ARITHMETIC AND COMPUTATIONAL ASPECTS OF MODULAR FORMS OVER GLOBAL FIELDS

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Abstract

Arithmetic and computational aspects of modular forms over global fields

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This thesis consists of two parts. In the first part, we present a positive characteristic analogue of Shimura's theorem on the special values of modular forms at CM points. More precisely, we show using Hayes' theory of Drinfeld modules that the special value at a CM point of an arithmetic Drinfeld modular form of arbitrary rank lies in the Hilbert class field of the CM field up to a period, independent of the chosen modular form.This is achieved via Pink's realization of Drinfeld modular forms as sections of a sheaf over the compactified Drinfeld modular curve.

In the second part of the thesis, we present various computational and algorithmic aspects both for the classical theory (over $\mathbb{C}$) and function field theory. First, we implement the rings of quasimodular forms in SageMath and give some applications such as the symbolic calculation of the derivative of a classical modular form. Second, we explain how to compute objects associated with a Drinfeld modules such as the exponential, the logarithm, and Potemine's set of basic J-invariants. Lastly, we present a SageMath package for computing with Drinfeld modular forms and their expansion at infinity using the nonstandard A-expansion theory of López and Petrov.

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À ma chère Alexandra

Contents

Introduction

The theory of classical modular forms has been widely used for various arithmetic applications. A famous example is the instrumental role they played in the proof of Fermat's last theorem. The statement that classical modular form are a convenient tool for studying arithmetic is justified in part from two facts. First, one may use Katz-interpretation of modular forms in order to connect algebro-geometric objects with analytic objects. For instance, a theorem by Shimura states that their special values at CM points are linked with the finite abelian extensions of quadratic imaginary field [Shi75]. Second, classical modular forms benefits from numerous algorithmic properties. This allows explicit experimentation and, more generally, their implementation in computer algebra softwares such as SageMath [S^+23].

These two general concepts have been widely studied in the classical setting, that is for modular forms over $\mathbb{C}$. In this thesis, we are interested in expanding both of these concepts in the global function field of positive characteristic setting. Therefore, we have made the choice of separating this thesis in two main parts. The first part being more theoretically focused aims to prove a function field analogue of Shimura's theorem on special values of modular forms at CM points. The second part aims to develop computational tools for computing in both the classical setting and the function field settings.

I. Special values of Drinfeld modular forms

In the first part of this thesis, we use the theory of CM Drinfeld modules and Katz-like interpretation of Drinfeld modular forms in order to prove a function field analogue of Shimura's theorem on the special values of modular forms at CM points. The key of our proof is to formulate in a proper algebro-geometric language the usual notions of Drinfeld modules and Drinfeld modular forms. These notions are (more or less explicitly) already present in literature but in this thesis we present them in a more uniform way, comparing all the different definitions. We note that an analogue of Shimura's result is already known in a specific case by the work of Chang in rank two [Cha12] and by the work of Chen and Gezmiş in arbitrary ranks [CG23].

Classical setting. Recall that a classical (or elliptic) modular form of weight $k \in \mathbb{Z}$ for a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the complex upper half plane, satisfying two additional properties. The first property is a modularity invariance under the action of the group Γ:

$$f\left(\frac{aw+b}{cw+d}\right) = (cw+d)^{-k} f(w)$$

for all $w \in \mathcal{H}$ and all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ. The second property is a growth condition at the cusps of $\mathcal{H}$, known as the holomorphicity at infinity. In particular, if $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ for $N \geq 1$, this last property requires f to have a Fourier expansion of the form

$$f = \sum_{n \geq 0} a_n(f) q^n, \quad q = e^{2\pi i w}.$$

If the first nonzero Fourier coefficient of f is equal to 1, we say that f is normalized. A crucial fact about the space of all modular forms of fixed weight is that it forms a $\mathbb{C}$-vector space of finite dimension. Moreover, there are linear operators acting on this finite space called the Hecke operators. Any modular form which is a simultaneous eigenfunction for all Hecke operators is called an eigenform. By the theory of Hecke operators, one may show that the coefficients of any normalized eigenform f generates a number field $K_f/\mathbb{Q}$.

Then, a result attributed to Shimura states that if f is any weight k normalized eigenform and w is a CM point, i.e. a point in $\mathcal{H} \cap \mathbb{Q}(\sqrt{d})$ for some $d < 0$, then

$$\frac{f(w)}{\mathbf{\Omega}_w^k} \in K_f H_{\mathbb{Q}(\sqrt{d})}$$

where $H_{\mathbb{Q}(\sqrt{d})}$ is the maximal abelian unramified extension of $\mathbb{Q}(\sqrt{d})$ and $\mathbf{\Omega}_w \in \mathbb{C}^\times$ does not depends on f [Shi71, §6.8], [Shi75]. This proof was generalized by Urban for nearly holomorphic modular forms [Urb14, §2.6]. One goal of this thesis is to prove a function field analogue of this result in the setting of Drinfeld modular forms, inspired by the proofs in loc. cit.

Drinfeld setting. We first describe the context in which we will be working. Let X be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$ of q elements and let

∞ be a closed point on X. We let K be the function field of X and define A to be the ring of function of X which are regular everywhere outside ∞. We let K_∞ be the completion of K at ∞ and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. Throughout this thesis, this context will often be refered to as the *Drinfeld setting*. To put things in perpective, one can think of the explicit example $X = \mathbb{P}^1_{\mathbb{F}_q}$. In this case, A is identified with the univariate polynomial ring $\mathbb{F}_q[T]$ and $K_\infty = \mathbb{F}_q((1/T))$ is the completion of $K = \mathbb{F}_q(T)$ at the place $1/T$.

The notion of Drinfeld modular forms were first introduced by Goss in 1980 [Gos80a]. As in the classical case, Drinfeld modular forms are functions defined over a certain rigid analytic space, the Drinfeld periods domain, satisfying a modular invariance property under arithmetic subgroups of $\mathrm{GL}_r(K)$ where $r \geq 2$ is an integer known as the rank. The Drinfeld periods domain of rank r is defined by

$$\Omega^r(\mathbb{C}_\infty) := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}.$$

This set admits a structure of rigid analytic space and is the analogue of the complex upper half plane $\mathcal{H}$. In the case where $X = \mathbb{P}^1_{\mathbb{F}_q}$, we have the identification

$$\Omega^r(\mathbb{C}_\infty) \longleftrightarrow \{\text{Homothety classes of rank } r \text{ projective } A\text{-module inside } \mathbb{C}_\infty\},$$

which is to be seen as the analogue of the correspondence

$$\mathcal{H} \longleftrightarrow \{\text{Homothety classes of rank 2 free } \mathbb{Z}\text{-modules in } \mathbb{C}\}.$$

Via linear fractional transformations, we may define an action of the group $\mathrm{GL}_r(K_\infty)$ on $\Omega^r(\mathbb{C}_\infty)$ denoted $\gamma(\cdot) : w \mapsto \gamma(w)$ for all $\gamma \in \mathrm{GL}_r(K_\infty)$. A weak Drinfeld modular form f of rank r for $\mathrm{GL}_r(A)$ is a rigid analytic function $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ satisfying $f(\gamma(w)) = j(\gamma, w)^k f(w)$ for some automorphy factor $j(\gamma, w) \in \mathbb{C}_\infty^\times$. A Drinfeld modular form, is a weak modular form which is holomorphic at infinity, meaning that it satisfy a Fourier-like expansion of the form

$$f = \sum_{n \geq 0} a_n(f) u^n$$

for some analytic parameter $u : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ which is translation invariant, called the parameter at infinity.

An important aspect of Drinfeld modular forms is that they also possess an algebraic interpretation à la Katz which behaves well under base change. This interpretation involves the crucial notion of Drinfeld modules. Emerged from the work of Drinfeld [Dri74], this function field concept is often observed as the analogue of elliptic curves. In short, a *Drinfeld A-module of rank r* over $\mathbb{C}_\infty$ is a morphism $\phi : A \to \mathbb{C}_\infty\{\tau\}$ where $\mathbb{C}_\infty\{\tau\}$ is the endomorphism ring of polynomial in τ, the q-Frobenius, such that

$$\phi : a \mapsto \phi_a = \iota(a) + g_{1,a}(\phi)\tau + \cdots + g_{r-1,a}(\phi)\tau^{r-1} + g_{r,a}(\phi)\tau^{\deg(a)r}$$

where ι is the inclusion $A \hookrightarrow \mathbb{C}_\infty$, the coefficients $g_{i,a}(\phi)$ lies in $\mathbb{C}_\infty$, $g_{r,a}(\phi) \neq 0$ and $\deg(a) := |A/(a)|$. One can show that the set of Drinfeld A-modules over $\mathbb{C}_\infty$ of rank r corresponds bijectively to the set of projective A-submodules $\Lambda \subset \mathbb{C}_\infty$ of projective rank r. Any such submodule is called a *A-lattice*. This bijection, which is more precisely a correspondence of categories, is known as the Drinfeld uniformization, to be seen as the analogous notion in the elliptic curves case.

One can then define Drinfeld modular forms algebraically as sections of the k-th tensor of a sheaf $\boldsymbol{\omega}$ on the Drinfeld modular variety M_N^r over K of level $N \trianglelefteq A$. This variety is essentially the isomorphism classes of pairs of Drinfeld A-modules together with an added structure, namely a level-N structure. The modular variety M_N^r admits various interesting properties. First, as shown by Drinfeld, it is a smooth affine variety over K of dimension $r-1$ [Dri74]. Next, after base change over $\mathbb{C}_\infty$, it possess a decomposition in connected components

$$M_N^r \times_K \mathbb{C}_\infty \cong \bigsqcup_{s\in S(N)} M_s^r \tag{1}$$

for some finite set $S(N)$ depending on N. Via Drinfeld uniformization, one may show that the $\mathbb{C}_\infty$-valued points of each connected components corresponds to $M_s^r(\mathbb{C}_\infty) \cong \Gamma_s \setminus \Omega^r(\mathbb{C}_\infty)$ as rigid-analytic spaces for some arithmetic subgroup Γ_s of $\mathrm{GL}_r(K)$. Lastly, in 2013, Pink constructed a Satake compactification of M_N^r which is unique up to isomorphism [Pin13]. He showed moreover that the sheaf $\boldsymbol{\omega}$ extends to this compactification. Therefore, this allowed him to define the set of algebraic Drinfeld modular forms which are holomorphic at infinity and he showed how we may identify them with a direct sum of their analytic counterparts.

In section 3.4.3, following mainly Goss' ideas [Gos80a] we will reformulate Pink's algebraic definition of Drinfeld modular forms as a rule on triples $f : (\phi, \alpha_N, \omega) \mapsto f(E, \alpha_N, \omega) \in R$ where ϕ is a Drinfeld module over a A-field R, α_N a level-N structure and ω is a nonzero section of the sheaf $\boldsymbol{\omega}$. For such rule to be called an algebraic Drinfeld modular form, we require three conditions: namely that f depends only on the isomorphism classes of the triple, that f behave well under base changes, and that scaling the section ω by a unit in R results in simply scaling the value at the triple by that unit to a power of negative k. We prove that this definition agrees with Pink's definition (see theorem 3.4.11). Our definition of algebraic Drinfeld modular forms will be interpreted in this thesis as a direct analogue of Katz' version of classical modular forms. We note that the base change property of our definition together with the complex multiplication theory of Drinfeld module will be crucial for proving an analogue of Shimura's result.

Next, the theory of complex multiplication of Drinfeld modules will be of crucial importance for us. Indeed, in 1979 Hayes showed that any such Drinfeld A-module is defined over a Hilbert class field of the CM field [Hay79]. More precisely, a Drinfeld A-module over $\mathbb{C}_\infty$ is said to be CM if the field $K_\phi := \mathrm{End}(\phi) \otimes K$ has degree exactly r over K and if the place ∞ is inert in it. This theory of complex multiplication must also be translated to the points of the Drinfeld period domain. A point $w = [w_1 : \ldots : w_{r-1} : 1]$ in the period domain $\Omega^r(\mathbb{C}_\infty)$ is said to have CM if the field

$$K_w := K(w_1, \ldots, w_{r-1})$$

is of degree r over K and ∞ is inert in it. When A is $\mathbb{F}_q[T]$, Hamahata showed that the A-lattice $\Lambda_w := \oplus_{i=1}^r A w_i$ give rise to a CM Drinfeld module ϕ_w [Ham03]. In this thesis, we generalize this fact for arbitrary A (not necessarily for $A = \mathbb{F}_q[T]$). In this situation, the Drinfeld modular variety of level N may have some distinct connected components, indexed by $S(N)$. Thus, we will show in section 4.5 that for any CM point w and any representative $s \in S(N)$, the associated Drinfeld A-module ϕ_w^s has CM by the field K_w. One has to be careful in this case as the ring A may not be a principal ideal domain in general.

Recall that the main goal of the present thesis is to prove a function field analogue of Shimura's result for Drinfeld modular forms. This goal is achieved in section 4.6 which constitutes the main contribution of this thesis. In this section, we will prove the following theorem:

Theorem. *Let $s \in S(N)$ and let $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ be a Drinfeld modular form of weight k and rank r for Γ_s which is arithmetic over a finite extension K_f/K. Let $w = (w_1 : \ldots : w_r) \in \Omega^r(\mathbb{C}_\infty)$ be a CM point and let K_w be the field $K(w_1, \ldots, w_r)$. Then, there exists a period $\mathbf{\Omega}_w \in \mathbb{C}_\infty^\times$ such that*

$$\frac{f(w)}{\mathbf{\Omega}_w^k} \in H_w(N) K_f$$

were $H_w(N) := H_w(\phi_w^s[N])$ and $\phi_w^s[N]$ denote the group of N-torsion points of ϕ_w^s, that is the set of roots in $\overline{K}$ of the polynomials $\phi_{w,a}^s$ for all $a \in N$.

Via a Galois arguement, we specialize this theorem to the level 1 case and obtain an analogue of Shimura's result:

Theorem. *Let $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ be a Drinfeld modular form of weight k for $\mathrm{GL}_r(A)$ which is arithmetic over K. Let $w \in \Omega^r(\mathbb{C}_\infty)$ be a CM point. Then, there exists a period $\mathbf{\Omega}_w \in \mathbb{C}_\infty$ such that $f(w)/\mathbf{\Omega}_w^k \in H_w$.*

The above theorems will be proved in section 4.6. The key ingredients of the proof is to first use the fact that Drinfeld A-modules having complex multiplication by K_w are defined over the Hilbert class field H_w. Next, we may write the form f algebraically and by evaluating it at a CM Drinfeld A-module we find that its value must lie in H_w. Finally, by base changing to $\mathbb{C}_\infty$ and using the transformation property of a modular forms, we obtain the period $\mathbf{\Omega}_w$ which is independent of the form f.

As mentioned in the beginning, this last theorem is already known by Chen and Gezmiş in the case when A is $\mathbb{F}_q[T]$ [CG23]. Their proof involves analytic methods and is dependent on the fact that A is generated by T as a $\mathbb{F}_q$-algebra. We note also that they make use of a more general notion of CM points as they don't require the place ∞ to be inert. Then, instead of lying in a Hilbert class field, the special value is an arbitrary algebraic element in $\overline{K}$, up to a specific period.

II. Computations: classical case and function field case

Depending on the nature of an unsolved problem, a researcher in mathematics will often resort to experimentation to better understand said problem. For example, the L-functions and Modular Forms Database, abreviated LMFDB, contains large data set on the nontrivial zeros of the famous Riemann ζ function [LMF23]. This is where computer algebra softwares such as SageMath becomes useful. SageMath is a mathematical software built on top of many already existing libraries such as NumPy and SimPy. Moreover, it is free and open-source, meaning that anybody can install the software, access its source code and examine the algorithms. In number theory, SageMath offers a wide variety of functionality such as computing with number fields, with elliptic curves, and with modular forms.

In the second part of this thesis, we enhanced and expanded the functionalities of SageMath. In particular, we implemented the graded ring of quasimodular forms, we contributed to the implementation of Drinfeld module and we developed a SageMath external package for computing with Drinfeld modular forms and their expansion at infinity.

(Quasi)modular forms. First introduced by Kaneko and Zagier in [KZ95], quasimodular forms are holomorphic function on the complex upper half plane which are close of being modular forms, but fail to satisfy the modular invariance property. The weight 2 Eisenstein series E_2 is a well known example:

$$(cw+d)^{-2}E_2(\gamma w) = E_2(w) - \frac{6}{\pi i}\left(\frac{c}{cw+d}\right)$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. Other examples arise when computing the derivative of a modular form. More generally, a quasimodular form of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ is an holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

$$(cw+d)^{-k}f(\gamma w) = \sum_{i=0}^{p} f_i(w)\left(\frac{c}{cw+d}\right)^i$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $f_i : \mathcal{H} \to \mathbb{C}$ are holomorphic. Then, one can prove that the ring of all quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ is simply

$$\tilde{\mathcal{M}}_\bullet^{\mathrm{ell}} := \mathbb{C}[E_2, E_4, E_6].$$

where E_k is the normalized k-th Eisenstein series. More generally, if Γ is any congruence subgroup, we have

$$\tilde{\mathcal{M}}_\bullet^{\mathrm{ell}}(\Gamma) = \mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)[E_2]$$

where $\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma) := \oplus_{k\in\mathbb{Z}}\mathcal{M}_k(\Gamma)$ is the graded ring of all modular forms, which is finitely generated [DR73, p. 303].

We implemented the ring $\tilde{\mathcal{M}}_\bullet^{\mathrm{ell}}(\Gamma)$ in SageMath when Γ is $\Gamma_0(N)$ or $\Gamma_1(N)$ and the functionalities are now included as of version 9.5 of the software. We explain in this thesis two main functionalities of this implementation, but we invite the interested reader to try the implementation themself and read the documentation in SageMath's reference manual.

The first main functionality concerns an algorithm for converting any (quasi)modular forms as an homogeneous polynomial in the generators of the ring. More precisely, suppose that $\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)$ is generated by a finite set $\{g_1, \ldots, g_n\}$, then we give a systematic procedure that, for any (quasi)modular form f, computes a multivariate polynomial $P_f(Y, X_1, \ldots X_n)$ such that

$$f = P_f(E_2, g_1, \ldots, g_n).$$

We note that the exact ring structure of $\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)$ is unknown in general, but algorithms for computing a generating set in some specific cases are known [Rus14].

The second property is the computation of the derivative $D : f \mapsto \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}w} f$. In general, the derivative of a modular form is not a modular form, but a quasimodular form. The calculation essentially relies on computing the Serre derivative

$$\theta_k(f) := D(f) + \frac{k}{12} E_2 f$$

and then isolating back the derivative.

Drinfeld $\mathbb{F}_q[T]$-modules. It is well-known that j-invariants of elliptic curves over $\mathbb{C}$ having complex multiplication by a quadratic imaginary L field generates the Hilbert class field of L [Sil94, Chap. II]. A similar fact holds for Drinfeld $\mathbb{F}_q[T]$-modules of arbitrary rank [Gek83], [Ham03]. In rank two, the j-invariant of a Drinfeld module

$$\phi : T \mapsto \gamma(T) + g_1\tau + g_2\tau^2$$

is defined by formula $j(\phi) := g_1^{q+1}/g_2$. It is well defined since g_2 is nonvanishing. The j-invariant is constructed so that if ψ is another Drinfeld module isomorphic to ϕ, then $j(\phi) = j(\psi)$. Higher ranks j-invariants are also defined due to the work of Potemine [Pot98]. In this case, instead of a single j-invariant, Potemine defined a finite family named the *basic J-invariants* and showed that two Drinfeld $\mathbb{F}_q[T]$-modules are isomorphic over an algebraic closure of the base field if and only if all the basic J-invariants agree. We will explain how to compute all such J-invariants. The core idea consists in the fact that basic J-invariants are parametrised by integral points of a convex subset of $\mathbb{R}^r$.

On another note, recall that the category of Drinfeld A-module of rank r is equivalent to the category of rank r discrete projective A-submodule of $\mathbb{C}_\infty$. This bijection is essentially induced by a unique function $e_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ which is $\mathbb{F}_q$-linear, surjective and nonconstant. Moreover, it admits an expansion of the form

$$e_\phi(z) = z + \sum_{i\geq 1} \alpha_i z^{q^i} \in \mathbb{C}_\infty[[z]].$$

The function e_ϕ is named the exponential of ϕ. We will explain a procedure that, given any Drinfeld module ϕ, returns a power series approximation of e_ϕ. An interesting applications of such procedure is to compute the Goss polynomial associated with a Drinfeld module. *Goss polynomial* are a special class of polynomials introduced by Goss which are useful when computing the expansion at infinity of a Drinfeld modular form.

Drinfeld modular forms. Lastly, we present in this thesis a SageMath external package for computing with Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[T])$, namely `drinfeld-modular-forms`. The main goal of this package is to manipulate Drinfeld modular forms as formal objects and to compute their expansion at infinity when the rank is two.

This goal is achieved by representing a Drinfeld modular form as an homogeneous polynomial in a set of generators for the graded ring of all Drinfeld modular forms. More precisely, by [BBP18c, Theorem 17.5 (a)], we know that the graded ring of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[T])$ of type 0, denoted $\mathcal{M}_\bullet^r(\mathrm{GL}_r(\mathbb{F}_q[T]))$, is generated by r forms:

$$\mathcal{M}_\bullet^r(\mathrm{GL}_r(\mathbb{F}_q[T])) \cong \mathbb{C}_\infty[g_1, \dots, g_r].$$

These forms $g_i : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ are of weight $q^i - 1$ and may be defined as the coefficients of a certain universal Drinfeld $\mathbb{F}_q[T]$-module over $\Omega^r(\mathbb{C}_\infty)$. When $1 \leq i \leq r-1$, the modular forms g_i may be seen (after normalization) as the Drinfeld Eisenstein series of weight $q^i - 1$ and g_r is the Drinfeld discriminant function, usually denoted Δ.

In the specific case where the rank is equal to two, we are able to compute expansion at infinity through the work of López. Contextually, López proved in [Ló10] a *nonstandard* expansion for the Drinfeld discriminant function:

$$-\tilde{\pi}^{1-q^2}\Delta(w) = \sum_{a \text{ monic}} a^{q(q-1)} u_a^{q-1}, \tag{2}$$

where $u_a : w \mapsto u(aw)$ and u is the parameter at infinity. It is nonstandard in the sense that the sum ranges over monic elements of $\mathbb{F}_q[T]$ instead of positive integers. This allows the computation of its expansion at infinity. Moreover, a similar nonstandard expansion is known for the Drinfeld Eisenstein series [Gek88, §6]. Hence, our package provides methods for computing these expansions. The computations are done *lazily*, meaning that we don't need to input any precision parameter and the coefficients are computed only on demand.

Based on the work of López, Petrov formalized the concept of nonstandard expansion and introduced the A-expansion [Pet13]. In short, a Drinfeld modular form is said to admit an A-expansion if it can be written as a sum over the monic elements of $A = \mathbb{F}_q[T]$, similar to equation (2). He moreover showed that there exists an infinite family of modular forms (more precisely eigenform) which satisfies an A-expansion. We will show in section 7.3 an algorithm for computing the i-th coefficient of any forms which admits an A-expansion. Using Sturm-type bounds for Drinfeld modular forms, we may now convert the forms in the Petrov family as an homogeneous polynomial in g_1 and Δ.

Organization of this thesis

This thesis is seperated in two parts. The first part is divided in 4 chapters and consists in proving the analogue of Shimura's theorem. The first chapter is a review of the classical theory, meant to prove Shimura's theorem using Katz theory of modular forms over $\mathbb{C}$. The second chapter defines Drinfeld modules, their complex uniformization and the Drinfeld modular varieties. The third chapter covers both the analytic and algebraic theory of Drinfeld modular forms. Then, the last chapter of part one defines complex multiplication theory for Drinfeld modules, culminating in a proof of the analogue of Shimura's theorem.

The second part of this thesis concerns software implementations and their applications. In the first chapter, we explain and present an implementation of quasimodular forms in SageMath. The second chapter aims to explain some algorithms implemented for computing with Drinfeld modules. Finally, the goal of the last chapter is to showcase a SageMath package developed by the author for computing with Drinfeld modular forms over $\mathrm{GL}_r(A)$ and their exansion at infinity. We note that the two last chapters of the second part depend on the first part.

Part I

Special Values of Drinfeld Modular Forms

We hope that the study of these forms will add not only to our knowledge of function fields but also to our understanding of modular forms in general.

David Goss, [Gos80b]

Chapter 1

Review of the Classical Theory

1.1 Elliptic modular forms

1.1.1 Analytic theory

In this section, we recall without proofs the notion of elliptic modular forms and their properties, i.e. modular forms over the field of complex numbers $\mathbb{C}$. Some standard references about this topic includes [Ser77] and [DS05].

Let $\mathcal{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$ be the complex upper half plane and consider the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ consisting of 2×2 matrices with integer coefficients and determinant 1. We define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ via fractional linear transformations

$$\gamma(w) := \frac{aw + b}{cw + d},$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ and $w \in \mathcal{H}$. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is said to be a *congruence subgroup* if Γ contains the *principal congruence subgroup*

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

Let k be an integer and let Γ be a congruence subgroup. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ and holomorphic function $f : \mathcal{H} \to \mathbb{C}$, we define the $|_k$*-operator*:

$$(f|_k\gamma)(w) := (j(\gamma, w))^{-k} f(\gamma(w))$$

where $j(\gamma, w) := cw + d$.

Definition 1.1.1. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is said to be a *weak (elliptic) modular form of weight k for* Γ if $f|_k\gamma = f$ for any γ in Γ.

From now on, we will assume for simplicity that the subgroup Γ contains the translation matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. For example, this is the case for the modular group and the following two groups:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\};$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \text{ and } c \equiv 0 \bmod N \right\}.$$

By definition, this implies that any weak modular forms of weight k for Γ is 1-periodic and so admits a Fourier expansion of the form

$$f(w) = \sum_{n \in \mathbb{Z}} a_n(f) e^{2\pi i n w},$$

for some sequence of complex numbers $(a_n(f))_{n\in\mathbb{Z}}$. For any $w \in \mathcal{H}$, we set $q_w := e^{2\pi i w}$.

Definition 1.1.2. A weak modular form f of weight k for Γ is said to be a *modular form* if $a_n(f|_k\gamma) = 0$ for all negative integer n and all matrices γ in Γ. The $\mathbb{C}$-vector space of all modular forms is denoted by $\mathcal{M}_k^{\mathrm{ell}}(\Gamma)$. The *ring of modular forms for* Γ is defined by

$$\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma) := \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k^{\mathrm{ell}}(\Gamma).$$

Proposition 1.1.3. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then, the space of weight k modular forms for Γ is finite dimensional over $\mathbb{C}$.*

This important but not trivial fact can be proved in at least two ways. First, one can prove the valence formula of a modular form f which is a formula that roughly relates the weight of f with the order of vanishing of f at the cusps of Γ. From this formula we then deduce that f is determined by a finite number of Fourier coefficients and thus the whole space must be finite dimensional. The finite dimensionality of weight k spaces may also be proved using a more geometric approach using the Riemann–Roch theorem. This last approach also gives formulas to compute the exact dimension of a given space, cf. [DS05, Chapter 3].

Example 1.1.4. Let $k \geq 4$ be an even integer. The weight k Eisenstein series is defined by

$$G_k(w) := \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(cw+d)^k},$$

where $w \in \mathcal{H}$. The sum defining G_k is uniformly and absolutely convergent on $\mathcal{H}$. The absolute convergence allows us to rearange the terms of of the series without changing its value and thus proving the invariance under the $|_k$-operator. Moreover, the function admits the following expansion:

$$G_k(w) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q_w^n.$$

where ζ is the Riemann ζ function and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. One may normalize G_k so that its first nonzero coefficient is 1:

$$E_k(w) := \frac{1}{2\zeta(k)}G_k(w).$$

To end this section, we recall some facts which are specific to the case $\Gamma = \Gamma_1(N)$. First, we have the following decomposition

$$\mathcal{M}_k^{\mathrm{ell}}(\Gamma_1(N)) = \bigoplus_{\chi}\mathcal{M}_k^{\mathrm{ell}}(N,\chi)$$

where the sum is taken over the characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ and $\mathcal{M}_k^{\mathrm{ell}}(N,\chi)$ is the space of forms in $\mathcal{M}_k^{\mathrm{ell}}(\Gamma_1(N))$ such that $f|_k\gamma = \chi(d)f$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d_0\end{smallmatrix}\right) \in \Gamma_0(N)$ satisfying $d_0 \equiv d \pmod N$. Next, the space $\mathcal{M}_k^{\mathrm{ell}}(\Gamma_1(N))$ admits two sequences of linear maps T_n and $\langle n\rangle$, together forming the family of *Hecke operators*. A modular form which is a simultaneous eigenvector for each Hecke operators is called an eigenform. Moreover, we say that it is normalized if it first nonzero Fourier coefficient is one.

Proposition 1.1.5. *Let f be a normalized eigenform of weight k and level N, and let $(a_n)_{n\geq 0}$ be the coefficients of its expansion at infinity. Then the field*

$$\mathbb{Q}_f := \mathbb{Q}((a_n)_{n\geq 0})$$

is a finite extension of $\mathbb{Q}$.

A proof of the above proposition may be found in [Hid93, Corollary 5.4.2]. We say that $\mathbb{Q}_f$ is the *field of coefficients* of f.

1.1.2 Katz's interpretation

In [Kat73], Katz gave an algebro-geometric interpretation of modular forms. One novelty behind this interpretation is that it allows the definition of modular forms over arbitrary rings. We give in this section a quick review of this algebro-geometric interpretation.

Definition 1.1.6. Let S be a scheme. An elliptic curve over S is a proper smooth morphism of scheme

$$p : E \longrightarrow S$$

whose fibers are geometrically connected curves of genus 1, together with a section $e \in E(S)$. We write $\boldsymbol{\omega}_{E/S} := p_*(\Omega^1_{E/S})$ where $\Omega^1_{E/S}$ is the invertible sheaf of invariant differential 1-forms on S.

Fix E/S an elliptic curve and let $N \geq 1$ be any integer which is invertible in $H^0(S, \mathcal{O}_S)$, where $\mathcal{O}_S$ is the structure sheaf of S. The group-scheme of N-torsion points of E is denoted by $E[N]$. A *level N-structure* is an isomorphism of group schemes

$$\alpha_N : E[N] \xrightarrow{\sim} \underline{(\mathbb{Z}/N\mathbb{Z})^2},$$

where $\underline{(\mathbb{Z}/N\mathbb{Z})^2}$ is the constant group scheme with fibers $(\mathbb{Z}/N\mathbb{Z})^2$.

Definition 1.1.7. A *weak modular form of weight k and level N over S* is a rule F which assigns to each pair $(E/S, \alpha_N)$, where E is an elliptic curve over S and α_N is a level-N structure, a global section of $\boldsymbol{\omega}^{\otimes k}_{E/S}$:

$$F(E/S, \alpha_N) \in H^0(S, \boldsymbol{\omega}^{\otimes k}_{E/S})$$

such that

1. F depends only on the isomorphism class of $(E/S, \alpha_N)$;
2. F commutes with arbitrary base change $g : S' \to S$.

Remark 1.1.8. If $N = 1$, a weak modular form will be a rule F which to any elliptic curve E/S assigns a section

$$F(E/S) \in H^0(S, \boldsymbol{\omega}^{\otimes k}_{E/S})$$

satisfying the two properties above.

Let R be a commutative ring containing the multiplicative inverse of N. In what follows, we want to specialize definition 1.1.7 to the case $S = \mathrm{Spec}(R)$. In this case, we have

$$H^0(S, \boldsymbol{\omega}_{E/S}^{\otimes k}) = R\omega^{\otimes k}$$

for some choice of nonzero section ω generating the global sections of the invertible sheaf $\boldsymbol{\omega}_{E/S}$. Therefore, for any weak modular form F over R we have:

$$F(E, \alpha_N) = f(E, \alpha_N, \omega)\omega^{\otimes k} \tag{1.1}$$

for some $f(E, \alpha_N, \omega) \in R$. This gives us a rule on triple

$$(E, \alpha_N, \omega) \mapsto f(E, \alpha_N, \omega) \in R$$

which depends only on the isomorphism class of (E, α_N, ω) and commutes with arbitrary base change. Moreover, given any unit $\lambda \in R^\times$, we will have

$$F(E, \alpha_N) = f(E, \alpha_N, \lambda\omega)(\lambda\omega)^{\otimes k} = \lambda^k f(E, \alpha_N, \lambda\omega)\omega^k.$$

Combining the above calculation with (1.1), we get

$$f(E, \alpha_N, \lambda\omega) = \lambda^{-k} f(E, \alpha_N, \omega).$$

This allows a reformulation of definition 1.1.7:

Definition 1.1.9. A *weak modular form of weight k and level N over R* is a rule which assigns to each triple (E, α_N, ω) where

- E is an elliptic curve over R;
- α_N is a level-N structure;
- ω is a nonzero section of $\boldsymbol{\omega}_{E/S}$;

an element $f(E, \alpha_N, \omega) \in R$ satisfying the following conditions:

1. f depends only on the isomorphism class of (E, α_N, ω);
2. f commutes with arbitrary base change;
3. For any unit λ in R, we have $f(E, \alpha_N, \lambda\omega) = \lambda^{-k} f(E, \alpha_N, \omega)$.

Remarks 1.1.10. 1. As in remark 1.1.8, the level 1 case is essentially the same definition, without the level-N structure.

2. In [Kat73], Katz also defined the q-expansion of any weak modular forms over a ring R using the *Tate curve*. This curve is an elliptic curve over $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R$ where q is a formal parameter at the cusp, see §3 of chapter V of Silverman's book [Sil94] for more details. Loosely speaking, a modular form over R is a weak modular form whose value at the Tate curve lies in $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R$.

1.1.3 Analytic vs Katz modular forms

Level 1 case. A well known result from the theory of elliptic curves is that for any elliptic curves E over $\mathbb{C}$ there exists a $\mathbb{Z}$-lattice, i.e. a free $\mathbb{Z}$-submodule $\Lambda \subset \mathbb{C}$ of rank 2, such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ as complex Lie groups. Moreover, this lattice Λ is unique up to homothety. This fact is known as the uniformization theorem of elliptic curves. More precisely, given a $\mathbb{Z}$-lattice Λ, there exists a Λ-invariant meromorphic function $\wp_\Lambda : \mathbb{C} \to \mathbb{C}$, called the *Weierstrass $\wp$-function*, such that the following map

$$z + \Lambda \longmapsto [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1]$$

induces a complex Lie group isomorphism $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ where E is an elliptic curve which is given by the equation

$$Y^2 = 4X^3 - g_2(\Lambda)X - g_3(\Lambda),$$

with $g_2(\Lambda), g_3(\Lambda) \in \mathbb{C}$. The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ can be defined explicitely using Eisenstein series and depends on Λ. Conversely, given an elliptic curve $E/\mathbb{C}$ as above together with a nonzero differential ω, one may recover the lattice by considering the periods of ω:

$$\Lambda(E, \omega) := \left\{ \int_\nu \omega : \nu \in H_1(E(\mathbb{C}), \mathbb{Z}) \right\}.$$

Taking ω to be the differential dX/Y on E, one may observe that it pulls back to the Λ-invariant holomorphic differential $\mathrm{d}z$ on $\mathbb{C}/\Lambda$. One sees that replacing ω by $\lambda\omega$ for some nonzero $\lambda \in \mathbb{C}^\times$ has the effect of rescaling the lattice Λ by a factor of λ. Next, note that by choosing a $\mathbb{Z}$-basis $\{\nu_1, \nu_2\}$ of the homology group $H_1(E, \mathbb{Z})$ we have

$$\Lambda(E, \omega) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$$

where $w_i := \int_{\nu_i} \omega$. We may moreover suppose without loss of generality that $\mathrm{Im}(w_1/w_2) > 0$ (i.e. $w_1/w_2 \in \mathcal{H}$) so that we have

$$w_2^{-1}\Lambda(E, \omega) = \Lambda_w := \mathbb{Z}w \oplus \mathbb{Z}$$

where $w := w_1/w_2 \in \mathcal{H}$. One may show that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we will have

$$\Lambda_{\gamma(w)} = (cw+d)\Lambda_w.$$

From the above discussion, if we have a level 1 Katz modular form $f : (E, \omega) \mapsto f(E, \omega)$ over $\mathbb{C}$, then we can define an elliptic modular form $f_\infty : \mathcal{H} \to \mathbb{C}$ by $f_\infty(w) := f(\Lambda_w, \mathrm{d}z)$.

Level N case. Let $E/\mathbb{C}$ be an elliptic curve uniformized by the lattice $\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ and let $N > 1$ be an integer. Given a tuple $(E/\mathbb{C}, \alpha_N)$ where $\alpha_N : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$ is a level-N structure. The choice of such level-N structure is equivalent to the choice of an ordered basis of $E[N]$. Moreover, using that E is uniformized by Λ, we get the following group isomorphism:

$$E[N] \cong \langle w_1/N + \Lambda \rangle \times \langle w_2/N + \Lambda \rangle.$$

From this isomorphism, we consider the Weyl pairing

$$e_N : E[N] \times E[N] \to \mu_N$$

where μ_N is the group of N-th roots of unity. Recall that the Weyl pairing is defined by

$$e_N(P, Q) := e^{2\pi i \det(\gamma)/N},$$

where γ is a 2×2 matrix with coefficients in $\mathbb{Z}/N\mathbb{Z}$ such that

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \gamma \begin{pmatrix} w_1/N + \Lambda \\ w_2/N + \Lambda \end{pmatrix}.$$

Definition 1.1.11. We define $\mathcal{E}(N)/\mathbb{C}$ to be the set of isomorphism classes of pairs $(E, (P, Q))$ where $E/\mathbb{C}$ is an elliptic curve and P and Q are two N-torsion points of E such that $e_N(P, Q) = e^{2\pi i/N}$.

Proposition 1.1.12. *The following map is a bijection:*

$$\begin{aligned} \Gamma(N) \setminus \mathcal{H} &\longrightarrow \mathcal{E}(N) \\ [w] &\longmapsto (\mathbb{C}/\Lambda_w, (w/N + \Lambda_w, 1/N + \Lambda_w)). \end{aligned}$$

By the above discussion, an algebraic weak modular form f over $\mathbb{C}$ as given by definition 1.1.9 can be seen as a function on triples

$$f : (E/\mathbb{C}, (P, Q), \omega) \mapsto f(E/\mathbb{C}, (P, Q), \omega) \in \mathbb{C},$$

where ω is a nonzero differential on E. Via proposition 1.1.12, we define a function $f_\infty : \mathcal{H} \to \mathbb{C}$ by

$$f_\infty(w) := f(\mathbb{C}/\Lambda_w, (w/N + \Lambda_w, 1/N + \Lambda_w), \mathrm{d}z).$$

For similar reasons as in the level 1 case, the function f_∞ is an analytic weak modular form for $\Gamma(N)$.

1.2 CM elliptic curves

Let $K/\mathbb{Q}$ be an imaginary quadratic field with ring of integer $\mathcal{O}_K$ and fix an embedding $K \hookrightarrow \mathbb{C}$. An elliptic curve $E/\mathbb{C}$ is said to have complex multiplication (or CM) by $\mathcal{O}_K$ if $\mathrm{End}(E) \cong \mathcal{O}_K$. Letting $\mathrm{Ell}(\mathcal{O}_K)$ denote the isomorphism classes of elliptic curves over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_K$, one may show that the ideal class group $\mathrm{Cl}(\mathcal{O}_K)$ is in one-to-one correspondence with $\mathrm{Ell}(\mathcal{O}_K)$. Indeed, given a fractional ideal $\mathfrak{a}$, we may view it as a lattice in $\mathbb{C}$ and consider the elliptic curve $\mathbb{C}/\mathfrak{a}$. In particular, we have that $E/\mathbb{C}$ has *CM* by $\mathcal{O}_K$ if and only if $E(\mathbb{C}) \cong \mathbb{C}/\Lambda_w$ where $\Lambda_w = \mathbb{Z} + w\mathbb{Z}$ and $w \in K$. In this case, we say that w is a *CM point* of $\mathcal{H}$.

The theory of CM elliptic curve is important from an arithmetic viewpoint because of the following result:

Theorem 1.2.1. *Let $E/\mathbb{C}$ be an elliptic curve with complex multiplication by $\mathcal{O}_K$ and j-invariant $j(E)$, then $\mathbb{Q}(j(E))$ is the Hilbert class field of K.*

This theorem may be viewed as a *rank* 2 (the $\mathbb{Z}$-rank of the lattice attached to an elliptic curve) version of the Kronecker-Weber theorem which states that any finite abelian extension of $\mathbb{Q}$ is contained within some cyclotomic field. A consequence of theorem 1.2.1 is that any elliptic curve with CM by $\mathcal{O}_K$ is isomorphic with one that is defined over the Hilbert class field of K.

1.3 Special values at CM points

The goal of this section is to give a proof of the following theorem:

Theorem 1.3.1. *Let $w \in \mathcal{H}$ be a CM point and let $f : \mathcal{H} \to \mathbb{C}$ be a normalized eigenform of weight k, level $N \geq 1$ defined over a number field $\mathbb{Q}_f$, then there exists a nonzero period*

$\boldsymbol{\Omega_w} \in \mathbb{C}^\times$ *such that*

$$\frac{f(w)}{\mathbf{\Omega}_w^k} \in \mathbb{Q}_f H_{\mathbb{Q}(w)}.$$

Proof. Since $w \in \mathcal{H}$ is a CM point, it corresponds to a elliptic curve $E_w/\mathbb{C}$ having CM by $\mathbb{Q}(w)$. By theorem 1.2.1, we may suppose that E_w is defined over the Hilbert class field of $\mathbb{Q}(w)$, denoted H_w.

Next, since f is defined over $\mathbb{Q}_f$, it will correspond to an algebraic modular form f_0 as in definition 1.1.9. Let $g : H_w \to \mathbb{C}$ be an embedding, then the pullback of the sheaf $\boldsymbol{\omega}_{E_w/H_w}$ by g is $g^*\boldsymbol{\omega}_{E_w/H_w} = \mathbf{\Omega}_w \mathrm{d}z$ for some nonzero period $\mathbf{\Omega}_w \in \mathbb{C}^\times$. Therefore, we have

$$g^* f_0(E_w, \alpha_N, \omega) = f_0(g^* E_w, g^* \alpha_N, \mathbf{\Omega}_w \mathrm{d}z) = \mathbf{\Omega}_w^{-k} f(w) \in \mathbb{Q}_f H_{\mathbb{Q}(w)}.$$

□

This theorem was first proven by Shimura in [Shi75] using different techniques. For reference, the proof described above can be found in [Urb14, §2.6]. The above proof will constitutes a base frame for our proof in the Drinfeld setting, which can be found in section 4.6.

To conclude this section, we note that this theorem is interesting from an arithmetic point of view because it relates modular forms, a notion of analytic flavor, with the Hilbert class field of a quadratic imaginary field. In arithmetic, the Hilbert class field is an interesting tool since its Galois group over the base field is isomorphic to the ideal class group of the base field.

Chapter 2

Drinfeld Modules

The origin of Drinfeld modules first date back to the work of Carlitz in 1935 who introduced a function field analogue of cyclotomic polynomials [Car35]. Later, Drinfeld formalized the concept by introducing what he called *elliptic modules* [Dri74]. Drinfeld used them in order to prove a special cases of the Langlands conectures for GL_2 for a function field.

Throughout this chapter and the subsequent ones, we let $X/\mathbb{F}_q$ be a smooth projective curve over a field of cardinailty q and let K be the function field of X.We also fix ∞ to be a closed point on X and define A to be the ring of functions which are regular outside infinity. We let K_∞ to be the completion of K with respect to ∞ and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞.

2.1 The ring of $\mathbb{F}_q$-linear polynomials

Let F be any field of characteristic p. We denote by $\tau : \alpha \mapsto \alpha^q$ the Frobenius endomorphism. We define $F\{\tau\}$ to be the endomorphism ring of elements of the form

$$\alpha_0\tau^0 + \alpha_1\tau + \cdots + \alpha_n\tau^n,$$

for $n \geq 0$ and $\alpha_i \in F$. Note that the multiplication in $F\{\tau\}$ is the composition of morphisms, so this ring is not commutative in general:

$$\tau\alpha = \alpha^q\tau. \tag{2.1}$$

An element of $F\{\tau\}$ will be called a *τ-polynomial over F*.

Proposition 2.1.1. *The set $F\{\tau\}$ is in bijection with the set of $\mathbb{F}_q$-linear polynomials over F, that is the set of univariate polynomials P over F such that $P(\alpha + c\beta) = P(\alpha) + cP(\beta)$ for every $\alpha, \beta \in F$ and $c \in \mathbb{F}_q$.*

Proof. The bijection is given by the identity on the scalars and by sending τ^i to X^{q^i} where X is the variable of a univariate polynomial ring over F. It is now clear that any τ-polynomial under this map is sent to a $\mathbb{F}_q$-linear polynomial. Moreover, one can prove that any $\mathbb{F}_q$-linear polynomial is of the form $\sum_{i=0}^{n} \alpha_i X^{q^i}$, see for example Prop. 1.1.5 of [Gos96]. □

Remark 2.1.2. The elements of these two sets will sometimes be identified. However, one needs to be careful here as the ring structures are incompatible. In order to differentiate them, if $P(\tau)$ is a τ-polynomial, we will write $P(X)$ to denote its representation as a $\mathbb{F}_q$-linear polynomial. Moreover, if $P(X)$ is a $\mathbb{F}_q$-linear polynomial, we will write $P(\tau)$ to denote its reprentation as a τ-polynomial. In particular, for two such polynomials P and Q, the notation $P(\tau)Q(\tau)$ will mean the multiplication with the twisted rule (2.1) and the notation $P(X)Q(X)$ will mean the usual polynomial multiplication.

Definition 2.1.3. Let P be a τ-polynomial over F given by

$$P(\tau) = \alpha_0\tau^0 + \alpha_1\tau + \cdots + \alpha_n\tau^n, \quad \alpha_i \in F,\ \alpha_n \neq 0.$$

We define $\deg_\tau(P(\tau)) := n$. Note that we have $\deg(P(X)) = q^{\deg_\tau(P(\tau))}$, where deg is the usual degree of a univariate polynomial. We also define the derivative $D_X : F\{\tau\} \to F$ by

$$D_X P(\tau) := \frac{\mathrm{d}}{\mathrm{d}X} P(X).$$

Since F is of characteristic q, we have $D_X P(\tau) = a_0$.

Proposition 2.1.4. *We have $F\{\tau\}^\times = F^\times$.*

Proof. Let $P(\tau)$ be a unit in $F\{\tau\}$ and suppose that $\deg_\tau(P) > 0$. Then there exists $Q(\tau)$ such that $(P \cdot Q)(\tau) = \tau^0$. Letting $d_P := \deg_\tau(P)$ and $d_Q := \deg_\tau(Q)$, we see that $0 = \deg_\tau(P \cdot Q) = d_Q^{d_P}$, hence we must have $d_Q = 0$ and $Q(\tau) = c\tau^0$ for some nonzero $c \in F$. However, since $\tau c = c^q \tau$, this implies that $\deg_\tau(P \cdot Q) = \deg_\tau(P) > 0$, a contradiction. □

Proposition 2.1.5. *Every left ideal of $F\{\tau\}$ is principal. Moreover, if F is perfect, that is τ is an automorphism of F, then every right ideal of $F\{\tau\}$ is principal.*

Proof. This is corollaries 1.6.3 and 1.6.6 in [Gos96]. □

Remarks 2.1.6. 1. The ring $F\{\tau\}$ is in fact the endomorphism ring of the additive group scheme $\mathbb{G}_a/F$;

2. The divisibility properties of this ring were first studied by Ore in [Ore33]. More generally, given a ring R, Ore defined the ring $R[X;\sigma,\delta]$ as the ring of univariate polynomial in X over R satisfying the following rule:

$$Xr = \sigma(r)X + \delta(r),\ \forall r \in R,$$

where $\sigma : R \to R$ is an automorphism and $\delta : R \to R$ is a σ-derivation, i.e. a map satisfying the σ-twisted Leibniz rule:

$$\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'.$$

In our case, σ is the q-Frobenius and the σ-derivation δ is the zero map. In literature, this ring is sometime called a *skew polynomial ring* or an *Ore polynomial ring.*

2.2 Analytic theory

Definition 2.2.1. An *A-lattice in* $\mathbb{C}_\infty$ is a finitely generated A-submodule $\Lambda \subset \mathbb{C}_\infty$ which is discrete for the topology of $\mathbb{C}_\infty$. We define the *rank* of Λ, denoted $\mathrm{rank}(\Lambda)$ to be its rank as a finitely generated projective submodule of $\mathbb{C}_\infty$.

Definition 2.2.2. Let Λ_1 and Λ_2 be two A-lattices of the same rank. We define a *morphism from* Λ_1 *to* Λ_2 to be an element $c \in \mathbb{C}_\infty$ such that $c\Lambda_1 \subset \Lambda_2$. If we have equality, that is $c\Lambda_1 = \Lambda_2$, then we say that Λ_1 and Λ_2 are *homothetic.* The category of A-lattices of rank r in $\mathbb{C}_\infty$ with the given notion of morphism will be denoted by $\mathrm{LAT}_{r,A}(\mathbb{C}_\infty)$.

Example 2.2.3. Assume that A has class number equal to 1. In particular, the ring A will be a principal ideal domain and thus every A-lattice of rank $r > 0$ in $\mathbb{C}_\infty$ will be of the form

$$\Lambda = w_1A + w_2A + \cdots + w_rA$$

for some K_∞-linearly independent elements $w_i \in \mathbb{C}_\infty$, $1 \le i \le r$.

Definition 2.2.4. Let Λ be an A-lattice. For every $z \in \mathbb{C}_\infty$, We define the *exponential attached to* Λ by the product

$$e_\Lambda(z) := z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right).$$

Proposition 2.2.5. *1. $e_\Lambda(z)$ converges for every $z \in \mathbb{C}_\infty$;*

2. $e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a nonconstant entire function. In particular, it is surjective;

3. e_Λ is $\mathbb{F}_q$-linear.

Proof. See propositions 4.2.4 and 4.2.5 of [Gos96]. □

Corollary 2.2.6. *The function $e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ induces an isomorphism of abelian groups*

$$\mathbb{C}_\infty/\Lambda \cong \mathbb{G}_a(\mathbb{C}_\infty). \tag{2.2}$$

Proof. By definition, we have $\ker(e_\Lambda) = \Lambda$. The result follows from the first isomorphism theorem. □

Remark 2.2.7. Returning to the classical case, we recall that any complex elliptic curve satisfies $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ for some rank 2 lattice $\Lambda \subset \mathbb{C}$. Thus, we will see (2.2) as an analogue of this isomorphism. However, we point out that there is a crucial difference here as $\mathbb{C}/\Lambda$ is not the whole complex space but a torus.

Definition 2.2.8. Let $a \in A$ be any nonzero element and $\Lambda \subset \mathbb{C}_\infty$ any A-lattice. We define the following quantity

$$\phi_a^\Lambda(X) := aX \prod_{\substack{\lambda \in a^{-1}\Lambda/\Lambda \\ \lambda \neq 0}} \left(1 - \frac{X}{e_\Lambda(\lambda)}\right).$$

Theorem 2.2.9. *Let $\Lambda \subset \mathbb{C}_\infty$ be a A-lattice and let $a \in A$ be any nonzero element. Then*

1. $\phi_a^\Lambda(X)$ is a $\mathbb{F}_q$-linear polynomial;

2. For $z \in \mathbb{C}_\infty$, we have the following functional equation

$$e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z)).$$

Proof. See theorem 4.3.1 in [Gos96]. □

Definition 2.2.10. The map $\phi^\Lambda : A \to \mathbb{C}_\infty\{\tau\}$, $a \mapsto \phi_a^\Lambda$ is called the *Drinfeld module associated to* Λ.

Proposition 2.2.11. *Let ϕ^{Λ_1} and ψ^{Λ_2} be two Drinfeld modules associated to the A-lattices Λ_1 and Λ_2 respectively. Let $c \in \mathbb{C}_\infty$ be such that $c\Lambda_1 \subset \Lambda_2$ (i.e. c is a morphism from Λ_1 to Λ_2). Then, the element c corresponds to a τ-polynomial $P_c(\tau)$ with coefficient in $\mathbb{C}_\infty$ such that*

1. $P_c(\tau)\phi_a^{\Lambda_1}(\tau) = \phi_a^{\Lambda_2}(\tau)P_c(\tau)$ *for all* $a \in A$;
2. $D_X P_c(\tau) = c$.

Proof. See [Gos96, Proposition 4.3.5]. □

Example 2.2.12. In this example, we let $A = \mathbb{F}_q[T]$ and K be its fraction field. In 1935, before the development of the notion of Drinfeld modules, Carlitz proved that the infinite product

$$z \prod_{\substack{\lambda \in A \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right)$$

converges [Car35]. Moreover, one can show that there exists a (not necessarily unique) normalization period $\tilde{\pi} \in \mathbb{C}_\infty$ such that the exponential $e_{\tilde{\pi}A}$ possesses the following expansion:

$$e_{\tilde{\pi}A}(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i},$$

where D_i is the product of all monic polynomials of degree equal to i. The period $\tilde{\pi}$ is defined up to a $(q-1)$-root of the polynomial $T^q - T$ in $\overline{K}_\infty$. Throughout the rest of this thesis, we fix such period and call it the *Carlitz period*. The Drinfeld module attached to the lattice $\tilde{\pi}A$ will be called the *Carlitz module* and we have

$$\phi_T^{\tilde{\pi}A}(\tau) = T + \tau.$$

A detailed exposition for this explicit example can be found in [Gos96, Chapter 3].

2.3 Algebraic theory

In this section, we fix a morphism of schemes $j : S \to \mathrm{Spec}(A)$ so that S is a A-scheme. We will denote its structural sheaf by $\mathcal{O}_S$. For any $a \in A$, we define $\deg(a) := |A/(a)|$.

Definition 2.3.1. A *Drinfeld module of rank r over S* is a pair (L, ϕ) consisting of a line bundle L over S and a ring homomorphism

$$\begin{aligned} \phi : A &\longrightarrow \mathrm{End}_S(L, +) \\ a &\longmapsto \phi_a \end{aligned}$$

subject to the following condition: there exists a trivialisation of L by open affine A-subschemes $\mathrm{Spec}(B)$ of S such that

$$\phi_a|_{\mathrm{Spec}(B)} = \sum_{i=0}^{r\deg(a)} b_i \tau^i$$

with $b_i \in B$ such that

1. $b_0 = i(a)$ where $i : A \to B$ is the associated morphism of ring;
2. $b_{r\deg(a)}$ is a unit in B.

If $S = \mathrm{Spec}(B)$ is an affine scheme for an A-algebra B, we say that the pair (L, ϕ) is *defined over B*.

Definition 2.3.2. A field F with a A-algebra structure given by a morphism $i : A \to F$ will be called a *A-field*. The kernel of i is called the *characteristic* of F. If $\ker(i) = (0)$ (i.e. i is injective), then we say that F has *generic characteristic*, otherwise, we say that F is *finite* and F has *finite characteristic*.

Example 2.3.3 (Drinfeld modules over A-fields)**.** Let F be a A-field. Since any line bundle over $S = \mathrm{Spec}(F)$ is trivial, a Drinfeld module over F will only be determined by the morphism $\phi : A \to F\{\tau\}$. Moreover, since the $\mathbb{F}_q$-algebra A is finitely generated, we simply need to know the values of the morphism ϕ at a set of generators. For example, if $A = \mathbb{F}_q[T]$, then a Drinfeld module ϕ of rank r over F is uniquely determined by

$$\phi_T = i(T)\tau^0 + g_1\tau + g_2\tau^2 + \cdots + g_r\tau^r,$$

where each g_i are elements of F.

Definition 2.3.4. Let (L, ϕ) and (L', ϕ') be two Drinfeld module over S. A *morphism from (L, ϕ) to (L', ϕ')* is an element $P \in \mathrm{Hom}_{\mathbb{F}_q}(L, L')$ such that $P\phi = \phi' P$. A nonzero morphism is called an *isogeny*.

In the affine case, where ϕ and ϕ' are defined over a A-field F, then a morphism from ϕ to ϕ' is simply a τ-polynomial P in $F\{\tau\}$ such that $P(\tau)\phi_a(\tau) = \phi'_a(\tau)P(\tau)$ for all $a \in A$. Furthermore, if $A = \mathbb{F}_q[T]$, then the commuting condition needs to be verified only at the generator $a = T$.

Proposition 2.3.5. *Let $\phi : \mathbb{F}_q[T] \to F\{\tau\}$ be a rank r Drinfeld $\mathbb{F}_q[T]$-module over a $\mathbb{F}_q[T]$-field F given by*

$$T \mapsto g_0(\phi) + g_1(\phi)\tau + \cdots + g_r(\phi)\tau^r.$$

Then, for any $c \in F^\times$ and any $i \in \{0, \ldots, r\}$, we have $g_i(c^{-1}\phi c) = c^{q^i-1}g_i(\phi)$.

Proof. This is proven by a direct calculation. Indeed, let $c \in F^\times$ and let $i \in \{1, \ldots, r\}$, then we have

$$c^{-1}g_i(\phi)\tau^i c = c^{-1}g_i(\phi)c^{q^i}\tau^i = c^{q^i-1}g_i(\phi)\tau^i,$$

which shows that $c^{-1}\phi c = g_0(\phi) + c^{q-1}g_1(\phi)\tau + \cdots + c^{q^r-1}g_r(\phi)\tau^r$. □

Definition 2.3.6. We define $\mathrm{DM}_{r,A}(S)$ to be the category where the objects are Drinfeld A-modules of rank r over an A-scheme S and the arrows are the morphisms as given by definition 2.3.4.

2.4 Uniformization of Drinfeld A-modules over $\mathbb{C}_\infty$

2.4.1 Analytic uniformization

In this section, our Drinfeld modules will be over $\mathbb{C}_\infty$. Recall that in section 2.2, for any A-lattice Λ we defined the associated Drinfeld module, denoted ϕ^Λ. This construction gives us a functor

$$\begin{aligned} \mathcal{U} : \mathrm{LAT}_r(\mathbb{C}_\infty) &\longrightarrow \mathrm{DM}_{r,A}(\mathbb{C}_\infty) \\ \Lambda &\longmapsto \phi^\Lambda \\ (c\Lambda \subset \Lambda') &\longmapsto P_c(\tau) \end{aligned}$$

where $P_c(\tau)$ is the τ-polynomial given by proposition 2.2.11.

Theorem 2.4.1 (Analytic Uniformization)**.** *The functor $\mathcal{U}$ is an equivalence of categories.*

Proof. We refer the reader to theorem 4.6.9 of [Gos96] for the proof of that theorem. □

Remark 2.4.2. One main aspect of the proof of theorem 2.4.1 involves constructing the logarithm associated to ϕ. We will define and construct the logarithm in this remark for the special case $A = \mathbb{F}_q[T]$. Let $\phi : A \to \mathbb{C}_\infty\{\tau\}$ be a rank r Drinfeld module over $\mathbb{C}_\infty$ given by

$$\phi : T \mapsto T + g_1\tau + \cdots + g_r\tau^r.$$

Then, we know that the exponential of ϕ, if it exists, is a power series of the form

$$e_\phi(z) = z + \sum_{i\geq 1} \alpha_i z^{q^i}.$$

Supposing that the exponential exists, then there exists a compositional inverse, called the logarithm of ϕ:

$$\log_\phi(z) = z + \sum_{i\geq 1} \beta_i z^{q^i}.$$

Fixing a nonzero element a in A, we apply $\log_\phi$ on both sides of the functional equation

$$e_\phi(az) = \phi_a(e_\phi(z))$$

and we obtain

$$a \log_\phi(z) = \log_\phi(\phi_a(z)).$$

Comparing the coefficients on both sides of the above equation yields a recursive procedure:

$$a\beta_i = \sum_{n+m=i} \beta_n g_m^{q^n}.$$

In particular, this procedure allows the construction of the logarithm using only the knowledge of the coefficients $g_1, \ldots, g_r$ (it does not depend on the exponential). We may then *define* the exponential of ϕ to be the compositional inverse of the logarithm constructed as so and then define $\Lambda_\phi := \ker(e_\phi)$. This last object will be the lattice associated with ϕ, showing the surjectivity of the map $\Lambda \mapsto \phi^\Lambda$.

2.4.2 The Drinfeld period domain

An important use of analytic uniformization is to describe the isomorphism classes of Drinfeld modules as a quotient of a Drinfeld period domain. To put things in perspective, we recall that, in the classical theory, the set of isomorphism classes of elliptic curves over $\mathbb{C}$ is in one-to-one correspondence with the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$. We will see in the coming sections that a similar fact holds true for Drinfeld modules.

Definition 2.4.3. We define the *Drinfeld period domain of rank r over* $\mathbb{C}_\infty$ to be

$$\Omega^r(\mathbb{C}_\infty) := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}.$$

Remarks 2.4.4.

1. The space $\Omega^r(\mathbb{C}_\infty)$ admits the structure of a rigid analytic space and we may therefore perform analysis on it. In the next chapter, we will study Drinfeld modular forms which will be rigid analytic function $f : \Omega(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ satisfying a transformations properties under a group action.

2. Any element $w \in \Omega^r(\mathbb{C}_\infty)$ is represented uniquely of the form $w = [w_1 : \ldots : w_{r-1} : 1]$. Hence, from now on, we will identify the elements of $\Omega^r(\mathbb{C}_\infty)$ with the set of column vectors
$$w = (w_1, \ldots, w_{r-1}, 1)^{\mathrm{T}}$$
such that $w_1, \ldots, w_{r-1}, 1$ are K_∞-linearly independent. If $r = 2$, then we simply have $\Omega^2(\mathbb{C}_\infty) = \mathbb{C}_\infty \setminus K_\infty$.

Definition 2.4.5. For any $\gamma \in \mathrm{GL}_r(K)$ and $w \in \Omega^r(\mathbb{C}_\infty)$, we define

$$\gamma(w) := j(\gamma, w)^{-1}\gamma w$$

where γw is the usual matrix multiplication and $j(\gamma, w)$ is the last entry of this multiplication.

We will see in the following chapter that the map $(\gamma, w) \mapsto \gamma(w)$ defines an action of $\mathrm{GL}_r(K)$ on $\Omega^r(\mathbb{C}_\infty)$.

2.4.3 Isomophism classes of Drinfeld $\mathbb{F}_q[T]$-modules over $\mathbb{C}_\infty$

Fix $r \geq 2$ an integer and let $A = \mathbb{F}_q[T]$. In this specific case, we will describe the set of isomorphism classes of rank r Drinfeld module over $\mathbb{C}_\infty$ as a quotient of the Drinfeld period domain.

We first observe that since A is a principal ideal domain, any A-lattice Λ will be free of rank r over A, that is

$$\Lambda = Aw_1 \oplus \cdots \oplus Aw_{r-1} \oplus Aw_r \cong A^r \tag{2.3}$$

as A-modules for some $w_i \in \mathbb{C}_\infty$ which are A-linearly independent. Note that the discreteness condition of Λ in $\mathbb{C}_\infty$ implies that $w_1, \ldots, w_r$ are linearly independent over the completion K_∞. Moreover, any A-lattice Λ of the form (2.3) is homothetic to the lattice

$$\Lambda' = A\tfrac{w_1}{w_r} \oplus \cdots \oplus A\tfrac{w_{r-1}}{w_r} \oplus A.$$

For any, $w \in \Omega^r(\mathbb{C}_\infty)$, we will denote by Λ_w the associated A-lattice.

Proposition 2.4.6. *Let w and w' be two point in $\Omega^r(\mathbb{C}_\infty)$. Then, Λ_w and $\Lambda_{w'}$ are isomorphic if and only if there exists $\gamma \in \mathrm{GL}_r(A)$ such that $\gamma(w) = w'$.*

Proof. Suppose there exists $c \in \mathbb{C}_\infty$ such that $c\Lambda_w = \Lambda_{w'}$ where $w = (w_1, \dots, w_{r-1}, w_r)^{\mathrm{T}}$ and $w' = (w'_1, \dots, w'_{r-1}, w'_r)^{\mathrm{T}}$ where $w_r = w'_r = 1$. Then, since the w_i and w'_i are A-linearly independent, this is equivalent to the existence of a matrix $\gamma := (a_{i,j})_{1\leq i,j\leq r}$ in $\mathrm{GL}_r(A)$ such that

$$\begin{cases} cw'_i = \sum_{j=1}^r a_{i,j}w_j, \quad \text{for} \;\; 0 \leq i \leq r-1; \\ \quad c = \sum_{j=1}^r a_{r,j}w_j \end{cases} \tag{2.4}$$

We may rewrite these equations in the form $cw' = \gamma w$ and observe that $c = j(\gamma, w)$. Therefore we get $w' = \gamma(w)$. We also get the converse of the proposition. □

Corollary 2.4.7. *For any $\gamma \in \mathrm{GL}_r(A)$ we have $\Lambda_{\gamma(w)} = j(\gamma, w)^{-1}\Lambda_w$.*

Proof. It is a direct consequence of equations (2.4). □

Definition 2.4.8. Let L be a A-field. We denote the set of isomorphism classes of Drinfeld modules over L of rank r by $M_1^r(L)$.

Corollary 2.4.9. *We have a one-to-one bijection between the two sets:*

$$M_1^r(\mathbb{C}_\infty) \longleftrightarrow \mathrm{GL}_r(A) \setminus \Omega^r(\mathbb{C}_\infty).$$

Proof. By uniformization, the isomorphism classes of Drinfeld module over $\mathbb{C}_\infty$ corresponds to the isomorphism classes of A-lattices which is given by proposition 2.4.6. □

2.5 Drinfeld modular varieties

In light of the last section, we now define the Drinfeld modular curve with level stucture. To some extent, corollary 2.4.9 described the level-1 Drinfeld modular curve. In parallel with the classical theory, we can add extra structure to the isomorphism classes in order to obtain the level-N Drinfeld modular curve, where N is an ideal of A. This new object is more precisely a fine moduli space which can be understood via the notion of representable functors.

Definition 2.5.1. Let SCH be the category of schemes and SETS be the category of sets. A functor $\mathcal{F} : \mathrm{SCH} \to \mathrm{SETS}$ is said to be *representable by a scheme* X if $\mathcal{F}$ is isomorphic to the functor $\mathrm{Hom}_{\mathrm{SCH}}(-, X)$. In this case, we will have

$$\mathcal{F}(S) = \mathrm{Hom}_{\mathrm{SCH}}(S, X).$$

for any scheme S.

Remark 2.5.2. The set $\mathrm{Hom}_{\mathrm{SCH}}(S, X)$ is often denoted simply by $X(S)$ and called the *set of S-valued points of X*. If $S = \mathrm{Spec}(R)$ for a ring R, we use $X(R) := X(S)$ and we say that it is the *set of R-valued points of X*.

For the rest of this section, we fix N to be any nonzero proper ideal of A.

Definition 2.5.3. Let S be a scheme over K and let $E = (L, \phi)$ be a Drinfeld module of rank r over S. A *level-N structure on E* is a A-linear isomorphism of group schemes over S

$$\alpha_N : \underline{(N^{-1}/A)^r} \xrightarrow{\sim} \phi[N] := \bigcap_{a \in N} \ker(\phi_a)$$

where $\underline{(N^{-1}/A)^r}$ is the constant group scheme over S with fibers $(N^{-1}/A)^r$.

Definition 2.5.4. For any $r \geq 1$ and any nonzero proper ideal N of A, we define the functor $\mathcal{F}_N^r : \mathrm{SCH} \to \mathrm{SETS}$ sending any scheme S over K to the set of isomorphism classes of pairs (E, α) where E is a Drinfeld module of rank r over S and α is a level-N structure.

Theorem 2.5.5 (Drinfeld)**.** *Let $r \geq 1$ and let N be a nonzero proper ideal of A. Then the functor $\mathcal{F}_N^r$ is representable by a scheme M_N^r over K. Furthermore, M_N^r is a smooth affine variety of dimension $r - 1$ of finite type over K.*

Proof. This is proposition 5.3 of [Dri74]. □

Definition 2.5.6. The K-scheme M_N^r will be called the *Drinfeld modular variety of rank r of level N*.

A consequence of the representability of the functor $\mathcal{F}_N^r$ by a scheme M_N^r is the existence of a universal object, called the universal Drinfeld module over M_N^r and denoted $(\mathcal{E}, \alpha_{N,\mathcal{E}})$, where $\mathcal{E} = (\mathcal{L}, \varphi)$ is a Drinfeld module over M_N^r and $\alpha_{N,\mathcal{E}} : (N^{-1}/A)^r \cong \varphi[N]$ is a level-N structure. In particular, this object yields the following useful property: for any A-field

F and any morphism $\iota : \mathrm{Spec}(F) \to M_N^r$, the pullback of $(\mathcal{E}, \alpha_{N,\mathcal{E}})$ by ι corresponds to a Drinfeld module of rank r over R together with a level-N structure.

The above construction describe the Drinfeld modular variety of level N, but one may also define it in a more general context, which we will briefly describe over the next few lines.

Definition 2.5.7. We denote the profinite completion of A by $\hat{A} := \prod_{\mathfrak{p}} A_{\mathfrak{p}}$ and the associated ring of finite adèles by $\mathbb{A}_K^f := \hat{A} \otimes_A K$. The *principal congruence subgroup of level* N is defined to be

$$\mathcal{K}(N) := \ker(\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(A/N)),$$

where $\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(A/N)$ is simply the reduction modulo N.

Definition 2.5.8. A subgroup $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$ is called *fine* if there exists a prime ideal $\mathfrak{p}$ such that the image of $\mathcal{K}$ in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent (i.e. for every element γ in the image, $\gamma - 1$ is nilpotent).

More generally, one can construct the Drinfeld modular variety for any fine subgroup $\mathcal{K}$ of $\mathrm{GL}_r(\hat{A})$. Denoted by $M_{\mathcal{K}}^r$, this variety over K is normal integral and affine, see [Pin13, §1] for more details. It turns out that, via analytic uniformization of Drinfeld modules, the set of $\mathbb{C}_\infty$-valued points of this variety can be made explicit:

Proposition 2.5.9. *Suppose that $\mathcal{K}$ is fine, then, we have an isomorphism of rigid analytic spaces*

$$M_N^r(\mathbb{C}_\infty) \cong \mathrm{GL}_r(K) \setminus (\Omega^r(\mathbb{C}_\infty) \times \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}).$$

Proposition 2.5.10. *Let $\mathcal{S}(\mathcal{K})$ be a set of double coset representatives of*

$$\mathrm{GL}_r(K) \setminus \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}$$

and set $\Gamma_s := \mathrm{GL}_r(K) \cap s\mathcal{K}s^{-1}$ for any $s \in \mathcal{S}(\mathcal{K})$. Then we have the following decomposition of connected components

$$M_{\mathcal{K}}^r \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{C}_\infty) = \bigsqcup_{s \in \mathcal{S}(\mathcal{K})} M_s$$

where $M_s(\mathbb{C}_\infty) \cong \Gamma_s \setminus \Omega^r(\mathbb{C}_\infty)$ as rigid spaces.

The proofs of the two last propositions may be found in various references, such as [Dri74, §6], [Gek86, Chapter II], [Hub13] or [Pin13, §1].

For the rest of the section, we use theorem 2.5.5 together with propositions 2.5.10 and 2.5.9 in order to describe how the associated lattice of a Drinfeld module scales with respect to the group action. We first observe that the isomorphism classes of pairs (ϕ, α_N); ϕ is a Drinfeld module over $\mathbb{C}_\infty$ and α_N a level-N structure; is parametrized by a disjoint union of rigid spaces of the form $\Gamma_s \setminus \Omega^r(\mathbb{C}_\infty)$. More precisely, if $(w, s) \in \Omega^r(\mathbb{C}_\infty) \times \mathrm{GL}_r(\mathbb{A}_K^f)$, then we may define the A-lattice

$$\Lambda_w^s := w^*(K^r \cap s\hat{A}^r)$$

where $w^* : K_\infty^r \to \mathbb{C}_\infty$ is the K_∞-linear map

$$(a_1, \ldots, a_{r-1}, a_r) \longmapsto a_1 w_1 + \cdots + a_{r-1} w_{r-1} + a_r.$$

This A-lattice Λ_w^s comes equipped with a level-N structure

$$\alpha_N^s : (N^{-1}/A)^r \xrightarrow{\sim} N^{-1}\Lambda_w^s/\Lambda_w^s.$$

In particular, if w, w' lie in $\Omega^r(\mathbb{C}_\infty)$ and s is a certain representative of $\mathrm{GL}_r(K)\backslash\mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}(N)$, then Λ_w^s and $\Lambda_{w'}^s$ will be homothetic if and only if $w' = \gamma(w)$ for some $\gamma \in \Gamma_s$. Furthermore, from a similar calculation as in proposition 2.4.6, we get the following:

Proposition 2.5.11. *Let s be a representative of the double coset* $\mathrm{GL}_r(K) \setminus \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}(N)$ *and let $w, w' \in \Omega^r(\mathbb{C}_\infty)$. Then,*

1. *Λ_w^s and $\Lambda_{w'}^s$ are homothetic if and only if $w' = \gamma(w)$ for some $\gamma \in \Gamma_s$;*
2. *For any $\gamma \in \Gamma_s$, we have $\Lambda_{\gamma(w)}^s = j(\gamma, w)^{-1}\Lambda_w^s$.*

Therefore, if we endow the quotient $\mathbb{C}_\infty/\Lambda_w^s$ with a Λ_w^s-invariant differential $\mathrm{d}z$, then, replacing w by $\gamma(w)$ for some $\gamma \in \Gamma_s$ will have the effect of replacing $\mathrm{d}z$ by $j(\gamma, w)^{-1}\mathrm{d}z$.

Remark 2.5.12. In the next chapter, we will define Drinfeld modular forms both analytically and algebraically. The above proposition will be useful for our purpose, more precisely in section 3.4.3, in order to pass from one definition to the other.

Chapter 3

Drinfeld Modular Forms

Drinfeld modular forms were first introduced by Goss in his seminal thesis [Gos80b]. He studied both the analytic and algebraic versions of Drinfeld modular forms. He moreover defined explicit rank 2 examples such as the Eisentein series and proved their holomorphicity at infinity. A few years later, Gekeler studied the coefficients expansions at infinity of some explicit rank two forms [Gek88]. Despite the fact that Goss defined them for arbitrary ranks (i.e. ranks > 2), it tooks nearly thirty years until the higher rank theory was further developed. In particular, we cite the works of Basson, Breuer and Pink [Bas14], [BBP18a], [BBP18b], [BBP18c] and the work of Gekeler [Gek17]. We note that a particular breakthrough is attributed to Pink for the construction of a Satake compactification of the Drinfeld modular variety of rank r [Pin13]. Pink used this compactification in order to define algebraic Drinfeld modular forms as global sections of a certain sheaf on this variety.

In this chapter, we cover both the analytic and algebraic (à la Pink) definition of Drinfeld modular forms of arbitrary ranks. We will also give some classical examples in order to shed a bit of light on this notion.

3.1 Rigid analysis on $\mathbb{C}_\infty$

The goal of this section is to give some intuition about the theory of rigid analytic spaces which was originally developed by Tate [Tat71]. Our main reference will be the notes of Schneider [Sch98].

The idea behind rigid analysis is to solve the problem that the topology of a nonarchimedean complete field (take for example $\mathbb{Q}_p$) is totally disconnected. This implies that we have a large supply of functions which are locally constant. Roughly, Tate's idea was to *rigidify* the topology in order to develop a good notion of analyticity. We will specialize the definitions over $\mathbb{C}_\infty$, but the theory also generalize for arbitrary nonarchimedean complete fields.

First, we define the Tate algebra of n variables

$$\mathbb{C}_\infty\langle X_1, \dots, X_n\rangle := \{\sum_{i_1,\dots,i_n \geq 0} \alpha_{i_1,\dots,i_n} X_1^{i_1} \cdots X_n^{i_n} : \alpha_{i_1,\dots,i_n} \to 0 \text{ as } |i_1 + \dots + i_n| \to \infty\}.$$

By nonarchemedean analysis, this ring describes the multivariate power series converging on the n-dimensional polydisk

$$B^n := \{(z_1, \dots, z_n) \in \mathbb{C}_\infty^n : \max\{|z_i| \leq 1\}\}.$$

The Tate algebra enjoy many interesting properties. For instance, it is noetherian and a unique factorisation domain, Also, the maximal ideals of $\mathbb{C}_\infty\langle X_1, \dots, X_n\rangle$ are in bijection with B^n. An *affinoid algebra* $\mathcal{A}$ is consists of the Tate algebra quotiented by a (finitely generated) ideal. For the rest of the section, we let $\mathcal{X} := \mathrm{MaxSpec}(\mathcal{A})$ consisting in the set of maximal ideals of an affinoid algebra $\mathcal{A}$. Moreover, for $f \in \mathcal{A}$ and a maximal ideal $x \in \mathcal{X}$, we define $f(x) := f + x \in \mathcal{A}/x$ which we view as an element of $\mathbb{C}_\infty$ after fixing an embedding $A/x \hookrightarrow \mathbb{C}_\infty$.

Definition 3.1.1. For any element $g, f_1 \dots, f_m \in \mathcal{A}$, we define

$$\mathcal{X}(\frac{f_\cdot}{g}) := \{x \in \mathcal{X} : \max_i\{|f_i(x)|\} \leq |g(x)|\}.$$

Any subset of $\mathcal{X}$ defined as above is called a *rational subdomain*.

The rational subdomains play a important role in rigid analysis for defining the so-called *Grothendieck topology*. This last concept is not an actual topology, but a category equipped with a notion of open set called *admissible open* and a notion of open covering called *admissible covering*. We refer the reader to Schneider's notes for definitions [Sch98]. One can define a presheaf on the rational subdomains of $\mathcal{X}$, denoted $\mathcal{O}_\mathcal{X}$, such that

$$\mathcal{O}_\mathcal{X}(\mathcal{X}(\frac{f_\cdot}{g})) := \mathcal{A}(\frac{f_\cdot}{g})$$

where $\mathcal{A}(\frac{f_\cdot}{g}) := \mathcal{A}\langle Y_1, \dots, Y_n\rangle / \langle gY_1 - f_1, \dots, gY_n - f_n\rangle$. We note that $\mathcal{A}\langle Y_1, \dots, Y_n\rangle$ is the multivariate power series ring in the variables $Y_1, \dots, Y_n$ with coefficients in $\mathcal{A}$ tending towards zero.

An important result by Tate is that the presheaf $\mathcal{O}_{\mathcal{X}}$ extends formally to a sheaf on $\mathcal{X}$ with respect to the Grothendieck topology. We illustrate this in the case when $n = 1$. Suppose that $\mathcal{X} = \mathrm{MaxSpec}(\mathbb{C}_\infty\langle X_1\rangle)$, then the rational subdomain $\mathcal{X}(\frac{1}{X_1})$ is sent to the annulus:

$$\begin{aligned}\mathcal{O}_{\mathcal{X}}(\mathcal{X}(\frac{1}{X_1})) &= \mathbb{C}_\infty\langle X_1, X_1^{-1}\rangle \\ &= \{\sum_{i\in\mathbb{Z}} \alpha_i X_1^i : |a_i| \to 0 \text{ as } |i| \to \infty\}\end{aligned}$$

and the global sections are whole Tate algebra $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = \mathbb{C}_\infty\langle X_1\rangle$.

Definition 3.1.2. An *affinoïd subdomain* consists in the set $\mathcal{X} = \mathrm{MaxSpec}(\mathcal{A})$ for some affinoïd algebra $\mathcal{A}$ endowed with the Grothendieck topology together with the structural sheaf $\mathcal{O}_{\mathcal{X}}$.

In analogy with the notion of complex variety, these affinoïd subdomains can be seen as the building blocks for a general rigid analytic space. More precisely, a set $\mathcal{X}$ is a *rigid analytic space* over $\mathbb{C}_\infty$ if it admits an admissible covering $\mathcal{X} = \cup_i U_i$ where $(U_i, \mathcal{O}_{\mathcal{X}}|_{U_i})$ is isomorphic to an affinoïd space. An example of a rigid analytic space is $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ for $r \geq 2$.

3.2 The Drinfeld period domain of rank r

We recall the Drinfeld period domain of rank r over $\mathbb{C}_\infty$ defined by:

$$\Omega^r(\mathbb{C}_\infty) := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{K_\infty\text{-rational hyperplanes}\}.$$

Let H be a K_∞-rational hyperplane. Then, H is defined by a linear form $\ell_H : K_\infty^r \to K_\infty$,

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_{r-1} \\ w_r \end{pmatrix} \longmapsto h_1 w_1 + \cdots + h_{r-1} w_{r-1} + h_r w_r.$$

We consider the following norm on $\mathbb{C}_\infty$

$$|w| := \max_{1\leq i\leq r}\{|w_i|\},$$

and for any $w \in \mathbb{C}_\infty^{r-1}$ we define

$$h(w) := \frac{1}{|w|} \inf\{|\ell_H(w)| : H \text{ is a } K_\infty\text{-rational hyperplane}\}.$$

Note that h is invariant under scalar multiplication: $h(\mu w) = h(w)$ for any $\mu \in \mathbb{C}_\infty$, hence it is well defined on $\Omega^r(\mathbb{C}_\infty)$. To some extent, the function h measure the distance of a period $w \in \Omega^r(\mathbb{C}_\infty)$ to all K_∞-hyperplanes.

Definition 3.2.1. For any positive integer n, we define

$$\Omega^r_n(\mathbb{C}_\infty) := \{w \in \Omega^r(\mathbb{C}_\infty) : h(w) \geq |\tilde{\pi}|^n\}.$$

Theorem 3.2.2. *For any positive integer n, the set $\Omega^r_n(\mathbb{C}_\infty)$ is an affinoïd subdomain of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. Moreover, the collection $\{\Omega^r_n(\mathbb{C}_\infty) : n \geq 1\}$ is an admissible covering of $\Omega^r(\mathbb{C}_\infty)$.*

Proof. See Proposition 1 of [SS91]. □

Remark 3.2.3. Observe that definition 3.2.1 is equivalent to

$$\Omega^r_n(\mathbb{C}_\infty) = \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \bigcup_H H(|\tilde{\pi}|^n)$$

where the union ranges over the set of K_∞-hyperplanes and

$$H(|\tilde{\pi}|^n) = \{w \in \Omega^r(\mathbb{C}_\infty) : |\ell_H(w)| \leq |\tilde{\pi}|^n\}.$$

Given two hyperplanes H_1 and H_2 defined by ℓ_{H_1} and ℓ_{H_2}, one can show that $\ell_{H_1} \equiv \ell_{H_2} \pmod{\tilde{\pi}^n}$ (coefficientwise) if and only if $H_1(|\tilde{\pi}|^n) = H_2(|\tilde{\pi}|^n)$ [SS91, Lemma 2]. In particular, one may obtain that $\Omega^r_n(\mathbb{C}_\infty)$ is defined only by a finite number of hyperplanes, which is a key point in proving that it is an affinoïd domain.

An important consequence of theorem 3.2.2 is that Ω^r_n is a rigid analytic space over K_∞. Therefore, if $U \subset \Omega^r(\mathbb{C}_\infty)$ is an admissible open, then a function is *holomorphic on U* if it is a section of the structure sheaf at U of $\Omega^r(\mathbb{C}_\infty)$.

3.3 Analytic Drinfeld modular forms

3.3.1 Weak modular forms

Fixing a nonzero constant $\xi \in \mathbb{C}_\infty$, we observe that any element of $\Omega^r(\mathbb{C}_\infty)$ is uniquely represented by a column vector $w = (w_1, \ldots, w_{r-1}, \xi)^t$ in $\mathbb{C}^r_\infty$ where $w_1, \ldots, w_{r-1}, \xi$ are all K_∞-linearly independent. Via the usual matrix multiplication, we have an action of $\mathrm{GL}_r(K_\infty)$ on $\mathbb{C}^r_\infty$. For any $\gamma \in \mathrm{GL}_r(K_\infty)$, we denote the automorphy factor by

$$j(\gamma, w) := \xi^{-1}(\text{last entry of } \gamma w).$$

Since ξ is nonzero, the last entry of γw is also nonzero, hence $j(\gamma, w) \in \mathbb{C}_\infty^\times$. Thus, we define an action of $\mathrm{GL}_r(K_\infty)$ on $\Omega^r(\mathbb{C}_\infty)$ by

$$\gamma(w) := j(\gamma, w)^{-1}\gamma w.$$

Lemma 3.3.1. *For any $\gamma_1, \gamma_2 \in \mathrm{GL}_r(K_\infty)$, we have*

$$j(\gamma_1\gamma_2, w) = j(\gamma_1, \gamma_2(w)) \cdot j(\gamma_2, w).$$

Proof. We will denote the last entry of a vector v in $\mathbb{C}_\infty^r$ by $\mathrm{Proj}_r(v)$. We have directly

$$\begin{aligned} j(\gamma_1, \gamma_2(w)) \cdot j(\gamma_2, w) &= \xi^{-1}\mathrm{Proj}_r\left(\gamma_1 j(\gamma_2, w)^{-1}\gamma_2 w\right) \cdot j(\gamma_2, w) \\ &= \xi^{-1}\mathrm{Proj}_r(\gamma_1\gamma_2 w) \cdot j(\gamma_2, w)^{-1} j(\gamma_2, w) \\ &= j(\gamma_1\gamma_2, w). \end{aligned}$$

□

A direct application of this lemma implies that:

Proposition 3.3.2. *The map*

$$\begin{aligned} \mathrm{GL}_r(K_\infty) \times \Omega^r(\mathbb{C}_\infty) &\longrightarrow \Omega^r(\mathbb{C}_\infty) \\ (\gamma, w) &\longmapsto \gamma(w) \end{aligned}$$

defines an action of $\mathrm{GL}_r(K_\infty)$ *on* $\Omega^r(\mathbb{C}_\infty)$.

Definition 3.3.3. A subgroup Γ of $\mathrm{GL}_r(K_\infty)$ is said to be *arithmetic* if Γ is commensurable with $\mathrm{GL}_r(A)$, that is $\Gamma \cap \mathrm{GL}_r(A)$ has finite index in both Γ and $\mathrm{GL}_r(A)$.

Definition 3.3.4. Let Γ be an arithmetic subgroup of $\mathrm{GL}_r(K_\infty)$ and let k and m be two integers. We say that an holomorphic function $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ is a *weak modular form of rank r, weight k and type m for* Γ if for every γ in Γ, we have the following invariance condition:

$$f(\gamma(w)) = \det(\gamma)^{-m} j(\gamma, w)^{-k} f(w).$$

The space of weak modular forms of weight rank r, weight k and type m for Γ will be denoted by $\mathcal{W}_{k,m}^{\mathrm{an},r}(\Gamma)$. If $m = 0$, we will simply denote it by $\mathcal{W}_k^{\mathrm{an},r}(\Gamma)$.

Proposition 3.3.5. *Let Γ be an arithmetic subgroup of $\mathrm{GL}_r(K_\infty)$ and let k and m be two integers.*

1. *If $m \equiv m'$ modulo $|\det(\Gamma)|$, then $\mathcal{W}_{k,m}^{\mathrm{an},r}(\Gamma) = \mathcal{W}_{k,m'}^{\mathrm{an},r}(\Gamma)$;*
2. *If $k \not\equiv rm$ modulo $|\Gamma \cap \mathbb{F}_q^\times|$, then $\mathcal{W}_{k,m}^{\mathrm{an},r}(\Gamma) = 0$.*

where we see the elements of $\mathbb{F}_q^\times$ in $\mathrm{GL}_r(K_\infty)$ as constant multiples of the identity matrix.

Proof. The first statement follows from the assumption that the group Γ is arithmetic and thus its determinant, $\det(\Gamma) := \{\det(\gamma) : \gamma \in \Gamma\}$, is a subgroup of $\mathbb{F}_q^\times$. This implies that the order of $\det(\Gamma)$ divides $q-1$ and thus the type depends only on the class of m modulo $|\det(\Gamma)|$.

For the second statement, we notice that if $\gamma = c \in \mathbb{F}_q^\times$, then $j(\gamma, w) = c$ and $\gamma(w) = w$. Hence, we have on one hand $f(\gamma(w)) = f(w)$ and on the other hand

$$f(\gamma(w)) = c^{-rm+k} f(w).$$

Therefore, $f \equiv 0$ unless $k \equiv rm$ modulo $|\Gamma \cap \mathbb{F}_q^\times|$. $\square$

3.3.2 Expansion at infinity

In this subsection, we will define what it means to be holomorphic at infinity, which is the main condition to add in order to define modular forms. Let $r \geq 2$ and consider the algebraic subgroup U of GL_r over K consisting of matrices of the form

$$\left(\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & \mathrm{Id}_{r-1} & \\ 0 & & & \end{array}\right).$$

We note that, seeing the elements of K^{r-1} as row vectors, we have an isomorphism of K-vector spaces

$$\begin{aligned} i : K^{r-1} &\longrightarrow U(K) \\ v &\longmapsto \begin{pmatrix} 1 & v \\ 0 & \mathrm{Id}_{r-1} \end{pmatrix}. \end{aligned}$$

The matrices in $U(K)$ may be seen as translations matrices. Thus, just like the classical theory, it would be natural for Drinfeld modular forms to be translation-invariant. Fortunately, for any arithmetic subgroups $\Gamma \leq \mathrm{GL}_r(K)$, if we define $\Gamma_U := \Gamma \cap U(K)$, then we have the following:

Proposition 3.3.6. *Every weak modular form f for Γ is Γ_U-invariant.*

Proof. This result follows from the fact that for every $\gamma \in \Gamma_U$ and $w \in \Omega^r(\mathbb{C}_\infty)$, we have $\det(\gamma) = 1$ and $j(\gamma, w) = 1$. $\square$

Next, consider the pre-image:

$$\Lambda' := i^{-1}(\Gamma_U) \subseteq K^{r-1}.$$

We see that Λ' is a A-lattice since it is commensurable with A^{r-1} (as Γ is commensurable with $\mathrm{GL}_r(A)$). For any $w \in \Omega^r(\mathbb{C}_\infty)$ we denote by w_1 and w' the elements of $\mathbb{C}_\infty$ and $\Omega^{r-1}(\mathbb{C}_\infty)$ respectively such that $w = \left(\begin{smallmatrix} w_1 \\ w' \end{smallmatrix}\right)$.

Definition 3.3.7. Let $w \in \Omega(\mathbb{C}_\infty)$. Under the same notation as above, we define the *parameter at infinity* to be the function:

$$u_w := u_{w'}(w_1) := e_{\tilde{\pi}w'\Lambda'}(w_1)^{-1}.$$

Recall that $\tilde{\pi}$ is the Carlitz period. Its presence is for normalization purpose. Note that this parameter depends on both w_1 and w' (hence it is a function of w). It plays a similar role to the function $q = e^{2\pi i w}$ in the theory of elliptic modular forms.

Proposition 3.3.8. *For any Γ_U-invariant holomorphic function $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$, there exists a unique sequence of holomorphic functions $f_n : \Omega^{r-1}(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ such that*

$$\sum_{n\in\mathbb{Z}} f_n(w')u_{w'}(w_1)^n \tag{3.1}$$

converges to $f(w_1, w')$ on some neighborhood of infinity and uniformly on every affinoïd subset.

Proof. See Proposition 5.4 in [BBP18b]. $\square$

The expansion (3.1) is called the *u-expansion* of a Γ_U-invariant holomorphic function f. If $f_n = 0$ for every negative integers n, then we say that f is *holomorphic at infinity.*

3.3.3 Definition of an analytic modular form

Let Γ be an arithmetic subgroup of $\mathrm{GL}_r(K)$ and γ be any element of Γ. For any integers k and m and any holomorphic function $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ we define

$$f|_{k,m}\gamma(w) := \det(\gamma)^m j(\gamma, w)^k f(\gamma(w)).$$

By lemma 3.3.1, this defines an action of Γ on the set of holomorphic functions on $\Omega^r(\mathbb{C}_\infty)$.

Definition 3.3.9. Let Γ be an arithmetic subgroup of $\mathrm{GL}_r(K)$. A *modular form of weight k, type m for* Γ is a weak modular form such that $f|_{k,m}\gamma$ is holomorphic at infinity for any element γ of Γ. The $\mathbb{C}_\infty$-vector space of all such forms is denoted $\mathcal{M}^{\mathrm{an},r}_{k,m}(\Gamma)$. If $m = 0$, we simply write $\mathcal{M}^{\mathrm{an},r}_{k}(\Gamma)$.

Example 3.3.10. Suppose that $r = 2$ and $A = \mathbb{F}_q[T]$. Let k be any integer which is a multiple of $q - 1$. Then, we define the weight k Drinfeld Eisenstein series of rank 2 by

$$E_k(w) := \sum_{\substack{c,d \in \mathbb{F}_q[T] \\ (c,d) \neq (0,0)}} \frac{1}{(cw + d)^k}.$$

One may show that E_k is a Drinfeld modular form of rank 2 for $\mathrm{GL}_2(\mathbb{F}_q[T])$, see for example [Gos80a, §1.7] or [Gek88, (5.9)].

Example 3.3.11. Let $r \geq 2$ and assume that $A = \mathbb{F}_q[T]$. Let $w \in \Omega^r(\mathbb{C}_\infty)$ written uniquely of the form $w = (w_1, \ldots, w_{r-1}, 1)^{\mathrm{T}}$ and consider

$$\Lambda_w := Aw_1 \oplus \cdots \oplus Aw_{r-1} \oplus A$$

the associated A-lattice. By uniformization, we define

$$\begin{aligned} \phi^w : A &\longrightarrow \mathbb{C}_\infty\{\tau\} \\ T &\longmapsto T + g_1(w)\tau + \cdots + g_r(w)\tau^r \end{aligned}$$

the corresponding Drinfeld module of rank r. For any $\gamma \in \mathrm{GL}_r(\mathbb{F}_q[T])$, corollary 2.4.7 tells us that $\Lambda_{\gamma(w)} = j(\gamma, w)^{-1}\Lambda_w$ and, from proposition 2.2.11, the element $j(\gamma, w)^{-1} \in \mathbb{C}_\infty^\times$ defines an isomorphism between ϕ^w and $\phi^{\gamma(w)}$:

$$j(\gamma, w)^{-1}\phi^w j(\gamma, w) = \phi^{\gamma(w)}.$$

By equating the coefficients on both side, we observe that $g_i(\gamma(w)) = j(\gamma, w)^{q^i - 1} g_i(w)$ for all i between 1 and r.

Remark 3.3.12. The forms $\{g_i\}_{1\leq i\leq r}$ are sometime called the *coefficients forms*. More generally, for any $a \in \mathbb{F}_q[T]$ and any $i \in \{1,\dots,r\}$, we define the function $g_{a,i} : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ to be the i-th coefficient of the τ-polynomial $\phi_a^w(\tau)$:

$$\phi_a^w(\tau) = a + g_{a,1}(w)\tau + \cdots + g_{a,r}(w)\tau^r.$$

3.4 Algebraic modular forms

3.4.1 Invariant differentials and the Lie algebra

Let $\phi : A \to \mathbb{C}_\infty\{\tau\}$ be a Drinfeld module over $\mathbb{C}_\infty$. Recall that ϕ induces a A-module action on $\mathbb{G}_a$, the additive group scheme over $\mathbb{C}_\infty$, given by $(a,z) \mapsto \phi_a(z)$ for any $a \in A$ and $z \in \mathbb{C}_\infty$. By definition of the additive group scheme, $\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}_\infty[z])$ for some variable z and therefore we may consider the module of ϕ-invariant differential $\Omega^{\mathrm{inv}}_{\phi/\mathbb{C}_\infty}$ which will be generated over $\mathbb{C}_\infty$ by a differential $\mathrm{d}z$ such that $\mathrm{d}(\phi_a(z)) = a\mathrm{d}z$ for any $a \in A$. Next, we may also interpret this differential $\mathrm{d}z$ as an invariant differential on the quotient of $\mathbb{C}_\infty/\Lambda$ where Λ is the lattice associated with ϕ. Indeed, recall that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}_\infty/\Lambda & \xrightarrow{a} & \mathbb{C}_\infty/\Lambda \\ \big\downarrow{\scriptstyle e_\phi} & & \big\downarrow{\scriptstyle e_\phi} \\ \mathbb{C}_\infty & \xrightarrow{\phi_a} & \mathbb{C}_\infty. \end{array}$$

where the top arrow is the multiplication by a map and e_ϕ is the exponential defined by ϕ. Then, by pulling back the differential $\mathrm{d}z$ to $\mathbb{C}_\infty/\Lambda$, we get a differential $\mathrm{d}z$ on $\mathbb{C}_\infty/\Lambda$ such that $a\mathrm{d}z = \mathrm{d}(az)$ and $\mathrm{d}(z+\lambda) = \mathrm{d}z$ for any $\lambda \in \Lambda$.

Next, to define the notion of modular forms in the manner of Pink, we will need the dual notion of the module of differentials, which is the Lie algebra associated with a Drinfeld module. For any Drinfeld module $E = (L,\phi)$ over a scheme S, there exists an invertible sheaf $\mathrm{Lie}(E)$ defined as the Lie algebra of the line bundle L seen as a group scheme. The general construction of this sheaf can be found in [Dem63]. In the affine case, if F is a A-field and ϕ is a Drinfeld module over F, then the Lie algebra over F is

$$\mathrm{Lie}(\phi) = \ker(\mathbb{G}_a(F[\varepsilon]) \to \mathbb{G}_a(F))$$

where ε is such that $F[z]/(z^2) = F[\varepsilon]$ and the map $\mathbb{G}_a(F[\varepsilon]) \to \mathbb{G}_a(F)$ is the projection $x + \varepsilon y \mapsto x$. One may show that $\mathrm{Lie}(\phi)$ is isomorphic to the dual of the module of invariant differential forms. A good reference about this topic is chapter 3 of [Mil13].

3.4.2 Pink's definition

Let N be a nonzero ideal of A and consider M_N^r, the Drinfeld modular variety of rank r for the principal congruence subgroup of level N. Recall that we have an object $(\mathcal{E}, \alpha_{N,\mathcal{E}})$ where $\mathcal{E} := (\mathcal{L}, \varphi)$ is the universal Drinfeld module of rank r over M_N^r whose fiber at each point is the Drinfeld module that corresponds to that point and $\alpha_{N,\mathcal{E}}$ is a level-N structure. We define $\boldsymbol{\omega}_N$ to be the dual of the Lie algebra of $\mathcal{E}$, in other words:

$$\boldsymbol{\omega} := \mathrm{Hom}_{\mathcal{O}_S\text{-mod}}(\mathrm{Lie}(\mathcal{E}), \mathcal{O}_S).$$

We also recall that the adelic principal congruence subgroup is

$$\mathcal{K}(N) = \ker(\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(A/N)).$$

Definition 3.4.1 (version 1)**.** Let k be any integer and N be a nonzero ideal of A. The space of *(algebraic) weak modular forms of weight k for $\mathcal{K}$* is defined to be the global sections of the k-th tensor of the sheaf $\boldsymbol{\omega}$:

$$\mathcal{W}_k^{\mathrm{alg},r}(N) := H^0(M_N^r, \boldsymbol{\omega}^{\otimes k}).$$

Remark 3.4.2. Pink also defines Drinfeld modular forms for arbitrary open compact subgroups of $\mathrm{GL}_r(\mathbb{A}_K^f)$. Let $\mathcal{K}$ be such subgroup. If $\mathcal{K}$ is fine (its image in $\mathrm{GL}_r(A/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ is idempotent), then we have

$$\mathcal{W}_k^{\mathrm{alg},r}(\mathcal{K}) := H^0(M_{\mathcal{K}}^r, \boldsymbol{\omega}^{\otimes k}).$$

When $\mathcal{K}$ is not fine, then, we may choose a sufficiently divisible ideal N of A such that $\mathcal{K}(N) \subset \mathcal{K}$ and then Pink shows that the quotient $\mathcal{K}/\mathcal{K}(N)$ acts on the space $\mathcal{W}_k^{\mathrm{alg},r}(N)$, so that he defines

$$\mathcal{W}_k^{\mathrm{alg},r}(\mathcal{K}) := \mathcal{W}_k^{\mathrm{alg},r}(N)^{\mathcal{K}}.$$

Pink moreover shows that this definitions is independent of the choice of N. For more details, we refer the reader to definition 5.4 in [Pin13].

Using the decomposition of $M_N^r \times_K \mathbb{C}_\infty$ into connected components given by proposition 2.5.10, one has:

Proposition 3.4.3. *Let $S(N)$ be a set of double coset representatives of*

$$\mathrm{GL}_r(K) \setminus \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}(N).$$

Then we have the following decomposition:

$$\mathcal{W}_k^{\mathrm{alg},r}(N) \otimes_K \mathbb{C}_\infty \cong H^0(M_N^r \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{C}_\infty), \boldsymbol{\omega}^{\otimes k}) \cong \bigoplus_{s \in S} H^0(M_s, \boldsymbol{\omega}^{\otimes k}),$$

where each M_s is a connected component of $M_N^r \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{C}_\infty)$.

We denote by π_s the projection of $M_N^r \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{C}_\infty)$ onto the connected component M_s. We denote by $\boldsymbol{\omega}_{\mathrm{an}}$ the invertible sheaf on the rigid analytic space $M_N^r(\mathbb{C}_\infty)$ which comes from $\boldsymbol{\omega}$.

Proposition 3.4.4. *The pullback by π_s induces an isomorphism*

$$H^0(M_s(\mathbb{C}_\infty), \boldsymbol{\omega}_{\mathrm{an}}^{\otimes k}) \cong \mathcal{W}_k^{\mathrm{an},r}(\Gamma_s).$$

Proof. See section 10 of [BBP18b]. □

In analogy to the classical case, there exists a *Satake compactification* of the Drinfeld modular variety, that we shall denote $\overline{M}_N^r$. In [Pin13], Pink gave an abstract charaterization of this compactification and proved that it is unique up to isomorphism. Moreover, the universal Drinfeld module $\mathcal{E}$ extends to a *generalized* Drinfeld module, denoted $\overline{\mathcal{E}}$, which is an extended notion of a Drinfeld module allowing variation of the rank. This allows us to consider the sheaf $\overline{\boldsymbol{\omega}}$ to be the dual of of the Lie algebra of $\overline{\mathcal{E}}$ and define algebraic modular forms.

Definition 3.4.5. Let k be any integer. The space of *(algebraic) modular forms of weight k, rank r and level N* is

$$\mathcal{M}_k^{\mathrm{alg},r}(N) := H^0(\overline{M}_N^r, \overline{\boldsymbol{\omega}}^{\otimes k}).$$

For any s in S, we define the space of algebraic modular forms of weight k and rank r for Γ_s by

$$\mathcal{M}_k^{\mathrm{alg},r}(\Gamma_s) := H^0(\overline{M}_s, \overline{\boldsymbol{\omega}}^{\otimes k})$$

where $\overline{M}_s$ is a connected component of $\overline{M}_N^r$.

Proposition 3.4.6. *We have an isomorphism of $\mathbb{C}_\infty$-modules $\mathcal{M}_k^{\mathrm{alg},r}(\Gamma_s) \cong \mathcal{M}_k^{\mathrm{an},r}(\Gamma_s)$.*

Proof. This is lemma 10.7 of [BBP18b]. □

Theorem 3.4.7. *We have an isomorphism of $\mathbb{C}_\infty$-modules:*

$$\mathcal{M}_k^{\mathrm{alg},r}(N) \otimes_K \mathbb{C}_\infty \cong \bigoplus_{s \in S} \mathcal{M}_k^{\mathrm{an},r}(\Gamma_s).$$

Proof. See theorem 10.9 of [BBP18b]. □

3.4.3 Modular forms over A-algebras

As explained in the beginning of this chapter, Goss also introduced an algebraic theory of Drinfeld modular forms [Gos80a, Definition 1.4.1]. In essence, his definition is modeled after Katz's version of classical modular forms. In this section, we consider a reformulation of definiton 3.4.1 inspired by Goss' definiton. This definition will be useful for studying the special values of Drinfeld modular forms at CM points.

Let R be a A-algebra and consider the affine A-scheme $S = \mathrm{Spec}(R)$. For any pairs (E, α_N) where $E = (L, \phi)$ is a Drinfeld A-module over R and α_N a level-N structure, there exists a unique morphism

$$\iota : \mathrm{Spec}(R) \to M_N^r$$

such that $\iota^*(\mathcal{E}, \alpha_{N,\mathcal{E}}) = (E, \alpha_N)$. Let ω be a basis of $\iota^*\boldsymbol{\omega}$, so that under this choice of basis we have

$$H^0(\mathrm{Spec}(R), \boldsymbol{\omega}^{\otimes k}) \cong R\omega^{\otimes k}.$$

Thus, if F is a weak modular form of weight k and level N in the sense of definition 3.4.1, then there exists $f(E, \alpha_N, \omega) \in R$ such that

$$\iota^* F(\mathcal{E}, \alpha_{N,\mathcal{E}}) = f(E, \alpha_N, \omega)\omega^{\otimes k}.$$

Proposition 3.4.8. *For any triple (E, α_N, ω) and f as above, the rule*

$$f : (E, \alpha_N, \omega) \mapsto f(E, \alpha_N, \omega) \in R$$

satisfies the following three properties:

1. *The value $f(E, \alpha_N, \omega) \in R$ depends only on the isomorphism class of (E, α_N, ω);*

2. *For any morphism $g : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$, R' an A-algebra, we have*

$$f(g^*(E, \alpha_N, \omega)) = g^* f(E, \alpha_N, \omega);$$

3. For all $\mu \in R^\times$, we have

$$f(E, \alpha_N, \mu\omega) = \mu^{-k} f(E, \alpha_N, \omega).$$

Proof. Let F be an algebraic weak modular form of weight k and level N and consider the rule f defined as above. The first and second properties are clear from the definitions. Next, to prove the third property, we take a unit μ in R and consider the basis $\mu\omega$ of the global sections of $\boldsymbol{\omega}$. On one hand, we have

$$\iota^* F = f(E, \alpha_N, \omega)\omega^{\otimes k},$$

and on the other hand we have

$$\iota^* F = f(E, \alpha_N, \mu\omega)(\mu\omega)^{\otimes k} = \mu^k f(E, \alpha_N, \mu\omega)\omega^{\otimes k}.$$

Therefore, the combination of these two expressions yields

$$f(E, \alpha_N, \omega) = \mu^k f(E, \alpha_N, \mu\omega).$$

□

Definition 3.4.9 (Version 2)**.** Let k be any integer and N a nonzero ideal of A. A *(algebraic) weak modular forms of weight k, rank r and level N over an A-algebra R* is a rule f which to each triple (E, α_N, ω) where

(a) $E = (L, \phi)$ is a Drinfeld A-module over R;

(b) α_N is a level-N structure;

(c) ω is a nonzero section of $\boldsymbol{\omega}$;

assigns an element $f(E, \alpha_N, \omega) \in R$ satisfying the conditions 1, 2 and 3 of proposition 3.4.8. We will denote the set of such modular forms by $\mathcal{G}_k^r(N, R)$.

Remark 3.4.10. We define a *level 1* weak modular form over R as rule which to each pair (E, ω), where $E = (L, \phi)$ is a Drinfeld A-module over R and ω is a nonzero section of $\boldsymbol{\omega}$, assigns an element $f(E, \omega) \in R$ satisfying the aforementioned conditions.

Theorem 3.4.11. *If R/K is an extension of A-fields, then we have*

$$\mathcal{G}_k^r(N, R) \cong \mathcal{W}_k^{\mathrm{alg},r}(N) \otimes_K R$$

as R-modules.

Proof. If $F \in \mathcal{W}_k^{\mathrm{alg},r}(N) \otimes_K R$, then one may obtain an element of $\mathcal{G}_k^r(N,R)$ via the method described above proposition 3.4.8. Conversely, let f be a rule in $G_k^r(N,R)$ and fix a scheme morphism $\iota : \mathrm{Spec}(R) \to M_N^r$. The pullback of $(\mathcal{E}, \alpha_{N,\mathcal{E}})$ by ι is a pair (E, α_N) where E is a Drinfeld module over R and α_N is a level-N structure. Then, we obtain an element of $\mathcal{W}_k^{\mathrm{alg},r}(N) \otimes_K R$ by setting

$$F(E, \alpha_N) := f(E, \alpha_N, \omega)\omega^{\otimes k}.$$

We note that F is indeed independent of ω. $\square$

Recall that $S(N)$ is defined to be a set of double coset representative of

$$\mathrm{GL}_r(K) \setminus \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}(N).$$

Corollary 3.4.12. *We have the isomorphism of $\mathbb{C}_\infty$-modules:*

$$\mathcal{G}_k^r(N, \mathbb{C}_\infty) \cong \bigoplus_{s \in S(N)} \mathcal{W}_k^{\mathrm{an},r}(\Gamma_s).$$

Proof. This follows from taking $R = \mathbb{C}_\infty$ in theorem 3.4.11 and applying theorem 3.4.7. $\square$

We note that for any $s \in S(N)$, we may see $\mathcal{M}_k^{\mathrm{an},r}(\Gamma_s)$ as a $\mathbb{C}_\infty$-submodule of $\mathcal{G}_k^r(N, \mathbb{C}_\infty)$. Thus, letting $p_s : \mathcal{G}_k^r(N, \mathbb{C}_\infty) \to \mathcal{M}_k^{\mathrm{an},r}(\Gamma_s)$ denote the projection map, we make the following definition:

Definition 3.4.13. Let $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ be an analytic Drinfeld modular form of weight k, rank r for Γ_s. We say that f is *arithmetic* if there exists a finite extension K_f/K such that

$$p_s^{-1}(f) \in \mathcal{G}_k^r(N, K_f).$$

We also say that f is *arithmetic over K_f*.

For the rest of this section, we will explain how we may pass from the algebraic version to the analytic version of a modular form. Let $f \in \mathcal{G}_k^r(N, \mathbb{C}_\infty)$. Recall from section 2.5 that for any $s \in S(N)$ and $w \in \Omega^r(\mathbb{C}_\infty)$ there is a Drinfeld module ϕ_w^s over $\mathbb{C}_\infty$ whose associated A-lattice is

$$\Lambda_w^s = w^*(K^r \cap s\hat{A}^r).$$

Moreover, this lattice comes with a level-N structure $\alpha_N^s : (N^{-1}/A)^r \xrightarrow{\sim} N^{-1}\Lambda_w^s/\Lambda_w^s$. Then, we define a function $f_\infty^s : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ by setting

$$f_\infty^s(w) := f(\phi_w^s, \alpha_N^s, \mathrm{d}z)$$

where $\mathrm{d}z$ is the invariant differential of ϕ^s_w. We recall that replacing w by $\gamma(w)$ for any $\gamma \in \Gamma_s$ have the effect of replacing $\mathrm{d}z$ by $j(\gamma, w)^{-1}\mathrm{d}z$. A straightforward computation gives

$$\begin{aligned} f^s_\infty(\gamma(w)) &= f(\phi^s_{\gamma(w)}, \alpha^s_N, j(\gamma, w)^{-1}\mathrm{d}z) \\ &= j(\gamma, w)^k f(\phi^s_w, \alpha^s_N, \mathrm{d}z) \\ &= j(\gamma, w)^k f^s_\infty(w). \end{aligned}$$

Therefore, we have proved:

Proposition 3.4.14. *For any weak algebraic modular form f of weight k, rank r and level N over $\mathbb{C}_\infty$ and for any choice of representative $s \in S(N)$, the associated function $f^s_\infty : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ is a weak analytic modular form of weight k and rank r for Γ_s.*

Remark 3.4.15. Unlike the classical theory, there is no analogue of the Tate curve for Drinfeld modular forms. As a consequence, we use Pink's compactification of the Drinfeld modular variety in order to define algebraic modular forms which are holomorphic at infinity.

Chapter 4

Complex Multiplication

4.1 Drinfeld $\mathcal{O}$-modules and the Picard group action

Definition 4.1.1. An *order* $\mathcal{O}$ in K is a subring of A such that $\mathrm{Frac}(\mathcal{O}) = K$. If $\mathcal{O} = A$, then we say that $\mathcal{O}$ is the *maximal order*. A *fractional $\mathcal{O}$-ideal* is a nonzero noetherian $\mathcal{O}$-submodule of K. The set of all nonzero fractional $\mathcal{O}$-ideal is denoted by $I(\mathcal{O})$. For any fractional $\mathcal{O}$-ideal $\mathfrak{a}$, we set

$$\mathfrak{a}^* := \{x \in K : x\mathfrak{a} \subset \mathcal{O}\}.$$

If $\mathfrak{a}\mathfrak{a}^* = \mathcal{O}$, we say that $\mathfrak{a}$ is *invertible* and we define $I^*(\mathcal{O})$ to be the group of all invertible fractional ideals. Note that if $\mathcal{O}$ is a Dedekind domain, then $I^*(\mathcal{O}) = I(\mathcal{O})$. We let $P(\mathcal{O})$ be the group of fractional $\mathcal{O}$-ideals which are principal and we define the Picard group of $\mathcal{O}$ to be the quotient $\mathrm{Pic}(\mathcal{O}) := I^*(\mathcal{O})/P(\mathcal{O})$.

In chapter 2, we considered Drinfeld modules over a A-field F as a morphism $\phi : A \to F\{\tau\}$ as they were initially defined by Drinfeld in [Dri74]. In [Hay79], Hayes consider slightly more general objects by replacing the ring A by any order $\mathcal{O}$ in K.

Definition 4.1.2. A *Drinfeld $\mathcal{O}$-module of rank r over a $\mathcal{O}$-field F* (a field equipped with a morphism $i : \mathcal{O} \to F$) is a morphism $\phi : \mathcal{O} \to F\{\tau\}$ such that for every $a \in \mathcal{O}$ we have

$$\phi_a = i(a)\tau + g_{a,1}(\phi)\tau + \cdots + g_{a,r}(\phi)\tau^r,$$

where $g_{a,i}(\phi) \in F$ and $g_{a,r}(\phi)$ is nonzero. A *morphism* between two Drinfeld $\mathcal{O}$-modules ϕ and ψ over F of the same rank is an element $P \in F\{\tau\}$ such that $P\phi_a = \psi_a P$ for every $a \in \mathcal{O}$.

Hayes also developed the analytic theory by proving a uniformization theorem between Drinfeld $\mathcal{O}$-modules and $\mathcal{O}$-lattices which are discrete projective $\mathcal{O}$-submodules of $\mathbb{C}_\infty$. We will omit the details as the theory is very similar to the one covered in sections 2.2 and 2.4.1. For any $\mathcal{O}$-lattice Λ we denote by e_Λ and ϕ^Λ the attached exponential and the Drinfeld $\mathcal{O}$-module respectively.

Let $\mathfrak{a}$ be any invertible fractional ideal. For any $\mathcal{O}$-lattice Λ of rank r, we let $\mathfrak{a}*\Lambda := \mathfrak{a}^{-1}\Lambda$. If $\mathfrak{a} = x\mathcal{O}$ is principal, then we observe that $\mathfrak{a} * \Lambda = x^{-1}\Lambda \cong \Lambda$. Therefore, the map

$$I^*(\mathcal{O}) \times (\mathfrak{a}, \Lambda) \mapsto \mathfrak{a} * \Lambda \tag{4.1}$$

induces an action of $\mathrm{Pic}(\mathcal{O})$ on the isomorphism classes of $\mathcal{O}$-lattice of rank r.

Our next goal is to define an action of the Picard group of $\mathcal{O}$ on the set of isomorphism classes of Drinfeld $\mathcal{O}$-modules of rank r and show that it is compatible with the action given by (4.1).

Let $\mathcal{O}$ be an order of K and F be a $\mathcal{O}$-field of generic characteristic which is a finite extension of K. Let $\phi : \mathcal{O} \to F\{\tau\}$ be a Drinfeld $\mathcal{O}$-module of rank r and let $\mathfrak{a}$ be a nonzero fractional ideal of $\mathcal{O}$. We define $I_{\phi,\mathfrak{a}}$ to be the left ideal generated by the set $\{\phi_a : a \in \mathfrak{a}\}$. By proposition 2.1.5, left ideals are principal, hence $I_{\phi,\mathfrak{a}}$ is generated by a unique monic τ-polynomial denoted $\phi_\mathfrak{a}$. We observe that the ideal $I_{\phi,\mathfrak{a}}$ is fixed by multiplication on the right by elements of the form ϕ_b for any $b \in \mathcal{O}$. Indeed, since $\mathfrak{a}$ is an ideal of $\mathcal{O}$ we have $\mathfrak{a}b \subset \mathfrak{a}$ for any $b \in \mathcal{O}$ and thus the commutativity of $\mathcal{O}$ implies $\phi_a\phi_b = \phi_b\phi_a$ for any $a \in \mathfrak{a}$. Therefore, for any $a \in \mathcal{O}$, there exists $\psi_a \in F\{\tau\}$ such that $\phi_\mathfrak{a}\phi_a = \psi_a\phi_\mathfrak{a}$. The association $a \mapsto \psi_a$ for all $a \in \mathcal{O}$ defines a Drinfeld $\mathcal{O}$-module ψ. Note that its rank is also r since the τ-polynomial $\phi_\mathfrak{a}$ defines a isogeny between ϕ and ψ. We define

$$\mathfrak{a} * \phi := \psi.$$

Lemma 4.1.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of $\mathcal{O}$ and ϕ be a Drinfeld $\mathcal{O}$-module of rank r. We have:*

1. *if $\mathfrak{a} = a\mathcal{O}$ is principal, then $\mathfrak{a} * \phi \cong \phi$;*
2. *$\mathfrak{a} * (\mathfrak{b} * \phi) = (\mathfrak{a}\mathfrak{b}) * \phi$.*

Proposition 4.1.4. *Let Λ be a $\mathcal{O}$-lattice and ϕ^Λ be the associated Drinfeld $\mathcal{O}$-module. For an invertible fractional ideal $\mathfrak{a}$ of $\mathcal{O}$, we have*

$$\phi^{\mathfrak{a}*\Lambda} = \mathfrak{a} * \phi^\Lambda.$$

4.2 Field of definition of Drinfeld modules

In this section, we suppose that $\phi : \mathcal{O} \to \mathbb{C}_\infty\{\tau\}$ is a Drinfeld $\mathcal{O}$-module of rank r of generic characteristic.

Definition 4.2.1. We say that a subfield $F \subset \mathbb{C}_\infty$ is a *field of definition* of ϕ if there exists a Drinfeld $\mathcal{O}$-module ϕ' isomorphic to ϕ such that ϕ' has coefficients in F. If $F \subset F'$ for any field of definition F', we say that F is *minimal.*

In what follows, we will give an explicit description of a minimal field of definition for ϕ. We consider the ring $K\left[\{X_i\}_{i\geq 1}\right]$ and its fraction field $K\left(\{X_i\}_{i\geq 1}\right)$. We have a graduation on $K(\{X_i\}_{i\geq 1})$ defined by

$$\mathrm{grad}(X_i) := q^i - 1, \quad i \geq 0.$$

Definition 4.2.2. The *field of formal invariants*, denoted $K\left(\{X_i\}_{i\geq 1}\right)_0$, consists of the homogeneous elements of degree zero. More precisely, we have

$$K\left(\{X_i\}_{i\geq 1}\right)_0 := \left\{ \frac{f}{g} \in K\left(\{X_i\}_{i\geq 1}\right) : \mathrm{grad}(f) = \mathrm{grad}(g) \right\}.$$

For any $a \in \mathcal{O}$ we consider the τ-polynomial:

$$\phi_a(\tau) = a + g_{a,1}(\phi)\tau + g_{a,2}(\phi)\tau^2 + \cdots + g_{a,r\deg(a)}(\phi)\tau^{r\deg(a)}.$$

Setting $g_{a,i}(\phi) := 0$ if $i > r\deg(a)$, we have a substitution morphism

$$\begin{aligned} S_{a,\phi} : K\left[\{X_i\}_{i\geq 1}\right] &\longrightarrow \mathbb{C}_\infty \\ X_i &\longmapsto g_{a,i}(\phi). \end{aligned}$$

Definition 4.2.3. We define

$$V_{a,\phi} := \left\{ \frac{f}{g} \in K\left[\{X_i\}_{i\geq 1}\right] : S_{a,\phi}(g) \neq 0 \right\}.$$

This allows us to extend the domain of the morphism $S_{a,\phi}$ to $V_{a,\phi}$.

Definition 4.2.4. For any $a \in \mathcal{O}$, the *field of invariants of ϕ at a* is defined by

$$I_a(\phi) := S_{a,\phi}(V_{a,\phi}).$$

Proposition 4.2.5. *Let a be any element of $\mathcal{O}$. Then, the field $I_a(\phi)$ satisfies the following properties:*

1. *$I_a(\phi)$ depends only on the isomorphism class of ϕ.*
2. *For any field of definition F of ϕ, we have $I_a(\phi) \subset F$, i.e. $I_a(\phi)$ is minimal.*
3. *If $\deg(a) > 0$, then $I_a(\phi)$ is a field of definition of ϕ.*

Proof. This is proposition 6.4 and theorem 6.5 of [Hay79]. □

Corollary 4.2.6. *For any $a \in A$, the field $I_a(\phi)$ is independent of a.*

Proof. Let a and b be two nonconstant polynomials. By point 2 of proposition 4.2.5, we have simultaneously $I_a(\phi) \subset I_b(\phi)$ and $I_b(\phi) \subset I_a(\phi)$, therefore they must be equal. □

Proposition 4.2.7. *The Galois group $\mathrm{Gal}(\mathbb{C}_\infty/K)$ acts on the isomorphism classes of rank r Drinfeld $\mathcal{O}$-modules over $\mathbb{C}_\infty$ and the Galois action commutes with the action of $\mathrm{Pic}(\mathcal{O})$.*

Proof. See proposition 8.1 in [Hay79]. The action of an automorphism σ on a Drinfeld module ϕ is given coefficient-wise:

$$\sigma(\phi_a) = \sigma(a) + \sigma(g_1)\tau + \cdots + \sigma(g_{r-1})\tau^{r-1} + \sigma(g_r)\tau^r.$$

□

Definition 4.2.8. A finite Galois extension $H_{\mathcal{O}}/K$ such that ∞ splits completely and

$$\mathrm{Gal}(H_{\mathcal{O}}/K) \cong \mathrm{Pic}(\mathcal{O})$$

is said to be a *Hilbert class field of $\mathcal{O}$*.

Remark 4.2.9. Any Hilbert class field of $\mathcal{O}$ is unique up to isomorphism.

Theorem 4.2.10. *Let $\phi : \mathcal{O} \to \mathbb{C}_\infty\{\tau\}$ be a rank 1 Drinfeld module. Then, the field of invariants of ϕ is a Hilbert class field of $\mathcal{O}$.*

Proof. See proposition 8.4 of [Hay79]. □

4.3 The ring of endomorphism

Let ϕ be a rank r Drinfeld A-module over an A-field F. In this section, we present some properties of the endomorphism ring of ϕ, defined by

$$\mathrm{End}_F(\phi) := \{P \in F\{\tau\} : P\phi_a = \phi_a P \text{ for all } a \in A\}.$$

Proposition 4.3.1. *Suppose that F has generic characteristic, then* $\mathrm{End}_F(\phi)$ *is a commutative ring.*

Proof. Since F is of generic characteristic, we may embed it in $\mathbb{C}_\infty$. Hence, without loss of generality we suppose that $F = \mathbb{C}_\infty$. By uniformization, we let Λ be the associated lattice of ϕ. Then, the ring $\mathrm{End}(\Lambda) \subset \mathbb{C}_\infty$ is obviously commutative, from which we deduce the commutativity of $\mathrm{End}(\phi)$ as we have $\mathrm{End}(\Lambda) \cong \mathrm{End}_{\mathbb{C}_\infty}(\phi)$. □

Proposition 4.3.2. $\mathrm{End}_F(\phi)$ *is a projective A-module.*

Proof. The A-module structure comes from the fact that for any a in A, the τ-polynomial ϕ_a is an endomorphism of ϕ. Indeed, for all b in A, the fact that $\phi : A \to F\{\tau\}$ is a $\mathbb{F}_q$-algebra morphism together with the fact that A is commutative gives us:

$$\phi_a\phi_b = \phi_{ab} = \phi_{ba} = \phi_b\phi_a,$$

thus $\phi_a \in \mathrm{End}_F(\phi)$. We therefore have an action of A on $\mathrm{End}(\phi)$ simply by the multiplication of τ-polynomials $(a, P) \mapsto \phi_a(\tau)P(\tau)$. Next, $F\{\tau\}$ is torsion free, hence the same property holds for $\mathrm{End}_F(\phi)$. Since A is a Dedekind domain, this is equivalent to saying that $\mathrm{End}_F(\phi)$ is projective. □

Theorem 4.3.3. *Let ϕ be a rank r Drinfeld module over F. Then the rank of* $\mathrm{End}_F(\phi)$ *is at most r^2.*

Proof. We give a sketch of the proof, following theorem 4.7.8 in [Gos96]. First, we will need the following lemma whose proof may be found in Goss' book [Gos96, Proposition 4.6.2]:

Lemma 4.3.4. *Let V be a finite dimensional K-vector space of dimension d. Then, any discrete A-submodule of V is finitely generated over A and have an A-rank at most d.*

Next, we first show that $\mathrm{End}_F(\phi)$ is finitely generated. Suppose that it is not, so we may fix an infinite sequence $\{e_i\}_{i\geq 1}$ of K-linearly independent elements. Then, setting

$$V_i := Ke_1 \oplus \cdots \oplus Ke_i,$$

we obtain a strictly increasing sequence of K-vector spaces of dimension i. By lemma 4.3.4, we have that $E_i := V_i \cap \mathrm{End}_F(\phi)$ is finitely generated of rank i over A. However, we observe that, for any $a \in A \setminus \{0\}$, there exists an injection $a^{-1}E_i/E_i \hookrightarrow a^{-1}\mathrm{End}_F(\phi)/\mathrm{End}_F(\phi)$ (indeed, if $e \in a^{-1}E_i$ and $e \in \mathrm{End}_F(\phi)$, then $e \in E_i$), resulting in a contradiction.

To prove that the rank of $\mathrm{End}_F(\phi)$ is bounded by r^2, one needs to prove that, for a prime to the characteristic of F, the natural map

$$\mathrm{End}_F(\phi) \otimes_A A/(a) \to \mathrm{End}_A(\phi[a]) \tag{4.2}$$

is injective, where $\phi[a] := \{a \in A : \phi_a = 0\}$ is the set of a-torsion points of ϕ. A crucial fact about this set of torsion points is that we have $\phi[a] \cong (A/(a))^r$ as A-module [Gos96, Remarks 4.5.5]. Therefore, we will have $\mathrm{rank}_{A/(a)}(\mathrm{End}(\phi[a])) = r^2$. □

Recall that an isogeny between two Drinfeld modules ϕ and ψ is a nonzero morphism $P : \phi \to \psi$. The next result does not only concern endomorphisms of a Drinfeld module.

Proposition 4.3.5. *Let $P : \phi \to \psi$ be an isogeny between two Drinfeld modules ϕ and ψ. Then there exists an isogeny $\hat{P} : \psi \to \phi$ such that $\hat{P}P = \phi_a$ and $P\hat{P} = \psi_a$ for some nonzero a in A.*

Proof. This is proposition 4.7.13 of [Gos96]. □

If ϕ is defined over an A-field F of generic characteristic, then the above result together with the fact that $\mathrm{End}_F(\phi)$ is commutative implies that $\mathrm{End}_F(\phi) \otimes_A K$ is a finite extension of K of degree at most r^2. In fact, there exists a better bound on its degree:

Proposition 4.3.6. *If F has generic characteristic, then* $\mathrm{End}_F(\phi)$ *has rank at most r.*

Proof. See corollary 3.20 of [Fli13]. □

4.4 Complex multiplication of Drinfeld modules

Definition 4.4.1. A Drinfeld A-module ϕ over $\mathbb{C}_\infty$ is said to be *singular* if $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$ strictly contains A. Furthermore, if the field

$$K_\phi := \mathrm{End}_{\mathbb{C}_\infty}(\phi) \otimes_A K$$

is of degree r as an extension of K and if ∞ is inert in K_ϕ, then we say that ϕ *has complex multiplication (or CM) by K_ϕ.*

Remark 4.4.2. The ring $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$ is an order of K_ϕ.

Theorem 4.4.3. *Let ϕ be a rank r CM Drinfeld A-module over $\mathbb{C}_\infty$. Then, the minimal field of definition of ϕ is the Hilbert class field of* $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$.

Proof. Let $\phi : A \to \mathbb{C}_\infty\{\tau\}$ be a rank r CM Drinfeld A-module. Observe that for any $a \in A$, we have $\phi_a \in \mathrm{End}_{\mathbb{C}_\infty}(\phi)$. Moreover, since the map defining any Drinfeld module is injective, we can extend ϕ to a Drinfeld $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$-module

$$\psi : \mathrm{End}_{\mathbb{C}_\infty}(\phi) \longrightarrow \mathbb{C}_\infty\{\tau\}$$

such that $\psi_a = \phi_a$ for any $a \in A$. Next, we claim that ψ has rank 1. Indeed, let v_∞ be the valuation of K associated to ∞ and consider v'_∞ its extension to K_ϕ. Then, since $[K_\phi : K] = r$ and ∞ is inert in K_ϕ, we have $v_\infty = v'_\infty/r$. In particular, we compute

$$\deg_\tau(\psi_a(\tau)) = rd_\infty v_\infty(a) = d_\infty v'_\infty(a),$$

where d_∞ is the degree of ∞, proving the claim. By theorem 4.2.10, the minimal field of definition of ψ is the Hilbert class field of $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$. □

Remark 4.4.4. We remark that in the definition of complex multiplication we assume that ∞ is inert in K_ϕ, i.e. ∞ is still a prime in K_ϕ. From the proof of the above theorem, we see that this condition is important so that we can, in some way, reduce the rank of ϕ to one. Then, via Hayes's theory of rank one Drinfeld modules, this rank reduction trick implies that the minimal field of definition is the Hilbert class field of the CM field.

4.5 CM points of $\Omega^r(\mathbb{C}_\infty)$

The goal of this section is to define the notion of CM points in the Drinfeld period domain of rank r and relate them to CM Drinfeld module by generalizing ideas of Hamahata. More precisely, when A is $\mathbb{F}_q[T]$, Hamahata proves that given $w = (w_1, \ldots, w_{r-1}, 1)$ in $\Omega^r(\mathbb{C}_\infty)$ such that $K_w = K(w_1, \ldots, w_{r-1})$ has degree exactly r over K and such that ∞ is inert in K_w, then the Drinfeld A-module associated with the A-lattice

$$\Lambda_w = Aw_1 \oplus \cdots \oplus Aw_{r-1} \oplus A$$

has CM by K_w [Ham03, Theorem 3.6].

Definition 4.5.1. Let $w \in \Omega^r(\mathbb{C}_\infty)$ and let $K_w := K(w_1, \ldots, w_r)$. Then w is a *CM point* if and only if K_w/K has degree r and ∞ is inert in K_w. The field K_w is called the *CM field.*

In our situation, the ring A is arbitrary and, in order to relate points of the period domain to Drinfeld modules, we use the rigid analytic description of the Drinfeld modular variety:

$$M_N^r(\mathbb{C}_\infty) \cong \bigsqcup_{s \in \mathcal{S}(N)} \Gamma_s \setminus \Omega^r(\mathbb{C}_\infty)$$

where $\mathcal{S}(N)$ is a set of double coset representative of $\mathrm{GL}_r(K) \setminus \mathrm{GL}_r(\mathbb{A}_K^f)/\mathcal{K}(N)$. Recall from section 2.5 that for any pairs $(w, s) \in \Omega^r(\mathbb{C}_\infty) \times \mathcal{S}(N)$ there exists a Drinfeld module ϕ_w^s whose associated A-lattice is $\Lambda_w^s = w^*(K^r \cap s\hat{A}^r)$. If we write w uniquely as a column vector $w = (w_1, \ldots, w_{r-1}, 1)^{\mathrm{T}}$ for some $w_i \in \mathbb{C}_\infty$ which are K_∞-linearly independent, then the lattice Λ_w^s satisfies

$$\Lambda_w^s \otimes_A K = Kw_1 \oplus \cdots \oplus Kw_{\ell-1} \oplus K.$$

In this case, we define the following field

$$K_w := K(w_1, w_2, \ldots, w_{r-1}).$$

Proposition 4.5.2. *If $r = \ell$ is a prime number and ϕ_w^s is singular, then the extension K_w/K has degree ℓ.*

Proof. Suppose that ϕ_w^s is singular and $r = \ell$ a prime number. Let Λ_w^s be the associated lattice which satisfies

$$\Lambda_w^s \otimes_A K = Kw_1 \oplus \cdots \oplus Kw_{\ell-1} \oplus K.$$

Since ϕ_w^s is singular, there exists $\mu \in \mathbb{C}_\infty \setminus A$ such that $\mu\Lambda_w^s \subset \Lambda_w^s$. Hence, we have a system of ℓ linear equations:

$$\begin{cases} \mu w_i = \sum_{j=1}^{\ell} a_{i,j} w_j, & \text{for } \ 0 \le i \le \ell - 1; \\ \quad \mu = \sum_{j=1}^{\ell} a_{r,j} w_j \end{cases} \tag{4.3}$$

for $a_{i,j} \in K$, $1 \le i, j \le r$, where we set $w_\ell := 1$. We may deduce two results from these equations. First, we see that μ is a linear combination of the w_i, and thus we get the following inclusion

$$\mathrm{End}_{\mathbb{C}_\infty}(\Lambda_w^s) \otimes K \subset Kw_1 \oplus \cdots \oplus Kw_{\ell-1} \oplus K. \tag{4.4}$$

Second, we claim that we must have $[K(\mu) : K] = \ell$. Indeed, the equations (4.3) are equivalent to saying that μ is an eigenvalue of the matrix $\gamma := (a_{i,j})_{1 \le i,j \le \ell}$ with eigenvector w. This implies that $[K(\mu) : K]$ divides ℓ, as μ is a root of the characteristic polynomial of γ. Since $\mu \notin K$ and ℓ is prime, this proves the claim.

Now, by (4.4) and the definition of the element μ, we have the following inclusions of K-vector spaces:

$$K(\mu) \subset \mathrm{End}_{\mathbb{C}_\infty}(\Lambda_w^s) \otimes K \subset Kw_1 \oplus \cdots \oplus Kw_{\ell-1} \oplus K.$$

We may deduce that they all must be equal, as $K(\mu)$ and $\oplus_i^\ell Kw_i$ are of dimension ℓ over K. Since $w_i \in K(\mu)$ for all i, we have $K(\mu) = K_w$ and therefore K_w/K is of degree ℓ. □

Proposition 4.5.3. *Let $w \in \Omega^r(\mathbb{C}_\infty)$ such that K_w/K is a finite extension of degree r and ∞ is inert in K_w, then for any choice of $s \in \mathcal{S}(N)$, the Drinfeld module ϕ_w^s has CM by K_w.*

Proof. Suppose that K_w/K is of degree r and choose $\mu \in K_w \setminus K$. Then, because the w_i are K-linearly independent, we have

$$\begin{aligned} K_w &= Kw_1 \oplus \cdots \oplus Kw_{r-1} \oplus K \\ &= \Lambda_w^s \otimes_A K, \end{aligned}$$

and so $\mu(\Lambda_w^s \otimes_A K) \subset \Lambda_w^s \otimes_A K$. As a A-module, the lattice Λ_w^s will be generated by $w_1, \ldots, w_{r-1}, w_r, \ldots, w_{r+t}$ for some $t \geq 0$ and w_{r+j} in $\mathbb{C}_\infty$, $j \in \{0, \ldots, t\}$. Therefore, for each i between 1 and $r+t$ we get

$$\mu w_i = \sum_{j=1}^{r+t} a_{i,j} w_j.$$

for $a_{i,j} \in K$. Since K is the fraction field of A, we can choose $c \in A$ with enough factors such that the product $ca_{i,j}$ lies in A for every i and j. Thus $c\mu$ is in $\mathrm{End}(\Lambda_w^s) \otimes K$ and so $\mathrm{End}(\Lambda_w^s)$ is an order of K_w with rank r. □

Remark 4.5.4. Propositions 4.5.2 and 4.5.3 may be viewed as a generalisation for arbitrary ring A (not necessarily $A = \mathbb{F}_q[T]$) of Hamahata's result [Ham03, Theorem 3.6].

4.6 Special values of Drinfeld modular forms

We are now ready to pesent the main contribution of this thesis, mainly an analogue of Shimura's result on the special values of modular forms at CM points. The strategy of our proof is to follow the lines of the classical result, that we described in section 1.3.

Let N be a nonzero ideal of A and consider the subgroups Γ_s such that

$$M_N^r(\mathbb{C}_\infty) \cong \bigsqcup_{s \in \mathcal{S}(N)} \Gamma_s \setminus \Omega^r(\mathbb{C}_\infty).$$

We fix a representative $s \in \mathcal{S}(N)$. For any Drinfeld module ϕ over an extension F/K, we recall that $\phi[N]$ denotes its group of N-torsion points, i.e. the set of roots in $\overline{K}$ of ϕ_a for all $a \in N$.

Theorem 4.6.1. *Let $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ be a Drinfeld modular form of weight k rank r for Γ_s which is arithmetic over a finite extension K_f/K. Let $w \in \Omega^r(\mathbb{C}_\infty)$ be a CM point. Then there exists a period $\mathbf{\Omega}_w \in \mathbb{C}_\infty^\times$ such that*

$$\frac{f(w)}{\mathbf{\Omega}_w^k} \in H_w(N)K_f$$

were $H_w(N) := H_w(\phi_w^s[N])$ and H_w is the Hilbert class field of the CM field K_w.

Proof. Let $w \in \Omega^r(\mathbb{C}_\infty)$ be a CM point. Let ϕ_w^s be the associated Drinfeld module together with its associated level-N structure $\alpha_N^s : (N^{-1}/A)^r \xrightarrow{\sim} \phi_w^s[N]$. By proposition 4.5.3, the Drinfeld module ϕ_w^s has CM by K_w and thus, by theorem 4.4.3, its minimal field of definition is the Hilbert class field of $\mathrm{End}_{\mathbb{C}_\infty}(\phi)$, denoted H_w. We may therefore assume that it is defined over H_w, otherwise we replace it by an isomorphic Drinfeld module defined over H_w. In particular, the pair (ϕ_w^s, α_N^s) will be defined over $H_w(N)$.

Next, since f is arithmetic over K_f, it corresponds to an algebraic modular form $F \in G_k(N, K_f)$ such that $p_s(F) = f$ where $p_s : \mathcal{G}_k^r(N, K_f) \twoheadrightarrow \mathcal{M}_k^{\mathrm{an},r}(\Gamma_s)$ is the projection. Fixing an embedding $K_f \hookrightarrow H_w(N)K_f$, we see F as an element of $G_k(N, H_w(N)K_f)$ and so we may evaluate it at the triple $(\phi_w^s, \alpha_N, \omega)$ for any nonzero section ω of $\overline{\boldsymbol{\omega}}$:

$$F(\phi_w^s, \alpha_N, \omega) \in H_w(N)K_f.$$

Let $H_w(N)K_f \hookrightarrow \mathbb{C}_\infty$ be an embedding and consider the induced scheme morphism

$$g : \mathrm{Spec}(\mathbb{C}_\infty) \to \mathrm{Spec}(H_w(N)K_f).$$

The pullback of ω by g is a nonzero section of the invertible sheaf $g^*\overline{\boldsymbol{\omega}} = \mathbb{C}_\infty \mathrm{d}z$ where $\mathrm{d}z$ is the invariant differential of ϕ_w^s. Therefore, we have $g^*\omega = \mathbf{\Omega}_w \mathrm{d}z$ for some nonzero constant $\mathbf{\Omega}_w$ of $\mathbb{C}_\infty$. By the weight k property of an algebraic modular form, we have the following relation:

$$F(\phi_w^s, \alpha_N, \omega) = \frac{F(\phi_w^s, \alpha_N, \mathrm{d}z)}{\mathbf{\Omega}_w^k}.$$

Since $p_s(F) = f$ and $f(w) = F(\phi_w^s, \alpha_N, \mathrm{d}z)$, we conclude that $f(w)/\mathbf{\Omega}_w^k \in H_w(N)K_f$. $\square$

Remark 4.6.2. 1. It is still unknown whether the set of CM points is dense in $\Omega^r(\mathbb{C}_\infty)$. This fact would imply a converse of the above theorem.

2. This result states only the existence of the period $\mathbf{\Omega}_w$. It could be interesting to construct it explicitely. We will see in the next section that for $A = \mathbb{F}_q[T]$, one can obtain an explicit value for the period.

3. One can relax the notion of complex multiplication of a Drinfeld module by simply assuming that the field $\mathrm{End}(\phi) \otimes K$ is of degree r over K, without requiring that ∞ is inert in it. In this case, the same proof implies that $f(w)/\mathbf{\Omega}_w^k \in F_w K_f$ where F_w is an extension of $K_w = \mathrm{End}(\phi_w^s) \otimes K$.

As a consequence of theorem 4.6.1, we are now able to prove the analogue of Shimura's result:

Corollary 4.6.3. *Let $f : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ be a Drinfeld modular form of weight k for $\mathrm{GL}_r(A)$ which is arithmetic over K. Let $w \in \Omega^r(\mathbb{C}_\infty)$ be a CM point. Then, there exists a period $\mathbf{\Omega}_w \in \mathbb{C}_\infty$ such that $f(w)/\mathbf{\Omega}_w^k \in H_w$.*

Proof. Since f is of level 1, there exists an ideal $N \subset A$ such that $\mathcal{K}(N)$ is fine and f is of level N (see remark 3.4.2). We will view f as an element of $\mathcal{G}(N, K)$. Let $w \in \Omega^r(\mathbb{C}_\infty)$ be a CM point, then for any representative $s \in S(N)$ and any level-N structure $\alpha_N : (N^{-1}/A) \xrightarrow{\sim} \phi_w^s[N]$ we have $f(\phi_w^s, \alpha_N, \omega) \in H_w(N)$ where ω is a nonzero section of $\overline{\boldsymbol{\varpi}}$. Then, by the base change property of algebraic modular forms, for any automorphism $\sigma \in \mathrm{Gal}(H_w(N)/H_w)$ we have

$$\sigma(f(\phi_w^s, \alpha_N, \omega)) = f(\phi_w^s, \sigma \circ \alpha_N, \omega).$$

We note that ϕ_w^s has CM by K_w, so it is defined over H_w and therefore it is fixed by σ. The same fact applies to the section ω. We also observe that since f is of level 1, its value at a triple $(\phi_w^s, \alpha_N, \omega)$ does not depends on the choice of the level structure. Therefore we have $\sigma(f(\phi_w^s, \alpha_N, \omega)) = f(\phi_w^s, \alpha_N, \omega)$ for all $\sigma \in \mathrm{Gal}(H_w(N)/H_w)$ which implies

$$f(\phi_w^s, \alpha_N, \omega) \in H_w.$$

Finally, we obtain the desired result by pulling back to $\mathbb{C}_\infty$. $\square$

4.7 The case $A = \mathbb{F}_q[T]$

In this section, we fix $A = \mathbb{F}_q[T]$ and we consider modular forms over the full group $\mathrm{GL}_r(A)$. Results in the direction of theorem 4.6.1 for this specific case have already been known in rank two by Chang [Cha12, §2.2] and in arbitrary ranks by Chen and Gezmiş [CG23, §6.2]. We will present the result of Chen and Gezmiş in this section. Beforehand, we need some preliminary definitions and results.

First, let $w = [w_1 : \ldots : w_{r-1} : 1]$ be a point in $\Omega^r(\mathbb{C}_\infty)$ and let Λ_w be the associated lattice. We denote the associated Drinfeld module by

$$\phi^w : T \mapsto T + g_1(w)\tau + \cdots + g_{r-1}(w)\tau^{r-1} + g_r(w)\tau^r.$$

where $g_i : \Omega^r(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ are Drinfeld modular forms of weight $q^i - 1$ and type 0 for $\mathrm{GL}_r(A)$. For any $1 \leq i \leq r-1$, we set $g_i^{\mathrm{new}} := \tilde{\pi}^{1-q^i} g_i$. In the rank two case, this normalization has the effect of removing the transcendental parts in the coefficient of expansion at infinity of the form g_1.

Second, through the work of Gekeler [Gek17, Theorem 3.8], there exists a Drinfeld modular form of weight $(q^r - 1)/(q-1)$ and type 1 for $\mathrm{GL}_r(A)$, denoted h_r which satisfies the relation:

$$h_r^{q-1}(w) = \frac{(-1)^r}{T} g_r^{\mathrm{new}}(w).$$

The importance of the coefficient forms g_i and the function h_r comes from the following result:

Proposition 4.7.1. *Let $\mathcal{M}_\bullet^r$ be the graded ring of Drinfeld modular forms of rank r for $\mathrm{GL}_r(A)$ and arbitrary type. Then*

$$\mathcal{M}_\bullet^r \cong \mathbb{C}_\infty[g_1, \ldots, g_{r-1}, h_r]$$

as $\mathbb{C}_\infty$-algebra. Moreover, the subring $\mathcal{M}_\bullet^{r,0} \subset \mathcal{M}_\bullet^r$ of forms of type 0 is

$$\mathcal{M}_\bullet^{r,0} \cong \mathbb{C}_\infty[g_1, \ldots, g_{r-1}, g_r].$$

The proof of the above result may be found in [BBP18c, Theorem 17.5]. Using this result, we say that a Drinfeld modular form f of rank r is *arithmetic* if f lies in $\overline{K}[g_1^{\mathrm{new}}, \ldots, g_{r-1}^{\mathrm{new}}, h_r]$.

Next, Chen and Gezmiş consider a relaxed notion of CM Drinfeld modules in the sense that they don't require the place ∞ to be inert in the CM field. More precisely, a Drinfeld module ϕ over $\mathbb{C}_\infty$ is said to be CM if $\mathrm{End}(\phi) \otimes_A K$ has degree r over K. The effect of considering this definition is that the minimal base field of a CM Drinfeld module will not be the Hilbert class field of the CM field in general. Assuming this relaxed notion of CM Drinfeld modules, we define the notion of CM points in $\Omega^r(\mathbb{C}_\infty)$ accordingly.

From now on, we fix a CM point w in $\Omega^r(\mathbb{C}_\infty)$ and let ϕ^w be the associated CM Drinfeld module. We moreover choose an element $\lambda_w \in \mathbb{C}_\infty^\times$ so that $g_r(w)\lambda_w^{1-q^r} = 1$. Then, Chen and Gezmiş proves the following theorem [CG23, Theorem 6.5]:

Theorem 4.7.2. *Let f be an arithmetic Drinfeld modular form of weight k and type m for* $\mathrm{GL}_r(A)$. *Then we have*

$$\frac{f(w)}{\mathbf{\Omega}_w^k} \in \overline{K},$$

where $\mathbf{\Omega}_w := \lambda_w/\tilde{\pi}$.

In particular, this theorem states that the special value of a Drinfeld modular form at a CM point is transcendental over $\overline{K}$. Moreover, Chen and Gezmiş make use of the above theorem in order to prove the transcendence of some special functions at CM points [CG23, Theorem 6.11]. Our result, theorem 4.6.1, indicate that their result might be generalizable for arbitrary ring of function A and therefore obtain a more precise description of the period $\mathbf{\Omega}_w$.

Part II

Computations: Classical Case and Function Field case

The idea behind digital computers may be explained by saying that these machines are intended to carry out any operations which could be done by a human computer.

Alan Turing, [Tur50]

Chapter 5

Graded Rings of (Quasi)modular Forms

In this chapter, we give a brief review of quasimodular forms in SageMath and then explain some related algorithmic aspects. In particular we showcase a SageMath implementation of the graded ring of modular and quasimodular forms in the software. More precisely, we enhanced significantly the implementation rings of classical modular forms and completely added the support of quasimodular forms.

5.1 Background material

5.1.1 Quasimodular forms

Let $k \geq 4$ be an integer. We recall that the (normalized) weight k Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(cz+d)^k}. \tag{5.1}$$

The sum defining E_k is uniformly and absolutely convergent for every $z \in \mathcal{H}$ and $k \geq 4$. Moreover, it admits the following expansion at infinity:

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where B_k is the k-th Bernoulli number defined by the expansion

$$\frac{xe^x}{e^x - 1} = \sum_{k\geq 0} \frac{B_k}{k!} x^k.$$

One may generalize (5.1) to the case $k = 2$. However, in this case the defining sum is not absolutely convergent anymore, and so the value of the summation depends of its ordering. Therefore, we choose the following ordering:

$$G_2(z) := \sum_{d \in \mathbb{Z} \setminus \{0\}} \frac{1}{d^2} + \sum_{c \in \mathbb{Z} \setminus \{0\}} \sum_{d \in \mathbb{Z}} \frac{1}{(cz+d)^2}$$

which brings us to the definition:

Definition 5.1.1. For $z \in \mathcal{H}$, the *(normalized) weight* 2 *Eisenstein series* is defined by

$$E_2(z) := \frac{1}{2\zeta(2)} G_2(z) = 1 + \frac{1}{2\zeta(2)} \sum_{c \in \mathbb{Z} \setminus \{0\}} \sum_{d \in \mathbb{Z}} \frac{1}{(cz+d)^2}.$$

Remark 5.1.2. In what follows, we will loose the adjective "normalized" and simply use the terminology "weight k Eisenstein series" in order to denote the series E_k for any even $k \geq 2$.

Proposition 5.1.3. *Let z be in the complex upper half plane. Then, we have:*

1. *the series $E_2(z)$ admits the expansion: $E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$;*
2. *for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$, we have $(E_2|_2\gamma)(z) = E_2(z) - \frac{6}{\pi i}\left(\frac{c}{cz+d}\right)$.*

We see from the above propostion that the weight 2 Eisenstein series is very close to being a modular form. For that reason, it is called a *quasimodular form*. The following definition is taken from [Zag08, section 5.3]:

Definition 5.1.4. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let k and $p \geq 0$ be two integers. An holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is said to be a *quasimodular form of weight k and depth at most p for* Γ if there exists holomorphic functions $f_i : \mathcal{H} \to \mathbb{C}$, $i \in \{0, \ldots, p\}$, such that

$$(f|_k\gamma)(z) = \sum_{i=0}^{p} f_i(z) \left(\frac{c}{cz+d}\right)^i \tag{5.2}$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ and any z in $\mathcal{H}$. The *depth* of f is the largest i for which f_i is not identically zero. The set of all quasimodular form of weight k and depth at most p for Γ is denoted $\widetilde{\mathcal{M}}_k^{\leq p}(\Gamma)$. We will also use the following notation

$$\widetilde{\mathcal{M}}_k(\Gamma) := \sum_{p=0}^{\infty} \widetilde{\mathcal{M}}_k^{\leq p}(\Gamma)$$

in order to denote the set of quasimodular form of fixed weight and arbitrary depth. The graded ring of quasimodular forms is denoted

$$\widetilde{\mathcal{M}}_{\bullet}(\Gamma) := \bigoplus_{k\geq 0} \widetilde{\mathcal{M}}_k(\Gamma).$$

Remark 5.1.5. The terminology "quasimodular forms" was first introduced by Kaneko and Zagier in [KZ95]. Their original definition was crafted via the theory of nearly holomorphic modular forms. More precisely, a *nearly (or almost) holomorphic modular form of weight k and depth at most p* is a function $F : \mathcal{H} \to \mathbb{C}$ which is invariant under the $|_k$-operator and can be written as a polynomial in $Y := (-4\pi\mathrm{Im}(z))^{-1}$ with holomorphic coefficients:

$$F(z) = f_0(z) + f_1(z)Y + \cdots + f_p(z)Y^p.$$

Kaneko and Zagier then define a quasimodular form to be the constant term $f_0(z)$ of this polynomial. The map sending a nearly holomorphic modular form to its constant term defines a bijection between the two sets. One advantage of working with nearly holomorphic modular forms is that we keep the $|_k$-invariance property at the cost of loosing the holomorphic properties. On the quasimodular side, we keep the holomorphic properties while loosing the $|_k$-invariance.

Proposition 5.1.6. *Let f be a quasimodular form of weight k and depth at most p for Γ and let f_i, $i \in \{0, \ldots, p\}$, be the holomorphic functions defined by (5.2). Then each f_i is a quasimodular form of weight $k - 2i$ for Γ of depth at most $p - i$.*

Proof. See proposition 3.3 of [Roy12]. □

Corollary 5.1.7. *Under the notations of proposition 5.1.6, if f has depth exactly p then the function f_p is a modular form of weight $k - 2p$.*

Proof. This follows simply by the fact that any quasimodular form of depth 0 is a modular form. □

Theorem 5.1.8. *Let Γ be a congruence subgroup and k and $p \geq 0$ be two integers, then we have*

$$\widetilde{\mathcal{M}}_k^{\leq p}(\Gamma) \cong \bigoplus_{j=0}^{p} \mathcal{M}_{k-2j}^{\mathrm{ell}}(\Gamma) E_2^j.$$

Proof. We proceed by induction on the depth. First, we observe that, by defintion, any weight k quasimodular form of depth 0 is in fact a modular form. Next, let f be a quasimodular form of weight k and depth p for Γ and let f_i, $i \in \{0, \dots, p\}$, be the holomorphic function satisfying (5.2). Then, we claim that the function

$$h(z) := f(z) + (-1)^{p-1} \left(\frac{i\pi}{6}\right)^p f_p(z) E_2(z)^p$$

has depth at most $p-1$. Indeed, a simple calculation yields

$$\begin{aligned}
(h|_k\gamma)(z) &= (f|_k\gamma)(z) + (-1)^{p-1}(cz+d)^{-k} \left(\frac{i\pi}{6}\right)^p f_p(\gamma z) E_2(\gamma z)^p \\
&= (f|_k\gamma)(z) + (-1)^{p-1}(cz+d)^{-k+2p} f_p(\gamma z) \left(\frac{i\pi}{6} E_2|_2\gamma(z)\right)^p \\
&= \sum_{j=0}^{p} f_j(z) \left(\frac{c}{cz+d}\right)^j + (-1)^{p-1} f_p(z) \left(\frac{i\pi}{6} E_2(z) - \frac{c}{cz+d}\right)^p,
\end{aligned}$$

where on the last equality we have used the fact that f_p is a modular form of weight $k-2p$. Expanding the last term using the binomial theorem, we see that h have depth at most $p-1$ and, by the induction hypothesis, it must lie in $\oplus_{j=0}^{p-1} \mathcal{M}_{k-2j}^{\mathrm{ell}}(\Gamma) E_2^j$. This gives us

$$f(z) = h(z) + (-1)^p \left(\frac{i\pi}{6}\right)^p f_p(z) E_2(z)^p \in \bigoplus_{j=0}^{p} \mathcal{M}_{k-2j}^{\mathrm{ell}}(\Gamma) E_2^j,$$

as desired. □

Corollary 5.1.9. *Every weight k quasimodular form for Γ have depth at most $\lceil \frac{k}{2} \rceil$.*

Proof. For any $j \geq \lceil \frac{k}{2} \rceil$, we have $k - 2j \leq 0$. Thus $\mathcal{M}_{k-2j}^{\mathrm{ell}}(\Gamma) = 0$ and the result follows from theorem 5.1.8. □

Corollary 5.1.10. $\widetilde{\mathcal{M}}_\bullet(\Gamma) = \mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)[E_2]$.

Definition 5.1.11. For any holomorphic function $f : z \mapsto f(z)$, we define the derivative

$$Df := \frac{1}{2\pi i} \frac{\mathrm{d}f}{\mathrm{d}z}.$$

If f is a weight k modular form for Γ which admits a q-expansion of the form $\sum_n a_n(f) q^n$, then we have $Df = q\frac{\mathrm{d}f}{\mathrm{d}q}$.

Definition 5.1.12. For any modular form f of weight k for Γ, we define the *weight k Serre derivative* by

$$\theta_k f := Df - \frac{k}{12}E_2 f.$$

Proposition 5.1.13. *θ_k sends any weight k modular form to a weight $k+2$ modular form.*

Proof. One need to simply verify that $\theta_k f$ is $|_k$-invariant for every γ in Γ, which is a straightforward computation. $\square$

Via theorem 5.1.8 and the definition of the operator θ_k, one can prove:

Proposition 5.1.14. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For any k and $p \geq 0$, we have*

$$D(\widetilde{\mathcal{M}}_k^{\leq p}(\Gamma)) \subset \widetilde{\mathcal{M}}_k^{\leq p+1}(\Gamma).$$

5.1.2 Graded rings of (quasi)modular forms

We recall the ring of modular forms:

$$\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma) := \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_k^{\mathrm{ell}}(\Gamma).$$

This ring is in fact a graded $\mathbb{Z}$-algebra where the graduation is given by the weight of a modular form f.

Proposition 5.1.15. *$\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)$ is of finite type.*

Proof. This is théorème 3.4 of [DR73, p. 303]. $\square$

In the case of the full modular group, we recall that $\mathcal{M}_\bullet^{\mathrm{ell}}(\mathrm{SL}_2(\mathbb{Z}))$ is generated by E_4 and E_6 and these two Eisenstein series are algebraically independent, meaning that

$$\mathcal{M}_\bullet^{\mathrm{ell}}(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6].$$

For a general congruence subgroup Γ, this is not necessarily the case as there can be some relations between the generators of the ring. In other words, we will have

$$\mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma) \cong \mathbb{C}[g_{1,k_1}, g_{2,k_2}, \ldots, g_{n,k_n}]/I_\Gamma$$

where for each $i \in \{1, \ldots, n\}$, g_{i,k_i} is a modular form in $\mathcal{M}_{k_i}(\Gamma)$ and I_Γ is some ideal in $\mathbb{C}[g_{1,k_1}, g_{2,k_2}, \ldots, g_{n,k_n}]$. Even thought the exact structure of these modular forms rings remains to be discovered, some advancement have been made in that direction during the past years. For example, in

Example 5.1.16. We consider $\mathcal{M}^{\mathrm{ell}}_{\bullet}(\Gamma_0(3))$. Then in section 5.5 we will see that $\mathcal{M}^{\mathrm{ell}}_{\bullet}(\Gamma_0(3))$ is generated by three modular forms $g_{1,2}, g_{2,4}$ and $g_{3,6}$. One may also verify using computational method that we have the following example:

$$-g_{1,2}^4 + 6g_{1,2}^2 g_{2,4} - 8g_{1,2}g_{3,6} + 3g_{2,4}^2 = 0.$$

5.1.3 Sturm bounds

We end the section on background material by stating an essential tool when computing with modular forms. This tool is the Sturm bound and it allows to determine whether any form in $\mathcal{M}^{\mathrm{ell}}_k(\Gamma_1(N))$ is identically zero or not simply by computing a finite number of its Fourier coefficients.

Theorem 5.1.17 (Sturm bound)**.** *Let Γ be a congruence subgroup containing $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and let f be a form in $\mathcal{M}^{\mathrm{ell}}_k(\Gamma)$ which admits the q-expansion $f = \sum_{n\geq n_0} a_n(f)q^n$ for some $n_0 \in \mathbb{Z}_{\geq 0}$. If we have*

$$n_0 > \left\lfloor \frac{k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{12} \right\rfloor$$

then $f \equiv 0$.

Proof. This result can be proven by studying the valence formula of a modular form. A proof can be found in the original paper of Sturm [Stu87, Theorem 1]. □

In the specific case where $\Gamma = \Gamma_1(N)$ for some integer N, then one may get a better bound for modular forms with fixed character:

Proposition 5.1.18. *Let χ be a Dirichlet character modulo N and let f be a form in $\mathcal{M}^{\mathrm{ell}}_k(N, \chi)$ with q-expansion $f = \sum_{n\geq n_0} a_n(f)q^n$ for some $n_0 \in \mathbb{Z}_{\geq 0}$. If we have*

$$n_0 > \left\lfloor \frac{kN}{12} \prod_{p|N} (1 + 1/p) \right\rfloor$$

then $f \equiv 0$.

Proof. See corrolary 9.20 of [Ste07]. We note that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N}(1 + 1/p)$, see for example [DS05, §1.2]. □

Definition 5.1.19. Given a congruence subgroup Γ, the quantity

$$B_k(\Gamma) := \left\lfloor \frac{k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{12} \right\rfloor$$

is called the *Sturm bound of weight k for Γ*.

5.2 Classical modular forms in SageMath

5.2.1 Spaces of modular forms

In this section, we give a brief overview of modular forms in SageMath and their implementation. A recurring challenge when implementing a general mathematical object into a computer program is finding the right data to use in order to represent the object with most of its properties in the most efficient way. For example, to represent a univariate polynomial

$$a_0 + a_1X + \cdots + a_iX^i + \cdots + a_dX^d, \quad a_i \in \mathbb{Q}$$

one could think of at least two ways of doing it. First, by storing the coefficients in an ordered list

$$[a_0, a_1, \ldots, a_i, \ldots, a_d]$$

where the coefficient a_i is at position i in the list. Second, one could store a list of tuples

$$[(a_{n_0}, n_0), \ldots, (a_{n_i}, n_i), \ldots, (a_{n_d}, n_d)]$$

where a_{n_i} is the i-th nonzero coefficient of the polynomial and n_i is the degree of its corresponding monomial. These two approach equally represent the same polynomial. However, we observe here that the first approach can become problematic for high degree polynomials having a lot of zero coefficients (e.g. $1 + X^{10^5}$) as it would require to store a lot of zero coefficients. Thus, in this case, it is much more efficient in terms of data storage to use the second approach.

In light of the above example, we may wonder what would be an acceptable implementation of a modular form in a software. If Γ is either $\mathrm{SL}_2(\mathbb{Z}), \Gamma_0(N)$ or $\Gamma_1(N)$, then using the Sturm bound, one could simply represent a modular form as a finite number of Fourier coefficients. However, similarly to polynomials, this is not ideal as the sturm bound can be relatively large. For example, the sturm bound of the space $\mathcal{M}_{12}^{\mathrm{ell}}(\Gamma_1(17))$ is 289[1]. Another

[1] Computed using SageMath with the command `Gamma1(17).sturm_bound(12)`.

approach is to use the fact that the spaces of modular forms for Γ are finite dimensional $\mathbb{Q}$-vector spaces. Hence, by fixing a basis of $\mathcal{M}_k^{\mathrm{ell}}(\Gamma)$, one has the isomorphism of $\mathbb{Q}$-vector spaces:

$$\mathcal{M}_k^{\mathrm{ell}}(\Gamma) \cong \mathbb{Q}^d$$

for some integer d. Fortunately, one can compute explicit basis for the spaces $\mathcal{M}_k^{\mathrm{ell}}(\Gamma)$ when $\Gamma = \mathrm{SL}_2(\mathbb{Z}), \Gamma_0(N)$ or $\Gamma_1(N)$ (and even for $\mathcal{M}_k^{\mathrm{ell}}(N,\chi)$ for some Dirichlet character χ). Therefore, it makes sense to say that the datum needed to represent a modular form is a triple (k, N, v), where k and N are two integers representing the weight and the level respectively and v is vector of dimension d with coefficients in $\mathbb{Q}$. This approach is the main idea behind current implemention in SageMath.

SageMath session 5.2.1.
In the following session, we create the space $\mathcal{M}_5^{\mathrm{ell}}(\Gamma_1(3))$ which is of dimension 2 and then consider the two basis elements of this space so that $\mathcal{M}_5^{\mathrm{ell}}(\Gamma_1(3)) \cong \mathbb{Q}^2$.

```
sage: M = ModularForms(Gamma1(3), 5)
sage: M.basis()
[
1 - 90*q^2 - 240*q^3 - 3744*q^5 + O(q^6),
q + 15*q^2 + 81*q^3 + 241*q^4 + 624*q^5 + O(q^6)
]
sage: f = M.0  # first basis element
sage: g = M.1  # second basis element
```

One may use the method `element` on any modular forms to get its representation in $\mathbb{Q}^2$:

```
sage: f.element()
(1, 0)
sage: g.element()
(0, 1)
sage: ((1/2)*f + (7/11)*g).element()
(1/2, 7/11)
```

It is also possible to construct any combination of f and g:

```
sage: M([1, 1])
1 + q - 75*q^2 - 159*q^3 + 241*q^4 - 3120*q^5 + O(q^6)
```

```
sage: M([1, 1]) == f + g
True
```

The above session shows that it is possible to create spaces of modular forms in SageMath and manipulate then as formal symbolic objects. Moreover, the implementation does not depends on the q-expansion and is computed simply "on demand". Finally, we end this section by mentioning that algorithms for computing basis of modular forms spaces are well known and implemented in a variety of computer algebra system such as SageMath and PARI/GP. Most of these algorithms uses modular symbols methods in order to compute the cuspidal subspace. We refer the reader to [Ste07, §2.3 & §5.3] and [BBB$^+$21, §4 & §5] for more about the subject.

5.2.2 Rings of modular forms

Rings of modular forms are also implemented in SageMath. However, the elements of these rings were simply stored as mere q-expansion. Thus, as of version 9.5 of SageMath, we implemented formal ring objects for modular forms ring. More precisely, if Γ is either $\mathrm{SL}_2(\mathbb{Z}), \Gamma_0(N)$ or $\Gamma_1(N)$, then an element F of $\mathcal{M}^{\mathrm{ell}}_{\bullet}(\Gamma)$ is now represented in SageMath as a sequence $F = (F[k])_{k\in\mathbb{Z}}$ where $F[k]$ is a form in $\mathcal{M}^{\mathrm{ell}}_{k}(\Gamma)$ for finitely many $k \in \mathbb{Z}$ and zero otherwise. The form $F[k_i]$ will be called a *homogeneous component of F of weight k_i*. Observe that in general F is not modular form as it may have mixed weight components. Moreover, note that, as for the implementation of a polynomial, we do not store unecessarily zero elements.

SageMath session 5.2.2.
In the following session, we create the ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and consider its generators E_4 and E_6.

```
sage: M = ModularFormsRing(1)
sage: E4 = M.0
sage: E6 = M.1
sage: D = (E4^3 - E6^2)/1728
sage: F = E4 + E6 + D
sage: F[4]
1 + 240*q + 2160*q^2 + 6720*q^3 + 17520*q^4 + 30240*q^5 + O(q^6)
```

```
sage: F[6]
1 - 504*q - 16632*q^2 - 122976*q^3 - 532728*q^4 - 1575504*q^5
    + O(q^6)
sage: F[12]
q - 24*q^2 + 252*q^3 - 1472*q^4 + 4830*q^5 + O(q^6)
sage: F[8]
0
```

This new representation required the implementation of the *Parent/Element framework* for the ring of modular forms. This SageMath's specific development framework allows the implementation of mathematical objects that contains elements and defines the algebraic operations. A simple example of a Parent/Element structure is the ring of integers: the ring as a whole together with its ring operation maps is a *parent* and the integers are the *elements*. In other words, one goal of this framework is to model a general algebraic structure.

SageMath session 5.2.3.
In this session, we first create the space $\mathcal{M}^{\mathrm{ell}}_{\bullet}(\Gamma_1(5))$, which is a parent:

```
sage: M = ModularFormsRing(Gamma1(5))
sage: M
Ring of Modular Forms for Congruence Subgroup Gamma1(5) over
    Rational Field
sage: isinstance(M, Parent)
True
```

Then, we create an element of this ring:

```
sage: F = M.0
sage: F
1 + 60*q^3 - 120*q^4 + 240*q^5 + O(q^6)
sage: F in M  # containement check
True
```

In object-oriented computer programming terminology, we say that `ModularFormsRing` is a *class* which *inherits* from the class `Parent` (inheritance means gaining the properties of some other class). In the above session, `M` is an *object*, which is more precisely an *instance* of the class `ModularFormsRing` (any object is an instance of some class).

5.3 Quasimodular forms rings in SageMath

Now that the rings of modular forms follows the Parent/Element framework, it becomes relatively easy to implement rings of quasimodular forms in SageMath. Indeed, recall that for any congruence subgroup Γ, we have

$$\widetilde{\mathcal{M}}_\bullet(\Gamma) = \mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)[E_2].$$

In other words, any elements F of the ring of quasimodular forms may be represented as a univariate polynomial in E_2 over the ring of modular forms:

$$F = F_0 + F_1 E_2 + \cdots + F_p E_2^p$$

where $p \geq 0$ and $F_i \in \mathcal{M}_\bullet^{\mathrm{ell}}(\Gamma)$.

SageMath session 5.3.1.
In this session, we create the ring of quasimodular form for the full modular group:

$$\tilde{\mathcal{M}}_\bullet(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_2, E_4, E_6].$$

We first create the ring and access its generators:

```
sage: QM = QuasiModularForms(SL2Z)
sage: QM.ngens()  # number of generators
3
sage: E2 = QM.0
sage: E4 = QM.1
sage: E6 = QM.2
```

We may then perform any simple algebraic manipulations with these generators:

```
sage: F = E6 + E4*E2 + E2^3
sage: F
3 - 360*q - 18792*q^2 - 189216*q^3 - 950760*q^4 - 3171312*q^5
   + O(q^6)
```

We emphasise that a graded quasimodular forms is not necessarily homogeneous in the weight:

```
sage: G = E2 + E4 + E6
```

```
sage: G[2]  # weight 2 component
1 - 24*q - 72*q^2 - 96*q^3 - 168*q^4 - 144*q^5 + O(q^6)
sage: G[4]  # weight 4 component
1 + 240*q + 2160*q^2 + 6720*q^3 + 17520*q^4 + 30240*q^5 + O(q^6)
sage: G[6]  # weight 6 component
1 - 504*q - 16632*q^2 - 122976*q^3 - 532728*q^4 - 1575504*q^5 + O(q^6)
```

The congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ are also supported:

```
sage: QM = QuasiModularForms(Gamma1(7))
sage: QM.ngens()
13
sage: QM.0  # the first generator is always E2
1 - 24*q - 72*q^2 - 96*q^3 - 168*q^4 - 144*q^5 + O(q^6)
sage: QM.0*QM.1 + QM.3 + QM.6*QM.0^2
2 - 72*q + 361*q^2 + 3168*q^3 + 9288*q^4 + 21700*q^5 + O(q^6)
```

All the new features of this new implementation were not covered in the above session. The interested reader may consult the SageMath reference manual[2]. In the subsequent section, we will be interested in giving an application of the implementation of quasimodular forms.

5.4 Computing derivatives

The first application of quasimodular forms in SageMath that we present consist of the symbolic calculation of the derivative of any modular form. The derivation operator is defined by:

$$Df := \frac{1}{2\pi i}\frac{\mathrm{d}f}{\mathrm{d}z} = q\frac{\mathrm{d}f}{\mathrm{d}q}.$$

In general this derivative sends a modular form to a quasimodular form since we have the following relation

$$Df = \theta_k(f) + \frac{k}{12}E_2 f,$$

[2] https://doc.sagemath.org/html/en/reference/modfrm/index.html#quasimodular-forms

where θ_k is the Serre derivative of a modular form f of weight k defined by

$$\theta_k : f \longmapsto Df - \frac{k}{12}E_2 f.$$

Recall that θ_k sends any modular form of weight k to a modular form of weight $k+2$. Thus, most of the computation behind the derivative Df is hidden behind the computation of $\theta_k(f)$. The computation of the Serre derivative is relatively straightforward and mostly require the computation of the q-expansion of the given modular form up to a sufficient precision (which is essentially the Sturm bound).

SageMath session 5.4.1.
In this session, we illustrate the algorithm for computing the Serre derivative.

```
65 sage: M4 = ModularForms(SL2Z, 4)  # weight 4 space
66 sage: B = SL2Z.sturm_bound(6) + 1  # Sturm bound + 1
67 sage: E2 = eisenstein_series_qexp(2, prec=B, normalization='constant')  # weight 2 Eisenstein series
68 sage: E4 = M4.0.q_expansion(prec=B)
69 sage: q = E4.parent().gen()  # generator of the power series ring
70 sage: thetaE4 = q*E4.derivative() - (4/12)*E2*E4
71 sage: thetaE4
72 -1/3 + 168*q + O(q^2)
```

The variable `thetaE4` above is simply stored as a q-expansion object and not an actual modular form. To transform it into a modular form, one has to first create the space of weight 6 and then coerce the form into the space:

```
73 sage: M6 = ModularForms(SL2Z, 6)  # weight 6 space
74 sage: serre_E4 = M6(thetaE4)  # this coerce the q-expansion into a modular form object
75 sage: serre_E4
76 -1/3 + 168*q + 5544*q^2 + 40992*q^3 + 177576*q^4 + 525168*q^5 + O(q^6)
```

The process that happens at line 74 is that SageMath consider the q-expansion up to the Sturm bound for each basis element of the space and then create the vector space generated by these q-expansion vector. Next, it simply finds the right linear combination corresponding to the q-expansion `thetaE4`. In the specific case of this session, $M_6(\mathrm{SL}_2(\mathbb{Z}))$ is of dimension 1, thus generated by E_6, and we have $\theta_4(E_4) = -E_6/3$. This last identity was well known by Ramanujan amongst many others [Ram00].

The computation presented above can now be performed in a single command using the method `serre_derivative`:

```
sage: E4 = ModularForms(SL2Z, 4).0  # weight 4 Eisenstein series
sage: f = E4.serre_derivative()
sage: f
-1/3 + 168*q + 5544*q^2 + 40992*q^3 + 177576*q^4 + 525168*q^5 + O(q^6)
sage: serre_E4 == E4.serre_derivative()
True
sage: f in ModularForms(SL2Z, 6)
True
```

In light of the implementation of the Serre derivative, we may now easily compute the derivative of any modular form.

SageMath session 5.4.2.
As for the implementation of the Serre derivative, we illustrate here the idea behind the implementation of the derivative of a modular form.

```
sage: QM = QuasiModularForms(SL2Z)  # Ring of quasimodular forms
sage: E4 = QM.1  # Weight 4 Eisenstein series
sage: E2 = QM.weight_2_eisenstein_series()
sage: DE4 = E4.serre_derivative() + (E4.weight()/12)*E2*E4
sage: DE4
240*q + 4320*q^2 + 20160*q^3 + 70080*q^4 + 151200*q^5 + O(q^6)
```

The above computations are simplified by calling the method `derivative`:

```
sage: QM = QuasiModularForms(SL2Z)  # Ring of quasimodular forms
sage: E4 = QM.1  # Weight 4 Eisenstein series
```

```
sage: E4.derivative()
240*q + 4320*q^2 + 20160*q^3 + 70080*q^4 + 151200*q^5 + O(q^6)
sage: DE4 == E4.derivative()
True
```

5.5 Algebraic relations between rings generators

Recall that $\mathcal{M}_{\bullet}^{\mathrm{ell}}(\Gamma)$ is a finitely generated ring and therefore, by fixing a generating set $\{g_0, \ldots, g_n\} \subset \mathcal{M}_{\bullet}^{\mathrm{ell}}(\Gamma)$, there exists a surjective map

$$\begin{aligned}\mathbf{\Phi}_{\Gamma} : \mathbb{C}[X_0, \ldots, X_n] &\longrightarrow \mathcal{M}_{\bullet}^{\mathrm{ell}}(\Gamma) \\ X_i &\longmapsto g_i\end{aligned}$$

for some $n \geq 1$. For every $0 \leq i \leq n$, we set $\deg(X_i)$ to be the weight of g_i. Hence, the pull back of any homogeneous spaces of weight k is contained inside the subring of homogeneous polynomials of degree k. In the case of the full modular group, we have $n = 2$ and this map is a bijection. The kernel $I_{\Gamma} := \ker(\mathbf{\Phi}_{\Gamma})$ is called the *ideal of relations*. For our last application, we will see how we may compute an element in the fiber over any element $f \in \mathcal{M}_{\bullet}^{\mathrm{ell}}(\Gamma)$. In other words, given any $f \in \mathcal{M}_{\bullet}^{\mathrm{ell}}(\Gamma)$, we will find a polynomial $P(X_0, \ldots, X_n)$ in $\mathbb{Q}[X_0, \ldots, X_n]$ such that

$$P(g_0, \ldots, g_n) = f.$$

Example 5.5.1. In this example, we write the Eisenstein series of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$ in terms of E_4 and E_6. We know that

$$\mathcal{M}_{\bullet}^{\mathrm{ell}}(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6].$$

Thus, the map $\mathbf{\Phi}_{\mathrm{SL}_2(\mathbb{Z})} : \mathbb{C}[X_0, X_1] \to \mathcal{M}_{\bullet}^{\mathrm{ell}}(\mathrm{SL}_2(\mathbb{Z}))$ sending $X_0 \mapsto E_4$ and $X_1 \mapsto E_6$ is a bijection and there exists a unique polynomial $P(X_0, X_1)$ such that

$$P(E_4, E_6) = E_{12}.$$

The polynomial P must be homogeneous of degree 12 so we deduce that, for some constants c_0 and c_1,

$$P(X_0, X_1) = c_0 X_0^3 + c_1 X_1^2, \tag{5.3}$$

as the only degree 12 monomials are X_0^3 and X_1^2 (recall that, by definition, X_0 and X_1 have degree 4 and 6 respectively). Now, finding the constants c_0 and c_1 is only a matter of linear algebra. Indeed, recall that the Sturm bound of $\mathcal{M}^{\text{ell}}_\bullet(\mathrm{SL}_2(\mathbb{Z}))$ is $B_{12}(\mathrm{SL}_2(\mathbb{Z})) = \lfloor k/12 \rfloor = 1$ and thus by computing the first two coefficients of E_4, E_6 and E_{12} and by substituing everything in (5.3) we get the linear system:

$$\begin{pmatrix} 1 \\ \frac{65520}{691} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 720 & -1008 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

The unique solution to this linear system is given by $c_0 = 441/691$ and $c_1 = 250/691$ and therefore we have

$$E_{12} = \frac{441}{691} E_4^3 + \frac{250}{691} E_6^2.$$

The calculation carried out in example 5.5.1 can be generalized for any modular form f and for any congruence subgroup Γ. A crucial difference is that the polynomial P is not unique in the general case, but this does not affect the procedure.

Procedure 5.5.2. *Let f be a modular form of weight k for a congruence subgroup Γ.*

1. *Compute the Sturm bound $B_k = \lfloor k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/12 \rfloor$;*

2. *Initialize the column vector*

$$\boldsymbol{F} := (a_0(f), \dots, a_{B_k+1}(f))^{\mathrm{T}}$$

   where $a_i(f)$ is the i-th coefficient of f;

3. *Compute $G := \{g_0, \dots, g_n\}$ a generating set of the ring $\mathcal{M}^{\text{ell}}_\bullet(\Gamma)$ and define k_i :=weight of g_i;*

4. *Compute $\{h_0, \dots, h_m\}$ the set of all monomials of weight k given by multiplying elements of G.*

5. *For every $i \in \{0, \dots, m\}$ define the column vector*

$$\boldsymbol{H}_i := (a_0(h_i), \dots, a_{B_k+1}(h_i))^{\mathrm{T}};$$

6. *Solve the linear system:*

$$\big(\boldsymbol{H}_0, \dots, \boldsymbol{H}_m | \boldsymbol{F}\big); \tag{5.4}$$

7. *For any vector* $(c_0, \ldots, c_m)^{\mathrm{T}}$ *in the solutions set of the system 5.4, we have*

$$f = c_0 h_0 + \cdots + c_m h_m.$$

SageMath session 5.5.3.
In the following session, we write the modular discriminant $\Delta \in \mathcal{M}^{\mathrm{ell}}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and the Eisenstein series weight 16 in terms E_4 and E_6, the two generators of $\mathcal{M}^{\mathrm{ell}}_{\bullet}(\mathrm{SL}_2(\mathbb{Z}))$.

```
sage: M = ModularFormsRing(SL2Z)
sage: D = ModularForms(SL2Z, 12).0  # modular discriminant
sage: M(D).to_polynomial("E4,E6")  # E4 and E6 are the names of the variables
1/1728*E4^3 - 1/1728*E6^2
sage: E16 = EisensteinForms(SL2Z, 16).0  # Eisenstein series of weight 16
sage: M(E16).to_polynomial("E4,E6")
1617/3617*E4^4 + 2000/3617*E4*E6^2
```

We also implemented the inverse operation: given a multivariate polynomial $P(X_0, \ldots, X_n)$, the method `from_polynomial` replaces the variables X_i by the generator g_i and returns a graded modular form.

```
sage: E2, E4 = polygens(QQ, 2, "E2,E4")
sage: M.from_polynomial(E2)
1 + 240*q + 2160*q^2 + 6720*q^3 + 17520*q^4 + 30240*q^5 + O(q^6)
sage: M.from_polynomial(E4)
1 - 504*q - 16632*q^2 - 122976*q^3 - 532728*q^4 - 1575504*q^5 + O(q^6)
sage: M.from_polynomial((E2^3 - E4^2)/1728) == D
True
```

SageMath session 5.5.4.
In general, the linear system 5.4 does not possesses a unique solution. More precisely, for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, there might be algebraic relations between the generators of the ring. We illustrate this assertion in the following session.

```
sage: M = ModularFormsRing(Gamma0(7))
sage: M.ngens()
5
sage: g0 = M.0
sage: (g0^4).to_polynomial()
9481/260*x0^2*x2 + x0*x3 - 698/13*x0*x4 + 7599/260*x1*x2 +
   3578/65*x2^2
sage: x0, x1, x2, x3, x4 = polygens(QQ, 5, "x")
sage: M.from_polynomial(x0^4) == g0^4
True
```

In the above computation, we have found that the five generators of $\mathcal{M}^{\mathrm{ell}}_\bullet(\Gamma_0(7))$ satisfy the relation

$$g_0^4 = \frac{9481}{260} g_0^2 g_2 + g_0 g_3 - \frac{698}{13} g_0 g_4 + \frac{7599}{260} g_1 g_2 + \frac{3578}{65} g_2^2.$$

SageMath session 5.5.5.

In this session, we compute three generators for the ring $\mathcal{M}^{\mathrm{ell}}_\bullet(\Gamma_0(3))$, denoted g_0, g_1 and g_2 and compute that

$$g_0^4 = 6g_0^2 g_1 - 8g_0 g_2 + 3g_1^2.$$

```
sage: M = ModularFormsRing(Gamma0(3))
sage: M.ngens()  # number of generators
3
sage: g0, g1, g2 = M.0, M.1, M.2
sage: (g0^4).to_polynomial()
6*x0^2*x1 - 8*x0*x2 + 3*x1^2
```

Chapter 6

Computing with Drinfeld $\mathbb{F}_q[T]$-Modules

The goal of this chapter is to explore some algorithmic aspects of Drinfeld modules. First, we explain how to compute power series approximation of the exponential and the logarithm of any Drinfeld $\mathbb{F}_q[T]$-module over an extension of $\mathbb{F}_q(T)$. Next, we will cover the basic J-invariants theory introduced by Potemine [Pot98] and explain how one may compute them systematically. Lastly, we present a SageMath implementation of those procedures which will be merged in an upcoming version of SageMath.

From now on, we will specialize ourself to the explicit case $A = \mathbb{F}_q[T]$ and $K = \mathbb{F}_q(T)$. Most of the derived properties of this special case comes from two facts. First, A is of class number one, which implies that every A-lattices are *freely* generated. Next, as a $\mathbb{F}_q$-algebra, A is generated by a single element, namely $T \in A$, which gives us that any Drinfeld $\mathbb{F}_q[T]$-module ϕ is determined by the image ϕ_T

$$\phi : T \mapsto \gamma(T) + g_1(\phi)\tau + \cdots + g_{r-1}(\phi)\tau^{r-1} + g_r(\phi)\tau^r$$

where the coefficients $g_i(\phi)$ lives in an A-field F given by $\gamma : A \to F$.

A note on Drinfeld modules in SageMath In May 2023, version 10.0 of SageMath was released and now includes Drinfeld $\mathbb{F}_q[T]$-modules. The development started in 2022 by Leudière [1]. The author of this thesis provided comments and reviewed the code so that it could be merged in the software. Subsequently, this new implementation was enhanced by the addition of new features. In particular, sections 6.1 and 6.2 describes the algorithms

[1] See the GitHub discussion: `https://github.com/sagemath/sage/issues/33713` and the Pull Request: `https://github.com/sagemath/sage/pull/35026`.

which will be merged in version 10.1 of the software. An in depth presentation and tutorial of theses features can be found as parts of [ACLM23][2].

6.1 Exponentials and logarithms

In this section, we suppose that F contains K and $\gamma : A \to F$ is injective. We recall that for any A-lattice Λ, the associated exponential function is defined by

$$e_\Lambda(x) := x \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{x}{\lambda}\right). \tag{6.1}$$

This function is $\mathbb{F}_q$-linear, nonconstant and surjective. Moreover, it admits a series expansion of the form

$$e_\Lambda(x) = x + \alpha_1(\Lambda)x^q + \alpha_2(\Lambda)x^{q^2} + \cdots + \alpha_i(\Lambda)x^{q^i} + \cdots .$$

where $\alpha_i(\Lambda) \in F$. We will see this series as a power series in $F[[x]]$. Given the associated Drinfeld module ϕ^Λ, we have the important functional equation:

$$e_\Lambda(ax) = \phi_a^\Lambda(e_\Lambda(x)). \tag{6.2}$$

By Drinfeld uniformization, we will also say that e_Λ is the exponential of ϕ^Λ. If Λ is not specified, we will denote it by e_ϕ.

Definition 6.1.1. The *logarithm* of the lattice Λ is the compositional inverse of the exponential, denoted $\log_\Lambda$. We will denote the q^i-th coefficient of the logarithm by $\beta_i(\Lambda)$

Remark 6.1.2. The compositional inverse of the exponential exists since its x-valuation is exactly 1.

Next, following [Gek88, (2.6)] we explain a recursive procedure that compute, for any Drinfeld module

$$\phi : a \mapsto \gamma(a) + g_1(a)\tau + \cdots + g_{r-1}(a)\tau^{r-1} + g_r(a)\tau^r,$$

the logarithm of Λ^ϕ. Taking $\log_{\Lambda^\phi}$ on both sides of the functional equation (6.2), we get

$$a \log_{\Lambda^\phi}(x) = \log_{\Lambda^\phi}(\phi_a(x)).$$

[2] Joint work with Caruso, Leudière and Musleh

Expanding the logarithm as a power series, we compare the coefficients on both sides, from which we derive the following recursive sequence:

$$a\beta_k = \sum_{i+j=k} \beta_i g_j(a)^{q^i}.$$

We may rewrite it as

$$\beta_k = \frac{1}{a - a^{q^k}} \sum_{i=0}^{k-1} \beta_i g_{k-i}(a)^{q^i}$$

where $\beta_0 = 0$, $\beta_1 = 1$ and $g_0(a) = \gamma(a)$. The resulting sequence $(\beta_k)_{k\geq 0}$ defines the power series for the logarithm.

Conversly, one may compute the power series of the exponential by computing the compositional inverse of the logarithm. Comparing the coefficients on both side of the equation

$$e \circ \log(x) = x,$$

we find the recurrence:

$$\alpha_k = -\sum_{i=0}^{k-1} \alpha_i \beta_{k-i}^{q^i}$$

where $\alpha_0 = 0$.

6.2 The set of basic J-invariants

In this section, we continue to specialize the theory to the case $A = \mathbb{F}_q[T]$ and $K = \mathrm{Frac}(A)$. For any element w in $\Omega^r(\mathbb{C}_\infty)$ represented by $w = [w_1 : \ldots : w_{r-1} : 1]$, we set $\Lambda_w := Aw_1 + \cdots + Aw_{r-1} + A$. As explained in section 2.4.3, we have the following proposition:

Proposition 6.2.1. *Let $w, w' \in \Omega^r(\mathbb{C}_\infty)$. Then, $\Lambda_w \cong \Lambda_w'$ if and only if there exists $\gamma \in \mathrm{GL}_r(A)$ such that $w' = \gamma w$.*

Now, let $(F, i : \mathbb{F}_q[T] \to F)$ be a $\mathbb{F}_q[T]$-field. For any rank r Drinfeld $\mathbb{F}_q[T]$-module ϕ over F and any integer $1 \leq k \leq r$, we define $g_k(\phi) \in F$, to be the k-th coefficient of the τ-polynomial $\phi_T(\tau)$. Note that by definition we always have $g_r(\phi) \neq 0$.

Definition 6.2.2. Let ϕ be a rank 2 Drinfeld $\mathbb{F}_q[T]$-module. The *j-invariant of ϕ* is defined by

$$j(\phi) := g_1(\phi)^{q+1}/g_2(\phi).$$

Lemma 6.2.3. *Let ϕ and ψ be two Drinfeld $\mathbb{F}_q[T]$-modules of same ranks and let $P \in F\{\tau\}$ be a nonzero morphism from ϕ to ψ. Then P is an isomorphism if and only if $P = c\tau^0$ for $c \in F^\times$.*

Proof. If $P \neq 0$ is an isomorphism, then it is equivalent to saying that there exists $Q \in F\{\tau\}$ such that $P \cdot Q = \tau^0$ which is equivalent to $\deg_\tau(P) = 0$. □

Proposition 6.2.4. *Let ϕ and ψ be two Drinfeld $\mathbb{F}_q[T]$-modules of rank 2 over F. Then ϕ and ψ are isomorphic over $\overline{F}$ if and only if $j(\phi) = j(\psi)$.*

Proof. By lemma 6.2.3, we may suppose that ϕ is isomorphic to ψ via $P = c\tau^0$ for a nonzero $c \in F$. Then we have $c\phi_T = \psi_T c$. By comparing the coefficients on both sides, we then obtain that ϕ and ψ are isomorphic if and only if $g_1(\phi) = g_1(\psi)c^{q-1}$ and $g_2(\phi) = g_2(\psi)c^{q^2-1}$. Next, the invariance under isomorphisms for the j-invariant comes from the following direct calculation:

$$j(\phi) = \frac{g_1(\phi)^{q+1}}{g_2(\phi)} = \frac{g_1(\psi)^{q+1}c^{(q-1)(q+1)}}{g_2(\psi)c^{q^2-1}} = \frac{g_1(\psi)^{q+1}}{g_2(\psi)} = j(\psi).$$

Conversely, suppose that $j(\phi) = j(\psi)$. Then, one may choose $\xi \in \overline{F}$ such that

$$\xi^{q^2-1} = g_2(\psi)^{-1}g_2(\phi).$$

We claim that ξ defines an isomorphism between ϕ and ψ. To see this, first we notice that $g_1(\psi) = 0$ is equivalent to $g_1(\phi) = 0$ (otherwise their j-invariant differ) and thus we suppose that $g_1(\psi) \neq 0$. Then, by definition of u and the assumption that $j(\phi) = j(\psi)$, we have

$$\xi^{q^2-1} = \frac{g_2(\phi)}{g_2(\psi)} = \frac{g_1(\phi)^{q+1}}{g_1(\psi)^{q+1}}.$$

Therefore, we obtain $\xi^{q-1}g_1(\psi) = g_1(\phi)$ and $\phi \cong \psi$ via ξ. □

Remark 6.2.5. Along the lines of the above proof, we showed the rank two version of a more general fact: let ϕ be any rank r Drinfeld $\mathbb{F}_q[T]$-module over F, then for any $c \in F^\times$ and any intger $1 \leq k \leq r$, we have $g_k(c^{-1}\phi c) = c^{q^k-1}g_k(\phi)$ so that g_k are *modular of weight* $q^k - 1$. Moreover, we have showed that if $j(\phi) = j(\psi)$, then any solution in $\overline{F}$ of the equation

$$X^{q^2-1} = g_2(\psi)^{-1}g_2(\phi)$$

defines an isomorphism over $\overline{F}$ between ϕ and ψ.

The above result is already well-known by Gekeler [Gek83] and its generalization to higher ranks is due to Potemine in [Pot98]. Instead of considering a single j-invariant, Potemine defined a finite family of j-invariants, called the *set of basic J-invariants.*

Definition 6.2.6. For any integer $1 \leq l \leq r-1$, we consider a tuple $(k_1, k_2, \ldots, k_l)$ with $1 \leq k_1 < \cdots < k_l \leq r-1$ which admits nonnegative integers $\delta_1, \ldots, \delta_l, \delta_r$ such that

$$\delta_1(q^{k_1}-1) + \delta_2(q^{k_2}-1) + \cdots + \delta_l(q^{k_l}-1) = \delta_r(q^r-1).$$

Then, the $J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}$*-invariant* of ϕ is defined by

$$J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}(\phi) := \frac{g_{k_1}(\phi)^{\delta_1} \cdot g_{k_1}(\phi)^{\delta_2} \cdot \ldots \cdot g_{k_l}(\phi)^{\delta_l}}{g_{k_r}(\phi)^{\delta_r}}.$$

Furthermore, if we have the following conditions:

1. $0 \leq \delta_i \leq (q^r-1)/(q^{\gcd(i,r)}-1)$ for all $1 \leq i \leq l$;
2. $\gcd(\delta_1, \delta_2, \ldots, \delta_l, \delta_r) = 1$.

then $J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}$ is said to be *basic*. For any $1 \leq k \leq r-1$, we define the *j_k-invariant of ϕ*:

$$j_k(\phi) := J^{\delta_k}_{k}(\phi) = \frac{g_k(\phi)^{(q^r-1)/(q^{\gcd(k,r)}-1)}}{g_r(\phi)^{(q^k-1)/(q^{\gcd(k,r)}-1)}}.$$

The *j-invariant of ϕ* is $j(\phi) := (j_1(\phi), j_2(\phi), \ldots, j_{r-1}(\phi))$ and we denote by $J(\phi) \subset F$ to be the multiset of all basic J-invariants of ϕ. For any two Drinfeld modules ϕ and ψ, we write $J(\phi) = J(\psi)$ if and only if $J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}(\phi) = J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}(\psi)$ for any integers k_i and δ_j as above.

Remark 6.2.7. We recall that a *multiset* means a set which allows repetitions. Even if their value might be equal, we distinguish two basic J-invariants by their parameters $(k_1, \ldots, k_l)$ and $(\delta_{k_1}, \ldots, \delta_{k_l}, \delta_r)$.

Proposition 6.2.8. *The multiset of all basic J-invariants is finite.*

Proof. We first note that we have a bijection between $J(\phi)$ and the set of points in $\mathbb{R}^r$ with coprime integer coordinates that satisfies the equation:

$$\delta_1(q-1) + \delta_2(q^2-1) + \cdots + \delta_{r-1}(q^{r-1}-1) = \delta_r(q^r-1) \tag{6.3}$$

together with the inequalities

$$0 \leq \delta_i \leq (q^r-1)/(q^{\gcd(i,r)}-1), \quad 1 \leq i \leq r. \tag{6.4}$$

This bijection is given by sending $J^{\delta_1,\ldots,\delta_r}_{k_1,\ldots,k_l}(\phi)$ to the point:

$$(\delta_1, \delta_2, \ldots, \delta_{r-1}, \delta_r) \in \mathbb{Z}^r$$

where δ_{k_i} is at position k_i and $\delta_j = 0$ if $j \neq k_i$ for all i. Next, we observe that the equation (6.3) together with the inequalities (6.4) defines a convex polyhedron in $\mathbb{R}^r$. Therefore, this identifies the multiset of basic J-invariants with a subset of the finite set of points with integer coordinates inside this polyhedron. □

Example 6.2.9. If $r = 2$, then, for arbitrary value of q, the unique basic J-invariant of ϕ is $j_1(\phi) = g_1(\phi)^{q+1}/g_2(\phi)$.

If $r = 3$ and $q = 2$, then

$$J(\phi) = \left\{ j_1 = J_1^7,\ j_2 = J_2^7,\ J_{1,2}^{1,2},\ J_{1,2}^{4,1},\ J_{1,2}^{5,3},\ J_{1,2}^{6,5},\ J_{1,2}^{7,7} \right\}.$$

Remark 6.2.10. By the proof of proposition 6.2.8, the problem of computing all the basic J-invariants is reduced to the the problem of finding lattices points inside a polyhedron. This can be achieved by many approaches, for example by using Barvinok's algorithm [Bar94]. Algorithms for computing lattices points inside a polyhedron are already included in SageMath.

Theorem 6.2.11. *Let ϕ and ψ be two rank r Drinfeld $\mathbb{F}_q[T]$-modules over F. Then, ϕ and ψ are isomorphic if and only if $J(\phi) = J(\psi)$.*

Proof. This is theorem 2.2 of [Pot98]. □

Remark 6.2.12. Potemine also showed that if F is separably closed, then the tuple $j(\phi)$ determines a finite number of the isomorphism classes of Drinfeld $\mathbb{F}_q[T]$-modules. More precisely, if ϕ and ψ are two Drinfeld $\mathbb{F}_q[T]$-modules of rank r having the same j-invariant $j(\phi) = j(\psi)$, then there exists a Drinfeld $\mathbb{F}_q[T]$-module ϕ' isomorphic to ϕ such that

$$g_k(\psi) = \xi_k g_k(\phi'),$$

where ξ_k is such that $\xi_k^{(q^r-1)/(q^{\gcd(k,r)}-1)} = 1$ for any $1 \leq k \leq r-1$ [Pot98, Theorem 2.2].

6.3 SageMath implementation

As mentioned in the beginning of this chapter, Drinfeld $\mathbb{F}_q[T]$-modules are now included in SageMath as of version 10.0. The implementation is thorouhgly documented, hence the reader is refered to the SageMath reference manual for a complete description of the features. We first present a simple example explaining how to create a Drinfeld module in SageMath.

SageMath session 6.3.1.
In this session, we create the Carlitz $\mathbb{F}_5[T]$-module and compute its image at some elements in $\mathbb{F}_5[T]$.

```
sage: A = GF(5)['T']
sage: K.<T> = Frac(A)
sage: phi = DrinfeldModule(A, [T, 1])
sage: phi
Drinfeld module defined by T |--> t + T
sage: phi(T)
t + T
sage: phi(T^2)
t^2 + (T^5 + T)*t + T^2
```

In the following two sessions, we showcase the implementation of the methods described in section 6.1 and 6.2. We note however that these are not yet merged in the official release of SageMath, but will be available in an upcoming release[3].

SageMath session 6.3.2.
In this session, we compute the exponential and the logarithm of a Drinfeld module.

```
sage: A = GF(2)['T']
sage: K.<T> = Frac(A)
sage: phi = DrinfeldModule(A, [T, 1, T, T+1])
sage: log = phi.logarithm()
sage: log
z + ((1/(T^2+T))*z^2) + (((T^3+T^2+1)/(T^6+T^5+T^3+T^2))*z^4)
   + O(z^8)
sage: exp = phi.exponential()
sage: exp
z + ((1/(T^2+T))*z^2) + (((T^5+T^3+1)/(T^8+T^6+T^5+T^3))*z^4)
   + O(z^8)
```

[3] See the GitHub Pull Requests: `https://github.com/sagemath/sage/pull/35260` and `https://github.com/sagemath/sage/pull/35057`.

The logarithm and the exponential are *lazy* power series. This means that their coefficients are computed only on demand and it is not required to input any precision parameter. One can indeed verify that one is the compositional inverse of the other:

```
sage: log.compose(exp)
z + O(z^8)
sage: exp.compose(log)
z + O(z^8)
```

SageMath session 6.3.3.
In this session, we compute all the possible basic J-invariants of the rank two Drinfeld $\mathbb{F}_7[T]$-module $T \mapsto T + \tau + (T+1)\tau^2$.

```
sage: A = GF(7)['T']
sage: K.<T> = Frac(A)
sage: phi = DrinfeldModule(A, [T, 1, T+1])
sage: phi.basic_j_invariant_parameters()
[((1,), (8, 1))]
```

The output of the method `basic_j_invariants_parameters` is a list of pairs of lists

$$[((1, 2, \dots, r-1), (\delta_1, \delta_2, \dots, \delta_{r-1}, \delta_r))]$$

corresponding to all the possible parameters for the basic J-invariants. We present a example in rank three.

```
sage: A = GF(3)['T']
sage: K.<T> = Frac(A)
sage: phi = DrinfeldModule(A, [T, T, T^2 + 1, T + 2])
sage: phi.basic_j_invariant_parameters()
[((1,), (13, 1)), ((1, 2), (1, 3, 1)), ((1, 2), (5, 2, 1)),
   ((1, 2), (6, 5, 2)), ((1, 2), (7, 8, 3)), ((1, 2), (8, 11,
   4)), ((1, 2), (9, 1, 1)), ((1, 2), (11, 7, 3)), ((1, 2),
   (13, 13, 5)), ((2,), (13, 4))]
```

To obtain the value of the basic J-invariant for any parameter, one simply needs to use the method `j_invariant` and pass the parameter as argument:

```
sage: phi.j_invariant([(1, 2), (1, 3, 1)])
(T^7 + T)/(T + 2)
sage: phi.j_invariant([(1, 2), (5, 2, 1)])
(T^9 + 2*T^7 + T^5)/(T + 2)
```

Chapter 7

Computing with Drinfeld Modular Forms for $\mathrm{GL}_r(\mathbb{F}_q[T])$

In contrast with the classical theory, very little is known for the computational aspects of Drinfeld modular forms. In this chapter, we explore some algorithmic aspects of Drinfeld modular forms via the López-Petrov A-expansion theory. More precisely, we explain how to compute the expansion at infinity of a special rank 2 family of eigenform called the *Petrov family*.

We also present a SageMath external package offering an implementation the graded ring of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[T])$. This implemenation allows symbolic calculation with Drinfeld modular forms and is partly modeled after the quasimodular forms implementation explained in chapter 5.

7.1 Goss polynomials

An important tool for studying expansion at infinity of Drinfeld modular forms is the sequence of Goss polynomials. Introduced by Goss, these polynomials shows up in the computation of the expansion of the Drinfeld eisenstein series.

Let $\Lambda \subset \mathbb{C}_\infty$ be a A-lattice of rank r. We recall the associated exponential function

$$e_\Lambda(w) := w \prod_{\lambda}(1 - z/\lambda) = \sum_{i=0}^{\infty} \alpha_i(\Lambda) w^{q^i}$$

where $w \in \mathbb{C}_\infty$ and $\alpha_i(\Lambda) \in \mathbb{C}_\infty$. Moreover, we let

$$u_\Lambda(w) := e_\Lambda(w)^{-1} = \sum_{\lambda\in\Lambda} \frac{1}{w-\lambda}$$

and

$$S_{k,\Lambda}(w) := \sum_{\lambda\in\Lambda} \frac{1}{(w+\lambda)^k}.$$

We note that $S_{1,\Lambda} = u_\Lambda$.

Proposition 7.1.1. *There exists a polynomial $G_k = G_{k,\Lambda}$ such that*

$$S_{k,\Lambda} = G_{k,\Lambda}(u_\Lambda).$$

Proof. This proof is due to Goss in [Gos80a, §1.7]. First, we recall the Girard–Newton formula. Let

$$P(X) = X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$$

be any polynomial over an arbitrary field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of its roots (non necessarily distincts) in an algebraic closure of F. If we define S_k to be the sum of the k-th power of the roots of P:

$$S_k := S_k(P) := \sum_{i=1}^n \alpha_i^k,$$

then, the well-known *Girard–Newton identity* states that

$$S_k = \begin{cases} -(a_1S_{k-1} + \cdots + a_{k-1}S_1 + ka_k), & \text{if } k \leq n; \\ -(a_1S_{k-1} + \cdots + a_{n-1}S_{k-n+1} + a_nS_{k-n}), & \text{if } k > n. \end{cases} \tag{7.1}$$

Next, we first assume that Λ is a finite of dimension m over $\mathbb{F}_q$. Then, for any $w \in \mathbb{C}_\infty$, we consider the polynomial

$$P(x) := -e_\Lambda(w)^{-1}e_\Lambda(X-w) = e_\Lambda(w)^{-1}e_\Lambda(X) + 1.$$

Note that P is of degree q^m and its the roots are exactly $\{z + \lambda : \lambda \in \Lambda\}$. Thus, by setting $\tilde{P}(X) := P(X^{-1})X^{q^m}$, we obtain a new polynomial with roots $\{1/(w+\lambda) : \lambda \in \Lambda\}$. The coefficients of $\tilde{P}$ are given by

$$\tilde{P}(X) = X^{q^m} - \sum_{i=0}^m u_\Lambda \alpha_i X^{q^m - q^i}.$$

Since it is monic, we may apply the Girard-Newton formula to get

$$S_{k,\Lambda} = u_\Lambda \left(S_{k-1} + \sum_{i=1}^{\log_q(k)} \alpha_i S_{k-q^i} \right). \tag{7.2}$$

This proves, by recursion, that $S_{k,\Lambda}$ is a polynomial in t_Λ.

Lastly, if $\Lambda \subset \mathbb{C}_\infty$ is infinite then we define $S_{k,\Lambda}$ by the same formula. It will be a meromorphic function on $\mathbb{C}_\infty$ with poles at most at Λ. Also, we express Λ as an infinite union $\Lambda = \cup_i \Lambda_i$ where each Λ_i are finite over $\mathbb{F}_q$. Then the function $S_{k,\Lambda}$ will be the limit of the S_{k,Λ_i}. □

Corollary 7.1.2. *The polynomial $G_{k,\Lambda}$ satisfies the following properties:*

1. $G_k(X) = X \left(G_{k-1}(X) + \sum_{i=1}^{\lfloor \log_q(k) \rfloor} \alpha_i G_{k-q^i}(X) \right)$;
2. G_k *is monic of degree* k;
3. *If* $k \leq q$ *then* $G_k(X) = X^k$;
4. *If* $p = \mathrm{char}(\mathbb{F}_q)$, *then* $G_{pk} = (G_k)^p$;
5. $X^2 G_k'(X) = k G_{k+1}(X)$.

Proof. Properties 1, 2 and 3 follows from equation 7.2. Property 4 is obtained by using the properties of a finite characteristic ring:

$$G_{pk}(u_\Lambda(w)) = \sum_{\lambda \in \Lambda} \frac{1}{(w-\lambda)^{pk}} = \left(\sum_{\lambda \in \Lambda} \frac{1}{(w-\lambda)^k} \right)^p = (G_k(u_\Lambda))^p.$$

Finally, we obtain the last property by direct calculations. We have on one hand

$$\frac{\mathrm{d}}{\mathrm{d}z} S_{k,\Lambda}(z) = -k S_{k+1,\Lambda}(z) = -k G_{k+1,\Lambda}(u(z));$$

and on the other hand

$$\frac{\mathrm{d}}{\mathrm{d}z} u(z) = \frac{\mathrm{d}}{\mathrm{d}z} e(z)^{-1} = -e(z)^{-2} = -u(z)^2.$$

□

Definition 7.1.3. The polynomial $G_{k,\Lambda}$ is called the *k-th Goss polynomial* of the lattice Λ.

Remark 7.1.4. The goal of this remark is to do a classical calculation in order to showcase the idea behind Goss polynomials. Classically, the function $w \mapsto \sin(\pi w)$ is zero whenever $w \in \mathbb{Z}$. By computing its logarithmic derivative we get

$$\pi \frac{\cos(\pi w)}{\sin(\pi w)} = \frac{1}{w} + \sum_{\lambda=1}^{\infty} \left(\frac{1}{w+\lambda} + \frac{1}{w-\lambda} \right). \tag{7.3}$$

One may roughly imagine this sum as being equal to $\sum_{\lambda \in \mathbb{Z}} \frac{1}{w+\lambda} = S_{1,\mathbb{Z}}$, although this last sum does not make sense since we don't have absolute convergence. Setting $q_w := e^{2\pi i w}$ and writing the left hand side of equation (7.3) in terms of q_w, we obtain the Fourier series:

$$\frac{1}{w} + \sum_{\lambda=1}^{\infty} \left(\frac{1}{w+\lambda} + \frac{1}{w-\lambda} \right) = i\pi - 2i\pi \sum_{n=0}^{\infty} q_w^n$$

and differentiating $k-1$ times (k an even integer), we get

$$\sum_{\lambda \in \mathbb{Z}} \frac{1}{(w+\lambda)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q_w^n. \tag{7.4}$$

The above expansion can then be utilized in order to find the Fourier expansion of the classical weight k Eisenstein series. We will see in the coming sections that the Goss polynomials will play a similar role for the Drinfeld Eisenstein series. In fact, one may think of these polynomials as a function field analogue of the right hand side of equation (7.4).

Proposition 7.1.5. *Let Λ be any A-lattice with exponential function $e_\Lambda(x) = \sum_{i=0}^{\infty} \alpha_i x^{q^i}$ for $x \in \mathbb{C}_\infty$ and $\alpha_i \in \mathbb{C}_\infty$. Then, for any $k \geq 1$, we have*

$$G_{k+1,\Lambda}(X) = \sum_{j \leq k} \sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} X^{j+1}$$

where $\sum_{\underline{i}}$ runs over the set of multi indices $\underline{i} = (i_0, \ldots, i_s)$ (of arbitrary length) satisfying

1. $i_0 + \cdots + i_s = j$
2. $i_0 + i_1 q + \cdots + i_s q^s = k$

and

$$\binom{j}{\underline{i}} := \frac{j!}{i_0! \cdots i_s!}, \qquad \alpha^{\underline{i}} := \alpha_0^{i_0} \cdots \alpha_s^{i_s}.$$

Proof. This is obtained via the study of the generating function:

$$G_\Lambda(U, X) := \sum_{k \geq 0} G_{k,\Lambda}(X) U^k.$$

See section 3 of [Gek88] for the details. □

The above proposition is not very useful in practice. In order to compute any Goss polynomial $G_{k,\Lambda}$, one can first compute the first $\lfloor \log_q(k) \rfloor$ coefficients of the exponential e_{ϕ^Λ} as explained in section 6.1 and then use the recurrence relation

$$G_k(X) = X\left(G_{k-1}(X) + \sum_{i=1}^{\lfloor \log_q(k) \rfloor} \alpha_i G_{k-q^i}(X)\right).$$

7.2 A-expansion theory of López–Petrov

Let f be a Drinfeld modular form of rank r for an arithmetic subgroup Γ. We recall that from proposition 3.3.8, the form f possesses a u-expansion of the form

$$f(w) = \sum_{n \in \mathbb{Z} \geq 0} f_n(w')u(w)^n$$

where $w = \binom{w_1}{w'} \in \Omega^r(\mathbb{C}_\infty)$ and $u(w) = e_{\tilde{\pi}w'\Lambda'}(w_1)^{-1}$. Unfortunately, the computation of u-expansion of a Drinfeld modular form is less understood compared to the classical theory, even in rank 2. In [Gek88], Gekeler computed the coefficients of multiple rank 2 Drinfeld modular forms such as the Drinfeld discrimant and the Eisenstein series. In [Ló10], López built upon the work of Gekeler to prove what he called a nonstandard expansion for the Drinfeld discriminant. In [Pet13], Petrov expanded the work of López by proving that an infinite family of Drinfeld modular forms admits a nonstandard expansion. Thus, the goal of this section is to give a brief overview of this expansion theory of López–Petrov.

For the rest of this section, we suppose the rank to be equal to two. In this specific case, the parameter at infinity is given by

$$u(w) = \frac{1}{e_{\tilde{\pi}\mathbb{F}_q[T]}} = \frac{1}{\tilde{\pi}} \sum_{d \in \mathbb{F}_q[T]} \frac{1}{w+d}.$$

where $\tilde{\pi}\mathbb{F}_q[T]$ is the lattice associated with the Carlitz module $\rho : T \mapsto T + \tau$. For any nonzero $a \in \mathbb{F}_q[T]$, we define $u_a : w \mapsto u(aw)$.

Lemma 7.2.1. *For any nonzero a in $\mathbb{F}_q[T]$, we have $u_a = u^{|a|} + O(u^{|a|+1})$ where $|a| = q^{\deg a}$.*

Proof. Fix $a \in \mathbb{F}_q[T]$ nonzero. We define the *a-inverse cyclotomic polynomial* by

$$\begin{aligned} f_a(X) &:= \rho_a(X^{-1})X^{|a|} \\ &= 1 + (\text{higher terms in } X). \end{aligned}$$

Then, using the fact that $\rho_a(e_{\tilde{\pi}\mathbb{F}_q[T]}(\tilde{\pi}w)) = e_{\tilde{\pi}\mathbb{F}_q[T]}(\tilde{\pi}aw)$, we have

$$u_a = \frac{u^{|a|}}{f_a(u)} = \frac{u^{|a|}}{1 + (\text{higher terms in } u)} = u^{|a|}(1 + O(u)).$$

□

Next, recall that the Drinfeld Eisenstein series of weight $k \in (q-1)\mathbb{Z}$ and rank 2 is defined by

$$E_k^2(w) := \sum_{\substack{a,b \in \mathbb{F}_q[T] \\ (a,b) \neq (0,0)}} \frac{1}{(aw+b)^k}.$$

Using the theory of Goss polynomials, we are now able to prove their holomorphicity at infinity:

Proposition 7.2.2. *Let k an integer in $(q-1)\mathbb{Z}$. Then, the weight k Drinfeld Eisenstein series of rank 2 is holomorphic at infinity.*

Proof. Let A^+ be the set of monic polynomials in $\mathbb{F}_q[T]$. We first compute:

$$\begin{aligned}
E_k^2(w) &= \sum_{\substack{a,b \in \mathbb{F}_q[T] \\ (a,b) \neq (0,0)}} \frac{1}{(cw+d)^k} \\
&= \sum_{c \in A \setminus \{0\}} \frac{1}{c^k} - \sum_{a \in A^+} \sum_{b \in A} \frac{1}{(aw+b)^k} \\
&= \sum_{c \in A \setminus \{0\}} \frac{1}{c^k} - \tilde{\pi}^k \sum_{a \in A^+} G_{k,\tilde{\pi}\mathbb{F}_q[T]}(u_a(w)) \\
&= \tilde{\pi}^k E_k^1(\tilde{\pi}\mathbb{F}_q[T]) - \tilde{\pi}^k \sum_{a \in A^+} G_{k,\tilde{\pi}\mathbb{F}_q[T]}(u_a(w))
\end{aligned}$$

where the last equality is given by proposition 7.1.1 and $E_k^1(\tilde{\pi}\mathbb{F}_q[T])$ is the Eisenstein series of the Carlitz module

$$E_k^1(\tilde{\pi}\mathbb{F}_q[T]) = \sum_{\lambda \in \tilde{\pi}\mathbb{F}_q[T]} \frac{1}{\lambda^k}.$$

Then, the desired result follows from lemma 7.2.1 and the fact that $G_{k,\tilde{\pi}\mathbb{F}_q[T]}$ is a polynomial.

□

Definition 7.2.3. For any $k \equiv 0 \mod (q-1)$, we define by δ_k to be the first coefficient of E_k^2. Then, the *normalized Drinfeld Eisenstein series of weight k* is

$$g_k := (\delta_k)^{-1} E_k^2.$$

In [Ló10], López asked whether there exist Drinfeld modular forms which admit an expansion of the form $\sum_{a\in A^+} G_{k,\tilde{\pi}\mathbb{F}_q[T]}(c_a u_{aw})$. This nonstandard expansion theory was later formalized by Petrov in [Pet13].

Definition 7.2.4. A Drinfeld modular form $f : \Omega^2(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ of rank 2 is said to have an *A-expansion of exponent $n \geq 1$* if there exists elements $c_0(f), c_a(f) \in \mathbb{C}_\infty$ such that

$$f = c_0(f) + \sum_{a\in A^+} c_a(f) G_{n,\tilde{\pi}\mathbb{F}_q[T]}(u_a)$$

Proposition 7.2.5. *Let $(c_a)_{a\in A^+}$ be a sequence of elements in $\mathbb{C}_\infty$ having polynomial growth in $|a|$ for all but finitely many a. Then, the series*

$$\sum_{a\in A^+} c_a G_{n,\tilde{\pi}\mathbb{F}_q[T]}(u_a)$$

converges to a well-defined function on $\{w \in \Omega^2(\mathbb{C}_\infty) : |w_i| > i\}$.

Proof. See section 1 of [Pet13]. □

Proposition 7.2.6. *Let $n \geq 1$ be a fixed integer and let f be a Drinfeld modular form which admits an A-expansion of exponent n. Then the A-expansion is unique.*

Proof. See theorem 3.1 of [Ló11]. □

Remark 7.2.7. The above theorem states the unicity of the A-expansion *once the exponent is fixed*. This means that there could be two A-expansions of different exponents which converges to the same Drinfeld modular forms. However, it is conjectured by Petrov that the exponent is unique [Pet13, Remark 1.2].

We now look at some concrete examples of Drinfeld modular forms which admits an A-expansion, other than the Eisenstein series.

Definition 7.2.8. The *Drinfeld discriminant function* is the function $\Delta : \Omega^2(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ defined by the leading coefficient of

$$\phi^w : T \mapsto T + g_1(w)\tau + \Delta(w)\tau^2$$

were $w \in \Omega^2(\mathbb{C}_\infty)$ and ϕ^w is the Drinfeld module associated to the lattice $\Lambda_w = A + Aw$.

Remark 7.2.9. The analytic definition of a Drinfeld module gives us that

$$\phi_T^w(X) = TX \prod_{\substack{\lambda \in T^{-1}\Lambda_w/\Lambda_w \\ \lambda \neq 0}} \left(1 - \frac{X}{e_{\Lambda_w}(\lambda)}\right).$$

Therefore, since $\Delta(w)$ is defined to be the leading coefficient of the above polynomial, we derive the product formula

$$\Delta(w) = T \prod_{\substack{u,v \in T^{-1}A/A \\ (u,v) \neq (0,0)}} e_{\Lambda_w}^{-1}(uw+v). \tag{7.5}$$

Proposition 7.2.10 (López). *For any $w \in \Omega^2(\mathbb{C}_\infty)$, the Drinfeld discrimant function satisfies*

$$-\tilde{\pi}^{1-q^2}\Delta(w) = \sum_{a \in A^+} a^{q(q-1)} u_a^{q-1}.$$

Proof. See section 5 of [Ló10]. The idea of the proof is to first define the following function:

$$D(w) := \sum_{\substack{(u,v) \in T^{-1}A/A \\ \neq (0,0)}} \left(\frac{u^{q(q-1)}}{(uw+v)^{q-1}} + \sum_{\substack{(a,b) \in A \\ \neq (0,0)}} \frac{(av-bu)^{q(q-1)}}{(az+b)^{q(q-1)}((a-u)z+(b-v))^{q-1}} \right)$$

and then show that it satisfies $D(1/w) = w^{q^2-1}D(w)$ and

$$D(w) = \frac{-\tilde{\pi}^{q-1}}{T^{(q-1)^2}} \sum_{a \in A^+} a^{q(q-1)} u_a^{q-1}.$$

Therefore, by knowning that the space of cusps forms (modular forms whose constant term is zero) of weight q^2-1 is of dimension 1, it must be a multiple of the Drinfeld discrimant. $\square$

Remark 7.2.11. The definition of the function $D(w)$ may look totally out of the blue, but it is in fact a direct modification of the logarithmic derivative of the discrimant:

$$\frac{\Delta'(w)}{\Delta(w)} = \sum_{\substack{(u,v) \in T^{-1}A/A \\ \neq (0,0)}} \left(\frac{-u}{uw+v} + \sum_{\substack{(a,b) \in A \\ \neq (0,0)}} \frac{av-bu}{(az+b)((a-u)z+(b-v))} \right).$$

The above equation is obtained by elementary calculations via the product formula (7.5). After some manipulations, one can then obtain the expansion

$$\tilde{\pi}^{-1}\frac{\Delta'(w)}{\Delta(w)} = \sum_{a \in A^+} au(aw). \tag{7.6}$$

See [Ló10] for the details. We note that expansion (7.6) is also known by Gekeler [Gek88, (8.2)].

Proposition 7.2.12 (Petrov)**.** *Let k and n be two positive integers such that $k-2n$ is a positive multiple of $q-1$ and $n \leq p^{v_p(k-n)}$. Then*

$$f_{k,n} := \sum_{a \in A^+} a^{k-n} G_{n,\tilde{\pi}\mathbb{F}_q[T]}(u_a)$$

is a Drinfeld modular form of weight k and type $m \equiv n \pmod{q-1}$ for $\mathrm{GL}_2(A)$.

Proof. This is theorem 1.3 of [Pet13]. It is obtained by generalizing López' ideas. □

Corollary 7.2.13. *Setting $\Delta_0 := -\tilde{\pi}^{1-q^2}\Delta$, we have:*

$$\Delta_0 = f_{q^2-1,q-1} \quad \textit{and} \quad g_1 = 1 - \delta_{q-1} f_{q-1,q-1}.$$

Proof. Corollary 7.1.2 gives us that $G_{q-1,\tilde{\pi}\mathbb{F}_q[T]}(u_a) = u_a^{q-1}$, from which we obtain the result. □

Definition 7.2.14. The family of all the forms $f_{k,n}$ satisfying the above proposition will be called the *Petrov family*.

Remark 7.2.15. The Petrov family is particularly interesting when stuying Hecke operator action on Drinfeld modular forms. More precisely, if $\mathfrak{p}$ is a nonzero prime ideal of A generated by a unique monic polynomial $\wp$ and $f \in \mathcal{M}_{k,m}^{\mathrm{an},2}(\mathrm{GL}_2(A))$ then the $\mathfrak{p}$-th Hecke operator is defined by

$$T_\mathfrak{p} f(w) := \wp^k f(\wp w) + \sum_{\beta \in S_\mathfrak{p}} f\left(\frac{w+\beta}{\wp}\right).$$

It is well-known in the classical case that a Hecke eigenform is determined up to a constant by its eigensystem [DS05, Theorem 5.8.2]. However, this fact is not true in the Drinfeld case. Indeed, Goss showed that g_1, $g_1^q\Delta$ and Δ all have the same eigensystem: $\{\lambda_\mathfrak{p} = \wp^{q-1}\}$ [Gos80a, Corollaries 2.2.4, 2.2.5]. In light of this fact, Petrov showed that any form in the Petrov family is an eigenform with eigensystem $\lambda_\mathfrak{p} = \wp^n$. He moreover showed that any eigenform f of weight k which admits an A-expansion of exponent n is of the form:

$$f = \sum_{a \in A^+} a^{k-n} G_{n,\tilde{\pi}\mathbb{F}_q[T]}(u_a).$$

In particular, this proves that an eigenform having an A-expansion with fixed exponent is determined by its eigensystem and its weight [Pet13, Theorem 2.6].

7.3 Expansion at infinity

Let f be a Drinfeld modular forms which admits an A-expansion of the form

$$f = c_0(f) + \sum_{a \in A^+} c_a(f) G_n(u_a).$$

The goal of this section is to compute the i-th coefficient of its expansion at infinity:

$$f := \sum_{i \geq 0} c_i(f) u^i.$$

For simplicity, we assume that $c_0(f) = 0$. By corollary 7.1.2, we define the integer m_n such that X^{m_n} is the largest power that divides G_n and we denote by $G^{(m_n)}$ the coefficient of X^{m_n}. Thus, by lemma 7.2.1, there exists $d_i(a,n)$ in K such that

$$G_n(u_a) = G^{(m_n)} u^{m_n q^{\deg(a)}} + \sum_{i > m_n q^{\deg(a)}} d_i(a,n) u^i. \tag{7.7}$$

In the following calculations, we substitute expansion (7.7) in the A-expansion of the form f and manipulate its terms in order to obtain an explicit formula for the coefficients $c_i(f)$:

$$\begin{aligned}
f &= \sum_{a \in A^+} c_a(f) G_n(u_a) \\
&= \sum_{d \geq 1} \sum_{\substack{a \in A^+ \\ \deg(a) = d}} c_a(f) G_n(u_a) \\
&= \sum_{d \geq 1} \sum_{\substack{a \in A^+ \\ \deg(a) = d}} c_a(f) \Big(G^{(m_n)} u^{m_n q^d} + \sum_{i > m_n q^d} d_i(a,n) u^i \Big) \\
&= \sum_{d \geq 1} \Bigg(\sum_{\substack{a \in A^+ \\ \deg(a) = d}} c_a(f) \Bigg) G^{(m_n)} u^{m_n q^d} + \sum_{d \geq 1} \sum_{i > m_n q^d} \Bigg(\sum_{\substack{a \in A^+ \\ \deg(a) = d}} c_a(f) d_i(a,n) \Bigg) u^i.
\end{aligned} \tag{7.8}$$

The first part of the above equality may be written as

$$\sum_{d \geq 1} \Bigg(\sum_{\substack{a \in A^+ \\ \deg(a) = d}} c_a(f) \Bigg) G^{(m_n)} u^{m_n q^d} = \sum_{i \geq 1} \mathbb{1}_{\mathcal{I}_n}(i) \Bigg(\sum_{\substack{a \in A^+ \\ \deg(a) = \log_q(i/m_n)}} c_a(f) \Bigg) G^{(m_n)} u^i \tag{7.9}$$

where $\mathcal{I}_n := \{m_n q^d : d \geq 1\}$ and $\mathbb{1}_{\mathcal{I}}$ is the indicator function:

$$\mathbb{1}_{\mathcal{I}}(i) := \begin{cases} 1, & \text{if } i \in \mathcal{I}; \\ 0, & \text{otherwise.} \end{cases}$$

for some set of indices $\mathcal{I}$. Similarly, the second part of equality (7.8) may be written as

$$\sum_{d\geq 1}\sum_{i>m_n q^d}\left(\sum_{\substack{a\in A^+\\ \deg(a)=d}} c_a(f)d_i(a,n)\right)u^i = \sum_{i\geq 1}\sum_{d=1}^{B(i,n)} \mathbb{1}_{(m_n q^d,\infty)}(i)\left(\sum_{\substack{a\in A^+\\ \deg(a)=d}} c_a(f)d_i(a,n)\right)u^i, \tag{7.10}$$

where $B(i,n) := \max\{\log_q\lfloor i/m_n\rfloor, 1\}$ and $(m_n q^d,\infty)$ is simply the open interval $\{i : i > m_n q^d\}$. Combining equations (7.9) and (7.10), we obtain the following theorem:

Theorem 7.3.1. *Under the above notations, the i-th coefficient of the modular form $f = \sum_{a\in A^+} c_a(f)G_n(u_a)$ is*

$$c_i(f) = \mathbb{1}_{\mathcal{I}_n}(i)\left(\sum_{\substack{a\in A^+\\ \deg(a)=\log_q(i/m_n)}} c_a(f)\right)G^{(m_n)} + \sum_{d=1}^{B(i,n)} \mathbb{1}_{(m_n q^d,\infty)}(i)\left(\sum_{\substack{a\in A^+\\ \deg(a)=d}} c_a(f)d_i(a,n)\right).$$

The above formula describe an algorithm for computing the expansion at infinity of any Drinfeld modular forms which admits an A-expansion. In particular, we may now compute the expansion of any Eisenstein series and any forms in the Petrov family. This includes the modular discriminant function and Gekeler's h function.

Let us mention that computing the expansion at infinity of a Drinfeld modular form is particularly interesting since it allows us to determine whether two forms are equal or not. Indeed, as well as in the classical case, the spaces of Drinfeld modular forms of fixed weight admits Sturm-type bounds:

Proposition 7.3.2. *Let $f \in \mathcal{M}_{k,m}^{\mathrm{an},2}$ which admits the expansion $f = \sum_{i\geq 0}^{\infty} c_i(f)u^i$. If $c_i(f) = 0$ for $0 \leq i \leq \lceil k/(q+1)\rceil + 1$, then f is identically zero.*

Proof. See [Gek88, (5.17)] for the details. □

The key ingredient of the proof is to use a valence formula for Drinfeld modular form. A good reference for Sturm-type bounds in the Drinfeld case is [AW22].

7.4 `drinfeld-modular-forms` SageMath package

We present in this section an external SageMath package for computing with Drinfeld modular forms and their expansion at infinity. It is *external* in the sense that it does not comes automatically with SageMath and must be installed by the user in order to use its functionalities. For this reason, we have tried to make it as easy as possible to install:

```
sage: pip install drinfeld-modular-forms
```

The above command should be run inside SageMath. We note that this package has been tested on SageMath version 9.8 and above. It is not guaranteed to work on previous versions. After the execution of the above command, the package should be installed and can be imported using:

```
sage: from drinfeld-modular-form import *
```

We will present the main algorithms provided by this package, but there are plenty more functionalities and we refer the reader to the documentation which is hosted here:

`https://davidayotte.github.io/drinfeld_modular_forms`

SageMath session 7.4.1.
In this session, we create the graded ring of Drinfeld modular forms of type 0 for $\mathrm{GL}_2(A)$:

```
sage: from drinfeld_modular_forms import *
sage: A = GF(3)['T']  # Fq[T]
sage: K.<T> = Frac(A)
sage: M = DrinfeldModularFormsRing(K, 2)  # rank 2
sage: M.gens()  # generators
[g1, g2]
```

The above generator $g_1(w)$ and $g_2(w)$ corresponds to the coefficients forms defined by the Drinfeld $\mathbb{F}_q[T]$-module over $\Omega^r(\mathbb{C}_\infty)$:

$$\phi^w : T \mapsto T + g_1(w)\tau + g_2(w)\tau^2.$$

We note that g_2 is the Drinfeld discriminant function Δ.

It is possible to compute a basis of the K-vector space of Drinfeld modular forms of any weight using the method `basis_of_weight`:

```
sage: M.basis_of_weight(3^2 - 1)
[g2, g1^4]
sage: M.basis_of_weight(2*11)
[g1^3*g2^2, g1^7*g2, g1^11]
```

Using the method `inject_variables` it is possible to quickly assign variables to the generators in your session:

```
sage: M.inject_variables(); # Define the variables g1 and g2
sage: g1.weight()
2
sage: g2.weight()
8
sage: g1 == M.coefficient_form(1)
True
sage: g2 == M.coefficient_form(2)
True
```

Next, one main feature of this package consist in the possibility of computing the expansion at infinity of the coefficient forms and any algebraic relations between them:

```
sage: g1.expansion(’u’)
1 + ((2*T^3+T)*u^2) + O(u^7)
sage: g2.expansion(’u’)
u^2 + 2*u^6 + O(u^8)
sage: F = T*g1*g2 + g2^2 + T*g1
sage: F.expansion(’u’)
T + ((2*T^4+T^2+T)*u^2) + ((2*T^4+T^2+1)*u^4) + 2*T*u^6 + O(u
  ^7)
```

It is important to note that, as for the case of exponential and logarithm of Drinfeld modules, the returned expansions above are *lazy* power series. Recall that such series is an object in SageMath which aims to represents a power series exactly. In other words, the returned expansion stores the procedure for computing the coefficients and computes them only if it needs to (and thus the name "lazy"). In our case, the procedure for computing the coefficients is given by theorem 7.3.1. Therefore, depending on the power of a computing machine, it is possible to query any coefficients at any precision:

```
sage: g1[3^4 - 1] # 80-th coefficient
T^30 + 2*T^28 + T^12 + 2*T^10 + T^4 + 2*T^2
```

Conversely, if an expansion at infinity is known up to a sufficient precision, one can convert it into a Drinfeld modular form:

```
sage: M.from_expansion([1, 0, 2*T^3 + T], 3 - 1)
g1
sage: M.from_expansion([0, 0, K.one()], 3^2 - 1)
g2
```

This is achieved via a similar algorithm as procedure 5.5.2, which writes any classical modular form in terms of polynomial relations of ring generators. As one see, it is important to input the weight of the expected Drinfeld modular form. This is explained by the fact that the algorithm uses the Sturm bound of the space of Drinfeld modular forms of fixed weight and then perform linear algebra to find the right algebraic relation.

SageMath session 7.4.2.
In this second session, we present how to compute the forms in the Petrov family. Recall that the Petrov family consist of the forms defined by

$$f_{k,n} = \sum_{a \in A^+} a^{k-n} G_n(u_a).$$

Under some conditions on k and n, Petrov showed that the above A-expansion defines a Drinfeld modular form of weight k and type $m \equiv n \pmod{q-1}$. In our package, we compute the expansion at infinity of this form via the function `compute_petrov_expansion(k, n, A)`.

```
sage: from drinfeld_modular_forms import *
sage: q = 3
sage: A = GF(q)['T']
sage: f8=compute_petrov_expansion(q^2 - 1, q - 1, A, 'u')
sage: f8
u^2 + 2*u^6 + O(u^8)
sage: f26=compute_petrov_expansion(q^3 - 1, q - 1, A, 'u')
sage: f26
u^2 + ((2*T^18+2*T^12+2*T^6+2)*u^6) + O(u^8)
```

Since we know the weight of the form $f_{k,n}$ and the Sturm bound of the space $\mathcal{M}_k^{\mathrm{an}}(\mathrm{GL}_2(A))$, we can convert these expansions as an algebraic expression of g_1 and g_2 using the method `from_expansion`:

```
sage: M.from_expansion(f8, 8)  # weight = 8
g2
sage: M.from_expansion(f26, 26)  # weight = 26
g1^9*g2 + (-T^18 - T^12 - T^6)*g1*g2^3
```

We compile here some results of computations with $q = 3$ for the sequence $(f_{q^n-1,q-1})_{n\geq 2}$:

$$\begin{aligned}
f_{q^2-1,q-1} &= g_2\\
f_{q^3-1,q-1} &= g_1^9 g_2 + \left(2T^{18} + 2T^{12} + 2T^6\right) g_1 g_2^3\\
f_{q^4-1,q-1} &= g_1^{36} g_2 + c_{28,3}^{(4)} \cdot g_1^{28} g_2^3 + c_{4,9}^{(4)} \cdot g_1^4 g_2^9 + c_{0,10}^{(4)} \cdot g_2^{10}\\
f_{q^5-1,q-1} &= g_1^{117} g_2 + c_{109,3}^{(5)} \cdot g_1^{109} g_2^3 + c_{85,9}^{(5)} \cdot g_1^{85} g_2^9 + c_{81,10}^{(5)} \cdot g_1^{81} g_2^{10} + c_{13,27}^{(5)} \cdot g_1^{13} g_2^{27}\\
&\quad + c_{9,28}^{(5)} \cdot g_1^9 g_2^{28} + c_{1,30}^{(5)} \cdot g_1 g_2^{30}.
\end{aligned}$$

The coefficients $c_{i,j}^{(n)}$ are polynomials in $\mathbb{F}_3[T]$ of relatively large degree. In the case $n = 4$, we have explicitely:

$$c_{28,3}^{(4)} = 2T^{72} + 2T^{66} + 2T^{60} + 2T^{54} + 2T^{48} + 2T^{42} + 2T^{36} + 2T^{30} + 2T^{24} + 2T^{18} + 2T^{12} + 2T^6$$

$$\begin{aligned}
c_{4,9}^{(4)} &= T^{126} + T^{120} + T^{114} + T^{108} + 2T^{102} + 2T^{96} + 2T^{90} + 2T^{84} + 2T^{72} + 2T^{66} + 2T^{60}\\
&\quad + 2T^{54} + T^{48} + T^{42} + T^{36} + T^{30}
\end{aligned}$$

$$\begin{aligned}
c_{0,10}^{(4)} &= T^{129} + 2T^{127} + T^{123} + 2T^{121} + T^{117} + 2T^{115} + T^{111} + 2T^{109} + 2T^{105} + T^{103} + 2T^{99}\\
&\quad + T^{97} + 2T^{93} + T^{91} + 2T^{87} + T^{85} + 2T^{81} + 2T^{75} + T^{73} + 2T^{69} + T^{67} + 2T^{63} + T^{61}\\
&\quad + 2T^{57} + T^{55} + T^{51} + 2T^{49} + T^{45} + 2T^{43} + T^{39} + 2T^{37} + T^{33} + 2T^{31} + T^{27}.
\end{aligned}$$

We haven't been able to determine a pattern here, but we hope that these functions will help in better understanding the forms in the Petrov family.

SageMath session 7.4.3.

For the last session, we create the ring of Drinfeld modular forms of arbitrary type for $\mathrm{GL}_2(A)$. This ring is generated by g_1, the first coefficient form and by Gekeler's h function, which is a $(q-1)$-th root of the modular discriminant.

```
sage: from drinfeld_modular_forms import *
sage: q = 4
sage: A = GF(q)['T']
sage: K.<T> = Frac(A)
sage: M = DrinfeldModularFormsRing(K, 2, has_type=True)
```

```
sage: M.gens()
[g1, h]
```

Through the work of López, we know that $h = f_{q+1,1}$ and thus we may compute its expansion using theorem 7.3.1.

```
sage: M.inject_variables(); # Define the variables g1 and h
sage: h_exp = h.expansion('u')
sage: h_exp
u + O(u^8)
sage: h_exp[0:24]  # all coefficients in the range 0 <= i < 24
[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, T^4 + T, 0, 0, 0, 0,
    0, 1, 0, 0, 0, 0]
```

We check that the form h is a $(q-1)$-th root of the coefficient form g_2:

```
sage: M.coefficient_form(2).expansion('u')
u^3 + O(u^10)
sage: (h^(q - 1)).expansion('u')
u^3 + O(u^10)
sage: h^(q - 1) == M.coefficient_form(2)
True
```

We point out that all the computations performed above were done in the rank 2 case. To this day, expansion at infinity of higher rank Drinfeld modular forms are still quite mysterious. We cite the work of Basson who has obtained A-expansions type expansion for the higher rank Drinfeld Eisenstein series [Bas14, Proposition 3.5.3] and computed the first coefficient of the coefficients forms [Bas14, §5.3.5].

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