On Bézier Curves and Surfaces

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ABSTRACT

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The Bézier curve is a continuous parametric curve that is used in numerous applications, such as automobile design and road design. I begin by surveying the fundamental properties of the Bézier curve and provide a variety of examples and calculations. I also survey, but more succinctly, the notion of the Bézier surface, which also has numerous applications in computer-aided design software, such as aircraft design. I end by presenting the special types of points that a cubic Bézier curve (with four control points) can have. The configuration of planar control points that I present can be used by computer engineers to produce various shapes with respect to the necessary design requirements and to minimize fluctuations of the curve itself.

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Chapter 1

Introduction

In geometry, there are a variety of curves with wonderful properties that are employed in real-world applications. Certain geometric curves tend to be increasingly difficult to compute mathematically and require a high level of mathematical knowledge to manipulate in order to acquire significant information of their geometry, such as determining the area of the curve as well as its curvature. Furthermore, these curves can have many singularities, discontinuities and fluctuations that are not desirable in many real-world applications, such as creating the complex bodywork of an aircraft, or automobile, or designing a race track. These constructions are examples of designs that require curves that are smooth, hardly fluctuate and are locally controllable. For example, an automotive designer utilizing a computer-aided design software must use a type of curve that preserves smoothness when manipulating the endpoints of the curve and this remains unchanged under rigid transformations of the Cartesian plane.

Mathematically, these properties are called endpoint interpolation and affine invariance, which sparks the discussion of which curve is optimal for this situation. Pierre Bézier, a mathematician and engineer who worked for Renault from 1933 to 1975 as a tool designer and then as a managing staff director for technical development, spent many years of his life focusing his research on computer numerical control and mathematical modelling. His goal was to develop a parametric curve with polynomial components that could aid in the construction of the bodywork of Renault automobiles and that can be easily used by anyone in computer-aided design and computer graphics. This curve is called nowadays Bézier curve, which is a continuous parametric curve with polynomial components with respect to the parameter $t \in [0, 1]$ and that is fully dependent on the quantity of planar control points. For example, a quadratic Bézier curve will have three control points and the components $x(t), y(t)$ will have parametric equations of maximal degree two.

A Bézier curve with many control points will have components represented by parametric equations that have a very high degree, which therefore entails many difficult computations, particularly when computing the cusps of a high degree Bézier curve by finding the roots of the derivative of its parametric form. Therefore, the curve should be universally used with the lowest degree possible without degenerating into a straight line ($n > 1$). Furthermore, the shape of the Bézier curve is fully dictated by a set of $(n - 2)$ control points on the Cartesian plane, in which the curve always commences at the first control point and terminates at the last control point. These control points form the control polygon of the Bézier curve and, generally, altering the shape of the polygon changes the shape of the curve. The control polygon must encompass the curve and will always fluctuate more than the curve itself, the latter called the variation diminishing property. This property will be especially significant throughout the study of the Bézier curve and its special types of points, which include the cusps, the self-intersection points and the inflection points. If the curve has a loop or an inflection point, then the curve will have a greater amount of fluctuations even if the curve will satisfy the variation diminishing property.

Furthermore, the idea that a curve is interpolated by n-planar control points can be extended to surfaces, also known as Bézier surfaces. These surfaces are interpolated by a grid of control points, which can be thought of as m-control points in the horizontal direction and n-control points in the vertical direction. These control points interpolate smooth, regular surfaces in $\mathbb{R}^3$ and can be used to create many nice, smooth shapes on CAD programs. Though, as the equation of a Bézier surface is dependent on the quantity of control points on the grid, the surface when the degree $(m \times n)$ is increased will lead to higher-degree polynomials. This can lead to difficult computations when calculating the Gaussian curvature, mean curvature and principle curvatures of the Bézier surface as well as calculating the first and second fundamental forms of the surface.

In this thesis, I survey the basic properties of a Bézier curve of degree n and investigate the length of a Bézier curve and its curvature. Next, I discuss some properties of a Bézier surface of degree $m \times n$ as well as its curvatures and the first and second fundamental forms of the surface. I end by presenting special types of points that a cubic Bézier curve can have with respect to a set of four control points $\{(P_0, P_1, P_2, P_3)\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ for $\lambda, \mu \in \mathbb{R}^2$. These special points include a cusp, a self-intersection point and a maximum of two inflection points. Finally, I show how the λ, μ plane is split up depending on whether the cubic Bézier curve with control points $\{(P_0, P_1, P_2, P_3)\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ has a cusp, a self-intersection point, one real inflection point, two real inflection points or none of these points. In my exposition, I am closely following the text *An Integrated Introduction to Computer Graphics and Geometric Modeling* by Ronald Goldman [2], and *Computational Geometry - Curve and Surface Modeling* by Su Bu-qing and Liu Ding-yuan [1]. I am closely following the text by Ronald Goldman when discussing the fundamentals of Bézier curves and surfaces and I am closely following the text by Su-Bu-qing and Liu Ding-yuan when discussing the geometric properties of the cubic Bézier curve and its special types of points. For the discussion of the special points, I chose the method of the paper, [4] As I overview Bézier curves and surfaces, I provide some examples, calculations and remarks addressing some of their properties. Specifically, I give the curvature for a Bézier curve of degree n, the first and second fundamental forms for the $m \times n$ Bézier surface and I give the details of the proof of a theorem in [1, 4] that describes which values $(\lambda, \mu) \in \mathbb{R}^2$ correspond to a cusp, a self-intersection point, up to two real inflection points or none of these points for a cubic Bézier curve.

Chapter 2

Bézier Curves

A Bézier curve is a continuous parametric curve, denoted $\vec{r}(t) = (x(t), y(t))$, which interpolates a fixed integer number, greater than two, of control points on the Cartesian plane. These control points, form a polygon, called the control polygon, which form the segments $\overline{\mathrm{P_0P_1}}, \overline{\mathrm{P_1P_2}}, ..., \overline{\mathrm{P_{n-1}P_n}}$. The Bézier curve follows the shape of the control polygon due to the convex hull property and, moreover, the curve follows its control points smoothly. Furthermore, the Bézier curve is inscribed inside the convex hull of its control polygon at all times and can be scaled, however preferable, by changing the location of its control points.

Examples of Bézier curves of degrees one ($n = 1$, two control points), two ($n = 2$, three control points) and four ($n = 4$, five control points) are shown below:

Definition 2.1. *Control Points:* A Bézier curve is a planar curve in parametric form defined by a set of $n + 1$ $(n \geq 0)$ control points denoted P_i, $i = 0, 1, 2, \ldots, n$ in the plane $\mathbb{R}^2$. These points are called control points of the Bézier curve.

Definition 2.2. *Bernstein Polynomials: [2]* Let $i = 0, 1, 2, \ldots, n$. We call the i-th Bernstein polynomial of degree $n \in \mathbb{N}$ the polynomial $B_{i,n}(t) = \frac{n!}{i!(n-i)!} \cdot (1-t)^{n-i} \cdot t^i$, viewed as a function of t, where $t \in [0, 1]$.

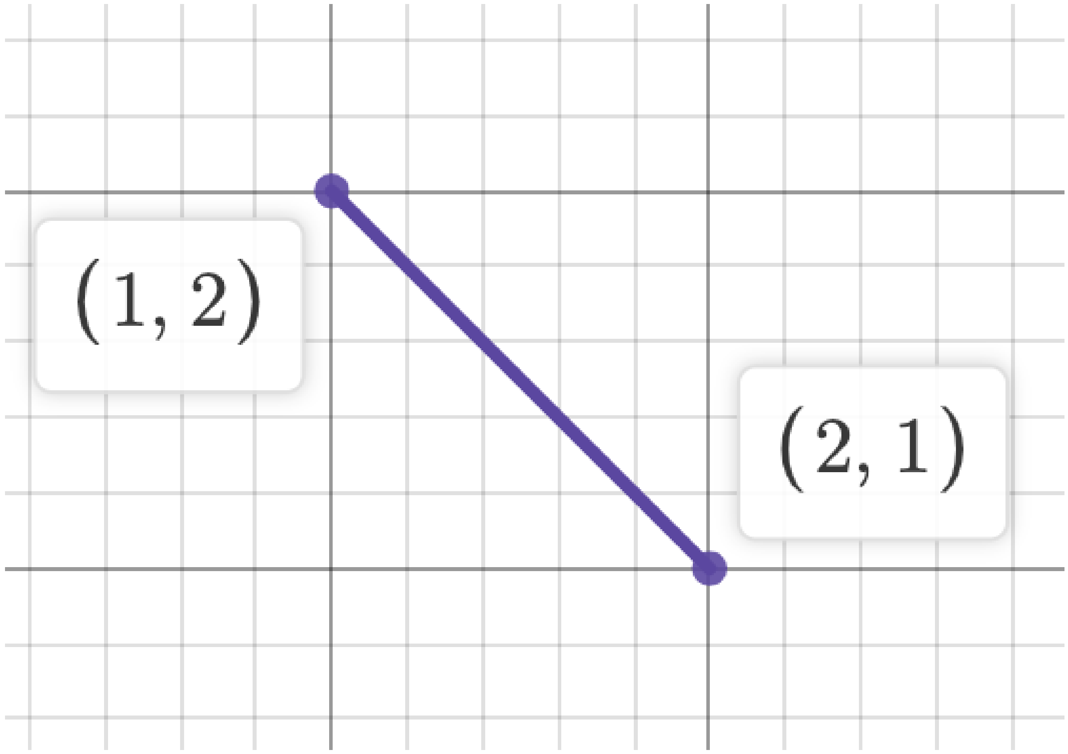

Figure 2.1: Example of a linear Bézier curve, where $\vec{r}(t) = (1 + t, 2 - t),\ t \in (0, 1)$.

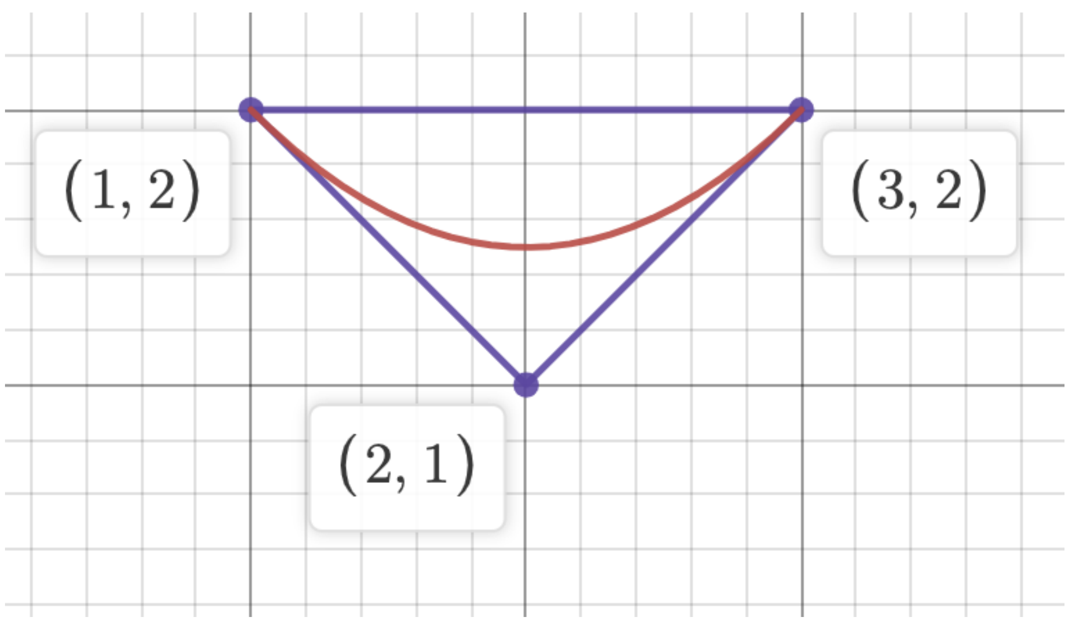

Figure 2.2: Example of a quadratic Bézier curve, where $\vec{r}(t) = (3-2t, 2t^2-2t+2),\ t \in (0, 1)$.

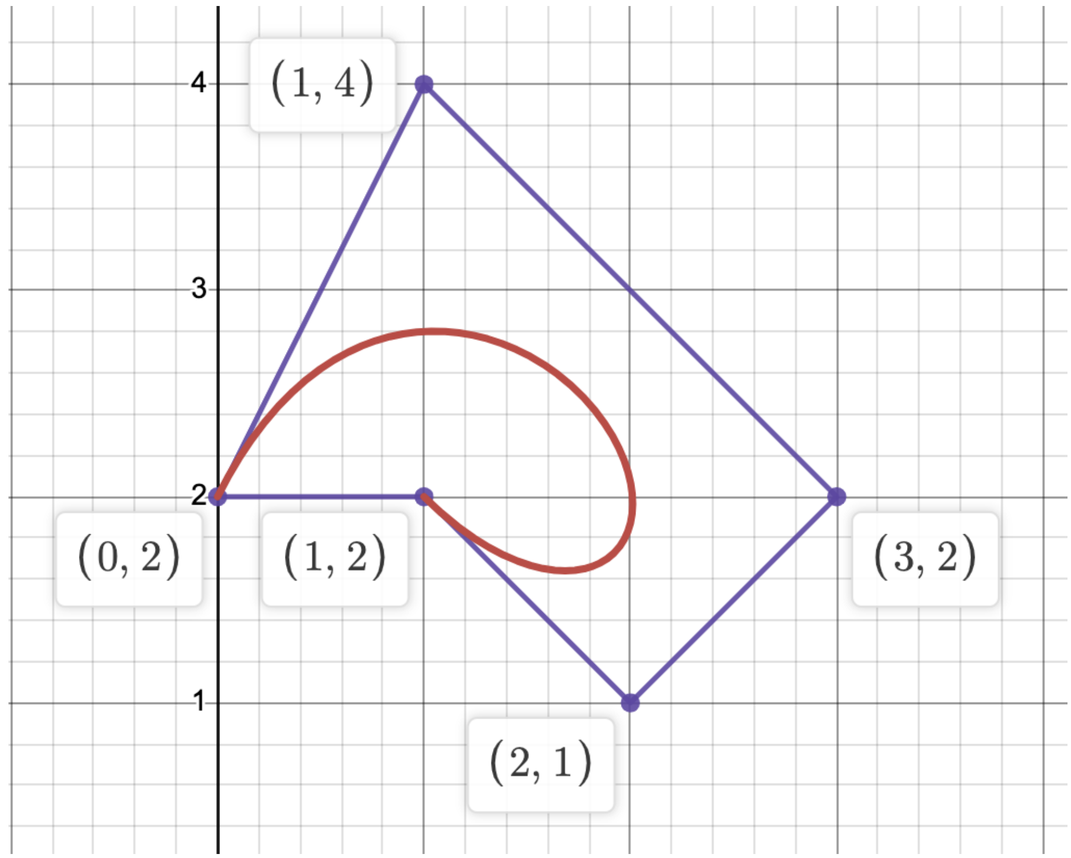

Figure 2.3: Example of a quartic Bézier curve, where $\vec{r}(t) = (1 + 4t - 12t^3 + 7t^4, 2 - 4t + 12t^2 - 4t^3 - 4t^4),\ t \in (0, 1)$.

Below, we list the properties of Bernstein polynomials:

a. **Non-Negativity:** $\forall t \in [0,1],\ B_{i,n}(t) \geq 0,\ i = 0, 1, 2, \ldots, n.$

b. **Symmetry:** $\forall t \in [0,1],\ B_{n-i,n}(1-t) = B_{i,n}(t),\ i = 0, 1, 2, \ldots, n.$

c. **Partition of unity:** Utilizing the binomial expansion, $\forall t \in [0,1],\ \sum_{i=0}^{n} B_{i,n}(t) = ((1-t)+t)^n = 1,\ i = 0, 1, 2 \ldots, n$, while note that for $t = 0$ or $t = 1$ we have,

$$B_{i,n}(0) = \delta_{i,0} = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases} \tag{2.1}$$

$$B_{i,n}(1) = \delta_{i,n} = \begin{cases} 1, & i = n \\ 0, & i \neq n. \end{cases} \tag{2.2}$$

d. **Differentiation:** $\frac{dB_{i,n}(t)}{dt} = n[B_{i-1,n-1}(t) - B_{i,n-1}(t)]$, for any $t \in (0,1)$. For $i = 0, 1, 2, \ldots, n-1$, the first derivative $\frac{dB_{i,n}(t)}{dt}$ is called the i^{th} hodograph of the Bézier curve. For a fixed Bézier curve, one can select any $i = 0, 1, 2, \ldots, n-1$.

The Bernstein polynomial, also called the blending function, can also be represented in a normalized form. This form is useful for t-values on any finite interval $a \leq t \leq b$, which is represented below.

Remark 2.1. *Let $a \leq t \leq b,\ a, b \in \mathbb{R}$. Then, the normalized Bernstein Polynomial is defined as $B_{i,n}(t) = \frac{n!}{i!(n-i)!} \cdot (b-t)^{n-i} \cdot \frac{(t-a)^i}{(b-a)^n},\ \forall i = 0, 1, 2, \ldots, n.$*

Definition 2.3. *The Bézier curve in parametric form is defined $\forall t \in [0,1]$ by the position vector $\vec{r}(t) = \sum_{i=0}^{n} P_i B_{i,n}(t)$, $i = 0, 1, 2 \ldots n$, where*

a. $r(t)$ is a point lying on the plane $\forall t \in [0,1]$;

b. n is the degree of the Bézier curve;

c. $B_{i,n}(t)$ is the i-th Bernstein polynomial, $i = 0, 1, 2 \ldots n$, $\forall t \in [0, 1]$;

d. P_i is the i-th control point of the Bézier curve, $i = 0, 1, 2 \ldots n$, given by its two Cartesian coordinates.

Remark 2.2. *In view of the linear independence of the Bernstein Polynomials, note that they form a basis of functions for a Bézier curve due to the fact that the Bézier curve is generated by them.*

Higher-order derivatives of the Bézier curve are required to gain knowledge revolving around the characteristics of the curve, including its curvature, which will be discussed in great detail throught this text. To add, higher-order derivatives are needed when joining two smooth Bézier curves of degree n and to detect any cusps or inflection points for a given curve of degree n.

Lemma 2.1. *Higher order derivatives of the Bézier curve may be computed recurrently as shown below:*

Set $R_i = n\Delta P_i = n(P_{i+1} - P_i)$, $i = 0, 1, ..., n$.
$\therefore \vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t) = \sum_{i=0}^{n-1} R_i B_{i,n-1}(t)$.

$\Delta R_i = n(\Delta P_{i+1} - \Delta P_i) = n(P_{i+2} - P_{i+1}) = n(P_{i+2} - P_{i+1} - P_{i+1} + P_i) = n(P_{i+2} - 2P_{i+1} + P_i)$.

$\therefore \vec{r}\,''(t) = \frac{d^2\vec{r}(t)}{dt^2} = (n-1)\sum_{i=0}^{n-2} B_{i,n-2}(R_{i+1} - R_i) = n(n-1)\sum_{i=0}^{n-2} B_{i,n-2}(P_{i+2} - 2P_{i+1} + P_i)$.

$\Delta\Delta R_i = \Delta(R_{i+1} - R_i) = (R_{i+2} - R_{i+1} - R_{i+1} + R_i) = (R_{i+2} - 2R_{i+1} + R_i)$.
$\therefore \Delta\Delta R_i = n(\Delta P_{i+2} - 2\Delta P_{i+1} + \Delta P_i) = n(P_{i+3} - 3(P_{i+2} - P_{i+1}) - P_i)$.

$\therefore \vec{r}'''(t) = \frac{d^3\vec{r}(t)}{dt^3} = n(n-1)(n-2)\sum_{i=0}^{n-3} B_{i,n-3}(t)(P_{i+3} - 3(P_{i+2} - P_{i+1}) - P_i).$

The control points corresponding to the k^{th} derivative of the Bézier curve are $\Delta^{k-1}R_i = n\Delta^{k-1}P_i$, $i = 0, 1 \ldots, n$.

Continuing this procedure, it can be shown below:

$\vec{r}^{(k)}(t) = \frac{d^k\vec{r}(t)}{dt^k} = n(n-1)(n-2)(n-3) \cdot \ldots \cdot (n-k+1) \sum_{i=0}^{n-k} B_{i,n-k}(t)\Delta^{k-1}P_i,$ *where $k \in \mathbb{N}^+$ and the degree of the resulting polynomial is equal to $n-k$.*

A crucial property of the Bézier curve is that it commences at the first control point and terminates at the last control point, which allows control over the location of the start and the end of the curve, particularly useful in computer programming. This is denoted as the interpolation property, which is stated below.

Lemma 2.2. *The Bézier curve has the endpoint interpolation property, such that $\vec{r}(0) = P_0$ & $\vec{r}(1) = P_n$.*

Proof. For this proof, one must show the third property of Definition 2.2.

Recall that $B_{i,n}(t) = \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}$, $t \in [0,1]$ & $i = 0, 1, 2, \ldots, n$

At $t = 0$, $B_{i,n}(0) = \binom{n}{i} \cdot 0^i \cdot 1^{n-i} = \binom{n}{i} \cdot 0^i$.

At $i \neq 0$, $B_{i,n}(0) = \binom{n}{i} \cdot 0^i = 0$.

Note that $B_{i,n}(0)$ depends on the value of i. The value for $i = 0$ is chosen by continuity: $\lim_{t\to 0^+} \binom{n}{0} \cdot t^0 = \lim_{t\to 0^+} t^0 = 1$.

$$\therefore B_{i,n}(0) \;=\; \begin{cases} 1, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

$\therefore \vec{r}(0) = \sum_{i=0}^{n} P_i B_{i,n}(0) = P_0 B_{0,n}(0) + P_1 B_{1,n}(0) + ... + P_{n-1} B_{n-1,n}(0) + P_n B_{n,n}(0) = P_0 \cdot 1 + P_1 \cdot 0 + ... + P_n \cdot 0 = P_0$

Similarly, for $t = 1$ and $i = n$, $\lim_{t\to 1^-} \binom{n}{n} \cdot (1-t)^0 = \lim_{t\to 1^-} (1-t)^0 = 1$.

At $t = 1,\; B_{i,n}(1) = \binom{n}{i} \cdot 1^i \cdot 0^{n-i} = \binom{n}{i} \cdot 0^{n-i}$

At $i \neq n,\; B_{i,n}(1) = \binom{n}{i} \cdot 0^{n-i} = 0$.

$$\therefore B_{i,n}(1) \;=\; \begin{cases} 1, & i = n \\ 0, & i \neq n. \end{cases}$$

$\therefore \vec{r}(1) = \sum_{i=0}^{n} P_i B_{i,n}(1) = P_0 B_{0,n}(1) + P_1 B_{1,n}(1) + ... + P_{n-1} B_{n-1,n}(1) + P_n B_{n,n}(1) = 0 + 0 + ... + 0 + P_n \cdot 1 = P_n$.

□

To complement the smoothness of a Bézier curve of degree n, the curve must be tangent to the first and last segments of the control polygon. This condition is denoted as the tangential property. This allows for the curve to be smooth on both ends, which gives the curve a desirable "curved" appearance and makes concatenation of Bézier curves produce a new (smooth) Bézier curve.

Lemma 2.3. *Let P_0 and P_1 be two control points creating a line segment denoted $\overline{P_0P_1}$. Then $\overline{P_0P_1}$ is tangent to the Bézier curve at the control point P_0.*

Proof. Utilizing the derivative of the normalized Bernstein Polynomial of Remark 2.1, assume $t = 0$.

$\therefore \vec{r}'(0) = \frac{n}{b-a} \cdot \sum_{i=0}^{n-1} B_{i,n-1}(0) \cdot (P_{i+1} - P_i)$ and $B_{i,n-1}(0) = \frac{(n-1)!}{i! \cdot (n-i-1)!} \cdot (b)^{n-i-1} \cdot \frac{(-a)^i}{(b-a)^{n-1}}$.

At $i = 0$, $\vec{r}'(0) = \frac{n}{b-a} \cdot B_{0,n-1}(0) \cdot (P_1 - P_0)$ and at $n = 1$, $B_{0,0}(0) = 1$.

$\Rightarrow \vec{r}'(0) = \frac{1}{b-a} \cdot (B_{0,0}(0)) \cdot (P_1 - P_0) = \frac{1}{b-a} \cdot (P_1 - P_0)$.

By inspection, $\frac{1}{b-a} = m \in \mathbb{R}$. This results in $r'(0) = m \cdot (P_1 - P_0)$, concluding that the tangent vector at $t = 0$ is in the direction of $m \cdot (P_1 - P_0)$.

□

Lemma 2.4. *Let P_{n-1} and P_n be two control points creating a line segment denoted $\overline{P_{n-1}P_n}$. Then $\overline{P_{n-1}P_n}$ is tangent to the Bézier curve at control point P_n.*

Proof. Referring to the proof above, let $t = 1$.

$\therefore \vec{r}'(1) = \frac{n}{b-a} \cdot \sum_{i=0}^{n-1} B_{i,n-1}(1) \cdot (P_{i+1} - P_i)$ and $B_{i,n-1}(1) = \frac{(n-1)!}{i! \cdot (n-i-1)!} \cdot (b-1)^{n-i-1} \cdot \frac{(1-a)^i}{(b-a)^{n-1}}$.

At $i = n-1$, $r'(1) = \frac{n}{b-a} \cdot B_{n-1,n-1}(1) \cdot (P_n - P_{n-1})$. $B_{n-1,n-1} = \frac{(1-a)^{n-1}}{(b-a)^{n-1}} = \frac{(1-a)}{(b-a)^{n-1}}$.

$\Rightarrow \vec{r}'(1) = \frac{n}{b-a} \cdot \frac{1-a}{(b-a)^{n-1}} \cdot (P_n - P_{n-1})$.

By inspection, $\frac{n}{b-a} \cdot \frac{1-a}{(b-a)^{n-1}} = m \in \mathbb{R}$. This results in $r'(1) = m \cdot (P_n - P_{n-1})$, concluding that the tangent vector at $t = 1$ is in the direction of $m \cdot (P_n - P_{n-1})$.

□

Proposition 2.1. *The Bézier curve always lies within the convex hull of its control points.*

Proof. All points on the Bézier curve have the form $\sum_{i=0}^{n} a_i b_i = a_0 b_0 + a_1 b_1 + ... + a_{n-1} b_{n-1} + a_n b_n$, where $a_i = B_{i,n}(t)$, for some t, and $b_i = P_i$.

$\therefore$ By partition of unity, $\sum_{i=0}^{n} a_i \equiv 1$.

$\therefore$ Utilizing the non-negativity property, $a_i \geq 0 \Rightarrow \sum_{i=0}^{n} a_i \geq 0$ on the interval $0 \leq B_{i,n}(t) \leq 1, \ \forall t \in [0, 1]$.

□

Remark 2.3. *The Bézier curve is a continuous, smooth curve. The curve has end-points* P_0,P_n *and its shape is determined by points* (P_1,P_2,...,P_{n-1}). *Changing the order of the control points will guarantee a change in the shape of curve. An example is shown below:*

Example 2.1. *Let n=3 with control points* (P_0, P_1, P_2, P_3).

Therefore, $\vec{r}(t) = \sum_{i=0}^{3} P_i B_{i,n}(t) = P_0 B_{0,n}(t) + P_1 B_{1,n}(t) + P_2 B_{2,n}(t) + P_3 B_{3,n}(t)$, *where* $B_{n,i} = \frac{n!}{i! \cdot (n-i)!} \cdot t^i \cdot t^{n-i}$ *so that:*

$$B_{0,3} = \binom{3}{0} \cdot t^0 \cdot (1-t)^3 = (1-t)^3$$
$$B_{1,3} = \binom{3}{1} \cdot t^1 \cdot (1-t)^2 = 3 \cdot t \cdot (1-t)^2$$
$$B_{2,3} = \binom{3}{2} \cdot t^2 \cdot (1-t)^1 = 3 \cdot t^2 \cdot (1-t)$$
$$B_{3,3} = \binom{3}{3} \cdot t^3 \cdot (1-t)^0 = t^3.$$

Let $\vec{r}_1(t)$ *be the parametric form of a Bézier curve with control points* (P_0, P_1, P_2, P_3) *and let* $\vec{r}_2(t)$ *be the parametric form of a Bézier curve with control points* (P_0, P_1, P_3, P_2).

$\therefore\ \vec{r}_1(t) = P_0 \cdot (1-t)^3 + 3P_1 \cdot t(1-t)^2 + 3P_2 \cdot t^2(1-t) + P_3 \cdot t^3$ & $\vec{r}_2(t) = P_0 \cdot (1-t)^3 + 3P_1 \cdot t(1-t)^2 + P_2 \cdot t^3 + 3P_3 \cdot t^2(1-t)$.

Let $(P_0, P_1, P_2, P_3) = ((2,0),(2,2),(5,2),(7,1))$ *and let* $t = 1$.

$\therefore\ \vec{r}_1(1) = (2,0) \cdot (0)^3 + 3(2,2) \cdot (0)^2 + (5,2)(0) + (7,1)$ & $\vec{r}_2(1) = (2,0) \cdot (0)^3 + 3(2,2) \cdot (0)^2 + (7,1) \cdot (0) + (5,2)$.

$\therefore\ \vec{r}_1(1) = (7,1)$ *and* $\vec{r}_2(1) = (5,2)$.

$\therefore\ \vec{r}_1(1)$ & $\vec{r}_2(1)$ *give different points* $(r_1 \neq r_2)$, *which conveys the fact that altering the order of the control points will ultimately alter the shape of the Bézier curve. To note, we have used De Casteljau's algorithm, which will be discussed throughout the study of the Bézier curve.*

Definition 2.4. *Invariance under Affine Transformations: Any given points located on the Bézier curve will remain on the curve under any affine transformation, which* *includes rotations, translations, shearing, bending & reflections. In other words, the curve will preserve its shape under any transformation* $(x, y) \mapsto ax + by + c$, *for any fixed* $a, b, c \in \mathbb{R}$.

Let A *be any given affine transformation,* $P_i = (x_i, y_i)$ *be any control point on the Bézier curve & let* $B_{i,n}(t)$ *be the Bernstein Polynomial.*

Then, one can mathematically represent invariance under the affine transformation A *as the equality below for a Bézier curve of degree n:*

$$A(\vec{r}(t)) = A(\sum_{i=0}^{n} P_i B_{i,n}(t)) = \sum_{i=0}^{n} A(P_i) B_{i,n}(t). \tag{2.3}$$

Proof. Let the affine transformation $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given as $A(x, y) = (a_1x + a_2y + a_3, b_1x + b_2y + b_3)$, with $a_i, b_i \in \mathbb{R}$, $i = 1, 2, 3$ and $(x, y) \in \mathbb{R}^2$.

Throughout this proof, one must use property three from Definition 2.2, which states:

$$\sum_{i=0}^{n} B_{i,n}(t) = 1. \tag{2.4}$$

Then,

$$\text{A}(\vec{r}(t)) = (a_1 \sum_{i=0}^{n} x_i B_{i,n}(t) + a_2 \sum_{i=0}^{n} y_i B_{i,n}(t) + a_3, b_1 \sum_{i=0}^{n} x_i B_{i,n}(t) + b_2 \sum_{i=0}^{n} y_i B_{i,n}(t) + b_3)$$

$= (a_1 \sum_{i=0}^{n} x_i B_{i,n}(t) + a_2 \sum_{i=0}^{n} y_i B_{i,n}(t) + a_3 \sum_{i=0}^{n} B_{i,n}(t), b_1 \sum_{i=0}^{n} x_i B_{i,n}(t) +$

$b_2 \sum_{i=0}^{n} y_i B_{i,n}(t) + b_3 \sum_{i=0}^{n} B_{i,n}(t)$

$= (\sum_{i=0}^{n}(a_1 x_i + a_2 y_i + a_3) B_{i,n}(t), \sum_{i=0}^{n}(b_1 x_i + b_2 y_i + b_3) B_{i,n}(t))$

$= \sum_{i=0}^{n}(a_1 x_i + a_2 y_i + a_3, b_1 x_i + b_2 y_i + b_3) B_{i,n}(t)$

$= \sum_{i=0}^{n} A(P_i) B_{i,n}(t).$

□

Remark 2.4. *Assume two Bézier curves of degree* $(n-1) \in \mathbb{N}$ *have control points* $(P_0, P_1, ..., P_{n-1}, P_n)$ & $(Q_0.Q_1, ..., Q_{n-1}, Q_n)$. *To join the two smooth and continuous Bézier curves, one must refer to property four from Definition 2.2 shown below:*

$$\frac{d}{dt}\vec{r}(t) = n \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} (P_{i+1} - P_i). \tag{2.5}$$

Utilizing property three from Definition 2.2 with a modification:

$$B_{i,n-1}(0) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

$$B_{i,n-1}(1) = \begin{cases} 1, & i = n-1 \\ 0, & i \neq n-1. \end{cases}$$

As well, one must refer to Lemma 2.1 to compute higher order derivatives and to obtain a general expression for $B_{i,n-k}(0)$ & $B_{i,n-k}(1)$, *where* $k \in \mathbb{Z}^+$ *is the order of the derivative of the Bézier curve. These expressions are shown below:*

$$B_{i,n-k}(0) = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$$B_{i,n-k}(1) = \begin{cases} 1 & i = n-k \\ 0 & i \neq n-k. \end{cases}$$

Let $\vec{r}_1(t) = \sum_{i=0}^{n} P_i B_{i,n}(t)$ *&* $\vec{r}_2(t) = \sum_{i=0}^{n} Q_i B_{i,n}(t)$*. Utilizing these position vectors, one must compute* $r_1^{(k)}(t)$ & $r_2^{(k)}(t)$ *at* $t = 0$ & $t = 1$ *and then set* $r_1^{(k)}(0) = r_2^{(k)}(1)$*. The derivative of orders* $k = 0, 1, 2, 3$ *are shown below:*

$k = 0 : \vec{r}_1(0) = \vec{r}_2(1) \Rightarrow \sum_{i=0}^{n} P_i B_{i,n}(0) = \sum_{i=0}^{n} Q_i B_{i,n}(1) \Rightarrow P_0 = Q_n,$

$k = 1 : \vec{r}\,'_1(0) = \vec{r}\,'_2(1) \Rightarrow \sum_{i=0}^{n-1} n(P_{i+1} - P_i) B_{i,n-1}(0) = \sum_{i=0}^{n-1} n(Q_{i+1} - Q_i) B_{i,n-1}(1) \Rightarrow P_1 - P_0 = Q_n - Q_{n-1}.$

Utilizing $k = 0$*:* $P_1 - Q_n = Q_n - Q_{n-1} \Rightarrow P_1 = 2Q_n - Q_{n-1},$

$k = 2 : \vec{r}\,''_1(0) = \vec{r}\,''_2(1) \Rightarrow \sum_{i=0}^{n-2} n(n-1)(P_{i+2} - 2P_{i+1} + P_i) B_{i,n-2}(0) = \sum_{i=0}^{n-2} n(n-1)(Q_{i+2} - 2Q_{i+1} + Q_i) B_{i,n-2}(1) \Rightarrow P_2 - 2P_1 + P_0 = Q_n - 2Q_{n-1} + Q_{n-2}.$

Utilizing $k = 1$ & $k = 0$*:* $P_2 - 2(2Q_n - Q_{n-1}) + Q_n = Q_n - 2Q_{n-1} + Q_{n-2} \Rightarrow P_2 = 4Q_n - 4Q_{n-1} + Q_{n-2} = 4(Q_n - Q_{n-1}) + Q_{n-2},$

$k = 3 : \vec{r}\,'''_1(0) = \vec{r}\,'''_2(1) \Rightarrow \sum_{i=0}^{n-3} n(n-1)(n-2)(P_{i+3} - 3(P_{i+2} - P_{i+1}) - P_0) B_{i,n-3}(0) = \sum_{i=0}^{n-3} n(n-1)(n-2)(Q_{i+3} - 3(Q_{i+2} - Q_{i+1}) - Q_0) B_{i,n-3}(1) \Rightarrow P_3 - 3(P_2 - P_1) - P_0 = Q_n - 3(Q_{n-1} - Q_{n-2}) - Q_{n-3}.$

Utilizing $k = 0,\ k = 1$ & $k = 2$*:* $P_3 - 3(4(Q_n - Q_{n-1}) - 2Q_n + Q_{n-1}) - Q_n = Q_n - 3(Q_{n-1} - Q_{n-2}) - Q_{n-3} \Rightarrow P_3 = 8Q_n - 12Q_{n-1} + 3Q_{n-2} - Q_{n-3} = 4(2Q_n - 3Q_{n-1}) + 3Q_{n-2} - Q_{n-3}$.

One can compute up derivatives to the n^{th} *order using this algorithm, but it will be a tedious process.*

Lemma 2.5. *Let a Bézier curve of degree* $n \in \mathbb{N}, n \geq 2$*, have* $(n+1)$ *control points* $\{P_0, P_1, ..., P_{n-1}, P_n\}$*. One may increase the degree of the Bézier curve to* $(n+1)$ *without altering the shape of the Bézier curve by determining a new set of control points* $\{R_0, R_1, ..., R_n, R_{n+1}\}$*. By maintaining an identical Bézier curve, one must set* $P_0 = R_0$ & $P_n = R_{n+1}$*. The set of control points* $\{R_1, ..., R_{n-1}, R_n\}$ *may be computed utilizing the recursive formula below for* $i \in [1, n]$:

$$R_i = \left(\frac{i}{n+1}\right) P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i. \tag{2.6}$$

By inspection, this formula represents the i^{th} *control point* R_i *as the linear combination of the control points* P_{i-1} & P_i *due to the fact that* R_i *has the following form for two points of a set S:*

$$\forall_{\lambda \in [0,1]} \wedge \forall_{x_1, x_2 \in S} \quad y = \lambda x_1 + (1 - \lambda) x_2. \tag{2.7}$$

As degree elevation corresponds to i data points, one can consider this representation as a piecewise-linear interpolation. This definition will be quite useful when discussing the Variation Diminishing Property, which is a significant property of the Bézier curve.

Remark 2.5. *The Bézier curve can be considered as a limit curve of the degree* *elevation of control polygons* $(\gamma_1, \gamma_2, ..., \gamma_{n-1}, \gamma_n)$. *This implies that the control polygons created by recursively performing the process of degree elevation to control points* $\{P_0, P_1, ..., P_{n-1}, P_n\}$ *converge uniformly to the initial Bézier curve, denoted* $\mathcal{B}$. *This convergence is stated below:*

$$\lim_{i \to \infty} \gamma_i = \mathcal{B}. \tag{2.8}$$

Remark 2.6. *To preserve the shape of the Bézier curve when the degree of the curve is increased by one, one must use the fact that* $(1-t)+t = 1$. *Then, multiplying the position vector* $\vec{r}(t)$ *by the factor* $(1-t)+t$ *will increase the degree of the curve by one. A short example is provided below:*

Example 2.2. *A Bézier curve of degree one and degree two are shown below:*

$$\vec{r}_1(t) = (1-t)P_0 + tP_1. \tag{2.9}$$

$$\vec{r}_2(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2. \tag{2.10}$$

Multiplying (2.9) *by the factor* $(1-t)+t$ *yields the equation below:*

$$(1-t)\vec{r}_1(t) + t\vec{r}_1(t) = (1-t)^2 P_0 + t(1-t)P_1 + t(1-t)P_0 + t^2 P_1. \tag{2.11}$$

Then, grouping (2.11) *in terms of* (2.10) *will yield:*

$$(1-t)\vec{r}_1(t) + t\vec{r}_1(t) = (1-t)^2 P_0 + t(1-t)(P_0 + P_1) + t^2 P_1.$$

Then,

$$(1-t)\vec{r}_1(t) + t\vec{r}_1(t) = (1-t)^2 P_0 + 2t(1-t)(\frac{P_0}{2} + \frac{P_1}{2}) + t^2 P_1.$$

Therefore, $P_0 = R_0$, $\frac{P_0}{2} + \frac{P_1}{2} = R_1$ *and* $P_1 = R_2$. *This is identical to using the* *recursive formula* (2.6) *for the values* $i = 1, 2$ *and* $n = 1$.

It can be observed that Bézier curves can be represented as a combination of their control points as well as functions solely dependent on t, denoted $B_{i,n}(t)$. Though, there are other forms of the curve that can be discussed. Obviously, the degree of $B_{i,n}(t)$ depends on the degree of the Bézier curve, which allows for complicated polynomials of higher degrees. One can write these polynomials $B_{i,n}(t)$ as symmetric polynomials, in which $B_{i,n}(t)$ is a function of many variables $B_{i,n}(t_1, t_2, ..., t_n)$ with a maximum degree of one (linearity). Let us view several properties of this notion below, denoted as the "polar form" or "blossoming" of polynomial, which deals with homogenizing polynomials in the following definition.

Definition 2.5. *Let* $P(t)$ *be our polynomial of degree* n *in which* $P(t) = c_0 + c_1 t + c_2 t^2 + ... + c_n t^n, n \in \mathbb{N}$ & $c_i \in \mathbb{R}$. *Then, one can convert* $P(t)$ *into the blossoming polynomial* $p(t_1, t_2, ..., t_n)$ *with the following properties:*

a. ***Symmetry**: Any permutation of* $p(t_1,t_2,...,t_n)$ *does not alter the polynomial. For example,* $p(t_1,t_2)=t_1t_2=t_2t_1$ *or* $p(t_1,t_2,t_3)=t_1t_2t_3=t_2t_1t_3=t_3t_1t_2=t_3t_2t_1$.

b. ***Diagonal**: Let* $t_1=t_2=...=t_n=t$, $n \in \mathbb{N}$. $\therefore p(t, t, ..., t) = P(t)$.

c. ***Multi-affine**: Let* $p(\vec{t}_N) := p(t_1, t_2, ..., \lambda t_s + (1-\lambda)t_v, \dots, t_n)$, *for any* s, v *such that* $1 \leq s, v \leq n$ *and for any* $\lambda \in [0, 1]$.
$\therefore p(\vec{t}_N) = p(t_1, t_2, ..., \lambda t_s, ..., t_n) + p(t_1, t_2, ..., (1-\lambda)t_v, ..., t_n) = \lambda p(t_1, t_2, ..., t_s, ..., t_n) + (1-\lambda)p(t_1, t_2, ..., t_v, ..., t_n)$.
The polynomial p *in the variables* $(t_1, \dots, t_n)$ *is called the blossom of the polynomial* $P(t)$.

Remark 2.7. *Let* $P(t) = c_0 + c_1 t + c_2 t^2 + ... + c_k t^k + ... + c_n t^n$ *be a polynomial of degree* $n \in \mathbb{N}$ *such that* $k \in \mathbb{Z}^+ \leq n$. *To compute the blossom* $p(t_1, ..., t_n)$ *of each* *monomial of $P(t)$, which will be denoted $m_{\{k,n\}}(t)$, one can utilize the formula below:*

$$m_{\{k,n\}}(t_1, ..., t_n) = p(t_1, ..., t_n) = \binom{n}{k}^{-1} \sum_{j \subset \{1,...,n\}} \prod_{j=\{i_1,...,i_k\}} t_j. \qquad (2.12)$$

Example 2.3. *Let $\{n, k\} = \{3, 2\}$. In other words, this is identical to $P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ and selecting the monomial $m_{\{3,2\}}(t) = t^2$. Utilizing the formula above to blossom a polynomial of degree $k \leq n$, one obtains the computations below:*

$$m_{\{3,2\}}(t_1, t_2, t_3) = p(t_1, t_2, t_3) = \binom{3}{2}^{-1} \sum_{j \subset \{1,2,3\}} \prod_{j=\{i_1,i_2\}} t_j,$$

$$m_{\{3,2\}}(t_1, t_2, t_3) = p(t_1, t_2, t_3) = \frac{1}{3} \sum_{j \subset \{1,2,3\}} t_{i_1} t_{i_2},$$

$$m_{\{3,2\}}(t_1, t_2, t_3) = p(t_1, t_2, t_3) = \frac{1}{3}(t_1 t_2 + t_1 t_3 + t_2 t_3).$$

One can show that this is the correct blossom of t^2 for $P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ by utilizing the properties of Definition 2.5, shown below:

Symmetry: Any permutation of the blossom $p(t_1, t_2, t_3)$ will not alter the expression $\frac{1}{3}(t_1 t_2 + t_1 t_3 + t_2 t_3)$. This can be immediately observed by the reader.

Diagonal: Let $t_1 = t_2 = t_3 = t$. Therefore $p(t, t, t) = \frac{1}{3}(t \cdot t + t \cdot t + t \cdot t) = t^2$.

Multi-affine: Let $\lambda \in [0,1]$ and $u, v \in \mathbb{R}$ be two arbitrary variables. Therefore, one must show that $p(\lambda u + (1-\lambda)v, t_2, t_3) = \lambda p(u, t_2, t_3) + (1-\lambda)p(v, t_2, t_3)$. The brief proof of the multi-affine property of the blossom is provided below:

$$p(\lambda u + (1-\lambda)v, t_2, t_3) = \frac{(\lambda u+(1-\lambda)v)t_2+(\lambda u+(1-\lambda)v)t_3+\lambda(t_2t_3)+(1-\lambda)(t_2t_3)}{3}$$

$$p(\lambda u + (1-\lambda)v, t_2, t_3) = \frac{\lambda(ut_2+ut_3+t_2t_3)+(1-\lambda)(vt_2+vt_3+t_2t_3)}{3}$$

$$p(\lambda u + (1-\lambda)v, t_2, t_3) = \lambda\frac{(ut_2+ut_3+t_2t_3)}{3} + (1-\lambda)\frac{(vt_2+vt_3+t_2t_3)}{3}$$

$$p(\lambda u + (1-\lambda)v, t_2, t_3) = \lambda p(u, t_2, t_3) + (1-\lambda)p(v, t_2, t_3).$$

Proposition 2.2. *[2] [Subdivision] Let a Bézier curve of degree n defined on $t \in [a,b]$ be given. Then, one can represent its control points $\{P_0, P_1, \ldots, P_n\}$ as the values of the blossomed polynomials $p(t_1, t_2, ..., t_{j-1}, t_j, t_{j+1}, ..., t_{n-1}, t_n)$ at specific points as indicated by the i-functions below for $i = 0, 1, 2, \ldots, n$:*

$$P_i(j) = \begin{cases} a, & \text{if } (j+i) \leq n \\ b, & \text{if } (j+i) > n. \end{cases}$$

In other words, for an n^{th} degree Bézier curve, the control points P_i can be given by:

$P_0 = p(a,a,a,...,a,a)$ $P_1 = p(a,a,a,...,a,b)$ $P_2 = p(a,a,a,...,a,b,b)$ $P_3 = p(a,a,a,...,a,b,b,b)$

$$\ldots$$

$P_{n-3} = p(a,a,a,b,...,b,b)$ $P_{n-2} = p(a,a,b,,...,b,b)$ $P_{n-1} = p(a,b,b,...,b,b,b)$ $P_n = p(b,b,b,...,b,b)$.

Example 2.4. *A brief example of applying Proposition 2.2 is by selecting any closed compact interval* $[a, b]$*. Let* $a = 1$*,* $b = 2$ *and let the Bézier curve be of degree four* $(n + 1 = 4 \Rightarrow n = 3)$*. Then, the control points will have the form below:*

$$P_0 = (1, 1, 1) \qquad P_1 = (1, 1, 2) \qquad P_2 = (1, 2, 2) \qquad P_3 = (2, 2, 2).$$

Remark 2.8. *Bézier curves are defined on a closed interval* $t \in [a, b]$*. We have gradually commenced our discussion of De Casteljau's algorithm by showing an example on an interval* $t \in [a, b]$*. For example, throughout integral calculus, it was observed that an integral of a function* $f(x)$ *over the domain* $a \leq t \leq b$ *can be separated into the addition of two integrals of* $f(x)$ *over the domains* $a \leq t \leq c$ *and* $c \leq t \leq b$*,* $c \in \mathbb{R}$*. Prior to introducing an example on Bézier subdivision, introducing some prerequisites and the notion of affine combinations is mandatory.*

Utilizing our endpoints $a \in \mathbb{R}$ *&* $b \in \mathbb{R}$ *as well as* $c \in \mathbb{R}$*, we may use the following formula to compute the blossom* $p(t_1, t_2, ..., t_{n-1}, c) \in \mathbb{R}$ *:*

$$P_c(t) = p(t_1, t_2, ..., t_{n-1}, c) = \frac{(b-c)p(t_1, t_2, ..., t_{n-1}, a) + (c-a)p(t_1, t_2, ..., t_{n-1}, b)}{b-a}. \tag{2.13}$$

Definition 2.6. *Affine combinations of polynomials are linear combinations of polynomials* p_i *such that its coefficients sum up to one:*

$$p = \sum_{i=1}^{n} \alpha_i p_i = \alpha_1 p_1 + \alpha_2 p_2 + ... + \alpha_{n-1} p_{n-1} + \alpha_n p_n$$

such that

$$\sum_{i=1}^{n} \alpha_i = 1.$$

Remark 2.9. *With regards to Definition 2.6,* $\forall \alpha_i \in [0, 1]$*, p is a convex combination of the points* $\{p_1, p_2, ..., p_{n-1}, p_n\}$*.*

Example 2.5. *Let a Bézier curve of degree four* $(n = 4)$ *take on the values* $a = 0$ & $b = 3$ *such that* $t \in [0, 3]$ *with the following "polar" control points:*

$$p(0, 0, 0, 0),\ p(0, 0, 0, 3),\ p(0, 0, 3, 3),\ p(0, 3, 3, 3)\ \&\ p(3, 3, 3, 3).$$

As previously mentioned, these blossoms were obtained by utilizing the piece-wise form stated in Proposition 2.2.

The Bézier subdivision problem will be represented as:

- $p(0, 0, 0, 0)$, $p(0, 0, 0, t)$, $p(0, 0, t, t)$, $p(0, t, t, t)$ & $p(t, t, t, t)$ *for* $t \geq 0$, $a = 0$ & $c = t$.

- $p(t, t, t, t)$, $p(t, t, t, 3)$, $p(t, t, 3, 3)$, $p(t, 3, 3, 3)$ & $p(3, 3, 3, 3)$ *for* $t \leq 3$, $c = t$ & $b = 3$.

Utilizing the three axioms of blossoming polynomials, an affine combination of control points with the domain $0 \leq t \leq 3$ must be represented as a linear combination with coefficients $(1-\frac{c-a}{b-a}) = (1-\frac{t}{3})$ & $\frac{c-a}{b-a} = \frac{t}{3}$. Note that the goal is to obtain the control point $p(t,t,t,t) = P(t)$.

A picture of this situation is provided below:

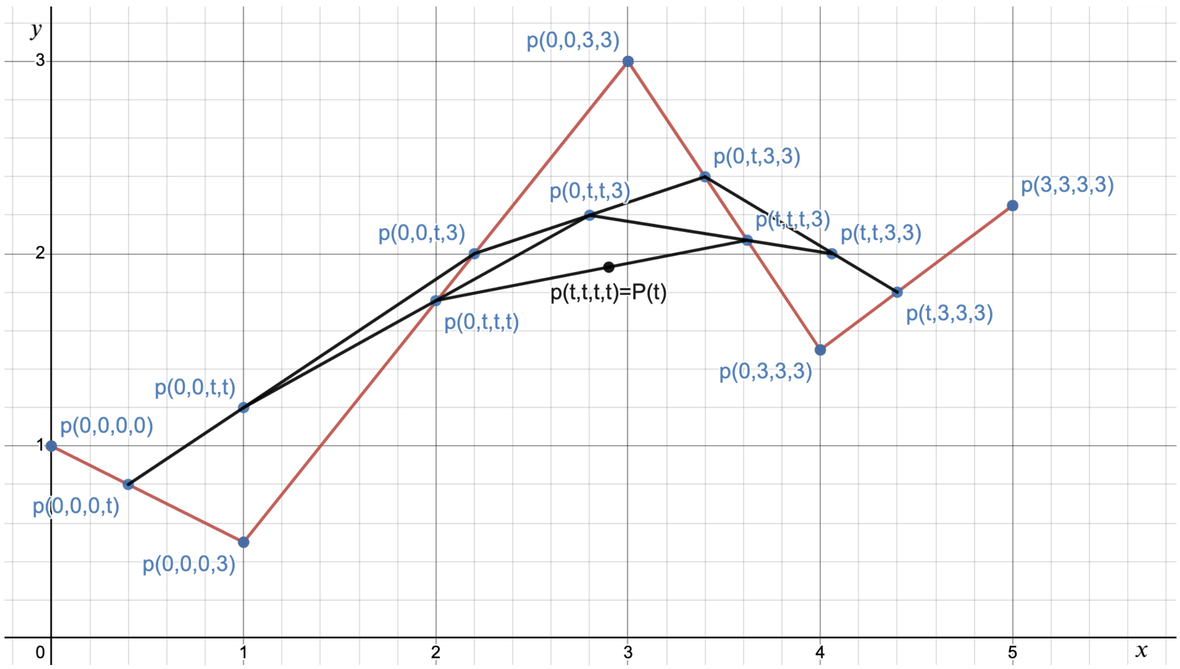

Figure 2.4: Subdivision problem of a Bézier curve of degree $n = 4$, $t \in [0,3]$.

The combinations of Figure 2.4 are shown below:

- $(1-\frac{t}{3})p(0,0,0,0)+\frac{t}{3}p(0,0,0,3) = p(0,0,0,0)+p(0,0,0,t) = p(0,0,0,t)$

- $(1-\frac{t}{3})p(0,0,0,3)+\frac{t}{3}p(0,0,3,3) = p(0,0,0,3-t)+p(0,0,t,t) = p(0,0,t,3)$

- $(1-\frac{t}{3})p(0,0,3,3)+\frac{t}{3}p(0,3,3,3) = p(0,0,3-t,3-t)+p(0,t,t,t) = p(0,t,3,3)$

- $(1-\frac{t}{3})p(0,3,3,3)+\frac{t}{3}p(3,3,3,3)=p(0,3-t,3-t,3-t)+p(t,t,t,t)=$ $p(t,3,3,3)$.

Now, the affine combinations of the control points above with coefficients $(1-\frac{t}{3})$ & $\frac{t}{3}$ *will be:*

- $(1-\frac{t}{3})p(0,0,0,t)+\frac{t}{3}p(0,0,t,3)=p(0,0,0,t-\frac{t^2}{3})+p(0,0,\frac{t^2}{3},t)=$ $p(0,0,0,t-\frac{t^2}{3})+p(0,0,t,\frac{t^2}{3})=p(0,0,t,t)$

- $(1-\frac{t}{3})p(0,0,t,3)+\frac{t}{3}p(0,t,3,3)=p(0,0,t-\frac{t^2}{3},3-t)+p(0,\frac{t^2}{3},t,t)=$ $p(0,0,t-\frac{t^2}{3},3-t)+p(0,t,\frac{t^2}{3},t)=p(0,t,t,3)$

- $(1-\frac{t}{3})p(0,t,3,3)+\frac{t}{3}p(t,3,3,3)=p(0,t-\frac{t^2}{3},3-t,3-t)+p(\frac{t^2}{3},t,t,t)=$ $p(0,t-\frac{t^2}{3},3-t,3-t)+p(\frac{t^2}{3},t,t,t)=p(t,t,3,3)$.

Continuing the algorithm, two control points will be computed with respect to polar value t*:*

- $(1-\frac{t}{3})p(0,0,t,t)+\frac{t}{3}p(0,t,t,3)=p(0,0,t-\frac{t^2}{3},t-\frac{t^2}{3})+p(0,\frac{t^2}{3},\frac{t^2}{3},t)=$ $p(0,0,t-\frac{t^2}{3},t-\frac{t^2}{3})+p(0,t,\frac{t^2}{3},\frac{t^2}{3})=p(0,t,t,t)$

- $(1-\frac{t}{3})p(0,t,t,3)+\frac{t}{3}p(t,t,3,3)=p(0,t-\frac{t^2}{3},t-\frac{t^2}{3},3-t)+p(\frac{t^2}{3},\frac{t^2}{3},t,t)=$ $p(0,t-\frac{t^2}{3},t-\frac{t^2}{3},3-t)+p(t,\frac{t^2}{3},\frac{t^2}{3},t)=p(t,t,t,3)$.

Finally, one control point will be obtained:

- $(1-\frac{t}{3})p(0,t,t,t)+\frac{t}{3}p(t,t,t,3) = p(0,t-\frac{t^2}{3},t-\frac{t^2}{3},t-\frac{t^2}{3})+p(\frac{t^2}{3},\frac{t^2}{3},\frac{t^2}{3},t) = p(0,t-\frac{t^2}{3},t-\frac{t^2}{3},t-\frac{t^2}{3})+p(t,\frac{t^2}{3},\frac{t^2}{3},\frac{t^2}{3}) = p(t,t,t,t) = P(t).$

Lemma 2.6. *Recall from Definition 2.5 that* $P(t) = c_0 + c_1 t + c_2 t^2 + ... + c_k t^k + ... + c_n t^n,\ n \in \mathbb{N},\ k \in \mathbb{Z}^+,\ k \leq n\ \&\ c_i \in \mathbb{R}$.

Recall the Taylor series of an infinitely differentiable real function $f(x)$ is formally the infinite sum at a point of the interior of the domain denoted x_0 in terms of derivatives of $f(x)$ at x_0 as follows:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)(x-x_0)^2}{2}+\frac{f'''(x_0)(x-x_0)^3}{6}$$

$$+...+\sum_{k}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}.$$

The Maclaurin series is simply the Taylor series at the particular point $x_0 = 0$. This formula is stated below:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}0)}{k!}x^k = f(0) + xf'(0) + \frac{x^2 f''(0)}{2} + \frac{x^3 f'''(0)}{6} + ... + \sum_{k}^{\infty} \frac{x^k f^{(k)}(0)}{k!}.$$

Utilizing the Maclaurin theorem in terms of the Maclaurin series above, we have:

$$P(t) = P(0)+tP'(0)+\frac{t^2 P''(0)}{2}+...+\frac{t^k P^{(k)}(0)}{k!}+\frac{t^{k+1}P^{(k+1)}(0)}{(k+1)!}+...+\frac{t^{n-1}P^{(n-1)}(0)}{(n-1)!}+\frac{t^n P^{(n)}(0)}{n!}.$$

Comparing the Maclaurin theorem and the $P(t)$ from Definition 2.5, one observes that $c_k = \frac{P^{(k)}(0)}{k!}$.

As well, one may write c_k in terms of the blossom $p(0,0,...,0,\gamma,\gamma,...,\gamma)$, where $\gamma = (1,0)$ is a vector & $p(0,0,...,0,\gamma,\gamma,...,\gamma) = \frac{c_k}{\binom{n}{k}}$. Note that there are $(n-k)$ zero terms & k γ-terms.

Remark 2.10. *The dual function $P_k = p(0,0,..,0,1,1,,..,1)$, where P_k contains $(n-k)$ zeros and k ones. This dual function can also be denoted as the k^{th} control point of the Bézier curve.*

Lemma 2.7. *The polar form of the polynomial $P(t+h)$ can be represented as the formula below for $t \in \mathbb{R}$, $n \in \mathbb{N}$, $(n-k)$ t-terms, k 1-terms & n $(t+h)$-terms:*

$$P(t+h) = p(t+h,...,t+h) = \sum_{k=0}^{n} \binom{n}{k} p(t,...,t,1,...,1)h^k.$$

A proof by utilizing the properties in Definition 2.5 will convey that the formula above holds for $\forall n \in \mathbb{N}$.

Proof. Using the properties from Definition 2.5, one can identify a pattern as one continuously decomposes the blossom $p(t+h,...,t+h)$ n times. Several decompositions are shown below:

First decomposition:

$$p(t+h,...,t+h) = p(t,t+h,...,t+h) + p(h,t+h,...,t+h)$$

$$= p(t,t,t+h,...,t+h) + p(t,h,t+h,...,t+h)$$

$$+p(h,t,t+h,...,t+h)+p(h,h,t+h,...,t+h)$$

$$=p(t,t,t+h,...,t+h)+2p(h,t,t+h,...,t+h)$$

$$+p(h,h,t+h,...,t+h)$$

$$=\binom{2}{0}p(t,t,t+h,...,t+h)+\binom{2}{1}p(h,t,t+h,...,t+h)$$

$$+\binom{2}{2}p(h,h,t+h,...,t+h).$$

Second decomposition:

$$p(t+h,...,t+h)=p(t,t,t,t+h,...,t+h)+p(t,t,h,t+h...,t+h)$$

$$+2[p(h,t,t,t+h,...,t+h)+p(h,t,h,t+h,...,t+h)]$$

$$+p(h,h,t,t+h,...,t+h)+p(h,h,h,t+h,...,t+h)$$

$$=p(t,t,t,t+h,...,t+h)+3p(t,t,h,t+h,...,t+h)$$

$$+3p(h,h,t,t+h,...,t+h)+p(h,h,h,t+h,...,t+h)$$

$$=\binom{3}{0}p(t,t,t,t+h,...,t+h)+\binom{3}{1}p(t,t,h,t+h,...,t+h)$$

$$+\binom{3}{2}p(h,h,t,t+h,...,t+h)+\binom{3}{3}p(h,h,h,t+h,...,t+h).$$

Third decomposition:

$$p(t+h,...,t+h) = p(t,t,t,t,t+h,...,t+h) + p(t,t,t,h,t+h...,t+h)$$

$$+3[p(t,t,h,t,t+h,...,t+h) + p(t,t,h,h,t+h,...,t+h)]$$

$$+3[p(h,h,t,t,t+h,...,t+h) + p(h,h,t,h,t+h,...,t+h)]$$

$$+p(h,h,h,t,t+h,...,t+h) + p(h,h,h,h,t+h,...,t+h)$$

$$= p(t,t,t,t,t+h,...,t+h) + 4p(t,t,t,h,t+h...,t+h)$$

$$+6p(t,t,h,h,t+h,...,t+h) + 4p(h,h,h,t,t+h,...,t+h)$$

$$+p(h,h,h,h,t+h,...,t+h)$$

$$= \binom{4}{0}p(t,t,t,t,t+h,...,t+h) + \binom{4}{1}p(t,t,t,h,t+h...,t+h)$$

$$+\binom{4}{2}p(t,t,h,h,t+h,...,t+h) + \binom{4}{3}p(h,h,h,t,t+h,...,t+h)$$

$$+\binom{4}{4}p(h,h,h,h,t+h,...,t+h).$$

One can decompose the blossom n times, but that will be a tedious process. A pattern can be established, which is shown below for $m \in \mathbb{Z}^+$, $(m-k)$ t-terms, k h-terms and $(n-m)$ $(t+h)$-terms:

$$\sum_{k=0}^{m}\binom{m}{k}p(t,...,t,h,...,h,t+h,...,t+h)=\sum_{k=0}^{m}\binom{m}{k}p(t,...,t,1,...,1,t+h,...,t+h)h^k.$$

Setting $m=n$, one immediately obtains the formula in Lemma 2.7 which is provided below for $(n-k)$ t-terms, k h-terms and zero $(t+h)$-terms:

$$P(t+h)=\sum_{k=0}^{n}\binom{n}{k}p(t,...,t,h,...,h)=\sum_{k=0}^{n}\binom{n}{k}p(t,...,t,1,...,1)h^k.$$

□

Remark 2.11. *The k^{th} derivative of the polynomial $P(t)$ is computed as*

$P^{(k)}(t)=k!\binom{n}{k}p(t,t,...,t,1,1,...,1)=\frac{n!}{(n-k)!}p(t,t,...,t,1,1,...,1)$, *where there are $(n-k)$ t-terms & k 1-terms.*

Proof. A formula that will be particularly useful throughout this proof is stated below:

$$P(t+h)=\sum_{k=0}^{n}\frac{P^{(k)}(t)}{k!}h^k,\ t\in[0,1]\ \&\ h\in\mathbb{R}\ is\ arbitrary\ infinitesimal. \tag{2.14}$$

To briefly show that the formula above holds, one must utilize equation (2.14) in terms of the polynomial $P(x)$ of degree n. This formula is shown below:

$$P(x)=\sum_{k=0}^{n}\frac{P^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Let $x_0=t$ and $x=t+h$. Then, one obtains the desired formula below:

$$P(t+h) = \sum_{k=0}^{n} \frac{P^{(k)}(t)}{k!}(t+h-t)^k = \sum_{k=0}^{n} \frac{P^{(k)}(t)}{k!}h^k. \tag{2.15}$$

Utilizing property two of Definition 2.5, one obtains below:

$P(t+h) = p(t+h, t+h, ..., t+h)$, where $p(t+h)$ is the polar form of $P(t+h)$ with n-elements.

Another formula is provided below for $(n-k)$ t-terms & k 1-terms:

$$P(t+h) = \sum_{k=0}^{n} \binom{n}{k} p(t, ..., t, 1, ..., 1)h^k. \tag{2.16}$$

The formula above greatly resembles the binomial expansion of $(t+1\cdot h)^n$, which is provided below:

$$\textstyle\sum_{k=0}^{n} \binom{n}{k} t^{n-k} 1^k h^k = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} h^k.$$

Comparing the equations (2.15) and (2.16), one obtains the following equalities:

$P(t)+hP'(t)+h^2\frac{P''(t)}{2!}+...+h^n\frac{P^n(t)}{n!} = \binom{n}{0}p(t, .., t)+\binom{n}{1}p(t, ..., t, 1)h+\binom{n}{2}p(t, ..., t, 1, 1)+ ... + \binom{n}{n}p(1, ..., 1)h^n,$

$$\begin{cases} P(t) = 0!\binom{n}{0}p(t,...,t) \\ P'(t) = 1!\binom{n}{1}np(t,...,t,1) \\ P''(t) = 2!\binom{n}{2}p(t,...,t,1,1) \\ \qquad\qquad \vdots \\ P^{(n-1)}(t) = (n-1)!\binom{n}{n-1}p(t,1,...,1) \\ P^{(n)}(t) = n!\binom{n}{n}p(1,...,1). \end{cases}$$

One can notice that a pattern determined from the equalities above and the k^{th} derivative for $k \geq 0$. The k^{th} derivative of the blossom $P(t)$ is provided below for $(n-k)$ t-terms and k 1-terms:

$$P^{(k)}(t) = k!\binom{n}{k}p(t,...,t,1,...,1).$$

□

Remark 2.12. Variation Diminishing Property: *Consider a discrete set of control points* $\{P_0, P_1, ..., P_n\}$*. Denote their convex hull by* γ *and the Bézier curve associated to them by* $\mathcal{B}$*. Let* $\mathcal{L}$ *denote a line which may intersect* $\mathcal{B}$ *at some point(s). Bézier curves have the very useful Variation Diminishing Property which says that* $\mathcal{L}$ *intersects* $\mathcal{B}$ *at most as many times as* $\mathcal{L}$ *intersects* γ*.*

The Variation Diminishing property shows that any Bézier curve of any degree does not fluctuate very much while it is smoother than its control polygon. Furthermore, the idea of proving that Bézier curves have the Variation Diminishing Property is to show that the recursive equation (2.6) implies the point-wise limit (2.8). However, the proof is difficult and its difficulty is too elevated for this text.

Remark 2.13. *The discrete set of control points may not always lie on the Bézier curve.*

Definition 2.7. *Arc-length: The distance between two points on any given curve* $\mathcal{C} \in \mathbb{R}^n$. *A formula of the arc-length function will be provided below for a parametrization of* $\mathcal{C}$ *such that* $\vec{r}(t) = (x_1(t), x_2(t), ..., x_n(t))$ *with once differentiable components, the starting value of* $t = t_0 = a$ *lying on* $\mathcal{C}$ & $t \in [a, b]$ *is an arbitrary variable:*

$$S(t) = \int_a^t \|\vec{r}'(u)\| \, du = \int_a^t \sqrt{(\frac{dx_1}{du})^2 + (\frac{dx_2}{du})^2 + ... + (\frac{dx_n}{du})^2} \, du. \tag{2.17}$$

Example 2.6. *Providing a simple example of the utilization of the arc-length function* (2.17)*, one can derive the arc-length of a semi-circle of radius* ℓ *and centered at the origin below:*

The equation of a semi-circle centered at the origin is $x^2+y^2 = \ell^2$ *for* $x_1(t) = x(t)$, $x_2(t) = y(t)$ & $t \in [0, \pi]$.

In polar coordinates, one can represent this equation as $\vec{r}(t) = (\ell cos(t), \ell sin(t)),\ t \in [0, \pi]$ *below:*

$$\therefore \vec{r}'(t) = (-\ell sin(t), \ell cos(t)).$$

The norm of $\vec{r}'(t)$ *above is calculated below:*

$$\begin{aligned} \|\vec{r}'(t)\| &= \sqrt{(-\ell sin(t))^2 + (\ell cos(t))^2} \\ &= \sqrt{(\ell^2(sin(t))^2 + \ell^2(cos(t))^2} \end{aligned}$$

$$= \sqrt{\ell^2((sin(t))^2 + (cos(t))^2)}$$
$$= \sqrt{\ell^2}$$
$$= \ell.$$

Then, $S(t) = \int_0^t \ell \, du = u\ell\Big|_0^t = t\ell - (0)\ell = t\ell.$

Furthermore, the length of the semi-circle is $S(b) = S(\pi) = \pi\ell.$

Definition 2.8. *Curvature: Let* $\vec{r} : I \longrightarrow \mathbb{R}^n$ *be a twice differentiable regular curve parametrized by arc-length (i.e.* $\|\vec{r}\,'(s)\| = 1$*) in* $\mathbb{R}^3$*. The curvature of this curve at a point, denoted* $\kappa(s)$*, describes the rate of change in which the curve changes direction at the point* $\vec{r}(s)$*. Additionally, one can describe this measurement as the rate at which the velocity vector is turning.*

For a parametrization by arclength of the position vector with twice differentiable components, one can derive several key formulae which are listed below:

$$Unit\ Tangent\ Vector = \vec{T}(s) = \vec{r}\,'(s),$$

$$Unit\ Normal\ Vector = \vec{N}(s) = \frac{\vec{T}'(s)}{\|\vec{T}'(s)\|},$$

$$Curvature = \kappa(s) = \|\frac{d\vec{T}(s)}{ds}\|.$$

Let $\vec{r} : J \longrightarrow \mathbb{R}^n$ *be a twice differentiable regular curve in* $\mathbb{R}^n$ *(i.e.* $\|\vec{r}\,'(t)\| \neq 0$*) not necessarily parametrized by arc-lengh. As previously defined, let curvature of this curve be denoted* $\kappa(t)$*.*

For a parametrization of the position vector with twice differentiable components, one can derive several key formulae which are listed below:

$$\textit{Unit Tangent Vector} = \vec{T}(t) = \tfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|},\ \vec{r}(t)\ \textit{differentiable for}\ t \in [0,1]\ \&\ \vec{r}'(t) \neq \vec{0},$$

$$\textit{Unit Normal Vector} = \vec{N}(t) = \tfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|},\ \vec{T}(t)\ \textit{differentiable for}\ t \in [0,1]\ \&\ \vec{T}'(t) \neq \vec{0},$$

$$\textit{Curvature} = \kappa(t) = \|\frac{d\vec{T}(t)}{dt}\| \cdot \frac{1}{\|\vec{r}'(t)\|}.$$

Remark 2.14. *The curvature, denoted κ, can also be represented as the formula below for a regular and twice differentiable curve parameterized by $\vec{r}(t) \subset \mathbb{R}^n$ where $n = 2, 3$:*

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}. \tag{2.18}$$

Definition 2.9. *The curvature κ can also be represented in terms of the unit tangent vector $\vec{T}(t)$. The curvature of a regular curve is the l^2 norm of the second derivative of the parametrization $\vec{r}(t)$ from Definition 2.7 and an arc-length parameter denoted s, which is shown below for $s = s(t)$:*

$$\kappa(s) = \|\tfrac{d^2\vec{r}(s(t))}{ds^2}\|.$$

A simple proof will show that this is equivalent to the formula below:

$$\kappa(s) = \|\tfrac{d\vec{T}(s)}{ds}\|.$$

Proof. By the Fundamental Theorem of Calculus and the Arc-Length function in Definition 2.7, $\frac{d}{dt}S(t) = \|\vec{r}'(t)\| > 0$.

Utilizing the chain rule, $\frac{d}{ds}(\frac{d\vec{r}(s(t))}{ds}) = \frac{d}{ds}(\frac{d\vec{r}(s(t))}{dt} \cdot \frac{dt}{ds}) = \frac{d}{ds}(\frac{d\vec{r}(s(t))}{dt} \cdot (\frac{ds}{dt})^{-1})$.

Utilizing $\frac{d}{dt}S(t)$, the above expressions equate to $\frac{d}{ds}(\frac{d\vec{r}(s(t))}{dt} \cdot \frac{1}{\|\vec{r}'(s(t))\|}) = \frac{d}{ds}(\frac{\vec{r}'(s(t))}{\|\vec{r}'(s(t))\|}) = \frac{d}{ds}\vec{T}(s(t))$.

Then,

$$\kappa(s) = \|\frac{d^2\vec{r}(s)}{ds^2}\| = \|\frac{d\vec{T}(s)}{ds}\|.$$

□

Lemma 2.8. *$\vec{T}'(s)$ and $\vec{T}(s)$ are orthogonal vectors parameterized by the arc-length parameter s, in other words, $\langle\vec{T}'(s) , \vec{T}(s)\rangle = 0$, implies that $\|\vec{T}'(s)\times\vec{T}(s)\| = \|\vec{T}'(s)\|$ for $s = S(t)$.*

Proof. A quick proof of the left-hand side is immediate, which is shown below:

$\|\vec{T}(s)\| = 1$ as $\vec{T}(s)$ is a unitary vector $\iff$ $\|\vec{T}(s)\|^2 = \langle\vec{T}(s) , \vec{T}(s)\rangle = 1$,

$$\begin{aligned}\frac{d}{dt}\|\vec{T}(S(t))\|^2 &= \frac{d}{dt}\langle\vec{T}(S(t)) , \vec{T}(S(t))\rangle \\ &= \langle\vec{T}'(S(t)) , \vec{T}(S(t))\rangle + \langle\vec{T}(S(t)) , \vec{T}'(S(t))\rangle \\ &= \langle\vec{T}'(S(t)) , \vec{T}(S(t))\rangle + \langle\vec{T}'(S(t)) , \vec{T}(S(t))\rangle,\end{aligned}$$

$\frac{d}{dt}\|\vec{T}(S(t))\|^2 = \langle 2\vec{T}'(S(t)) , \vec{T}(S(t))\rangle = 2\langle\vec{T}'(S(t)) , \vec{T}(S(t))\rangle = \frac{d}{dt}(1) = 0$,

$\therefore \langle\vec{T}'(S(t)) , \vec{T}(S(t))\rangle = 0 \implies \vec{T}'(S(t))$ & $\vec{T}(S(t))$ are orthogonal vectors $\implies$

The angle between these two vectors equals $\theta = \frac{\pi}{2}$ rad.

For the right-hand side of the lemma, one must utilize the trigonometric formula involving the cross product between two vectors for $\theta \in [0, 2\pi]$, shown below:

$$\|\vec{T}'(S(t)) \times \vec{T}(S(t))\| = \|\vec{T}'(S(t))\| \cdot \|\vec{T}(S(t))\| \cdot \sin(\theta).$$

For $\theta = \frac{\pi}{2}$, the proof is finished and the formula above reduces to the equality below:

$$\|\vec{T}'(S(t)) \times \vec{T}(S(t))\| = \|\vec{T}'(S(t))\| \cdot \|\vec{T}(S(t))\| = \|\vec{T}'(S(t))\|.$$

□

Remark 2.15. *Proof of the formula* κ *in Remark 2.14.*

Proof. Utilizing the formula from Definition 2.9 and setting $S'(t) = \|\vec{r}\,'(t)\|$, one obtains the formula below:

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}\,'(t)\|}.$$

One must also utilize the formula of the unit tangent vector not in terms of arc-length from Definition 2.8 to commence the proof, also shown below:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}.$$

The formula above can be written as $\vec{r}\,'(t) = \vec{T}(t) \cdot \|\vec{r}\,'(t)\| \implies \vec{r}\,'(t) = \vec{T}(t) \cdot S'(t)$.

$\therefore$ Differentiating the above formula by the product rule yields the equality below:

$$\vec{r}''(t) = \vec{T}'(t) \cdot S'(t) + \vec{T}(t) \cdot S''(t),$$

and, thus,

$$\vec{r}'(t) \times \vec{r}''(t) = [\vec{T}(t) \cdot S'(t)] \times [\vec{T}'(t) \cdot S'(t) + \vec{T}(t) \cdot S''(t)],$$

$$\vec{r}'(t) \times \vec{r}''(t) = [\vec{T}'(t) \cdot S'(t) + \vec{T}'(t) \cdot S'(t)] + [\vec{T}(t) \cdot S'(t) + \vec{T}(t) \cdot S''(t)],$$

$$\vec{r}'(t) \times \vec{r}''(t) = (S'(t))^2 \cdot [\vec{T}(t) \times \vec{T}'(t)] + S'(t) \cdot S''(t) \cdot [\vec{T}(t) \times \vec{T}(t)].$$

Recall that for any vector $\vec{a} \in \mathbb{R}^n$, $\vec{a} \times \vec{a} = \vec{0}$

$$\implies \vec{r}'(t) \times \vec{r}''(t) = (S'(t))^2 \cdot [\vec{T}(t) \times \vec{T}'(t)] + S'(t) \cdot S''(t) \cdot \vec{0} = (S'(t))^2 \cdot [\vec{T}(t) \times \vec{T}'(t)].$$

Taking the l^2 norm above, one yields the equality below:

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|(S'(t))^2 \cdot [\vec{T}(t) \times \vec{T}'(t)]\| = (S'(t))^2 \cdot \|\vec{T}(t) \times \vec{T}'(t)\|,$$

$$S'(t) = \|\vec{r}'(t)\| \in \mathbb{R} \implies \|S'(t)\| = S'(t).$$

$\therefore$ Utilizing Lemma 2.8 yields the following result below:

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = (S'(t))^2 \cdot \|\vec{T}(t) \times \vec{T}'(t)\| = (S'(t))^2 \cdot \|\vec{T}'(t)\|,$$

$\therefore \|\vec{T}'(t)\| = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{(S'(t))^2} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^2}.$

Utilizing the formula from Definition 2.9 yields the desired formula below:

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

$\square$

Theorem 2.1. *The curvature of a quadratic (degree two) Bézier curve with control points* (P_0, P_1, P_2) *is* $\kappa(t) = \frac{\|(P_2-P_1)\times(P_0-P_1)\|}{[(1-t)^2\|P_1-P_0\|^2+2t(1-t)\langle P_1-P_0\ ,P_2-P_1\rangle+t^2\|P_2-P_1\|^2]^{\frac{9}{2}}}$.

Proof. The numerator, which will be denoted N_1 and the denominator, which will be denoted N_2 of the general formula κ will be calculated separately. A degree two Bézier curve will yield the following position vector $\vec{r}(t)$ below:

$$\vec{r}(t) = \sum_{i=0}^{2} P_i B_{i,2}(t) = P_0 B_{0,2}(t) + P_1 B_{1,2} + P_2 B_{2,2} = P_0(1-t)^2 + 2P_1 t(1-t) + P_2 t^2,$$

$$= P_0(1-t)^2 + P_1 t(1-t) + P_1 t(1-t) + P_2 t^2 = (1-t)[P_0(1-t) + P_1 t] + t[P_1(1-t) + P_2 t].$$

Note that $\vec{r}(t)$ is constructed in the form above to simply the expression later on in the proof when utilizing inner products as well as cross products.

$$\vec{r}'(t) = \frac{d}{dt}\vec{r}(t) = -[P_0(1-t) + P_1 t] + (1-t)[-P_0 + P_1] + [P_1(1-t) + P_2 t] + t[-P_1 + P_2],$$

$$\vec{r}'(t) = -2P_0(1-t) + 2P_1(1-t) - 2P_1 t + 2P_2 t = 2(1-t)[P_1 - P_0] + 2t[P_2 - P_1],$$

$$\vec{r''}(t) = \frac{d^2}{dt^2}\vec{r''}(t) = -2[P_1 - P_0] + 2[P_2 - P_1] = 2[P_2 - 2P_1 + P_0],$$

$$N_1 = \|(2(1-t)[P_1 - P_0] + 2t[P_2 - P_1]) \times (2[P_2 - 2P_1 + P_0])\|,$$

$$N_1 = \|(2(1-t)[P_1 - P_0] \times 2[P_2 - 2P_1 + P_0]) + (2t[P_2 - P_1] \times 2[P_2 - 2P_1 + P_0])\|,$$

$$N_1 = \|4((1-t)[P_1 - P_0] \times [P_2 - 2P_1 + P_0]) + 4(t[P_1 - P_0] \times [P_2 - 2P_1 + P_0])\|,$$

$$N_1 = \|4(1-t)([P_1 - P_0] \times [P_2 - 2P_1 + P_0]) + 4t([P_1 - P_0] \times [P_2 - 2P_1 + P_0])\|,$$

$$N_1 = \|4(1-t)([P_1 - P_0] \times [(P_2 - P_1) + (P_0 - P_1)]) + 4t([P_1 - P_0]\times, \\ [(P_2 - P_1) + (P_0 - P_1)])\|,$$

$$N_1 = \|4(1-t)[(P_1 - P_0) \times (P_2 - P_1)] + 4t[(P_2 - P_1) \times (P_0 - P_1)]\|,$$

$$N_1 = \|{-4}(1-t)[(P_2 - P_1) \times (P_1 - P_0)] + 4t[(P_2 - P_1) \times (P_0 - P_1)]\|,$$

$$N_1 = \|4(1-t)[(P_2 - P_1) \times (P_0 - P_1)] + 4t[(P_2 - P_1) \times (P_0 - P_1)]\|,$$

$$N_1 = \|4[(P_2 - P_1) \times (P_0 - P_1)]\|,$$

$\|\vec{r}\,'(t)\| = (\|\vec{r}\,'(t)\|^2)^{\frac{3}{2}} = \langle 2(1-t)[P_1 - P_0] + 2t[P_2 - P_1]\ ,\ 2(1-t)[P_1 - P_0] + 2t[P_2 - P_1]\rangle^{\frac{3}{2}}$,

$$\|\vec{r}\,'(t)\| = (4(1-t)^2\|P_1 - P_0\|^2 + 8t(1-t)\langle P_1 - P_0\ ,\ P_2 - P_1\rangle + 4t^2\|P_2 - P_1\|^2)^{\frac{3}{2}},$$

$$N_2 = (4(1-t)^2\|P_1 - P_0\|^2 + 8t(1-t)\langle P_1 - P_0\ ,\ P_2 - P_1\rangle + 4t^2\|P_2 - P_1\|^2)^{\frac{9}{2}}.$$

$\therefore \kappa = \frac{N_1}{N_2}$ gives the desired result.

Note that N_1 can also be written as the following expression below:

$$N_1 = 4\|(P_2 \times P_0) - (P_2 \times P_1) - (P_1 \times P_0)\|.$$

$\square$

One can use the same technique as above to find the curvature of a Bézier curve of degree $n \geq 3$. Though, it is guaranteed that this process will be especially tedious due to the fact that long polynomials will require manipulation. Therefore, one must express this curvature in terms of an n-degree Bézier curve.

Remark 2.16. *To write the formula for the curvature, denoted by κ, of an n-degree Bézier curve, one must recall the first-order and second-order derivatives of the position vector $\vec{r}(t)$. These derivatives are shown below:*

$$\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = n\sum_{i=0}^{n-1} B_{n-1,i}(t)(P_{i+1} - P_i)$$

$$\frac{d^2}{dt^2}\vec{r}(t) = \vec{r}\,''(t) = n(n-1)\sum_{i=0}^{n-2} B_{n-2,i}(t)(P_{i+2} - 2P_{i+1} + P_i).$$

As previously discussed in Lemma 2.1, the n^{th}-order derivative of the position vector of the Bézier curve, denoted $\vec{r}(t)$, is dependent on the n^{th} order derivative of the Berstein polynomial, denoted $B_{i,n}(t)$.

Utilizing the formula $\kappa(t)$ in Remark 2.14, one can represent the curvature of a regular Bézier curve of order two or three ($n = 2, 3$) in terms of the sums above. This is particularly useful if one desires to compute the curvature of a Bézier curve of a higher degree. Following the same strategy, we can infer the following general formula for the curvature of a Bézier curve of order n at each point $\vec{r}(t)$ on the curve:

Proposition 2.3. *The formula of the curvature of a Bézier curve of order* n *at the point* $\vec{r}(t)$ *of the curve is:*

$$\kappa(t) = \frac{\|\sum_{i=0}^{n-1}\sum_{j=0,j\neq i}^{n-2} n^2(n-1)B_{n-1,i}(t)B_{n-2,j}(t)[(P_{i+1}-P_i)\times(P_{j+2}-2P_{j+1}+P_j)]\|}{\|\sum_{i=0}^{n-1} nB_{i,n}(t)(P_{i+1}-P_i)\|^3}.$$

Proof. The proof follows from direct computation as described above.

□

Chapter 3

Bézier Surfaces

The properties above discuss the intricacies as well as the properties of Bézier curves of degree n. Though, the properties of the Bézier surface, which is defined as the Cartesian product of the Bernstein polynomials of two orthogonal Bézier curves, have yet to be considered. This surface is denoted as a Tensor-Product Surface.

So far, the previous properties and their proofs follow from the study of the position vector of a univariate Bézier curve, defined as $\vec{r}(t) = \sum_{i=0}^{n} P_i B_{i,n}(t)$ of degree n and depending on the parameter $t \in [0,1]$. We now consider an extension of these properties for Bézier surfaces in $\mathbb{R}^3$. We start with the following defintion.

Definition 3.1. *Consider the points* $P_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$ *in* $\mathbb{R}^3$

$$P_{i,j} = \{(P_{0,0}, P_{0,1}, \ldots, P_{0,n+1}), (P_{1,0}, P_{1,1}, ..., P_{1,n+1}), ..., (P_{m+1,0}, P_{m+1,1}, \ldots, P_{m+1,n+1})\},$$

$0 \leq i \leq m+1$, $0 \leq j \leq n+1$, *which we will call control points. Introducing a bivariate position vector with two variables* $(u,v) \in [0,1] \times [0,1]$ *of degree* $m \times n$*, we define the*

Bézier surface as the image of the position vector $\vec{r}(u,v)$:

$$\vec{r}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{i,j}B_{i,m}(u)B_{j,n}(v). \tag{3.1}$$

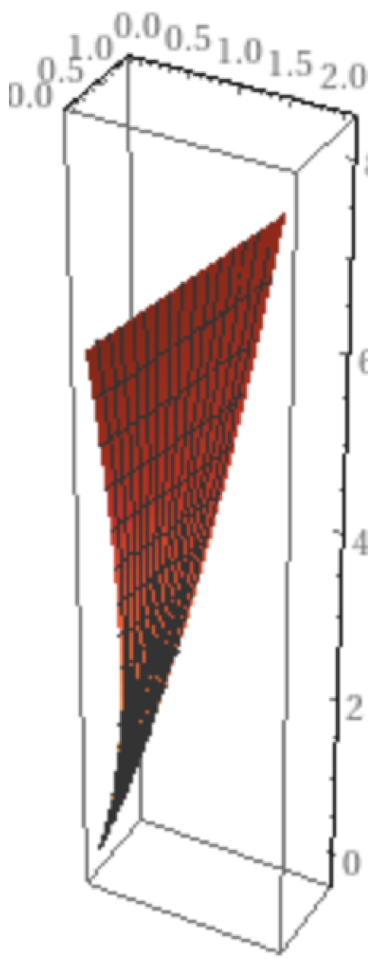

Figure 3.1: Example of a 2×3 Bézier surface, where $\vec{r}(u,v) = (2uv^2, 2v(u+1)(1-v), 2v(2v+uv+1))$, $u \in (0,1)$ and $v \in (0,1)$.

Note that the components of the vector $\vec{r} : D \to \mathbb{R}^3$ are polynomials in two variables defined on $D = [0,1] \times [0,1] \subset \mathbb{R}^2$. Usually, the domain of a Bézier surface of degree $m \times n$ is the unit square, though D can be any rectangle $[a,b] \times [c,d]$, $a \leq b$ & $c \leq d$. Furthermore, one can select control points such that the surface represented by $\vec{r}(u, v)$ has no cusps nor self-intersections.

Definition 3.2. *Isoparametric curves of a surface: Let (u, v) be surface parameters in which $u = u_0 \in [0, 1]$ and $v = v_0 \in [0, 1]$. Isoparametric curves are obtained when one of the two surface parameters are left constant while the other varies between $[0, 1]$.*

Therefore, u and v are images of *vertical* and, respectively, *horizontal* lines on the surface. This leads to the following remark below.

Remark 3.1. *Isoparametric curves provide a relationship between Bézier curves and Bézier surfaces. A Bézier surface patch can be transformed into two Bézier curves utilizing the isoparametric curves below:*

$\vec{r}(u_0, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i,m}(u_0) B_{j,n}(v)$. *Grouping the terms inside the sum, one can set* $a_j = \sum_{i=0}^{m} P_{i,j} B_{i,m}(u_0)$. $\therefore \vec{r}(u_0, v) = \sum_{j=0}^{n} a_j B_{j,n}(v)$.

$\vec{r}(u, v_0) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i,m}(u) B_{j,n}(v_0)$. *Like above, one can set* $b_i = \sum_{j=0}^{n} P_{i,j} B_{j,n}(v_0)$. $\therefore \vec{r}(u, v_0) = \sum_{i=0}^{m} b_i B_{i,m}(u)$.

Remark 3.2. *Noticing the position vector for the tensor product of two Bézier curves, one can deduce the following properties below:*

- *The Bézier surface (or patch) is contained in the convex hull of its extreme points (control points).*
- *The Bézier sub-patch interpolates four points.*
- *The tangent plane at any control point interpolates the given control point as well as two neighboring points.*

Definition 3.3. *Tangent plane: A plane which is orthogonal to the normal vector at any point of the surface if the normal vector exists. The tangent plane to a surface consists of all tangent lines to any point $\vec{r}(u,v)$ lying on the surface.*

Formulae for determining the tangent plane at any given point $\vec{r}(u,v)$ for the parametric form $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ are given below for a point $P = \vec{r}(u_0, v_0) \in \mathbb{R}^3$:

$$\vec{r}_u(u,v) = \left(\tfrac{\partial x(u,v)}{\partial u}, \tfrac{\partial y(u,v)}{\partial u}, \tfrac{\partial z(u,v)}{\partial u}\right)$$

$$\vec{r}_v(u,v) = \left(\tfrac{\partial x(u,v)}{\partial v}, \tfrac{\partial y(u,v)}{\partial v}, \tfrac{\partial z(u,v)}{\partial v}\right)$$

$$\vec{N}(u,v) = \tfrac{\vec{r}_u(u,v)\times\vec{r}_v(u,v)}{\|\vec{r}_u(u,v)\times\vec{r}_v(u,v)\|}$$

$$\textit{Tangent plane: } \langle \vec{r}(u,v) - \vec{r}(u_0,v_0), \vec{N}(u_0,v_0)\rangle = 0.$$

Note that $\vec{N}(u,v)$ denotes the unit normal at any given point $\vec{r}(u,v)$. To add, the vectors $\vec{r}_u$ & $\vec{r}_v$ are tangent to the surface patch at the point $\vec{r}(u,v)$. The collection of all tangent vectors at a point of the surface is called the tangent plane of the surface at that point.

Remark 3.3. *A Bézier surface of degree $m \times n$ is smooth almost everywhere except at isolated points.*

Definition 3.4. *First Fundamental Form induced by $\mathbb{R}^3$: Let M be a regular surface patch parameterized by the curve $\vec{r} = \vec{r}(u,v) \subset \mathbb{R}^3$, in other words, $\vec{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. Furthermore, let A be a point and A' be a point infinitesimally close to A, in which A and A' are points in the tangent space T_pM of M for $p = \vec{r}(u,v) \in M$. Also,* *let du and dv be chosen directions of a Bézier surface patch. Therefore, the First Fundamental Form measures the distances along paths of the surface patch. One can express the differential of* $\vec{r}$ *as the equality below for* $u \in [0, 2\pi]$ *and* $v \in [0, 2\pi]$:

$$d\vec{r} = \tfrac{\partial \vec{r}}{\partial u} du + \tfrac{\partial \vec{r}}{\partial v} dv = \vec{r}_u du + \vec{r}_v dv.$$

The differential above corresponds to the speed of the curve $\vec{r}(u, v)$ *by taking the dot product of* $A = d\vec{r}$ & $A' = d\vec{r}^T$, *shown below:*

$$AA' = d\vec{r}d\vec{r}^T = \langle d\vec{r}, d\vec{r} \rangle = \langle \vec{r}_u du + \vec{r}_v dv, \vec{r}_u du + \vec{r}_v dv \rangle = \|\vec{r}_u\|^2 du^2 + 2\vec{r}_u \cdot \vec{r}_v dudv + \|\vec{r}_v\|^2 dv^2.$$

Let $AA' = ds^2,\ E = \|\vec{r}_u\|^2,\ F = \vec{r}_u \cdot \vec{r}_v,\ G = \|\vec{r}_v\|^2$. *This reduces to the equality below, which is the First Fundamental Form of a surface in terms of the arc-length parameter s giving norms of vectors in the tangent plane, hence length of curves on the surface:*

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

In matrix form, the First Fundamental Form, also known as the intrinsic metric, can be written in the quadratic form below:

$$ds^2 = \vec{x}^T A \vec{x} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

Remark 3.4. *The First Fundamental Form can be represented as the following inner product with respect to* $(\vec{v}_1, \vec{v}_2) \in T_pM$ *shown below for* $\vec{v}_1 = \vec{v}_2$ *and any point* $p = r(u, v) \in M$:

$$\mathbb{I}_p(\vec{v}_1, \vec{v}_2) = \langle \vec{v}_1, \vec{v}_2 \rangle.$$

Remark 3.5. *Matrix A is a positive definite & symmetric matrix. In other words,* $\begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$ *is strictly positive.*

Lemma 3.1. *The formula ds^2 is consistent with the definition of the length of a curve on the surface $\gamma(t) = \vec{r}(u(t), v(t))$, $t \in [0,1]$.*

Proof. The derivative of the Arc-Length formula in Definition 2.7 is $\frac{dS(t)}{dt} = \|\vec{r}\,'(t)\|$.

Furthermore, one must use the differential of $\vec{r}$ from Definition 3.4, which is $d\vec{r} = \vec{r}_u du + \vec{r}_v dv$.

Utilizing the equalities above, one obtains the following below:

$\|\frac{d\vec{r}}{dt}\| = \|\vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt}\|$

$\|\frac{d\vec{r}}{dt}\| = \sqrt{(\vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt}) \cdot (\vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt})}$

$\|\frac{d\vec{r}}{dt}\| dt = \sqrt{\langle \vec{r}_u, \vec{r}_u \rangle du^2 + 2\langle \vec{r}_u, \vec{r}_v \rangle dudv + \langle \vec{r}_v, \vec{r}_v \rangle dv^2} dt$

$\|d\vec{r}\| = \sqrt{E(\gamma(t))du^2 + 2F(\gamma(t))dudv + G(\gamma(t))dv^2}$, where (E, F, G) are the coefficients of the First Fundamental Form.

This implies that $dS(t) = \sqrt{E(\gamma(t))du^2 + 2F(\gamma(t))dudv + G(\gamma(t))dv^2}$.

Then, setting $s = S(t)$, $ds = \sqrt{E(\gamma(t))du^2 + 2F(\gamma(t))dudv + G(\gamma(t))dv^2}$.

$\therefore ds^2 = Edu^2 + 2Fdudv + Gdv^2$, completing the proof.

□

Remark 3.6. *The First Fundamental Form is altered when the surface patch in Definition 3.4, is altered.*

Example 3.1. *Determine the First Fundamental Form of the unit sphere parameterized by $\vec{r}(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$, $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $v \in [0, 2\pi]$.*

Taking partial derivatives, one obtains the expressions below:

$$\vec{r}_u = (-\sin(u)\cos(v), -\sin(u)\sin(v), \cos(u))$$

$$\vec{r}_v = (-\cos(u)\sin(v), \cos(u)\cos(v), 0).$$

Calculating the coefficients of the First Fundamental Form, one obtains the expressions below:

$$E = \|\vec{r}_u\|^2 = \sin^2(u)\cos^2(v) + \sin^2(u)sin^2(v) + \cos^2(u) = \sin^2(u) + \cos^2(u) = 1$$

$$F = \vec{r}_u \cdot \vec{r}_v = \sin(u)\cos(v)\cos(u)\sin(v) - \sin(u)\cos(v)\cos(u)\sin(v) = 0$$

$$G = \|\vec{r}_v\|^2 = \cos^2(u)\sin^2(v)\cos^2(u)\cos^2(v) = \cos^2(u).$$

$\therefore$ *The First Fundamental Form of $\vec{r}$ is shown below:*

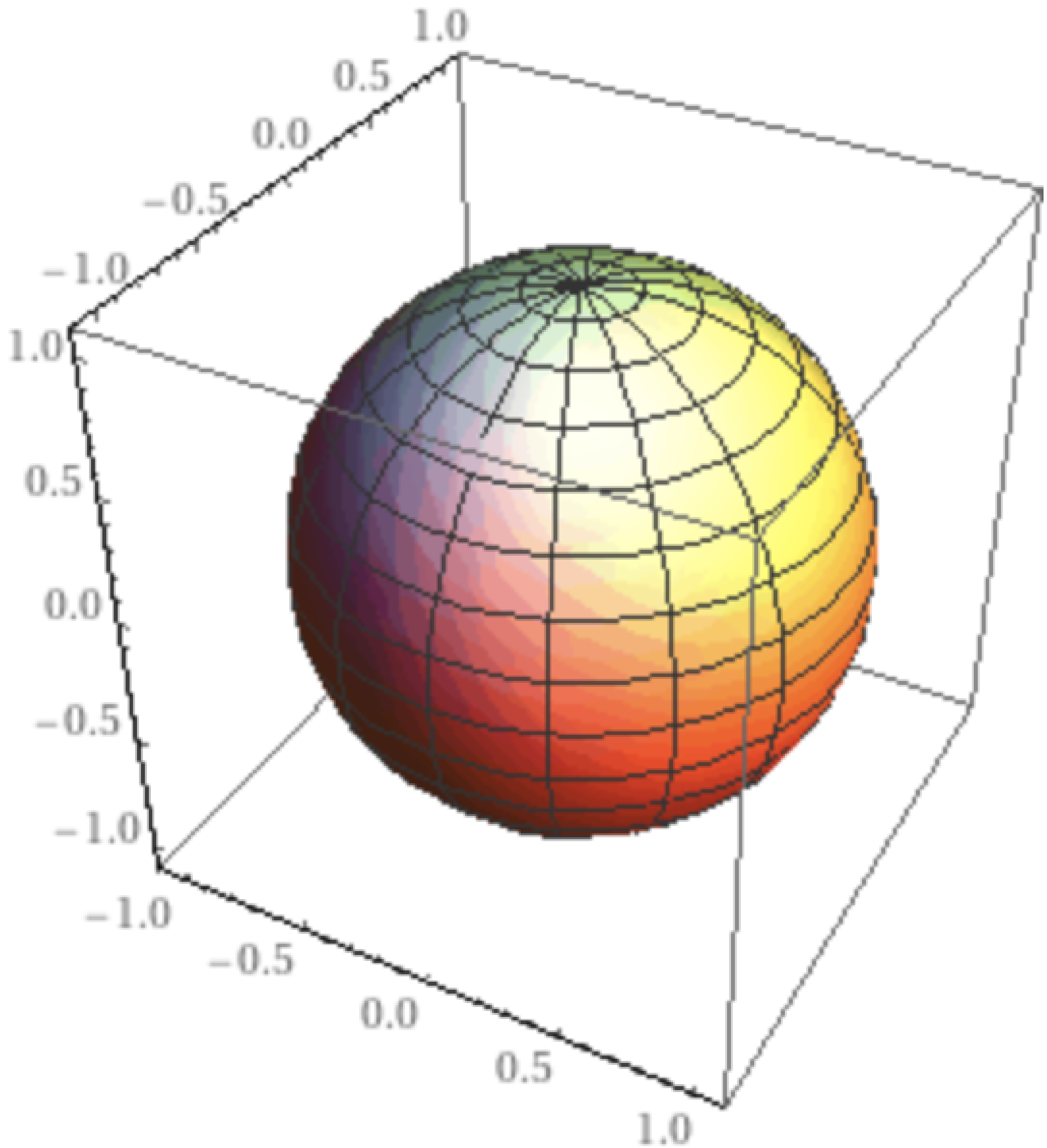

Figure 3.2: Unit sphere, where $\vec{r}(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$, $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $v \in [0, 2\pi]$.

$$ds^2 = du^2 + \cos^2(u)dv^2.$$

Lemma 3.2. *Utilizing Lemma 2.1, one can evaluate the first-order partial derivatives of the multivariate formula (3.1) provided in Definition 3.1, shown below:*

$$\vec{r}_u(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{n} n(P_{i+1,j} - P_{i,j})B_{i,m-1}(u)B_{j,n}(v)$$

$$\vec{r}_v(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n-1} n(P_{i,j+1} - P_{i,j})B_{i,m}(u)B_{j,n-1}(v).$$

Calculating the coefficients of the First Fundamental Form will require four indices, in other words, $i \neq k \wedge j \neq l$ for $i,j,k,l \in \mathbb{N} \cup \{0\}$. These dot products, given in Definition 3.1, are generalized below:

$$E = n^2 \sum_{i=0,j=0,k=0,l=0}^{m-1,n,m-1,n} B_{i,m-1}(u)B_{k,m-1}(u)B_{j,n}(v)B_{l,n}(v)\langle P_{i+1,j} - P_{i,j}, P_{k+1,l} - P_{k,l}\rangle$$

$$F = n^2 \sum_{i=0,j=0,k=0,l=0}^{m-1,n,m,n-1} B_{i,m-1}(u)B_{k,m}(u)B_{j,n}(v)B_{l,n-1}(v)\langle P_{i+1,j} - P_{i,j}, P_{k,l+1} - P_{k,l}\rangle$$

$$G = n^2 \sum_{i=0,j=0,k=0,l=0}^{m,n-1,m,n-1} B_{i,m}(u)B_{k,m}(u)B_{j,n-1}(v)B_{l,n-1}(v)\langle P_{i+1,j} - P_{i,j}, P_{k+1,l} - P_{k,l}\rangle.$$

Remark 3.7. *Utilizing the coefficients above, one can generalize the First Fundamental Form in terms of the arc-length parameter s of an $m \times n$ Bézier surface patch. This is shown below:*

$$ds^2 = n^2[\sum_{i=0,j=0,k=0,l=0}^{m-1,n,m-1,n} B_{i,m-1}(u)B_{k,m-1}(u)B_{j,n}(v)B_{l,n}(v)\langle P_{i+1,j} - P_{i,j}, P_{k+1,l} - P_{k,l}\rangle du^2$$

$$+ 2\sum_{i=0,j=0,k=0,l=0}^{m-1,n,m,n-1} B_{i,m-1}(u)B_{k,m}(u)B_{j,n}(v)B_{l,n-1}(v)\langle P_{i+1,j} - P_{i,j}, P_{k,l+1} - P_{k,l}\rangle dudv$$

$+ \sum_{i=0,j=0,k=0,l=0}^{m,n-1,m,n-1} B_{i,m}(u)B_{k,m}(u)B_{j,n-1}(v)B_{l,n-1}(v)\langle P_{i+1,j}-P_{i,j}, P_{k+1,l}-P_{k,l}\rangle dv^2].$

Definition 3.5. *Regular Surface: Let $M \subset R^3$ be a surface parameterized by the curve $\vec{r}: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. This surface is called a regular surface if and only if $\vec{r_u} \times \vec{r_v} \neq \vec{0}$ exists.*

Definition 3.6. *Orientable Surface: Let $M \subset R^3$ be a regular surface. M is said to be orientable if it allows a smooth differentiable field of unit normal vectors globally defined on the entire surface.*

Definition 3.7. *Gauss Map: Let $p = \vec{r}(u, v) \in M$ be an arbitrary point of an orientable surface M and let $\pm\vec{N}_p(r(u, v))$ be the two possible directions of the unit normal vectors at the given point, in which the formula $\vec{N}(u, v)$ is described in Definition 3.3. The map $N : M \longrightarrow \mathbb{S}^2$, $N(p) = N_p$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, is called the Gauss Map of M.*

Definition 3.8. *Weingarten Map: Let M be an orientable surface and let N be the Gauss Map defined above. Furthermore, let $p = \vec{r}(u, v) \in M$ be an arbitrary point. The linear map $\mathbb{W}_p : T_pM \longrightarrow T_pM$, such that $\mathbb{W}_p = -dN_p$, is called the Weingarten Map. This is also the negative differential of the Gauss map.*

Definition 3.9. *Second Fundamental Form: In terms of the Weingarten map of Definition 3.8, the inner product below gives the Second Fundamental Form for vector $\vec{v}_{1,2} \in T_pM$ and a point $p = \vec{r}(u, v) \in M$:*

$$\mathbb{II}_p(\vec{v}_1, \vec{v}_2) = \langle \mathbb{W}_p(\vec{v}_1), \vec{v}_2\rangle = -\langle DN_p(\vec{v}_1), \vec{v}_2\rangle. \tag{3.2}$$

The Second Fundamental Form describes how curved the surface is.

Utilizing the inner product (3.2), one can determine the coefficients of the Second Fundamental Form in the quadratic form $\mathbb{II} = edu^2 + 2fdudv + gdv^2$*. These coefficients are shown below in terms of the Weingarten map:*

$$e = -\langle DN_p(\vec{r}_u), \vec{r}_u \rangle = \langle \vec{N}, \vec{r}_{uu} \rangle$$

$$f = -\langle DN_p(\vec{r}_u), \vec{r}_v \rangle = \langle \vec{N}, \vec{r}_{uv} \rangle$$

$$g = -\langle DN_p(\vec{r}_v), \vec{r}_v \rangle = \langle \vec{N}, \vec{r}_{vv} \rangle.$$

In matrix form, the Second Fundamental Form can be written in the quadratic form below:

$$\mathbb{II}(u, v) = \vec{x}^T A \vec{x} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

Example 3.2. *Determine the Second Fundamental Form of the unit sphere parameterized by* $\vec{r}(u, v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$*,* $u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *&* $v \in [0, 2\pi]$*.*

Taking partial derivatives, one obtains the expressions below:

$$\vec{r}_u = (-\sin(u)\cos(v), -\sin(u)\sin(v), \cos(u))$$

$$\vec{r}_v = (-\cos(u)\sin(v), \cos(u)\cos(v), 0).$$

First, one must calculate $\vec{r}_u \times \vec{r}_v$, which is shown below:

$$\det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin(u)\cos(v) & -\sin(u)\sin(v) & \cos(u) \\ -\cos(u)\sin(v) & \cos(u)\cos(v) & 0 \end{pmatrix} = -\cos^2(u)\cos(v)\hat{\mathbf{i}} - \cos^2(u)\sin(v)\hat{\mathbf{j}} - \cos(u)\sin(u)\hat{\mathbf{k}}.$$

Calculating $\|\vec{r}_u \times \vec{r}_v\|$, one obtains:

$$\sqrt{\cos^4(u)\cos^2(v) + \cos^4(u)\sin^2(v) + \cos^2(u)\sin^2(u)} = |\cos(u)|\sqrt{\cos^2(u) + \sin^2(u)} = \cos(u).$$

$\therefore$ Combining the above expressions will give the following result:

$$\vec{N}(u, v) = -\cos(u)\cos(v)\hat{\mathbf{i}} - \cos(u)\sin(v)\hat{\mathbf{j}} - \sin(u)\hat{\mathbf{k}}.$$

Calculating the second order partial derivatives will yield the equalities below:

$$\vec{r}_{uu} = -(\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$$

$$\vec{r}_{uv} = \vec{r}_{vu} = (\sin(u)\sin(v), -\sin(u)\cos(v), 0)$$

$$\vec{r}_{vv} = -(\cos(u)\cos(v), \cos(u)\sin(v), 0).$$

Finally, calculating coefficients (e, f, g) will give the results below:

$$\mathrm{e} = \vec{N}\cdot\vec{r}_{uu} = \cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) + \sin^2(u) = \cos^2(u) + \sin^2(u) = 1 = E$$

$$\text{f} = \vec{N} \cdot \vec{r}_{uv} = -\cos(u)\cos(v)\sin(u)\sin(v) + \sin(u)\cos(u)\sin(v)\cos(v) = 0 = F$$

$$\text{g} = \vec{N} \cdot \vec{r}_{vv} = \cos^2(u)\cos^2(v) + \cos^2(u)\sin^2(v) = \cos^2(u) = G.$$

$\therefore$ The Second Fundamental Form is shown below:

$$\mathbb{II}(u, v) = du^2 + \cos^2(u)dv^2.$$

Lemma 3.3. *Utilizing Lemma 2.1, one can evaluate the second-order partial derivatives of the multivariate formula in Definition 3.1, shown below:*

$$\vec{r}_{uu}(u, v) = \sum_{i=0}^{m-2}\sum_{j=0}^{n} n(n-1)(P_{i+2.j} - 2P_{i+1,j} + P_{i,j})B_{i,m-2}(u)B_{j,n}(v)$$

$$\vec{r}_{uv}(u, v) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} n^2(P_{i+1,j+1} - P_{i+1,j} - P_{i,j+1} + P_{i,j})B_{i,m-1}(u)B_{j,n-1}(v)$$

$$\vec{r}_{vv}(u, v) = \sum_{i=0}^{m}\sum_{j=0}^{n-2} n(n-1)(P_{i,j+2} - 2P_{i,j+1} + P_{i.j})B_{i,m}(u)B_{j,n-2}(v).$$

Let the unit normal vector denoted $\vec{N}$, defined in Definition 3.3, the first-order partial derivatives $\vec{r}_u, \vec{r}_v$ and the coefficients of the First Fundamental Form (E, F, G) described in Lemma 3.2. One can utilize the definition of the cross product of $\vec{r}_u$ & $\vec{r}_v$, shown below:

$$\vec{r}_u \times \vec{r}_v = det(\vec{r}_u, \vec{r}_v) = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \vec{r}_{u_1} & \vec{r}_{u_2} & \vec{r}_{u_3} \\ \vec{r}_{v_1} & \vec{r}_{v_2} & \vec{r}_{v_3} \end{pmatrix}. \tag{3.3}$$

Furthermore, one can utilize the following equation below for $\theta \in (0, 2\pi]$, where θ is the angle between the vectors $\vec{r}_u$ & $\vec{r}_v$:

$$\|\vec{r}_u \times \vec{r}_v\| = \|\vec{r}_u\| \cdot \|\vec{r}_v\| \cdot \sin(\theta). \tag{3.4}$$

The cross product formula in terms of equation (3.4) can be simplified as the equation below in terms of coefficients E, F, G for $\theta \in (0, 2\pi]$:

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{E \cdot G} \cdot \sin(\theta). \tag{3.5}$$

Utilizing equations (3.3), (3.4) and (3.5), the unit normal vector, denoted $\vec{N}$, can be written as the following equation below:

$$\vec{N} = \tfrac{1}{\sqrt{E \cdot G} \cdot \sin(\theta)} \cdot det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \vec{r}_{u1} & \vec{r}_{u2} & \vec{r}_{u3} \\ \vec{r}_{v1} & \vec{r}_{v2} & \vec{r}_{v3} \end{pmatrix}.$$

One can also write $\vec{N}$ more formally in terms of four indices. In other words, $i \neq k \land j \neq l$ for $i, j, k, l \in \mathbb{N} \cup \{0\}$. This formula is shown below:

$$\vec{N} = \frac{\sum_{i=0,j=0}^{m-1,n}(P_{i+1,j} - P_{i,j})B_{i,m-1}(u)B_{j,n}(v) \times \sum_{k=0,l=0}^{m,n-1}(P_{i,j+1} - P_{i,j})B_{i,m}(u)B_{j,n-1}(v)}{\|\sum_{i=0,j=0}^{m-1,n}(P_{i+1,j} - P_{i,j})B_{i,m-1}(u)B_{j,n}(v) \times \sum_{k=0,l=0}^{m,n-1}(P_{i,j+1} - P_{i,j})B_{i,m}(u)B_{j,n-1}(v)\|}.$$

Lemma 3.4. *Calculating the coefficients of the Second Fundamental Form will require an additional two indices, in other words, $p \neq q$ for $p, q \in \mathbb{N} \cup \{0\}$. These dot products, given in Definition 3.9, are generalized below:*

$$e = n(n-1)\sum_{p=0,q=0}^{m-2,n} B_{p,m-2}(u)B_{q,n}(v)\langle \vec{N}, P_{p+2,q} - 2P_{p+1,q} + P_{p,q}\rangle$$

$$f = n^2 \sum_{p=0,q=0}^{m-1,n-1} B_{p,m-1}(u)B_{q,n-1}(v)\langle \vec{N}, P_{p+1,q+1} - P_{p+1,q} - P_{p,q+1} + P_{p,q}\rangle$$

$$g = n(n-1)\sum_{p=0,q=0}^{m,n-2} B_{p,m}(u)B_{q,n-2}(v)\langle \vec{N}, P_{p,q+2} - 2P_{p,q+1} + P_{p,q}\rangle.$$

Theorem 3.1. *Utilizing the coefficients of Lemma 3.4, one can generalize the Second Fundamental Form in terms of $\vec{N}$ of an $(m \times n)$ Bézier surface patch as follows:*

$$\begin{aligned}\mathbb{II}(u,v) = n[&(n-1)\textstyle\sum_{p=0,q=0}^{m-2,n} B_{p,m-2}(u)B_{q,n}(v)\langle \vec{N}, P_{p+2,q} - 2P_{p+1,q} + P_{p,q}\rangle du^2 \\ &+2n\textstyle\sum_{p=0,q=0}^{m-1,n-1} B_{p,m-1}(u)B_{q,n-1}(v)\langle \vec{N}, P_{p+1,q+1} - P_{p+1,q} - P_{p,q+1} + P_{p,q}\rangle dudv \\ &+(n-1)\textstyle\sum_{p=0,q=0}^{m,n-2} B_{p,m}(u)B_{q,n-2}(v)\langle \vec{N}, P_{p,q+2} - 2P_{p,q+1} + P_{p,q}\rangle dv^2].\end{aligned}$$

Remark 3.8. *It is tedious to generalize the fomulae Gaussian Curvature, Mean Curvature and the Principal Curvatures of an $m \times n$ Bézier surface due to the lengthiness of the formulas and the large degrees of the polynomials being worked with. Therefore, one can give the general formulas of these curvatures by taking a limited number of control points in the u direction and the v direction.*

Remark 3.9. *The Weingarten map, defined in Definition 3.8, can also be represented below in matrix form in terms of coefficients a, b, c, d:*

$$\mathbb{W}_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \mathbb{I}^{-1}\mathbb{II}. \tag{3.6}$$

Multiplying the matrices on the right-hand side of equation (3.6)*, one will obtain the coefficients* a, b, c, d *in terms of the coefficients of the First and Second Fundamental Forms shown below:*

$$a = \frac{Ge - Ff}{EG - F^2}$$

$$b = \frac{Gf - Fg}{EG - F^2}$$

$$c = \frac{Ef - Fe}{EG - F^2}$$

$$d = \frac{Eg - Ff}{EG - F^2}.$$

Definition 3.10. *Gaussian Curvature: Let* κ_p *denote the Gaussian curvature of a Bézier surface at an arbitrary point* $p = r(u, v) \in M$*. One can write* κ_p *in terms of* $\mathbb{W}_p$ *in Remark 3.9, which is shown below:*

$$\kappa_p = \det(\mathbb{W}_p) = \frac{\det(\mathbb{II})}{\det(\mathbb{I})} = \frac{eg - f^2}{EG - F^2}.$$

Definition 3.11. *Mean Curvature: Let* H_p *denote the Mean curvature of a Bézier surface at an arbitrary point* $p = \vec{r}(u, v) \in M$*. One can write* H_p *in terms of* $\mathbb{W}_p$ *in Remark 3.9, which is shown below:*

$$H_p = \frac{1}{2} trace(\mathbb{W}_p) = \frac{1}{2} \frac{Ge + Eg - 2Ff}{EG - F^2}.$$

Definition 3.12. *Principal Curvatures: Let* (λ_1, λ_2) *denote the minimum and maximum principal curvatures, respectively, of a Bézier surface. One can find* λ_1, λ_2 *as the eigenvalues of* $\mathbb{W}_p$ *in Remark 3.9, where* I *denotes the identity* 2×2 *matrix:*

$$\det(\mathbb{W}_p - \lambda I) = 0. \tag{3.7}$$

Referring to equation (3.7)*, the result will provide the eigenvalues* $\lambda_{1,2} = (\lambda_1, \lambda_2)$ *of the Weingarten matrix in terms of the coefficients* (E, F, G) *&* (e, f, g)*, shown below for* $\lambda_1 = min(\lambda_1, \lambda_2)$ & $\lambda_2 = max(\lambda_1, \lambda_2)$:

$$\lambda_{1,2}^2 - \frac{Ge - 2Ff + Eg}{EG - F^2}\lambda_{1,2} + \frac{(Ge - Ff)(Eg - Ff) - (Gf - Fg)(Ef - Fe)}{(EG - F^2)^2} = 0.$$

In terms of κ_p *and* H_p *defined in Definition 3.10 and Definition 3.11, respectively, the equation above can be written in the form below:*

$$\lambda_{1,2}^2 - 2H_p\lambda_{1,2} + \kappa_p = 0. \tag{3.8}$$

The principles curvatures $\lambda_{1,2} = (\lambda_1, \lambda_2)$*, which are roots of equation* (3.8)*, are presented below:*

$$\lambda_1 = H_p - \sqrt{H_p^2 - \kappa_p}$$

$$\lambda_2 = H_p + \sqrt{H_p^2 - \kappa_p}.$$

Remark 3.10. *The Gaussian Curvature, denoted* κ_p*, can be represented as the product of* λ_1 *&* λ_2*.*

Remark 3.11. *The Mean Curvature, denoted H_p, can be represented as the average of λ_1 & λ_2.*

Example 3.3. *Determine the Gaussian Curvature, the Mean Curvature and the Principle Curvatures of the unit sphere parameterized by*
$\vec{r}(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u)),\ u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ & $v \in [0, 2\pi]$.

From the worked examples above, one notices that the coefficients of the First Fundamental Form and the Second Fundamental Form are equivalent, in other words, $e = E, f = F, g = G$. This gives the following Weingarten map provided in Remark 3.9 below:

$$\mathbb{W}_p = \begin{pmatrix} \frac{EG-F^2}{EG-F^2} & \frac{GF-FG}{EG-F^2} \\ \frac{EF-FE}{EG-F^2} & \frac{EG-F^2}{EG-F^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As stated in Definition 3.10, the Weingarten map above will provide the Gaussian Curvature of the unit sphere, shown below:

$$\kappa_p = det(\mathbb{W}_p) = (1)(1) - (0)(0) = 1.$$

The Mean Curvature of the unit sphere is also shown below in terms of the Weingarten map above:

$$H_p = \frac{1}{2} trace \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(1+1) = 1.$$

The principal curvatures, provided in Definition 3.12, can now be solved by solving the quadratic equation in terms of κ_p & H_p below:

$$\lambda_{1,2}^2 - 2H_p\lambda_{1,2} + \kappa_p = \lambda_{1,2}^2 - 2\lambda_{1,2} + 1 = 0. \tag{3.9}$$

Solving the quadratic equation (3.9) *will give the roots, denoted as the principle curvatures, shown below:*

$$\lambda_1 = \lambda_2 = 1.$$

One can utilize Remarks 3.10 & 3.11 to check that κ_p, H_p *&* $\lambda_{1,2}$ *are correct.*

Example 3.4. *Let* $(P_{0,0}, P_{1,0}, P_{0,1}, P_{1,1}) \in \mathbb{R}^3$ *be the set of non-planar control points of a* (2×2) *Bézier surface patch, in other words,* $i = 0, 1$ *and* $j = 0, 1$ *and* $z \neq 0$. *Compute the First and Second Fundamental Forms, the Gaussian Curvature, the Mean Curvature and the Principal Curvatures of this surface patch by utilizing the control points listed below:*

$$P_{0,0} = (2, 1, 1)$$

$$P_{0,1} = (2, 3, 1)$$

$$P_{1,0} = (1, 1, 2)$$

$$P_{1,1} = (1, 3, 3). \tag{3.10}$$

The Tensor-Product formula described in Definition 3.1 for an $m \times n$ *Bézier surface is shown below for* $i = 0, 1$ *&* $j = 0, 1$:

$$\vec{r}(u, v) = \sum_{i=0}^{1} B_{i,1}(u) \sum_{j=0}^{1} P_{i,j} B_{j,1}(v)$$

$$\vec{r}(u, v) = P_{0,0}B_{0,1}(u)B_{0,1}(v) + P_{0,1}B_{0,1}(u)B_{1,1}(v) + P_{1,0}B_{1,1}(u)B_{0,1}(v) + P_{1,1}B_{1,1}(u)B_{1,1}(v)$$

$$\vec{r}(u,v) = (1-u)(1-v)P_{0,0} + v(1-u)P_{0,1} + u(1-v)P_{1,0} + uvP_{1,1}.$$

Using the set of non-planar control points of (3.10)*, one can represent the Tensor-Product formula as the vector* $\vec{r}(u,v)$ *below:*

$$\vec{r}(u,v) = (2-u, 1+2v, 1+u+uv).$$

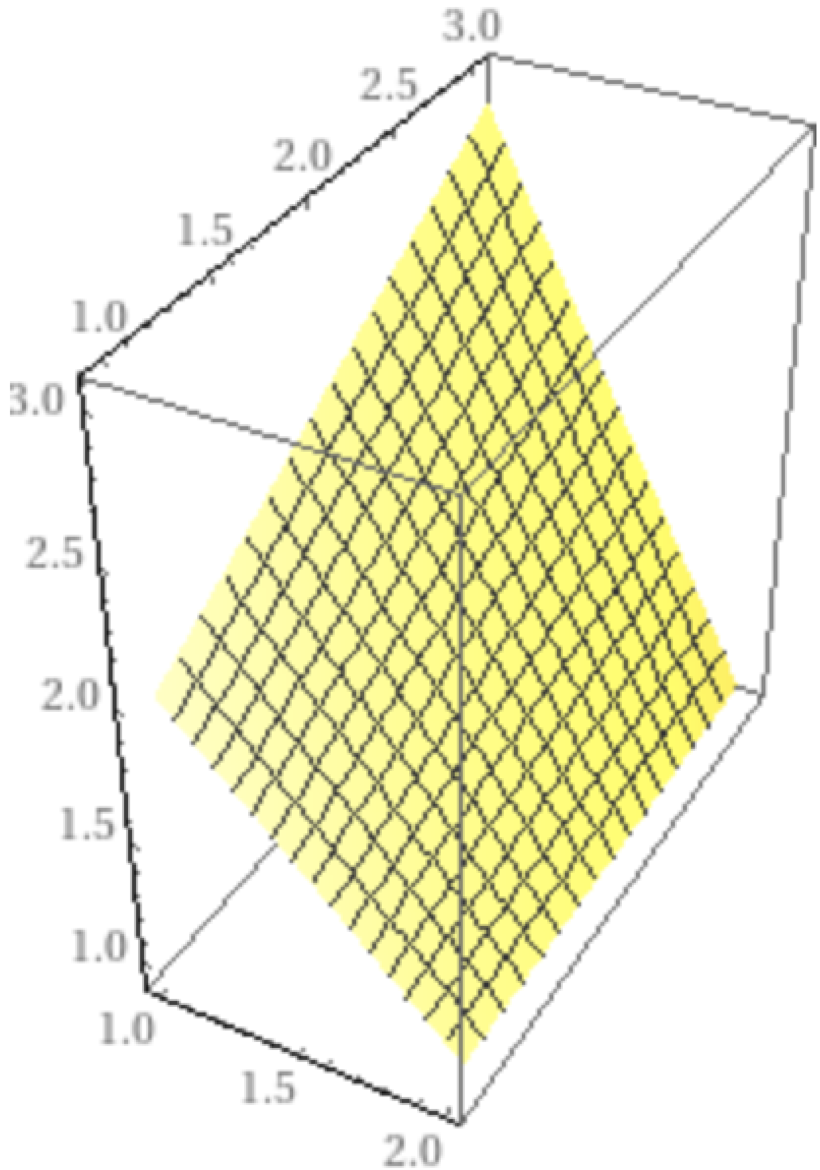

Figure 3.3: Example of a 2×2 Bézier surface, where $\vec{r}(u,v) = (2-u, 1+2v, 1+u+uv)$, $u \in (0,1)$ and $v \in (0,1)$.

In order to compute the First Fundamental Form of the Bézier surface patch, the first-order derivatives of the vector $\vec{r}(u,v)$ must be evaluated. These vectors are provided below:

$$\vec{r}_u = (-1, 0, v+1)$$

$$\vec{r}_v = (0, 2, u).$$

One can now determine the First Fundamental Form with the first-order derivatives listed above. The necessary computations are shown below:

$$E = \|\vec{r}_u\|^2 = v^2 + 2v + 2$$

$$F = \vec{r_u}^T \ \vec{r_v} = uv + u$$

$$G = \|\vec{r}_v\|^2 = 4 + u^2$$

$\therefore$ *The First Fundamental Form is* $ds^2 = I(u,v) = (v^2 + 2v + 2)du^2 + 2u(v + 1)dudv + (4 + u^2)dv^2$.

The cross-product, its L^2 *norm and the unit normal vector* $\vec{N}(u,v)$ *of the* (2×2) *Bézier surface patch are shown below:*

$$\vec{r_u} \times \vec{r_v} = det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & v+1 \\ 0 & 2 & u \end{pmatrix} = -2(v+1)\hat{\mathbf{i}} + u\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\|\vec{r_u} \times \vec{r_v}\| = \sqrt{4(v+1)^2 + u^2 + 4} = \sqrt{4v^2 + 8v + u^2 + 8}$$

$$\therefore \vec{N}(u,v) = \frac{(-2(v+1), u, -2)}{\sqrt{4v^2+8v+u^2+8}}.$$

In order to compute the Second Fundamental Form of the Bézier surface patch, the second-order derivatives of the vector $\vec{r}(u,v)$ must be evaluated. These vectors are provided below:

$$\vec{r}_{uu} = (0,0,0)$$

$$\vec{r}_{uv} = \vec{r}_{vu} = (0,0,1)$$

$$\vec{r}_{vv} = (0,0,0).$$

One can now determine the Second Fundamental Form with the second-order derivatives and the unit normal vector listed above. The necessary computations are shown below:

$$e = \vec{N}^T \vec{r}_{uu} = 0$$

$$f = \vec{N}^T \vec{r}_{uv} = \frac{-2}{\sqrt{4v^2+8v+u^2+8}}$$

$$g = \vec{N}^T \vec{r}_{vv} = 0.$$

$\therefore$ *The Second Fundamental Form is* $\mathbb{II}(u,v) = \frac{-4}{\sqrt{4v^2+8v+u^2+8}} dudv.$

The matrix representations of $\mathbb{I}(u,v)$ *and* $\mathbb{II}(u,v)$*, denoted* $A_{\mathbb{I}}$ *and* $A_{\mathbb{II}}$*, respectively,* *and their determinants, are provided below:*

$$A_{\mathbb{I}} = \begin{pmatrix} v^2+2v+2 & u(v+1) \\ u(v+1) & 4+u^2 \end{pmatrix}$$

$$det(A_{\mathbb{I}}) = u^2+4v^2+8v+8$$

$$A_{\mathbb{II}} = \begin{pmatrix} 0 & \frac{-4}{\sqrt{4v^2+8v+u^2+8}} \\ \frac{-4}{\sqrt{4v^2+8v+u^2+8}} & 0 \end{pmatrix}$$

$$det(A_{\mathbb{II}}) = \frac{-16}{4v^2+8v+u^2+8}.$$

To determine the coefficients of the Weingarten map, one must determine the inverses of $A_{\mathbb{I}}$*. This matrix is provided below:*

$$A_{\mathbb{I}}^{-1} = \frac{1}{u^2+4v^2+8v+8} \begin{pmatrix} 4+u^2 & -u(v+1) \\ -u(v+1) & v^2+2v+2 \end{pmatrix}.$$

The coefficients of the Weingarten map, given in Remark 3.9, are provided below for the (2×2) *Bézier surface patch:*

$$\mathbb{W}_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{u^2+4v^2+8v+8} \begin{pmatrix} 4+u^2 & -u(v+1) \\ -u(v+1) & v^2+2v+2 \end{pmatrix} \begin{pmatrix} 0 & \frac{-4}{\sqrt{4v^2+8v+u^2+8}} \\ \frac{-4}{\sqrt{4v^2+8v+u^2+8}} & 0 \end{pmatrix}$$

$$a = \frac{4u(v+1)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}$$

$$b = \frac{-4(4+u^2)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}$$

$$c = \frac{-4(v^2+2v+2)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}$$

$$d = \frac{4u(v+1)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}.$$

The Gaussian Curvature κ_p, provided in Definition 3.10 is given below by utilizing the determinants above:

$$\kappa_p = det(\mathbb{W}_p) = \frac{-16}{(4v^2+8v+u^2+8)^2} < 0.$$

The Mean Curvature H_p, provided in Definition 3.11, is given below by utilizing the Weingarten map above:

$$H_p = \frac{1}{2}trace(\mathbb{W}_p) = \frac{4u(v+1)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}.$$

The Principal Curvatures (λ_1, λ_2), provided in Definition 3.12, are evaluated below by utilizing κ_p and H_p above and the quadratic formula:

$$\lambda_{1,2}^2 - 2H_p\lambda_{1,2} + K_p = 0$$

$$\lambda_{1,2}^2 - \frac{8u(v+1)}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}\lambda_{1,2} - \frac{16}{(4v^2+8v+u^2+8)^2} = 0.$$

$$\lambda_{1,2} = \frac{8u(v+1) \pm \sqrt{64u^2v^2+128u^2v+128u^2+256v^2+512v+512}}{2(4v^2+8v+u^2+8)^{\frac{3}{2}}}$$

$$\lambda_{1,2} = \frac{4[u(v+1) \pm \sqrt{(v^2+2v+2)(u^2+4)}]}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}$$

$$\lambda_1 = \frac{4[u(v+1)-\sqrt{(v^2+2v+2)(u^2+4)}]}{(4v^2+8v+u^2+8)^{\frac{3}{2}}} \text{ and } \lambda_2 = \frac{4[u(v+1)+\sqrt{(v^2+2v+2)(u^2+4)}]}{(4v^2+8v+u^2+8)^{\frac{3}{2}}}.$$

Note that κ_p is strictly negative, therefore the signs of λ_1 and λ_2 must be opposite for $(u,v) \in [0,1]$, which is the case above. Furthermore, the reader can check that the Principal Curvatures obtained above are correct by Remark 3.10 and Remark 3.11, in other words, $\kappa_p = \lambda_1 \lambda_2$ and $H_p = \frac{\lambda_1+\lambda_2}{2}$.

Chapter 4

Singularities of Cubic Bézier Curves

Recall the position vector of a cubic Bézier curve $\vec{r}: [0,1] \to \mathbb{R}^2$ with four control points $\{P_0, P_1, P_2, P_3\}$ as below:

$$\vec{r}(t) = \sum_{i=0}^{3} P_i B_{i,3}(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3. \qquad (4.1)$$

Lemma 4.1. *The formulas to convert the derivatives of a parametric curve $\vec{r}(t) = (x(t), y(t)) \subset \mathbb{R}^2$, for t into some open interval, into the derivatives of the curve as a function $y = f(x)$ are:*

$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{(\frac{dx}{dt})}.$$

Remark 4.1. *We determine the first and second derivative of the parametric curve of equation* (4.1) *as shown below:*

$$\vec{r}'(t) = -3(1-t)^2 P_0 + 3(3t-1)(t-1)P_1 + 3t(2-3t)P_2 + 3t^2 P_3$$

$$\vec{r}''(t) = 6(1-t)P_0 + 6(3t-2)P_1 + 6(1-3t)P_2 + 6tP_3.$$

Thus, using $\frac{dy}{dx}$*, the first derivative of the cubic Bézier curve as a function of* x *can be expressed below as the quotient of its parametric equations by taking* $P_0 = (x_0, y_0), P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$*:*

$$\frac{y'(t)}{x'(t)} = \frac{-(1-t)^2 y_0 + (3t-1)(t-1)y_1 + t(2-3t)y_2 + t^2 y_3}{-(1-t)^2 x_0 + (3t-1)(t-1)x_1 + t(2-3t)x_2 + t^2 x_3}. \tag{4.2}$$

Definition 4.1. *Cusp: A cusp, also denoted as a sharp corner or a turning point, is a singularity in which the tangent changes direction. This occurs when* $\dfrac{dx}{dt} = 0$ *at some particular value of* t^**. If* $\dfrac{dy}{dt} \neq 0$ *at that same value of* t^**, the graph has a vertical tangent at that point. For a cubic Bézier curve, if the cusp occurs at* t^**, the derivative function of the position vector will have an infinite discontinuity at the point* t^* *as in one of the cases described below, and the curve has a vertical tangent*

a. $\lim_{t\to t^{*-}} \frac{y'(t)}{x'(t)} = +\infty$ & $\lim_{t\to t^{*+}} \frac{y'(t)}{x'(t)} = -\infty$

b. $\lim_{t\to t^{*-}} \frac{y'(t)}{x'(t)} = -\infty$ & $\lim_{t\to t^{*+}} \frac{y'(t)}{x'(t)} = +\infty$.

If both $\dfrac{dx}{dt}$ *and* $\dfrac{dy}{dt}$ *are zero at some value of* t*, then the cubic Bézier curve has a cusp if, besides the previous possible limits, we have*

$$\lim_{t\to t^{*-}} \frac{y'(t)}{x'(t)} = \lim_{t\to t^{*+}} \frac{y'(t)}{x'(t)} = L \in \overline{\mathbb{R}}$$

and the curve has a tangent of slope L *at that point.*

Remark 4.2. *The value* $t^* \in (0,1)$ *is determined by evaluating when condition below holds for some* $t \in (0,1)$:

$$\|\vec{r}'(t)\|^2 = \vec{r}'(t)^T \vec{r}'(t) = 0, \tag{4.3}$$

or reduced to the condition below for some $t \in (0,1)$:

$$\vec{r}'(t) = \vec{0}, \tag{4.4}$$

after which one needs to evaluate the limit.

An example of a cubic Bézier curve with a cusp at $t^* = \frac{1}{2}$ is provided below:

Remark 4.3. *For any set of four control points, the resulting cubic Bézier curve can only have at most one cusp.*

Example 4.1. *Let* $\{P_0, P_1, P_2, P_3\} = \{(0,0),(1,1),(0,1),(1,0)\}$. *Therefore, using equation* (4.1) *and Definition 4.1, the Bézier curve and its first derivative can be represented as the position vector* $\vec{r}(t) = (x(t), y(t))$ *below:*

$$\vec{r}(t) = (4t^3 - 6t^2 + 3t, 3t - 3t^2) \tag{4.5}$$

$$\vec{r}'(t) = 3(4t^2 - 4t + 1, 1 - 2t). \tag{4.6}$$

Now, as $\vec{r}'(t) \neq \vec{0}$, *one can solve equation* (4.4) *to determine the roots of the equation* (4.6). *This is shown below* :

$$(4t^2 - 4t + 1, 1 - 2t) = \vec{0},$$

$$\begin{cases} 4t^2 - 4t + 1 = (2t-1)^2 = 0 \\ 1 - 2t = 0. \end{cases}$$

Both the quadratic equation and the linear equation above give the solution $t^* = \frac{1}{2}$*, where we will show that the curve* $\vec{r}(t)$ *has a cusp. Plugging* t^* *into equation* (4.5)*, one obtains the point* $\vec{r}(\frac{1}{2}) = (x(\frac{1}{2}), y(\frac{1}{2})) = (\frac{1}{2}, \frac{3}{4})$.

Furthermore, the limit as t *approaches to the cusp point from the right and the left of* $\frac{dy}{dx}$ *shows that there is an infinite discontinuity. This is observed below using Lemma 4.1,* $\frac{dy}{dx}$ *and equation* (4.6)*:*

$\frac{dy}{dx} = \frac{1-2t}{4t^2-4t+1} = \frac{-1}{2t-1}, \ \forall_{t\in(0,\frac{1}{2})\cup(\frac{1}{2},1)}.$

Then,

$\lim_{t\to\frac{1}{2}^-} \frac{dy}{dx} = \lim_{t\to\frac{1}{2}^-} \frac{-1}{2t-1} = +\infty$ & $\lim_{t\to\frac{1}{2}^+} \frac{dy}{dx} = \lim_{t\to\frac{1}{2}^+} \frac{-1}{2t-1} = -\infty.$

Therefore, as $\vec{r}(t)$ *is a function (any vertical line intersects the curve at one point only), then there is an infinite discontinuity for* $\frac{dy}{dx}$ *at the point* $(\frac{1}{2}, \frac{3}{4})$ *corresponding to the value* $t^* = \frac{1}{2}$*. The curve and its corresponding non-convex control polygon are shown below:*

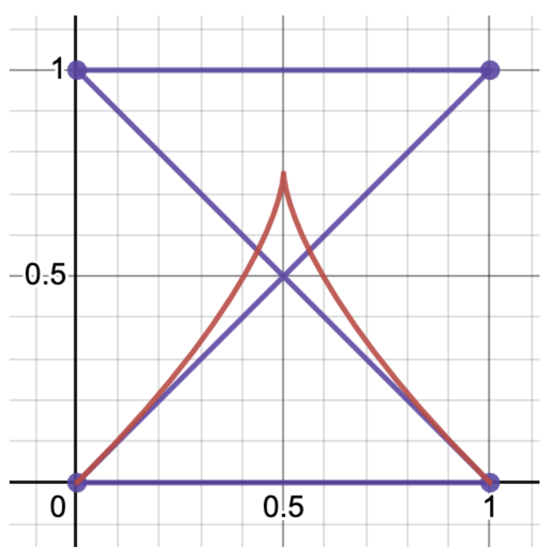

Figure 4.1: Example of a Bézier curve with a cusp, where $\vec{r}(t) = (4t^3-6t^2+3t, 3t-3t^2)$, $t \in (0, 1)$.

Another example of a cubic Bézier curve with a cusp at $t^* = \frac{3}{7}$ is provided below:

Example 4.2. *Let* $\{P_0, P_1, P_2, P_3\} = \{(0,1), (3,1), (1, \frac{8}{5}), (1,0)\}$. *Therefore, using equation* (4.1) *and Definition 4.1, the Bézier curve and its first derivative can be represented as the position vector* $\vec{r}(t) = (x(t), y(t))$ *below:*

$$\vec{r}(t) = (7t^3 - 15t^2 + 9t, 1 + \frac{9t^2}{5} - \frac{14t^3}{5}) \tag{4.7}$$

$$\vec{r}\,'(t) = (21t^2 - 30t + 9, \frac{18t}{5} - \frac{42t^2}{5}). \tag{4.8}$$

Now, as $\vec{r}\,'(t) \neq \vec{0}$, *one can solve equation* (4.4) *to determine the roots of the equation* (4.8). *This is shown below :*

$(21t^2 - 30t + 9, \frac{18t}{5} - \frac{42t^2}{5}) = \vec{0}$,

$$\begin{cases} 21t^2 - 30t + 9 = 3(7t^2 - 10t + 3) = 3(1-t)(3-7t) = 0 \\ \frac{18t}{5} - \frac{42t^2}{5} = t(\frac{18}{5} - \frac{42t}{5}) = 0. \end{cases}$$

The first quadratic equation gives the solutions $t = 1$, $t^* = \frac{3}{7}$ *and the second quadratic equation gives the solutions* $t = 0$, $t^* = \frac{3}{7}$. *We will dismiss the solutions* $t = 0$ *and* $t = 1$ *due to these parametric values not being part of the domain of the cubic curve. If we would allow these parametric values as part of the domain, then* $\vec{r}(0)$ *and* $\vec{r}(1)$ *will equate to* P_0 *and* P_3, *respectively, which cannot occur. Now, we will show that the curve* $\vec{r}(t)$ *has a cusp. Plugging* t^* *into equation* (4.7), *one obtains the point* $\vec{r}(\frac{3}{7}) = (x(\frac{3}{7}), y(\frac{3}{7})) = (\frac{81}{49}, \frac{272}{245})$.

Furthermore, the limit as t approaches to the cusp point from the right is equivalent to the limit as t approaches to the cusp point from the left of $\frac{dy}{dx}$ *and is finite. This is observed below using Lemma 4.1,* $\frac{dy}{dx}$ *and equation* (4.8) *:*

$\frac{dy}{dx} = \frac{\frac{18t}{5} - \frac{42t^2}{5}}{21t^2 - 30t + 9} = \frac{2t}{5(1-t)}, \ \forall_{t \in (0, \frac{3}{7}) \cup (\frac{3}{7}, 1)}.$

Then,

$\lim_{t \to \frac{3}{7}^-} \frac{dy}{dx} = \lim_{t \to \frac{3}{7}^-} \frac{2t}{5(1-t)} = \frac{3}{10}$ & $\lim_{t \to \frac{3}{7}^+} \frac{dy}{dx} = \lim_{t \to \frac{3}{7}^+} \frac{2t}{5(1-t)} = \frac{3}{10}.$

Therefore, the curve $\vec{r}(t)$ has a cusp at the point $(\frac{81}{49}, \frac{272}{245})$ corresponding to the value $t^* = \frac{3}{7}$. The curve (in red) and its corresponding non-convex control polygon (in blue) are shown below:

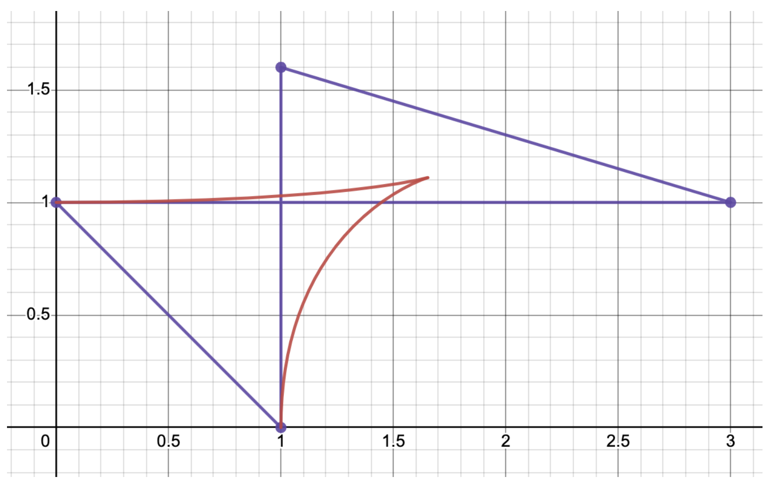

Figure 4.2: Example of a Bézier curve with a cusp, where $\vec{r}(t) = (7t^3 - 15t^2 + 9t, 1 + \frac{9t^2}{5} - \frac{14t^3}{5})$, $t \in (0, 1)$.

Definition 4.2. *Loop: Let $t_1, t_2 \in (0, 1)$ be two parametric values such that $t_1 \neq t_2$ and $t_1 < t_2$. These two values correspond to the self-intersection point of the parametric equations $x = x(t)$ & $y = y(t)$ if and only if t_1, t_2 satisfy the simultaneous conditions below:*

$$\begin{cases} x(t_1) - x(t_2) = 0 \\ y(t_1) - y(t_2) = 0. \end{cases} \tag{4.9}$$

If a Bézier curve with the above data satisfies equations (4.9), then the curve has a loop.

Remark 4.4. *For any set of four control points, the resulting cubic Bézier curve can have at most one loop. As previously discussed, this loop will correspond to the two values* $t_1, t_2 \in (0, 1)$ *giving the same point on the curve.*

An example of a cubic Bézier curve with a loop, self-intersecting at the values $t_1, t_2 = \{\frac{-2}{29}(3\sqrt{3} - 7), \frac{2}{29}(3\sqrt{3} + 7)\}$, is shown below:

Example 4.3. *Let* $\{P_0, P_1, P_2, P_3\} = \{(0, 1), (4, 1), (1, 3), (1, 0)\}$. *Therefore, using equation* (4.1), $\vec{r}(t)$ *can be written as the position vector below:*

$$\vec{r}(t) = (12t - 21t^2 + 10t^3, 1 + 6t^2 - 7t^3). \tag{4.10}$$

Next, using equations (4.9) *and* (4.10), *one obtains the system below:*

$$\begin{cases} -12(t_2 - t_1) + 21(t_2^2 - t_1^2) - 10(t_2^3 - t_1^3) = 0 \\ 7(t_2^3 - t_1^3) - 6(t_2^2 - t_1^2) = 0. \end{cases}$$

Dividing each equation by the trivial solution $t_2 - t_1 = 0$ *and using elementary algebra to factor to cubic and quadratic expressions, the simplified system of equations is provided below:*

$$\begin{cases} 21(t_2 + t_1) - 10(t_2^2 + t_1 t_2 + t_1^2) = 12 \\ 7(t_2^2 + t_1 t_2 + t_1^2) - 6(t_2 + t_1) = 0. \end{cases} \tag{4.11}$$

Solving equations (4.11) *using substitution will give the following equations below:*

$$
\begin{cases} t_2^2 + t_1 t_2 + t_1^2 = \frac{24}{29} \\ t_2 + t_1 = \frac{28}{29}. \end{cases} \tag{4.12}
$$

Then, isolating the linear equation of equation (4.12) *for* t_1 *or* t_2 *and then using the quadratic formula gives the pair of solutions* $(t_1, t_2) = (\frac{-2}{29}(3\sqrt{3} - 7), \frac{2}{29}(3\sqrt{3} + 7))$. *These solutions correspond to the point* $\vec{r}(\frac{14\pm6\sqrt{3}}{29}) = (x(\frac{14+6\sqrt{3}}{29}), y(\frac{14-6\sqrt{3}}{29})) = (\frac{28952}{24389}, \frac{26325}{24389})$. *Therefore, the Bézier curve defined by the position vector* $\vec{r}(t)$ *has a loop. The curve (in red) and its corresponding non-convex control polygon (in blue) are shown below:*

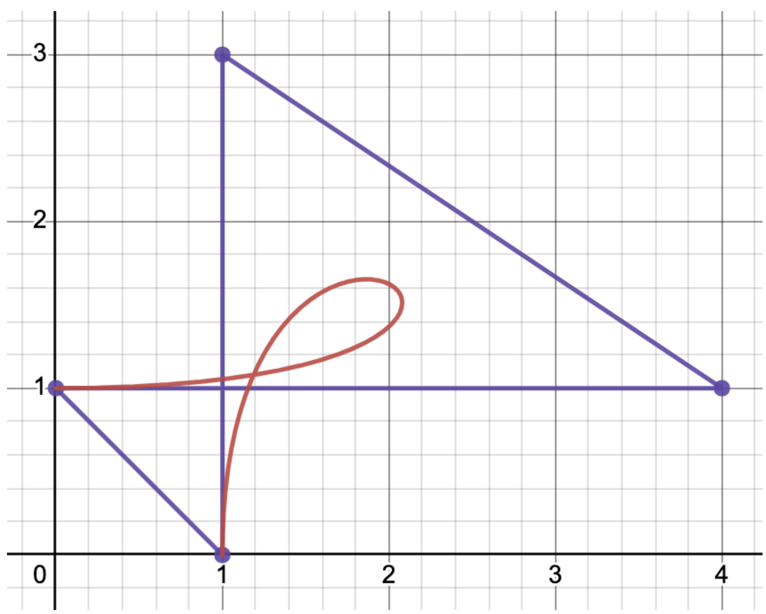

Figure 4.3: Example of a Bézier curve with a loop, where $\vec{r}(t) = (12t - 21t^2 + 10t^3, 1 + 6t^2 - 7t^3)$, $t \in (0, 1)$.

Definition 4.3. *Inflection Point: An inflection point, denoted* $(x_0, y_0) \in \mathbb{R}^2$, *is a point on a curve in which the curvature changes sign and, thus, is a simple (or odd-order) root of the second derivative function* $\frac{d^2y}{dx^2}$.

Remark 4.5. *Note that equation* $\frac{d^2y}{dx^2} = 0$ *at a point* $(x(t_0), y(t_0))$ *is equivalent* *to* $\vec{r}'(t_0) \times \vec{r}''(t_0) = \vec{0}$ *as* $\vec{r}'(t) \times \vec{r}''(t) = (0, 0, x'(t)y''(t) - y'(t)x''(t))$ *and* $\frac{d^2y}{dx^2} = \frac{x'(t)y''(t)-y'(t)x''(t)}{(x'(t))^3}$.

At inflection points, the curvature is zero and the sign of the second derivative $\frac{d^2y}{dx^2}$ *changes. Therefore, the possible inflection points for a parametric curve* $\vec{r}(t) = (x(t), y(t))$ *can be determined by solving for* t_0 *the root of the equation* $\vec{r}'(t) \neq \vec{0}$ *:*

$$\vec{r}'(t) \times \vec{r}''(t) = \vec{0}. \tag{4.13}$$

Then one has to study the sign change of the last coordinate of the vector $\vec{r}'(t) \times \vec{r}''(t)$ *around the root* t_0.

Remark 4.6. *Regarding a cubic Bézier curve with four control points, there can be a maximum of two inflection points for* $t \in (0, 1)$. *The number of inflection points is dependent on the location of these control points. This will be proved later in Theorem 4.1.*

An example of a cubic curve with one inflection point at the value $t = 2 - \sqrt{3} \in (0, 1)$ is shown below:

Example 4.4. *Let* $\{P_0, P_1, P_2, P_3\} = \{(0, 1), (\frac{1}{2}, 1), (1, 2), (1, 0)\}$. *Therefore, using equations* (4.1) *and* (4.13)*, the position vector and the first and second order derivatives of* $\vec{r}(t)$ *can be written as the vectors below:*

$$\vec{r}(t) = (\frac{3t}{2} - \frac{t^3}{2}, 1 + 3t^2 - 4t^3). \tag{4.14}$$

$$\vec{r}'(t) = \frac{3}{2}(1 - t^2, 4t - 8t^2). \tag{4.15}$$

$$\vec{r}''(t) = -3(t, 8t - 2). \tag{4.16}$$

Then, the cross product of the first and second order derivatives of $\vec{r}(t)$ *is provided below:*

$$\vec{r}'(t) \times \vec{r}''(t) = -\tfrac{9}{2} \cdot \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1-t^2 & 4t-8t^2 & 0 \\ t & 8t-2 & 0 \end{pmatrix} = \tfrac{-9}{2}(0, 0, -2(t^2-4t+1)).$$

Solving equation (4.13) *for the formula above is equivalent to solving the quadratic equation below:*

$$t^2 - 4t + 1 = 0. \tag{4.17}$$

The solutions to equation (4.17) *are* $t = 2 \pm \sqrt{3}$. *Though, as* $2 + \sqrt{3} \notin [0, 1]$, *one must select* $t = 2 - \sqrt{3}$. *Therefore,* $\vec{r}(t)$ *has only one inflection point. The location of this inflection point is determined using equation* (4.14) *at* $t = 2-\sqrt{3}$, *which gives the point* $\vec{r}(2-\sqrt{3}) = (6\sqrt{3}-10, 48\sqrt{3}-82)$. *The curve and its corresponding non-convex control polygon are shown below:*

An example of a cubic curve with two inflection points at the value $t = \frac{5}{11} - \frac{\sqrt{3}}{11} \in (0, 1)$ and $t = \frac{5}{11} + \frac{\sqrt{3}}{11} \in (0, 1)$ is shown below:

Example 4.5. *Let* $\{P_0, P_1, P_2, P_3\} = \{(0, 1), (2, 1), (1, \frac{3}{2}), (1, 0)\}$. *Therefore, using equations* (4.1) *and* (4.13), *the position vector and the first and second order derivatives of* $\vec{r}(t)$ *can be written as the vectors below:*

$$\vec{r}(t) = (4t^3 - 9t^2 + 6t, -\frac{5}{2}t^3 + \frac{3}{2}t^2 + 1). \tag{4.18}$$

$$\vec{r}'(t) = 3(4t^2 - 6t + 2, t - \frac{5}{2}t^2). \tag{4.19}$$

$$\vec{r}''(t) = 3(8t - 6, 1 - 5t). \tag{4.20}$$

Then, the cross product of the first and second order derivatives of $\vec{r}(t)$ *is provided below:*

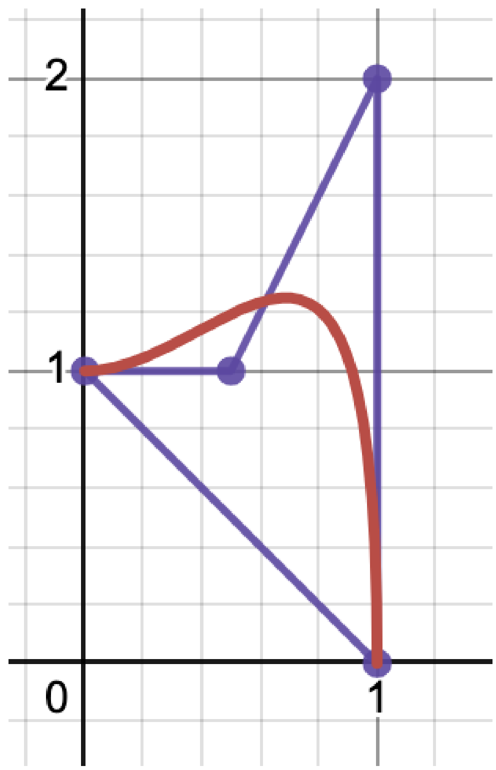

Figure 4.4: Example of a Bézier curve with one inflection point, where $\vec{r}(t) = (\frac{3t}{2} - \frac{t^3}{2}, 1 + 3t^2 - 4t^3)$, $t \in (0, 1)$.

$$\vec{r}'(t) \times \vec{r}''(t) = 9 \cdot \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4t^2 - 6t + 2 & t - \frac{5}{2}t^2 & 0 \\ 8t - 6 & 1 - 5t & 0 \end{pmatrix} = 9(0, 0, 11t^2 - 10t + 2).$$

Solving equation (4.13) for the formula above is equivalent to solving the quadratic equation below:

$$11t^2 - 10t + 2 = 0. \tag{4.21}$$

The solutions to equation (4.21) are $t = \frac{5}{11} \pm \frac{\sqrt{3}}{11} \in (0, 1)$. *Therefore,* $\vec{r}(t)$ *has two inflection points. The location of these inflection points are determined using equation (4.18) at* $t = \frac{5}{11} - \frac{\sqrt{3}}{11}$ *and* $t = \frac{5}{11} + \frac{\sqrt{3}}{11}$, *which gives the points* $\vec{r}(\frac{5}{11} + \frac{\sqrt{3}}{11}) \approx (1.218, 0.989)$ *and* $\vec{r}(\frac{5}{11} - \frac{\sqrt{3}}{11}) \approx (1.093, 1.067)$. *The curve and its corresponding non-convex control polygon are shown below:*

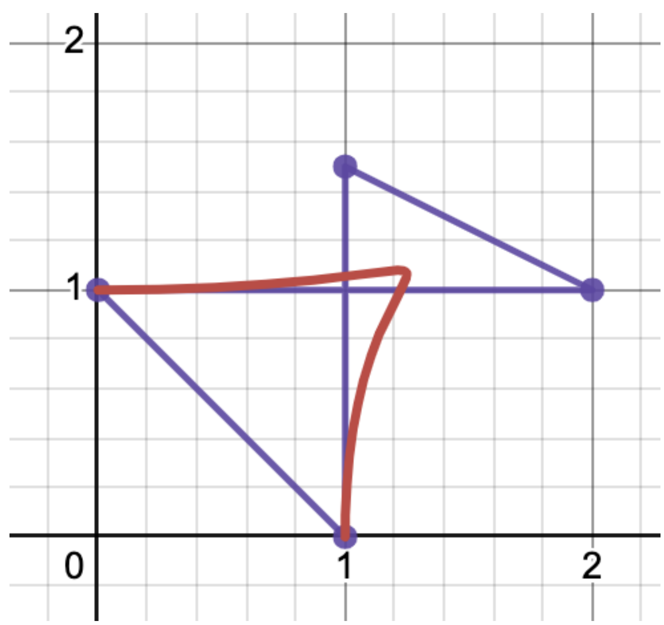

Figure 4.5: Example of a Bézier curve with exactly two real inflection points, where $\vec{r}(t) = (4t^3 - 9t^2 + 6t, -\frac{5}{2}t^3 + \frac{3}{2}t^2 + 1)$, $t \in (0, 1)$.

Remark 4.7. *Many cubic Bézier curves have zero inflection points, no loop nor cusp. Such a curve is also known as an arch. For instance, the control points*

$\{P_0, P_1, P_2, P_3\} = \{(0,1), (-1,1), (1,-1), (1,0)\}$ correspond to the position vector of a Bézier curve $\vec{r}(t) = (-5t^3 + 9t^2 - 3t, 5t^3 - 6t^2 + 1)$ which is an arch. The curve (in red) and its corresponding convex control polygon (in blue) are shown below:

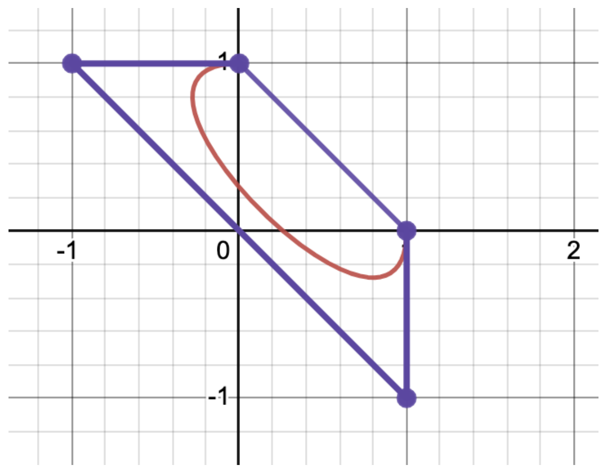

Figure 4.6: Example of a Bézier curve with an arch, where $\vec{r}(t) = (-5t^3 + 9t^2 - 3t, 5t^3 - 6t^2 + 1)$, $t \in (0,1)$.

Throughout the study of the singularities of cubic Bézier curves, one can introduce the characteristic point $R = (\lambda, \mu) \in \mathbb{R}^2$, for which λ and μ are solely dependent on the location of the control points $\{P_1, P_2\} = \{(\lambda, 1), (1, \mu)\}$. By fixing the control points $\{P_0, P_3\} = \{(0,1), (1,0)\}$, one can classify the types of singularities for a cubic Bézier curve with control points $\{P_0, P_1, P_2, P_3\}$ by observing the cases below. In other words, one can show that the (λ, μ)-Cartesian plane is split up into various regions depending on the location of the characteristic point R. Note that the values λ, μ cannot take on the value 0, or simultaneously 1, as the four points must be distinct $P_0 \neq P_1 \neq P_2 \neq P_3$.

As previously discussed, the inflection points of the cubic Bézier curve, if any, correspond to curvature which changes sign, and in particular we have point(s) of vanishing curvature. One should use the equation $\vec{r}'(t) \times \vec{r}''(t) = 0$ in Cartesian coordinates due to less tedious computations. With respect to the parametric equations $x = x(t)$ and $y = y(t)$, one must introduce the formula below to determine the quantity of inflection points and their respective locations:

$$x'(t)y''(t) - x''(t)y'(t) = 0. \tag{4.22}$$

Note that $\vec{r}(t) = (x(t), y(t))$, $\vec{r}'(t) = (x'(t), y'(t))$ and $\vec{r}''(t) = (x''(t), y''(t))$.

Furthermore, the position vector of the cubic Bézier curve as well as its derivatives corresponding to the set of control points $\{P_0, P_1, P_2, P_3\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ are provided below:

$$\vec{r}(t) = (3t^3\lambda - 2t^3 - 6t^2\lambda + 3t^2 + 3t\lambda, (1-t)(3t^2\mu - 2t^2 + t + 1)). \tag{4.23}$$

$$\vec{r}'(t) = (-3(1-t)(3\lambda t - 2t - \lambda), -9t^2\mu + 6t^2 + 6t\mu - 6t). \tag{4.24}$$

$$\vec{r}''(t) = (18\lambda t - 12t - 12\lambda + 6, 12t - 18\mu t + 6\mu - 6). \tag{4.25}$$

Plugging in equations (4.24) and (4.25) into equation (4.22) will give the following:

$$-3(1-t)(3\lambda t - 2t - \lambda))(12t - 18\mu t + 6\mu - 6) - (18\lambda t - 12t - 12\lambda + 6)(-9t^2\mu + 6t^2 + 6t\mu - 6t) = 0.$$

$$t^2(-18\lambda + 54\lambda\mu - 18\mu) + t(36\lambda - 54\lambda\mu) + (18\lambda\mu - 18\lambda) = 0.$$

Therefore, in the quadratic form, one may write this equation as the roots of the function $F(t)$ below:

$$F(t) = At^2 + Bt + C = t^2(-18\lambda + 54\lambda\mu - 18\mu) + t(36\lambda - 54\lambda\mu) + (18\lambda\mu - 18\lambda). \tag{4.26}$$

Therefore, the discriminant function $\triangle = B^2 - 4AC$ and functions A, B and C of $F(t)$ are provided below using equation (4.26) :

$$\begin{cases} A = 18(3\lambda\mu - \lambda - \mu) \\ B = 18\lambda(2 - 3\mu) \\ C = 18\lambda(\mu - 1) \\ \triangle = 324\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4). \end{cases} \tag{4.27}$$

In addition, depending on the number of inflection points on the curve, the signs of the curvature at the endpoints $t = 0$ and $t = 1$ must be considered. Though these endpoints are not considered in the domain of the curve, $\vec{r}(t)$ will still allow the parametric values $t = 0$ and $t = 1$ as its components are polynomials. Using the Cartesian form to determine the curvature of the cubic Bézier curve $\vec{r}(t)$, one must solely compute the numerator (in the limiting sense) of $\kappa(0)$ and $\kappa(1)$, as shown below:

$$F(0) = A(0)^2 + B(0) + C = C. \tag{4.28}$$

$$F(1) = A(1)^2 + B(1) + C = A + B + C. \tag{4.29}$$

The previous notations A, B, C, Δ and the product $F(0)F(1) = AC + BC + C^2$ will be considered in the theorem below.

Remark 4.8. *The cubic Bézier curve can either have up to two real inflection points, a cusp point, a loop or none of these points due to the Variation Diminishing property,* 2.12. *Heuristically, if the cubic curve has a cusp and a loop simultaneously, then a given line may intersect the curve more times than it will intersect its control polygon, which would violate the Variation Diminishing property.*

Remark 4.9. *By the Tangential property, 2.3, the control polygon of a cubic Bézier curve with a cusp point can never be convex. If the control polygon, with four control* *points, is a convex quadrilateral, then this curve will never be tangent to the segments* $\overline{P_0P_1}$ *and* $\overline{P_2P_3}$*, violating the Tangential property. If the curve has a loop, then the shape of its control polygon will be identical to the control polygon of the cubic curve with a cusp point. Then, the control polygon will only be convex in the case where the cubic curve has no cusp, no loop nor inflection points.*

Theorem 4.1. *[1] [4] Let* $\{P_0, P_1, P_2, P_3\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ *be the control points of a cubic Bézier curve for* $\lambda \in \mathbb{R}\backslash\{0\}$*,* $\mu \in \mathbb{R}\backslash\{0\}$*, and* λ, μ *not simultaneously equal to* 1*.*

The following three types of special points can occur for such a Bézier curve of degree $n = 3$*: a cusp, denoted* (t^*)*, a loop point, and one or two inflection points. The existence of these points depends based on the five cases relative to* A, B, C *and* $\triangle$ *are shown below:*

Cusp, no inflection points nor loops:

$$\begin{cases} (A, B, C) \neq \mathbf{0} \\ \triangle = 0 \\ 0 < \frac{-B}{2A} < 1. \end{cases}$$

Loop, no cusp nor inflection points:

$$\begin{cases} (A, B, C) \neq \mathbf{0} \\ \triangle < 0 \\ 0 < \frac{-B \pm \sqrt{-3\triangle}}{2A} < 1. \end{cases}$$

Exactly two inflection points, no cusp nor loop:

$$\begin{cases} (A,B,C) \neq \mathbf{0} \\ \triangle > 0 \\ AC + BC + C^2 > 0 \\ 0 < \frac{-B \pm \sqrt{\triangle}}{2A} < 1. \end{cases}$$

Exactly one inflection point, no cusps nor loops:

$$\begin{cases} (A,B,C) \neq \mathbf{0} \\ \triangle > 0 \\ AC + BC + C^2 < 0. \end{cases}$$

No inflection points, no cusps nor loops:

$$\begin{cases} (A,B,C) \neq \mathbf{0} \\ \triangle < 0 \\ AC + BC + C^2 > 0 \\ \frac{-B \pm \sqrt{-3\triangle}}{2A} \leq 0 \ \ or \ \ \frac{-B \pm \sqrt{-3\triangle}}{2A} \geq 1. \end{cases}$$

Proof. **Case 1: Cusp**.

The cubic Bézier curve has a cusp, denoted t^*, for $t^* \in (0,1)$ when t^* is associated with the quantities below for $(A,B,C) \neq \mathbf{0}$:

$$\begin{cases} (A,B,C) \neq \mathbf{0} \\ t^* = \frac{-B}{2A} \\ \triangle = 0. \end{cases} \tag{4.30}$$

To obtain the quantities of (4.30), one must use the condition $\vec{r}\,'(t) = \vec{0}$ to determine the cusp of the cubic Bézier curve. The computations are shown below:

$$\vec{r}'(t) = (-3(1-t)(3\lambda t - 2t - \lambda), -9t^2\mu + 6t^2 + 6t\mu - 6t) = (0,0),$$

$$\begin{cases} (1-t)(3\lambda t - 2t - \lambda) = 0 \\ t(3t\mu - 2t - 2\mu + 2) = 0. \end{cases} \tag{4.31}$$

The first equation gives solutions $t = 1$ and $3\lambda t - 2t - \lambda = 0$ and the second equation gives solutions $t = 0$ and $3t\mu - 2t - 2\mu + 2 = 0$. These solutions re-written in terms of λ and μ of t^* are shown below:

$$\begin{cases} t = 1 \\ t = 0 \\ \lambda = \frac{2t^*}{3t^*-1} \\ \mu = \frac{2(t^*-1)}{3t^*-2}. \end{cases} \tag{4.32}$$

One can dismiss the first two solutions as they are not part of the domain of the parametric value t^*. Isolating the quantities of λ and μ for t, one obtains below:

$$\begin{cases} t^* = \frac{\lambda}{3\lambda-2} \\ t^* = \frac{2(\mu-1)}{3\mu-2}, \end{cases} \tag{4.33}$$

Then, eliminating the parameter t^* between μ and λ of (4.33) and then using cross multiplication, one obtains:

$$\frac{\lambda}{3\lambda - 2} = \frac{2(\mu - 1)}{3\mu - 2},$$

$$\lambda(3\mu - 2) = 2(\mu - 1)(3\lambda - 2),$$

$$3\mu\lambda - 2\lambda = 6\mu\lambda - 4\mu - 6\lambda + 4. \tag{4.34}$$

Then, for equation (4.34), cancelling out like-terms and placing all terms to one side of (4.34) gives below:

$$3\lambda\mu - 4\lambda - 4\mu + 4 = 0. \tag{4.35}$$

To note, one can notice that equation (4.35) is equivalent to $\triangle = 0$, which is a hyperbola in the λ, μ plane. The standard form of the hyperbola is given below:

$$(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}. \tag{4.36}$$

Then, there is a unique root of the equation $At^2+Bt+C = 0$ as $\triangle = 0$. Therefore, the parametric value $t^* \in (0, 1)$ is shown below by plugging in $\triangle = 0$ into $t = \frac{-B\pm\sqrt{\triangle}}{2A}$, the solutions of the equation $At^2 + Bt + C = 0$:

$$t^* = \frac{-B}{2A} = \frac{\lambda(3\mu - 2)}{2(3\lambda\mu - \lambda - \mu)}. \tag{4.37}$$

Now, the derivative $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ in terms of the configuration $\{(0, 1), (\lambda, 1), (1, \mu), (1, 0)\}$ is provided below:

$$\frac{y'(t)}{x'(t)} = \frac{t(3t\mu - 2t - 2\mu + 2)}{(1 - t)(3\lambda t - 2t - \lambda)}.$$

Using the quantities $\lambda = \frac{2t}{3t-1}$ and $\mu = \frac{2(t-1)}{3t-2}$ from (4.28), the derivative $\frac{y'(t,t^*)}{x'(t,t^*)}$ can be represented in the form below:

$$\frac{y'(t, t^*)}{x'(t, t^*)} = \frac{2t(t\left(\frac{\mu-1}{t^*}\right) - (\mu - 1))}{(1 - t)(\frac{\lambda t}{t^*} - \lambda)},$$

Which can also be written as the fraction below:

$$\frac{y'(t,t^*)}{x'(t,t^*)} = \frac{2t(\mu-1)(\frac{t}{t^*}-1)}{\lambda(1-t)(\frac{t}{t^*}-1)}.$$

Taking the limit as t approaches t^*, one must examine the factor $(t-t^*)$ of the derivative $\frac{y'(t,t^*)}{x'(t,t^*)}$. As $t^* \neq 0$, the derivative can be written below in terms of the factor $(t-t^*)$:

$$\frac{y'(t,t^*)}{x'(t,t^*)} = \frac{2t(\mu-1)(t-t^*)}{\lambda(1-t)(t-t^*)}.$$

One can notice that there is a factor $(t-t^*)$ on both the numerator and the denominator. When t approaches t^*, the factor $(t-t^*)$ approaches 0. In other words, this shows that the cusp point t^* is a removable discontinuity of $\frac{y'(t,t^*)}{x'(t,t^*)}$. Then, by Definition 4.1, the cubic curve $\vec{r}(t)$ can never be described as the graph of a function $y = f(x)$ with the configuration $\{(0,1),(\lambda,1),(1,\mu),(1,0)\}$ at any cusp point $t^* \in (0,1)$ as $x'(t,t^*) = y'(t,t^*) = 0$.

Then, the limit as t approaches to the cusp point t^* of $\frac{y'(t,t^*)}{x'(t,t^*)}$ is shown below for $\lambda, \mu \neq 0$:

$$\lim_{t\to t^*} \frac{2t(\mu-1)(t-t^*)}{\lambda(1-t)(t-t^*)} = \lim_{t\to t^*} \frac{2t(\mu-1)}{\lambda(1-t)} = \frac{2t^*(\mu-1)}{\lambda(1-t^*)} \in \mathbb{R},\ t^* \neq 0,1.$$

Or,

$$\lim_{t\to t^{*-}} \frac{2t(\mu-1)}{\lambda(1-t)} = \lim_{t\to t^{*+}} \frac{2t(\mu-1)}{\lambda(1-t)} = \frac{2t^*(\mu-1)}{\lambda(1-t^*)} \in \mathbb{R},\ t^* \neq 0,1.$$

As the two-sided limits above exist and are finite due to λ, μ being finite, there exists a cusp point, denoted t^*, when $\triangle = 0$ and $0 < \frac{-B}{2A} < 1$.

Case 2: Loop point.

The cubic Bézier curve has a loop, also called a self-intersection point, when the parametric values $t_1, t_2 \in (0,1),\ t_1 < t_2$ are associated with the quantities below:

$$\begin{cases} (A,B,C) \neq 0 \\ 0 < \frac{-B \pm \sqrt{-3\triangle}}{2A} < 1 \\ \triangle < 0. \end{cases} \tag{4.38}$$

Recall that a cubic Bézier curve has a loop for the parametric values $t_1, t_2 \in (0,1)$, $t_1 < t_2$ when the conditions below are satisfied for $x = x(t)$ and $y = y(t)$:

$$\begin{cases} x(t_1) - x(t_2) = 0 \\ y(t_1) - y(t_2) = 0. \end{cases}$$

The computations of applying these condition to determine at which points λ, μ correspond to a loop are provided below:

$$\begin{cases} (t_1^3 - t_2^3)(3\lambda - 2) + 3(t_1 - t_2)^2(1 - 2\lambda) + 3(t_1 - t_2)\lambda = 0 \\ (t_1^3 - t_2^3)(2 - 3\mu) + 3(\mu - 1)(t_1^2 - t_2^2) = 0, \end{cases} \tag{4.39}$$

Which is equivalent to the system below:

$$\begin{cases} (t_1 - t_2)(t_1^2 + t_1t_2 + t_2^2)(3\lambda - 2) + 3(t_1 - t_2)^2(1 - 2\lambda) + 3(t_1 - t_2)\lambda = 0 \\ (t_1 - t_2)(t_1^2 + t_1t_2 + t_2^2)(2 - 3\mu) + 3(\mu - 1)(t_1 - t_2)(t_1 + t_2) = 0. \end{cases} \tag{4.40}$$

Reducing the equations of (4.40) by dividing them by the trivial solution $t_1 - t_2 = 0$ will give the following equations:

$$\begin{cases} (t_1^2 + t_1 t_2 + t_2^2)(3\lambda - 2) + 3(t_1 + t_2)(1 - 2\lambda) = -3\lambda \\ (t_1^2 + t_1 t_2 + t_2^2)(2 - 3\mu) + 3(\mu - 1)(t_1 + t_2) = 0. \end{cases} \tag{4.41}$$

One can represent (4.41) in the matrix form $MX = N$, shown below:

$$\begin{pmatrix} 3\lambda - 2 & 3 - 6\lambda \\ 2 - 3\mu & 3\mu - 3 \end{pmatrix} \begin{pmatrix} t_1^2 + t_1 t_2 + t_2^2 \\ t_1 + t_2 \end{pmatrix} = \begin{pmatrix} -3\lambda \\ 0 \end{pmatrix}. \tag{4.42}$$

Using row-reduction on augmented matrix (4.42), one can determine a solution for the system (4.41) in terms of $t_1^2 + t_1 t_2 + t_2^2$ and $t_1 + t_2$. The row-reduction is shown below:

$$\left(\begin{array}{cc|c} 3\lambda - 2 & 3 - 6\lambda & -3\lambda \\ 2 - 3\mu & 3\mu - 3 & 0 \end{array}\right),$$

$$\frac{1}{2 - 3\mu} R_2$$

$$\left(\begin{array}{cc|c} 1 & \frac{3-6\lambda}{3\lambda-2} & \frac{-3\lambda}{3\lambda-2} \\ 1 & \frac{3\mu-3}{2-3\mu} & 0 \end{array}\right),$$

$$R_2 - R_1$$

$$\left(\begin{array}{cc|c} 1 & \frac{3-6\lambda}{3\lambda-2} & \frac{-3\lambda}{3\lambda-2} \\ 0 & \frac{-9\lambda\mu+3\mu+3\lambda}{(2-3\mu)(3\lambda-2)} & \frac{3\lambda}{3\lambda-2} \end{array}\right),$$

$$\frac{(2-3\mu)(3\lambda-2)}{-9\lambda\mu+3\mu+3\lambda}R_2$$

$$\left(\begin{array}{cc|c} 1 & \frac{3-6\lambda}{3\lambda-2} & \frac{-3\lambda}{3\lambda-2} \\ 0 & 1 & \frac{\lambda(2-3\mu)}{-3\lambda\mu+\mu+\lambda} \end{array}\right),$$

$$R_1 - \frac{3-6\lambda}{3\lambda-2}R_2$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{3\lambda-3\lambda\mu}{-3\lambda\mu+\mu+\lambda} \\ 0 & 1 & \frac{\lambda(2-3\mu)}{-3\lambda\mu+\mu+\lambda} \end{array}\right) \begin{pmatrix} t_1^2+t_1t_2+t_2^2 \\ t_1+t_2 \end{pmatrix}.$$

Then, in the matrix form $MX = N$, one obtains the system of equations below, represented in terms of functions $A, B, C, \triangle$:

$$\begin{cases} t_1^2+t_1t_2+t_2^2 = \frac{3\lambda(1-\mu)}{\lambda-3\lambda\mu+\mu} = \frac{3C}{A} \\ t_1+t_2 = \frac{\lambda(2-3\mu)}{-3\lambda\mu+\lambda+\mu} = \frac{-B}{A}. \end{cases} \tag{4.43}$$

Solving (4.43) for $t_1, t_2 = t_{1,2}$ in terms of functions $A, B, C, \triangle$, one must isolate the second equation for either t_1 or t_2, which gives the equation below:

$$t_1 = \frac{-B}{A} - t_2 = -\left(\frac{B}{A}+t_2\right).$$

Then, plugging t_1 into the first equation gives the following computations:

$$\left(-\left(\frac{B}{A}+t_2\right)\right)^2 - \left(\frac{B}{A}+t_2\right)t_2 + t_2^2 = \frac{3C}{A},$$

$$t_2^2 + \frac{t_2B}{A} + \frac{B^2}{A^2} = \frac{3C}{A},$$

$$t_2^2 + \frac{B}{A}t_2 + \left(\frac{B^2}{A^2} - \frac{3C}{A}\right) = 0. \tag{4.44}$$

Note that $t_1 = \min\{t_1, t_2\}$ and $t_2 = \max\{t_1, t_2\}$. Using the quadratic formula to determine the parametric values $t_{1,2}$ of equation (4.44) gives the computations below:

$$t_{1,2} = \frac{\frac{-B}{A} \pm \sqrt{\frac{B^2}{A^2} - 4\left(\frac{B^2}{A^2} - \frac{3C}{A}\right)}}{2},$$

$$t_{1,2} = \frac{\frac{-B}{A} \pm \frac{1}{A}\sqrt{-3B^2 + 12AC}}{2}. \tag{4.45}$$

Then, equation (4.45) is represented below for $t_{1,2} \in (0,1)$ solely in terms of functions $A, B, C, \triangle$ for $\triangle < 0$:

$$t_{1,2} = \frac{-B \pm \sqrt{-3\triangle}}{2A}. \tag{4.46}$$

Case 3: One inflection point.

The cubic Bézier curve has one real inflection point for $t \in (0,1)$ when t is associated with the quantities below for $(A, B, C) \neq \mathbf{0}$:

$$\begin{cases} F(0)F(1) = C(A + B + C) < 0 \\ \triangle > 0. \end{cases} \tag{4.47}$$

Note that in terms of the quantities of (4.47), $\triangle \neq 0$ due to the necessity that $\lambda \neq 0$, $\mu \neq 0$ or $4\lambda - 3\lambda\mu + 4\mu - 4 \neq 0$. If these conditions are not satisfied, then the cubic Bézier curve will degenerate into a straight line as plugging $\lambda = \mu = 0$ into $\triangle$ will give $\triangle = 0$ and taking $\lambda = \mu = 0$ will give a linear ($n = 1$) Bézier curve of control points $\{P_0, P_1\} = \{(0,1),(1,0)\}$, which is not possible. Also, if $4\lambda - 3\lambda\mu + 4\mu - 4 = 0$ then the cubic Bézier curve must only a have a cusp point, which cannot occur as the curve cannot have two special types of points simultaneously. In addition, $\triangle \nless 0$ as the quantity $\frac{-B-\sqrt{\triangle}}{2A}$ or the quantity $\frac{-B+\sqrt{\triangle}}{2A}$ (depending on which of these quantities are within the interval (0,1)) must be real.

If the cubic Bézier curve for $t \in (0,1)$ has exactly one real inflection point, then the sign of the curvature before the inflection point and after the inflection point will change. Therefore, one must determine the signs of the curvature at the endpoints $t = 0$ and $t = 1$ (in the limiting sense). This will give the condition below:

$$\kappa(0)\kappa(1) < 0.$$

The quantities $\kappa(0)$ and $\kappa(1)$, using $\kappa(t) = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$, are shown below:

$$\kappa(0) = \frac{\|\vec{r}\,'(0)\times\vec{r}\,''(0)\|}{\|\vec{r}\,'(0)\|^3} = \frac{C}{27|\lambda|^3} = \frac{C}{27\lambda^2|\lambda|},$$

$$\kappa(1) = \frac{\|\vec{r}\,'(1)\times\vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \frac{A+B+C}{27|\mu|^3} = \frac{A+B+C}{27\mu^2|\mu|}.$$

Then, for there to be one inflection point on the cubic Bézier curve on $t \in (0,1)$, either of the following cases must be satisfied:

$$\tfrac{C}{27\lambda^2|\lambda|} < 0 \text{ and } \tfrac{A+B+C}{27\mu^2|\mu|} > 0,$$

$$\tfrac{C}{27\lambda^2|\lambda|} > 0 \text{ and } \tfrac{A+B+C}{27\mu^2|\mu|} < 0.$$

The denominators of the quantities $\kappa(0)$ and $\kappa(1)$ are strictly positive due to $\lambda \neq 0$ and $\mu \neq 0$. Due to this, isolating for C and $A+B+C$ does not change the signs of the inequalities above. Then, this immediately gives the reduced conditions below:

$$C < 0 \text{ and } A+B+C > 0,$$

$$C > 0 \text{ and } A+B+C < 0.$$

The conditions above are equivalent to equations (4.28) and (4.29), respectively. Then, the condition above is equivalent to the condition below:

$$F(0)F(1) < 0.$$

Then, by (4.24) and (4.25), the condition above is equivalent to the condition below:

$$C(A+B+C) < 0. \tag{4.48}$$

Then, in terms of λ, μ, the region $C(A+B+C) < 0$ in standard form is shown below in terms of the curvature $\kappa(t)$:

$$(18\lambda\mu - 18\lambda)(18\lambda\mu - 18\mu) = 324\lambda\mu(1-\lambda)(1-\mu) < 0.$$

Which is equivalent to the condition below:

$$\lambda\mu(1-\lambda)(1-\mu) < 0.$$

In addition, though not necessary, there exists an inflection point when $A = \mathbf{0}$ when $\{A = \mathbf{0}\} \subset \{\lambda\mu(1-\lambda)(1-\mu) < 0\}$. This quantity is derived using the equation

$At^2 + Bt + C = 0$ below:

$$(0)t^2 + Bt + C = 0,$$

$$Bt + C = 0, \tag{4.49}$$

$$t = \frac{-C}{B}. \tag{4.50}$$

Note that when $A = 0$, equation (4.49) is linear.

Then, using (4.50), there exists an additional case regarding the existence of one real inflection point on a cubic Bézier curve for $t \in (0, 1)$, shown below:

$$\begin{cases} A = \mathbf{0} \\ 0 < \frac{-C}{B} < 1. \end{cases}$$

Case 4: Two inflection points.

The cubic Bézier curve has two real inflection points for $t \in (0, 1)$ when t is associated with the quantities below for $(A, B, C) \neq \mathbf{0}$:

$$\begin{cases} 0 < \frac{-B \pm \sqrt{\triangle}}{2A} < 1 \\ \triangle > 0. \end{cases} \tag{4.51}$$

The quantities $\frac{-B \pm \sqrt{\triangle}}{2A}$ of (4.51) are immediate when analyzing the roots of $At^2 + Bt + C = 0$. For the cubic Bézier curve to have two real inflection points for $t \in (0, 1)$, $At^2 + Bt + C = 0$ must correspond to two roots in the domain $(0, 1)$.

The quantities of (4.51) are sufficient but can be extended in terms of the product of (4.28) and (4.29). By definition, when the cubic Bézier curve on $t \in (0, 1)$ has two real inflection points, the curvatures at the endpoints (in the limiting sense) must have the same sign. The proof of how the product $\kappa(0)\kappa(1)$ implies the product $F(0)F(1)$ is given in Case 3.

Then, one obtains the condition below:

$$C(A+B+C) = AC + BC + C^2 = \lambda\mu(1-\lambda)(1-\mu) > 0.$$

As discussed, the quantities $\frac{-B\pm\sqrt{\triangle}}{2A} \in (0,1)$ implies that the cubic Bézier curve has two inflection points for $\triangle > 0$. Then, these quantities must be analyzed when $A > 0$ and $A < 0$.

For $A > 0:$

$$0 < \frac{-B \pm \sqrt{\triangle}}{2A} < 1,$$

$$0 < \frac{-B \pm \sqrt{\triangle}}{2} < A. \tag{4.52}$$

For $A < 0:$

$$0 < \frac{-B \pm \sqrt{\triangle}}{2A} < 1,$$

$$A < \frac{-B \pm \sqrt{\triangle}}{2} < 0. \tag{4.53}$$

Then, the quantities of (4.51) can also be represented as the following cases below:

$$\begin{cases} A > 0 \\ \triangle > 0 \\ AC + BC + C^2 > 0 \\ 0 < \frac{-B\pm\sqrt{\triangle}}{2} < A, \end{cases}$$

$$\begin{cases} A < 0 \\ \triangle > 0 \\ AC + BC + C^2 > 0 \\ A < \frac{-B \pm \sqrt{\triangle}}{2} < 0. \end{cases}$$

Case 5: No inflection points, no loop nor cusp (Arch).

The cubic Bézier curve has no cusp, denoted t^*, no loop, denoted $t_{1,2}$ and no inflection points for $t \in (0, 1)$ when t is associated with the quantities below for $(A, B, C) \neq \mathbf{0}$:

$$\begin{cases} (A, B, C) \neq \mathbf{0} \\ \triangle < 0 \\ AC + BC + C^2 > 0 \\ \frac{-B \pm \sqrt{-3\triangle}}{2A} \leq 0 \text{ or } \frac{-B \pm \sqrt{-3\triangle}}{2A} \geq 1. \end{cases}$$

First, a cubic curve with an arch has the property $F(0)F(1) = C(A+B+C) > 0$ due to no inflection points being present on $t \in (0, 1)$. In other words, $\kappa(0)$ and $\kappa(1)$ assume the same sign.

As the cubic curve cannot have a cusp point, $\triangle \neq 0$. Then, λ, μ cannot satisfy the equation below:

$$(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}.$$

Recall the condition $F(t) = At^2 + Bt + C = 0$, which detects the quantity and respective locations of the inflection points of the cubic Bézier curve, in which $A, B, C, \triangle$

are functions of (4.27). For there to be no real inflection points on the curve for $t \in (0, 1)$, the roots of the function $F(t)$ must not be real. These quantities are shown below:

$$\frac{-B \pm \sqrt{\triangle}}{2A} \in \mathbb{C}.$$

When the cubic Bézier curve is an arch, the function $F(t)$ must either be strictly greater than zero or strictly less than zero. In other words, the second derivative must either be strictly positive or strictly negative. These conditions are shown below:

$$F(t) < 0 \text{ or } F(t) > 0. \tag{4.54}$$

The first-order and second-order derivatives of the function $F(t) = At^2 + Bt + C$ are shown below for $A \neq \mathbf{0}$:

$$F'(t) = 2At + B, \tag{4.55}$$

$$F''(t) = 2A. \tag{4.56}$$

The goal is to show that no real inflection points are present when (4.54) is satisfied and when the parametric value t is outside the domain $t \in (0, 1)$. The unique critical point of $F(t)$ is shown below for $t \in \mathbb{R}$:

$$F'(t) = 2At + B = 0,$$

$$t^* = \frac{-B}{2A}.$$

Plugging t^* into $F(t)$ yields the computations below:

$$
\begin{aligned}
F(t^*) &= A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C, \\
F(t^*) &= \frac{B^2}{4A} - \frac{B^2}{2A} + C, \\
F(t^*) &= \frac{4AC - B^2}{4A}, \\
F(t^*) &= -\frac{B^2 - 4AC}{4A}, \\
F(t^*) &= \frac{-1}{4A}\triangle. \qquad (4.57)
\end{aligned}
$$

When $F(t) > 0$ for all t, then $F(t^*) > 0$ and when $F(t) < 0$ for all t, then $F(t^*) < 0$.

In addition, when $A > 0$, (4.56) is strictly positive. In other words, when $A > 0$, $F(t)$ is strictly concave up, which results in $t^* = \frac{-B}{2A}$ being a local minimum of the parabola $F(t)$. Then, when $A > 0$, $F(t)$ must be strictly positive, meaning that the parabola is strictly above the t-axis. Immediately, (4.57) gives $\triangle < 0$ due to no t-intercepts.

When $A < 0$, (4.56) is strictly negative. In other words, when $A < 0$, $F(t)$ is strictly concave down, which results in $t^* = \frac{-B}{2A}$ being a local maximum of the parabola $F(t)$. Then, when $A < 0$, $F(t)$ must be strictly negative, meaning that the parabola is strictly below the t-axis. Immediately, (4.57) also gives $\triangle < 0$ due to no t-intercepts.

As $\triangle < 0$, $F(t)$ cannot have any real solutions for $t \in \mathbb{R}$. Then, $\frac{-B \pm \sqrt{\triangle}}{2A} \in \mathbb{C}$.

Then, only **Case 2** is satisfied, which indicates the possible presence of a self-intersection point. For there to be no self-intersection on the domain $t \in (0, 1)$, the inequalities below must be satisfied in terms of the quantities $t_{1,2}$ of Definition 4.2:

$$\frac{-B \pm \sqrt{-3\triangle}}{2A} \leq 0 \text{ or } \frac{-B \pm \sqrt{-3\triangle}}{2A} \geq 1.$$

□

We will now consider the possible types of cubic Bézier curves based on the position on the given values (λ, μ) as a position of a point in the (λ, μ)-plane.

Corollary 4.1. *Let the control polygon of the cubic Bézier curve with control points* $\{P_0, P_1, P_2, P_3\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ *as in the previous theorem be denoted by* γ. *Let* $R = R(\lambda, \mu)$ *be called the characteristic point of the resulting Bézier curve.*

The distribution of the special types of points of the cubic Bézier curve with control points $\{P_0, P_1, P_2, P_3\} = \{(0,1), (\lambda, 1), (1, \mu), (1, 0)\}$ *are given below based on the position of* R *in one of the regions of the* (λ, μ)*-plane:*

$$\begin{cases} Cusp : R \in \bigcup_{i=1}^{2} C\_i \\ Loop : R \in \bigcup_{i=1}^{3} L\_i \\ One\ inflection\ point : R \in \bigcup_{i=1}^{4} O\_i \\ Two\ inflection\ points : R \in \bigcup_{i=1}^{3} T\_i \\ Arch : R \in \bigcup_{i=1}^{4} N\_i, \end{cases} \tag{4.58}$$

The regions of (4.58) are defined as follows:

$$
\begin{cases}
C\_1 \cup C\_2 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid (\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9} \setminus \{0 < \lambda < 1\}\} \\
L\_1 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 3\mu^2 - 3\mu + \lambda < 0 \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 > 0\} \\
L\_2 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 3\lambda^2 - 3\lambda + \mu < 0 \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 < 0\} \\
L\_3 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda > \frac{4}{3} \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 < 0\} \\
O\_1 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 0 < \lambda \leq 1 \ \mathit{and}\ \mu > 1\} \\
O\_2 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda < 0 \ \mathit{and}\ 0 < \mu \leq 1\} \\
O\_3 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 0 < \lambda \leq 1 \ \mathit{and}\ \mu < 0\} \\
O\_4 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda > 1 \ \mathit{and}\ 0 < \mu \leq 1\} \\
T\_1 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda < 0 \ \mathit{and}\ \mu > 1 \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 < 0\} \\
T\_2 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda > 1 \ \mathit{and}\ \mu < 0 \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 < 0\} \\
T\_3 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda > 1 \ \mathit{and}\ \mu > 1 \ \mathit{and}\ 4\lambda - 3\lambda\mu + 4\mu - 4 > 0\} \\
N\_1 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 0 < \lambda < 1 \ \mathit{and}\ 0 < \mu < 1\} \\
N\_2 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid \lambda < 0 \ \mathit{and}\ \mu < 0\} \\
N\_3 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 3\mu^2 - 3\mu + \lambda \geq 0 \ \mathit{and}\ \lambda < 0 \ \mathit{and}\ \mu > 1\} \\
N\_4 = \{(\lambda,\mu) \in \mathbb{R}^2 \mid 3\lambda^2 - 3\lambda + \mu \geq 0 \ \mathit{and}\ \lambda > 1 \ \mathit{and}\ \mu < 0\}
\end{cases}
\tag{4.59}
$$

The hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$ *(red curves) and the parabolas* $3\mu^2 - 3\mu + \lambda = 0$ *and* $3\lambda^2 - 3\lambda + \mu = 0$ *(blue curves) split the* λ, μ *plane in Figure 4.7 into various regions depending on the location of the characteristic point* R *in terms of* λ, μ *in* (4.59).

Remark 4.10. *The regions of* (4.58) *are represented by the picture below:*

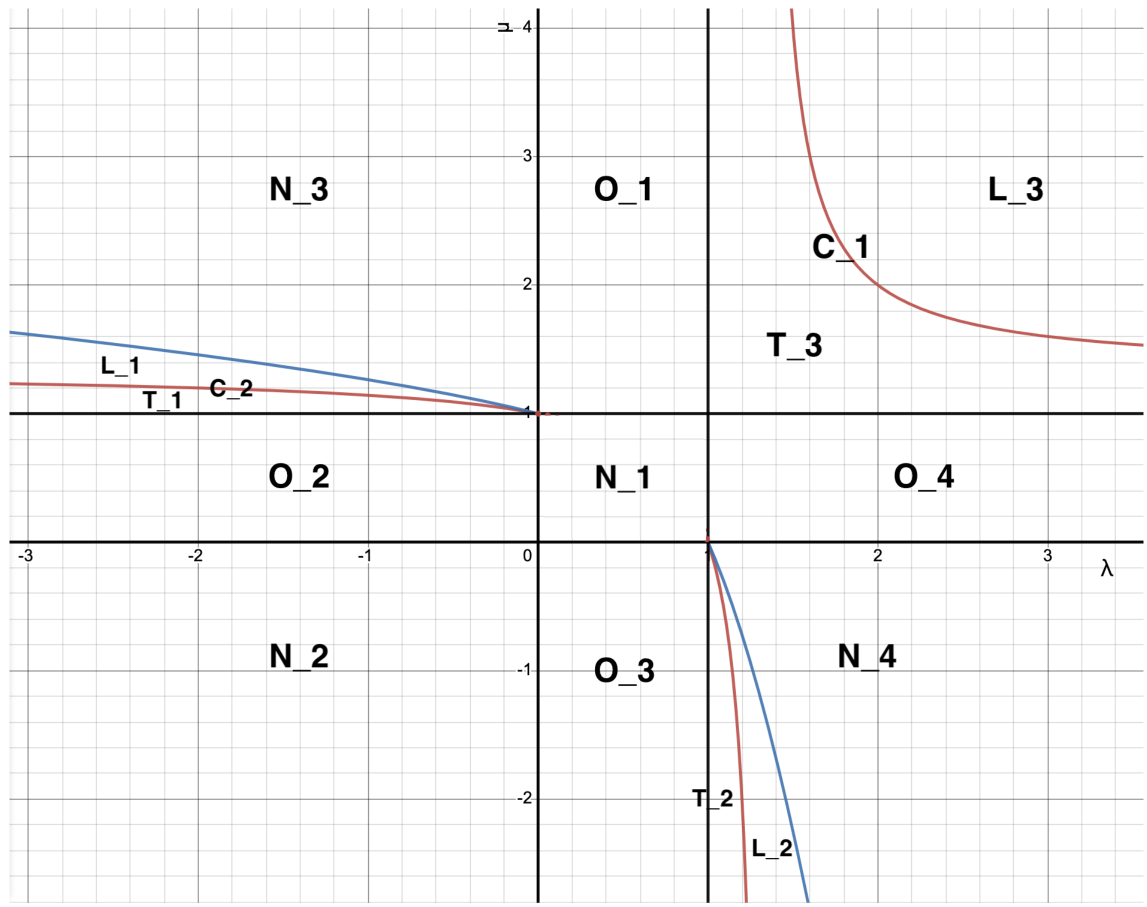

Figure 4.7: Distribution of the three types of special points of the λ, μ plane of a cubic Bézier curve with control points $\{(0,1),(\lambda,1),(1,\mu),(1,0)\}$.

Proof. The regions are determined by the conditions of Theorem 4.1 and are made explicit below:

Cusp point:

$$0 < \frac{-B}{2A} < 1,$$

$$\triangle = 0.$$

In terms of λ, μ, the quantities above become:

$$0 < \frac{\lambda(3\mu - 2)}{2(3\lambda\mu - \lambda - \mu)} < 1. \tag{4.60}$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) = 0.$$

When $A > 0$, (4.60) becomes:

$$0 < 3\lambda\mu - 2\lambda < 6\lambda\mu - 2\lambda - 2\mu,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) = 0.$$

Breaking down the above chain of inequalities gives:

$$\lambda(3\mu - 2) > 0,$$

$$3\lambda\mu - 2\lambda < 6\lambda\mu - 2\lambda - 2\mu,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) = 0.$$

Which becomes:

$$\lambda(3\mu - 2) > 0,$$

$$\mu(3\lambda - 2) > 0,$$

$$4\lambda - 3\lambda\mu + 4\mu - 4 = 0.$$

The two inequalities above and $\triangle = 0$ give the curve C_1, which is the upper section of the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$ in Figure 4.7.

When $A < 0$, the relations (4.60) become:

$$0 > 3\lambda\mu - 2\lambda > 6\lambda\mu - 2\lambda - 2\mu,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) = 0.$$

Breaking down the above chain of inequalities gives:

$$\lambda(3\mu - 2) < 0,$$

$$6\lambda\mu - 2\lambda - 2\mu < 3\lambda\mu - 2\lambda,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) = 0.$$

Which becomes:

$$\lambda(3\mu - 2) < 0,$$

$$\mu(3\lambda - 2) < 0,$$

$$4\lambda - 3\lambda\mu + 4\mu - 4 = 0.$$

The two inequalities above and $\triangle = 0$ give the curve C_2, which is the lower section of the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$ in Figure 4.7. Note that the curve C_2 excludes the values $\lambda, \mu \in (0, 1)$.

Self-intersection point:

$$0 < \frac{-B \pm \sqrt{-3\triangle}}{2A} < 1, \tag{4.61}$$

$$\triangle < 0.$$

Breaking down (4.61) gives the following inequalities:

$$\frac{-B - \sqrt{-3\triangle}}{2A} > 0,$$

$$\frac{-B + \sqrt{-3\triangle}}{2A} < 1,$$

$$\triangle < 0.$$

When $A > 0$, the inequalities above can be written as:

$$0 < \sqrt{-3\triangle} < -B, \tag{4.62}$$

$$0 < \sqrt{-3\triangle} < 2A + B. \tag{4.63}$$

Using the positivity of $-B$ and $2A + B$, another inequality is produced, which is shown below:

$$2A > -B > 0.$$

Which can be broken down into the following inequalities:

$$2A > -B,$$

$$B < 0.$$

Squaring both sides of (4.62) and (4.63) gives the inequalities below:

$$B^2 > -3\triangle > 0,$$

$$4A^2 + 4AB + B^2 > -3\triangle > 0,$$

$$2A > -B,$$

$$B < 0,$$

$$\triangle < 0.$$

Then, replacing $\triangle$ with $B^2 - 4AC$ and simplifying gives:

$$B^2 - 3AC > 0,$$

$$A^2 + AB + B^2 - 3AC = A(A + B - 3C) + B^2 > 0,$$

$$2A > -B,$$

$$B < 0,$$

$$\triangle < 0.$$

In terms of λ, μ, the inequalities above become:

$$\lambda^2 + 3\lambda\mu^2 - 3\lambda\mu > 0,$$

$$\mu^2 + 3\mu\lambda^2 - 3\lambda\mu > 0,$$

$$6\lambda\mu - 2\lambda - 2\mu > 3\lambda\mu - 2\lambda,$$

$$\lambda(2 - 3\mu) < 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) < 0.$$

Which becomes:

$$\lambda(\lambda + 3\mu^2 - 3\mu) > 0,$$

$$\mu(\mu + 3\lambda^2 - 3\lambda) > 0,$$

$$\mu(3\lambda - 2) > 0,$$

$$\lambda(3\mu - 2) > 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) < 0.$$

The four inequalities above and $\triangle < 0$ give the region L_3 in Figure 4.7.

When $A < 0$, one should use the quantities below:

$$\frac{-B + \sqrt{-3\triangle}}{2A} > 0,$$

$$\frac{-B - \sqrt{-3\triangle}}{2A} < 1,$$

$$\triangle < 0.$$

The inequalities above can be written as:

$$0 < \sqrt{-3\triangle} < B, \tag{4.64}$$

$$0 < \sqrt{-3\triangle} < -(2A + B). \tag{4.65}$$

Using the positivity of B and $-2A - B$, another inequality is produced, which is shown below:

$$2A < -B < 0.$$

The latter can be broken down into the following inequalities:

$$2A < -B,$$

$$B > 0.$$

Squaring both sides of (4.64) and (4.65) gives the inequalities below:

$$B^2 > -3\triangle > 0,$$

$$4A^2 + 4AB + B^2 > -3\triangle > 0,$$

$$2A < -B,$$

$$B > 0,$$

$$\triangle < 0.$$

Then, replacing $\triangle$ with $B^2 - 4AC$ and simplifying gives:

$$B^2 - 3AC > 0,$$

$$A^2 + AB + B^2 - 3AC = A(A + B - 3C) + B^2 > 0,$$

$$2A < -B,$$

$$B > 0,$$

$$\triangle < 0.$$

In terms of λ, μ, the inequalities above become:

$$\lambda^2 + 3\lambda\mu^2 - 3\lambda\mu > 0,$$

$$\mu^2 + 3\mu\lambda^2 - 3\lambda\mu > 0,$$

$$6\lambda\mu - 2\lambda - 2\mu < 3\lambda\mu - 2\lambda,$$

$$\lambda(2 - 3\mu) > 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) < 0.$$

These inequalities become:

$$\lambda(\lambda + 3\mu^2 - 3\mu) > 0,$$

$$\mu(\mu + 3\lambda^2 - 3\lambda) > 0,$$

$$\mu(3\lambda - 2) < 0,$$

$$\lambda(3\mu - 2) < 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) < 0.$$

The four inequalities above and $\triangle < 0$ give the regions L_1 and L_2 in Figure 4.7. In addition, note that L_1 is bounded by the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$ and the parabola $3\mu^2 - 3\mu + \lambda = 0$. Also, the region L_2 is bounded by the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$ and the parabola $3\lambda^2 - 3\lambda + \mu = 0$.

One inflection point:

$$C(A + B + C) < 0,$$

$$\triangle > 0.$$

Note that the proof of the derivation of the quantities above was provided in **Case 4** of Theorem 4.1. Recall that the region $C(A+B+C) = 324(1-\lambda)(1-\mu) < 0$ is rectangular and corresponds to regions O_1, O_2, O_3 and O_4 in Figure 4.7.

Two inflection points:

$$0 < \frac{-B \pm \sqrt{\triangle}}{2A} < 1,$$

$$\triangle > 0.$$

Breaking down the inequality above gives the following inequalities:

$$\frac{-B - \sqrt{\triangle}}{2A} > 0,$$

$$\frac{-B + \sqrt{\triangle}}{2A} < 1,$$

$$\triangle > 0.$$

When $A > 0$, the inequalities above can be written as:

$$0 < \sqrt{\triangle} < -B, \tag{4.66}$$

$$0 < \sqrt{\triangle} < 2A + B. \tag{4.67}$$

Using the positivity of $-B$ and $2A + B$, another inequality is produced, which is shown below:

$$2A > -B > 0.$$

This can be broken down into the following inequalities:

$$2A > -B,$$

$$B < 0.$$

Squaring both sides of (4.66) and (4.67) gives the inequalities below:

$$B^2 > \triangle > 0,$$

$$4A^2 + 4AB + B^2 > \triangle > 0,$$

$$2A > -B,$$

$$B < 0,$$

$$\triangle > 0.$$

Then, replacing $\triangle$ with $B^2 - 4AC$ and simplifying gives:

$$AC > 0, \tag{4.68}$$

$$A + B + C > 0,$$

$$2A > -B,$$

$$B < 0,$$

$$\triangle > 0.$$

Note that when $A > 0$, (4.68) reduces to $C > 0$.

In terms of λ, μ, the inequalities above become:

$$\lambda(\mu - 1) > 0,$$

$$\mu(\lambda - 1) > 0,$$

$$6\lambda\mu - 2\lambda - 2\mu > 3\lambda\mu - 2\lambda,$$

$$\lambda(2-3\mu) < 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) > 0.$$

This becomes:

$$\lambda(\mu - 1) > 0,$$

$$\mu(\lambda - 1) > 0,$$

$$\mu(3\lambda - 2) > 0,$$

$$\lambda(3\mu - 2) > 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) > 0.$$

The four inequalities above and $\triangle > 0$ corresponds to region T_3 in Figure 4.7.

When $A < 0$, one should use the quantities below:

$$\frac{-B+\sqrt{\triangle}}{2A} > 0,$$

$$\frac{-B-\sqrt{\triangle}}{2A} < 1,$$

$$\triangle > 0.$$

When $A < 0$, the inequalities above can be written as:

$$0 < \sqrt{\triangle} < B, \tag{4.69}$$

$$0 < \sqrt{\triangle} < -(2A+B). \tag{4.70}$$

Using the positivity of B and $-(2A+B)$, another inequality is produced, which is shown below:

$$2A < -B < 0.$$

This can be broken down into the following inequalities:

$$2A < -B,$$

$$B > 0.$$

Squaring both sides of (4.69) and (4.70) gives the inequalities below:

$$B^2 > \triangle > 0,$$

$$4A^2 + 4AB + B^2 > \triangle > 0,$$

$$2A < -B,$$

$$B > 0,$$

$$\triangle > 0.$$

Then, replacing $\triangle$ with $B^2 - 4AC$ and simplifying gives:

$$AC > 0, \tag{4.71}$$

$$A + B + C < 0,$$

$$2A < -B,$$

$$B > 0,$$

$$\triangle > 0.$$

Note that when $A < 0$, (4.71) reduces to $C < 0$.

Then, on terms of λ, μ, the inequalities above become:

$$\lambda(\mu - 1) < 0,$$

$$\mu(\lambda - 1) < 0,$$

$$6\lambda\mu - 2\lambda - 2\mu < 3\lambda\mu - 2\lambda,$$

$$\lambda(2 - 3\mu) > 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) > 0.$$

This becomes:

$$\lambda(\mu - 1) < 0,$$

$$\mu(\lambda - 1) < 0,$$

$$\mu(3\lambda - 2) < 0,$$

$$\lambda(3\mu - 2) < 0,$$

$$\lambda\mu(4\lambda - 3\lambda\mu + 4\mu - 4) > 0.$$

The four inequalities above and $\triangle > 0$ give the regions T_1 and T_2 in Figure 4.7. In addition, note that T_1 is bounded by the line $\mu = 1$ and the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$. Also, the region T_2 is bounded by the line $\lambda = 1$ and the hyperbola $(\lambda - \frac{4}{3})(\mu - \frac{4}{3}) = \frac{4}{9}$.

Also, the regions that have not been produced by any of the inequalities above are the regions $\bigcup_{i=1}^{4}$N_i.

□

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