth closed n-dimensional manifold, so $[\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))] \in \mathcal{N}_n$. We define $\beta([g]) = [\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))]$. Note that $\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is embedded in $\mathbb{R}^{n+q} = \mathbb{S}^{n+q} \setminus \infty$ since the point at infinity of $\mathbb{S}^{n+q}$ is mapped to the point at infinity of $T(\gamma^q)$ and $\mathbb{G}_q(\mathbb{R}^{r+q})$ doesn't contain it, so its preimage will be completely contained in $\mathbb{R}^{n+q}$.

Now that we have given a definition for the map β, we need to prove that it is well-defined. Let g_1 and g_2 be two homotopic maps. We want to show that $\beta([g_1]) \sim \beta[g_2] \in \mathcal{N}_n$. Let H be the homotopy between them. Then, we can approximate H with a homotopic map $\tilde{H}$ by Lemma III.3.7 and

Theorem III.3.11, a smooth map on $H^{-1}(T(\gamma_r^q) \setminus \{\infty\})$ which is transverse to $Z = \mathbb{G}_q(\mathbb{R}^{r+q})$. Then, by the same argument as before $\tilde{H}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is a smooth compact $(n+1)$-dimensional manifold. We can take $\tilde{g}_1 = \tilde{H}(\cdot, 0)$ and $\tilde{g}_2 = \tilde{H}(\cdot, 1)$ (they are homotopic to g_1, g_2, smooth and transverse to Z), so $\beta([g_1]) = \tilde{g}_1^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ and $\beta([g_2]) = \tilde{g}_2^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$. By Theorem III.3.10, $\partial\tilde{H}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is a proper subset of $\partial(\mathbb{S}^{n+q} \times [0,1]) = \mathbb{S}^{n+q} \times \{0,1\}$ (and since the preimage of H already has a boundary, also the preimage of $\tilde{H}$ must have a boundary), and thus $\partial\tilde{H}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) = \tilde{g}_1^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) \sqcup \tilde{g}_2^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) = \beta([g_1]) \sqcup \beta([g_2])$, so $\beta([g_1]) \sim \beta([g_2])$ in $\mathcal{N}_n$. Hence, β is well defined.

Our next goal is to prove that $\beta : \pi_{n+q}(T(\gamma^q)) \to \mathcal{N}_n$ is an homomorphism. Let $f, g \in \pi_{n+q}(T(\gamma^q))$. We want to show that $\beta([f * g]) = \beta([f]) \sqcup \beta([g]) = [\tilde{f}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))] \sqcup [\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))]$. We have that applying $f * g$ to $\mathbb{S}^{n+q}$ is the same thing as applying f to the closed northern hemisphere and g to the closed southern hemisphere, and then crushing the equator. We can perturb the sphere in order to send the equator to $t_0(\gamma^q)$. This way the only common point in the images of f and g would be the point at infinity, and thus $[\tilde{f}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))]$ and $[\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))]$ are be disjoint (they don't contain the preimage of the point at infinity). Hence, $\beta([f * g]) = [\tilde{f}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))] \sqcup [\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))]$ as we wanted.

To complete the proof, we want to show that β is the inverse of α. Since we are working within the category of abelian groups, it is enough to show that β is an isomorphism (and thus that it is surjective and injective) and that α is its right inverse (since it is invertible, the left and right inverses will exist and coincide).

Let's start by showing that the map is surjective. Let M be a manifold embedded in $\mathbb{R}^{n+q}$. We can choose the classifying map f for $\nu(M)$ to make $T(f) \circ t$ smooth and transverse to the zero section of γ^q Z. Moreover we have that $(T(f) \circ t)^{-1}(Z) = M$ since $T(f)^{-1}(Z)$ is the zero section of $\nu(M)$ $s_0(M)$ and $t^{-1}(s_0(M))$ is M. Thus, $\beta([T(f) \circ t]) = [M]$. This proves the surjectivity.

From this argument we also prove the claim that α is the right inverse of

β: for any $[M] \in \mathcal{N}_n$

$$[M] = [(T(f) \circ t)^{-1}(Z)] = \beta([T(f) \circ t]) = \beta(\alpha([M])),$$

so we get

$$\mathrm{id}_{\mathcal{N}_n} = \beta \circ \alpha.$$

To conclude the whole proof, we just need to show that β is injective. Let $g : \mathbb{S}^{n+q} \to T(\gamma_r^q) \subset \ker(\beta)$, so equivalently, $\beta([g]) = [\emptyset] \in \mathcal{N}_n$ (i.e. $\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q}))$ is a boundary). We want to show that $[g] = [\mathrm{const}_{t_0}]$, so that g is null-homotopic. We have that $\tilde{g}^{-1}(E(\gamma_r^q))$ is smooth and transverse to $\mathbb{G}_q(\mathbb{R}^{r+q})$ by construction. By assumption, $\tilde{g}^{-1}(\mathbb{G}_q(\mathbb{R}^{r+q})) := M$ is a boundary of some X.

The inclusion $M \hookrightarrow \mathbb{S}^{n+q}$ extends to an inclusion $X \hookrightarrow D^{n+q+1}$ for q large enough (by Theorem III.3.3 $q = n$ is enough). We can assume that $U = \tilde{g}^{-1}(T(\gamma_r^q) \setminus \{\infty\}) \subset \mathbb{S}^{n+q}$ is a tubular neighborhood of M (so $U \cong E(\nu(M))$) and that $\tilde{g} : U \to E(\gamma_r^q)$ is a bundle map. By the relative tubular neighborhood Theorem III.3.6, the tubular neighborhood U of M can be extended to a tubular neighborhood W in D^{n+q+1} and $\tilde{g}$ extends to a bundle map $\tilde{h} : W \cong E(\nu(X)) \to E(\gamma_r^q)$. Compactifying $\tilde{h}$ gives us a map

$$\begin{aligned} h : D^{n+q+1} &\to T(\gamma_r^q) \\ W &\xrightarrow{\sim} E(\gamma_r^q). \\ D^{n+q+1} \setminus W &\mapsto t_0(\gamma_r^q) \end{aligned}$$

Therefore, $\tilde{g}$ can be extended to the map $h : D^{n+q+1} \to T(\gamma^q)$ and thus g (which is homotopic to $\tilde{g}$) is null-homotopic.

This concludes our proof. □

To complete Thom's theorem, we actually have the following additional result.

Theorem III.3.12 (Thom)**.** *We have that $\mathcal{N}_*$ and $\pi_*(TO)$ are isomorphic as $\mathbb{Z}/2\mathbb{Z}$-algebras.*

Proof. We want to show that the isomorphism α respects the graded ring structure. In particular, we have to show that for any m, n, the following diagram commutes

$$\begin{array}{ccc} \mathcal{N}_m \times \mathcal{N}_n & \xrightarrow{\alpha\times\alpha} & \pi_m(TO) \times \pi_n(TO) \\ \times\big\downarrow & & \big\downarrow\cdot \\ \mathcal{N}_{m+n} & \xrightarrow[\alpha]{} & \pi_{m+n}(TO) \end{array}$$

Let M be an m-dimensional manifold embedded in $\mathbb{R}^{m+q}$ with tubular neighborhood $U \cong E(\nu_M)$ and N an n-dimensional manifold embedded in $\mathbb{R}^{n+r}$ with tubular neighborhood $V \cong E(\nu_N)$. We have that $M \times N$ can be embedded in $\mathbb{R}^{m+n+q+r}$ with tubular neighborhood $U \times V \cong E(\nu_M) \times E(\nu_N) = E(\nu_M \times \nu_N) \cong E(\nu_{M\times N})$. Let the Pontrjagin–Thom construction for M be $t_M : \mathbb{S}^{m+q} \to T(\nu_M)$ and $t_N : \mathbb{S}^{n+r} \to T(\nu_N)$ be the Pontrjagin–Thom construction for N. Moreover, let $T(f_M) : T(\nu_M) \to T(\gamma^q)$ and $T(f_N) : T(\nu_N) \to T(\gamma^r)$ be induced by the classifying maps $\nu_M \to \gamma^q$ and $\nu_N \to \gamma^r$. Then, $\alpha([M]) = [T(f_M) \circ t_M]$ and $\alpha([N]) = [T(f_N) \circ t_N]$. Thus,

$$\cdot((\alpha \times \alpha)([M], [N])) = \alpha([M]]) \cdot \alpha([N]) = [T(f_M) \circ t_M] \cdot [T(f_N) \circ t_N]$$

is the composition

$$\mathbb{S}^{m+n+q+r} \cong \mathbb{S}^{m+q} \wedge \mathbb{S}^{n+r} \xrightarrow{(T(f_M)\circ t_M)\wedge(T(f_N)\circ t_N)} T(\gamma^q) \wedge T(\gamma^r) \xrightarrow{\phi_{q,r}} T(\gamma^{q+r})$$

and since $(T(f_M)\circ t_M) \wedge (T(f_N)\circ t_N) = (T(f_M)\wedge T(f_N)) \circ (t_M \wedge t_N)$, $\alpha([M]]) \cdot \alpha([N])$ is the homotopy class of the composition

$$\mathbb{S}^{m+q} \wedge \mathbb{S}^{n+r} \xrightarrow{t_M\wedge t_N} T(\nu_M) \wedge T(\nu_N) \xrightarrow{T(f_M)\wedge T(f_N)} T(\gamma^q) \wedge T(\gamma^r) \xrightarrow{\phi_{q,r}} T(\gamma^{q+r}).$$

We need to show that this is homotopy equivalent to $\alpha([M] \times [N]) = \alpha([M \times N])$. First of all, let's study $t_M \wedge t_N$. Let $[(x, y)] \in \mathbb{S}^{m+q} \wedge \mathbb{S}^{n+r}$. Then if $x \notin U$, $t_M(x) = t_0(\nu_M)$, so $[(t_M(x), t_N(y))] = [(t_0(\nu_M), t_N(y))] = [(t_0(\nu_M), t_0(\nu_N))]$ which is the point at infinity of $T(\nu_M) \wedge T(\nu_N) \cong T(\nu_M \times \nu_N) \cong T(\nu_{M\times N})$. Similarly, if $y \notin V$, $[(t_M(x), t_N(y))] = t_0(\nu_{M\times N})$. For $(x, y) \in U \times V$, $t_M(x) \neq$

$t_0(\nu_M)$ and $t_N(y) \neq t_0(\nu_N)$ so $t_M \wedge t_N$ remains an isomorphism here. In particular, $(t_M \wedge t_N)|_{U\times V} : U \times V \xrightarrow{\cong} E(\nu_M) \times E(\nu_N) \cong E(\nu_{M\times N})$, so up to isomorphisms $t_M \wedge t_N$ is the Pontrjagin–Thom construction for $M \times N$.

For the classifying map, note that we have a bundle map $f_M \times f_N : \nu_M \times \nu_N \to \gamma^q \times \gamma^r$ and a classifying map $g : \gamma^q \times \gamma^r \to \gamma^{q+r}$. Thus, the classifying map that we will use for $M \times N$ will be $T(g \circ (f_M \times t_N)) = T(g) \circ T(f_M \times t_N)$. This way, the following diagram will commute up to homotopy

$$\begin{array}{ccc} T(\nu_M) \wedge T(\nu_N) & \xrightarrow{T(f_M)\wedge T(f_N)} & T(\gamma^q) \wedge T(\gamma^r) \\ \cong\downarrow & & \downarrow\cong \\ T(\nu_M \times \nu_N) & \xrightarrow[T(f_M\times f_N)]{} & T(\gamma^q \times \gamma^r) \\ =\downarrow & & \downarrow T(g) \\ T(\nu_{M\times N}) & \xrightarrow[T(g)\circ T(f_M\times t_N)]{} & T(\gamma^{q+r}) \end{array} .$$

Note that the vertical composition on the right is the map $\phi_{q,r}$ by definition.

In conclusion, we have that

$$\alpha([M \times N]) = [T(g) \circ T(f_M \times t_N) \circ t_M \wedge t_N] = [\phi_{q,r} \circ T(\gamma^q) \wedge T(\gamma^r) \circ t_M \wedge t_N]$$

as we wanted. □

Proving these result allows us to shift the problem to a problem in homotopy theory, that in general might be just as difficult, but in our particular case lets us complete our study, as we will show in the next sections.

III.4 The structure of the homotopy ring of the Thom spectrum

We saw how the cobordism ring $\mathcal{N}_*$ is isomorphic as a $\mathbb{Z}/2\mathbb{Z}$-algebra to the homotopy ring of the Thom spectrum $\pi_*(TO)$. This result is quite useful since we can study in depth the structure of $\pi_*(TO)$. In particular, we will define in a next section the injective Hurewicz map $h : \pi_*(TO) \to H_*(TO; \mathbb{Z}/2\mathbb{Z})$ and we will use it to study the image of $\pi_*(TO)$ in $H_*(TO; \mathbb{Z}/2\mathbb{Z})$. Thus, in this section our goal will be to investigate the structure of $H_*(TO;\mathbb{Z}/2\mathbb{Z})$. We will do that using the Thom isomorphism between the Grassmannian and the Thom space of the universal bundle, and its passage to colimits.

In order to do so, let's start by defining the homology and cohomology group of a prespectrum. As we did for the homotopy group of a prespectrum $T = \{T_q\}_q$, the homology of T with coefficients in $\mathbb{Z}/2\mathbb{Z}$

$$H_n(T;\mathbb{Z}/2\mathbb{Z}) = \operatorname{colim} \tilde{H}_{n+q}(T_q;\mathbb{Z}/2\mathbb{Z}),$$

where the colimit is taken over the maps

$$\tilde{H}_{n+q}(T_q;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\Sigma_*} \tilde{H}_{n+q+1}(\Sigma T_q;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sigma_{q*}} \tilde{H}_{n+q+1}(T_{q+1};\mathbb{Z}/2\mathbb{Z})$$

and the cohomology of T with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is

$$H^n(T;\mathbb{Z}/2\mathbb{Z}) = \lim \tilde{H}^{n+q}(T_q;\mathbb{Z}/2\mathbb{Z}),$$

where the limit is taken over the maps

$$\tilde{H}^{n+q+1}(T_{q+1};\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sigma_q{}^*} \tilde{H}^{n+q+1}(\Sigma T_q;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\Sigma^{-1}} \tilde{H}^{n+q}(T_q;\mathbb{Z}/2\mathbb{Z}).$$

Note that $H_n(T;\mathbb{Z}/2\mathbb{Z})$ and $H^n(T;\mathbb{Z}/2\mathbb{Z})$ are finite dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces, one the dual of the other (because for each q $H_{n+q}(T_q;\mathbb{Z}/2\mathbb{Z})$ and $H^{n+q}(T_q;\mathbb{Z}/2\mathbb{Z})$ are dual finite dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces).

We have that the Thom isomorphisms

$$\Phi^q : H^n(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z}) \to \tilde{H}^{n+q}(T(\gamma^q);\mathbb{Z}/2\mathbb{Z})$$

pass to limits and thus we get a stable Thom isomorphism

$$\Phi : H^n(BO;\mathbb{Z}/2\mathbb{Z}) \to H^n(TO;\mathbb{Z}/2\mathbb{Z})$$

by [May99, Section 3, Chapter 25]. Taking the dual map of Φ^q we get the maps

$$\Phi_q : \tilde{H}_{n+q}(T(\gamma^q);\mathbb{Z}/2\mathbb{Z}) \to H_n(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z})$$

that similarly pass to colimits in order to get the stable Thom isomorphism

$$\Phi : H_n(TO;\mathbb{Z}/2\mathbb{Z}) \to H_n(BO;\mathbb{Z}/2\mathbb{Z}).$$

The $\mathbb{Z}/2\mathbb{Z}$-algebra structure is preserved by the Thom isomorphism, as shown in [May99, Section 3, Chapter 25] so we get that

Proposition III.4.1. *The Thom isomorphism*

$$\Phi : H_*(TO; \mathbb{Z}/2\mathbb{Z}) \to H_*(BO; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$*-algebras.*

Where the structure of graded commutative $\mathbb{Z}/2\mathbb{Z}$-algebra for the homology ring $H_*(BO; \mathbb{Z}/2\mathbb{Z})$ is given by the H-structure of BO, whereas for $H_*(TO; \mathbb{Z}/2\mathbb{Z})$ is given by the passage to colimits of the maps ϕ_{m,n_*}.

A simple corollary coming from this proposition and Theorem III.1.9 ($H_*(BO; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[b_j | j \geq 1]$) is the following.

Corollary III.4.2. *Let* $a_j \in H_j(TO; \mathbb{Z}/2\mathbb{Z})$ *be the image of* $\Phi^{-1}(b_j)$*. Then* $H_*(TO; \mathbb{Z}/2\mathbb{Z})$ *is the polynomial algebra* $\mathbb{Z}/2\mathbb{Z}[a_j | j \geq 1]$.

Since the maps Φ_n are stable, and the diagrams

$$\begin{array}{ccc} H_n(\mathbb{G}_1; \mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\Phi_n} & H_{n+1}(T(\gamma^1); \mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\scriptstyle i_*} & & \downarrow \\ H_n(BO; \mathbb{Z}/2\mathbb{Z}) & \xleftarrow[\Phi_n]{} & H_n(TO; \mathbb{Z}/2\mathbb{Z}) \end{array},$$

where the second vertical map is the colimit map, commute for every $n \geq 0$.. Thus, the elements a_i come from $H_*(T(\gamma^1))$.

Moreover, recall from Lemma III.2.5 that we have an homotopy equivalence $j : \mathbb{P}^\infty \to T(\gamma^1)$. Thus $H_i(T(\gamma^1); \mathbb{Z}/2\mathbb{Z}) \cong H_i(\mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \{0, x_i\}$ for every i. Since j_* is an isomorphism, it will take generators to generators. From these two results, we get the following.

Corollary III.4.3. *For* $i \geq 0$, $j_*(x_{i+1})$ *maps to* a_i *in* $H_*(TO; \mathbb{Z}/2\mathbb{Z})$*, where* $a_0 = 1$.

In particular, $a_0 = 1$ because a_0 is the image under the colimit map of the non-zero element of $H_1(T(\gamma^1); \mathbb{Z}/2\mathbb{Z})$.

III.5 The Steenrod algebra

In order to understand completely the relation between the homotopy ring $\pi_*(TO)$ and the homology ring $H_*(TO; \mathbb{Z}/2\mathbb{Z})$ we need to study the Steenrod operations and the Steenrod algebra. Indeed, the algebras $H_*(TO;\mathbb{Z}/2\mathbb{Z})$ and $H^*(TO;\mathbb{Z}/2\mathbb{Z})$ are built from $\pi_*(TO)$ and Steenrod operations.

Let's recall the definition of these two objects. The Steenrod squares are a family of natural homomorphisms $\mathrm{Sq}^i : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X;\mathbb{Z}/2\mathbb{Z})$ where $i \geq 0$ and X is a topological space. The Sq^i satisfies:

- $\mathrm{Sq}^0 = \mathrm{id}$;
- For $x \in H^n(X;\mathbb{Z}/2\mathbb{Z})$, $\mathrm{Sq}^n(x) = x \smile x$;
- For $x \in H^n(X;\mathbb{Z}/2\mathbb{Z})$ and $i > n$, $\mathrm{Sq}^i(x) = 0$;
- (Cartan formula) $\mathrm{Sq}^k(x \smile y) = \sum_{i+j=k} \mathrm{Sq}^i(x) \smile \mathrm{Sq}^j(y)$.

Since these cohomology operations are natural and stable, we get that they pass to limits to define natural operations

$$\mathrm{Sq}^i : H^n(TO;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(TO;\mathbb{Z}/2\mathbb{Z}).$$

Note that this passage to limits does not preserve all the axioms above.

Consider the $\mathbb{Z}/2\mathbb{Z}$-algebra freely generated by all the Steenrod squares $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, \ldots$. The *Steenrod algebra* $\mathcal{A}$ is the quotient of this algebra by the Adem relations: for $i, j > 0$ and $i < 2j$, the Steenrod squares satisfy

$$\mathrm{Sq}^i \circ \mathrm{Sq}^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \circ \mathrm{Sq}^k .$$

The Steenrod algebra has as a basis the operations $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \circ \cdots \circ \mathrm{Sq}^{i_k}$ with $I = (i_1, \ldots, i_k)$ such that $i_j \geq 2i_{j+1}$ for $1 \leq j < k$.

Moreover, other than having the structure of an algebra, thanks to the composition

$$\begin{aligned} \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \\ \mathrm{Sq}^i \otimes \mathrm{Sq}^j &\mapsto \mathrm{Sq}^i \circ \mathrm{Sq}^j, \end{aligned}$$

and thanks to the Cartan formula we also get a coproduct

$$\begin{aligned} \psi : \mathcal{A} &\to \mathcal{A} \otimes \mathcal{A} \\ \mathrm{Sq}^k &\mapsto \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j, \end{aligned}$$

and thus $\mathcal{A}$ has a dual vector space $\mathcal{A}_*$ (with canonical basis the dual one to the one we have described). This $\mathcal{A}_*$ will be a commutative

We can explicitly see the structure of this algebra using the following theorem.

Theorem III.5.1. *For $r \geq 1$, let $I_r = (2^{r-1}, 2^{r-2}, \dots, 2, 1)$. Define ξ_r to be the basis element of $\mathcal{A}_*$ dual to Sq^{I_r}. Then $\mathcal{A}_*$ is the polynomial algebra $\mathbb{Z}/2\mathbb{Z}[\xi_r | r \geq 1]$.*

The reason why these elements are so important will be highlighted soon.

In the meantime, we have the following lemma, whose proof is pretty straightforward.

Lemma III.5.2. *For any space X and any prespectra T, $H^*(X; \mathbb{Z}/2\mathbb{Z})$ and $H^*(T; \mathbb{Z}/2\mathbb{Z})$ have a natural structure of $\mathcal{A}$-modules.*

Consider the A-module structure maps

$$\mathcal{A} \otimes H^*(X; \mathbb{Z}/2\mathbb{Z}) \to H^*(X; \mathbb{Z}/2\mathbb{Z}) \text{ and } \mathcal{A} \otimes H^*(T; \mathbb{Z}/2\mathbb{Z}) \to H^*(T; \mathbb{Z}/2\mathbb{Z})$$

for spaces X and prespectra T. These maps dualize to give $\mathcal{A}_*$-comodule structure maps

$$\gamma : H_*(X; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{A}_* \otimes H_*(X; \mathbb{Z}/2\mathbb{Z})$$

and

$$\gamma : H_*(T; \mathbb{Z}/2\mathbb{Z}) \to \mathcal{A}_* \otimes H_*(T; \mathbb{Z}/2\mathbb{Z}).$$

If T is an associative ring prespectrum, this latter coaction γ is an homomorphism of algebras.

In order to get back to the study of $H_*(TO; \mathbb{Z}/2\mathbb{Z}) \cong H_*(BO; \mathbb{Z}/2\mathbb{Z})$, we will use the homotopy equivalence $i : \mathbb{P}^\infty \hookrightarrow TO$ from Lemma III.2.5, so let's dissect the coaction γ for $X = \mathbb{P}^\infty$. To do so, we first need to see the A-module structure of $H^*(\mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]$.

Lemma III.5.3. *In $H^*(\mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z})$, we have that $\mathrm{Sq}^{I_r}(\alpha) = \alpha^{2^r}$ for $r \geq 1$ and $\mathrm{Sq}^I(\alpha) = 0$ for all other basis elements of $\mathcal{A}$.*

Dualizing this lemma gives us the following result, that uses that also $\mathbb{P}^\infty$ is an H-space, and thus by [Hat17, Section 3.C] its homology groups have a bilinear product.

Lemma III.5.4. *If we rewrite the coaction* $\gamma : H_*(\mathbb{P}^\infty;\mathbb{Z}/2\mathbb{Z}) \to \mathcal{A}_* \otimes H_*(\mathbb{P}^\infty;\mathbb{Z}/2\mathbb{Z})$ *as* $\gamma(x_i) = \sum_j a_{i,j} \otimes x_j$*, then*

$$a_{i,1} = \begin{cases} \xi_r & \text{if } i = 2^r \text{ for some } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that $a_{i,i} = 1$, dualizing $\mathrm{Sq}^0(\alpha^i) = \alpha^i$.

We can finally describe $H_*(TO;\mathbb{Z}/2\mathbb{Z})$ in terms of $\mathcal{A}_*$ with the following theorem.

Theorem III.5.5. *Let* N_* *be the algebra defined abstractly by*

$$N_* = \mathbb{Z}/2\mathbb{Z}[u_i | i > 1 \text{ and } i \neq 2^r - 1]$$

where u_i *has degree* i*. Define a homomorphism of algebras* f *by*

$$\begin{aligned} f : H_*(TO;\mathbb{Z}/2\mathbb{Z}) &\longrightarrow N_* \\ a_i &\longmapsto \begin{cases} u_i & \text{if } i \text{ is not of the form } 2^r - 1, \\ 0 & \text{if } i = 2^r - 1. \end{cases} \end{aligned}$$

Then the composite

$$g : H_*(TO;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\gamma} \mathcal{A}_* \otimes H_*(TO;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathrm{id}\otimes f} \mathcal{A}_* \otimes N_*$$

is an isomorphism of both $\mathcal{A}$*-modules and* $\mathbb{Z}/2\mathbb{Z}$*-algebras.*

Proof. Since both γ and f are both $\mathcal{A}$-comodules and $\mathbb{Z}/2\mathbb{Z}$-algebras morphisms, also g is a map of both $\mathcal{A}$-comodules and $\mathbb{Z}/2\mathbb{Z}$-algebras morphisms. We want to prove that this is also an isomorphism. Since the domain and the target have the same cardinality (they are both $\mathbb{Z}/2\mathbb{Z}$-algebras with one generator per degree, a_i for $H_*(TO;\mathbb{Z}/2\mathbb{Z})$ and

$$\begin{cases} (1 \otimes u_i) & \text{for } i \neq 2^r - 1 \\ (\xi_r \otimes 1) & \text{for } i = 2^r - 1 \end{cases}$$

for $\mathcal{A}_* \otimes N_*$), it is enough to show that generators are mapped to generators. Thanks to the last lemma, we know the image of γ for the infinite projective space, and combining this with the fact that $a_i = j_*(x_{i+1})$ by Corollary III.4.3, we get that

$$\gamma(a_i) = \gamma(j_*(x_{i+1})) = j_*(\gamma(x_{i+1})) =$$
$$= \begin{cases} \xi_r \otimes j_*(x_1) + 1 \otimes j_*(x_{i+1}) + \sum_{j\neq 1,i+1} a_{i,j} \otimes j_*(x_j) & \text{for } i = 2^r - 1 \\ 1 \otimes j_*(x_{i+1}) + \sum_{j\neq 1,i+1} a_{i,j} \otimes j_*(x_j) & \text{for } i \neq 2^r - 1 \end{cases}$$
$$= \begin{cases} \xi_r \otimes 1 + 1 \otimes a_i + \sum_{j\neq 1,i+1} a_{i,j} \otimes a_{j-1} & \text{for } i = 2^r - 1 \\ 1 \otimes a_i + \sum_{j\neq 1,i+1} a_{i,j} \otimes a_{j-1} & \text{for } i \neq 2^r - 1 \end{cases}.$$

We can ignore the terms that are decomposable in $\mathcal{A}_* \otimes H_*(TO; \mathbb{Z}/2\mathbb{Z})$, and we find

$$\gamma(a_i) = \begin{cases} \xi_r \otimes 1 + 1 \otimes a_{2^r-1} & \text{for } i = 2^r - 1 \\ 1 \otimes a_i & \text{for } i \neq 2^r - 1 \end{cases}.$$

When we apply $\mathrm{id} \otimes f$ to these elements, we get

$$(\mathrm{id} \otimes f)(\gamma(a_i)) = \begin{cases} \xi_r \otimes f(1) + 1 \otimes f(a_{2^r-1})) & \text{for } i = 2^r - 1 \\ 1 \otimes f(a_i) & \text{for } i \neq 2^r - 1 \end{cases} =$$
$$= \begin{cases} \xi_r \otimes 1 + 1 \otimes 0 & \text{for } i = 2^r - 1 \\ 1 \otimes u_i & \text{for } i \neq 2^r - 1 \end{cases}.$$

where $f(1) = 1$ since f is a morphism of algebras, as we wanted. □

We will show that this composition g actually helps us understand the structure of $\pi_*(TO)$ as we wanted. Indeed, if we see N_* as a subalgebra of $\mathcal{A}_* \otimes N_*$ (by considering its isomorphic algebra $\mathbb{Z}/2\mathbb{Z} \otimes N_* \subset \mathcal{A}_* \otimes N_*$), it can be proven that $g \circ h$ (where h is the Hurewicz injective morphism $h : \pi_*(TO) \to H_*(TO; \mathbb{Z}/2\mathbb{Z})$), maps $\pi_*(TO)$ to $\mathbb{Z}/2\mathbb{Z} \otimes N_*$. We have the following theorem, that helps us close the understanding of the structure of $\mathcal{N}_*$.

Theorem III.5.6. *$h : \pi_*(TO) \to H_*(TO; \mathbb{Z}/2\mathbb{Z})$ is a monomorphism and $g \circ h$ maps $\pi_*(TO)$ isomorphically onto N_*.*

III.6 Hurewicz homomorphism

In this section we will just discuss Theorem III.5.6. Before doing that, we need to actually define the Hurewicz homomorphism for prespectra.

The Hurewicz homomorphisms $\pi_{n+q}(T(\gamma^q)) \to \tilde{H}_{n+q}(T(\gamma^q);\mathbb{Z}/2\mathbb{Z})$ are defined in the following way. Consider the canonical generator $u_{n+q} \in H_{n+k}(\mathbb{S}^{n+k};\mathbb{Z}/2\mathbb{Z})$. Then the image of an homotopy class $[f] \in \pi_{n+q}(T(\gamma^q))$ will be the element $f_*(u_{n+q}) \in H_{n+q}(TO;\mathbb{Z}/2\mathbb{Z})$. Note that this is well defined because if $\hat{f} : \mathbb{S}^{n+k} \to T(\gamma^q)$ is homotopic to f, then $f_* \equiv \hat{f}_*$.

These maps pass to colimits since the following diagram commutes

$$\begin{array}{ccccc} \pi_{n+q}(T(\gamma^q)) & \xrightarrow{\Sigma} & \pi_{n+q+1}(\Sigma T(\gamma^q)) & \xrightarrow{\sigma_{q*}} & \pi_{n+q+1}(T(\gamma^{q+1})) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{H}_{n+q}(T(\gamma^q);\mathbb{Z}) & \xrightarrow{\Sigma_*} & \tilde{H}_{n+q+1}(\Sigma T(\gamma^q);\mathbb{Z}) & \xrightarrow{\sigma_{q*}} & \tilde{H}_{n+q+1}(T(\gamma^{q+1});\mathbb{Z}) \end{array},$$

where the vertical maps are Hurewicz homomorphisms, to give a stable Hurewicz homomorphism $\pi_n(TO) \to H_n(TO;\mathbb{Z})$. Since TO is an associative and commutative ring prespectrum, $h : \pi_*(TO) \to H_*(TO;\mathbb{Z}/2\mathbb{Z})$ is a well defined map between graded commutative rings.

Theorem III.6.1. *The homomorphism $h : \pi_*(TO) \to H_*(TO;\mathbb{Z}/2\mathbb{Z})$ is a monomorphism and $g \circ h$ maps $\pi_*(TO)$ isomorphically onto N_*.*

For the idea of the proof of this theorem, it can be checked [May99, Chapter 25, section 6].

However, we can see that the image of $g \circ h$ is actually in $\mathbb{Z}/2\mathbb{Z} \otimes N_*$. Indeed, the image of $\gamma \circ h$ is given by the following lemma.

Lemma III.6.2. *For $[\phi] \in \pi_*(TO)$, $\gamma(h([\phi])) = 1 \otimes h([\phi]) \subset \mathbb{Z}/2\mathbb{Z} \otimes H_*(TO;\mathbb{Z}/2\mathbb{Z})$.*

Thus, $(g \circ h)([\phi]) = (\mathrm{id} \otimes f) \circ (\gamma(h([\phi]))) = (\mathrm{id} \otimes f)(1 \otimes h([\phi])) = 1 \otimes f(h([\phi])) \in \mathbb{Z}/2\mathbb{Z} \otimes N_*$.

The idea of the proof of this lemma is that since all the Steenrod squares of any sphere are zero, checking on the generators the dual map to $\gamma \circ h$, this would be zero on all the elements of $\mathcal{A} \otimes N$ of the form $\mathrm{Sq}^I \otimes 1$, and it can be different to zero only on the generators $1 \otimes n_i$ (where n_i is some generator of the dual of N_*). Taking the dual gives our result.

III.7 Thom's theorem proof

We can finally see the proof of Thom's theorem that we stated in Chapter II, II.2.2.

Theorem III.7.1 (Thom)**.** *Let M be a closed smooth n-dimensional manifold. If all the Stiefel–Whitney numbers of M are zero, then M is the boundary of some smooth compact $(n+1)$-dimensional manifold with boundary.*

We are actually going to prove the following equivalent statement.

Theorem III.7.2. *Let M be a closed smooth n-dimensional manifold. If all the normal Stiefel–Whitney numbers of M are zero, then M is the boundary of some smooth compact $(n+1)$-dimensional manifold with boundary.*

Indeed, by Lemma I.2.15, we get that all the normal Stiefel–Whitney numbers are zero if and only if all the normal Stiefel–Whitney numbers are zero.

To prove this theorem, we are going to focus on the diagram

$$\begin{array}{ccccc} H^n(BO)\otimes\mathcal{N}_n & \xrightarrow{\mathrm{id}\otimes\alpha} & H^n(BO)\otimes\pi_*(TO) & \xrightarrow{\mathrm{id}\otimes h} & H^n(BO)\otimes H_n(TO) \\ \#\downarrow & & & & \downarrow\mathrm{id}\otimes\Phi \\ \mathbb{Z}/2\mathbb{Z} & & \xleftarrow[\langle\cdot,\cdot\rangle]{\hspace{6em}} & & H^n(BO)\otimes H_n(BO) \end{array},$$

where we are implying $\mathbb{Z}/2\mathbb{Z}$ coefficients in all the cohomology and homology groups, and we are denoting by $\#$ the map that assigns to each polynomia in the Stiefel–Whitney classes and each manifold their Stiefel–Whitney number. Indeed, all the Stiefel–Whitney classes of the canonical bundles are elements of $H^*(BO;\mathbb{Z}/2\mathbb{Z})$, thanks to the inclusions

$$\mathbb{Z}/2\mathbb{Z}[w_1(\gamma^q),\dots,w_q(\gamma^q)]\cong H^*(\mathbb{G}_q;\mathbb{Z}/2\mathbb{Z})\to H^*(BO;\mathbb{Z}/2\mathbb{Z}).$$

Thus also the polynomia with variables the Stiefel–Whitney classes are elements of the cohomology ring of BO. Moreover, once given an n-dimensional manifold M, its normal bundle ν_M is unique up to isotopy (for an embedding $M\subset\mathbb{R}^{n+q}$ with q sufficiently large) and the classifying map $f:\nu\to\gamma^q$ is also unique up to homotopy, thus

$$\begin{aligned} f^*(w_1(\gamma^n)^{r_1}\smile\cdots\smile w_n(\gamma^n)^{r_n}) =& f^*(w_1(\gamma^n))^{r_1}\smile\cdots\smile f^*(w_n(\gamma^n))^{r_n} = \\ =& w_1(\nu_M)^{r_1}\smile\cdots\smile w_n(\nu_M)^{r_n}. \end{aligned}$$

is unique. Adding the fact that also M's $\mathbb{Z}/2\mathbb{Z}$-fundamental class is unique, we can see that the map

$$\begin{aligned} \# : H^n(BO) \otimes \mathcal{N}_n &\to \mathbb{Z}/2\mathbb{Z} \\ w_1^{r_1}(\gamma^q) \smile \cdots \smile w_n^{r_n}(\gamma^q) \otimes [M] &\mapsto w_1^{r_1} \cdots w_n^{r_n}[\nu(M)] \end{aligned}$$

is clearly well-defined.

If we manage to show that the diagram above commutes, this will prove the Theorem III.7.2. Indeed, if we assume that every Stiefel–Whitney number of M is zero, then for each element w of $H^n(BO; \mathbb{Z}/2\mathbb{Z})$ $\#(w \otimes [M]) = 0$. But since we are assuming the diagram commutes, this means that for every w

$$\langle w, \Phi(h(\alpha([M])))\rangle = 0.$$

Since this pairing is the evaluation pairing of dual vector spaces, this implies that $\Phi(h(\alpha([M])))$ must be zero. But we have proved that Φ and α are isomorphisms, and h is a monomorphism, so we have that $[M]$ must be the zero element of $\mathcal{N}_n$, and so M is a boundary.

We just need to show the commutativity now. Let M be a smooth closed n-dimensional manifold. Let $\alpha([M]) = [T(f) \circ t]$ where $t : \mathbb{S}^{n+q} \to T(\nu_M)$ and $T(f) : T(\nu_M) \to T(\gamma^q)$ for q large enough. In the section about Thom spaces we saw that we have the following commutative diagram in homology:

$$\begin{array}{ccccc} \tilde{H}_{n+q}(\mathbb{S}^{n+q}; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{t_*} & \tilde{H}_{n+q}(T(\nu_M); \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{T(f)_*} & \tilde{H}_{n+q}(T(\gamma^q); \mathbb{Z}/2\mathbb{Z}) \\ & & \downarrow \Phi & & \downarrow \Phi \\ & & \tilde{H}_n(M; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow[f_*]{} & \tilde{H}_n(\mathbb{G}_q; \mathbb{Z}/2\mathbb{Z}) \end{array} .$$

Thus if we denote by u_{n+q} the fundamental class in the reduced homology $\tilde{H}_{n+q}(\mathbb{S}^{n+q}; \mathbb{Z}/2\mathbb{Z})$,

$$\begin{aligned} f_* \circ \Phi \circ t_*(u_{n+q}) = \Phi \circ T(f)_* \circ t_*(u_{n+q}) = \Phi \circ (T(f) \circ t)_*(u_{n+q}) = \\ = \Phi \circ h([T(f) \circ t]) = \Phi \circ h \circ \alpha([M]), \end{aligned}$$

by definition of h and of α.

We claim that $z = (\Phi \circ t_*)(u_{n+q}) \in \tilde{H}_n(M; \mathbb{Z}/2\mathbb{Z})$ is the fundamental class. Then, for $w \in H^*(BO; \mathbb{Z}/2\mathbb{Z})$:

$$\begin{aligned} w\#[M] =& \langle w(\nu_M), z\rangle = \langle f^* w(\gamma^q), (\Phi \circ t_*)(u_{n+q})\rangle = \\ =& \langle w(\gamma^q), (f_* \circ \Phi \circ t_*)(u_{n+q})\rangle = \\ =& \langle w(\gamma^q), \Phi \circ h \circ \alpha([M])\rangle \end{aligned}$$

as we wanted.

To conclude, we just need to prove the claim. To do so, we just need to show that z is mapped to the generator of $H_n(M, M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ by the map induced by the inclusion, for each point $x \in M$.

As we did in the section about the Thom space, we can see the homology of a quotient as the homology of the pair, and in particular the homology the Thom space $T(\nu) = D(\nu)/S(\nu)$ as the homology of the pair $(D(\nu), S(\nu))$. There we saw that the Thom isomorphism in homology is defined by the explicit formula

$$\begin{aligned} \Phi : H_{i+q}(D(\nu), S(\nu); \mathbb{Z}/2\mathbb{Z}) &\to H_i(M; \mathbb{Z}/2\mathbb{Z}) \\ a &\mapsto \pi_*(\mu \frown a) \end{aligned},$$

where $\mu \in H^q(D(\nu), S(\nu); \mathbb{Z}/2\mathbb{Z})$ is the fundamental class and π is the projection map of ν.

Take $x \in M$, and a neighborhood of this point $U \overset{\psi}{\cong} \mathbb{R}^n$. Denote by $B(U)$ the preimage of the unit ball $\psi^{-1}(D^n)$ and by $S(U)$ the preimage of the unit sphere $\psi^{-1}(\mathbb{S}^{n-1})$. Moreover, denote by V the space $B(U) \setminus S(U)$. Since $B(U)$ is diffeomorphic to a contractible space, it is contractible itself, so $\nu|_{B(U)}$ is trivial. This implies that $E(\nu|_{B(U)}) \cong B(U) \times \mathbb{R}^q$ and $D(\nu|_{B(U)}) \cong B(U) \times D^q$. We can compute the boundary of this space

$$\begin{aligned} \partial(B(U) \times D^q) &= (\partial B(U) \times D^q) \cup (B(U) \times \partial D^q) = \\ &\cong (S(U) \times D^q) \cup (B(U) \times \mathbb{S}^{q-1}) \end{aligned}.$$

An then consider the homotopy equivalence

$$\hat{t} : \mathbb{S}^{n+q} \to \frac{B(U) \times D^q}{\partial(B(U) \times D^q)} \cong \mathbb{S}^{n+q}$$

that sends the restriction of the tubular neighborhood W to $B(U)$ isomorphically to $D(\nu|_{B(U)}) = B(U) \times D^q$ and everything else will be crashed into $\mathbb{S}^{n+q} \setminus W$ to the boundary of $B(U) \times D^q$.

Denote by t the Pontrjagin-Thom construction for ν. Similarly to what we did for the map $\hat{t}$, this map will take W isomorphically to $D(\nu)$, and everything will be contracted.

We need to define some other maps. Denote by $\hat{\mu}$ the fundamental class of $H^q(D(\nu|_{B(U)}), S(\nu|_{B(U)}); \mathbb{Z}/2\mathbb{Z})$, by $\hat{\pi}_*$ the Künneth isomorphism

$$H_n(B(U) \times D^q, S(U) \times D^q; \mathbb{Z}/2\mathbb{Z}) \to H_n(B(U), S(U); \mathbb{Z}/2\mathbb{Z})$$

and by $\hat{\Phi}$ the map $a \mapsto \hat{p}_*(\hat{\mu} \frown a)$. Analogously, denote by $\tilde{\pi}$ the homotopy equivalence $(D(\nu), D(\nu_{M\setminus V})) \to (M, M \setminus V)$ and by $\tilde{\Phi}$ the map $a \mapsto \tilde{p}_*(\mu \frown a)$. In this case, we are using the relative cap product

$$H_{i+q}(D(\nu), S(\nu) \cup D(\nu|_{M\setminus V})) \otimes H^q(D(\nu), S(\nu)) \to H_i(D(\nu), D(\nu|_{M\setminus V})).$$

Observe that in particular, $\hat{\Phi}$ is the inverse of the suspension isomorphism $H_n(\mathbb{S}^n; \mathbb{Z}/2\mathbb{Z}) \to H_{n+q}(\mathbb{S}^{n+q}; \mathbb{Z}/2\mathbb{Z})$.

We have the commutative diagram

$$\begin{array}{ccccc}
\tilde{H}_{n+q}(\mathbb{S}^{n+q}) & \xrightarrow{\hat{t}_*} & H_{n+q}(B(U) \times D^q, \partial(B(U) \times D^q)) & \xrightarrow{\hat{\Phi}} & H_n(B(U), S(U)) \\
\Big\downarrow t_* & & \Big\downarrow & & \Big\downarrow \cong \\
 & \nearrow & H_{n+q}(D(\nu), S(\nu) \cup D(\nu|_{M\setminus V})) & \xrightarrow{\tilde{\Phi}} & H_n(M, M \setminus V) \\
 & & & & \Big\downarrow \cong \\
H_{n+q}(D(\nu), S(\nu)) & \xrightarrow{\Phi} & H_n(M) & \longrightarrow & H_n(M, M \setminus \{x\})
\end{array}$$

where the non named maps are inclusions and where we omitted the coefficients in $\mathbb{Z}/2\mathbb{Z}$. The composition on the top is an isomorphism by the discussion above (t is an homotopy equivalence and $\hat{\Phi}$ is an isomorphism), and so the image of the generator $u_{n+q} \in \tilde{H}_{n+q}(\mathbb{S}^{n+q})$ via those maps and the vertical maps on the right (which are excisions isomorphisms) will be a generator too. Since the diagram is commutative, this generator of $H_n(M, M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z})$ is also the image of $z = \Phi \circ t_*(u_{n+q}) \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ under the inclusion $H_n(M; \mathbb{Z}/2\mathbb{Z}) \to H_n(M, M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z})$, as we wanted. This proves our claim.

III.8 Generators of the unoriented cobordism ring

We have that Theorem III.5.6 proves how $\mathcal{N}_*$ is isomorphic as a $\mathbb{Z}/2\mathbb{Z}$-algebra to $N_* = \mathbb{Z}/2\mathbb{Z}[u_i | i > 1 \text{ and } i \neq 2^r - 1]$ where u_i has degree i, by Thom's theorem III.3.12. The interesting thing is that these generators are actually known for the unoriented cobordism ring. In order to find these generators, we use Thom's theorem on Stiefel–Whitney classes. Indeed, a manifold is a boundary if and only if all its Stiefel–Whitney numbers are zero. Therefore, if any closed smooth n-dimensional manifold has at least one Stiefel–Whitney different from 0, its cobordism class will not be the zero element of the n-th cobordism group $\mathcal{N}_n$, and thus it can be taken as the generator u_n.

In particular, recall that we have computed the Stiefel–Whitney of the projective space in Chapter I, section I.3. In Example I.3.2 we saw that for n odd each Stiefel–Whitney number of $\mathbb{P}^n$ is zero, and thus $[\mathbb{P}^n] = [\emptyset] \in \mathcal{N}_n$. However, the Stiefel–Whitney numbers of $\mathbb{P}^n$ are not all zero for n even, and so $[\mathbb{P}^n] \in \mathcal{N}_n$ is the generator u_n of order n in $\mathcal{N}_*$.

For n odd, since this n is not of the form $2^r - 1$, we can rewrite this as $n = 2^p(2q+1) - 1 = 2^{p+1}q + 2^p - 1$ for some $p, q > 0$. These p, q satisfy the inequality $2^{p+1}q > 2^p > 0$. Now, consider any $m, n > 0$ integers such that $m < n$. We can define the hypersurface $H_{n,m} \subset \mathbb{P}^n \times \mathbb{P}^m = \{([x_0, \dots, x_n], [y_0, \dots, y_n]) \in \mathbb{P}^n \times \mathbb{P}^m\}$ as the zero locus of the homogeneous polynomial $x_0y_0 + \cdots + x_my_m$. This hypersurface has dimension $m + n - 1$. We are going to take as generator u_n the element $[H_{2^{p+1}q, 2^p}] \in \mathcal{N}_{2^{p+1}q + 2^p - 1} = \mathcal{N}_n$. This is a manifold since it is a projective variety (its atlas is composed of the neighborhoods $U_i \times U_j = \{([x_0, \dots, x_n], [y_0, \dots, y_n]) \in \mathbb{P}^n \times \mathbb{P}^m |\ x_i \neq 0 \text{ and } y_j \neq 0\}$ and maps $([x_0, \dots, x_n], [y_0, \dots, y_n]) \mapsto \left(\left[\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right], \left[\frac{y_1}{y_j}, \dots, \frac{y_m}{y_j}\right]\right)$). it is an easy exercise in algebraic geometry to see that this is smooth and closed. The computation of its last Stiefel–Whitney number can be found in [Sto68, pages 79-80]. We see that

$$w_n[H_{2^{p+1}q, 2^p}] = -\binom{2^p(2q+1)}{2^p}$$

and since this is a binomial $\mod 2$, if $2^p(2q+1) = \sum_i n_i 2^i = 2^p + \sum_{i>p} n_i 2^i$

$$\begin{aligned}\binom{2^p(2q+1)}{2^p} &= \binom{n_p}{1} \prod_{i \neq p} \binom{k_i}{0} \mod 2 = \binom{n_p}{1} \mod 2 = \binom{1}{1} \mod 2 \\ &= 1 \neq 0\end{aligned}$$

and since this is not zero, $H_{2^{p+1}q, 2^p}$ is not a boundary, so its class is a generator of $\mathcal{N}_n$.

A

Appendix

The aim of this appendix is to give some details that are usually studied in a course about manifolds, but since are essential for this thesis we want to collect them in an appendix.

A.1 Manifolds and orientations

In this section we would like to recall the definition of smooth manifold, give the classical definition of orientability and orientation of a manifold without and with boundary and give some proprieties related to these topics.

Let's start by recalling the definition of smooth manifold.

Definition A.1.1. Let X be a second countable Hausdorff space. It is called a *smooth n-dimensional manifold without boundary* if it has a smooth structure on it. A *smooth structure* is a family of pairs $\{(U_i, \phi_i)\}_{i\in I}$ where U_i is an open set of X and ϕ_i is a homeomorphism of U_i onto an open subset of $\mathbb{R}^n$ such that:

- $\{U_i\}_{i\in I}$ is an open cover of X;
- For every $i, j \in I$, where $U_i \cap U_j \neq \emptyset$, the homeomorphisms $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are smooth maps between open sets of $\mathbb{R}^n$;

- the family $\{(U_i, \phi_i)\}_{i \in I}$ is maximal (i.e. it contains all the possible (U_i, ϕ_i) satisfying the first two conditions).

Even though we focus more on the unoriented cobordism theory, we often generalize to oriented manifolds and oriented vector bundles, so we would like to recall the definition of oriented manifold.

In order to do that, fix a commutative ring R and let M be an n-dimensional manifold.

First of all, recall that for any $x \in M$,

$$H_k(M, M \setminus \{x\}; R) \cong \begin{cases} R & \text{if } k = n \\ 0 & \text{else} \end{cases}.$$

Definition A.1.2. A *local R-orientation* of $x \in M$ is the choice of a generator $\mu_x \in H_n(M, M \setminus \{x\}; R) \cong R$.

An *R-orientation* is the choice of a local orientation $\mu_x \in H_n(M, M \setminus \{x\}; R)$ at every $x \in M$ such that at every x there exists a compact neighborhood K and an element $\mu \in H_n(M, M \setminus K; R)$ such that μ restricts to $\mu_y \in H_n(M, M \setminus \{y\}; R)$ for every $y \in K$.

We say that a manifold is *R-orientable* if there exists (at least) one R-orientation, but we don't choose one.

We are mainly interested in the cases were $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbb{Z}$. We have the following properties regarding these rings:

- Every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable (because there is a unique local $\mathbb{Z}/2\mathbb{Z}$-orientation);
- if a manifold is $\mathbb{Z}$-orientable, then it is R-orientable for all rings R. This orientation is induced by the canonical maps $\mathbb{Z} \to R$ that sends 1 to a generator of R.

The latter property gives us the reason why we usually call a manifold that is $\mathbb{Z}$-orientable just *orientable*.

We also need the following classical theorem, which allows us to define the fundamental class.

Theorem A.1.3. *Let M be an n-dimensional manifold together with an R-orientation $\{\mu_x \in H_n(M, M \setminus \{x\}; R)\}_{x \in M}$. Then there exists, for every* *compact $K \subset M$, a unique $\mu \in H_n(M, M \setminus K; R)$ restricting to μ_x for all $x \in K$.*

In particular, if M is compact, we can choose M to be K and $\mu \in H_n(M; R) = H_n(M, M \setminus M; R)$ is called the *fundamental class of M*, and is often denoted by $[M]$ or μ_M.

Now that we have seen some orientation theory about manifolds without boundaries, we want to focus on manifolds with boundary. As we did in Chapter I, we have used the definition of orientation of a manifold induced by the orientation of its tangent bundle. If we have an oriented manifold with boundary X, we have presented a definition of induced orientation of its boundary ∂X using the induced orientation of its tangent bundle.

In this appendix, we are going to stick to a definition more on the line of the classical orientation definition (with fundamental classes in the case of compact manifolds).

Definition A.1.4. A manifold with boundary X is *R-orientable* if $\mathring{X}$ is R-orientable and an R-orientation on X is an *R-orientation* on $\mathring{X}$.

It is obvious that if we are working with a manifold without boundary, $X = \mathring{X}$, so this definition applies in general.

Since the set ∂X is a submanifold of X, we want to see if an orientation on X induces an orientation on its boundary. We have the following result about this.

Proposition A.1.5. *If X is R-oriented, this orientation induces an R-orientation on ∂X.*

Before starting the proof, recall that the Remark II.1.4 tells us that $X \sim \mathring{X}$.

Proof. Let n be the dimension of X. Take any $x \in \partial X$ and let U be a chart of X that contains x. Without loss of generality (by taking a refinement of the atlas) U is homeomorphic to $\mathbb{H}^n$. In the same way, $V := \partial U = U \cap \partial X \cong \mathbb{R}^{n-1}$ and $U \setminus V = \mathring{U} \cong \mathbb{R}^n$. Let y be any point in $\mathring{U}$. We have that

$$H_n(\mathring{X}, \mathring{X} \setminus \mathring{U}; R) \cong H_n(\mathring{X}, \mathring{X} \setminus \{y\}; R) \cong H_n(X, X \setminus \{y\}; R) \cong H_n(X, X \setminus \mathring{U}; R)$$

where the first and the third isomorphism are due to the fact that $\mathring{U}$ is contractible and the one in the middle comes from the fact that $X \sim \mathring{X}$ and similarly $X \setminus \{y\} \sim \mathring{X} \setminus \{y\}$.

Now consider the long exact sequence of the triple $(X, X \setminus \mathring{U}, X \setminus U)$

$$H_n(X, X \setminus U) \to H_n(X, X \setminus \mathring{U}) \xrightarrow{\partial} H_{n-1}(X \setminus \mathring{U}, X \setminus U) \to H_{n-1}(X, X \setminus U)$$

where we omitted the coefficients in R. We have that the connecting homomorphism ∂ is an isomorphism, since $H_*(X, X \setminus U; R) \cong 0$. Indeed, we have that $X \sim X \setminus U$ (the proof is analogous to the one of $X \sim \mathring{X}$) and thus $H_*(X, X \setminus U; R) \cong H_*(X, X; R) \cong 0$.

Moreover, since $X \setminus U = (X \setminus \mathring{U}) \setminus V$ and $V \cong \mathbb{R}^{n-1}$ is contractible (without loss of generality we can contract it to $x \in \partial X \cap U = V$),

$$H_{n-1}(X \setminus \mathring{U}, X \setminus U; R) \cong H_{n-1}(X \setminus \mathring{U}, (X \setminus \mathring{U}) \setminus \{x\}; R).$$

Furthermore, we get another isomorphism using excision. Consider $\mathring{X} \setminus \mathring{U} \subset (X \setminus \mathring{U}) \setminus \{x\} \subset X \setminus \mathring{U}$. We have that these sets satisfy the conditions to use the *excision theorem* $\mathrm{cl}(\mathring{X} \setminus \mathring{U}) = X \setminus U = \mathrm{int}((X \setminus \mathring{U}) \setminus \{x\})$. Thus, since $\partial X = (X \setminus \mathring{U}) \setminus (\mathring{X} \setminus \mathring{U})$ and $\partial X \setminus \{x\} = ((X \setminus \mathring{U}) \setminus \{x\}) \setminus (\mathring{X} \setminus \mathring{U})$, we have

$$H_{n-1}(X \setminus \mathring{U}, (X \setminus \mathring{U}) \setminus \{x\}; R) \cong H_{n-1}(\partial X, \partial X \setminus \{x\}; R).$$

Lastly, since V is contractible and $V \subset \partial X$,

$$H_{n-1}(\partial X, \partial X \setminus \{x\}; R) \cong H_{n-1}(\partial X, \partial X \setminus V; R).$$

Now that we have shown that

$$H_n(\mathring{X}, \mathring{X} \setminus \mathring{U}; R) \cong H_{n-1}(\partial X, \partial X \setminus V; R)$$

it is clear that an R-orientation on X (which is an R-orientation on $\mathring{X}$) gives us an R-orientation on ∂X.

□

Let X be compact. We get that if the hypothesis above are satisfied, the orientation on X induces the choice of a generator $\mu_{\partial X}$ of $H_{n-1}(\partial X; R)$: indeed, since ∂X in an $(n-1)$-orientable manifold (without boundary), an orientation gives us a fundamental class in the classical way. We could wonder whether there exists a generator of $H_n(X, \partial X; R)$ such that $\partial_n(\mu_X) = \mu_{\partial X}$ where $\partial_n : H_n(X, \partial X; R) \to H_{n-1}(\partial X; R)$ is the n-th connecting morphism of the long exact sequence of the pair $(X, \partial X)$. The following result answers this question.

Proposition A.1.6. *Let X be a compact and R-oriented manifold with boundary. Let $\mu_{\partial X}$ be the fundamental class of ∂X given by the induced R-orientation on ∂X. Then there exists a unique generator $\mu_X \in H_n(X, \partial X; R)$ such that $\partial_n(\mu_X) = \mu_{\partial X}$.*

Proof. First of all, since X is a manifold with boundary, $H_n(X; R) = 0$. Looking at the long exact sequence coming from the pair $(X, \partial X)$

$$0 = H_n(X; R) \to H_n(X, \partial X; R) \xrightarrow{\partial} H_{n-1}(\partial X; R) \to H_{n-1}(X; R) \to \dots$$

by exactness we get that the connecting homomorphism ∂ is injective.

Let V be the open collar neighborhood of ∂X coming from Theorem II.1.3 and denote by Y the set $X \setminus V$. We have that Y is closed in X (and thus compact in X, since X is compact) and it is a deformation retraction of $\mathring{X}$. Thanks to this and the homotopy equivalence between X and $\mathring{X}$, we get the following chain of isomorphisms:

$$H_n(\mathring{X}, \mathring{X} \setminus Y; R) \cong H_n(X, X \setminus Y; R) \cong H_n(X, X \setminus \mathring{X}; R) = H_n(X, \partial X; R).$$

Hence, since Y is compact in X, it is also compact in $\mathring{X}$, and thus the R-orientation on $\mathring{X}$ determines a fundamental class $\tilde{\mu} \in H_n(\mathring{X}, \mathring{X} \setminus Y; R)$. Let $\mu_X \in H_n(X, \partial X; R)$ be the image of $\tilde{\mu}$ under the isomorphisms above. This is clearly a fundamental class in the tipical way: $\mu_X \in H_n(X, X \setminus \mathring{X}; R)$ restricts to a generator of $H_n(X, X \setminus \{y\}; R)$ for every $y \in \mathring{X}$, thanks to the commuting diagram below

$$\begin{array}{ccc} H_n(\mathring{X}, \mathring{X} \setminus Y; R) & \xrightarrow{\cong} & H_n(X, X \setminus \mathring{X}; R) \\ \downarrow & & \downarrow \\ H_n(\mathring{X}, \mathring{X} \setminus \{y\}; R) & \xrightarrow[\cong]{} & H_n(X, X \setminus \{y\}; R) \end{array}$$

where the vertical map on the left is the map coming from the definition of an R-orientation, that sends $\tilde{\mu}$ to a generator.

Using the isomorphisms from the proof of Proposition A.1.5, we get that for every $x \in \partial X$, $\partial \mu_X$ restricts to the generator of $H_{n-1}(\partial X, \partial X \setminus \{x\}; R)$ given by the induced R-orientation, and thus $\partial \mu_X = \mu_{\partial X}$. □

Bibliography

[Ben21] Riccardo Benedetti. *Lectures on differential topology*. Vol. 218. Providence, RI: American Mathematical Society (AMS), 2021. ISBN: 978-1-4704-6674-9. DOI: 10.1090/gsm/218.

[Bre93] Glen E. Bredon. *Topology and geometry*. English. Vol. 139. Grad. Texts Math. New York: Springer-Verlag, 1993. ISBN: 0-387-97926-3.

[Hat17] Allen Hatcher. *Vector Bundles and K-Theory*. English. Version 2.2. http://www.math.cornell.edu/~hatcher. 2017.

[Hir76] Morris W. Hirsch. *Differential topology*. English. Vol. 33. Grad. Texts Math. Springer, Cham, 1976. DOI: 10.1007/978-1-4684-9449-5.

[Lan02] Serge Lang. *Introduction to differentiable manifolds*. English. 2nd edition. Universitext. New York, NY: Springer, 2002. ISBN: 0-387-95477-5. DOI: 10.1007/b97450.

[May99] J. P. May. *A concise course in algebraic topology*. English. Chicago, IL: University of Chicago Press, 1999. ISBN: 0-226-51182-0.

[MS05] John W. Milnor and James D. Stasheff. *Characteristic classes*. English. Vol. 32. Texts Read. Math. New Delhi: Hindustan Book Agency, 2005. ISBN: 81-85931-52-6.

[Muk15] Amiya Mukherjee. *Differential topology*. 2nd edition. New Delhi: Hindustan Book Agency; Cham: Birkhäuser/Springer, 2015. ISBN: 978-3-319-19044-0. DOI: 10.1007/978-3-319-19045-7.

[Mun84] James R. Munkres. *Elements of algebraic topology*. English. Advanced Book Program. Redwood City, California etc.: Addison-Wesley Publishing Company, Inc. IX, 454 p. (1984). 1984.

[Sto68] R. E. Stong. *Notes on cobordism theory*. English. Math. Notes (Princeton). Princeton University Press, Princeton, NJ, 1968.

[Wal60] C. T. C. Wall. "Determination of the cobordism ring". English. In: *Ann. Math. (2)* 72 (1960), pp. 292–311. ISSN: 0003-486X. DOI: 10.2307/1970136.