Broadcasting in highly connected graphs

Aram Khanlari

A Thesis

in

The Department

of

Computer Science and Software Engineering

Presented in Partial Fulfillment of the Requirements

for the Degree of

Master of Computer Science (Computer Science) at

Concordia University

Montréal, Québec, Canada

August 2023

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: **Aram Khanlari**

Entitled: **Broadcasting in highly connected graphs**

and submitted in partial fulfillment of the requirements for the degree of

Master of Computer Science (Computer Science)

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the Final Examining Committee:

______________________________ Chair
Dr. Denis Pankratov

______________________________ External Examiner
Dr.

______________________________ Examiner
Dr. Lata Narayanan

______________________________ Supervisor
Dr. Hovhannes A. Harutyunyan

Approved by ______________________________
Joey Paquet, Chair
Department of Computer Science and Software Engineering

____________ 2023 ______________________________
Mourad Debbabi, Dean
Faculty of Engineering and Computer Science

Abstract

Broadcasting in highly connected graphs

Aram Khanlari

Throughout history, spreading information has been an important task. With computer networks expanding, fast and reliable dissemination of messages became a problem of interest for computer scientists. Broadcasting is one category of information dissemination that transmits a message from a single originator to all members of the network. In the past five decades the problem has been studied by many researchers and all have come to demonstrate that despite its easy definition, the problem of broadcasting does not have trivial properties and symmetries. For general graphs, and even for some very restricted classes of graphs, the question of finding the broadcast time and scheme remains NP-hard. This work uses graph theoretical concepts to explore mathematical bounds on how fast information can be broadcast in a network. The connectivity of a graph is a measure to assess how separable the graph is, or in other words how many machines in a network will have to fail to disrupt communication between all machines in the network.

We initiate the study of finding upper bounds on broadcast time $b(G)$ in highly connected graphs. In particular, we give upper bounds on $b(G)$ for k-connected graphs and graphs with a large minimum degree.

We explore 2-connected (biconnected) graphs and broadcasting in them. Using Whitney's open ear decomposition in an inductive proof we propose broadcast schemes that achieve an upper bound of $\lceil \frac{n}{2} \rceil$ for classical broadcasting as well as similar bounds for multiple originators. Exploring further, we use a matching-based approach to prove an upper bound of $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$ for all k-connected graphs. For many infinite families of graphs, these bounds are tight.

Discussion of broadcasting in highly connected graphs leads to an exploration of dependence between the minimum degree in the graph and the broadcast time of the latter. By using similar

techniques and arguments we show that if all vertices of the graph are neighboring linear numbers of vertices, then information dissemination in the graph can be achieved in $\lceil \log n \rceil + C$ time.

To the best of our knowledge, the bounds presented in our work are a novelty. Methods and questions proposed in this thesis open new pathways for research in broadcasting.

Keywords: Broadcasting, 2-connected graphs, k-connected graphs, minimum degree, diameter

Acknowledgments

First, I want to thank my supervisor professor, Hovhannes Harutyunyan. His support, counsel and encouragement made this an inspiring experience for me.

I wish to extend my deep gratitude towards Ararat Harutyunyan for his invaluable assistance and guidance throughout my research and Louisa Harutyunyan for her immense support and care.

I am extremely thankful and lucky to be where I am: To all my teachers and professors, mentors and motivators in life, every circumstance and enlightenment that brought me here. I thank you.

My colleagues in the research labs and classmates from Concordia. Discussions and lunch breaks with you were enriching and lovely. I am thankful to my students: passing knowledge to others is what ignites my fire. Through teaching I learn.

My dearest friends: whether laughing in the streets of Yerevan or creating memories together in Montreal, I love you and am thankful for your existence in my life. Anyone who has you by their side has a Master's degree in life.

Above all, my deepest gratitude goes towards my family. My mother's great acts of support, my father's kind words of wisdom, my sister's shear sweetness, my grandparents care, my aunts' encouragement, my cousins' love and my extended family's help were pillars for my journey. I love you and thank you with all my heart.

And this message is for my future self reading this: zoom out, you got this, love you.

Contents

List of Figures

List of Tables

Notation Table

$G(V, E)$	An undirected loop-free graph
$V(G)$	Set of vertices of graph G
$E(G)$	Set of edges of graph G
n	Number of vertices in graph G
m	Number of edges in graph G
$u \in V$	A vertex in graph G
$b(u, G)$	Broadcast time of vertex u in graph G
$b(G)$	Broadcast time of graph G
$D(G)$	Diameter of graph G
$deg(i)$	Degree of vertex i
$\delta(G)$	Minimum degree in graph G
$\Delta(G)$	Maximum degree in graph G

All logarithms in this work are base two ($\log x = \log_2 x$), unless otherwise specified.

Chapter 1

Introduction

Since the beginning of time, the spreading of information has been a crucial part of life for all species. The evolution and circle of life highly depend on transmitting genetic information. Bacteria and all eukaryotes perform a procedure called binary fission to reproduce and transmit genetic information. A similar procedure called mitosis takes place in all cells in the human body. Passing information around has been a crucially important task for humanity as well. From homing pigeons to fiber-optic cables, the means of dissemination of information have evolved over time, but the purpose remained unchanged: sharing valuable information in the safest and fastest manner.

This phenomenon finds its depiction in the mathematical model of information dissemination. With the invention and rise of computers, humanity entered the era of information, and safe and fast transmission of messages in a network became a major question.

In the past decades, computer scientists have tried to overcome theoretical, physical, and technological obstacles to ensure the reliable spread of information in networks. This, of course, has many applications, including but not limited to High-Performance Computing (J.-K. Lee, Hong, & Li, 2021; Rocher-Gonzalez, Escudero-Sahuquillo, García, & Quiles, 2017), parallel computing (Varvarigos & Bertsekas, 1995), multiprocessor systems (Chen, Shin, & Kandlur, 1990), etc.

Our work concerns broadcasting: the dissemination of information from a single originator to all members in a network. Much like eukaryotes and mitosis, computers depend on sending and receiving information all the time. On top of this, failures and malfunctions in computers happen as well. The concept of separating set(vertex cut) of a network depicts this phenomenon in graph theory. Connectivity is a measure to assess how separable the graph is, or in other words, how many machines in a network will have to fail to disrupt communication between all machines. Through such definitions and using mathematical tools, we shed light on broadcasting in graphs that are highly connected or possess other interesting properties.

We initiate the study of finding upper bounds on broadcast time $b(G)$ in highly-connected graphs. In particular, we give upper bounds on $b(G)$ for k-connected graphs and graphs with a large minimum degree.

This thesis is structured as follows: In Chapter 3, we present a theorem on the broadcast time of 2-connected graphs and the inductive proof using Whitney's theorem for open ear decomposition. Chapter 4 discusses a generalization of the two-connected graphs to k-connected graphs. We examine an infinite family of k-connected graphs, based on which we propose a theorem on broadcast time in k-connected graphs. We prove the theorem using consecutive matchings of certain sizes which exist by Kőnig's theorem in each time unit. In Chapter 5 we consider graphs with a relaxed condition and give upper bounds based on δ minimum degree in a graph. We draw conclusions and suggest pathways of future research in Chapter 6.

Chapter 2

Literature Review and Preliminaries

2.1 Spreading information in networks

Spreading information in networks can be done in various ways. These are some of the protocols to disseminate information in a graph where nodes of the network are represented by vertices and links by edges.

(1) Routing: A vertex sends the information i to another vertex. (One-to-One)

(2) Multi-casting: A vertex sends the information i to some of the other vertices. (One-to-Many)

(3) Broadcasting: A vertex sends the information i to all the other vertices. (One-to-All)

(4) Gossiping: All vertices send some information i_k to all other vertices. (All-to-All)

These methods serve different goals and have various applications. Our work concentrates on broadcasting. Broadcasting is a problem in which a sender, usually called the *originator*, has a piece of information in a network and wishes to inform all network members of this message. This is achieved by placing a series of calls over the network's communications links while respecting the following conditions (Hedetniemi, Hedetniemi, & Liestman, 1988):

(1) Time units are discrete

(2) A call takes place when an informed vertex informs one of its uninformed neighbors

(3) Each call involves exactly two adjacent (neighboring) vertices (a sender and a receiver),

(4) Each call requires one time unit,

(5) Each vertex can participate in only one call in each time unit,

(6) In one time unit, multiple calls can be performed in parallel.

The process ends when all vertices are informed. An informed vertex v is idle in time unit t if v has no uninformed neighbors and does not make any calls in time t. The set of calls used to distribute the message from originator v to all other vertices is called a *broadcast scheme* for vertex v. The broadcast scheme is a spanning tree rooted at v and all the communication lines are labeled with the transmission time. This is also known as a *broadcast tree*.

Under the general purpose of message dissemination, various modifications are possible by placing constraints on the originator, receiver, and message sets, on the network topology, the regulations of message transmissions, and/or the information about the network known to individual network members. Here are some of the categories:

- Single-port or multi-port models discussed by Fraigniaud and Lazard (1994) and H. A. Harutyunyan and Liestman (2001b)
- Multiple message broadcasting studied by Bar-Noy and Kipnis (1994) and Bruck, Cypher, and Ho (1992)
- Fault-tolerant broadcasting studied by Ahlswede, Gargano, Haroutunian, and Khachatrian (1996)
- Vertex disjoint and edge disjoint paths discussed by Farley (2004)
- Radio broadcasting discussed by Alon, Bar-Noy, Linial, and Peleg (1991)
- Universal lists broadcasting discussed by Diks and Pelc (1996)
- Messy broadcasting discussed by Ahlswede, Haroutunian, and Khachatrian (1994)

Of course, in real-life networks message spreading takes place very differently from all these models, especially from the classical model, but from the perspective of research, these constraints help understand the nature of the phenomenon and achieve theoretical results which can be applied to networks in real life.

2.2 Classical broadcasting

In our work, we use the terms network and graph interchangeably where the nodes of the network are graph vertices, and communication links between nodes are edges. Let $G = (V, E)$ be an undirected loop-free graph representing a network. The broadcast time of a vertex $v \in V$, denoted by $b(v, G)$, is the minimum time required to complete broadcasting when the originator is v. The broadcast time of the graph G, denoted by $b(G)$, is the maximum among all vertices' broadcast times i.e.

$$b(G) = \max_{v \in V(G)} \{b(v, G)\} \tag{1}$$

Given a graph $G = (V, E)$ on n vertices, while broadcasting from any originator, at each time unit the number of informed vertices can at most double, hence $b(G) \geq \lceil \log_2 n \rceil$. Also, as Fraigniaud and Lazard (1994) state, another lower bound for broadcasting is the diameter $D(G)$ defined as the maximum distance between any pair of vertices of the graph.

On the other hand, at least one vertex must be informed in each round, leading to the upper bound of $n - 1$. Overall for an arbitrary graph G on n vertices with diameter $D(G)$

$$\max\{\lceil \log n \rceil, D(G)\} \leq b(G) \leq n - 1 \tag{2}$$

Generally in broadcasting there are two main directions of research:

(1) Minimum broadcast graph problem: Given a positive integer n as the number of vertices, design a graph with the least cost (minimum number of edges) that achieves $\lceil \log n \rceil$ broadcast time or determine the broadcast function $B(n)$.

(2) Broadcast time problem: Finding the broadcast time or finding an optimal broadcast scheme of a given graph and general graphs with different topologies and characteristics.

These problems are discussed in 2.2.1 and 2.2.2, respectively.

2.2.1 Minimum broadcast graph problem

A graph G on n vertices is called a *broadcast graph (bg)* if $b(G) = \lceil \log n \rceil$. A *bg* with the minimum number of edges is called a *minimum broadcast graph (mbg)*. The minimum number of edges is called the *broadcast function* and is denoted by $B(n)$. From the perspective of network design, *mbg* minimizes the number of links (and thus the cost) of building the network which achieves the best (minimum) broadcast time.

The problem was proposed by Farley, Hedetniemi, Mitchell, and Proskurowski (1979). The authors also presented the values and corresponding graphs of $B(n)$ for $n \leq 15$ and $n = 2^k$ and introduced hypercubes as the first infinite family of mbg's. There are two other known infinite families of mbg's. Knödel graphs introduced in (Knödel, 1975) were proved to be mbg's for $n = 2^{\Delta}$, and $n = 2^{\Delta} - 2$ by Dinneen, Fellows, and Faber (1991); Khachatrian and Harutounian (1990) and recursive circulant graphs as a non-isomorphic alternative to hypercubes for mbg's on $n = 2^k$ vertices (Park & Chwa, 1994).

The values of $B(n)$ are also known for

- $n \leq 16$ (Farley, 1979)
- $n = 17$ (Mitchell & Hedetniemi, 1980)
- $n = 18, 19$ (Bermond, Hell, Liestman, & Peters, 1992b; Xiao & Wang, 1988)
- $n = 20, 21, 22$ (Maheo & Saclé, 1994),
- The values of $B(n)$ for $n =$ 23, 24, and 25 still remain unknown.
- $n = 26$ (Saclé, 1996; Zhou & Zhang, 2001)
- $n = 27, 28, 29$ (Saclé, 1996)
- $n = 30$ (Bermond, Hell, Liestman, & Peters, 1992b)
- $n = 58, 59, 61$ (Saclé, 1996)

The values of $B(n)$ are also known for some $n = 2^k - 1$, such as

- $n = 31$ (Bermond, Hell, Liestman, & Peters, 1992b)

- $n = 63$ (Labahn, 1994)
- $n = 127$ (H. A. Harutyunyan, 2008)
- $n = 1023, 4095$ (Shao, 2006)

However, there is no infinite family of mbg's for all such n known yet.

Bermond, Hell, Liestman, and Peters (1992a); Liestman and Peters (1988) explored bounded degree minimum broadcast graphs. Construction of minimum broadcast graphs for the k -broadcasting model was studied by H. A. Harutyunyan and Liestman (2001a).

2.2.2 Minimum broadcast time problem

The formal definition of the decision problem is as follows:

Minimum Broadcast Time (MBT) from (Garey & Johnson, 1983) Problem [ND49]: *Given a graph $G = (V, E)$ with a subset $V_0 \subseteq V$, and a positive integer K. Can a message be "broadcast" from the base set V_0 to all other vertices in time K, i.e., is there a sequence $V_0, E_1, V_1, E_2, V_2, \cdots, E_K, V_K$ such that each $V_i \subseteq V$, each $E_i \subseteq E$, $V_K = V$, and, for $1 \leq i \leq K$,*

(1) each edge in E_i has exactly one endpoint in V_{i-1}

(2) no two edges in E_i share a common endpoint

(3) $V_i = V_{i-1} \cup \{v : \{u, v\} \in E_i\}$?

The base set V_0 is the set of originators. When $|V_0| = 1$, broadcasting has one originator and the problem remains NP-complete. This decision problem was proven to be NP-complete in (Slater, Cockayne, & Hedetniemi, 1981) using a reduction from the three-dimension matching (3DM) problem.

The problem of finding $b(v, G)$ and $b(G)$ are both NP-Hard for arbitrary graphs and originators (Garey & Johnson, 1983; Slater, Cockayne, & Hedetniemi, 1981). Besides, the problem remains NP-Hard in more restricted families such as bounded degree graphs (Dinneen, 1994) and 3-regular planar graphs (Jakoby, Reischuk, & Schindelhauer, 1998; Middendorf, 1993).

Like many NP-complete problems, researchers approached the Minimum broadcast time problem from different directions. The three mains are: Exact algorithms, Approximation algorithms, and Heuristics.

In the rest of this section, some studies related to each of these are discussed. Also, since there are several papers dealing with finding the broadcast time of graphs, we mention a few survey papers from which the reader can trace back all the previous works (Fraigniaud & Lazard, 1994; H. A. Harutyunyan, Liestman, Peters, & Richards, 2013; Hedetniemi, Hedetniemi, & Liestman, 1988; Hromkovič, Klasing, Monien, & Peine, 1996). As a recent study of the complexity of the problem, we also refer the reader to Fomin, Fraigniaud, and Golovach (2023).

Exact algorithms

This problem is solved by exact algorithms for some families of graphs, which include, but are not limited to

- Trees (Proskurowski, 1981; Slater, Cockayne, & Hedetniemi, 1981),
- Grids and Tori (Farley & Hedetniemi, 1978),
- Cube Connected Cycles (Liestman & Peters, 1988),
- Shuffle Exchange (Hromkovič, Jeschke, & Monien, 1993),
- Unicyclic (containing a single cycle) graphs (H. A. Harutyunyan & Maraachlian, 2007, 2008),
- Tree of cycles (H. A. Harutyunyan & Maraachlian, 2009b),
- Fully connected trees (Gholami, Harutyunyan, & Maraachlian, 2022; H. A. Harutyunyan & Maraachlian, 2009a),
- Necklace graph (H. Harutyunyan, Laza, & Maraachlian, 2009),
- other tree-like graphs like tree of cliques, 2-restricted cacti (Maraachlian, 2010),
- k-cacti with constant k (Čevnik & Žerovnik, 2017), and
- graph families listed in 2.4

Some exact approaches formulated the optimization version of the problem and proposed dynamic programming algorithm (Scheuermann & Wu, 1984) and integer linear programming models (De Sousa et al., 2018; Ivanova, 2019).

Approximation algorithms

One of the first approximation algorithms gives an additive $(\sqrt{|V|})$-approximation algorithm for finding the broadcast time of any graph (Kortsarz & Peleg, 1995). These results were improved by Ravi (1994) who proposed a $(\frac{\log^2|V|}{\log\log|V|})$-approximation algorithm. The approach in this paper uses a graph theoretical concept of *poise* of a spanning tree is defined as $P(T) = D(T) + \Delta(T)$, where D is the diameter and Δ is the maximum degree in the tree. The poise of a graph G, denoted $P(G)$, is defined as the minimum poise of all its spanning trees. Determining the $P(G)$ is an NP-hard problem of its own, but the author gives $O(\log|V|)$ approximation algorithm for finding the poise of the graph and shows that $b(G) = O\left(\frac{\log|V|}{\log\log|V|} \cdot P(G)\right)$ which yields the mentioned approximation algorithm.

Bar-Noy, Guha, Naor, and Schieber (1998) worked on the multicasting problem in which the aim is to inform a subset T of vertices. Using linear programming they gave an $O(\log|T|)$-approximation for multicasting which can be generalized to a $(\log|V|)$-approximation algorithm for broadcasting.

Later, Elkin and Kortsarz (2005) introduced a combinatorial algorithm with the same approximation ratio and then improved it to a sublogarithmic $O(\frac{\log|T|}{\log\log|T|})$-approximation algorithm for multicasting (or a $O(\frac{\log|V|}{\log\log|V|})$-approximation solution for broadcasting) in (Elkin & Kortsarz, 2006). This algorithm currently offers the best approximation known for this problem.

There are several results on the lower bound of the possible approximation ratio as well. With the assumption of $P \neq NP$ the inapproximability of the problem was shown by polynomial time reduction from $E3 - SAT$ by Schindelhauer (2000). The author showed, that it is NP-hard to approximate $b(u)$ within a factor of $\frac{57}{56} - \epsilon$ for arbitrary $\epsilon > 0$. Elkin and Kortsarz (2005) improved this to a factor of $3 - \epsilon$ for general multicast models.

Another direction of approximation algorithms is tailoring the algorithms for a specific family or class of graphs. Examples of work in this direction can be found for graphs with intersecting cycles (Bhabak & Harutyunyan, 2015), k-path graphs (Bhabak & Harutyunyan, 2019), graphs with known broadcast time of the base graph (Bhabak & Harutyunyan, 2022), flower graphs (Ehresmann, 2021), Harary-like graphs (Bhabak, Harutyunyan, & Kropf, 2017; Bhabak, Harutyunyan, & Tanna, 2014), etc.

Heuristics

Many heuristics have been proposed for the broadcasting problem. This is a non exhaustive list:

- Matching-based approach (Beier & Sibeyn, 2000)
- Coloring-based approach (Beier & Sibeyn, 2000)
- The Round-Heuristic (Beier & Sibeyn, 2000)
- Heuristics for directed graphs (Elkin & Kortsarz, 2005)
- Random and semi-random heuristic (H. A. Harutyunyan & Wang, 2010)
- Deep Heuristic for arbitrary graphs (H. A. Harutyunyan, Hovhannisyan, & Magithiya, 2022)
- Tree-based heuristic (H. Harutyunyan & Shao, 2006) introducing the concept of "bright border"
- A notable work among all the heuristics is the derivation of a very important recurrent relation described by Scheuermann and Wu (1984), which gave motivation for our work in Chapter 4.

2.3 Other models of broadcasting

2.3.1 Multiple originator broadcasting

This model considers multiple originators and a critical problem here is determining the number of originators that are required to complete the process in a specified amount of time. Farley and Proskurowski (1981) discussed the problem in trees in at most t time units for an arbitrary t and gave a linear algorithm for decomposing a tree into a minimum number of subtrees such that broadcasting can be completed in at most t time units in each subtree. This model closely relates to the NP-complete problem of $k - SPST$(k shortest path spanning tree) defined by Farley, Fragopoulou, Krumme, Proskurowski, and Richards (2000), which is: Given a graph G with the length function l, k sources $s_1, ..s_k \in V$, and a positive integer K is there a spanning tree T of G whose cost (the sum of edge lengths on the paths from all sources to all vertices) does not exceed K? The authors have shown that $2 - SPST$ is NP-complete (Farley, Fragopoulou, Krumme, Proskurowski, & Richards, 2000). Also Chia, Kuo, and Tung (2007) show two-originator broadcast time of the Grid, and the k-originator broadcast time of the complete k-partite graphs and the Hypercubes. In general, this model can be a subproblem for broadcasting with classical constraints. In chapter 3, we use this idea to show bounds.

2.3.2 Multiple message broadcasting

In real-life networks when communicating large amounts of data, information is divided into sequence of packets, which are sent consecutively, but contain sequence numbers to ensure proper delivery. In this model, the originator has k messages $M_1, M_2, ...M_k$ and has to transmit them to all vertices. The problem of broadcasting multiple messages has been studied in several communication models. The telegraph model in directed graphs has been studied in Chinn, Hedetniemi, and Mitchell (1979); Cockayne (1979) and in classical undirected model has been studied in Bar-Noy and Kipnis (1994); Bar-Noy, Kipnis, and Schieber (2000); H. A. Harutyunyan (2000, 2006). Suderman (1999) discusses multiple message broadcasting in common topologies like trees, cycles, paths. In Gregor, Škrekovski, and Vukašinović (2018), the authors discuss the model for message dissemination in the three infinite families of minimum broadcast graphs: Hypercube, Knödel and Circulant graphs as well as Tori and other topologies. To further resemble real-life networks, the model has also been studied in models with latency (Bar-Noy & Kipnis, 1993; Karp, Sahay, Santos, & Schauser, 1993) and involving multiple ports (Bar-Noy & Ho, 1999).

2.3.3 Fault-tolerant broadcasting

Since graphs represent real-life networks, consideration of machine failures, link damages and viruses is important. The fault-tolerant model conducts broadcasting with k faulty edges or vertices.

A faulty edge stops transmission of information after some time unit, permanently or for some time (transiently). A failure of a node can again be permanent and transient, and can happen in either failure to send or receive messages. The model is a huge area of research and has many applications.

The model was introduced by Liestman (1985). The author discusses problems related to fault tolerant minimum broadcast graphs by introducing k-tolerant broadcast function $B_k(n)$ as the minimum number of edges in a graph supporting k-tolerant broadcasting from any originator in theoretically smallest possible time.

Pelc (1996) discusses message dissemination under this model, under different types: permanent edge failure, permanent vertex failure, permanent edge and vertex failure, transient faults, as well as distribution of failures and probabilistic fault model. This was further explored in Chau and Liestman (1985) and Ahlswede, Gargano, Haroutunian, and Khachatrian (1996). It is important to mention, that with all of the constraints discussed, the graph preserves its connectivity at all times. In Ahlswede, Gargano, Haroutunian, and Khachatrian (1996), authors discuss fault tolerant minimum broadcast networks where despite failure of k edges, any originator broadcasts in minimum broadcast time. Hromkovič, Klasing, Pelc, Ruzicka, and Unger (2005) have discussed the k-fault model and introduced algorithms for broadcasting under this model.

2.3.4 k-broadcasting

The model of k-broadcasting changes some conditions described in classical broadcasting, namely an informed vertex informs up to k of its neighboring vertices in each time unit. It can be clearly seen that classical broadcast is a special case of k-broadcasting where $k = 1$. k-broadcasting in general graphs is studied in Grigni and Peleg (1991); H. A. Harutyunyan and Liestman (2001a); König and Lazard (1994); S. Lee and Ventura (2001); Shao (2006). k-broadcasting in trees is studied in H. A. Harutyunyan and Liestman (2001b); H. A. Harutyunyan, Liestman, and Shao (2009); Labahn (1986, 1989); Proskurowski (1981).

2.3.5 Universal list broadcasting

This model of broadcasting, introduced by Diks and Pelc (1996); Rosenthal and Scheuermann (1987) every vertex is given a universal list of vertices, and follows the list, regardless of the originator. When a vertex v receives the message, it informs the neighbors by the order in the list.

This model has three submodels: The adaptive model, introduced by Rosenthal and Scheuermann (1987) allows the vertices to keep track of the neighbors from which it received a message from and skip them while going through the list to minimize redundant calls. Whereas in the non-adaptive model, introduced by Diks and Pelc (1996), vertices do not distinguish where the message comes from and that results in redundant calls and possibly overall worse broadcast time. In Fully adaptive model, introduced in Gholami and Harutyunyan (2022b), a vertex skips all of its uninformed neighbors while following a prescribed list.

Slater, Cockayne, and Hedetniemi (1981) proved that for any tree T, $b(T) = b_a(T)$, where $b_a(T)$ is the broadcast time of the tree under the adaptive universal list broadcasting model.

Universal list broadcasting was studied in multiple topologies:

- cycles, grids, complete graphs (Diks & Pelc, 1996; Gholami & Harutyunyan, 2022b; Kim & Chwa, 2005)
- tori (Diks & Pelc, 1996; Gholami & Harutyunyan, 2022b; H. A. Harutyunyan & Taslakian, 2004)
- paths, grids, hypercubes (Gholami & Harutyunyan, 2022a; Kim & Chwa, 2005)
- trees (Gholami & Harutyunyan, 2022b; H. A. Harutyunyan, Liestman, Makino, & Shermer, 2011)
- upper bounds on general graphs (Gholami & Harutyunyan, 2022a, 2022c; H. A. Harutyunyan, Liestman, Makino, & Shermer, 2011)

2.3.6 Messy broadcasting

This model of broadcasting was introduced by Ahlswede, Haroutunian, and Khachatrian (1994) and there is no centralized "orchestration" of vertices here. Each vertex has some knowledge about its neighbors and upon receiving the message informs a randomly-chosen neighbor. As the name suggests, this model can be very unpredictable and usually depicts the worst-case behavior of the network.

Model M_1: Each vertex knows the state of its neighbors (informed or uninformed) at any time unit. Once a vertex is informed it starts informing its uninformed neighbors.

Model M_2: Each vertex considers vertices it has received the message from to be informed. Once a vertex is informed it starts informing its neighbors which have not informed the vertex before.

Model M_3: Each node keeps a list of all neighbors to which it sent a message. Once informed, in each time unit the vertex sends the message to a neighbor not present in the list.

The exact value of the broadcast time under model M_i i.e. the maximum broadcast time of any vertex v of G over all possible broadcast schemes, for complete graphs, paths, cycles, and complete k-ary trees are known for all three models (H. A. Harutyunyan & Liestman, 1998). Li, Hart, Henry, and Neufeld (2008a) have studied the average-case messy broadcasting time of stars, paths, cycles, complete d-ary trees, and hypercubes. Comellas, Harutyunyan, and Liestman (2003) have studied complete bipartite graphs and multidimensional directed tori. Most papers concentrated on worst-case analysis of messy models, however discussion of average-case results can be found in Li, Hart, Henry, and Neufeld (2008b).

2.3.7 Radio broadcasting

This model depicts a real-life radio broadcasting where information is transmitted to all devices in a surrounding area and multiple incoming messages with the same frequency cause collision. Formally, the message transmitted by a vertex in given time unit is delivered to all of its neighbors. A vertex acting as a receiver successfully receives a message only if exactly one of its neighbors transmits in that time unit. If two or more neighbors of a vertex transmit simultaneously, then a collision occurs and none of the messages is heard by the vertex in that time unit. Research in this area was conducted by Chlebus, Gasieniec, Gibbons, Pelc, and Rytter (2002); Dessmark and Pelc (2002, 2007); Peleg (2007); Peleg and Radzik (2010) and others.

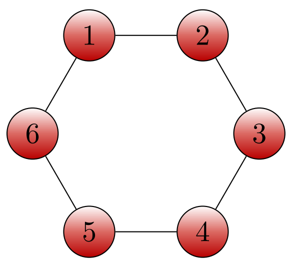
Other models of broadcasting can be found in Morosan (2007).

2.4 Broadcasting in different families of graphs

This section presents graph families, their properties, and classical broadcast times.

2.4.1 Cycle (Ring) C_n

A cycle C_n is a graph on n vertices, $V = \{1, \cdots, n\}$, and the edge set $E = \{(i, i+1) | 1 \leq i \leq n-1\} \cup \{(1, n)\}$. C_n has n edges, diameter of $\lfloor \frac{n}{2} \rfloor$ and is a 2-regular graph: meaning every vertex has degree 2. Since for every pair of vertices in $v(C_n)$ there are two vertex-disjoint paths, C_n is 2-connected. Also, $b(C_n) = \lceil \frac{n}{2} \rceil$. Figure 2.1 portrays C_6 and C_3.



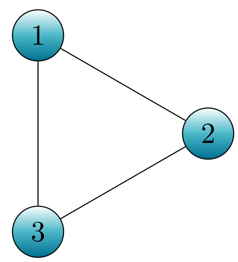


Figure 2.1: Cycles of even $n = 6$ and odd $n = 3$ size

2.4.2 Path graph P_n

A path P_n is a graph on n vertices, $V = \{1, \cdots, n\}$, and the following set of edges: $E = \{(i, i+1) | 1 \leq i \leq n-1\}$. P_n has $n-1$ edges, diameter of $n-1$. The first and the last vertex have a degree of 1 and all other vertices have degree 2. The vertex with the minimum broadcast time in a path graph is the midpoint (either of the midpoints if n is even, denoted by u) with $b(u, P_n) = \lceil \frac{n}{2} \rceil$. However, in general $b(P_n) = n - 1$. Figure 2.2 demonstrates P_6.

Figure 2.2: Path graph with $n = 6$

2.4.3 Star graph S_n

A star S_n is a graph on n vertices, $V = \{1, \cdots, n\}$, and the following set of edges: $E = \{(i, n) | 1 \leq i < n\}$. S_n has $n-1$ edges, a diameter of 2, and a maximum degree of $n-1$. Also,

$b(S_n) = n - 1$. Figure 2.3 portrays S_6.

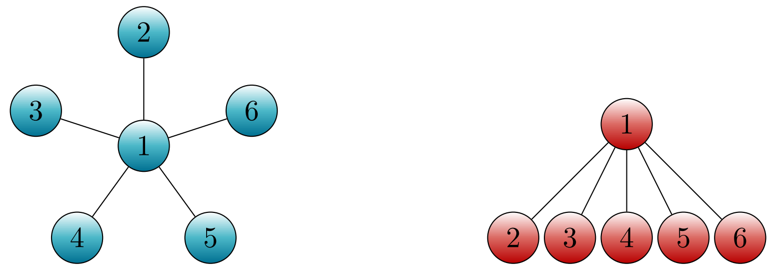


Figure 2.3: Star graph with $n = 6$

2.4.4 Complete graph K_n

A complete graph (clique) K_n is a graph on n vertices, $V = \{1, \cdots, n\}$, and all possible edges. K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, a diameter of 1, and is an $n-1$-regular graph. K_n is $n-1$-connected, since between any pair of vertices $u, v \in V(K_n)$ there is the direct edge and each of the remaining $n-2$ vertices is a mutual neighbor for u and v, resulting in $n-1$ vertex disjoint paths. $b(K_n) = \lceil \log n \rceil$. Figure 2.4 portrays K_6.

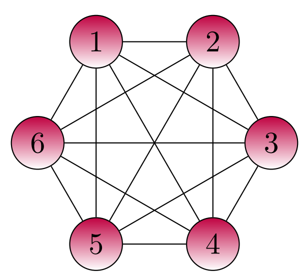


Figure 2.4: Complete graph with $n = 6$

2.4.5 Complete bipartite graph $K_{k,n-k}$

A complete bipartite graph consists of two partitions; bipartition P_1 with k vertices and bipartition P_2 with $n-k$ vertices. There are no edges between vertices from the same partition. All pairs of vertices $u \in P_1$ and $v \in P_2$ are connected by an edge. The degree of every vertex in partition P_1 equals $n-k$, and every vertex in P_2 has degree k. A complete bipartite graph consists of n vertices and $k \times (n-k)$ edges. In Section 4.1 we will show that $K_{k,n-k}$ is $\min\{k, n-k\}$-connected and discuss $b(K_{k,n-k})$. Figure 2.5 shows $K_{3,4}$.

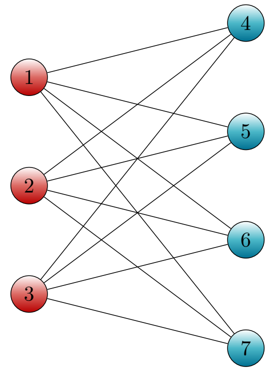


Figure 2.5: Complete bipartite graph with $m = 3, n = 4$

2.4.6 Fork graph $F_{n,k}$

A Fork (Broom) $F_{n,k}$ is a graph on n vertices, containing a path with $n-k$ vertices, where one of the leaves of the path is the center of a star graph with k leaves. $F_{n,k}$ has $n-1$ edges and a diameter of $n-k$. $b(F_{n,k}) = n-1$. Figure 2.6 demonstrates $F_{10,5}$.

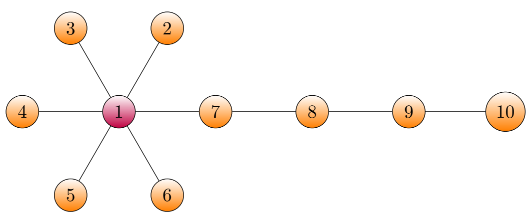


Figure 2.6: Fork graph with $n = 10$ and $k = 5$

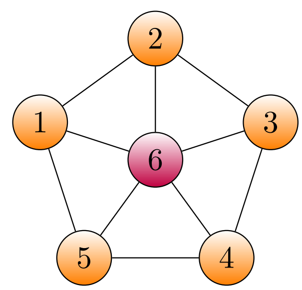


Figure 2.7: Wheel graph with $n = 6$

2.4.7 Wheel graph W_n

A wheel W_n is a graph on n vertices, $V = \{1, \cdots, n\}$, and the following set of edges: $E = \{(i,n)|1 \le i \le n-1\} \cup \{(i,i+1)|1 \le i \le n-2\} \cup \{(1,n-1)\}$. W_n has $2n-1$ edges, a diameter of 2 if $n > 4$, and a maximum degree of $n-1$. Wheel is a 3-connected graph, since between any pair of vertices, there are 3 vertex-disjoint paths. $b(W_n) = \left\lceil \frac{\sqrt{4n-3}+1}{2} \right\rceil$. Figure 2.7 portrays W_6.

2.4.8 Complete k-ary tree $T_{k,h}$

A k-ary tree is a rooted tree in which the number of children of each internal vertex is k. The degree of the root is k, the degrees of internal vertices are $k+1$, and the leaves' degrees are 1. A complete k-ary tree of height h, (denoted by $T_{k,h}$), is a rooted k-ary tree in which all leaves are on the same level h. $T_{k,h}$ has $\frac{k^{h+1}-1}{k-1}$ vertices, diameter of $2h$, and maximum degree of $k+1$. The broadcast time of a k-ary tree is $b(T_{k,h}) = kh + h - 1$ Figure 2.8 portrays $T_{5,2}$.

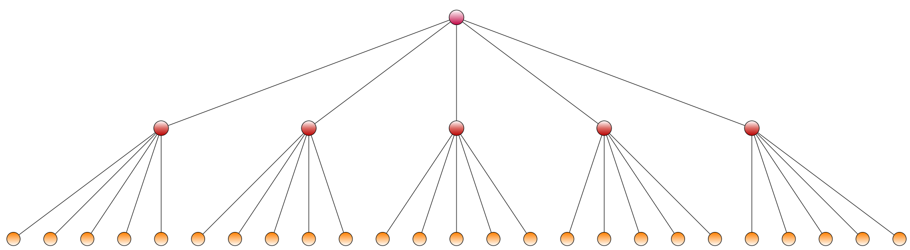


Figure 2.8: k-ary tree with $k = 5, h = 2$

2.4.9 Grid $G_{m\times n}$

A two-dimensional Grid (Lattice), $G_{m\times n}$, is a graph on $m \times n$ vertices. Each of the vertices is on integer coordinates of a Cartesian plane and there is an edge between vertices if they have Euclidean distance one (difference one in one coordinate). $G_{m\times n}$ has $(m-1)n+(n-1)m = 2mn-(m+n)$ edges, diameter of $m + n - 2$ and a maximum degree of 4. $G_{m\times n}$ is a 2-connected graph since between any pair of vertices there are two vertex disjoint paths. Farley and Hedetniemi (1978) showed that broadcast time $b(G_{m\times n}) = m + n - 2$. Figure 2.9 shows $G_{5\times 3}$. For more on Grids we refer the reader to (Adibi, 2021).

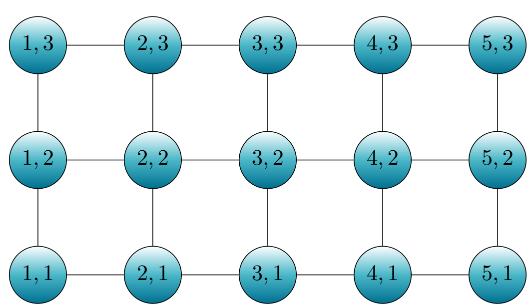


Figure 2.9: Grid with $m = 5, n = 3$

2.4.10 Torus $T_{m\times n}$

A Torus $T_{m\times n}$, similarly to a grid, is a graph on $m \times n$ vertices. It includes all edges of the grid and edges $((i,1),(i,n))$ and $((1,j),(m,j))$, $\forall i,j$ such that $1 \le i \le m, 1 \le j \le n$. Tori have $2mn$ edges, diameter of $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. Tori are 4-regular graphs and 4-connected graphs since there are four vertex-disjoint paths between any pair of vertices. $b(T_{m\times n}) = \frac{m+n}{2}$, if m and n are even, and $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ otherwise (Farley & Hedetniemi, 1978). $b(T_{3\times 3}) = 4$ is an exception to the formula. Figure 2.10 portrays $T_{5\times 3}$. For more on Tori we refer the reader to (Adibi, 2021).

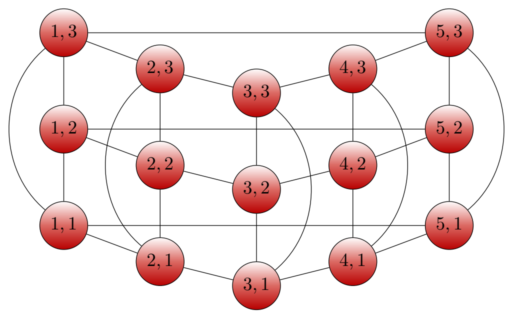


Figure 2.10: Torus with $m = 5, n = 3$

2.4.11 Hypercube Q_d

The d-dimensional hypercube, Q_d, is a graph on 2^d vertices. Each vertex is a d-bit binary string, and two vertices are adjacent if and only if their bit-wise difference is one. For example, the vertices v_2 and v_6 in H_3 are neighbors since their binary representations 010 and 110 differ only in the third bit. Q_d has $d \cdot 2^{d-1}$ edges, diameter d, is d-regular, d-connected and bipartite. Hypercube is one of the infinite families of broadcast graphs: $b(Q_d) = \lceil logn \rceil = d$. Figure 2.11 shows Q_3.

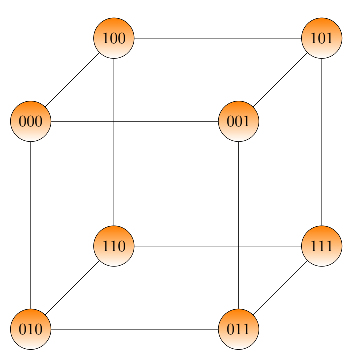


Figure 2.11: Hypercube with $d = 3$

2.4.12 Binomial tree BT_d

A binomial tree of dimension d is a tree on 2^d vertices with a recursive definition. The binomial tree of dimension 0 (BT_0) is a single vertex. BT_d has a root vertex of degree d whose children are roots of binomial trees $BT_{d-1}, BT_{d-2}, ..., BT_0$. Equivalently, BT_d can be defined as two copies of BT_{d-1} by connecting their roots by an edge and selecting one of the roots to be the root of the new tree. BT_d has $2^d - 1$ edges, a diameter $2d - 1$, and a maximum degree of d. The two roots of the previous dimension achieve minimum broadcast time $b(r, BT_d) = d = \log n$, but the overall broadcast time is $b(BT_d) = 2d - 1$. Figure 2.12 shows T_4.

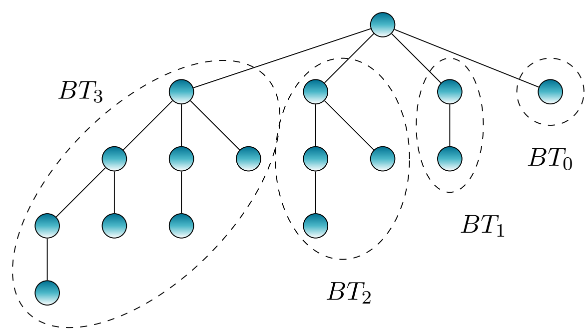


Figure 2.12: Binomial tree with $d = 4$

2.4.13 Cube-Connected Cycle CCC_d

The Cube Connected Cycle CCC_d of dimension d is generated by taking a hypercube of dimension d and replacing each of the 2^d vertices by a cycle of length d. For simplicity, we name vertices by the pair $\{\alpha, \beta\}$, where α denotes the vertex of the hypercube that derived the cycle the vertex belongs to, and β is the alphabetical index (a, b, c, etc.) of the vertex in the cycle. Two vertices are connected to each other in two cases: first, if they have the same α, and consecutive β indices($d-$th index is defined to be consecutive to index a), and second if they derive from neighboring vertices in the hypercube and have the same β. Cube-connected cycle is 3-regular on $d \cdot 2^d$ vertices and has $3 \cdot d \cdot 2^{d-1}$ edges. CCC_d is 3-connected for all $d \geq 3$ since removing three neighbors of a vertex will make the vertex disconnected. Due to their small degree, yet good connectivity, CCC_d's are commonly used in real-life network design (Preparata & Vuillemin, 1981). The diameter of CCC_d is $\lfloor \frac{5d}{2} \rfloor - 2$ and $b(CCC_d) = \lceil \frac{5d}{2} \rceil - 1$. Figure 2.13 shows CCC_3.

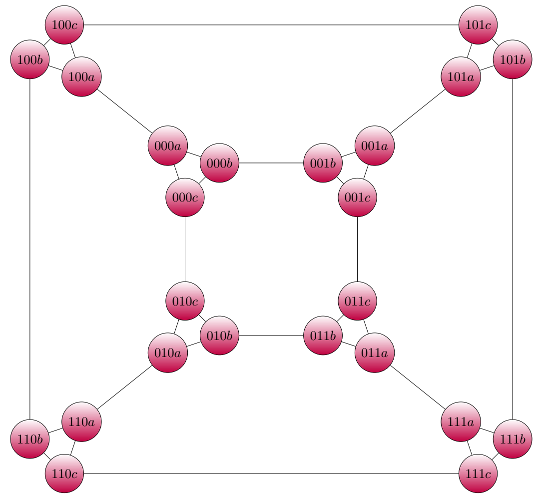


Figure 2.13: Cube-Connected Cycle with $d = 3$

2.4.14 Shuffle Exchange graph SE_d

The Shuffle-Exchange network SE_d of dimension d is a graph on 2^d vertices. Similar to Hypercube, its vertices represent binary strings of length d. The edges of SE_d are of two types: shuffle edges (wa, aw), and exchange edges $(wa, w\bar{a})$, where $w \in \{0,1\}^{d-1}, a, \bar{a} \in \{0,1\}, a \neq \bar{a}$. The SE_d has 2^d vertices, a diameter of $2d - 1$, and a maximum degree of 3. Since the all-zero and all-one vertices always have one neighbor, the graph cannot be 2-connected. Hromkovič, Jeschke, and Monien (1993) showed that $b(SE_d) = 2d - 1$. Figure 2.14 shows SE_3.

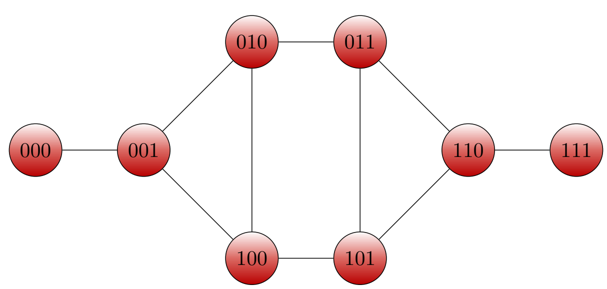


Figure 2.14: Shuffle Exchange graph with $d = 3$

2.4.15 Binary DeBruijn graph DB_d

The DeBruijn network DB_d of dimension d is a directed graph on 2^d vertices which represent binary strings of length d. The edges of DB_d are either shuffle edges (wa, aw), or shuffle-exchange edges $(aw, w\bar{a})$, where $w \in \{0,1\}^{k-1}, a, \bar{a} \in \{0,1\}$, and $a \neq \bar{a}$. The DB_d has 2^d vertices, a diameter of m, and a maximum degree 4. The broadcast time of DB_d is unknown. The tightest known bounds are $1.3171d \leq b(DB_d) \leq \frac{3}{2}(d+1)$ given by Bermond and Peyrat (1988); Klasing, Monien, Peine, and Stöhr (1994). Figure 2.15 shows BD_3 with transparent underlying directed edges.

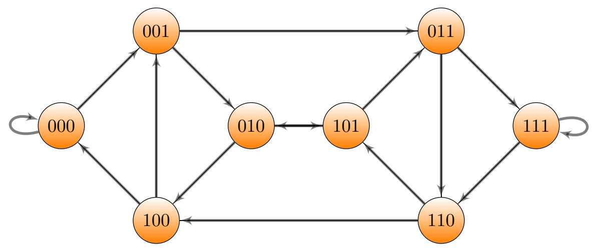


Figure 2.15: DeBruijn graph with $d = 3$

2.4.16 Wrapped Butterfly graph BF_d

The Wrapped Butterfly Network BF_d is a graph on $d \cdot 2^d$ vertices. The vertices are indexed by pair (l, i), where l is the level of the vertex (out uf d layers), and i is a binary string of length d, showing the position in the level (with 2^d positions in each layer). The edge set is constructed the following way: Each vertex (l, i) is adjacent to $(l + 1 \mod d, i)$ and to $(l + 1 \mod d, \bar{i})$. BF_d is a 4-regular graph with $d \cdot 2^{d+1}$ edges and diameter $\lfloor \frac{3d}{2} \rfloor$. Klasing, Monien, Peine, and Stöhr (1994) have shown that $1.7417d \leq b(BF_d) \leq 2d - 1$.

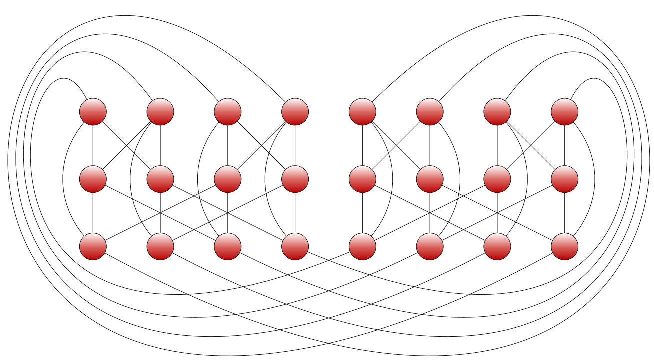
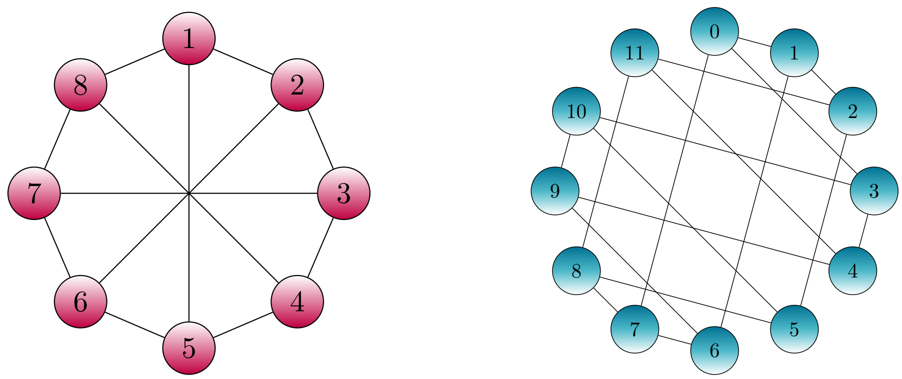


Figure 2.16: Wrapped Butterfly graph with $d = 3$

2.4.17 Recursive Circulant graph $G(n, d)$

The recursive circulant graph, defined in Park and Chwa (1994), has n vertices with a parameter d called jump. The edge set is defined as $E = \{(v, w)|$ there exists $i, 0 \leq i \leq \lceil \log_d n \rceil - 1$, such that $v + d^i \equiv w \mod n\}$.

A more interesting instance of the recursive circulant graph is $G(2^m, 4)$ since it has the same number of nodes, edges ($m \cdot 2^{m-1}$), and broadcast time ($b(G(2^m, 4)) = \lceil \log 2^m \rceil = m$) as the hypercube Q_m. $G(2^d, 4)$. has diameter $\lceil \frac{3m-1}{4} \rceil$, is $m = \lceil \log n \rceil$-regular and m-connected. Figure 2.17a shows $G(2^3, 4)$ which is isomorphic to the Wagner graph.



(a) Recursive Circulant graph with $n = 8, d = 4$ (b) Knödel graph with $\Delta = 3, n = 14$

Figure 2.17: Recursive Circulant and Knödel graphs

2.4.18 Knödel graph $KG_{\Delta,n}$

In Khachatrian and Harutounian (1990); Knödel (1975) and Bermond, Harutyunyan, Liestman, and Pérennes (1997), the Knödel graph $KG_{\Delta,n}$ is defined as a graph on n vertices, where $n \geq 6$ and is even. $E = \{(x, y)|x + y \equiv 2^\Delta - 1 \mod n\}$, where $1 \leq \Delta \leq \lfloor \log n \rfloor$. The Knödel graph $KG_{\Delta,2^\Delta}$ is a Δ-regular graph with diameter $\lceil \frac{\Delta+2}{2} \rceil$.

As mentioned in 2.2.1, the family of $KG_{\Delta,2^\Delta}$ is one of the three known infinite families of minimum broadcast graphs. This means that $b(KG_{\Delta,2^\Delta}) = \Delta = \log n$. It is also known that $b(KG_{\Delta,2^\Delta-2}) = \Delta$ ($\Delta \geq 2$) (Dinneen, Fellows, & Faber, 1991). Grigoryan and Harutyunyan (2013), showed that $2\lfloor \frac{1}{2} \lceil \frac{n-2}{2^\Delta-2} \rceil \rfloor + 1 \leq b(KG_{\Delta,n}) \leq \lceil \frac{n-2}{2^\Delta-2} \rceil + \Delta - 1$. Figure 2.17b showcases $KG_{3,14}$.

2.5 Connectivity in Graphs

The problem of broadcasting is defined on connected graphs, which means that any vertex is reachable from any other vertex in the graph. Equivalently, this means that there exists a path between any pair of vertices in the graph. A graph is connected if every pair of vertices in it is connected.

While the connectivity of a graph is important, in real life, when a network is simply connected it can be easily disconnected. Consider the Path graph presented in Section 2.4.2. Despite being a connected graph, the removal of a vertex that is not an endpoint will make it disconnected. Removal of a vertex from a graph is analogous to a machine failure in a real-life network. So if in Figure 2.2, machine 4 fails, then the network will become disconnected: i.e. there will not be a path between some pairs of machines such as 1 and 6. On the contrary, consider the Cycle graph in Figure 2.1. By removing a single vertex, the graph becomes a path graph on $n - 1$ vertices but stays connected. This brings us to the formal definition of connectivity.

Any graph G is said to be *k-vertex-connected (k-connected)* if it has at least $k + 1$ vertices, but does not contain a set of $k - 1$ vertices whose removal disconnects the graph (this set is defined as a *vertex-cut*). Connectivity of a graph $\kappa(G)$ is defined as the largest k such that G is k-connected.

For example, the path graph discussed earlier, has connectivity 1 and the cycle is 2-connected. Note, that 1-connectivity is identical to connectedness of a graph.

Menger (1927) shows an equivalent definition using the concept of internally vertex-disjoint paths. Two paths are internally vertex-disjoint if they do not have any internal vertex in common.

Theorem 2.5.1 (Menger (1927))**.** *A graph is k-vertex-connected if and only if every pair of vertices has at least k internally vertex-disjoint paths in between.*

It is evident that in a k-connected graph the minimum degree must be greater than or equal to k i.e. $\delta \geq k$. If $k = \delta$ the graph is called *maximally connected*.

Examples of 2-connected and higher k-connected graphs will be discussed further in this work and we will interchangeably use the definition or Menger's necessary and sufficient condition to show a graph's connectivity.

Chapter 3

Broadcasting in 2-connected graphs

Two-connected graphs, defined in the Section 2.5 include large families of graphs that serve as an underlying structure for networks in real life. Due to their definition, 2-connected graphs are also called non-separable, since the removal/failure of a single vertex will not affect the connectivity of the graph. Considering these properties, time bounds for the dissemination of information in these graphs become very interesting.

3.1 Broadcast time of 2-connected graphs

In this section, we prove the following theorem.

Theorem 3.1.1. *Let G be a 2-connected graph on n vertices. Then $b(G) \leq \lceil \frac{n}{2} \rceil$.*

The proof uses Whitney's theorem. An *open ear decomposition* of a graph G are subgraphs $C, P_1, P_2, ..., P_k$ of G such that C is a cycle and each ear P_i is a path such that the two endpoints of P_i are vertices on $C \cup P_1 \cup ... \cup P_{i-1}$, all the internal vertices of P_i have degree two and are disjoint from $C \cup P_1 \cup ... \cup P_{i-1}$ and $C \cup P_1 \cup ... \cup P_k = G$.

Theorem 3.1.2 (Whitney)**.** *Every 2-connected graph has an open ear decomposition. Moreover, C can be taken to be an arbitrary cycle.*

The general idea of the proof of our theorem is induction on the number of ears k. However, parity issues arise in certain cases so we have to tweak the induction hypothesis. As a result, we prove the following theorem instead.

Theorem 3.1.3. *Let G be a 2-connected graph on n vertices. Then the following holds.*

A. $b(G) \leq \lceil \frac{n}{2} \rceil$. Moreover, if n is odd, then there is a broadcast scheme for which at time $\lceil \frac{n}{2} \rceil - 1$ at most one vertex is uninformed.

B. Suppose G has two arbitrary originators. Then $b(G) \leq \lceil \frac{n}{2} \rceil - 1$. If n is odd, then there is a broadcast scheme such that at time $\lceil \frac{n}{2} \rceil - 2$ at most one vertex is uninformed.

C. Suppose G has three arbitrary originators and n is odd. Then $b(G) \leq \lceil \frac{n}{2} \rceil - 2$.

D. Suppose G has three arbitrary originators and n is even. Then there is a broadcast scheme such that at time $\lceil \frac{n}{2} \rceil - 2$ there is at most one uninformed vertex.

We note, that in statements B, C, and D we discuss broadcasting from two or three originators and the notation of $b(G)$ is abused. We use these cases to support statement A and thus they are always discussed in the context of single originator broadcasting. By specifying the number of originators we omit using the notations $b_2(G)$ or $b_3(G)$ and use $b(G)$ instead. If the number of originators is not mentioned we assume a single originator.

Proof. G admits a k-ear-decomposition if there is an open ear decomposition of G with at most k ears.

Proof by induction on number of ears k.

Base Case: If $k = 0$, then G is a cycle. (A) is obvious from 2.1. For (B), note that in any broadcast scheme, while there are at least two uninformed vertices, at least two vertices get informed in each time unit and there are $n-2$ vertices to inform. (C) and (D) follow from a similar argument.

Inductive Hypothesis: Assume the statements hold for a 2-connected graph G' on $k-1$ ears.

Inductive Step: Let G be a 2-connected graph, admitting a k-ear-decomposition. Consider the last ear, P_k. Denote the endpoints of P_k in the graph $G' := C \cup P_1 ... \cup P_{k-1}$ by u and v. Note that G' is a 2-connected graph on $k-1$ ears. Let x be the neighbor of u on P and y of v (see Figure 3.1). Let $|G'| = n_1$ and the number of vertices in the subpath xPy of P be n_2. Let $P_k = xPy$.

We note that by the proof of Whitney's theorem, the cycle C can be arbitrary. Thus, by taking C to be of maximum length, we may assume that $n_2 \leq n_1 - 2$. Moreover, $n_2 \geq 1$ for otherwise the result is trivial, and by choice of C we may assume that $n_1 \geq 4$.

We consider the following four cases:

Case 1: n_1 and n_2 are even.

Case 2: n_1 is even and n_2 is odd.

Case 3: n_1 is odd and n_2 is even.

Case 4: n_1 and n_2 are odd.

Let a, b and c be vertices of G', and a_1, b_1 and c_1 be vertices on P_k. The following lemmata will be used later in the proof.

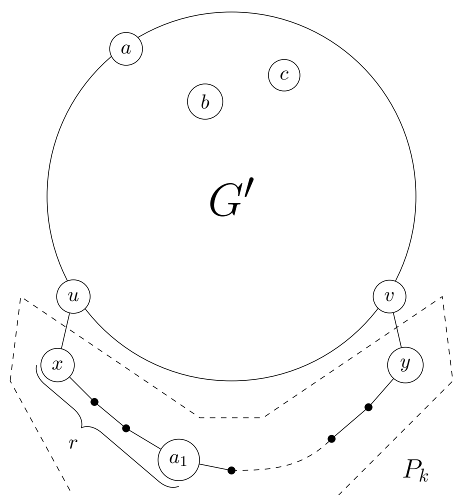


Figure 3.1: The 2-connected graph G

Lemma 3.1.1. *In a 2-connected graph G with three originators - one in G' (on n_1 vertices) and two in P_k (on n_2 vertices) $b(G) \leq \max\{\lceil \frac{n_1}{2} \rceil, \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2 - (r_1 + \lceil \frac{n_1}{2} \rceil)}{2} \rceil\}$ where r_1 is the closest distance of one of the originators on P_k from its closest vertex in G' and r_2 is the distance of the other originator from the other endpoint of P_k.*

Proof. Broadcasting will start in parallel in G' and P_k. By the inductive hypothesis (A), all vertices of G' will be informed by time $\lceil \frac{n_1}{2} \rceil$. By this time, there are three possible scenarios.

Scenario 1: broadcasting is finished in P_k. This means that $b(G) \leq \lceil \frac{n_1}{2} \rceil$.

Scenario 2: broadcasting is finished in the "outer legs" of P_k, but there are uninformed vertices on the subpath between the two originators. There are $n_2 - (r_1 + r_2)$ such vertices. The originators will start by informing these vertices and then inform their other neighbors on the path to G'. Informing the subpath from two ends will take $\lceil \frac{n_2-(r_1+r_2)}{2} \rceil + 1$ time. Since both r_1 and r_2 are at least 1, then $\lceil \frac{n_2-(r_1+r_2)}{2} \rceil + 1 \leq \lceil \frac{n_2-2}{2} \rceil + 1 = \lceil \frac{n_2}{2} \rceil - 1 + 1$. By the initial assumption $n_2 \leq n_1 - 2$ thus $\lceil \frac{n_2}{2} \rceil \leq \lceil \frac{n_1-2}{2} \rceil = \lceil \frac{n_1}{2} \rceil - 1 < \lceil \frac{n_1}{2} \rceil$.

Scenario 3: broadcasting continues in P_k until all vertices on the "long leg" are informed. Since $r_1 \leq r_2$ then $r_1 \leq \lceil \frac{n_2}{2} \rceil \leq \lceil \frac{n_1-2}{2} \rceil \leq \lceil \frac{n_1}{2} \rceil - 1$, so all the r_1 vertices on the shorter subpath will be informed by the time $\lceil \frac{n_1}{2} \rceil$. By that time there will be $\lceil \frac{n_1}{2} \rceil$ vertices on the longer path informed as well. There will be $n_2 - (r_1 + \lceil \frac{n_1}{2} \rceil)$ uninformed vertices remaining on the path P_k. Since both endpoints of the subpath are informed by the time $\lceil \frac{n_1}{2} \rceil$, informing the subpath will take $\lceil \frac{n_2-(r_1+\lceil \frac{n_1}{2} \rceil)}{2} \rceil$ time and result in $b(G) \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-(r_1+\lceil \frac{n_1}{2} \rceil)}{2} \rceil$.

□

This concludes the proof of the lemma, however, let us note these:

(1) The general idea of the proof is to show that the broadcast time i.e. $\max\{\lceil \frac{n_1}{2} \rceil, \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-(r_1+\lceil \frac{n_1}{2} \rceil)}{2} \rceil\}$ is less than or equal to the needed broadcast time. Table 3.1 summarizes the cases, the needed broadcast time (n.b.t) , and the conditions to make the broadcast time shown in Lemma 3.1.1 less than or equal to $\lceil \frac{n}{2} \rceil - 1$ (when n is odd) or $\lceil \frac{n}{2} \rceil - 2$ (when n is even) depending on what needs to be shown.

(2) The third column of Table 3.1 shows the conditions of n_2 such that n.b.t.$-\lceil \frac{n_1}{2} \rceil \geq 0$.

(3) In the case when n_1 is odd, since the assumption states that $n_1 \geq 4$, then $n_1 \geq 5$ and $\lceil \frac{n_1}{2} \rceil \geq 3$. This means that $r_1 + \lceil \frac{n_1}{2} \rceil \geq 4$. Finally, $\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-(r_1+\lceil \frac{n_1}{2} \rceil)}{2} \rceil \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-4}{2} \rceil = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 2$.

(4) In the case when n_1 and n_2 are even, $n_1 \geq 4$ which means that $\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-(r_1+\lceil \frac{n_1}{2} \rceil)}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2-(r_1+\frac{n_1}{2})}{2} \rceil \leq \frac{n_1}{2} + \lceil \frac{n_2-3}{2} \rceil = \frac{n_1}{2} + \frac{n_2}{2} - 1$. Moreover, since n_2 is even, then $n_2 - 3$ is

odd and thus at time unit $\frac{n}{2} - 2$ there is one uninformed vertex (in the middle of the subpath).

(5) In the case when n_1 is even and n_2 is odd, $n_1 \geq 4$ which means that $\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2 - (r_1 + \lceil \frac{n_1}{2} \rceil)}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2 - (r_1 + \frac{n_1}{2})}{2} \rceil \leq \frac{n_1}{2} + \lceil \frac{n_2 - 3}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2 - 1}{2} \rceil - 1 = \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 2$.

Case	n.b.t.	$\lceil \frac{n_1}{2} \rceil \leq$ n.b.t.	$\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2 - (r_1 + \lceil \frac{n_1}{2} \rceil)}{2} \rceil \leq$ n.b.t.
even n_1 even n_2	$\frac{n_1}{2} + \frac{n_2}{2} - 1$	$n_2 \geq 2$	$n_1 \geq 4, n_2 \geq 2$
even n_1 odd n_2	$\frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 2$	$n_2 \geq 3$	$n_1 \geq 4, n_2 \geq 2$
odd n_1 even n_2	$\lceil \frac{n_1}{2} \rceil + \frac{n_2}{2} - 2$	$n_2 \geq 4$	$n_1 \geq 4, n_2 \geq 2$
odd n_1 odd n_2	$\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 2$	$n_2 \geq 3$	$n_1 \geq 4, n_2 \geq 2$

Table 3.1

Lemma 3.1.2. *In a 2-connected graph G with three originators - two in G' (on n_1 vertices) and one in P_k (on n_2 vertices) $b(G) \leq \max\{\lceil \frac{n_1}{2} \rceil - 1, \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2 - (2r + \lceil \frac{n_1}{2} \rceil - 1)}{2} \rceil - 1\}$ where r is the closest distance of the originator on P_k from its closest vertex (u) in G'.*

Proof. Broadcasting will start in parallel in G' and P_k. The originator informs the shorter side of the path. Since there are two originators in G', by the inductive hypothesis (B), all vertices of G' will be informed by time $\lceil \frac{n_1}{2} \rceil - 1$. By this time, there are three possible scenarios.

Scenario 1: broadcasting is finished in P_k. This means that $b(G) = \lceil \frac{n_1}{2} \rceil - 1$.

Scenario 2: broadcasting is finished in G', but there are uninformed vertices on P_k. $r \leq \lceil \frac{n_2}{2} \rceil \leq \lceil \frac{n_1 - 2}{2} \rceil = \lceil \frac{n_1}{2} \rceil - 1$. This means that by the time broadcasting is complete in G', all vertices on the subpath to u are informed. There are $n_2 - (2r + \lceil \frac{n_1}{2} \rceil - 1)$ uninformed vertices remaining. Since both endpoints of the subpath are informed, then informing the vertices will take $\lceil \frac{n_2 - (2r + \lceil \frac{n_1}{2} \rceil - 1)}{2} \rceil = \lceil \frac{n_2 - (2r + \lceil \frac{n_1}{2} \rceil - 1)}{2} \rceil$. The overall broadcast time will be $b(G) \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2 - (2r + \lceil \frac{n_1}{2} \rceil - 1)}{2} \rceil - 1$.

□

This concludes the proof of the lemma, however, let us note these:

(1) The general idea of the proof is to show that the broadcast time i.e. $\max\{\lceil\frac{n_1}{2}\rceil - 1, \lceil\frac{n_1}{2}\rceil + \lceil\frac{n_2-(2r+\lceil\frac{n_1}{2}\rceil-1)}{2}\rceil - 1\}$ is less than or equal to the needed broadcast time. Table 3.2 summarizes the cases, the needed broadcast time, and the conditions to make the broadcast time shown in Lemma 3.1.2 less than or equal to $\lceil\frac{n}{2}\rceil - 1$ (when n is odd) or $\lceil\frac{n}{2}\rceil - 2$ (when n is even) depending on what needs to be shown.

(2) The third column of Table 3.2 shows the conditions of n_2 such that n.b.t.$-\lceil\frac{n_1}{2}\rceil - 1 \geq 0$.

(3) In the case when n_1 is odd, since the assumption states that $n_1 \geq 4$, then $n_1 \geq 5$ and $\lceil\frac{n_1}{2}\rceil \geq 3$. This means that $2r + \lceil\frac{n_1}{2}\rceil \geq 5$. Finally, $\lceil\frac{n_1}{2}\rceil - 1 + \lceil\frac{n_2-(2r+\lceil\frac{n_1}{2}\rceil-1)}{2}\rceil \leq \lceil\frac{n_1}{2}\rceil - 1 + \lceil\frac{n_2-5+1}{2}\rceil = \lceil\frac{n_1}{2}\rceil - 1 + \lceil\frac{n_2}{2}\rceil - 2 = \lceil\frac{n_1}{2}\rceil + \lceil\frac{n_2}{2}\rceil - 3$.

(4) In the case when n_1 and n_2 are even, $n_1 \geq 4$ which means that $\lceil\frac{n_1}{2}\rceil - 1 + \lceil\frac{n_2-(2r+\lceil\frac{n_1}{2}\rceil-1)}{2}\rceil = = \frac{n_1}{2} - 1 + \lceil\frac{n_2-(2r+\lceil\frac{n_1}{2}\rceil-1)}{2}\rceil \leq \frac{n_1}{2} + \lceil\frac{n_2-3}{2}\rceil - 1 = \frac{n_1}{2} + \frac{n_2}{2} - 2$.

(5) In the case when n_1 is even and n_2 is odd, $n_1 \geq 4$ meaning $\lceil\frac{n_1}{2}\rceil - 1 + \lceil\frac{n_2-(2r+\lceil\frac{n_1}{2}\rceil-1)}{2}\rceil = = \frac{n_1}{2} - 1 + \lceil\frac{n_2-(2r+\frac{n_1}{2}-1)}{2}\rceil \leq \frac{n_1}{2} - 1 + \lceil\frac{n_2-3}{2}\rceil = \frac{n_1}{2} - 1 + \lceil\frac{n_2-1}{2}\rceil - 1 = \frac{n_1}{2} + \lceil\frac{n_2}{2}\rceil - 3$.

Case	n.b.t.	$\lceil\frac{n_1}{2}\rceil - 1 \leq$ n.b.t.	$\lceil\frac{n_1}{2}\rceil + \lceil\frac{n_2-(r_1+\lceil\frac{n_1}{2}\rceil)}{2}\rceil \leq$ n.b.t.
even n_1 even n_2	$\frac{n_1}{2} + \frac{n_2}{2} - 1$	$n_2 \geq 2$	$n_1 \geq 4, n_2 \geq 2$
even n_1 odd n_2	$\frac{n_1}{2} + \lceil\frac{n_2}{2}\rceil - 2$	$n_2 \geq 3$	$n_1 \geq 4, n_2 \geq 2$
odd n_1 even n_2	$\lceil\frac{n_1}{2}\rceil + \frac{n_2}{2} - 2$	$n_2 \geq 2$	$n_1 \geq 4, n_2 \geq 2$
odd n_1 odd n_2	$\lceil\frac{n_1}{2}\rceil + \lceil\frac{n_2}{2}\rceil - 2$	$n_2 \geq 3$	$n_1 \geq 4, n_2 \geq 2$

Table 3.2

Now we go over the cases (1,2,3,4) in the inductive hypothesis and statements A,B,C,D to complete the proof of Theorem 3.1.3.

<u>Case 1:</u> n_1 and n_2 are even.

Proof of A. We have to show that $b(G) \leq \lceil \frac{n}{2} \rceil = \lceil \frac{n_1+n_2}{2} \rceil = \frac{n_1}{2} + \frac{n_2}{2}$.

<u>[Case even n_1 and even n_2 | one originator in G']</u>

Suppose the originator of G is $a \in G'$. By the inductive hypothesis (A), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil = \frac{n_1}{2}$ time units. After this broadcasting begins in P_k from u and v. Informing the path P_k from two endpoints will take $\frac{n_2}{2}$ time units which results in total of $\frac{n_1}{2} + \frac{n_2}{2} = \frac{n}{2}$ broadcast time for graph G.

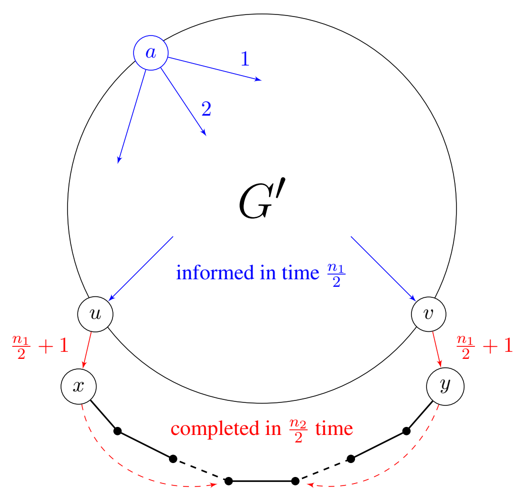


Figure 3.2: Case even n_1 and even n_2| one originator in G'

<u>[Case even n_1 and even n_2 | one originator in P_k]</u>

Now suppose the originator is the vertex a_1 on P_k. Without loss of generality, assume that from u and v, vertex u is closer to a_1 and $r := dist\{a_1, u\}$. Note that $1 \leq r \leq \frac{n_2}{2}$.

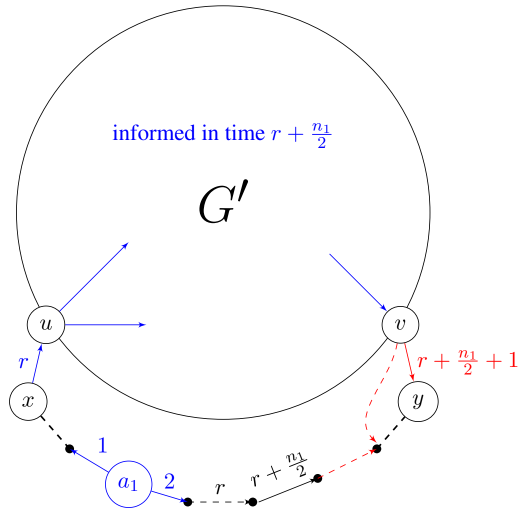


Figure 3.3: Case even n_1 and even $n_2|$ one originator in P_k

At time r the vertex u is informed. By the inductive hypothesis (A), at time $r + \frac{n_1}{2}$ all vertices in G' (including v) are informed. During the $r + \frac{n_1}{2}$ time units, $r + \frac{n_1}{2} - 1$ vertices are informed on the path from a_1 to v. Recall that r vertices on the path from a_1 to u were informed earlier. The remaining vertices to be informed are at most $n_2 - (r + r + \frac{n_1}{2} - 1) = n_2 - 2r - \frac{n_1}{2} + 1$. These vertices form a subpath of P_k, and the endpoints of this path are adjacent to two informed vertices (v and the vertex at distance $\frac{n_1}{2}$ from a_1). The additional time required to inform these vertices is $\lceil \frac{n_2}{2} - r - \frac{1}{2} \cdot \frac{n_1}{2} + \frac{1}{2} \rceil \leq \lceil \frac{n_2}{2} - r - 1 + 1 \rceil = \frac{n_2}{2} - r$.

Overall the broadcasting takes place in at most $r + \frac{n_1}{2} + \frac{n_2}{2} - r = \frac{n_1}{2} + \frac{n_2}{2} = \frac{n}{2}$ time units. This proves (A).

Proof of B. Consider three different locations of the two originators.

[Case even n_1 and even n_2 | two originators in G'] Let the originators be a and b. By the inductive hypothesis (B), $b(G') \leq \frac{n_1}{2} - 1$. This means that broadcasting in P_k begins in time unit $\frac{n_1}{2}$ at the latest. Informing the path P_k from two endpoints will take $\frac{n_2}{2}$ time. Thus, $b(G) = \frac{n_1}{2} - 1 + \frac{n_2}{2} = \frac{n}{2} - 1$.

[Case even n_1 and even n_2 | one originator in G' and one in P_k]

Suppose the one of the originators is in G' and the other is on P_k. Let these be a and b_1. Broadcasting in G' and P_k takes place in parallel. By the inductive hypothesis (A), the process in G' will terminate at time $\frac{n_1}{2}$.

b_1 informs the longer side of P_k first. r is distance of b_1 from u. $l = \frac{n_1}{2} - r$ is the number of time units broadcasting takes place in P_k after r time units. The number of uninformed vertices in P_k after time $\frac{n_1}{2}$ would be $n_2 - 2r - l$ and since broadcasting will take place from two endpoints, the total broadcast time would be $b(G) = \frac{n_1}{2} + \frac{n_2-2r-l}{2} = \frac{n_1}{2} + \frac{n_2}{2} - r - \frac{l}{2} = \frac{n_1}{2} + \frac{n_2}{2} - r - \frac{n_1}{4} + \frac{r}{2} \leq \frac{n_1}{2} + \frac{n_2}{2} - \frac{r}{2} - 1 \leq \frac{n}{2} - 1$.

[Case even n_1 and even n_2 | two originators in P_k]

The last arrangement of originators is a_1 and b_1.

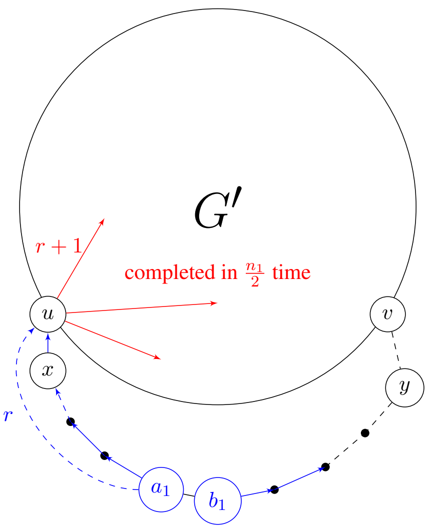


Figure 3.4: Case even n_1 and even n_2 | two originators in P_k

At time $r \leq \frac{n_2}{2} - 1$ at least one of u or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r+1$ and end in $r + \frac{n_1}{2}$ by the inductive hypothesis (A).

By that time, there are two possible cases for broadcasting in P_k.

First possible case is that all vertices in P_k will be informed before time $r + \frac{n_1}{2}$ (i.e. when broadcasting finishes in G'). In this case, the broadcasting takes place in $r + \frac{n_1}{2}$ time. Taking into account that $r \leq \frac{n_2}{2} - 1$, the overall broadcast time of G will be $b(G) \leq r + \frac{n_1}{2} \leq \frac{n_2}{2} + \frac{n_1}{2} - 1 = \frac{n}{2} - 1$.

The next scenario is that there are vertices in P_k which have not been informed by time unit $r + \frac{n_1}{2}$. Since $r \geq 1$, the worst possible positioning of a_1 and b_1 will be when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 2$. If the broadcasting in G' finishes in $r + \frac{n_1}{2}$ time units, it will continue in P_k until time

$n_2 - 2 = \frac{n_2}{2} + \frac{n_2}{2} - 2 \leq \frac{n_1}{2} + \frac{n_2}{2} - 2 = \frac{n}{2} - 2$ since $n_2 \leq n_1 - 2$.

Proof of C. Since n is even, statement C does not apply.

Proof of D. There are four different locations of the two originators.

[Case even n_1 and even n_2 | three originators in G']

Let the originators be a, b and c. By the inductive hypothesis (D), there is a broadcast scheme such that at time $\frac{n_1}{2} - 2$ there is at most one uninformed vertex in G'. Without loss of generality, let that vertex be v. Broadcasting in P_k begins in time unit $\frac{n_1}{2} - 2$ from u. At time $\frac{n_1}{2} - 1$ the vertex v is also informed and x is also informed. There are $n_2 - 1$ remaining vertices to be informed on P_k and that will take $\frac{n_2-1}{2}$ time. The overall time for the broadcasting will be $\leq \frac{n_1}{2} - 1 + \frac{n_2-1}{2} = \frac{n}{2} - 2$.

[Case even n_1 and even n_2 | two originators in G' and one in P_k]

From Table 3.2, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 2$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Since n_2 is even then $n_2 \geq 2$. Thus the case is complete.

[Case even n_1 and even n_2| one originator in G' and two in P_k]

From Table 3.1, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 2$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Since n_2 is even then $n_2 \geq 2$. Thus the case is complete.

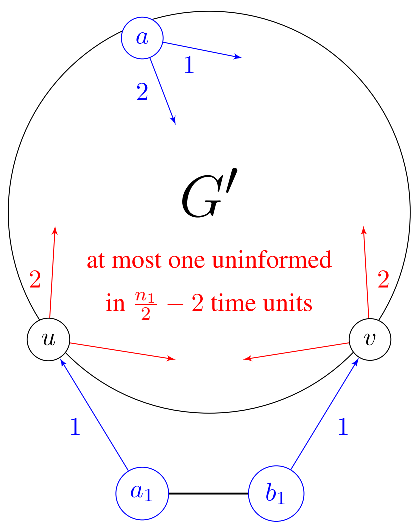


Figure 3.5: Case even n_1 and even n_2 | one originator in G' and two originators in P_k when $n_2 = 2$

[Case even n_1 and even n_2 three originators in P_k]

Consider the last case where the originators are a_1, b_1 and c_1. At time $r \leq \frac{n_2}{2} - 1$ at least one of u

or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r+1$ and end in $r+\frac{n_1}{2}$ by the inductive hypothesis (A).

Consider the special case when $r = \frac{n_2}{2} - 1$. This means that the originators are two neighbors located in the middle of P_k and the neighbor of the leftmost one. At the time $r+1$ vertices u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by inductive hypothesis (D), broadcasting in G is complete by time $r+1+\frac{n_1}{2}-1 = \frac{n_2}{2}+\frac{n_1}{2}-1$ and moreover at time $\frac{n}{2}-2$ at most one vertex of G is uninformed.

Now let $r \leq \frac{n_2}{2}-2$. Broadcasting in G' will begin in time $r+1$ and end in $r+\frac{n_1}{2}$ by the inductive hypothesis (A).

By that time, there are two possible cases for broadcasting in P_k.

Either all vertices in P_k will be informed before time unit $r+\frac{n_1}{2}$ (i.e. when broadcasting finishes in G') or not. In this case, the broadcasting takes place in $r+\frac{n_1}{2}$ time. Taking into account that $r \leq \frac{n_2}{2}-2$, the overall broadcast time will be $b(G) \leq \frac{n_2}{2}+\frac{n_1}{2}-2 = \frac{n}{2}-2$.

The next scenario is that there are vertices in P_k which have not been informed by the time unit $r+\frac{n_1}{2}$. Since $r \geq 1$, the worst possible positioning of a_1 ,b_1 and c_1 will be the when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 3$. If the broadcasting in G' finishes in $r+\frac{n_1}{2}$ time units, it will continue in P_k until time $n_2 - 3 = 2 \cdot \frac{n_2}{2} - 3 \leq \frac{n_1}{2}+\frac{n_2}{2}-3 = \frac{n}{2}-3$.

<u>**Case 2:**</u> n_1 is even and n_2 is odd.

Proof of A. We have to show that $b(G) \leq \lceil \frac{n}{2} \rceil = \lceil \frac{n_1+n_2}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil$.

<u>[Case even n_1 and odd n_2| one originator in G']</u>

Suppose the originator of G is $a \in G'$. By the inductive hypothesis (A), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil = \frac{n_1}{2}$ time units. After this broadcasting begins in P_k from u and v. Informing the path P_k from two endpoints will take $\lceil \frac{n_2}{2} \rceil$ time units which results in total of $\frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil = \lceil \frac{n}{2} \rceil$ broadcast time for graph G. Moreover, since n_2 is odd and u and v start informing at time $\frac{n_1}{2}$ at the latest, then in time unit $\lceil \frac{n}{2} \rceil - 1$ there will be at most one uninformed vertex (the middle vertex).

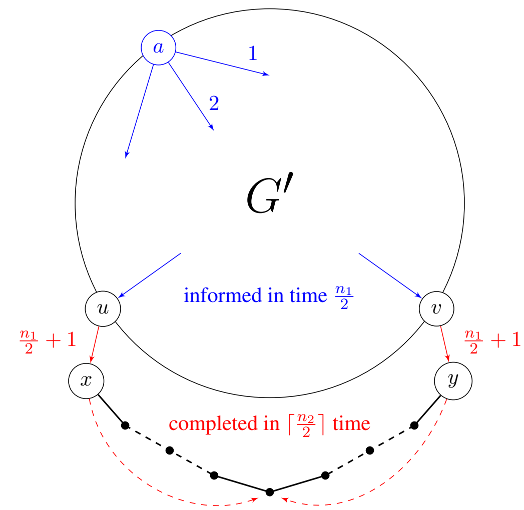


Figure 3.6: Case even n_1 and odd n_2| one originator in G'

[Case even n_1 and odd n_2| one originator in P_k]

Now suppose the originator is the vertex a_1 on P_k. Without loss of generality, assume that from u and v, vertex u is closer to a_1 and $r := dist\{a_1, u\}$. Note that $1 \leq r \leq \lceil \frac{n_2}{2} \rceil$. The vertex a_1 first informs its neighbor that is on the path to u; following this, in the second time unit, a_1 broadcasts to its other neighbor.

We note that at time r, the vertex u receives the message. By the inductive hypothesis (A), at time $r + \frac{n_1}{2}$ all vertices in G' are informed. During this time, every vertex on the path from a_1 to x is informed. There are r such vertices. There are also $r + \frac{n_1}{2} - 1$ other vertices on P_k that are informed. Thus, the number of not informed vertices on P_k is at most $n_2 - 2r - \frac{n_1}{2} + 1$. These vertices form a subpath of P_k, and the endpoints of this path are adjacent to two informed vertices. The additional time required to inform these vertices is $\lceil \frac{n_2+1}{2} - r - \frac{n_1}{4} \rceil \leq \lceil \frac{n_2+1}{2} - r - 1 \rceil = \frac{n_2+1}{2} - r - 1$. Thus, broadcast time is at most $\max\{\frac{n_2+1}{2} - r - 1, 0\} + r + \frac{n_1}{2}$. If $r < \lceil \frac{n_2}{2} \rceil$, then we can broadcast in time $r + \frac{n_2+1}{2} - r - 1 \leq \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 1$, and A is proved in this case. Remains to assume that $r = \lceil \frac{n_2}{2} \rceil$. This implies that at time $\lceil \frac{n_2}{2} \rceil + 1$, P_k is completely informed. Furthermore, we may assume that u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by inductive hypothesis (D), broadcasting in G is complete by time $\lceil \frac{n_2}{2} \rceil + 1 + \frac{n_1}{2} - 1$ and moreover at time $\lceil \frac{n_2}{2} \rceil + \frac{n_1}{2} - 1$ at most one vertex of G is uninformed. This proves (A).

Proof of B. Consider three different locations of the two originators.

[Case even n_1 and odd n_2| two originators in G']

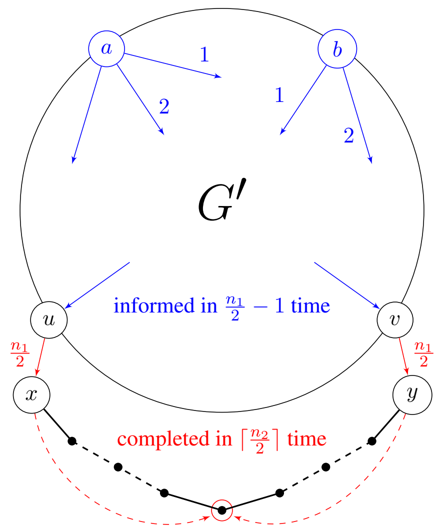


Figure 3.7: Case even n_1 and odd n_2| two originators in G'

Let the originators be a and b. By the inductive hypothesis (B), $b(G') \leq \frac{n_1}{2} - 1$. This means that broadcasting in P_k begins in time unit $\frac{n_1}{2}$ at the latest. Informing the path P_k from two endpoints will take $\lceil \frac{n_2}{2} \rceil$ time. Thus, $b(G) = \frac{n_1}{2} - 1 + \lceil \frac{n_2}{2} \rceil = \lceil \frac{n}{2} \rceil - 1$. Moreover, since the length of the path is odd, at time unit $\lceil \frac{n}{2} \rceil - 2$ only the middle vertex will be uninformed.

[Case even n_1 and odd n_2| one originator in G' and one in P_k]

Suppose one of the originators is a in G' and the other is b_1 on P_k. Broadcasting in G' and P_k takes place in parallel. By the inductive hypothesis (A), the process in G' will terminate at time $\frac{n_1}{2}$.

a_1 informs the longer side of P_k first. Let $r := dist\{a_1, u\}$. $l = \frac{n_1}{2} - r$ is the number of time units broadcasting takes place in P_k after r time units. The number of uninformed vertices in P_k after time $\frac{n_1}{2}$ would be $n_2 - 2r - l$ and since broadcasting will take place from two endpoints, the total broadcast time would be $b(G) = \frac{n_1}{2} + \lceil \frac{n_2 - 2r - l}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2}{2} - r - \frac{l}{2} \rceil = \frac{n_1}{2} + \lceil \frac{n_2}{2} - r - \frac{n_1}{4} + \frac{r}{2} \rceil \leq \frac{n_1}{2} + \lceil \frac{n_2}{2} - \frac{r}{2} \rceil - 1$ And since $r \geq 1$, then $b(G) \leq \lceil \frac{n}{2} \rceil - 1$.

[Case even n_1 and odd n_2| two originators in P_k]

The last arrangement of originators is a_1 and b_1, both in P_k. Consider the special case when $r = \lceil \frac{n_2}{2} \rceil - 1$. At time unit $r + 1$, there are three informed vertices in G'. This implies that at time $r + 1$,

P_k is informed. Furthermore, u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by inductive hypothesis (D), after time $\frac{n_1}{2} - 2$ there will be at most one uninformed vertex in G', thus broadcasting in G will be complete by time $r + 1 + \frac{n_1}{2} - 1 = r + \frac{n_1}{2} = \lceil \frac{n_2}{2} \rceil - 1 + \frac{n_1}{2} = \lceil \frac{n}{2} \rceil - 1$ and moreover, at time $\lceil \frac{n}{2} \rceil - 2$ at most one vertex of G is uninformed.

Now for $r \leq \lceil \frac{n_2}{2} \rceil - 2$ At time r at least one of u or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r+1$ and end in $r + \frac{n_1}{2}$ by the inductive hypothesis (A).

By that time, there are two possible cases for broadcasting in P_k.

First possible case is that all vertices in P_k will be informed before time $r + \frac{n_1}{2}$ (i.e. when broadcasting finishes in G'). In this case, the broadcasting takes place in $r + \frac{n_1}{2}$ time. Taking into account that $r \leq \lceil \frac{n_2}{2} \rceil - 2$, the overall broadcast time of G will be $b(G) \leq r + \frac{n_1}{2} \leq \lceil \frac{n_2}{2} \rceil + \frac{n_1}{2} - 2 = \lceil \frac{n}{2} \rceil - 2$. This also means that by the end of time $\lceil \frac{n}{2} \rceil - 2$ there will be no uninformed vertices.

The next scenario is that there are vertices in P_k which have not been informed by time unit $r + \frac{n_1}{2}$. Since $r \geq 1$, the worst possible positioning of a_1 and b_1 will be when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 2$. If the broadcasting in G' finishes in $r + \frac{n_1}{2}$ time units, it will continue in P_k until time $n_2 - 2 = \lceil \frac{n_2}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 - 2 \leq \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 3 = \lceil \frac{n}{2} \rceil - 3$ since $n_2 \leq n_1 - 2$.

Proof of C. There are four different locations of the two originators.

<u>[Case even n_1 and odd n_2| three originators in G']</u> Let the originators be a, b and c. By the inductive hypothesis (D), there is a broadcast scheme such that at time $\frac{n_1}{2} - 2$ there is at most one uninformed vertex in G'. Without loss of generality, let that vertex be v. Broadcasting in P_k begins in time unit $\frac{n_1}{2} - 2$ from u. At time $\frac{n_1}{2} - 1$ the vertex v is also informed and x is also informed. There are $n_2 - 1$ remaining vertices to be informed on P_k and that will take $\frac{n_2 - 1}{2} = \lceil \frac{n_2}{2} \rceil - 1$ time. The overall time for the broadcasting will be $\leq \frac{n_1}{2} - 1 + \lceil \frac{n_2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil - 2$.

<u>[Case even n_1 and odd n_2| two originators in G' and one in P_k]</u>

From Table 3.2, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 3$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Consider the case $n_2 = 1$: show that $b(G) \leq \lceil \frac{n}{2} \rceil - 2$.

Since $n_2 = 1$, then $\lceil \frac{n}{2} \rceil - 2 = \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 2 = \frac{n_1}{2} + 1 - 2 = \frac{n_1}{2} - 1$ and since there are two originators in G' by the inductive hypothesis (B), this holds. Then take $n_2 \geq 3$, the statement is true by Lemma 3.1.2.

[Case even n_1 and odd n_2| one originator in G' and two in P_k]

From Table 3.1, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 3$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Since there are two originators on P_k, $n_2 \geq 2$ and since it is odd, then $n_2 \geq 3$. Thus, the case is proved by Lemma 3.1.1.

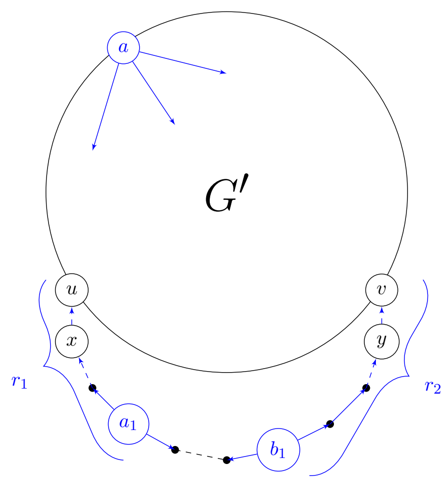


Figure 3.8: Case even n_1 and odd n_2| one originator in G' and two in P_k

[Case even n_1 and odd n_2| three originators in P_k]

Consider the last case where the originators are a_1, b_1 and c_1. At time $r \leq \lceil \frac{n_2}{2} \rceil - 1$ at least one of u or v will be informed. Without loss of generality let it be u.

Consider the special case when $r = \lceil \frac{n_2}{2} \rceil - 1$. This means that the originators are neighbors located in the middle of P_k(see Figure 3.9).

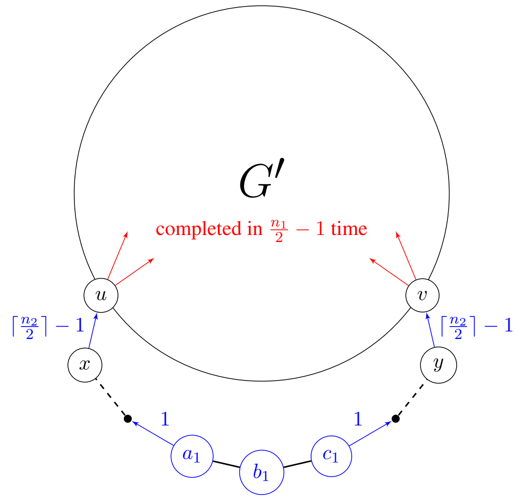


Figure 3.9: Case even n_1 and odd $n_2|$ three originators in P_k and $r = \lceil \frac{n_2}{2} \rceil - 1$

At the time r both u and v will be informed and by the inductive hypothesis (B), broadcasting in G' will take place in at most $\frac{n_1}{2} - 1$ time. This means that total broadcast time for G will be $b(G) \leq \lceil \frac{n_2}{2} \rceil - 1 + \frac{n_1}{2} - 1 = \lceil \frac{n}{2} \rceil - 2$

Now let $r \leq \lceil \frac{n_2}{2} \rceil - 2$. Broadcasting in G' will begin in time $r+1$ and end in $r + \frac{n_1}{2}$ by the inductive hypothesis (A).

By that time, there are two possible cases for broadcasting in P_k.

Either all vertices in P_k will be informed before time unit $r + \frac{n_1}{2}$ (i.e. when broadcasting finishes in G') or not. In this case, the broadcasting takes place in $r + \frac{n_1}{2}$ time. Taking into account that $r \leq \lceil \frac{n_2}{2} \rceil - 2$, the overall broadcast time will be $b(G) \leq \lceil \frac{n_2}{2} \rceil + \frac{n_1}{2} - 2 = \lceil \frac{n}{2} \rceil - 2$.

The next scenario is when there are vertices in P_k which have not been informed by the time unit $r + \frac{n_1}{2}$. Since $r \geq 1$, the worst possible positioning of a_1, b_1 and c_1 will be when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 3$. If the broadcasting in G' finishes in $r + \frac{n_1}{2}$ time units, it will continue in P_k until time $n_2 - 3 = 2 \cdot \lceil \frac{n_2}{2} \rceil - 1 - 3 \leq \frac{n_1}{2} + \lceil \frac{n_2}{2} \rceil - 4 = \lceil \frac{n}{2} \rceil - 4$.

Proof of D. Since n is odd, statement D does not apply.

<u>**Case 3:**</u> n_1 is odd and n_2 is even.

Proof of A. We have to show that $b(G) \leq \lceil \frac{n}{2} \rceil = \lceil \frac{n_1+n_2}{2} \rceil = \lceil \frac{n_1}{2} \rceil + \frac{n_2}{2}$.

[Case odd n_1 and even n_2 | one originator in G']

Suppose the originator of G is $a \in G'$. By the inductive hypothesis (A), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil$ time units. Moreover, since n_1 is odd, then at time unit $\lceil \frac{n_1}{2} \rceil - 1$ there is at most one uninformed vertex in G'. Let that vertex be v. At time unit $\lceil \frac{n_1}{2} \rceil$ vertices x and v are informed. Informing the path P_k from two endpoints will take $\lceil \frac{n_2-1}{2} \rceil = \frac{n_2}{2}$ time units which results in total of $\lceil \frac{n_1}{2} \rceil + \frac{n_2}{2} = \lceil \frac{n}{2} \rceil$ broadcast time for graph G. Moreover, since $n_2 - 1$ is odd and x and v start informing from both sides, then in time unit $\lceil \frac{n}{2} \rceil - 1$ there will be at most one uninformed vertex.

[Case odd n_1 and even n_2 | one originator in P_k]

Now suppose the originator is the vertex a_1 on P_k. Without loss of generality, assume that from u and v, vertex u is closer to a_1 and $r := dist\{a_1, u\}$. Note that $1 \leq r \leq \frac{n_2}{2}$. The vertex a_1 first informs its neighbor that is on the path to u; following this, in the second time unit, a_1 broadcasts to its other neighbor.

We note that at time r, the vertex u receives the message. By the inductive hypothesis (A), at time $r + \lceil \frac{n_1}{2} \rceil$ all vertices in G' are informed. During this time, every vertex on the path from a_1 to x is informed. There are r such vertices. There are also $r + \lceil \frac{n_1}{2} \rceil - 1$ other vertices on P_k that are informed. Thus, the number of uninformed vertices on P_k is at most $n_2 - 2r - \lceil \frac{n_1}{2} \rceil + 1$. These vertices form a subpath of P_k, and the endpoints of this path are adjacent to two informed vertices. The additional time required to inform these vertices is $\lceil \frac{n_2+1}{2} - r - \frac{n_1}{4} \rceil \leq \lceil \frac{n_2+1}{2} - r - 1 \rceil = \frac{n_2+1}{2} - r - 1$. Thus, broadcast time of G is at most $\max\{\frac{n_2+1}{2} - r - 1, 0\} + r + \lceil \frac{n_1}{2} \rceil$. If $r < \frac{n_2}{2}$, then we can broadcast in time $r + \frac{n_2+1}{2} - r - 1 \leq \lceil \frac{n_1}{2} \rceil + \frac{n_2}{2} - 1$, and A is proved in this case. Remains to assume that $r = \frac{n_2}{2}$. This implies that at time $\frac{n_2}{2} + 1$, P_k is completely informed. Furthermore, we may assume that u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by the inductive hypothesis (C), broadcasting in G is complete by time $\frac{n_2}{2} + 1 + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n}{2} \rceil - 1$. This completes the proof of A.

Proof of B. Consider three different locations of the two originators.

[Case odd n_1 and even n_2 | two originators in G'] Let the originators be a and b. By the inductive hypothesis (B), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil - 1$ time units. Moreover, since n_1 is odd, then at time unit $\lceil \frac{n_1}{2} \rceil - 2$ there is at most one uninformed vertex in G'. Let that vertex be v. At time unit $\lceil \frac{n_1}{2} \rceil - 1$ vertices x and v are informed. Informing the path P_k from two endpoints will take $\lceil \frac{n_2-1}{2} \rceil = \frac{n_2}{2}$ time units which results in total of $\lceil \frac{n_1}{2} \rceil - 1 + \frac{n_2}{2} = \lceil \frac{n}{2} \rceil - 1$ broadcast time for graph G. Moreover, since $n_2 - 1$ is odd and x and v start informing from both sides, then in time unit $\lceil \frac{n}{2} \rceil - 2$ there will be at most one uninformed vertex.

[Case odd n_1 and even n_2 | one originator in G' and one in P_k]

Suppose the one of the originators is in G' and the other is on P_k. Let these be a and b_1. Broadcasting in G' and P_k takes place in parallel. By the inductive hypothesis (A), the process in G' will terminate at time $\lceil \frac{n_1}{2} \rceil$.

b_1 informs the longer side of P_k first. $r := dist\{b_1, u\}$. The earliest possible to fully inform P_k from b_1 is $\frac{n_2}{2}$. Note that $1 \leq r \leq \frac{n_2}{2}$. Since the assumption is that $n_1 \geq n_2 + 2$, thus $\lceil \frac{n_1}{2} \rceil \geq \frac{n_2}{2} + 1 \geq r + 1$. This means that broadcasting in P_k will continue until time $\lceil \frac{n_1}{2} \rceil$. $l = \lceil \frac{n_1}{2} \rceil - r$ is the number of time units broadcasting takes place in P_k after r time units. The number of uninformed vertices in P_k after time $\lceil \frac{n_1}{2} \rceil$ would be $n_2 - 2r - l$ and since broadcasting will take place from two endpoints, the total broadcast time would be $b(G) = \lceil \frac{n_1}{2} + \max\{\frac{n_2-2r-l}{2}, 0\} \rceil = \lceil \frac{n_1}{2} + \frac{n_2}{2} - r - \frac{l}{2} \rceil = \lceil \frac{n_1}{2} + \frac{n_2}{2} - r - \lceil \frac{n_1}{4} \rceil + \frac{r}{2} \rceil$. Since $n_1 \geq 4$ and n_1 is odd, then $n_1 \geq 5$ and $\lceil \frac{n_1}{4} \rceil \geq 2$. Then $b(G) \leq \lceil \frac{n_1}{2} + \frac{n_2}{2} - \frac{r}{2} \rceil - 2 \leq \lceil \frac{n}{2} \rceil - 2$.

[Case odd n_1 and even n_2 | two originators in P_k]

The last arrangement of originators is a_1 and b_1. At time $r \leq \frac{n_2}{2} - 1$ at least one of u or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r + 1$ and end in $r + \lceil \frac{n_1}{2} \rceil$ by the inductive hypothesis (A). Moreover, since n_1 is odd, then at time unit $r + \lceil \frac{n_1}{2} \rceil - 1$ there is at most one uninformed vertex in G'.

By that time, there are two possible cases for broadcasting in P_k.

First possible case is that all vertices in P_k will be informed before time $r + \lceil \frac{n_1}{2} \rceil$ (i.e. when broadcasting finishes in G'). In this case, the broadcasting takes place in $r + \lceil \frac{n_1}{2} \rceil$ time. Taking into account that $r \leq \frac{n_2}{2} - 1$, the overall broadcast time of G will be $b(G) \leq r + \lceil \frac{n_1}{2} \rceil \leq \frac{n_2}{2} + \lceil \frac{n_1}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil - 1$ and at time unit $\lceil \frac{n}{2} \rceil - 1$ there is at most one uninformed vertex in G.

The next scenario is that there are vertices in P_k which have not been informed by time unit $r + \lceil \frac{n_1}{2} \rceil$.

Since $r \geq 1$, the worst possible positioning of a_1 and b_1 will be when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 2$. If the broadcasting in G' finishes in $r + \lceil \frac{n_1}{2} \rceil$ time units, it will continue by informing one vertex in each time unit in P_k until time $n_2 - 2 = \frac{n_2}{2} + \frac{n_2}{2} - 2 \leq \lceil \frac{n_1}{2} \rceil + \frac{n_2}{2} - 2 = \lceil \frac{n}{2} \rceil - 2$ since $n_2 \leq n_1 - 2$.

Proof of C. There are four different locations of the two originators.

[Case odd n_1 and even n_2 | three originators in G'] Let the originators be a, b and c. By the inductive hypothesis (C), broadcasting in G' will be complete by time $\lceil \frac{n_1}{2} \rceil - 2$. Broadcasting in P_k begins in time unit $\lceil \frac{n_1}{2} \rceil - 1$ from u. Informing the path P_k from two endpoints will take $\frac{n_2}{2}$ time units which results in total of at most $\lceil \frac{n_1}{2} \rceil - 2 + \frac{n_2}{2} = \lceil \frac{n}{2} \rceil - 2$ broadcast time for graph G.

[Case odd n_1 and even n_2 | two originators in G' and one in P_k]

From Table 3.2, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 2$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Since n_2 is even, then $n_2 \geq 2$.

[Case odd n_1 and even n_2| one originator in G' and two in P_k]

From Table 3.1, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 4$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Since n_2 is even consider the special case when $n_2 = 2$. Let the originators in G' be a and in P_k be a_1 and b_1. a starts broadcasting, a_1 informs u and b_1 informs v. After the first time unit, there are at least three originators in $G'(a, u, \text{and } v)$. By the inductive hypothesis (C), by the time unit $\lceil \frac{n_1}{2} \rceil - 2$ broadcasting is complete. Since $n_2 = 2$, at time $1 + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n_1}{2} \rceil - 1 + 1 - 1 = \lceil \frac{n_1}{2} \rceil - 1 + \frac{n_2}{2} - 1 \leq \lceil \frac{n}{2} \rceil - 2$ broadcasting in G will be complete.

Now take $n_2 \geq 4$, and the statement is true by Lemma 3.1.1.

[Case odd n_1 and even n_2 three originators in P_k]

Consider the last case where the originators are a_1, b_1 and c_1. At time $r \leq \frac{n_2}{2} - 1$ at least one of u or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r + 1$ and end in $r + \frac{n_1}{2}$ by the inductive hypothesis (A).

Consider the special case when $r = \frac{n_2}{2} - 1$. This means that the originators are two neighbors located in the middle of P_k and the neighbor of the leftmost one. At the time $r + 1$ vertices u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by inductive hypothesis (C), broadcasting in G is complete by time $r + 1 + \lceil \frac{n_1}{2} \rceil - 2 = \frac{n_2}{2} - 1 + 1 + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n}{2} \rceil - 2$.

Now let $r \leq \frac{n_2}{2} - 2$.

By that time, there are two possible cases for broadcasting in P_k.

Either all vertices in P_k will be informed before time unit $r + \lceil \frac{n_1}{2} \rceil$ (i.e. when broadcasting finishes in G')or not. In this case, the broadcasting takes place in $r + \lceil \frac{n_1}{2} \rceil$ time. Taking into account that $r \leq \frac{n_2}{2} - 2$, the overall broadcast time will be $b(G) \leq \frac{n_2}{2} + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n}{2} \rceil - 2$.

The next scenario is that there are vertices in P_k which have not been informed by the time unit $r + \lceil \frac{n_1}{2} \rceil$. Since $r \geq 1$, the worst possible positioning of a_1 ,b_1 and c_1 will be the when they are neighbors and a_1 is located at $r = 1$ distance from u. The broadcast time for P_k (without v informing from G') will be $n_2 - 3$. If the broadcasting in G' finishes in $r + \lceil \frac{n_1}{2} \rceil$ time units, it will continue in P_k until time $n_2 - 3 = 2 \cdot \frac{n_2}{2} - 3 \leq \lceil \frac{n_1}{2} \rceil + \frac{n_2}{2} - 3 = \lceil \frac{n}{2} \rceil - 3$.

Proof of D. Since n is odd, statement D does not apply.

Case 4: n_1 and n_2 are odd.

Proof of A. We have to show that $b(G) \leq \lceil \frac{n}{2} \rceil = \lceil \frac{n_1+n_2}{2} \rceil = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1$.

[Case odd n_1 and odd n_2| one originator in G'] Suppose the originator of G is $a \in G'$. By the inductive hypothesis (A), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil$ time units. Moreover, since n_1 is odd, then at time unit $\lceil \frac{n_1}{2} \rceil - 1$ there is at most one uninformed vertex in G'. Let that vertex be v. At time unit $\lceil \frac{n_1}{2} \rceil$ vertices x and v are informed. Informing the path P_k from two endpoints will take $\frac{n_2-1}{2} = \lceil \frac{n_2}{2} \rceil - 1$ time units which results in total of $\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil$ broadcast time for graph G.

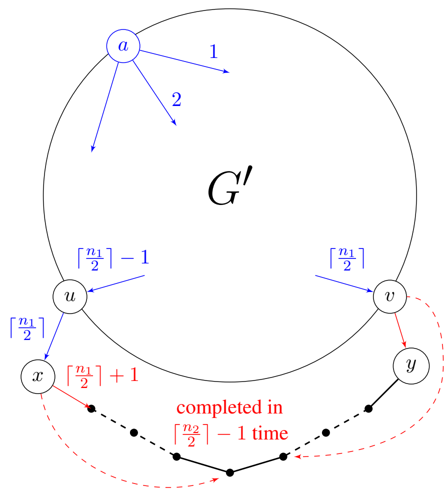


Figure 3.10: Case odd n_1 and odd $n_2|$ one originator in G'

[Case odd n_1 and odd $n_2|$ one originator in P_k] Now suppose the originator is the vertex a_1 on P_k. Without loss of generality, assume that from u and v, vertex u is closer to a_1 and $r := dist\{a_1, u\}$. Note that $1 \leq r \leq \lceil \frac{n_2}{2} \rceil$. The vertex a_1 first informs its neighbor that is on the path to u; following this, in the second time unit, a_1 broadcasts to its other neighbor.

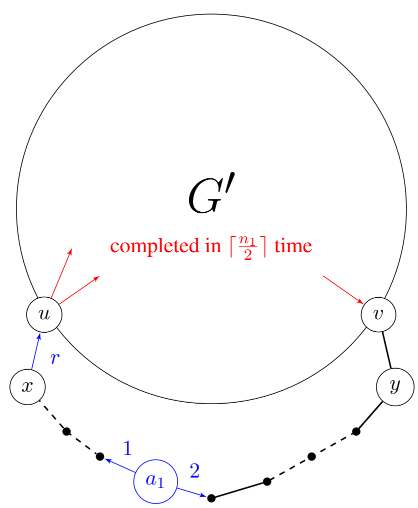


Figure 3.11: Case odd n_1 and odd $n_2|$ one originator P_k

We note that at time r, the vertex u receives the message. By the inductive hypothesis (A), at time $r + \lceil \frac{n_1}{2} \rceil$ all vertices in G' are informed. During this time, every vertex on the path from a_1 to x is informed. There are r such vertices. There are also $r + \lceil \frac{n_1}{2} \rceil - 1$ other vertices on P_k that are informed. Thus, the number of uninformed vertices on P_k is at most $n_2 - 2r - \lceil \frac{n_1}{2} \rceil + 1$. These vertices form a subpath of P_k, and the endpoints of this path are adjacent to two informed vertices. The additional time required to inform these vertices is $\lceil \frac{n_2+1}{2} - r - \frac{n_1}{4} \rceil \leq \lceil \frac{n_2+1}{2} - r - 1 \rceil = \frac{n_2+1}{2} - r - 1$. Thus, broadcast of G is at most $\max\{\frac{n_2+1}{2} - r - 1, 0\} + r + \lceil \frac{n_1}{2} \rceil$. If $r < \lceil \frac{n_2}{2} \rceil$, then we can broadcast in $r + \frac{n_2+1}{2} - r - 1 \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1$ time, and A is proved in this case. Remains to assume that $r = \lceil \frac{n_2}{2} \rceil$. This implies that at time $\lceil \frac{n_2}{2} \rceil + 1$, P_k is fully informed. Furthermore, we may assume that u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by (C), we can inform all vertices of G at time $\lceil \frac{n_2}{2} \rceil + 1 + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil$ time. This completes the proof of A.

Proof of B. We must show $b(G) \leq \lceil \frac{n}{2} \rceil - 1 = \lceil \frac{n_1+n_2}{2} \rceil - 1 = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 2$. Consider three different locations of the two originators.

<u>[Case odd n_1 and odd n_2| two originators in G']</u> Let the originators be a and b.

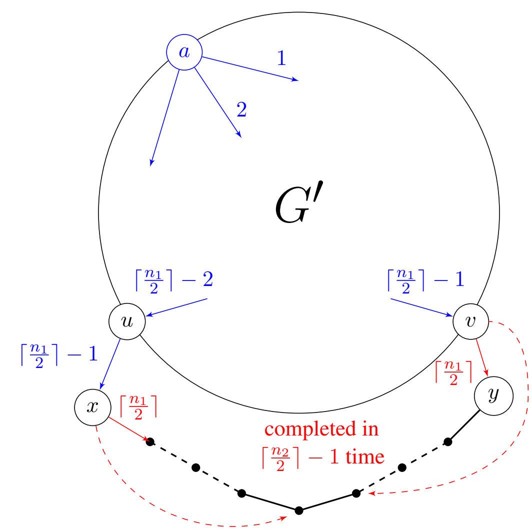


Figure 3.12: Case odd n_1 and odd n_2| two originators in G'

By the inductive hypothesis (B), vertices of G' are informed in $\lceil \frac{n_1}{2} \rceil - 1$ time units. Moreover, since n_1 is odd, then at time unit $\lceil \frac{n_1}{2} \rceil - 2$ there is at most one uninformed vertex in G'. Let that vertex be v. At time unit $\lceil \frac{n_1}{2} \rceil - 1$ vertices x and v are informed. Informing the path P_k from two endpoints will take $\frac{n_2-1}{2} = \lceil \frac{n_2}{2} \rceil - 1$ time units which results in total of $\lceil \frac{n_1}{2} \rceil - 1 + \lceil \frac{n_2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil - 1$ broadcast time for graph G.

<u>[Case odd n_1 and odd n_2| one originator in G' and one in P_k]</u>

Suppose one of the originators is a in G' and the other is b_1 on P_k. Broadcasting in G' and P_k takes place in parallel. By the inductive hypothesis (A), the process in G' will terminate at time $\lceil\frac{n_1}{2}\rceil$.

Since informing G' is not a priority, b_1 informs the longer side of P_k first. $r := dist\{b_1, u\}$. The earliest possible to fully inform P_k from b_1 is $\lceil\frac{n_2}{2}\rceil$. Note that $1 \leq r \leq \lceil\frac{n_2}{2}\rceil$. Since the assumption is that $n_1 \geq n_2 + 2$, thus $\lceil\frac{n_1}{2}\rceil \geq \lceil\frac{n_2}{2}\rceil + 1 \geq r + 1$. This means that broadcasting in P_k will continue until $l = \lceil\frac{n_1}{2}\rceil - r$ is the number of time units broadcasting takes place in P_k after r time units. The number of uninformed vertices in P_k after time $\lceil\frac{n_1}{2}\rceil$ would be $n_2 - 2r - l$ and since broadcasting will take place from two endpoints, the total broadcast time would be $b(G) = \lceil\frac{n_1}{2} + \max\{\frac{n_2-2r-l}{2}, 0\}\rceil = \lceil\frac{n_1}{2} + \frac{n_2}{2} - r - \frac{l}{2}\rceil = \lceil\frac{n_1}{2} + \frac{n_2}{2} - r - \frac{n_1}{4} + \frac{r}{2}\rceil \leq \lceil\frac{n_1}{2} + \frac{n_2}{2} - \frac{r}{2}\rceil - 1 \leq \frac{n}{2} - 1$.

[Case odd n_1 and odd n_2| two originators in P_k]

The last arrangement of originators is a_1 and b_1, both in P_k. Consider the special case when $r = \lceil\frac{n_2}{2}\rceil - 1$. At time unit $r+1$, there are three informed vertices in G'. This implies that at time $r+1$, P_k is informed. Furthermore, u, v, and the third vertex in G' are informed (minimum degree is at least two in G' and u must have a neighbor in G' which is not v). Then by inductive hypothesis (C), broadcasting in G' will finish in $\lceil\frac{n_1}{2}\rceil - 2$ units. Thus broadcasting in G will be complete by time $r + 1 + \lceil\frac{n_1}{2}\rceil - 2 = r + \lceil\frac{n_1}{2}\rceil - 1 = \lceil\frac{n_2}{2}\rceil - 1 + \lceil\frac{n_1}{2}\rceil - 1 = \frac{n}{2} - 1$.

Now for $r \leq \lceil\frac{n_2}{2}\rceil - 2$ At time r at least one of u or v will be informed. Without loss of generality let it be u. Broadcasting in G' will begin in time $r+1$ and end in $r + \lceil\frac{n_1}{2}\rceil$ by the inductive hypothesis (A).

By that time, there are two possible cases for broadcasting in P_k.

First possible case is that all vertices in P_k will be informed before time $r + \lceil\frac{n_1}{2}\rceil$ (i.e. when broadcasting finishes in G'). In this case, the broadcasting takes place in $r + \lceil\frac{n_1}{2}\rceil$ time. Taking into account that $r \leq \lceil\frac{n_2}{2}\rceil - 2$, the overall broadcast time of G will be $b(G) \leq r + \lceil\frac{n_1}{2}\rceil \leq \lceil\frac{n_2}{2}\rceil + \lceil\frac{n_1}{2}\rceil - 2 = \frac{n}{2} - 1$.

The next scenario is that there are vertices in P_k which have not been informed by time unit $r + \lceil\frac{n_1}{2}\rceil$. Since $r \geq 1$, the worst possible positioning of a_1 and b_1 will be when they are neighbors and a_1 is located at $r = 1$ distance from u - maximally far from v. The broadcast time for P_k (without v informing from G') will be $n_2 - 2$. If the broadcasting in G' finishes in $r + \lceil\frac{n_1}{2}\rceil$ time units, it will continue by informing one vertex in each time unit in P_k until time $n_2 - 2 = \lceil \frac{n_2}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 - 2 \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 2 = \frac{n}{2} - 1$ since $n_2 \leq n_1 - 2$.

Proof of C. Since n is even, statement C does not apply.

Proof of D. There are four different locations of the two originators.

[Case odd n_1 and odd n_2| three originators in G'] Let the originators be a, b and c.

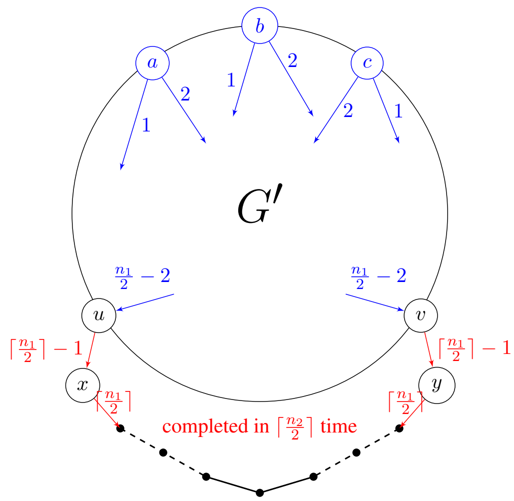


Figure 3.13: Case odd n_1 and odd n_2| three originators in G'

By the inductive hypothesis (C), broadcasting in G' will be complete by time $\lceil \frac{n_1}{2} \rceil - 2$. Broadcasting in P_k begins in time unit $\lceil \frac{n_1}{2} \rceil - 1$ from u. Informing the path P_k from two endpoints will take $\lceil \frac{n_2}{2} \rceil$ time units which results in total of at most $\lceil \frac{n_1}{2} \rceil - 2 + \lceil \frac{n_2}{2} \rceil = \lceil \frac{n}{2} \rceil - 1$ broadcast time for graph G. Moreover, since n_2 is odd and u and v start informing at time $\lceil \frac{n_1}{2} \rceil - 1$ at the latest, then in time unit $\lceil \frac{n}{2} \rceil - 2$ there will be at most one uninformed vertex (the middle vertex).

[Case odd n_1 and odd n_2| two originators in G' and one in P_k]

From Table 3.2, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 3$ the result holds. The initial assumption suggests that $n_1 \geq 4$ and $n_2 \geq 1$. Let us show that for the case $n_2 = 1$, $b(G) \leq \frac{n}{2} - 1$. Since $n_2 = 1$, then $\frac{n}{2} - 1 = \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 1 - 1 = \lceil \frac{n_1}{2} \rceil + 1 - 2 = \lceil \frac{n_1}{2} \rceil - 1$ and by the inductive hypothesis (B), this holds since there are two originators in G'. Since n_2 is odd, it remains to assume that $n_2 \geq 3$, which completes the proof of this case.

[Case odd n_1 and odd n_2| one originator in G' and two in P_k]

From Table 3.1, we can state that for all $n_1 \geq 4$ and for all $n_2 \geq 3$ the result holds. The initial assumption states that $n_1 \geq 4$ and $n_2 \geq 1$. Note, however, since there are two originators on P_k, then $n_2 \geq 2$ and since it's odd, then $n_2 \geq 3$. Thus the case is complete.

[Case odd n_1 and odd n_2| three originators in P_k]

Consider the last case where the originators are a_1, b_1 and c_1. At time $r \leq \lceil \frac{n_2}{2} \rceil - 1$ at least one of u or v will be informed. Without loss of generality let it be u.

Consider the special case when $r = \lceil \frac{n_2}{2} \rceil - 1$. This means that the originators are neighbors located in the middle of P_k(see Figure below).

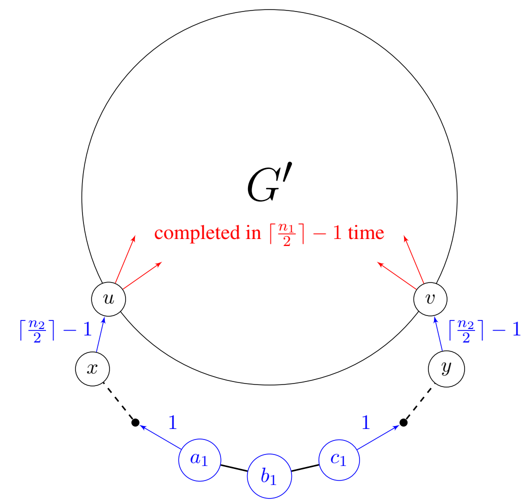


Figure 3.14: Case odd n_1 and odd n_2| three originators in P_k and $r = \lceil \frac{n_2}{2} \rceil - 1$

At the time r both u and v will be informed and since n_1 is odd, by the inductive hypothesis (B), at time $\lceil \frac{n_1}{2} \rceil - 2$ there is at most one uninformed vertex in G'. This means that total broadcast time for G will be $b(G) \leq \lceil \frac{n_2}{2} \rceil - 1 + \lceil \frac{n_1}{2} \rceil - 1 = \frac{n}{2} - 1$ and at time $\frac{n}{2} - 2$ there is at most one uninformed vertex.

Now let $r \leq \lceil \frac{n_2}{2} \rceil - 2$. Broadcasting in G' will begin in time $r + 1$ and end in $r + \lceil \frac{n_1}{2} \rceil$ by the inductive hypothesis (A). Moreover at time $r + \lceil \frac{n_1}{2} \rceil - 1$ there will be at most one uninformed vertex in G'.

By that time, there are two possible cases for broadcasting in P_k.

Either all vertices in P_k will be informed before time unit $r + \lceil \frac{n_1}{2} \rceil$ (i.e. when broadcasting finishes in G') or not. In this case, $b(G) \leq r + \lceil \frac{n_1}{2} \rceil$ time. Taking into account that $r \leq \lceil \frac{n_2}{2} \rceil - 2$, the overall broadcast time will be $b(G) \leq \lceil \frac{n_2}{2} \rceil + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n}{2} \rceil - 1$ and by the inductive hypothesis (A), at time $\lceil \frac{n}{2} \rceil - 2$ there will be at most one uninformed vertex in G'.

The next scenario is when there are vertices in P_k which have not been informed by the time unit $r + \lceil \frac{n_1}{2} \rceil$. Since $r \geq 1$, the worst possible positioning of a_1 ,b_1 and c_1 will be the when they are neighbors and a_1 is located at $r = 1$ distance from u. This will maximize the distance between the vertices and v. The broadcast time for P_k (without v informing from G') will be $n_2 - 3$. If the broadcasting in G' finishes in $r + \lceil \frac{n_1}{2} \rceil$ time units, it will continue in P_k until time $n_2 - 3 = 2 \cdot \lceil \frac{n_2}{2} \rceil - 1 - 3 \leq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil - 4 = \lceil \frac{n}{2} \rceil - 4$.

This completes the proof of the theorem. □

3.2 The bound is tight

In this section, we discuss 2-connected graphs for which the upper bound $\lceil \frac{n}{2} \rceil$ for broadcasting is best possible i.e. $b(G) = \lceil \frac{n}{2} \rceil$.

A trivial example of an infinite family of graphs matching the bound is the cycle C_n. As mentioned in Section 2.1, the broadcast time of C_n equals $\lceil \frac{n}{2} \rceil$.

Another example of an infinite family of graphs is the Unichordal graphs. These are cycles on n vertices for which there is an extra edge between two non-consecutive vertices. Let those two vertices be u and v. Note, that this graph is isomorphic to two cycles C^1 (of length n_1) and C^2 (of length n_2), which share the (u, v) edge in common. In each of the cycles, depending on parity, there is either one vertex a that is equidistant from u and v or two neighbor vertices b and c such that they have equal distance from u and v correspondingly.

Let us discuss broadcasting from these vertices. The case of b and c (i.e. even-length cycle) can have two different broadcast schemes.

In broadcast scheme 1, without loss of generality b is the originator and informs c in the first time unit. This results in u and v being informed in the same $\frac{n_2}{2}$-nd time unit. Starting in the next cycle together u and v finish broadcasting by time unit $\lceil \frac{n_1 - 2}{2} \rceil$ since they inform the remaining $n_1 - 2$ vertices by two in each time unit. This results in total broadcast time of $\frac{n_2}{2} + \lceil \frac{n_1}{2} \rceil - 1 = \lceil \frac{n_1 + n_2}{2} \rceil - 1$ and since u and v are counted twice in n_1 and n_2, then $n_1 + n_2 = n + 2$, which results in broadcast time of $\lceil \frac{n}{2} + \frac{2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil$.

In broadcast scheme 2, b informs the other neighbor first, resulting in u being informed in time unit $\frac{n}{2} - 1$. Since c is informed in the second time unit, v will be informed in time unit $2 + \frac{n_2}{2} - 1 = \frac{n_2}{2} + 1$. If there is at least two vertices to inform in the other cycle, then u should first inform the new cycle, since in that case at time unit $\frac{n_2}{2} + 1$ there will be 4 informed vertices in the new cycle including u and v, while if u informs v at time unit $\frac{n_2}{2}$, then there will be only three. So informing the cycle is more beneficial. Afterwards, broadcasting in the new cycle will be complete in $\lceil \frac{n_1-4}{2} \rceil$ time units. This means, that broadcasting will be complete in time $\frac{n_2}{2} + 1 + \lceil \frac{n_1-4}{2} \rceil = \frac{n_2}{2} + 1 + \lceil \frac{n_1}{2} \rceil - 2 = \lceil \frac{n_2}{2} + \frac{n_1}{2} \rceil - 1 = \lceil \frac{n_2+n_1}{2} \rceil - 1$, and since $n_1 + n_2 = n + 2$, then this will become $\lceil \frac{n+2}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil + 1 - 1 = \lceil \frac{n}{2} \rceil$.

The case of a (i.e. odd-length cycle), both directions of the first call result in symmetric broadcast schemes. Without loss of generality u is informed at time unit $\lceil \frac{n_1}{2} \rceil - 1$ and v is informed one time unit after by the other direction, so u informs one vertex in the other cycle. Having three originators in the new cycle, broadcasting is complete after $\lceil \frac{n_2-3}{2} \rceil$ time units. This results in total of $\lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2-1}{2} \rceil - 1 = \lceil \frac{n_1}{2} \rceil - 1 + \lceil \frac{n_2-1}{2} \rceil = \frac{n_1-1}{2} + \lceil \frac{n_2-1}{2} \rceil = \lceil \frac{n_1-1+n_2-1}{2} \rceil = \lceil \frac{n+2-2}{2} \rceil = \lceil \frac{n}{2} \rceil$ time units.

This means, that in any case, there exist at least two vertices in the Unichordal graph for which, and thus, for the whole graph, the broadcast time becomes $b(UG) = \lceil \frac{n}{2} \rceil$.

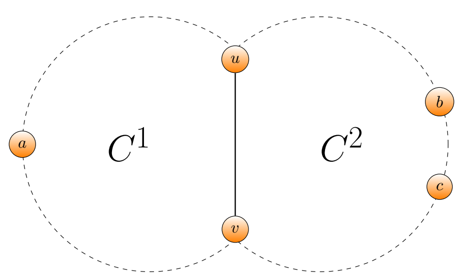


Figure 3.15: Unichordal graph with odd and even length cycles

Consider the Thagomizer graph TG_n presented in Figure 3.16. This graph is named after thagomizers, the spiked tails of the dinosaur named Stegosaurus. This graph is also known as the triangular book graph. Essentially this is a complete bipartite graph $K_{2,n-2}$ with an additional edge between the two vertices in the first partition. Vertices u and v are called the base vertices.

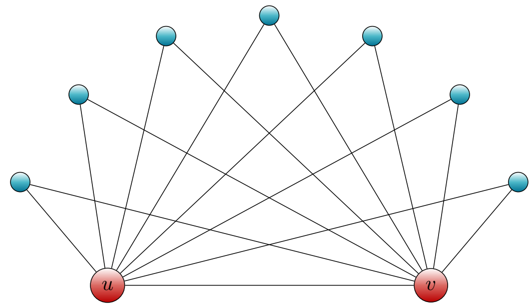


Figure 3.16: Thagomizer graph with $n = 9$

The graph is clearly 2-connected since the removal of the base vertices makes it disconnected.

Claim 3.2.1. *Broadcast time of the Thagomizer graph is* $b(TG_n) = \lceil \frac{n}{2} \rceil$.

Proof. In any broadcast scheme, regardless of the originator, after two time units four vertices including base vertices u and v are informed. After this, the base vertices are the only vertices that inform. In each time unit they inform 2 vertices. This means, that the remaining $n-4$ vertices will be informed in $\lceil \frac{n-4}{2} \rceil$ time units. Overall. the broadcast time of the Thagomizer graph is $b(TG_n) = 2 + \lceil \frac{n-4}{2} \rceil = 2 + \lceil \frac{n}{2} \rceil - 2 = \lceil \frac{n}{2} \rceil$. □

The Complete bipartite graph $K_{2,n-2}$, is a spanning subgraph of Thagomizer on n vertices. The only missing edge is (u, v), which can be omitted in the presented broadcast scheme, so for this graph as well $b(K_{2,n-2}) = \lceil \frac{n}{2} \rceil$.

Consider the Ladder graph L_m on $n = 2m$ vertices with $3m-2$ edges. This graph is isomorphic to the Grid $G_{2\times m}$ presented in 2.4.9. As mentioned, the broadcast time of this graph is $b(G_{2\times m}) = 2 + m - 2 = m$, where $m = \frac{n}{2}$. This broadcast time can be achieved from the endpoint vertices of the ladder. This is another example of an infinite family of two-connected graphs for which the $\lceil \frac{n}{2} \rceil$ upper bound for broadcast time is tight.

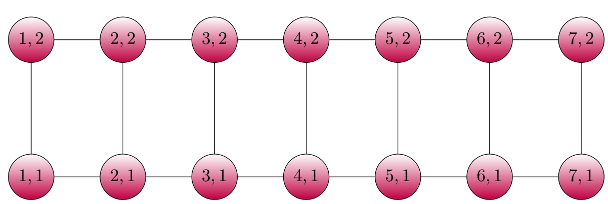


Figure 3.17: Ladder graph with $m = 7$

Chapter 4

Broadcasting in k-connected graphs

After seeing the upper bound for 2-connected graphs, we pose the following question.

If G is a k-connected graph ($k \geq 2$), is $b(G) \leq \lceil \frac{n}{k} \rceil$?

In the upcoming sections of this chapter, we discuss broadcasting in k-connected graphs and give an upper bound for broadcast time and examples of graphs that achieve it.

4.1 Complete bipartite graph $K_{k,n-k}$

The answer to the posed question in the preface of this chapter is negative. In this section, we present a counterexample of an infinite family of k-connected graphs for which $b(G) \nleq \lceil \frac{n}{k} \rceil$.

The complete bipartite graph $K_{k,n-k}$ ($k \leq n-k$) on n vertices is a graph with the partitions V_1 and V_2 such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, $|V_1| = k$, and $|V_2| = n-k$ and all possible edges of form (u,v) such that $u \in V_1$ and $v \in V_2$.

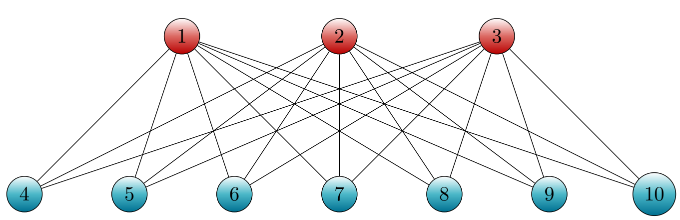


Figure 4.1: Complete bipartite graph $K_{3,7}$

We show that $K_{k,n-k}$ is k-connected. Consider three cases:

Case 1: $a, b \in V_1$

Since $n \geq 2k$, there are at least k vertices in V_2. There are k vertex disjoint paths of length 2 between a and b through k many vertices in V_2 $((a \sim v_i \sim b), \forall v_i \in V_2, 1 \leq i \leq k)$.

Case 2: $a \in V_1, b \in V_2$

The edge (a, b) exists in the graph, since it is complete bipartite. There are $k-1$ other vertex disjoint paths of length 3 between a and b through $k-1$ many vertices in V_1 and $k-1$ many vertices in V_2 $((a \sim v_i \sim u_i \sim b), \forall u_i \in V_1, v_i \in V_2, 1 \leq i \leq k, u_i \neq a, v_i \neq b)$.

Case 3: $a, b \in V_2$

There are k vertex disjoint paths of length 2 between a and b through k many vertices in V_1 $((a \sim u_i \sim b), \forall u_i \in V_1, 1 \leq i \leq k)$.

Equivalently, we can state that removal of k vertices of bipartition V_1 disconnects the graph but removal of any $k-1$ vertices does not. Thus $K_{k,n-k}$ is k-connected.

4.1.1 Broadcasting in $K_{k,n-k}$

Now let us discuss the broadcast time of the complete bipartite graph $K_{k,n-k}$.

Proposition 4.1.1. *For any complete bipartite graph $K_{k,n-k}$ such that $k \leq n-k$, $b(K_{k,n-k}) = \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.*

Proof. In GholamiNajarkola (2022) the author shows the following:

$$b(K_{k,n-k}) = \lceil \log k \rceil + 1 + \max\left\{ \left\lceil \frac{n-k-2^{\lceil \log k \rceil}}{k} \right\rceil, 0 \right\}$$

Which can be simplified to:

$$b(K_{k,n-k}) = \lceil \log k \rceil + \max\left\{ \left\lceil \frac{n-2^{\lceil \log k \rceil}}{k} \right\rceil, 1 \right\}$$

We show that the maximum function is not necessary. As long as $n - 2^{\lceil \log k \rceil} > 0$, then the ceiling of that fraction is at least 1. It is easy to see that $k \leq 2^{\lceil \log k \rceil} < 2k$ for all $k \in \mathbb{Z}^+$. We also noted that $k \leq n-k$, which implies $2k \leq n$, meaning that $2^{\lceil \log k \rceil} < 2k \leq n$. We get:

$$b(K_{k,n-k}) = \lceil \log k \rceil + \left\lceil \frac{n - 2^{\lceil \log k \rceil}}{k} \right\rceil$$

Since $k \leq 2^{\lceil \log k \rceil} < 2k$ for all $k \in \mathbb{Z}^+$

$$\lceil \log k \rceil + \left\lceil \frac{n-k}{k} \right\rceil \geq b(K_{k,n-k}) > \lceil \log k \rceil + \left\lceil \frac{n-2k}{k} \right\rceil$$
$$\lceil \log k \rceil + \left\lceil \frac{n}{k} \right\rceil - 1 \geq b(K_{k,n-k}) > \lceil \log k \rceil + \left\lceil \frac{n}{k} \right\rceil - 2$$
$$b(K_{k,n-k}) = \lceil \log k \rceil + \left\lceil \frac{n}{k} \right\rceil - 1$$

This means that broadcasting in $K_{k,n-k}$ takes place in $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$ time.

□

Next, we show, that this is, in fact, the upper bound for broadcasting in k-connected graphs.

4.2 An upper bound on broadcasting in k-connected graphs

In this section we prove the following theorem. The proof uses Kőnig's theorem on the equality of sizes of minimum vertex cover and maximum matching in bipartite graphs.

Theorem 4.2.1. *For any k-connected graph G, $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.*

Proof. Broadcasting in G takes place in three stages. By the end of *Stage 1*, at least k vertices are informed in G. During *Stage 2* at each time unit, k new vertices are being informed. *Stage 3* is the conclusive stage, where less than k vertices are left to be informed.

Let S be the set of currently informed vertices at some time t. Define U to be the set of uninformed vertices (i.e., $U = V - S$). Let the boundary of S, denoted by $\delta(S)$, be the set of uninformed vertices that are adjacent to at least one vertex of S i.e., $\delta(S) = \{v | v \notin S, uv \in E \text{ for some } u \in S\}$.

Claim 4.2.1. *Suppose $|S| \geq a$ and $|U| \geq a$ for some $a \leq k$. Then $|\delta(S)| \geq a$.*

Proof of Claim 1. This follows immediately from k-connectivity. Suppose that $|\delta(S)| \leq a - 1 \leq k - 1$. Then $G' = G - \delta(S)$ is connected by definition of k-connectivity. Let $u \in G' \setminus S$. Consider a shortest path P from u to S in G'. Then since P is a shortest path, the second to last vertex of P, say w, is an uninformed vertex adjacent to S, which is a contradiction. $\square$

Claim 4.2.2. *Let $G^* = (S, \delta(S))$ be the induced bipartite graph with bipartitions S and $\delta(S)$. Suppose $|S| = a$, and $|\delta(S)| = b$. Then G^* contains a matching of size at least $\min\{a, b, k\}$.*

Proof of Claim 2. We will show that $\tau(G^*) \geq \min\{a, b, k\}$, where τ is the size of the minimum vertex cover. Suppose that G^* contains a vertex cover X with $|X| < \min\{a, b, k\}$. Note that $G' = G - X$ is connected by definition, since $\min\{a, b, k\} \leq k$, thus $|X| < k$. Let $S' = S - X$ and $T' = \delta(S) - X$ and note that both are non-empty. Let $x \in T'$ and consider a shortest path P from x to S' in G'. Let $s \in S'$ and w be the last and second to last vertices of P, respectively. By the choice of P, $w \in \delta(S)$, and thus $w \in T'$. But now this implies that X does not cover the edge sw, a contradiction. Thus, $\tau \geq \min\{a, b, k\}$. Recall Kőnig's theorem: In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. By Kőnig's theorem $\tau \geq \min\{a, b, k\}$ implies that G^* contains a matching of size $\tau \geq \min\{a, b, k\}$. $\square$

Stage 1 of broadcasting takes place in such way. In the first time unit, $|S| = 1$ and in Claim 1, $a = 1$, thus $\delta(S) \geq 1$, so by Claim 2 there exists a matching M of size $\min\{a, b, k\} = a = 1$. Broadcasting happens through M. At the second time unit, $|S| = 2$, and in Claim 1 $a = 2$, thus $\delta(S) \geq 2$, so by Claim 2 there exists a matching of size $\min\{a, b, k\} = a = 2$.

Similarly, for each time unit $i \leq \lceil \log k \rceil$ there are $|S| = 2^i$ informed vertices and in Claim 1 $a = 2^i$, thus $\delta(S) \geq 2^i$, so by Claim 2 there exists a matching of size $\min\{a, b, k\} = 2^i$. This continues until $\min\{a, b, k\} \neq a$ anymore, which happens when $a > k$, meaning when $2^j \geq k + 1 \Rightarrow j \geq \lceil \log k \rceil + 1$. At time unit $\lceil \log k \rceil$, *Stage 1* is complete and $a = 2^{\lceil \log k \rceil} \geq k$ vertices are informed.

During *Stage 2* of broadcasting, $|S| = a \geq k$ and let us assume $|U| \geq k$, thus $b \geq k$ (by Claim 1) at some time t. Then at time $t+1$, $|S_{t+1}| \geq |S_t| + k$, since there exists a matching of size at least $\min\{a, b, k\} = k$ by Claim 2. Since k vertices have already been informed in *Stage 1*, there are at most $n - k$ vertices being informed in *Stage 2* and it takes at most $\lfloor \frac{n-k}{k} \rfloor$ time units.

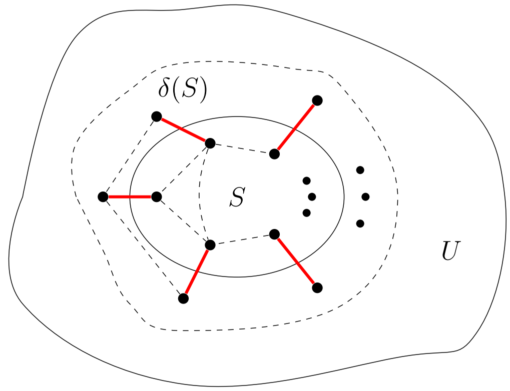


Figure 4.2: Stage 2: Matching of size k

Thus, we may assume that *Stage 3* begins i.e. $|U| < k$ and a matching of size $\geq k$ does not exist anymore. $|\delta(S)| = b < k$ and there are less than k uninformed vertices left. Since $a \geq k$ and

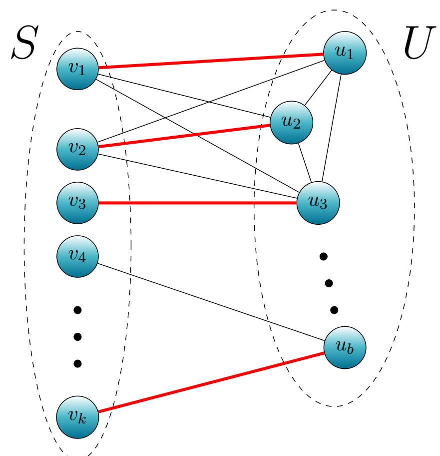


Figure 4.3: Stage 3: Matching of size b

$b < k$, by Claim 2, there exists a matching of size $\min\{a, b, k\} = b$. *Stage 3* will be complete in a single time unit - informing all vertices in $\delta(S)$ at once.

After completion of *Stage 3*, every vertex in G will be informed and the broadcast will terminate. Now we analyze the total broadcast time.

$k \mid n$: This means, that $n - k$ is also divisible by k, meaning, that all vertices are informed in *Stage 2* and *Stage 3* does not take place. Overall $b(G) \leq \lceil \log k \rceil + \lfloor \frac{n-k}{k} \rfloor = \lceil \log k \rceil + \frac{n}{k} - 1$.

$k \nmid n$: *Stage 3* takes place since there are $1 \leq b < k$ vertices informed during that. This means the total number of vertices informed during *Stage 2* are $n - k - b$. The overall broadcast time is at most $\lceil \log k \rceil + \lfloor \frac{n-k-b}{k} \rfloor + 1 = \lceil \log k \rceil + \lfloor \frac{n-b}{k} \rfloor - 1 + 1$ and since $1 \leq b < k$, then, $\lfloor \frac{n-b}{k} \rfloor \leq \lfloor \frac{n}{k} - \frac{1}{k} \rfloor \leq \lfloor \lceil \frac{n}{k} \rceil - \frac{1}{k} \rfloor = \lceil \frac{n}{k} \rceil + \lfloor -\frac{1}{k} \rfloor = \lceil \frac{n}{k} \rceil - \lceil \frac{1}{k} \rceil = \lceil \frac{n}{k} \rceil - 1$ and thus $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.

In general, for any k-connected graph G, $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. □

Corollary 4.2.1.1. *If a graph G is $\lceil \frac{n}{2} \rceil$-connected, then it is a broadcast graph.*

Proof. Proof is straightforward from the calculations of the theorem. For $\lceil \frac{n}{2} \rceil$-connected G,

$$b(G) \leq \left\lceil \log \left\lceil \frac{n}{2} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{2} \rceil} \right\rceil - 1 =$$

$$= \begin{cases} \lceil \log n \rceil - 1 + \left\lceil \frac{n}{\frac{n}{2}} \right\rceil - 1 = \lceil \log n \rceil & \text{if } n \text{ is even} \\ \lceil \log (n+1) \rceil - 1 + \left\lceil \frac{n}{\frac{n+1}{2}} \right\rceil - 1 = \lceil \log n \rceil & \text{if } n \text{ is odd} \end{cases}$$

Since $\lceil \log(n+1) \rceil = \lceil \log(n) \rceil + 1$ if and only if $n = 2^k$ for some $k \in \mathbb{Z}^+$ and thus n cannot be odd. □

Remark 1. *Note that $\lceil \frac{n}{2} \rceil$-connectivity cannot be lowered to $\frac{n}{2} - 1$. Consider the complete bipartite graph $K_{\frac{n}{2}-1, \frac{n}{2}+1}$ when $n = 2^k$ for some $k \in \mathbb{Z}^+$. Proposition 4.1.1 implies,*

$$b(K_{\frac{n}{2}-1, \frac{n}{2}+1}) = \left\lceil \log \left(\frac{n}{2} - 1 \right) \right\rceil + \left\lceil \frac{n}{\frac{n}{2} - 1} \right\rceil - 1 =$$
$$= \lceil \log (n-2) \rceil - 1 + \left\lceil \frac{2n}{n-2} \right\rceil - 1 = \log n + 1$$

Thus, the graph is not a broadcast graph.

Remark 2. *Note that $K_{\frac{n}{2}, \frac{n}{2}}$ for even n is $\lceil \frac{n}{2} \rceil$-connected broadcast graph.*

4.3 The bound is tight

In this section, we present k-connected graphs that achieve $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$ broadcast time.

4.3.1 General connected graphs

The general upper bound for all graphs in Equation 2 follows from Theorem 4.2.1 since the problem of broadcasting is defined for all connected graphs and when $k = 1$, by Theorem 4.2.1, $b(G) \le \lceil \log 1 \rceil + \lceil \frac{n}{1} \rceil - 1 = n - 1$.

As we have discussed in 2.4.2, the Path graph is one example where the $n - 1$ bound is tight. Another example of a connected graph that meets the bound is the Star graph discussed in 2.4.3. The same bound is tight for the Fork graph $F_{n,k}$ discussed in 2.4.6 and graphs listed in Grigoryan (2013).

4.3.2 2-connected graphs

The graphs presented in Section 3.2 are relevant here as well since the bound $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$ becomes $\lceil \log 2 \rceil + \lceil \frac{n}{2} \rceil - 1 = 1 + \lceil \frac{n}{2} \rceil - 1 = \lceil \frac{n}{2} \rceil$. For the biconnected graphs C_n, Unichordal, Thagomizer, and Ladder, the bound in Theorem 4.2.1 is tight.

4.3.3 k-connected graphs

As shown in Section 4.1.1, the complete bipartite graph $K_{k,n-k}$, which is a k-connected graph, has broadcast time $b(K_{k,n-k}) = \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$.

Consider the following construction for a k-connected graph that meets the bound. Given an integer k, take d copies of complete graph on k vertices $K_k^1, K_k^2, ..., K_k^d$, where d is an integer. $n = kd$. In each of the consecutive copies, connect the copies of corresponding vertices by an edge i.e. add the edges of form (u_i, v_i), s.t. $u_i \in K_k^j, v_i \in K_k^{j+1} \forall i \in \{1, ..., k\}, \forall j \in \{1, ..., d-1\}$.

The graph will be k-connected since removal of all the vertices in any K_k^j will disconnect the graph, and there is no set of $k - 1$ vertices that will do that. The diameter of this graph is d since from any vertex u_i in K_k^1 to all vertices in K_k^d except its copy v_i there is no shorter path. Note, that the construction yields the Ladder graph for $k = 2$.

We present a lower bound for broadcasting in this graph. Let the originator be a vertex u_1 in K_k^1. Any broadcast scheme will eventually do two things: inform all vertices on a diametral path from u_1 to $v_1 \in K_k^d$, and spread information inside K_k^d. These processes can take place in different order, but in any scheme, the number of time units used must be at least $\lceil \log k \rceil$ for informing a complete graph K_k^j, where $j \in \{1, ..., d\}$ or distributed within calls in different copies. Additionally, informing on the diametral path will take at least $d - 1$ time. Note, that here $d = \frac{n}{k}$ which results in $b(G) \geq \lceil \log k \rceil + \frac{n}{k} - 1$, which meets the bound in Theorem 4.2.1 tightly.

4.3.4 $\frac{n}{2}$-connected graphs

A specific example in the family of complete bipartite graphs is $K_{\frac{n}{2},\frac{n}{2}}$. This is a $\frac{n}{2}$-regular and maximally connected graph. Due to its bipartite definition, it cannot contain K_3 as a subgraph i.e. it is triangle-free.

Claim 4.3.1. *The triangle-free $\frac{n}{2}$-regular graph G on even n vertices is a broadcast graph.*

Proof. It can be easily verified, that for each even n, the triangle-free $\frac{n}{2}$-regular graph is unique. In fact this graph is $K_{\frac{n}{2},\frac{n}{2}}$. Being a $\frac{n}{2}$-connected graph, the broadcast time of this graph is upper bounded by $\lceil \log(\frac{n}{2}) \rceil + \lceil \frac{n}{n/2} \rceil - 1 = \lceil \log n \rceil - 1 + 2 - 1 = \lceil \log n \rceil$ (Theorem 4.2.1). And by the bounds in Equation 2, the lower bound meets the upper bound, thus $b(K_{\frac{n}{2},\frac{n}{2}}) = \lceil \log n \rceil$ $\square$

Similar to the previous example, consider the following graph.

Let n be a power of 2. Take 4 copies of cycles on $p = \frac{n}{4}$ vertices C^1 with vertices $\{a_1, a_2, ..., a_p\}$, C^2 with vertices $\{b_1, b_2, ..., b_p\}$, C^3 with vertices $\{c_1, ..., c_p\}$ and C^4 with vertices $\{d_1, ..., d_p\}$. The edge set of the graph is constructed in the following way. The edges in all the cycles remain as they are $\{(a_1, a_2), (a_2, a_3)...(a_p, a_1)\}$. For each a_i in C^1 we connect it to all b_j's from C^2 such that $i \neq j$. In simpler words, each vertex is connected to all vertices in the next cycle except its copy. Similarly, b_i's are connected to c_j's, c_i's to d_j's and finally, d_i's are connected to a_j's. Denote the graph by CC for circular cycles.

Each vertex in CC has 2 neighbors in its cycle, $p - 1$ neighbors from the next cycle and $p - 1$ neighbors from the previous cycle, resulting in the graph being $2 + p - 1 + p - 1 = 2p = \frac{n}{2}$-regular. CC is also $\frac{n}{2}$-connected since the removal of any fewer vertices will not result in a disconnected graph, but the removal of all neighbors of a vertex or removal of two non-consecutive cycles will.

Broadcasting in the CC graph takes place in the following way. Since the graph is very symmetric, we will discuss one originator. All other originators can follow this broadcast scheme. Let

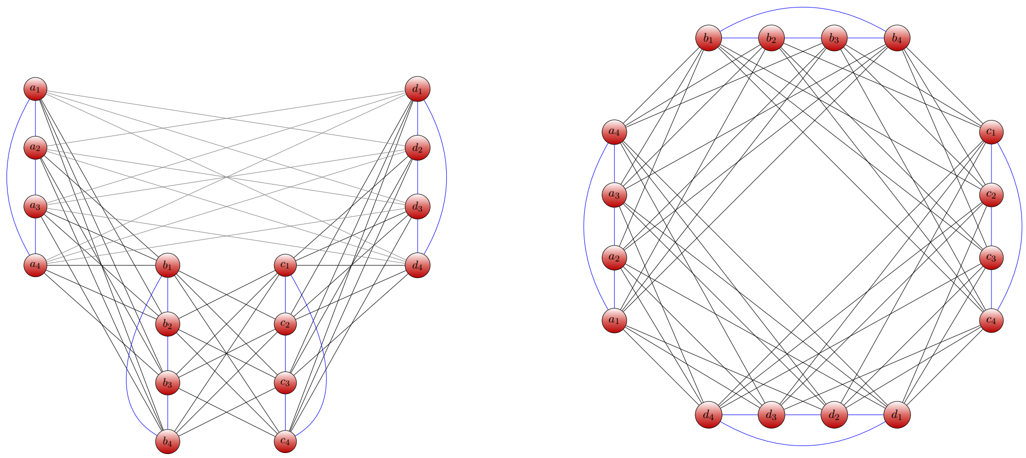


Figure 4.4: Different drawings of the CC graph on 16 vertices

the originator be b_2. In the first time unit, it informs c_1. During the second time unit b_2 informs a_1, and c_1 informs d_2. In the third time unit, only "local" calls take place. $a_1 \to a_2$, $b_2 \to b_1$, $c_1 \to c_2$, and $d_2 \to d_1$. After this time unit, vertices in C^1 inform C^2 and vice versa and vertices in C^3 inform C^4 and vice versa. Since almost all edges exist between these cycles, then we can always find a matching that duplicates the number of informed vertices. This means, that broadcasting takes place in $\log n$ time. Since CC is $\frac{n}{2}$-connected, then the upper bound on broadcasting by Theorem 4.2.1 will be $b(CC) \leq \lceil \log(\frac{n}{2}) \rceil + \lceil \frac{n}{n/2} \rceil - 1 = \lceil \log n \rceil - 1 + 2 - 1 = \lceil \log n \rceil$. Since this is the general lower bound for all graphs, then, CC is a broadcast graph and a graph for which the bound is tight.

4.3.5 $(n-1)$-connected graphs

The Complete graphs K_n are the only graphs which are $(n-1)$-connected. As mentioned in Section 2.4.4, $b(K_n) = \lceil \log n \rceil$ and the bound is tight since $(n-1)$-connectivity implies $\lceil \frac{n}{2} \rceil$-connectivity which by Corollary 4.2.1.1 ensures $\lceil \log n \rceil$ broadcast time.

4.4 Large minimum degree implies connectivity

Now we use Theorem 4.2.1 to obtain upper bounds on broadcast time of graphs with large minimum degree. We will use the following result.

Theorem 4.4.1. *(West, 2000) Let $1 \leq k \leq n-1$ be an integer. Then every graph G on n vertices with $\delta \geq \left\lceil \frac{n+k-2}{2} \right\rceil$, is k-connected.*

Proof. We prove this by contradiction. Assume G is a simple graph on n vertices with $\delta \geq \left\lceil \frac{n+k-2}{2} \right\rceil$ and is *not* k-connected ($1 \leq k \leq n-1$). This means, that the deletion of $k-1$ vertices (denoted by S) leaves a disconnected graph H. Take an arbitrary vertex $v \in V(H)$. v can have at most $k-1$ neighbors in S. This means, that $deg_H(v) \geq \delta(G) - k + 1 \geq \left\lceil \frac{n+k-2}{2} \right\rceil - \frac{2k-2}{2} = \left\lceil \frac{n-k}{2} \right\rceil$. From this fact, we can state, that each component of H has at least $1 + \left\lceil \frac{n-k}{2} \right\rceil$ vertices and due to being disconnected, there are at least two such components, meaning H has at least $n-k+2$ vertices. This, however, leads to a contradiction since $|V(H)| + |S| \geq n-k+2+k-1 = n+1 > n$. This means that G has to be k-connected. □

With this result, we propose the following theorem.

Theorem 4.4.2. *Let $1 \leq k \leq n-1$ be an integer. For any graph G on n vertices with $\delta \geq \left\lceil \frac{n+k-2}{2} \right\rceil$, broadcast time $b(G) \leq \lceil \log k \rceil + \left\lceil \frac{n}{k} \right\rceil - 1$.*

Proof. By Theorem 4.4.1, the graph G is guaranteed to be k-connected and by Theorem 4.2.1, its broadcast time is upper bounded by $\lceil \log k \rceil + \left\lceil \frac{n}{k} \right\rceil - 1$. □

This theorem covers a large class of graphs. Note, that the condition of the theorem requires the graph to have a high minimum degree, but that is not a necessary condition. The lower bound on the minimum degree of each vertex creates a dense graph and ensures connectivity and bounded broadcast time, so the natural question of whether dense and therefore highly connected graphs can have logarithmic broadcast time arises. A straightforward series of corollaries follow.

Corollary 4.4.2.1. *Any graph G on n vertices with $\delta \geq \left\lceil \frac{3n}{4} \right\rceil - 1$ is a broadcast graph.*

Proof. We use the fact that $\left\lceil \frac{n}{4} \right\rceil = \left\lceil \frac{\left\lceil \frac{n}{2} \right\rceil}{2} \right\rceil \; \forall n \in \mathbb{Z}^+$.

Let G be a graph G on n vertices with $\delta \geq \left\lceil \frac{3n}{4} \right\rceil - 1 = \left\lceil \frac{n}{2} + \frac{n}{4} \right\rceil - 1 = \left\lceil \frac{n + \left\lceil \frac{n}{2} \right\rceil - 2}{2} \right\rceil$. By Theorem 4.4.1, the graph is $\left\lceil \frac{n}{2} \right\rceil$-connected. By Corollary 4.2.1.1, the broadcast time is upper bounded by $\lceil \log n \rceil$ and the graph is a broadcast graph. □

This means, that if every vertex in the graph is connected to at least three-quarters of the vertices, then the broadcast time of the graph will be upper bounded by $\lceil \log n \rceil$, resulting in it being a broadcast graph. This result will be improved in Chapter 5.

Corollary 4.4.2.2. *Any graph G on n vertices with $\delta \geq \lceil \frac{5n}{8} \rceil - 1$ has broadcast time at most $\lceil \log n \rceil + 1$.*

Proof. Let G be a graph G on n vertices with $\delta \geq \lceil \frac{5n}{8} \rceil - 1 = \lceil \frac{n}{2} + \frac{n}{8} \rceil - 1 = \left\lceil \frac{n + \lceil \frac{n}{4} \rceil - 2}{2} \right\rceil$. By Theorem 4.4.1, the graph is $\lceil \frac{n}{4} \rceil$-connected.

$$b(G) \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{4} \rceil} \right\rceil - 1 \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + \left\lceil \frac{n}{\frac{n}{4}} \right\rceil - 1 \leq$$
$$\leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + 4 - 1 = \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + 3$$

Consider two cases for some $k \in \mathbb{Z}^+$ and $1 \leq x \in \mathbb{Z}^+$.

If $n = 2^k$, then $\left\lceil \log \left\lceil \frac{2^k}{4} \right\rceil \right\rceil + 3 = \lceil \log 2^{k-2} \rceil + 3 = (k-2) + 3 = k + 1 = \lceil \log 2^k \rceil + 1 = \lceil \log n \rceil + 1$.

Otherwise if $n = 2^k - x$, then $\left\lceil \log \left\lceil \frac{2^k - x}{4} \right\rceil \right\rceil + 3 = \lceil \log \lceil 2^{k-2} - \frac{x}{4} \rceil \rceil + 3 \leq \lceil \log \lceil 2^{k-2} - \frac{1}{4} \rceil \rceil + 3 = \lceil \log \lceil 2^{k-2} \rceil \rceil + 3 = k + 1 = \lceil \log(2^k - x) \rceil + 1 = \lceil \log n \rceil + 1$.

Thus, $b(G) \leq \lceil \log n \rceil + 1$. □

Corollary 4.4.2.3. *Any graph G on n vertices with $\delta \geq \lceil \frac{3n}{5} \rceil - 1$ has broadcast time at most $\lceil \log n \rceil + 2$.*

Proof. Let G be a graph G on n vertices with $\delta \geq \lceil \frac{3n}{5} \rceil - 1 = \lceil \frac{n}{2} + \frac{n}{10} \rceil - 1 = \left\lceil \frac{n + \lceil \frac{n}{5} \rceil - 2}{2} \right\rceil$. By Theorem 4.4.1, the graph is $\lceil \frac{n}{5} \rceil$-connected. We use the facts that $\lceil \frac{n}{5} \rceil \leq \lceil \frac{n}{4} \rceil, \forall n \in \mathbb{Z}^+$ and $\lceil \log \lceil \frac{n}{4} \rceil \rceil \leq \lceil \log n \rceil - 2$ as seen in Corollary 4.4.2.2.

$$b(G) \leq \left\lceil \log \left\lceil \frac{n}{5} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{5} \rceil} \right\rceil - 1 \leq \left\lceil \log \left\lceil \frac{n}{4} \right\rceil \right\rceil + \left\lceil \frac{n}{\lceil \frac{n}{5} \rceil} \right\rceil - 1 \leq$$
$$\leq \lceil \log n \rceil - 2 + 5 - 1 = \lceil \log n \rceil + 2$$

□

Chapter 5

Broadcast time and minimum degree

As we have seen in the previous chapter, broadcast time for k-connected graphs is upper bounded by $\lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. We also saw, that since a high minimum degree enforces some connectivity on the graph, there is a threshold number for a degree for each vertex in the graph to obtain a broadcast graph. In this chapter, we further explore the connection between the minimum degree and broadcast time of a graph.

5.1 Broadcasting in dense graphs

Consider the following conjecture:

Conjecture 5.1.1. *Let $\epsilon > 0$ be a fixed constant and let G be a connected graph on n vertices with minimum degree $\delta(G) \geq \epsilon n$ where n is sufficiently large. Then $b(G) \leq \lceil \log n \rceil + c_\epsilon$, where c_ϵ is some constant that only depends on ϵ.*

Remark 3. *Conjecture 5.1.1 is true for $\epsilon > \frac{1}{2}$ by Theorem 4.4.2.*

In fact, we believe that when $\epsilon = \frac{1}{2}$, the following is true.

Conjecture 5.1.2. *Let G be a graph on n vertices with minimum degree $\delta(G) \geq \frac{n}{2}$, where n is sufficiently large. Then G is a broadcast graph.*

Remark 4. *We note that $\delta \geq \frac{n}{2}$ cannot be relaxed, even if G is connected. Indeed, consider two copies of $K_{\frac{n}{2}}$'s where n is even and add a single edge between vertices u and v in each copy. The* *resulting graph (Figure 5.1) has minimum degree of $\frac{n}{2}-1$. Any originator $o \in V \setminus \{u,v\}$ will take at least 2 time units to inform a vertex in the other complete graph. After that broadcasting can be complete in $\lceil \log \lceil \frac{n}{2} \rceil \rceil = \lceil \log n \rceil - 1$ time units, resulting in overall $\lceil \log n \rceil + 1$ time.*

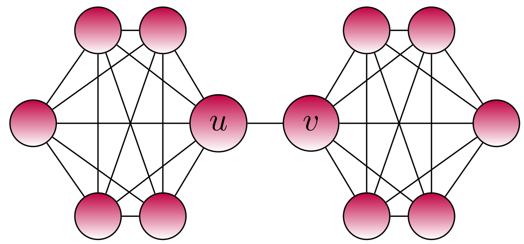


Figure 5.1: Two copies of K_6 connected by an edge

We prove a weaker version of Conjecture 5.1.1 where $\epsilon = \frac{1}{2}$ and $c_\epsilon = 3$.

Theorem 5.1.1. *Let G be a graph on n vertices with $\delta(G) \geq \frac{n}{2}$. Then $b(G) \leq \lceil \log n \rceil + 3$.*

Proof. We consider graphs on $n \geq 8$ vertices since otherwise $\lceil \log n \rceil + 3 \geq n-1$ (i.e., the upper bound in (2)). G is a simple connected graph and for every pair of non adjacent vertices u and v the intersection of their neighborhoods (each having cardinality of at least $\frac{n}{2}$) cannot be empty i.e. $u, v \in V(G), N(u) \cap N(v) \neq \emptyset$. This means, that the diameter of G is at most 2. If G is $\lceil \frac{n}{8} \rceil$-connected, then similar to Stage 1 of Theorem 4.2.1, broadcasting in arbitrary greedy manner will double the number of informed vertices at each time unit until time unit $\lceil \log \lceil \frac{n}{2} \rceil \rceil = \lceil \log n \rceil - 1$. There will always be a vertex to inform since $\delta \geq \frac{n}{2}$. After this, the minimum degree does not guarantee doubling of informed vertices, but since the graph is $\lceil \frac{n}{8} \rceil$-connected, there is a matching of size at least $\lceil \frac{n}{8} \rceil$. Informing the remaining at most $n - \lceil \frac{n}{2} \rceil$ vertices by matchings of size $\lceil \frac{n}{8} \rceil$ will take at most 4 time units resulting in $b(G) \leq \lceil \log n \rceil + 3$.

Now assume that G is not $\lceil \frac{n}{8} \rceil$-connected meaning, that there is a set X with $|X| \leq \lceil \frac{n}{8} \rceil$ such that $G \setminus X$ is disconnected.

Note that $G \setminus X$ must have exactly two components. Assume it has three or more components. Then, one of the components, say C_3, would have size at most $\frac{n}{3}$, and thus, any vertex in C_3 would have degree in G at most $\frac{n}{3} + \lceil \frac{n}{8} \rceil < \frac{n}{2}$, a contradiction.

Let C_1 and C_2 be the two components of $G \setminus X$. Note that by an identical argument as above, each C_i must contain at least $\frac{n}{2} - \lceil \frac{n}{8} \rceil$ vertices. This implies that in fact for each i, $\frac{n}{2} - \lceil \frac{n}{8} \rceil \leq$

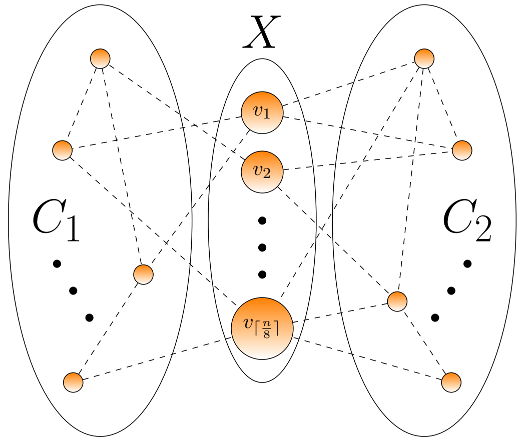


Figure 5.2: Graph G with $\delta(G) \geq \frac{n}{2}$ with 2 components

$|C_i| \leq \frac{n}{2}$. Thus, we may assume each component has size at most $\frac{n}{2}$.

First, note that independent of the location of the originator, since the diameter is at most 2, at time 2 at least one vertex of C_1 and at least one vertex of C_2 is informed. We now show that we need at most additional $\lceil \log n \rceil$ time such that all vertices of C_1 and C_2 are informed. Note that for each $v \in V(C_1)$, $\deg_{C_1}(v) \geq \frac{n}{2} - \lceil \frac{n}{8} \rceil = \lfloor \frac{3n}{8} \rfloor$. C_1 is a graph on at most $\frac{n}{2}$ vertices with $\delta \geq \lfloor \frac{3n}{8} \rfloor \leq \frac{\frac{n}{2} + \lfloor \frac{n+8}{4} \rfloor - 2}{2}$. By Theorem 4.4.1, C_1 is $\lfloor \frac{n+8}{4} \rfloor$-connected. Since $\frac{n}{4} \leq \lfloor \frac{n+8}{4} \rfloor \leq \frac{n}{2}$ is true $\forall n \in \mathbb{N}, n \geq 6$, the broadcast time of C_1 is upper bounded by:

$$\begin{aligned} b(C_1) &\leq \left\lceil \log \left\lfloor \frac{n+8}{4} \right\rfloor \right\rceil + \left\lceil \frac{n/2}{\lfloor (n+8)/4 \rfloor} \right\rceil - 1 \leq \\ &\leq \left\lceil \log \left(\frac{n}{2} \right) \right\rceil + \left\lceil \frac{n/2}{n/4} \right\rceil - 1 = \\ &= \lceil \log n \rceil - 1 + \left\lceil \frac{2n}{n} \right\rceil - 1 = \lceil \log n \rceil \end{aligned}$$

This shows that C_1 can be informed in $\lceil \log n \rceil$ time units. An identical argument applies for C_2. Thus, in time $\lceil \log n \rceil + 2$ all vertices of C_1 and C_2 are informed.

Now note that for every $v \in X$, there is a set N_v of neighbors of v such that $|N_v| \geq \frac{n}{8}$ and either $N_v \subset C_1$ or $N_v \subset C_2$ (in other words, at least half of the neighbors of v outside X are either in C_1 or C_2). Each such v can choose a distinct informer vertex from its N_v since $|X| \leq \frac{n}{8}$. Thus, it takes one more time to inform X, giving a total of $\lceil \log n \rceil + 3$ time.

Along with $\lceil \frac{n}{8} \rceil$-connected graphs, for all graphs G with $\delta(G) \geq \frac{n}{2}$, $b(G) \leq \lceil \log n \rceil + 3$.

□

5.2 Graphs of small minimum degree with large broadcast time

We want to examine the connection between minimum degree and broadcast time of a graph. Consider the following graph G. Take a Path graph presented in Section 2.4.2. Connect the endpoint vertices to the other neighbor of their neighbors. In other words, we get a path with two pendant K_3's. The minimum degree is 2 since every vertex has a degree of 2. The broadcast time of this graph will not be much different from the path graph. If the originator is one of the four endpoints, in the first time unit it informs the vertex on the path, then its other neighbor. The Path in the middle takes $n-5$ time to inform its endpoint. Then 2 time units are needed to inform the other pendant K_3. $b(G) = 1 + n - 5 + 2 = n - 2$.

This means, that minimum degree alone cannot be a sufficient condition for determining the broadcast time. The broadcast time, is lower bounded by the diameter of a graph. The diameter of the graph presented in before is $n-2$, so besides looking at minimum degree, we should also consider the diameter. However, these two concepts seem to be very related. A classical result of Erdős et al. gives an upper bound on diameter of graphs of minimum degree δ.

Theorem 5.2.1 (Erdős, Pach, Pollack, and Tuza (1989))**.** *Let G be a connected graph with n vertices and with minimum degree $\delta > 2$. Then diameter $D(G) \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor - 1$.*

As an example for the tightness of this bound the authors present the following graph. Let $k > 1$, $\delta > 5$, and $V(G) = V_0 \cup V_1 \cup \cdots \cup V_{3k-1}$, where:

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 2 (\text{mod} 3) \\ \delta & \text{if } i = 1 \text{ or } 3k-2 \\ \delta - 1 & \text{otherwise} \end{cases}$$

Let two distinct vertices $v \in V_i, u \in V_j$ be joined by an edge of G if and only if $|i-j| \leq 1$. An instance of such graph is presented in Figure 5.3. The diameter of this graph is $11 \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor - 1 = \left\lfloor \frac{3\times 30}{6+1} \right\rfloor - 1 = \lfloor \frac{90}{7} \rfloor - 1 = 12 - 1 = 11$. One diametral path is highlighted in blue.

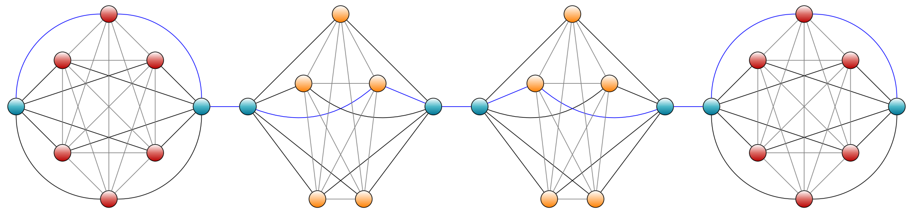


Figure 5.3: Maximal diameter graph with $n = 30$ and $\delta = 6$

Now consider broadcast time of this graph. Evidently, the diametral vertices will be the worst originators. Consider the following broadcast scheme from the leftmost vertex u of the graph. Let the vertex at $i = 3k - 3$ be called w. Since we want to minimize broadcast time, all calls will take place on the diametral path. Clearly by time unit $D(G) - 2$, w is informed. In time unit $D(G) - 1$ w informs any vertex in V_{3k-2}. In time unit $D(G)$ the rightmost vertex (v) and another vertex in V_{3k-2} are informed. Since V_{3k-2} is one of the largest sets and was informed the last, then once broadcasting is complete there, it will be complete everywhere else in the graph. It is easier to imagine $V_{3k-3} \cup V_{3k-2} \cup V_{3k-1}$ as a complete graph on $\delta + 2$ vertices with a single edge (w, v) missing. So after time unit $D(G)$, w, v and two other vertices are informed in this subgraph and this is equivalent to broadcasting in a complete graph after time unit 2. So overall broadcasting will take place in $D(G) + \lceil \log(\delta + 2) \rceil - 2$ time units. Since the diameter of this graph is maximal, then we get $b(G) \geq \lfloor \frac{3n}{\delta+1} \rfloor + \lceil \log(\delta + 2) \rceil - 3$.

We believe that the graph with maximal diameter is also the graph with the largest broadcast time. We conjecture the following.

Conjecture 5.2.1. *Let G be a graph on n vertices with minimum degree δ. Then broadcast time $b(G) \leq \left\lfloor \frac{3n}{\delta+1} \right\rfloor + \lceil \log \delta \rceil$.*

Chapter 6

Conclusion and Future work

6.1 Conclusion

This thesis is about broadcasting - the dissemination of information in networks using certain limitations. The problem of broadcast time has been of research interest for more than 4 decades and there are numerous results, but many open problems and questions still remain. To the best of our knowledge, our approach using k-connectivity and the minimum degree of graphs for giving upper bounds on general graphs presents novelty in the field.

In Chapter 3, we gave an upper bound for broadcasting in two-connected graphs. Using induction on the number of ears from Whitney's open ear decomposition, Theorem 3.1.3 claimed four important statements, which yielded the bound $b(G) \leq \lceil \frac{n}{2} \rceil$ for all two-connected graphs. These four statements, also give us bounds for broadcasting from multiple originators. Some of these statements also provide insights on the last time unit broadcasting in these graphs. Having one uninformed vertex remaining in the last time unit can have useful applications in some real-life examples. In this chapter, we also presented certain families of graphs (cycles, unichordal, thagomizer, and ladder graphs), for which the broadcast time is equal to the upper bound. It is very easy to see, that for some of these graphs the bound remains tight for multiple originator cases.

Chapter 4, generalizes the result in the previous chapter. Theorem 4.2.1 shows the upper bound for broadcasting in k-connected graphs by dividing the process into three stages and finding matching between informed vertices and their uninformed neighbors in every stage using Kőnig's theorem. From here we get, that for any k-connected graph G, $b(G) \leq \lceil \log k \rceil + \lceil \frac{n}{k} \rceil - 1$. Plugging different values of k, we see, that the bound is indeed tight for many infinite families of graphs (Path, Star, and Fork graphs, all examples seen in the previous chapter, complete bipartite graph $K_{k,n-k}$ with its extreme case of $K_{\frac{n}{2},\frac{n}{2}}$, our construction with copies of complete graphs and some complete graphs).

We note, that both proofs for Theorems 3.1.3 and 4.2.1 can be considered constructive and imply certain broadcast schemes. While in general these schemes can have similarities, they are very unlikely to yield the same broadcast trees. We also note, that the Whitney decomposition and the proofs mentioned above can be implemented as algorithms with greedy techniques.

It is important to mention here, that k-connected graphs are a very large class of graphs and within this class, different graphs and graph families have different broadcast times. A general lower bound of $\max\{\lceil \log n \rceil, D(G)\}$ presented in equation 2 should be mentioned here. As an example we can consider the d-dimensional hypercube. It is d-connected and has broadcast time d. This means, that graphs with low connectivity can have low broadcast time. This creates obstacles for discussing lower bounds in all k-connected graphs since there are graphs which achieve logarithmic broadcast time and graphs which meet the upper bound.

Toward the end of Chapter 4, we discuss a bound on the minimum degree of the graph that ensures k-connectivity. Corollary 4.4.2.1 shows that all graphs with minimum degree $\delta \geq \lceil \frac{3n}{4} \rceil - 1$ are broadcast graphs. We also show in Corollaries 4.4.2.2 and 4.4.2.3 that minimum degree can ensure relaxed broadcast graphs as well. Of course further we see that these bounds can be tightened.

Chapter 5 extends the idea of a minimum degree in a graph implying broadcast time introduced in the previous chapter. We propose Conjecture 5.1.1 and a specific case of it in Conjecture 5.1.2. We then prove a weaker version of Conjecture 5.1.1 where $\epsilon = \frac{1}{2}$ and $c_\epsilon = 3$. In the proof of Theorem 5.1.1, we decompose the graph into three units, two of which are internally well-connected and can finish broadcasting in $\lceil \log n \rceil$ time. Here, we used the argument for diameter to show that for any graph with $\delta \geq \frac{n}{2}$ broadcasting is complete in at most $\lceil \log n \rceil + 3$ time units. Towards the end of the chapter we present graphs with small minimum degree but high diameter to demonstrate the importance of diameter as a lower bound. Finally, we discuss a graph construction presented by Erdős et al. and show a lower bound on the broadcast time of that graph, which brings us to Conjecture 5.2.1.

6.2 Future Work

The results presented in this thesis open many doors for further consideration and research.

Theorem 3.1.3 presents an upper bound for two-connected graphs which is tight for some families of graphs seen in Section 3.2. A future direction of work will be to understand what characteristics these graphs have in common and maybe define a wider subclass of two-connected graphs for which the bound will be tight. Once this is done, a lower bound or a tighter upper bound can be deduced for the remaining graphs.

Similar steps can be applied toward the k-connected bound as well. We believe, that the list of examples of infinite families of k-connected graphs that meet the upper bound can be extended. Other factors such as the diameter, minimum degree, and other characteristics of the graph also have an effect on broadcast time.

Conjectures 5.1.1, 5.1.2 and 5.2.1 still remain open for further research. Results from Erdős et al. on diameter mentioned before can be very useful in the investigation of this. We believe that certain constraints/restrictions on diameter of a graph can open new pathways for research.

An interesting direction for research in this area is other models of broadcasting. Fault-tolerant broadcasting in particular is very interesting to us since the failure of any $k-1$ nodes in a k connected graph does not destroy connectivity. We firmly believe that this direction can be very promising when aggregated with the introduced bounds.

References

[1] Adibi, A. (2021). *Broadcasting in hyper-cylinder graphs* (Unpublished master's thesis). Concordia University. (Unpublished)

[2] Ahlswede, R., Gargano, L., Haroutunian, H., & Khachatrian, L. H. (1996). Fault-tolerant minimum broadcast networks. *Networks*, *27*(4), 293–307.

[3] Ahlswede, R., Haroutunian, H., & Khachatrian, L. H. (1994). Messy broadcasting in networks. In *Communications and Cryptography* (pp. 13–24). Springer.

[4] Alon, N., Bar-Noy, A., Linial, N., & Peleg, D. (1991). A lower bound for radio broadcast. *Journal of Computer and System Sciences*, *43*(2), 290–298.

[5] Bar-Noy, A., Guha, S., Naor, J., & Schieber, B. (1998). Multicasting in heterogeneous networks. In *Proceedings of the thirtieth annual ACM Symposium on Theory of computing* (pp. 448–453).

[6] Bar-Noy, A., & Ho, C.-T. (1999). Broadcasting multiple messages in the multiport model. *IEEE Transactions on Parallel and Distributed Systems*, *10*(5), 500–508.

[7] Bar-Noy, A., & Kipnis, S. (1993). Multiple message broadcasting in the postal model. In *[1993] proceedings seventh international parallel processing symposium* (pp. 463–470).

[8] Bar-Noy, A., & Kipnis, S. (1994). Broadcasting multiple messages in simultaneous send/receive systems. *Discrete Applied Mathematics*, *55*(2), 95–105.

[9] Bar-Noy, A., Kipnis, S., & Schieber, B. (2000). Optimal multiple message broadcasting in telephone-like communication systems. *Discrete Applied Mathematics*, *100*(1-2), 1–15.

[10] Beier, R., & Sibeyn, J. F. (2000). A powerful heuristic for telephone gossiping. In *Proceedings of the 7th International Colloquium on Structural Information and Communication*

Complexity (SIROCCO) (pp. 17–35).

[11] Bermond, J.-C., Harutyunyan, H. A., Liestman, A. L., & Pérennes, S. (1997). A note on the dimensionality of modified knö;del graphs. *Int. J. Found. Comput. Sci.*, *8*, 109-116.

[12] Bermond, J.-C., Hell, P., Liestman, A. L., & Peters, J. G. (1992a, 02). Broadcasting in bounded degree graphs. *SIAM J. Discrete Math.*, *5*, 10-24.

[13] Bermond, J.-C., Hell, P., Liestman, A. L., & Peters, J. G. (1992b). Sparse broadcast graphs. *Discrete Applied Mathematics*, *36*(2), 97–130.

[14] Bermond, J.-C., & Peyrat, C. (1988). Broascasting in de Bruijn networks. *Congressus Numerantium*, *66*, 283–292.

[15] Bhabak, P., Harutyunyan, H., & Kropf, P. (2017). Efficient Broadcasting Algorithm in Harary-like Networks. In *2017 46th International Conference on Parallel Processing Workshops (ICPPW)* (p. 162-170).

[16] Bhabak, P., & Harutyunyan, H. A. (2015). Constant Approximation for Broadcasting in k-cycle Graph. In *Conference on Algorithms and Discrete Applied Mathematics (CALDAM)* (pp. 21–32).

[17] Bhabak, P., & Harutyunyan, H. A. (2019). Approximation Algorithm for the Broadcast Time in k-Path Graph. *Journal of Interconnection Networks*, *19*(04), 1950006.

[18] Bhabak, P., & Harutyunyan, H. A. (2022). Approximation Algorithms in Graphs with Known Broadcast Time of the Base Graph. In *Conference on Algorithms and Discrete Applied Mathematics* (pp. 305–316).

[19] Bhabak, P., Harutyunyan, H. A., & Tanna, S. (2014). Broadcasting in Harary-Like Graphs. In *2014 IEEE 17th International Conference on Computational Science and Engineering* (p. 1269-1276).

[20] Bruck, J., Cypher, R., & Ho, C.-t. (1992, 12). Multiple message broadcasting with generalized fibonacci trees. *Proceedings of the Fourth IEEE Symposium on Parallel and Distributed Processing*.

[21] Chau, S.-C., & Liestman, A. L. (1985). Constructing minimal broadcast networks. *J. Combin. Inform. Syst. Sci*, *10*, 110–122.

[22] Chen, M.-S., Shin, K. G., & Kandlur, D. D. (1990). Addressing, routing, and broadcasting

in hexagonal mesh multiprocessors. *IEEE Transactions on Computers*, *39*(1), 10–18.

[23] Chia, M.-L., Kuo, D., & Tung, M.-F. (2007). The multiple originator broadcasting problem in graphs. *Discrete applied mathematics*, *155*(10), 1188–1199.

[24] Chinn, P., Hedetniemi, S., & Mitchell, S. (1979). Multiple-message broadcasting in complete graphs. In *Proceedings of the 10th SE conference on combinatorics, graph theory and computing. utilitas math. winnipeg* (pp. 251–260).

[25] Chlebus, B., Gasieniec, L., Gibbons, A., Pelc, A., & Rytter, W. (2002, 01). Deterministic broadcasting in ad hoc radio networks. *Distributed Computing*, *15*, 27-38.

[26] Cockayne, E. J. (1979). *Optimal multi-message broadcasting in complete graphs*. University of Victoria, Department of Mathematics.

[27] Comellas, F., Harutyunyan, H. A., & Liestman, A. L. (2003). Messy broadcasting in multidimensional directed tori. *Journal of Interconnection Networks*, *4*(01), 37–51.

[28] De Sousa, A., Gallo, G., Gutierrez, S., Robledo, F., Rodríguez-Bocca, P., & Romero, P. (2018). Heuristics for the minimum broadcast time. *Electronic Notes in Discrete Mathematics*, *69*, 165–172.

[29] Dessmark, A., & Pelc, A. (2002). Deterministic radio broadcasting at low cost. *Networks*, *39*(2), 88-97.

[30] Dessmark, A., & Pelc, A. (2007). Broadcasting in geometric radio networks. *Journal of Discrete Algorithms*, *5*(1), 187–201.

[31] Diks, K., & Pelc, A. (1996). Broadcasting with universal lists. *Networks*, *27*(3), 183–196.

[32] Dinneen, M. J. (1994). The complexity of broadcasting in bounded-degree networks. *arXiv preprint math/9411222*.

[33] Dinneen, M. J., Fellows, M. R., & Faber, V. (1991). Algebraic constructions of efficient broadcast networks. In *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes* (pp. 152–158).

[34] Ehresmann, A.-L. (2021). *Approximation algorithms for broadcasting in flower graphs* (Unpublished master's thesis). Concordia University. (Unpublished)

[35] Elkin, M., & Kortsarz, G. (2005). A combinatorial logarithmic approximation algorithm for the directed telephone broadcast problem. *SIAM Journal on Computing*, *35*(3), 672–689.

[36] Elkin, M., & Kortsarz, G. (2006). Sublogarithmic approximation for telephone multicast. *Journal of Computer and System Sciences*, *72*(4), 648–659.

[37] Erdős, P., Pach, J., Pollack, R., & Tuza, Z. (1989). Radius, diameter, and minimum degree. *Journal of Combinatorial Theory, Series B*, *47*(1), 73–79.

[38] Farley, A. M. (1979). Minimal broadcast networks. *Networks*, *9*(4), 313–332.

[39] Farley, A. M. (2004). Minimal path broadcast networks. *Networks*, *43*(2), 61–70.

[40] Farley, A. M., Fragopoulou, P., Krumme, D., Proskurowski, A., & Richards, D. (2000). Multi-source spanning tree problems. *Journal of Interconnection Networks*, *01*(01), 61-71.

[41] Farley, A. M., & Hedetniemi, S. T. (1978). Broadcasting in grid graphs. In *Proceedings 9th S.E. Conference on Combinatorics, Graph Theory, and Computing, Utilitas Mathematica* (pp. 275–288).

[42] Farley, A. M., Hedetniemi, S. T., Mitchell, S., & Proskurowski, A. (1979). Minimum broadcast graphs. *Discrete Mathematics*, *25*(2), 189–193.

[43] Farley, A. M., & Proskurowski, A. (1981). Broadcasting in trees with multiple originators. *SIAM Journal on Algebraic Discrete Methods*, *2*(4), 381-386.

[44] Fomin, F. V., Fraigniaud, P., & Golovach, P. A. (2023). *Parameterized complexity of broadcasting in graphs.*

[45] Fraigniaud, P., & Lazard, E. (1994). Methods and problems of communication in usual networks. *Discrete Applied Mathematics*, *53*(1), 79-133.

[46] Garey, M. R., & Johnson, D. S. (1983). *Computers and intractability. A guide to the theory of NP-Completeness.*

[47] Gholami, S., & Harutyunyan, H. A. (2022a). Broadcast Graphs with Nodes of Limited Memory. In *13th International Conference on Complex Networks (CompleNet).*

[48] Gholami, S., & Harutyunyan, H. A. (2022b). Fully-adaptive model for broadcasting with universal lists. In *2022 24th international symposium on symbolic and numeric algorithms for scientific computing (synasc)* (p. 92-99).

[49] Gholami, S., & Harutyunyan, H. A. (2022c). HUB-GA: A Heuristic for Universal lists Broadcasting using Genetic Algorithm. *Journal of Communications and Networks*.

[50] Gholami, S., Harutyunyan, H. A., & Maraachlian, E. (2022). Optimal Broadcasting in Fully

Connected Trees. *Journal of Interconnection Networks*.

[51] GholamiNajarkola, M. (2022). *Problems related to classical and universal list broadcasting* (Unpublished doctoral dissertation). Concordia University. (Unpublished)

[52] Gregor, P., Škrekovski, R., & Vukašinović, V. (2018, 11). Broadcasting multiple messages in the 1-in port model in optimal time. *Journal of Combinatorial Optimization*, *36*.

[53] Grigni, M., & Peleg, D. (1991). Tight bounds on mimimum broadcast networks. *SIAM Journal on Discrete Mathematics*, *4*(2), 207–222.

[54] Grigoryan, H. (2013). *Problems related to broadcasting in graphs* (Unpublished doctoral dissertation). Concordia University. (Unpublished)

[55] Grigoryan, H., & Harutyunyan, H. A. (2013). Broadcasting in the Knödel graph. *Ninth International Conference on Computer Science and Information Technologies Revised Selected Papers*, 1–6.

[56] Harutyunyan, H., Laza, G., & Maraachlian, E. (2009, 05). Broadcasting in necklace graphs. In (p. 253-256).

[57] Harutyunyan, H., & Shao, B. (2006). An efficient heuristic for broadcasting in networks. *Journal of Parallel and Distributed Computing*, *66*(1), 68–76.

[58] Harutyunyan, H. A. (2000). Multiple broadcasting in modified Knödel graphs. In *7th International Colloquium on Structural Information and Communication Complexity (SIROCCO)* (pp. 157–166).

[59] Harutyunyan, H. A. (2006). Minimum multiple message broadcast graphs. *Networks*, *47*(4), 218-224.

[60] Harutyunyan, H. A. (2008). An efficient vertex addition method for broadcast networks. *Internet Mathematics*, *5*(3), 211–225.

[61] Harutyunyan, H. A., Hovhannisyan, N., & Magithiya, R. (2022). Deep heuristic for broadcasting in arbitrary networks. In *2022 21st international symposium on parallel and distributed computing (ispdc)* (p. 1-8). Los Alamitos, CA, USA: IEEE Computer Society.

[62] Harutyunyan, H. A., & Liestman, A. L. (1998). Messy broadcasting. *Parallel Processing Letters*, *8*(02), 149–159.

[63] Harutyunyan, H. A., & Liestman, A. L. (2001a). Improved upper and lower bounds for

k-broadcasting. *Networks*, *37*(2), 94–101.

[64] Harutyunyan, H. A., & Liestman, A. L. (2001b). k-broadcasting in trees. *Networks*, *38*(3), 163–168.

[65] Harutyunyan, H. A., Liestman, A. L., Makino, K., & Shermer, T. C. (2011). Nonadaptive broadcasting in trees. *Networks*, *57*(2), 157–168.

[66] Harutyunyan, H. A., Liestman, A. L., Peters, J. G., & Richards, D. (2013). Broadcasting and gossiping. *Handbook of Graph Theory*, 1477–1494.

[67] Harutyunyan, H. A., Liestman, A. L., & Shao, B. (2009). A linear algorithm for finding the k-broadcast center of a tree. *Networks*, *53*(3), 287–292.

[68] Harutyunyan, H. A., & Maraachlian, E. (2007). Linear algorithm for broadcasting in unicyclic graphs. In *International Computing and Combinatorics Conference (COCOON)* (pp. 372–382).

[69] Harutyunyan, H. A., & Maraachlian, E. (2008). On broadcasting in unicyclic graphs. *Journal of Combinatorial Optimization*, *16*(3), 307–322.

[70] Harutyunyan, H. A., & Maraachlian, E. (2009a). Broadcasting in Fully Connected Trees. In *15th IEEE International Conference on Parallel and Distributed Systems, (ICPADS)* (pp. 740–745).

[71] Harutyunyan, H. A., & Maraachlian, E. (2009b). Linear Algorithm for Broadcasting in Networks With No Intersecting Cycles. In *Proceedings of the International Conference on Parallel and Distributed Processing Techniques and Applications, (PDPTA)* (pp. 296–301).

[72] Harutyunyan, H. A., & Taslakian, P. (2004). Orderly broadcasting in a 2d torus. In *Proceedings Eighth International Conference on Information Visualisation* (pp. 370–375).

[73] Harutyunyan, H. A., & Wang, W. (2010). Broadcasting algorithm via shortest paths. In *2010 IEEE 16th International Conference on Parallel and Distributed Systems* (pp. 299–305).

[74] Hedetniemi, S. M., Hedetniemi, S. T., & Liestman, A. L. (1988). A survey of gossiping and broadcasting in communication networks. *Networks*, *18*(4), 319–349.

[75] Hromkovič, J., Jeschke, C.-D., & Monien, B. (1993). Optimal algorithms for dissemination of information in some interconnection networks. *Algorithmica*, *10*(1), 24–40.

[76] Hromkovič, J., Klasing, R., Monien, B., & Peine, R. (1996). Dissemination of information in

interconnection networks (broadcasting & gossiping). In *Combinatorial Network Theory* (pp. 125–212). Springer.

[77] Hromkovič, J., Klasing, R., Pelc, A., Ruzicka, P., & Unger, W. (2005). Dissemination of information in communication networks - broadcasting, gossiping, leader election, and fault-tolerance. In *Texts in theoretical computer science. an eatcs series.*

[78] Ivanova, M. (2019). Optimization problems in communication networks and multi-agent path finding. *Ph.D. Thesis, The University of Bergen*.

[79] Jakoby, A., Reischuk, R., & Schindelhauer, C. (1998). The complexity of broadcasting in planar and decomposable graphs. *Discrete Applied Mathematics*, *83*(1-3), 179–206.

[80] Karp, R. M., Sahay, A., Santos, E. E., & Schauser, K. E. (1993). Optimal broadcast and summation in the logp model. In *Proceedings of the fifth annual acm symposium on parallel algorithms and architectures* (pp. 142–153).

[81] Khachatrian, L., & Harutounian, O. (1990). Construction of new classes of minimal broadcast networks. In *Conference on Coding Theory, Armenia* (pp. 69–77).

[82] Kim, J.-H., & Chwa, K.-Y. (2005). Optimal broadcasting with universal lists based on competitive analysis. *Networks*, *45*(4), 224–231.

[83] Klasing, R., Monien, B., Peine, R., & Stöhr, E. A. (1994). Broadcasting in butterfly and DeBruijn networks. *Discrete Applied Mathematics*, *53*(1-3), 183–197.

[84] Knödel, W. (1975). New gossips and telephones. *Discrete Mathematics*, *13*, 95.

[85] König, J.-C., & Lazard, E. (1994). Minimum k-broadcast graphs. *Discrete Applied Mathematics*, *53*(1-3), 199–209.

[86] Kortsarz, G., & Peleg, D. (1995). Approximation algorithms for minimum-time broadcast. *SIAM Journal on Discrete Mathematics*, *8*(3), 401–427.

[87] Labahn, R. (1986). The telephone problem for trees. *J. Inf. Process. Cybern.*, *22*(9), 475–485.

[88] Labahn, R. (1989). Extremal broadcasting problems. *Discrete Applied Mathematics*, *23*(2), 139–155.

[89] Labahn, R. (1994). A minimum broadcast graph on 63 vertices. *Discrete Applied Mathematics*, *53*(1-3), 247–250.

[90] Lee, J.-K., Hong, T., & Li, G. (2021). Traffic and overhead analysis of applied pre-filtering ACL firewall on HPC service network. *Journal of Communications and Networks*, *23*(3), 192-200.

[91] Lee, S., & Ventura, J. A. (2001). An algorithm for constructing minimal c-broadcast networks. *Networks*, *38*(1), 6-21.

[92] Li, C., Hart, T., Henry, K., & Neufeld, I. (2008b, 12). Average-case ”messy” broadcasting. *Journal of Interconnection Networks*, *9*, 487-505.

[93] Li, C., Hart, T. E., Henry, K. J., & Neufeld, I. A. (2008a). Average-case “messy” broadcasting. *Journal of Interconnection Networks*, *9*(04), 487–505.

[94] Liestman, A. L. (1985). Fault-tolerant broadcast graphs. *Networks*, *15*(2), 159–171.

[95] Liestman, A. L., & Peters, J. G. (1988). Broadcast networks of bounded degree. *SIAM journal on Discrete Mathematics*, *1*(4), 531–540.

[96] Maheo, M., & Saclé, J.-F. (1994). Some minimum broadcast graphs. *Discrete Applied Mathematics*, *53*(1-3), 275–285.

[97] Maraachlian, E. (2010). *Optimal broadcasting in treelike graphs* (Unpublished doctoral dissertation). Concordia University. (Unpublished)

[98] Menger, K. (1927). Zur allgemeinen kurventheorie. *Fundamenta Mathematicae*, *10*(1), 96-115.

[99] Middendorf, M. (1993). Minimum broadcast time is NP-Complete for 3-regular planar graphs and deadline 2. *Information Processing Letters*, *46*(6), 281–287.

[100] Mitchell, S., & Hedetniemi, S. (1980). A census of minimum broadcast graphs. *Journal of Combinatorics, Information, and System Sciences*, *5*, 141–151.

[101] Morosan, C. D. (2007). *Studies of interconnection networks with applications in broadcasting* (Unpublished doctoral dissertation). Concordia University. (Unpublished)

[102] Park, J.-H., & Chwa, K.-Y. (1994). Recursive circulant: A new topology for multicomputer networks. In *Proceedings of the International Symposium on Parallel Architectures, Algorithms and Networks (ISPAN)* (pp. 73–80).

[103] Pelc, A. (1996). Fault-tolerant broadcasting and gossiping in communication networks. *Networks*, *28*(3), 143–156.

[104] Peleg, D. (2007). Time-efficient broadcasting in radio networks. In A. Pelc (Ed.), *Distributed computing* (pp. 3–4). Berlin, Heidelberg: Springer Berlin Heidelberg.

[105] Peleg, D., & Radzik, T. (2010). Time-efficient broadcast in radio networks. In (pp. 311–334). Springer Verlag.

[106] Preparata, F. P., & Vuillemin, J. (1981). The cube-connected cycles: a versatile network for parallel computation. *Commun. ACM*, *24*, 300-309.

[107] Proskurowski. (1981). Minimum broadcast trees. *IEEE Transactions on Computers*, *C-30*(5), 363-366.

[108] Ravi, R. (1994). Rapid rumor ramification: Approximating the minimum broadcast time. In *Proceedings 35th Annual Symposium on Foundations of Computer Science (FOCS)* (pp. 202–213).

[109] Rocher-Gonzalez, J., Escudero-Sahuquillo, J., García, P. J., & Quiles, F. J. (2017). On the impact of routing algorithms in the effectiveness of queuing schemes in high-performance interconnection networks. In *IEEE 25th Annual Symposium on High-Performance Interconnects (HOTI)* (pp. 65–72).

[110] Rosenthal, A., & Scheuermann, P. (1987). Universal rankings for broadcasting in tree networks. In *Proceedings of the 25th Allerton Conference on Communication, Control and Computing* (pp. 641–649).

[111] Saclé, J.-F. (1996). Lower bounds for the size in four families of minimum broadcast graphs. *Discrete Mathematics*, *150*(1-3), 359–369.

[112] Scheuermann, P., & Wu, G. (1984). Heuristic algorithms for broadcasting in point-to-point computer networks. *IEEE Computer Architecture Letters*, *33*(09), 804–811.

[113] Schindelhauer, C. (2000). On the inapproximability of broadcasting time. In K. Jansen & S. Khuller (Eds.), *Approximation algorithms for combinatorial optimization* (pp. 226–237). Berlin, Heidelberg: Springer Berlin Heidelberg.

[114] Shao, B. (2006). On k-broadcasting in graphs. *Ph.D. Thesis, Concordia University*.

[115] Slater, P. J., Cockayne, E. J., & Hedetniemi, S. T. (1981). Information dissemination in trees. *SIAM Journal on Computing*, *10*(4), 692–701.

[116] Suderman, M. (1999). *Multiple message broadcasting* (Unpublished doctoral dissertation).

Simon Fraser University.

[117] Varvarigos, E. A., & Bertsekas, D. P. (1995). Dynamic broadcasting in parallel computing. *IEEE Transactions on Parallel and Distributed Systems*, *6*(2), 120–131.

[118] West, D. B. (2000). *Introduction to graph theory* (2nd ed.). Prentice Hall.

[119] Xiao, J., & Wang, X. (1988). A research on minimum broadcast graphs. *Chinese Journal of Computers*, *11*, 99–105.

[120] Zhou, J.-g., & Zhang, K.-m. (2001). A minimum broadcast graph on 26 vertices. *Applied Mathematics Letters*, *14*(8), 1023–1026.

[121] Čevnik, M., & Žerovnik, J. (2017). Broadcasting on cactus graphs. *Journal of Combinatorial Optimization*, *33*(1), 292-316.