A BRIEF INTRODUCTION TO KODAIRA DIMENSION AND IITAKA CONJECTURE

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Abstract

A brief Introduction to Kodaira dimension and Iitaka Conjecture
Emanuele Ronda
In the work *A brief introduction to Kodaira dimension and Iitaka Conjecture* by Emanuele Ronda, we define the concept of Kodaira dimension and state the Iitaka Conjecture $C_{n,m}$. We therefore prove some results on the Kodaira dimension and study, at different levels of detail, three, already known, instances of the Conjecture; namely:

i $C_{n,1}$ for base curves of general type and fibres of positive geometric genus;

ii $C_{2,1}$;

iii $C_{n,m}$ for base spaces of maximal Albanese dimension.

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I'll try be as concise as possible (but also have fun!)
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On a much more human level there are many people I would like to thank,

but as I will end up forgetting someone, I choose my christian democratic [1] way out of this and just say a generic: Thank you!

Bologna, 24th of June 2023

[1]This expression is used to refer to a person whose standpoint is inexistent. This is also the case of a forced impartiality.

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Introduction

In almost all branches of human knowledge, a very common approach to the study of the objects under consideration is to group those together with respect to some shared properties. In other words, we do not need to comprehend the differences between the Ross seal and the Weddell seal to know what a seal is!
This is a rather naive description of what the problem of classification is.
In mathematics, as in all the subjects, the classification problems play a key role, both the "easy" ones like the characterization of finite dimensional vector spaces over a fixed field and the hardest ones like the classification of the finite simple groups. The first one, for example, gives a powerful tool to work with, while the other had been one of the leading problems in the development of Group Theory in the 20th century.
In both these examples, we classify the objects (respectively vector spaces and finite simple groups) up to isomorphism. There obivously are many more. In Complex Differential Geometry for example, one can classify spaces up to bimeromorphic map which are invertible meromorphic maps that are proper over both factors.
The main focus in this discussion are some known instances of an open problem that arose in the study of the classification of algebraic varieties up to birational equivalence.
We define a birational map between two varieties X, Y to be a rational map (not necessarily defined everywhere)

$$f : X \dashrightarrow Y$$

such that there are open dense subsets $U \subseteq X, V \subseteq Y$, such that the restriction

$$f_{|_U} : U \to V$$

is a well-defined morphism, is bijective and its inverse is again a morphism. An example that explains well the situations we are interested in is the one of the nodal cubic. Consider k an algebraically closed field, and consider the vanishing locus in the plane of the following polynomial:

$$Y^2 - X^3 - X^2$$

The gradient of the polynomial is

$$\nabla(Y^2 - X^3 - X^2) = (-3X^2 - 2X, 2Y)$$

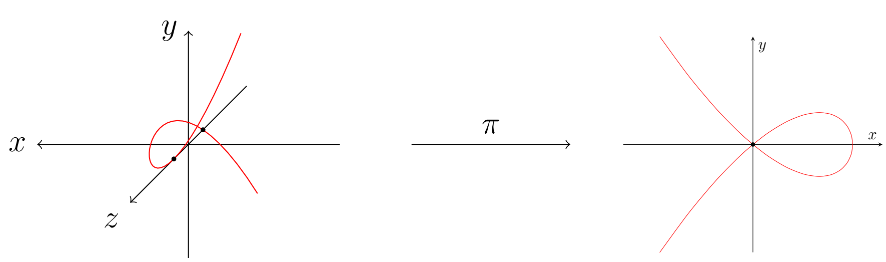
and so it is singular only at the origin, which is a point in the vanishing locus of the polynomial.
This means that the origin is a singular point of the curve.
In this case, the tangents are the bisectors of the plane. For every other point there is only one tangent line to the curve in that point. So if we denote by X the curve, we can consider the map

$$X \setminus \{(0,0)\} \to \mathbb{A}^2(k) \times \mathbb{P}^1(k)$$
$$(x,y) \longmapsto ((x,y), [-3x^2 - 2x, 2y])$$

that associates to each point the couple of the point itself and the tangent direction on it. It is well-defined.
Let Γ be the graph of such map and $\widetilde{X}$ its closure. Then the first projection

$$\Gamma \to X \setminus \{(0,0)\}$$

is a bijective continuous map of varieties. Thus an open dense subset of X is isomorphic to Γ which is open dense in B. We showed that X and B are the same "up to a bunch of points": indeed notice that $B \setminus \Gamma$ consists of two points, namely $(0,0,t_1), (0,0,t_2)$ where t_1, t_2 correspond to the bisectors of the plane. This example is summarized in the following picture.



What we have just studied is the construction of the blow-up of X at the origin. Blow-ups are the fundamental example of birational maps! Now that we have an understanding of some kind of what the concept of birational equivalence is, we can move on and talk a bit about the classification.
The classification of compact complex manifolds up to bimeromorphic equivalence began during the 19th century with the study of curves. Nowadays it is a very well understood problem: Riemann proved that compact Riemann surface is actually a projective algebraic curve, therefore, the concepts of bimeromorphic equivalence and rational equivalence are the same for such objects. From the algebraic point of view it has been proved that every irreducible projective algebraic curve is birational to a smooth projective curve: this implies that the problem of classification actually relies only on the smooth case. Moreover, the smooth projective curves are then classified through an important topological invariant: the genus.
The case of surfaces has a much more involved solution: there are examples of compact complex surfaces that are not algebraic; though many great mathematicians (Del Pezzo, Severi, Albanese and Zariski, just to name few of them) have attempted the solution between the end of the 19th and the beginning of the 20th century, a complete and correct solution only came in 1935 thanks to Walker. Moreover, in the case of surfaces the geometric genus is not sufficient anymore to give a satisfying classification, thus there have been introduced new invariants, like the plurigenera, the Kodaira dimension and the irregularity are introduced.
There is a clear pattern in this: the higher the dimension, the messier the picture!
The problem of resolution of singularities, of course over the complex numbers has been finally solved in 1964 by Hironaka. In the following decade, Iitaka and Ueno developed the theory of birational classification of algebraic varieties. Iitaka, in particular, introduced, in his doctoral thesis ([13]), the now-called Iitaka dimension of a line bundle over a projective algebraic variety: a number that measures, in the same time, both how big can be the image of the variety under a map canonically associated to the line bundle and how big are, asymptotically, the spaces of global sections of the tensor powers of the line bundle itself.
The existence of several line bundles over a given variety is a non-trivial problem, but for every variety we can identify two line bundles (that can coincide): the trivial one, associated to the structure sheaf of the variety and the canonical one, parametrizing the holomorphic differential forms of

maximal degree on the variety. The Iitaka dimension of the canonical bundle plays an important role in the birational classification of algebraic varieties, indeed it is a birational invariant. Iitaka at first called it canonical dimension of the variety, but it became more kown as Kodaira dimension, as an omage to Iitaka's doctoral supervisor, Kodaira.
In another of his papers ([14]), Iitaka, also posed several open problems. The most interesting one was about the Kodaira dimension. The Conjecture C states that: let

$$f : X \to Y$$

be an algebraic fibre space, which is a surjective morphism of smooth projective varieties with connected fibres. Then

$$\kappa(X) \geq \kappa(Y) + \kappa(F)$$

where F is a general (that is: taken in an open dense subset) fibre of f and $\kappa(V)$ is the Kodaira dimension of the variety V for every $V \in \{X, Y, F\}$.
As of today, the problem of classification has strayed a bit from Iitaka's approach and the most powerful tools to understand the problem are given by Mori's Minimal Model Program, but Iitaka's ideas and results still play a very important role in the classification problem. In particular Iitaka Conjecture is deeply linked to the Abundance Conjecture.
Therefore, many of the greatest algebraic and analytic geometers, like Mori, Fujita, Fujino, Kawamata, Viehweg, Kollár, Hacon, Birkar, Cao, Cascini, McKernan and many others, have kept working also on Iitaka Conjecture-related issues. However, regardless of its importance in the picture of the classification, Iitaka Conjecture is a deeply fascinating problem that, as all the great questions, has never lost and, hopefully, will never lose its power to amaze whomever comes in contact with it.
The main focus of this discussion, as it might be clear by now, is the Conjecture, or to be precise some instances of it that have been proven over the years.
As a matter of fact Iitaka did not state its conjecture in the terms we used: indeed, the definition of canonical bundle does not require the hypothesis of projective (hence algebraic) variety to be posed and thus Iitaka stated it in the more general setting of a compact Kählerian variety. In this setting, however, the inequality does not always hold, indeed, in a paper of his Ueno proposed a counterexample (for more details one can look at [29] Chapter IIII, Remark 15.3).
Another question one can ask about Iitaka Conjecture is that, since all the key

definitions are algebraic, whether this statement makes sense for varieties over a positive characteristics algebraically closed field. Everything makes sense in such setting, and there have been many developments in this direction, mainly due to Lei Zhang (see, for example, [33]).
We will focus on the setting in which we stated the Conjecture: an algebraic fibre space between two smooth projective varieties over the complex numbers. In this setting there have been many results, some of them still valid for non necessarily algebraic manifolds. An obviously non-comprehensive list is

i Ueno proved the inequality, in the stronger form of an equality, for Moishezon manifolds ([29]);

ii Fujita proved the inequality for Kähler fibre spaces over curves in [9];

iii Kawamata proved the inequality, in the stronger form of an equality, in case the base space is of general type in [15];

iv thanks to the developments in the study of the Minimal Model Theory, Birkar concluded that the Iitaka inequality is satisfied for domains of dimension at most 6 ([2]);

v Cao and Păun proved the inequality for fibrations over varieties of maximal Albanese dimension in [5];

vi Cao proved also that the Conjecture holds for fibrations over surfaces in [4].

We chose to focus our attention on three instances: a proof of the case of a fibration over a curve given by Lazarsfeld in [18] which is substantially Fujita's proof; a proof of the case surface-curve based only on algebraic methods given by Wessler in [32]; the outline and the main ideas of Cao and Păun's proof ([5]) in the exposition given by Hacon-Popa-Schnell in [10].
We preferred to use almost only algebraic methods, even though at some point we needed to introduce some analytic notions that are necessary. Many results are presented without proofs, in perfect accordance to the very italian principle of *"se una scena è complicata, non la famo, ma lo dimo."* [2] with the obvious interpretation in this case for those proofs that are lengthy, but not too important. Or also are too difficult.
The discussion is divided into three chapters. In the first one there are the

[2] "If a scene is too complicated we don't do that, but we say it."

main definitions of the varieties, divisors, linear systems, Iitaka and Kodaira dimension, ample and nef line bundles, Iitaka fibration, ample and nef vector bundles.

The second chapter contains: Castelnuovo–Mumford regularity theorems, vanishing theorems, Fujita theorem and Mourougane theorem.

The third chapter gives some general results on the Iitaka and Kodaira dimension and the proofs of those instances of $C_{n,m}$.

Chapter 1

Preliminaries

As usual, before going into deep detail with the study of the Iitaka Conjecture, we will need to recall and prove some basic facts and definitions about some stuff. Throughout the entire discussion we will be assuming as known the theory of Sheaf Cohomology. Some good references for this are the fourth chapter of [31] and the third chapter of [12]. We tried to recall (and give a good reference) for all the tools from Sheaf Cohomology that we used; sometimes, though, we used some results without mentioning them, mainly abouto higher direct image sheaves or flat morphisms. All of these results can be explicitly found in [12] Chapter III Sections 8 and 9.

1.1 Projective Complex Varieties

1.1.1 Projective Algebraic Varieties

In this section we will define what a *smooth projective complex variety* is. These will be the main objects on which all of the constructions will be made. For this part the main references have been the second section of the first chapter of [12]. We will always work over the complex numbers $\mathbb{C}$ and all the rings we consider will be commutative and unital.

Definition 1.1.1. A *graded ring* is a ring S together with a decomposition

$$S = \bigoplus_{d \in \mathbb{N}} S_d$$

in which each S_d is an abelian group and for every $d, e \in \mathbb{N}$:

$$S_d \cdot S_e \subseteq S_{d+e}.$$

An element of S_d is called an *homogeneous element of degree d*. An ideal

$$\mathfrak{a} \subseteq S$$

is said to be *homogeneous* if there exists a set of generators for $\mathfrak{a}$ in which every element is homogenous.

Now, we want to see, for a fixed $n \in \mathbb{N}$, the ring $\mathbb{C}[X_0, \dots, X_n]$ as a graded ring: for monomials in such ring we have an obvious definition of degree, hence, we define, for each $d \in \mathbb{N}$, S_d to be the set of $\mathbb{C}$-linear combinations of only monomials of degree d.
Now, let $f \in \mathbb{C}[X_0, \dots, X_n]$ be a fixed homogeneous polynomial; then we can define its zero locus in $\mathbb{P}^n(\mathbb{C})$ as

$$V(f) = \{P = [x_0, \dots, x_n] \in \mathbb{P}^n(\mathbb{C}) : f(P) = 0\}.$$

Of course, since f is homogeneous this definition is well-posed: indeed, for a fixed point $P \in \mathbb{P}^n(\mathbb{C})$, we can have two distinct homogeneous coordinates, namely

$$[x_0, \dots, x_n] = [\xi_0, \dots, \xi_n]$$

and there exists a non-zero $\lambda \in \mathbb{C}$ such that

$$x_i = \lambda \xi_i,\ 0 \leq i \leq n.$$

Set now

$$\underline{x} = (x_0, \dots, x_n),\ \underline{\xi} = (\xi_0, \dots, \xi_n)$$

then, by homogeneity of f:

$$f(\underline{x}) = \lambda^d f(\underline{\xi})$$

where d is the degree of f. This computation shows that the vanishing of f at a given point does not depend on the choice of homogeneous coordinates. In the same way as the one polynomial case we can define the vanishing locus of a family of homogeneous polynomials T:

$$V(T) = \{P \in \mathbb{P}^n : f(P) = 0,\ \forall f \in T\}.$$

Definition 1.1.2. A subset $Y \subseteq \mathbb{P}^n$ is an *algebraic set* if there exists a set T of homogeneous polynomials such that $Y = V(T)$.

Lemma 1.1.3. *The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

Proof. Omitted. Can be found in [12] Chapter 1, Section 2, Proposition 2.1 □

Now, observe that the previous Lemma implies that the algebraic subsets of $\mathbb{P}^n$ are canonically the closed subsets of a topological space. This way we can define a topology on $\mathbb{P}^n$: the *Zariski topology.*
We can give the following:

Definition 1.1.4. A *projective algebraic variety* is an irreducible algebraic subset of $\mathbb{P}^n$.

Definition 1.1.5. If Y is an algebraic subset of $\mathbb{P}^n$ we can consider the homogeneous ideal associated to Y, namely:

$$I(Y) = \{f \in \mathbb{C}[X_0, \dots, X_n] : f \text{ is homogeneous and } f(P) = 0 \; \forall P \in Y\},$$

and the coordinate ring associated to Y, which is

$$S(Y) = \frac{\mathbb{C}[X_0, \dots, X_n]}{I(Y)}.$$

Now, one may notice that on $\mathbb{P}^n$ we have two natural topologies that we can consider, namely the usual topology induced as a quotient of $\mathbb{C}^{n+1}$ and the Zariski topology. This can lead to many questions, like "Why have we chosen the Zariski topology to define our varieties?" or "Are the results one obtains using these two different topologies in some sense related?"
These are really important questions and, indeed, we will work almost only with the usual Hausdorff topology, but it felt easier to give these definitions in the Zariski topology. In the following subsection we will also define the complex manifolds and the complex analytic spaces, and state some classical results on both projectivity of such spaces and on the relation between the two approaches.

1.1.2 Comparison between Algebraic and Analytic Methods

We will state some results that link together the approach we outlined in the previous to the one we will actually use throughout the discussion. This will allow us to move in complete freedom among the different references that might use differing languages.
First of all, in the Hausdorff topology the concept that is the closest to the one of algebraic variety is the following:

Definition 1.1.6. A *complex analytic space* is a locally ringed space $(X, \mathcal{O}_X)$, which can be covered by open sets, each of which is isomorphic, as a locally space, to one of the following kind Y: let $U \subseteq \mathbb{C}^n$ be the polydisc

$$\{|z_i| < 1 : 1 \leq i \leq n\}$$

and let $f_1, \ldots, f_r$ holomorphic functions on U, then Y is the vanishing locus of the f_j's and has, as structure sheaf, the quotient

$$\mathcal{O}_Y = \frac{\mathcal{O}_U}{(f_1, \ldots, f_r)}.$$

An analytic complex space is said to be *normal* if, as a ringed space, each stalk of its structure sheaf is an integrally closed ring.

Now, we will state some classical results.

Definition 1.1.7. Let $(X, \mathcal{O}_X)$ be a ringed space then a *coherent sheaf* $\mathcal{F}$ over X is a sheaf of $\mathcal{O}_X$-modules such that for every $x \in X$, there exists an open neighborhood $U \subseteq X$ of x, natural numbers $n, m \in \mathbb{N}$ and morphisms φ, ψ such that the following sequence is exact:

$$\mathcal{O}_U^n \xrightarrow{\quad \varphi \quad} \mathcal{O}_U^m \xrightarrow{\quad \psi \quad} \mathcal{F}_{|U} \longrightarrow 0$$

Theorem 1.1.8. *Let X be a projective scheme over $\mathbb{C}$. Then there exists a complex analytic space X_h associated to X such that the assignation:*

$$h : X \longmapsto X_h$$

is functorial and such that h induces an equivalence of categories between the category of coherent sheaves over X and the category of coherent analytic *sheaves over X_h. Furthermore, for every coherent sheaf $\mathcal{F}$ on X, the natural maps*

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$$

are isomorphisms for every i.

Proof. Omitted. Can be found in [27]. □

Theorem 1.1.9. *If Y is a compact analytic subspace of the complex manifold $\mathbb{P}^n(\mathbb{C})$, then there is a subscheme $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ such that $X_h = Y$.*

Proof. Omitted. Can be found in [6]. □

Now the only thing that is still missing to tie together these different approaches is to link a variety as in Definition 1.1.4 to a scheme.

Proposition 1.1.10. *There is a natural fully-faithful functor*

$$t : \underline{Var}_{\mathbb{C}} \to \underline{Sch}_{\mathbb{C}}$$

from the category of varieties over $\mathbb{C}$ to the category of schemes over $\mathbb{C}$. For any variety V, its topological space is homeomorphic to the set of closed points of the topological space of $t(V)$ and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.

Proof. Omitted. See [12] Chapter II, Section 2, Proposition 2.6 □

We conclude with a definition: thanks to these results, one can see any projective algebraic variety as a complex analytic space, hence as a ringed space and just like for these spaces, there is a well-defined notion of smoothness or regularity.

Definition 1.1.11. Let $(X, \mathcal{O}_X)$ be a complex analytic space. A point $x \in X$ is said to be *regular* or *smooth* if there is an open neighborhood U of x such that U is isomorphic, as a locally ringed space, to an open ball of $\mathbb{C}^n$. If every point of X is smooth, then X is smooth.

All the varieties we consider will be smooth and projective. Actually some of the results still hold under weaker hypotheses, but we redirect the interested reader to the literature we used to get an appropriate discussion about these other cases.

1.1.3 Divisors

In this subsection we will define the concepts of Cartier divisor on a smooth projective variety X. The main references are [12] and [19].
So let X be a fixed smooth projective variety over $\mathbb{C}$. Then it is canonically defined its field of rational functions usually denoted as $\mathbb{C}(X)$. Let $\mathcal{M}_X$ be the constant sheaf associated to $\mathbb{C}(X)$. In particular, $\mathcal{M}_X$ contains as a subsheaf the structure sheaf of $\mathcal{O}_X$; hence we have an inclusion

$$\mathcal{O}_X^\times \subseteq \mathcal{M}_X^\times$$

of the multiplicative groups.

Definition 1.1.12. A *(Cartier) divisor* on X is a global section of the sheaf $\frac{\mathcal{M}_X^\times}{\mathcal{O}_X^\times}$. We denote the group of the divisors as $\mathrm{Div}(X)$.

More explicitly one can describe a fixed divisor D through the following data:

$$\{(U_i, f_i)\}_{i\in I},$$

where the U_i's are an open covering of X and $f_i \in \Gamma(U_i, \mathcal{M}_X^\times)$ for every $i \in I$ are such that for every $i, j \in I$ such that $U_{ij} = U_i \cap U_j \neq \emptyset$, there exists a section $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^\times)$. The section f_i is said to be a *local equation for the divisor* at each point of U_i.
A divisor D represented by $\{(U_i, f_i)\}_{i\in I}$ is *effective* if, for every $i \in I$, f_i is regular on U_i and we denote it as $D \geq 0$.
Of course, any global section $f \in \Gamma(X, \mathcal{M}_X^\times)$ determines a divisor

$$D = \mathrm{div}(f)$$

and it is called *principal*. Two divisors are said to be *linearly equivalent* if their difference[1] is a principal divisor.
If D is a divisor on X, and $f : Y \to X$ is a morphism, one can define, as long as certain conditions hold, a divisor f^*D on Y by pulling-back the local equations of D. This makes sense, only if (in case Y is reduced) the components of Y are not mapped into the support of D.
Now, if L is an invertible sheaf over X, let $s \in \Gamma(X, L)$ be fixed. For every

[1]Since the group $\frac{\mathcal{M}_X^\times}{\mathcal{O}_X^\times}$ is abelian we will use the additive notation regardless the fact that it canonically is described as a multiplicative group.

$U \subseteq X$ open on which the restriction of L is trivial, we can choose an isomorphism

$$\varphi_U : L_{|U} \cong \mathcal{O}_U$$

Making U vary over an open covering we get a collection

$$\{(U, \varphi_U(s_{|U})\}$$

As, the φ_U's, on the intersections of the opens we chose are defined up to an invertible section of $\mathcal{O}_U$, the above data defines a divisor.

Definition 1.1.13. Such a divisor is called the *divisor of zeroes of* s and is denoted as $\text{div}(s)$.

Remark 1.1.14. Usually, there are two different notions of divisors, namely, Weil divisors and Cartier divisors. In complete generality the two notions do not coincide, but, in the case of a smooth projective variety, which is the only case we will be considering, the two are equivalent. A good reference for this is the Section 6 of Chapter II of [12].

Let D be a fixed divisor of X. We will associate to it a line bundle $\mathcal{O}_X(D)$. Precisely, if D is represented by the data

$$\{(U_i, f_i)\}_{i \in I}$$

then, we can consider it to be an atlas for our line bundle $\mathcal{O}_X(D)$ with transition maps the g_{ij}'s that we defined in the explicit definition of D as represented by the sections f_i's.
From a more abstract point of view, one has the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{M}_X^\times \longrightarrow \frac{\mathcal{M}_X^\times}{\mathcal{O}_X^\times} \longrightarrow 0$$

and, by taking the sheaf cohomology of such sequence, we have the following exact sequence:

$$\Gamma(X, \mathcal{M}_X^\times) \longrightarrow \text{Div}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times)$$

At this point:

Definition 1.1.15. We define $\mathrm{Pic}(X)$ to be the group of line bundles over X up to isomorphism.

Lemma 1.1.16. *There is a canonical isomorphism* $H^1(X, \mathcal{O}_X^\times) = Pic(X)$.

Proof. Omitted. See [24] Chapter 2, Proposition 2.1.3. □

In particular, this last sequence implies the following equivalence

$$D_1 \equiv_{\text{lin}} D_2 \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2).$$

Let's finish this section with the following results:

Proposition 1.1.17. *The global sections of a line bundle over a projective variety are a finitely generated* $\mathbb{C}$*-vector space.*

Proof. Omitted. See [12] Chapter II, Section 5, Proposition 5.19. □

Theorem 1.1.18. *If* X *is irreducible, the map*

$$\frac{Div(X)}{\equiv_{lin}} \to Pic(X)$$

is an isomorphism.

Proof. Omitted. See [12] Chapter II, Section 6, Corollary 6.16. □

1.1.4 Linear Systems

In this subsection we will define and state some properties about the linear systems. As usual, the main references will be [12] and [19].
Before starting to talk about linear systems, we want to recall a result.

Proposition 1.1.19. *Let* X *be a smooth projective variety over* $\mathbb{C}$. *Let* D_0 *be a divisor on it and* $L = \mathcal{O}_X(D_0)$ *be the associated line bundle. Then:*

i for each non-zero global section of L *its divisor is an effective divisor linearly equivalent to* D_0;

ii every effective divisor linearly equivalent to D_0 *is the zero-divisor of a global section of* L;

iii two distinct s, s' sections have the same zero-divisor if and only if there is a non-zero complex number λ such that $s = \lambda s'$.

Proof. Omitted. See [12] Chapter II, Section 7, Proposition 7.7. □

Definition 1.1.20. A *complete linear system* on a smooth projective variety is defined as the set (possibly empty) of all effective divisors linearly equivalent to a given divisor. If the given divisor is D, the complete linear system is denoted as $|D|$.

By the Proposition, one has that if D is a divisor and L is its associated line bundle, we have a bijection:

$$|D| \cong \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))).$$

Using this last definition, we can also talk about the complete linear system associated to a line bundle L, denoted as $|L|$. Furthermore, for every subspace V of the global sections of L, we use the following notation:

$$|V| = \mathbb{P}(V).$$

Definition 1.1.21. Fixed a line bundle L on X, a *linear system* is a linear subspace of $|L|$.

Now, let us fix a projective algebraic variety X, a line bundle L on it and V a subspace of the global sections of L.
The evaluation of the sections of V gives rise to a morphism:

$$V \otimes_{\mathbb{C}} \mathcal{O}_X \to L$$

that, gives rise to another morphism

$$\psi_V : V \otimes_{\mathbb{C}} L^{\vee} \to \mathcal{O}_X.$$

Definition 1.1.22. The *base ideal* of $|V|$ is denoted as $\mathfrak{b}(|V|)$ and is defined as the image sheaf of ψ_V. We also define the *base locus* of V as the closed subset of X of the points at which all sections of V vanish and it is denoted as $\mathrm{Bs}(|V|)$.

Definition 1.1.23. One says that $|V|$ is *free* or *basepoint-free* if $\mathrm{Bs}(|V|)$ is empty. A divisor is said to be free if the correspondent linear system is. For a line bundle, if its associated complete linear system is basepoint-free, we say that is *globally generated* or *generated by global sections.*

Definition 1.1.24. Now, fixed a basis $s_0, \dots, s_r$ for V, we have a well defined morphism

$$\begin{aligned} \varphi_{|V|} : X \setminus \mathrm{Bs}(|V|) &\to \mathbb{P}^r \\ x &\longmapsto [s_0(x), \dots, s_r(x)]. \end{aligned}$$

Usually, we want to think about it as a rational map

$$\varphi_{|V|} : X \dashrightarrow \mathbb{P}^r.$$

Now we will conclude with a very important example of this kind of map.

Example 1.1.25. In the same situation as above, let $W \subseteq V$ be a subspace. Then

$$\mathrm{Bs}(|V|) \subseteq \mathrm{Bs}(|W|).$$

Thus both $\varphi_{|V|}$ and $\varphi_{|W|}$ are defined over $X \setminus \mathrm{Bs}(|W|)$. If, furthermore, one considers, the projection with center $\mathbb{P}\left(\frac{V}{W}\right)$, namely

$$\pi : \mathbb{P}(V) \setminus \mathbb{P}\left(\frac{V}{W}\right) \to \mathbb{P}(W),$$

one has that π is a map such that

$$\varphi_{|W|} = \pi \circ \varphi_{|V|}.$$

1.2 Iitaka Dimension and Kodaira Dimension

In this section we will define the Iitaka dimension of a line bundle and consequentely the Kodaira dimension of a variety. In the end we will present some results on both of them. The main reference for this part is [19].

1.2.1 Definitions

Definition 1.2.1. Let L be a line bundle on a smooth irreducible projective variety X. The *semigroup* of L is the following set

$$\mathbb{N}(L) = \mathbb{N}(X, L) = \{m \in \mathbb{N} : H^0(X, L^{\otimes m}) \neq 0\}.$$

When $\mathbb{N}(L) \neq \{0\}$ we also define the *exponent* of L, noted $e(L)$, as the greatest common divisor of all elements in $\mathbb{N}(L)$. If $L = \mathcal{O}_X(D)$ for some divisor D, we use the notations $\mathbb{N}(D)$ and $e(D)$.

Example 1.2.2. Let X be a smooth irreducible projective variety and L a fixed line bundle on it. Then, each of its tensor powers defines a (possibly zero) linear system, namely $|L^{\otimes m}|$ for every $m \in \mathbb{N}$.
For $m \in \mathbb{N}(L)$, we have that such linear system is non-trivial, so, by the construction we gave in Definition 1.1.24 (in the same notations), we have a rational map

$$\varphi_m = \varphi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

We denote by $Y_m \subseteq \mathbb{P}(H^0(X, L^{\otimes m}))$ the image, under the canonical projection, of the closure of the graph of φ_m.

Definition 1.2.3. In the same notation as above, assume also that X is normal. Then we define the *Iitaka dimension* of L as follows:

$$\kappa(L) = \kappa(X, L) = \max_{m \in \mathbb{N}(L)} \{\dim \varphi_m(X)\},$$

provided that $\mathbb{N}(L) \neq \{0\}$. Otherwise, we simply put $\kappa(L) = -\infty$. Furthermore, if X is not normal, we can consider its normalization

$$\nu : X` \to X,$$

and then define

$$\kappa(L) = \kappa(\nu^* L).$$

In the end, as usual, if $L = \mathcal{O}_X(D)$ for some divisor D, we simply use the notation $\kappa(D)$.

At this point we should prove that this definition is well-posed, which, in this case means, that the maximum actually exists. We won't be doing that immediately, but in a second moment (Lemma 1.2.12).
From this definition we clearly see that for every line bundle L over X:

$$\kappa(L) \in \{-\infty, 0, \ldots, \dim X\}.$$

Another result one can prove is the following

Lemma 1.2.4. *Let X, L be fixed as above and put $\kappa = \kappa(L)$. Then there are constants $a, b > 0$ such that*

$$a \cdot m^\kappa \leq \dim h^0(X, L^{\otimes m}) \leq b \cdot m^\kappa$$

for all m sufficiently large in $\mathbb{N}(L)$ and where $h^0(X, L^{\otimes m}) = \dim_{\mathbb{C}} H^0(X, L^{\otimes m})$.

Proof. Omitted. See [19] Corollary 2.1.38. □

In particular, we can now give another (equivalent) definition for the Iitaka dimension.

Definition 1.2.5. For L a line bundle:

$$\kappa(L) = \limsup_{m \longrightarrow +\infty} \frac{\log h^0(X, L^{\otimes m})}{\log m}$$

A special case of the Iitaka dimension is the case of in which L is the canonical bundle of X.

Definition 1.2.6. Let X be a smooth projective variety of dimension n and let TX be its holomorphic tangent bundle. So, I define the line bundle

$$\omega_X = \bigwedge^n (TX)^\vee$$

This line bundle is said to be the *canonical bundle* of X. Since by Theorem 1.1.18 in such hypotheses every line bundles is associated to a unique class of divisors modulo linear equivalence, we denote with K_X every divisor such that

$$\omega_X = \mathcal{O}_X(K_X)$$

and we refer to such divisors as *canonical divisors*.

Definition 1.2.7. The *Kodaira dimension* of X is the Iitaka dimension of its canonical bundle and is denoted as $\kappa(X)$.

Lemma 1.2.8. *The Kodaira dimension of a variety is a birational invariant.*

Proof. Omitted. See [24] Chapter 3, Section 2, Proposition 3.2.6. □

Definition 1.2.9. An *algebraic fibre space* is a surjective projective mapping $f : X \to Y$ of reduced irreducible varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Theorem 1.2.10. *An algebraic fibre space has connected fibres.*

Proof. Omitted. See [24] Chapter 3, Section 1, Lemma 3.1.5. □

Lemma 1.2.11. *Let $f : X \to Y$ be an algebraic fibre space and let L be a line bundle on Y. Then,*

$$H^0(X, f^*L^{\otimes m}) = H^0(Y, L^{\otimes m})$$

for all $m \in \mathbb{N}$.

Proof. In generality

$$H^0(X, f^*L^{\otimes m}) = H^0(Y, f_*f^*L^{\otimes m}).$$

But, then, by the Projection Formula and the fact that we are in fibre space

$$f_*f^*L^{\otimes m} = f_*\mathcal{O}_X \otimes L^{\otimes m} = L^{\otimes m};$$

hence our claim. □

1.2.2 Examples

Here we will state and prove some basic results about both the Iitaka dimension and the Kodaira dimension.

Lemma 1.2.12. *The definition of the Iitaka dimension of a line bundle L over a normal projective variety X is well-posed.*

Proof. To prove that the Iitaka dimension is well-defined we need to show that such a maximum exists.
Of course we can assume that $\mathbb{N}(L) \neq \{0\}$. Furthermore, up to change L with $L^{\otimes e}$, where $e = e(L)$, we can assume that L has exponent 1. In such case, there exists $\ell_0 \in \mathbb{N}$ such that for $\ell \geq \ell_0, \ell \in \mathbb{N}(L)$.
Let $m \in \mathbb{N}(L)$ be fixed, and let $k = \dim \varphi_m(X)$ and let $\ell \geq \ell_0$ also be fixed. Multiplying by a non-zero section of $H^0(X, L^{\otimes \ell})$ gives an inclusion

$$H^0(X, L^{\otimes m}) \subseteq H^0(X, L^{\otimes (m+\ell)}).$$

Thus, following the construction in Example 1.1.25, there is a rational map

$$\nu_\ell : \mathbb{P}(H^0(X, L^{\otimes (m+\ell)})) \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

such that, where they are all defined, $\varphi_m = \nu_\ell \circ \varphi_{m+\ell}$.
This implies, that

$$\dim \varphi_{m+\ell}(X) \geq \dim \varphi_m(X).$$

A consequence of this is that the sequence

$$\{\dim \varphi_m(X) : m \in \mathbb{N}\}$$

has an increasing subsequence. But it is bounded by $\dim X$, hence, the entire family is finite and admits a maximum. □

Remark 1.2.13. Lemma 1.2.12 also implies that there infinitely many $m \in \mathbb{N}(L)$ such that $\varphi_m(X)$ has as dimension the Iitaka dimension of L.

Example 1.2.14. The Iitaka dimension is absolutely non-stable under restrictions; indeed: let X be the blow-up of $\mathbb{P}^2$ at a point, H the pull-back of an hyperplane divisor, and E the exceptional divisor.
Let $L_1 = \mathcal{O}_X(H), L_2 = \mathcal{O}_X(H+E)$. They are both ample line bundles, so they have maximal Iitaka dimension. But, when restricted to E they have as Iitaka dimensions 0 and $-\infty$ respectively.
In this case the Iitaka dimension has decreased with the restriction, but this is not a general fact, indeed: if $X = \mathbb{P}^1 \times \mathbb{P}^1$, and $L = \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ the Iitaka dimension of L is $-\infty$, but when restricted to $Y = \{*\} \times \mathbb{P}^1$, it has maximal Iitaka dimension 1.

Example 1.2.15. We will now compute the Kodaira dimension of an irreducible projective algebraic curve. It will be used the following version of the Riemann-Roch Theorem whose proof can be found in the Chapter 4 of [12].

Theorem 1.2.16. *Let C be an irreducible projective algebraic curve of genus g and K its canonical divisor. For every divisor D on C:*

i $\deg(D) < 0 \Longrightarrow \ell(D) = 0$;

ii $\ell(0) = 1$;

iii $\deg(K) = 2g - 2$;

iv $\ell(D) - \ell(K - D) = \deg(D) + g - 1$.

where $\ell(D) = \dim H^0(C, \mathcal{O}_C(D))$.

Applying this theorem, we have 3 cases:

a $g = 0$, then the canonical divisor has negative degree and therefore all of its multiple too. This implies that $\mathbb{N}(K) = \{0\}$, hence $\kappa(\mathbb{P}^1) = -\infty$.

b $g = 1$ implies that the canonical divisor has degree 0, thus the canonical bundle is trivial, hence its powers are all isomorphic and have $\ell(nK) = 1$, thus $\kappa(C) = 0$.

c $g \geq 2$: applying Riemann-Roch to $mK, n \geq 1$, yields us that

$$\ell(mK) = (2m-1)(g-1).$$

Hence

$$\frac{\log(2m-1)(g-1)}{\log m} = \frac{\log(2m-1)}{\log m} + \frac{\log(g-1)}{\log m},$$

and the second summand is infinitesimal for $m \to +\infty$. Now, observe that

$$1 = \frac{\log m}{\log m} \leq \frac{\log(2m-1)}{\log m} \leq \frac{\log(2m)}{\log m} = \frac{\log 2}{\log m} + 1,$$

and taking the limit one shows $\kappa(C) = 1$.

Example 1.2.17. Let X, Y be irreducible projective varieties, and $f : X \to Y$ an algebraic fibre space. Then the induced homomorphism

$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is injective.
Indeed, let A be a line bundle on Y such that

$$f^*A = \mathcal{O}_X.$$

Then,

$$f_*f^*A = f_*\mathcal{O}_X = \mathcal{O}_Y.$$

But on the other hand

$$f_*f^*A = f_*\mathcal{O}_X \otimes A = \mathcal{O}_Y \otimes A = A,$$

hence, the map has trivial kernel.

Example 1.2.18. Let $f : X \to Y$ be a projective surjective morphism of normal varieties, and let

$$\mathbb{C}(Y) \subseteq \mathbb{C}(X)$$

be the corresponding inclusion of function fields. Then f is a fibre space if and only if $\mathbb{C}(Y)$ is algebraically closed in $\mathbb{C}(X)$.
To prove this statement we will need the following Theorem and Definition:

Theorem 1.2.19. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes. Then f factors into $g \circ f'$ where*

$$f' : X \to Y'$$

is a projective morphism with connected fibres, and

$$g : Y' \to Y$$

is a finite morphism.

Proof. Omitted. See [12] Capter III, Corollary 11.5. □

Definition 1.2.20. A factorization of a morphism like the one in Theorem 1.2.19 is called *Stein factorization.*

So, in this setting, consider the Stein factorization of f

$$X \xrightarrow{\varphi} Y' \xrightarrow{\nu} Y$$

where φ is a fibre space and ν is a finite morphism. Of course it is clear that f itself is a fibre space if and only if $Y' = Y$ and $\nu = \mathrm{id}_Y$.
At first observe that ν, being a finite morphism, gives rise to a finite (hence algebraic) intermediate extension:

$$\mathbb{C}(Y) \subseteq K = \mathbb{C}(Y') \subseteq \mathbb{C}(X).$$

Moreover, any of such extensions gives rise to a finite morphism from some variety Z:

$$Z \to Y.$$

Thus, ν is trivial if and only if there are no algebraic extensions over of $\mathbb{C}(Y)$ in $\mathbb{C}(X)$.

1.3 Positivity Notions for Line Bundles

In this subsection we will introduce some definitions and results regarding line bundles. The main reference is [19]. We will be considering as known the definitions and the notations of the Intersection Theory.

Definition 1.3.1. Let X be a projective variety and L a line bundle on it. We say that L is *very ample* if there exists a closed embedding

$$\varphi : X \hookrightarrow \mathbb{P}^n$$

for some $n \in \mathbb{N}$ such that

$$L = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1).$$

We say that L is *ample* if some of its tensor powers is very ample. A divisor D is either very ample or ample if the corresponding line bundle is.

We can give a nice cohomological characterization of ampleness:

Theorem 1.3.2. *Let L be a line bundle on a projective variety. The following are equivalent:*

i L is ample;

ii for any coherent sheaf $\mathcal{F}$ on X there exists a natural number $m_1 = m_1(\mathcal{F})$ such that

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0,\ \forall i > 0\,, m \geq m_1.$$

iii for any coherent sheaf $\mathcal{F}$ on X there exists a natural number $m_2 = m_2(\mathcal{F})$ such that $\mathcal{F} \otimes L^{\otimes m}$ is globally generated for every $m \geq m_2$;

iv there exists a natural number m_3 such that $L^{\otimes m}$ is very ample for every $m \geq m_3$.

Proof. Omitted. See [19] Theorem 1.2.6. □

Remark 1.3.3. Usually the second condition is referred to as *Serre's Vanishing*.

Corollary 1.3.4. *Let $f : X \to Y$ be a finite mapping of projective varieties and L an ample line bundle on X. Then f^*L is ample on Y. In particular, for any subvariety of X, the restriction to such subvariety of L is still ample.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Then, we have that

$$f_*(\mathcal{F} \otimes L^{\otimes m}) = f_*\mathcal{F} \otimes L^{\otimes m}$$

by the Projection Formula. Furthermore, $f_*\mathcal{F}$ is quasi-coherent (See [12], Chapter II, Proposition 5.8). Then, for every i

$$H^i(Y, \mathcal{F} \otimes L^{\otimes m}) = H^i(X, f_*\mathcal{F} \otimes L^{\otimes m}) = 0,$$

hence the thesis. □

Corollary 1.3.5. *Suppose that L is globally generated on a projective variety X and let*

$$\varphi = \varphi_{|L|} : X \to \mathbb{P} = \mathbb{P}(H^0(X, L))$$

be the map defined through the linear system $|L|$. Then L is ample and if and only if φ is a finite mapping, or equivalently, if

$$c_1(L) \cdot C > 0$$

for every irreducible curve $C \subseteq X$.

Proof. By Corollary 1.3.4, if φ is finite, then L is ample. In this case, for every $C \subseteq X$ irreducible curve $L_{|C}$ is still ample, hence

$$c_1(L) \cdot C = \deg(L_{|C}) > 0.$$

Conversely, if φ is not finite, there exists a subvariety $Z \subseteq X$ of positive dimension that is contracted to a point. As $L = \varphi^* \mathcal{O}_{\mathbb{P}}(1)$, then one sees that the restriction of L to Z is trivial; in particular it is not ample and for every irreducible curve $C \subseteq Z$ we have that

$$c_1(L) \cdot C = \deg(L_{|C}) = 0.$$

□

Another characterization of ampleness that generalizes Corollary 1.3.5 is the following result known as Nakai-Moishezon-Kleiman Criterion.

Theorem 1.3.6. *Let L be a line bundle on a projective variety X. Then L is ample if and only if*

$$c_1(L)^{\dim V} \cdot V > 0$$

for every irreducible subvariety of positive dimension V of X.

Proof. Omitted. See [19], Theorem 1.2.23. □

Corollary 1.3.7. *Let $f : Y \to X$ be a finite and surjective map of projective varieties, and let L be a line bundle on X. If f^*L is ample, then also L is.*

Proof. Let $V \subseteq X$ be a fixed irreducible subvariety. By surjectivity, there exists an irreducible subvariety $W \subseteq Y$ that is, finitely, mapped into V, namely

$$f(W) = V.$$

Indeed, to construct such a variety, one can consider $f^{-1}(V)$ and take any irreducible component. Then

$$c_1(f^*L)^{\dim W} \cdot W = \deg(f) c_1(L)^{\dim V} \cdot V$$

and by Theorem 1.3.6 we conclude. □

We conclude the discussion about ample line bundles with this result about the case in which X is not reduced or is reducible. The proof is omitted and one can find it in [19] Proposition 1.2.26.

Proposition 1.3.8. *Let X be a projective variety and L a line bundle on it.*

i L is ample on X if and only if L_{red} is ample on X_{red}.

ii L is ample on X if and only if the restriction of L to each irreducible component of X is nef on such component.

Now, we consider a slightly more general class of line bundles.

Definition 1.3.9. Let X be a projective variety. A line bundle L on X is said to be *numerically effective* or *nef*, if for every irreducible curve $C \subseteq X$

$$c_1(L) \cdot X \geq 0.$$

As usual, we extend the definition to divisors.

Remark 1.3.10. Every ample line bundle is nef and the tensor product of two nef line bundles is again nef.

I will conclude with some basic properties which I present without a proof.

Theorem 1.3.11. *Let X be a projective variety and L be a line bundle on it.*

i Let $f : Y \to X$ be a proper mapping. If L is nef, then f^*L is a nef line bundle on Y. In particular, restriction of nef line bundles are nef line bundles.

ii In the situation of i, if f is surjective and f^*L is nef, then L itself is nef.

iii L is nef if and only if L_{red} is on X_{red}.

iv L is nef if and only if is nef when restricted to each irreducible component of X.

1.4 The Iitaka Fibration

In this section we will prove an important theorem regarding fibred spaces, namely the Iitaka Fibration.

1.4.1 Semiample Line Bundles

Definition 1.4.1. A line bundle L on a smooth projective variety is said to be *semiample* if there exists an $m \in \mathbb{N}$ such that $L^{\otimes m}$ is globally generated. A divisor is said to be semiample if its associated line bundle is.

Definition 1.4.2. Let V a complex vector space, then, we define the *tensor algebra* $T(V)$ associated to V as

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}.$$

In such algebra consider the bilateral ideal I generated by the elements of the form $x \otimes y - y \otimes x$ for every $x, y \in V$. The quotient

$$S(V) = \frac{T(V)}{I},$$

is the *symmetric algebra* associated to V. This has canonically the structure of a graded algebra and we denote, for every $k \in \mathbb{N}$, as $S^k V$ the homogeneous elements of degree k.

Example 1.4.3. Let X be a smooth projective variety and L a semiample line bundle on it. Let furthermore m be such that $L^{\otimes m}$ be globally generated. For $k \geq 0$, $S^k H^0(X, L^{\otimes m})$ determines a free linear subsystem of $\left|L^{\otimes km}\right|$: indeed, there is a canonical injective map

$$S^k H^0(X, L^{\otimes m}) \hookrightarrow H^0(X, L^{\otimes km});$$

hence, the fact that it induces a subsystem. For the freedom, one can simply argue like this: as $L^{\otimes m}$ is globally generated, then for every $P \in X$ there exists an $s \in H^0(X, L^{\otimes m})$ which is non-zero on P, but then

$$s^k(P) \neq 0,$$

hence, the linear subsystem induced by $S^k H^0(X, L^{\otimes m})$ is free.
In the notation of Example 1.2.2, by Example 1.1.25, this subseries induces a finite map (a finite projection) ν_k, such that the following diagram is commutative:

$$\begin{array}{ccc} X & & \\ & \searrow^{\varphi_{km}} & \\ \varphi_m \downarrow & & Y_{km} \\ & \swarrow_{\nu_k} & \\ Y_m & & \end{array}$$

Furthermore, the ample line bundle $\mathcal{O}_{\mathbb{P}(H^0(X,L^{\otimes}))}(1)$ restricts to a very ample line bundle A_m on Y_m that, by the definition of φ_m, pulls back to $L^{\otimes m}$.

Now, for X a smooth projective variety and L a semiample line bundle, we define

$$M(X, L) = \{m \in \mathbb{N} : L^{\otimes m} \text{ is free } \}$$

and also define its exponent to be the greatest common divisor of all its elements.

Lemma 1.4.4. *Let X be a smooth projective variety and L a semiample line bundle. Fix $m \in M(X, L)$. In the notation of Example 1.4.3, for all k large enough, the composition*

$$X \xrightarrow{\varphi_{km}} Y_{km} \xrightarrow{\nu_k} Y_m$$

gives the Stein factorization φ_m. In particular, Y_{km} and φ_{km} are independent of the choice of k sufficiently large.

Proof. Consider

$$X \xrightarrow{\psi} V \xrightarrow{\mu} Y_m$$

the Stein factorization of φ_m, so ψ is a fibre space, V is normal and μ is finite. Let A_m be the ample line bundle on Y_m that pulls back to $L^{\otimes m}$. Since μ is finite, $B = \mu^* A_m$ is ample on V. This implies that $B^{\otimes k}$ is very ample for all k sufficiently large. On the other hand, $\psi^* B^{\otimes k} = L^{\otimes km}$ and

$$H^0(V, B^{\otimes k}) = H^0(X, L^{\otimes km})$$

by Lemma 1.2.11. But this implies that V is the image of X under $\varphi_{|L^{\otimes km}|}$. Therefore

$$V = Y_{km},\ \varphi_{km} = \psi$$

for all k sufficiently large. □

Theorem 1.4.5. *Let X be a smooth projective variety and L be a semiample line bundle on X. Then there is an algebraic fibre space*

$$\varphi : X \to Y$$

such that for all sufficiently large $m \in M(X, L)$,

$$Y_m = Y,\ \varphi_m = \varphi,$$

where φ_m, Y_m are, respectively, the morphism induced by $|L^{\otimes m}|$ and its image. Moreover there is an ample line bundle A on Y such that $\varphi^* A = L^{\otimes f}$, where f is the exponent of $M(X, L)$.

Proof. Up to change L with $L^{\otimes f}$ we may assume that the exponent is 1. This means that all sufficiently large powers of L are free.
Let $p, q \in \mathbb{N}$ relatively prime, large enough, integers such that for all $k \geq 1$,

$$Y_{kp} = Y_p,\ \varphi_{kp} = \varphi_p,\ Y_{kq} = Y_q,\ \varphi_{kq} = \varphi_q.$$

Such numbers exist by Lemma 1.4.4 and by the assumption of the exponent to be 1.
In particular we have that

$$Y_p = Y_{qp} = Y_q,\ \varphi_p = \varphi_{pq} = \varphi_q;$$

so they define the same fibre space which we will call

$$\varphi_q = \varphi_p = \varphi : X \to Y = Y_p = Y_q.$$

We know, by Lemma 1.4.4 that on Y are defined line bundles A_p and A_q such that

$$\varphi^* A_\ell = L^{\otimes \ell}, \ell \in \{p, q\}.$$

We can choose $r, s \in \mathbb{Z}$ such that $rp + sq = 1$, so, the pull-back through φ of

$$A = A_p^{\otimes r} \otimes A_q^{\otimes s}$$

is L.
Now, as φ is a fibred space, thus

$$\varphi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is injective by Example 1.2.17, hence $A_p = A^{\otimes p}$ and $A_q = A^{\otimes q}$ and A is ample.
We only need to show the independence of the choice of an element of $M(X, L)$.
Fix $c, d \geq 1$ integers. Then, $S^c H^0(Y, A^{\otimes p}) \otimes S^d H^0(Y, A^{\otimes q})$ determines a free linear subsystem of

$$H^0(Y, A^{\otimes cp+dq}) = H^0(X, L^{\otimes cp+dq})$$

and, by what we have shown in Example 1.4.3, φ can be factorized through φ_{cp+dq} followed by a finite mapping, thanks to this linear subsystem. Now, noticing that all sufficiently large integers can be written as $cp+dq$ for suitable $c, d \geq 1$ we can conclude. □

1.4.2 The Iitaka Fibration Theorem

Theorem 1.4.6. *Let X be a smooth projective variety and L a line bundle on X such that $\kappa(X, L) > 0$. Then for all sufficiently large $k \in \mathbb{N}(X, L)$, the rational mappings, induced by the linear systems $\left|L^{\otimes k}\right|$,*

$$\varphi_k : X \dashrightarrow Y_k,$$

are birationally equivalent to a fixed algebraic fibre space

$$\varphi_\infty : X_\infty \to Y_\infty$$

of normal varieties, and the restriction of L to a very general fibre of φ_∞ has Iitaka dimension 0. More precisely, there exists, for every sufficiently large $k \in \mathbb{N}(X, L)$, a commutative diagram:

$$\begin{array}{ccc} X_\infty & \xrightarrow{u_\infty} & X \\ {\scriptstyle \varphi_\infty}\downarrow & & \downarrow{\scriptstyle \varphi_k} \\ Y_\infty & \underset{\nu_k}{\dashrightarrow} & Y_k \end{array}$$

of rational maps and morphisms, where u_∞ *and* ν_k *are birational and* u_∞ *is a morphism and* Y_∞ *is such that* $\dim Y_\infty = \kappa(X, L)$*. Furthermore, if we set* $L_\infty = u_\infty^* L$*, and take* $F \subseteq X_\infty$ *a very general fibre of* φ_∞*, then*

$$\kappa(F, L_{\infty|_F}) = 0$$

and with "very general fibre" we mean that F *is the fibre of a point that belongs to the complement of countably many subavarieties of* Y_∞*.*

Proof. Fix $m \in \mathbb{N}(X, L)$ such that $\dim Y_m = \kappa(X, L)$: such an integer always exists by Lemma 1.2.12.
Let φ_ℓ be the rational map defined by the linear system $\left|L^{\otimes \ell}\right|$ for every $\ell \in \mathbb{N}$ and let Y_ℓ be its image.
The first thing we want to prove is that for all k sufficiently large, the rational map

$$\varphi_{km} : X \dashrightarrow Y_{km},$$

is birationally equivalent to a fixed algebraic fibre space

$$\psi_{(m)} : X_{(m)} \to Y_{(m)}$$

of normal varieties.
We consider a resolution of the indeterminacies of φ_m

$$u_m : X_{(m)} \to X$$

of X, by which I mean a birational morphism u_m such that

$$u_m^* \left|L^{\otimes m}\right| = |M_m| + F_m$$

where M_m is a globally generated line bundle on the normal variety $X_{(m)}$ and F_m is the fixed divisor of $|u_m^* L^{\otimes m}|$ and the map

$$\psi_m = \varphi_{|M_m|} : X_{(m)} \to Y_m \subseteq \mathbb{P}(H^0(X_{(m)}, M_m)) = \mathbb{P}(H^0(X, L^{\otimes m}))$$

is such that the following triangle is commutative:

$$\begin{array}{ccc} X_{(m)} & \xrightarrow{u_m} & X \\ & \searrow^{\psi_m} & \vdots \; \varphi_m \\ & & Y_m \end{array}$$

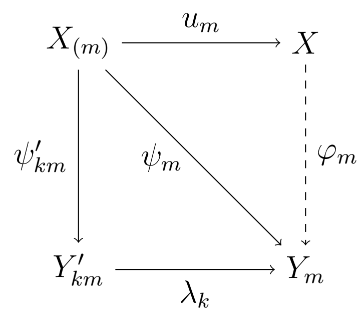
Now, consider the morphism induced by $\left|M_m^{\otimes k}\right|$:

$$\psi'_{km} : X_{(m)} \to Y'_{km}.$$

This map, being induced by a linear subsystem of $|u_m^* L^{\otimes m}|$, is such that there exists a finite map

$$\lambda_k : Y'_{km} \to Y_m$$

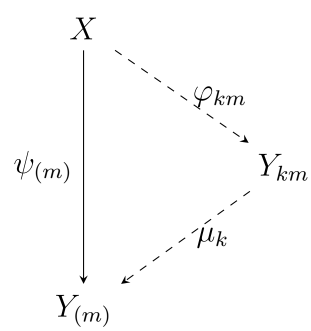
that makes the following diagram commutative (this is the same process as in Example 1.4.3):



Now, observe that M_m, being globally generated is trivially semiample, hence, by Theorem 1.4.5 for k sufficiently large, the morphisms ψ'_{km} stabilize to a fixed fibre space

$$\psi_{(m)} : X_{(m)} \to Y_{(m)}.$$

We know that $\left|M_m^{\otimes k}\right|$ is, for every (sufficiently large) k, a linear subsystem of $|u_m^* L^{\otimes m}|$, and therefore also of $|L^{\otimes m}|$. Moreover, $X_{(m)}$ is birational to X, hence, we can consider $\psi_{(m)}$ as a morphism from X and, therefore, get a factorization:



with μ_k generically finite. But, by Example 1.2.18, $\mathbb{C}(Y_{(m)})$ is algebraically closed in $\mathbb{C}(X)$ as $\psi_{(m)}$ is a fibre space, hence μ_k is birational, otherwise, ν_k would induce an algebraic extension of $\mathbb{C}(Y_{(m)})$. This implies that for every sufficiently large k, φ_{km} is birational to $\psi_{(m)}$. This proves the claim.
Now we claim that the fibre spaces

$$X_{(m)} \to Y_{(m)}$$

just constructed are all birationally equivalent and, to prove such thing we will construct a common model.
Up to change L with $L^{\otimes e}$ we can consider L to have exponent 1.
Now let $p, q \in \mathbb{N}$, large relatively prime such that

$$\dim Y_p = \dim Y_q = \kappa(X, L)$$

In the same notations used up till now, for m sufficiently large we have fibre spaces

$$\psi_{(p)} : X_{(p)} \to Y_{(p)}$$
$$\psi_{(q)} : X_{(q)} \to Y_{(q)}$$

and the morphisms are defined by the linear systems $\left|M_p^{\otimes p^{m-1}}\right|$ and $\left|M_q^{\otimes q^{m-1}}\right|$ respectively: even though in the previous discussion we did not choose these particular numbers, we proved that the choice is independent, in the sense of birational equivalence, of the multiplication times a sufficiently large element of $\mathbb{N}(X, L)$ and this is case for this choice of p, q.
Now, fix a normal variety X_∞ together with birational morphisms

$$v_p : X_\infty \to X_{(p)}$$
$$v_q : X_\infty \to X_{(q)}$$

such that $u_p \circ v_p = u_q \circ v_q$: such a variety always exists. Indeed, as u_p and u_q are birational morphisms, there are, for $\ell \in \{p, q\}$, open subsets $U_\ell \subseteq X$ such that the inverse of u_ℓ is well-defined on U_ℓ and is an isomorphism. Let

$$Z = X \setminus (U_p \cap U_q)$$

and we may consider

$$X_\infty = \mathrm{Bl}_Z(X)$$

the blow-up of X at Z. With this choice of X_∞ we have, canonically, birational morphisms

$$v_\ell : X \to X_{(\ell)},\ \ell \in \{p, q\}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} X_\infty & \xrightarrow{v_p} & X_{(p)} \\ {\scriptstyle v_q}\downarrow & \searrow^{u_\infty} & \downarrow{\scriptstyle u_p} \\ X_{(q)} & \xrightarrow[u_q]{} & X \end{array}$$

Now, let

$$M_{p,q} = v_p^* M_p^{\otimes p^{m-1}} \otimes v_q^* M_q^{\otimes q^{m-1}}.$$

This is a globally generated line bundle on X_∞, hence it induces a morphism

$$\varphi_{|M_{p,q}|} : X_\infty \to \mathbb{P}(H^0(X_\infty, M_{p,q})).$$

Denote with Y_∞ the normalization of the image of such morphism. Let

$$\varphi_\infty : X_\infty \to Y_\infty$$

be the induced map.
Let $\ell \in \{p, q\}$ be fixed. With an observation analogous to the one we did for u_m, as X_∞ is birational to X, we can consider φ_∞ to be a morphism from X and the same for $\psi_{(\ell)}$. Thus, we can find a finite morphism w_ℓ such that the following triangle is commutative:

$$\begin{array}{ccc} X & & \\ & \searrow^{\varphi_\infty} & \\ {\scriptstyle \psi_{(\ell)}}\downarrow & & Y_\infty \\ & \swarrow_{w_\ell} & \\ Y_{(\ell)} & & \end{array}$$

That implies that the morphisms w_p, w_q are such that the following diagram commutes:

$$\begin{CD}
X_{(p)} @<{v_p}<< X_\infty @>{v_q}>> X_{(q)} \\
@V{\psi_{(p)}}VV @V{\varphi_\infty}VV @VV{\psi_{(q)}}V \\
Y_{(p)} @<<{w_p}< Y_\infty @>>{w_q}> Y_{(q)}
\end{CD}$$

In particular, $\dim Y_\infty = \kappa(X, L)$ and therefore, w_p, w_q are generically finite. Since they factor the algebraic fibre spaces $\psi_{(p)} \circ v_p$ and $\psi_{(q)} \circ v_q$ respectively, they are birational. As, $\varphi_\infty \circ w_p$ is an algebraic fibre space and w_p is birational and generically finite, then φ_∞ is also a fibre space.
Since both $Y_{(p)}$ and $Y_{(q)}$ are projective, they are equipped with an ample line bundle. The pull-backs of these line bundles are then ample on Y_∞ and therefore their tensor product is as well. Let $A_{p,q} = A$ such line bundle. Then it pulls back to $M_{p,q}$ via φ_∞.
Now, fix positive integers c, d. Then

$$\begin{aligned}
H^0(X_\infty, M_{p,q}) &\subseteq H^0(X_\infty, u_p^* M_p^{\otimes(cp^{m-1})} \otimes u_q^* M_q^{\otimes(dq^{m-1})}) \\
&\subseteq H^0(X_\infty, u_\infty^* L^{\otimes(cp^m + dq^m)})
\end{aligned}$$

where the first inclusion comes from the multiplication by a fixed section of $u_p^* M_p^{\otimes((c-1)p^{m-1})} \otimes u_q^* M_q^{\otimes((d-1)q^{m-1})}$ and the second from the commutativity of the diagram defining X_∞.
But then we have a factorization that makes the following diagram commute:

$$\begin{array}{ccc}
X_\infty & \xrightarrow{\ u_\infty\ } & X \\
{\scriptstyle \varphi_\infty}\downarrow & & \vdots\,{\scriptstyle \varphi_{cp^m+dq^m}} \\
Y_\infty & \underset{\mu_{cp^m+dq^m}}{\dashleftarrow} & Y_{cp^m+dq^m}
\end{array}$$

furthermore, as $\varphi_\infty \circ u_\infty^{-1}$ (where it is defined) is a fibre space, $\mu_{cp^m+dq^m}$ is generically finite and birational. By taking, as ν_k, its inverse (as a birational map) we get the diagram in the statement.
The last thing to prove is the vanishing of the Iitaka dimension of the pull-back of L when restricted to a very general fibre.
Let $L_\infty = u_\infty^* L$. By, the above inclusions of cohomology groups we have that

for a very general fibre F of φ_∞

$$\kappa(F, L_{\infty|_F}) \geq 0.$$

So, let $y \in Y_\infty$ be a point and let F be its fibre. We can now assume that ν_k is defined and regular at y for every k sufficiently large and that $u_\infty(F)$ does not meet the indeterminacy locus of any φ_k. These are a countable number of conditions on y that lead us to the very general hypothesis of the statement of the Theorem.
In particular, the situation is described in the following diagram:

$$\begin{array}{ccc} X_\infty & \xrightarrow{u_\infty} & X \\ \varphi_\infty \downarrow & & \downarrow \varphi_k \\ Y_\infty & \overset{\nu_k}{\dashrightarrow} & Y_k \end{array}$$

and the, by the conditions on y, all maps, when restricted to F or $F' = u_\infty(F)$ are well-defined.
Under these assumptions, the image under φ_k of F' is again a point, but then, the restriction map

$$\rho_k : H^0(X_\infty, L_\infty^{\otimes k}) \to H^0(F, L_{\infty|_F}^{\otimes k})$$

has rank 1 for every k sufficiently large. To prove this we just have to notice that in this situation, we have that the restriction

$$\varphi_k : F' \to Y_k$$

is a constant map, but this map is defined through the global sections of $L^{\otimes k}$: let $s_0, \ldots, s_{n_k}$ be a basis for the global sections of $L^{\otimes k}$, thus φ_k in some homogeneous coordinates is represented by

$$\underline{x} \longmapsto [s_0(\underline{x}), \ldots, s_{n_k}(\underline{x})].$$

The restriction to F' is constant, so for every $\underline{x}, \underline{\xi} \in F'$, there exists $\lambda \in \mathbb{C}^\times$ such that for every $0 \leq i \leq n_k$

$$s_i(\underline{x}) = \lambda s_i(\underline{\xi}).$$

Thus, for a fixed k, the restriction of φ_k induces a non-zero map

$$F' \to \mathbb{C},$$

hence the restriction map has rank 1.
Now let B be a very ample line bundle on Y_∞. Then there exists a large positive integer m_0 such that

$$H^0(X_\infty, L_\infty^{\otimes m_0} \otimes \varphi_\infty^* B^\vee) \neq 0.$$

Indeed, $A^{\otimes m_1} \otimes B^\vee$ has a non-zero section for m_1 large enough. On the other hand $M_{p,q}^{\otimes m_1}$ is canonically a subsheaf of $L_\infty^{\otimes (p^m+q^m)m_1}$, hence

$$L_\infty^{\otimes m_1} \otimes \varphi_\infty^* B^\vee \supseteq \varphi_\infty^*(A^{\otimes m_1} \otimes B^\vee);$$

thus the non-vanishing.
Now, we have that

$$L_\infty^{\otimes m_1} \otimes \varphi_\infty^* B^\vee \supsetneq \mathcal{O}_{X_\infty},$$

thus,

$$\varphi_\infty^* B \subsetneq L_\infty^{\otimes m_0}.$$

Therefore, for $r > 0$ and fixed k, we have the following commutative diagram

$$\begin{array}{ccc} H^0(X_\infty, L_\infty^{\otimes k} \otimes \varphi_\infty^* B^{\otimes r}) & \longrightarrow & H^0(X_\infty, L_\infty^{\otimes (k+rm_0)}) \\ \Big\downarrow \beta_{k+rm_0} & & \Big\downarrow \rho_{k+rm_0} \\ H^0(F, (L_\infty^{\otimes k} \otimes \varphi_\infty^* B^{\otimes r})_F) & \longrightarrow & H^0(X_\infty, (L_\infty^{\otimes (k+rm_0)})_F) \end{array}$$

where the horizontal maps are inclusions and the vertical maps are induced by the restrictions.
Observe that β_{k+rm_0} can be identified with the map

$$\alpha_{k+rm_0} : H^0(Y_\infty, \varphi_{\infty *} L_\infty^{\otimes k} \otimes B^{\otimes r}) \to H^0(Y_\infty, \varphi_{\infty *} L_\infty^{\otimes k} \otimes B^{\otimes r}) \otimes \mathbb{C}(y),$$

obtained by evaluating the sections of $\varphi_{\infty *} L_\infty^{\otimes k} \otimes B^{\otimes r}$ at y; here with $\mathbb{C}(y)$ is the residue field of Y_∞ at y.
By Theorem 1.3.2 for r large enough $\varphi_{\infty *} L_\infty^{\otimes k} \otimes B^{\otimes r}$ is globally generated.

Therefore α_{k+rm_0} and also β_{k+rm_0} are surjective, but, by the commutativity of the above diagram, the rank of β_{k+rm_0} is also 1. This implies, that

$$\dim H^0(F, (L_\infty^{\otimes k} \otimes \varphi_\infty^* B^{\otimes r})_F) = 1$$

but, the restriction of $\varphi_\infty^* B^{\otimes r}$ to F is trivial by Corollary 1.3.4, then

$$h^0(F, L_{\infty|_F}^{\otimes k}) = 1$$

for every k. This implies that the Iitaka dimension of L restricted to F is zero: indeed, the canonical maps through which we define the Iitaka dimension have, as target space, the projective space associated to the $H^0(F, L_{\infty|_F}^{\otimes r})$, but these spaces are all 1-dimensional, thus their projective spaces are just points. □

Definition 1.4.7. In the setting of Theorem 1.4.6, $\varphi_\infty : X_\infty \to Y_\infty$ is the *Iitaka fibration* associated to L. We define the Iitaka fibration associated to a divisor D as the fibration associated to the line bundle associated to D. The Iitaka fibration of an irreducible variety X is the Iitaka fibration associated to the canonical bundle of a smooth model of X.

We conclude this section with a couple of Remarks.
The first two steps of the proof of Theorem 1.4.6 we presented are, nowadays, dated; indeed, there exist a proof which relies on the finite generation of the canonical ring, which is

$$R(X) = \bigoplus_{n \in \mathbb{N}} H^0(X, \omega_X^{\otimes n}).$$

This result has both an algebraic proof given by Birkar-Cascini-Hacon-McKernan in [3] and an analytic proof given by Siu in [28].
A natural question to ask is whether there exist an effective bound for the stabilization of the Iitaka fibration. For general type varieties there is an explicit bound depending only on the dimension found by Hacon-McKernan in [11]. More generally, the dependance requires more than the dimension, examples of this kind of results have been found by Mori-Fujino [8] and Pacienza [23].

1.5 Positivity Notions for Vector Bundles

In this section we will present some definitions and results about the positivity of vector bundles of rank larger than 1. A necessary tool to understand and even define such notions is the construction of the projective bundle associated to a vector bundle E over a variety X. The theory regarding such construction is developed in the 7-th section of Chapter II of [12]. The main reference for this section is [18].

Definition 1.5.1. A vector bundle E on a projective variety X is *ample* (respectively *nef*) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (respectively nef) on the projective bundle $\mathbb{P}(E)$.

Notice that if E is a line bundle, then $\mathbb{P}(E)$ is canonically isomorphic to X, and the canonical bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is isomorphic to E itself. This shows that the above definition actually generalizes the rank 1 case.

Proposition 1.5.2. *Let E be a vector bundle on a projective variety X. Then:*

1. *If E is ample (or nef) every quotient bundle of Q of E is as well.*
2. *Let $f : X \to Y$ be a finite mapping. If E is ample (or nef), then the pull-back f^*E is also ample (or nef).*

Proof. A surjective map $E \to Q$ induces an inclusion $\mathbb{P}(Q) \subseteq \mathbb{P}(E)$ such that

$$\mathcal{O}_{\mathbb{P}(Q)}(1) = \mathcal{O}_{\mathbb{P}(E)}(1)_{|Q}$$

Now, the restriction of an ample (or nef) line bundle is again ample (or nef), we conclude the ampleness (or nefness) of $\mathcal{O}_{\mathbb{P}(Q)}(1)$ and therefore the same property for Q.
For the second statement, just notice that f gives rise to a finite map

$$F : \mathbb{P}(f^*E) \to \mathbb{P}(E)$$

such that

$$\mathcal{O}_{\mathbb{P}(f^*E)}(1) = F^*\mathcal{O}_{\mathbb{P}(E)}(1)$$

and by the corresponding results for line bundles we can conclude. □

Proposition 1.5.3. *Let E be a vector bundle on a projective variety X. Then:*

i *E is ample (or nef) if and only if* E_{red} *is ample (or nef) on* X_{red}.

ii *E is ample (or nef) if and only if its restriction to any irreducible component of X is.*

iii *If* $f : X \to Y$ *is a finite surjective map and* f^*E *is ample, then E is ample too.*

iv *If* $f : X \to Y$ *is a surjective map and* f^*E *is nef, then E is nef too.*

Proof. It just is an application of the corresponding results for line bundles. □

Theorem 1.5.4. *Let E be a vector bundle on the projective variety X. The following are equivalent:*

i *E is ample;*

ii *Given any coherent sheaf* $\mathcal{F}$ *on X, there is a positive integer* $m_1 = m_1(\mathcal{F})$ *such that*

$$H^i(X, S^m E \otimes \mathcal{F}) = 0, \forall\, i > 0,\ m \geq m_1$$

iii *Given any coherent sheaf* $\mathcal{F}$ *on X, there is an integer* $m_2 = m_2(\mathcal{F})$ *such that* $S^m E \otimes \mathcal{F}$ *is globally generated for every* $m \geq m_2$.

iv *For any ample divisor H on X there is a positive integer* $m_3 = m_3(H)$ *such that* $S^m E$ *is a quotient of copies of* $\mathcal{O}_X(H)$ *for every* $m \geq m_3$.

v *For some ample divisor H on X there is a positive integer* $m_3 = m_3(H)$ *such that* $S^m E$ *is a quotient of copies of* $\mathcal{O}_X(H)$ *for every* $m \geq m_3$.

Proof. Omitted. See [18] Theorem 6.1.10. □

Proposition 1.5.5. *Let* E_1, E_2 *be vector bundles over X. Then:*

i *The direct sum* $E_1 \oplus E_2$ *is ample if and only both summands are ample.*

ii *If F is an extension of* E_2 *by* E_1*, and if both* E_1, E_2 *are ample, also F is.*

Proof. Omitted. See [18] Proposition 6.1.13. □

Proposition 1.5.6. *A vector bundle E over X is ample if and only if $S^k E$ is ample for some k. This is also equivalent to ask that $S^k E$ is ample for every k.*

Proof. Omitted. See [18] Theorem 6.1.15. □

Proposition 1.5.7. *If E, F are ample vector bundles over X, so is $E \otimes F$. In particular tensor powers of ample line bundles are ample.*

Proof. Omitted. See [18] Corollary 6.1.16. □

Theorem 1.5.8. *Let X be a projective variety.*

i Quotients and arbitrary pull-backs of nef vector bundles on X are nef. Given a vector bundle E on X and a surjective morphism $f : Y \to X$ of projective varieties, if f^*E is nef, so is E.

ii Direct sums and extensions of nef vector bundles are nef.

iii A vector bundle E on X is nef if and only if $S^k E$ is nef for some (or equivalently all) $k \geq 1$.

iv Tensor products and exterior products of nef bundles are nef. If E is nef and F is ample, then $E \otimes F$ is ample.

v If E is a vector bundle and B an ample divisor on X such that $S^m E \otimes \mathcal{O}_X(B)$ is nef for every m sufficiently large, than E itself is nef.

Proof. Omitted. See [18] Theorem 6.2.12 and Example 6.2.13. □

Chapter 2

Positivity Results for Direct Image Vector Bundles

Now we will state and prove under some assumptions some classical results about the positivity of direct image vector bundles. These results will be crucial in the proof of $C_{n,1}$.

2.1 Castelnuovo-Mumford Regularity

In this section we will study Mumford's theorems on regularity.

2.1.1 Koszul Complex

At first we will introduce an important object associated to a vector bundle E on a smooth projective variety X. The main references are, as usual, [12] and [19].
Let A be a ring and let $f_1, \dots, f_n \in A$ be fixed. The *Koszul complex* associated to a free A-module K_1 of rank n is defined as follows: in degree 0 we have A; in degree 1 we have K_1; for $m > 1$ in degree m there is $K_m = \bigwedge^m K_1$.
Fixed a basis $e_1, \dots, e_n$ of K_1, we have that for every $m \geq 1$, K_m has as a basis the elements

$$e_{i_1} \wedge \dots \wedge e_{i_m}, 1 \leq i_j \leq n$$

for every j and they are all distinct. In order to define the boundary maps of the complex it suffices to define those on a basis and then they can be extended by A-linearity. Put

$$d : K_m \to K_{m-1}$$

$$e_{i_1} \wedge \cdots \wedge e_{i_m} \longmapsto \sum_{j=1}^{m} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_m}.$$

Lemma 2.1.1. *The Koszul complex defined above is an actual complex.*

Proof. In the same notations as above, consider for K_1 and a fixed $e \in K_1$, the following sequence of free A-modules

$$0 \longrightarrow A \xrightarrow{\ell_e} K_1 \xrightarrow{\ell_e} K_2 \xrightarrow{\ell_e} \cdots \xrightarrow{\ell_e} K_n \xrightarrow{\ell_e} 0$$

where ℓ_e is the homomorphism

$$\ell_e : K_m \to K_{m+1}$$

$$x \longmapsto e \wedge x.$$

As for every $0 \leq m \leq n$ and every $x \in K_m$ we have that

$$\ell_e \circ \ell_e(x) = e \wedge (e \wedge x) = 0,$$

thus this sequence is a co-complex. On the other hand we can consider its dual which is a complex, namely

$$0 \longrightarrow K_n^\vee \xrightarrow{\ell_e^\vee} \cdots \xrightarrow{\ell_e^\vee} K_2^\vee \xrightarrow{\ell_e^\vee} K_1^\vee \xrightarrow{\ell_e^\vee} A \longrightarrow 0$$

Now, with the choice of $e = \sum_{i=1}^{n} f_i e_i$, we can choose isomorphisms

$$\varphi_m : K_m \to K_m^\vee$$

for every $0 \leq m \leq n$ in order such that the for all m's the following square commutes

$$\begin{array}{ccc} K_m & \xrightarrow{d} & K_{m-1} \\ \downarrow{\scriptstyle \varphi_m} & & \downarrow{\scriptstyle \varphi_{m-1}} \\ K_m^\vee & \xrightarrow[\ell_e^\vee]{} & K_{m-1}^\vee \end{array}$$

therefore, what we called the "Koszul complex" is isomorphic to a complex, hence is itself a complex. □

We denote the Kozsul complex as $K_\bullet(f_1, \dots, f_n; K_1)$.
Notice that the choice of n elements of A is equivalent to the choice of an A-module homomorphism

$$\varphi : K_1 \to A;$$

so we may also use the notation $K_\bullet(\varphi; K_1)$.
Once we defined what we mean by the Kozsul complex of a free A-module with respect to a fixed homomorphism of A-modules, we can easily generalize the situation to a locally free $\mathcal{O}_X$-module on a ringed space $(X, \mathcal{O}_X)$.
We conclude with a result without proof and a remark that will be useful in the following discussion.

Proposition 2.1.2. *Let X, E as above, and let $\mathbb{P}(E)$ be associated the projective bundle. Let $g : Y \to X$ be any morphism. Then to give a morphism of Y to $\mathbb{P}(E)$ over X is equivalent to give a surjective morphism $g^*E \to L$, where is a line bundle.*

Proof. Omitted. See [12] Chapter II, Proposition 7.12. □

Remark 2.1.3. By Proposition 2.1.2, to give a section of X over $\mathbb{P}(E)$ is equivalent to find a surjection onto a line bundle L from E.

2.1.2 Mumford Regularity Theorems

Let V be a complex vector space of dimension $r + 1$ and let $\mathbb{P} = \mathbb{P}(V)$ its associated r-dimensional projective space.

Definition 2.1.4. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}$ and let m be an integer. $\mathcal{F}$ is *m-regular in the sense of Castelnuovo-Mumford* if for every $i > 0$:

$$H^i(\mathbb{P}, \mathcal{F}(m - i)) = 0.$$

Theorem 2.1.5. *In the notations above, let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}$. Then, for every $k \in \mathbb{N}$:*

i $\mathcal{F}(m + k)$ is globally generated.

ii The natural maps

$$H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}(k)) \to H^0(\mathbb{P}, \mathcal{F}(m+k))$$

are surjective.

iii $\mathcal{F}$ is $(m+k)$-regular.

Proof. For $k = 0$, the second and the third statement are trivial.
At first we will prove the second statement and the third statement for $k = 1$, then, by induction, it will hold for every other k.
In the notations above consider V and the projective bundle $\mathbb{P} = \mathbb{P}(V)$. By the definition of the Koszul complex, to define one we only need a section

$$\sigma : \mathbb{P} \to \mathbb{P}$$

and by Remark 2.1.3, it suffice to find a surjection onto a line bundle

$$V_{\mathbb{P}} \to L$$

where $V_{\mathbb{P}} = \underline{V} \otimes \mathcal{O}_{\mathbb{P}}$ and $\underline{V}$ is the sheaf of locally constant V-valued functions on $\mathbb{P}(V)$. Canonically, we have a surjection

$$V_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1),$$

and we twist it via $\mathcal{O}_{\mathbb{P}}(-1)$, obtaining a surjection (the tensor product is right-exact):

$$V_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}.$$

Then the Koszul complex is

$$0 \longrightarrow (\bigwedge^{r+1} V_{\mathbb{P}})(-r-1) \longrightarrow \cdots \longrightarrow V_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

where we used the fact that if $k \in \mathbb{N}$, E a vector bundle and F line bundle, then it holds that

$$\bigwedge^k (E \otimes F) = \bigwedge^k E \otimes F^{\otimes k},$$

to write

$$\bigwedge^k (V_{\mathbb{P}}(-1)) = \bigwedge^k (V_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-1)) = \bigwedge^k (V_{\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(-k) = (\bigwedge^k V_{\mathbb{P}})(-k)$$

for every $1 \leq k \leq r+1$.
Twisting by $\mathcal{F}(m+1)$ yields

$$\cdots \longrightarrow \bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-1) \longrightarrow V_{\mathbb{P}} \otimes \mathcal{F}(m) \longrightarrow \mathcal{F}(m+1) \longrightarrow 0$$

Now, notice that there exists a canonical isomorphism

$$\underline{V} \cong (\underline{\mathbb{C}})^{r+1}$$

thus

$$V_{\mathbb{P}} \cong (\underline{\mathbb{C}})^{r+1} \otimes \mathcal{O}_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}^{r+1}$$

and this shows that $V_{\mathbb{P}}$ is free. Therefore also its exterior powers are free. Let r_i be the rank of $\bigwedge^{i+1} V_{\mathbb{P}}$ then

$$\bigwedge^{i+1} V_{\mathbb{P}} \otimes \mathcal{F}(m-i) = \mathcal{O}_{\mathbb{P}}^{r_i} \otimes \mathcal{F}(m-i) = \mathcal{F}(m-i)^{\oplus r_i}.$$

This implies that

$$H^i(\mathbb{P}, \bigwedge^{i+1} V_{\mathbb{P}} \otimes \mathcal{F}(m-i)) = H^i(\mathbb{P}, \mathcal{F}(m-i)^{\oplus r_i}) = H^i(\mathbb{P}, \mathcal{F}(m-i))^{\oplus r_i} = 0.$$

By taking the cohomology of the exact sequence

$$\bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-1) \longrightarrow V_{\mathbb{P}} \otimes \mathcal{F}(m) \longrightarrow \mathcal{F}(m+1) \longrightarrow 0$$

one finds that the map

$$H^0(\mathbb{P}, V_{\mathbb{P}} \otimes \mathcal{F}(m)) \to H^0(\mathbb{P}, \mathcal{F}(m+1))$$

is surjective. Moreover

$$H^0(\mathbb{P}, V_{\mathbb{P}} \otimes \mathcal{F}(m)) = H^0(\mathbb{P}, \mathcal{F}(m))^{\oplus(r+1)} = H^0(\mathbb{P}, \mathcal{F}(m)) \otimes \mathbb{C}^{r+1} =$$
$$= H^0(\mathbb{P}, \mathcal{F}(m)) \otimes V = H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)):$$

the second statement for $k = 1$ is proved.
In order to prove the third statement, we need to show that

$$H^i(\mathcal{F}(m-(i-1))) = 0.$$

Fix $i > 0$, consider the Koszul complex associated to the surjection

$$V_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}$$

and twist it by $\mathcal{F}(m+1-i)$, obtaining

$$\cdots \to \bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-i-1) \to V_{\mathbb{P}} \otimes \mathcal{F}(m-i) \to \mathcal{F}(m+1-i) \to \; 0$$

Observe that

$$H^i(\mathbb{P}, V_{\mathbb{P}} \otimes \mathcal{F}(m-i)) = H^i(\mathbb{P}, \mathcal{F}(m-i))^{\oplus(r+1)} = 0$$

and that

$$H^{i+1}(\mathbb{P}, \bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-i-1)) = H^{i+1}(\mathbb{P}, \mathcal{F}(m-i-1))^{\oplus r_2} = 0.$$

Thus, by taking cohomology of the exact sequence

$$\bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-i-1) \longrightarrow V_{\mathbb{P}} \otimes \mathcal{F}(m-i) \longrightarrow \mathcal{F}(m+1-i) \longrightarrow \; 0$$

in degree i:

$$0 = H^i(\mathbb{P}, V_{\mathbb{P}} \otimes \mathcal{F}(m-i)) \to H^i(\mathbb{P}, \mathcal{F}(m+1-i)) \to H^{i+1}(\mathbb{P}, \bigwedge^2 V_{\mathbb{P}} \otimes \mathcal{F}(m-i-1)) = 0$$

hence, $H^i(\mathbb{P}, \mathcal{F}(m+1-i)) = 0$. The third statement is proven for $k = 1$. Now, let $k > 1$ such that the second and the third statement are true for $k-1$. Then, $\mathcal{F}$ is $m+k-1$-regular, and by the above proof is also $(m+k)$-regular and, therefore, the map

$$H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \to H^0(\mathbb{P}, \mathcal{F}(m+k))$$

is surjective. By the induction principle, the two statements are true for every $k \in \mathbb{N}$.
Now, as

$$H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \to H^0(\mathbb{P}, \mathcal{F}(m+k))$$

for all k's, we have an induced morphism

$$\underline{H^0(\mathbb{P}, \mathcal{F}(m))} \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{F}(m)$$

where $\underline{H^0(\mathbb{P}, \mathcal{F}(m))}$ is the locally constant sheaf associated to $H^0(\mathbb{P}, \mathcal{F}(m))$. Before going on with the proof, it is necessary to make an observation: given an exact sequence of sheaves

$$\mathcal{G}_1 \longrightarrow \cdots \longrightarrow \mathcal{G}_n$$

on $\mathbb{P}$, there exists an $\ell_0 \in \mathbb{N}$ such that for every $\ell \geq \ell_0$ the sequence of sections

$$\mathcal{G}_1(\ell) \longrightarrow \cdots \longrightarrow \mathcal{G}_n(\ell)$$

is again exact. To prove this claim, consider, for every $1 \leq i \leq n$ the sheaf

$$\mathcal{H}_i = \ker(\mathcal{G}_i \to \mathcal{G}_{i+1}) = \operatorname{im}(\mathcal{G}_{i-1} \to \mathcal{G}_i).$$

Then there are short exact sequences

$$0 \longrightarrow \mathcal{H}_i \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{H}_{i+1} \longrightarrow 0$$

As for every $\ell \in \mathbb{N}$, the sheaf $\mathcal{O}_{\mathbb{P}}(\ell)$ is locally free it also is flat, hence the following sequence is exact

$$0 \longrightarrow \mathcal{H}_i(\ell) \longrightarrow \mathcal{G}_i(\ell) \longrightarrow \mathcal{H}_{i+1}(\ell) \longrightarrow 0$$

By Serre's Vanishing (Theorem 1.3.2) there exists an $\ell_i > 0$ such that $H^i(\mathbb{P}, \mathcal{H}_i(\ell)) = 0$ for every $\ell \geq \ell_i$. This implies that

$$0 \longrightarrow H^0(\mathbb{P}, \mathcal{H}_i(\ell)) \longrightarrow H^0(\mathbb{P}, \mathcal{G}_i(\ell)) \longrightarrow H^0(\mathbb{P}, \mathcal{H}_{i+1}(\ell)) \longrightarrow 0$$

is exact. Now, let ℓ_0 to be the maximum of the the ℓ_i. Then, for every $\ell \geq \ell_0$ the last sequence is exact for every $1 \leq i \leq n$, therefore so is

$$\mathcal{G}_1(\ell) \longrightarrow \cdots \longrightarrow \mathcal{G}_n(\ell)$$

for such ℓ's.
Going back to the main statement, let

$$\mathcal{K} = \ker(\underline{H^0(\mathbb{P}, \mathcal{F}(m))} \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{F}(m))$$

and let $\mathcal{C}$ be the cokernel of such map. Then the following is exact

$$0 \longrightarrow \mathcal{K} \longrightarrow \underline{H^0(\mathbb{P}, \mathcal{F}(m))} \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{C} \longrightarrow 0$$

therefore there exists an $\ell_0 > 0$ such that for every $\ell \geq \ell_0$ the twisted sequence is exact as well. But, on the other hand, by Serre's Vanishing Theorem, exists $\ell_1 > 0$ such that for $\ell \geq \ell_1$, $\mathcal{F}(m+\ell)$ is globally generated. Thus, for ℓ greater both than ℓ_0 and ℓ_1, the twisted sequence is exact and $\mathcal{C}(\ell) = 0$. Twisting

$$0 \longrightarrow \mathcal{C}(\ell) \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}}(-\ell)$ implies that $\mathcal{C}$ is itself 0, thus

$$\underline{H^0(\mathbb{P}, \mathcal{F}(m))} \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{F}(m)$$

is surjective. This proves that the first statement is true for $k = 0$. Now, if $k \geq 1$, $\mathcal{F}$ is $(m+k)$-regular, thus, $\mathcal{F}(m+k)$ is globally generated. □

Remark 2.1.6. By taking a closer look to the proof of this last Theorem, one may notice that the crucial hypotheses are the fact that $\mathbb{P}$ (in the notations of the proof) is a projective variety and the fact that $\mathcal{O}_{\mathbb{P}}(1)$ is an ample line bundle. Thus we give the following definition and a result which is presented without proof. For more details, see [19], Proposition 1.8.5.

Definition 2.1.7. Let X be a projective variety and B an ample line bundle over X, that is generated by its global sections. A coherent sheaf $\mathcal{F}$ on X is *m-regular with respect to B* if

$$H^i(\mathbb{P}, \mathcal{F} \otimes B^{\otimes(m-i)}) = 0$$

for $i > 0$.

Theorem 2.1.8. *In the notations of Definition 2.1.7, for every $k \geq 0$:*

i $\mathcal{F} \otimes B^{\otimes(m+k)}$ is globally generated.

ii The natural maps

$$H^0(X, \mathcal{F} \otimes B^{\otimes m}) \otimes H^0(X, B^{\otimes k}) \to H^0(X, \mathcal{F} \otimes B^{\otimes(m+k)})$$

are surjective.

iii $\mathcal{F}$ is $(m+k)$-regular with respect to B.

2.2 Kollár Vanishing Theorem

In this section we will present, without proofs, some results about the vanishing of the cohomology groups of positive degree of determined sheaves on smooth projective varieties. The main reference is [19].

Theorem 2.2.1. *Let X be a smooth irreducible projective complex variety and A an ample divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + A)) = 0, \ \forall i > 0.$$

Proof. Omitted. See [19] Theorem 4.2.1 □

This result have been later generalized by Kawamata and Viehweg obtaining another vanishing result. In order to state such result, we need the following:

Definition 2.2.2. Let X be a smooth projective variety over $\mathbb{C}$, then a line bundle L is said to be *big* if

$$\kappa(X, L) = \dim X$$

A divisor D is called big if $\mathcal{O}_X(D)$ is.

Theorem 2.2.3. *Let X be a smooth projective variety over $\mathbb{C}$ and let D be a nef and big divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Proof. Omitted. See [19] Theorem 4.3.1. □

We now present, without proof, two results due to Kollár. For the proofs, one can see [16].

Theorem 2.2.4. *Let X be a smooth projective complex variety and L a semiample divisor on X. Given $k \geq 1$, fix any divisor $D \in |kL|$. Then the homomorphisms*

$$H^i(X, \mathcal{O}_X(K_X + mL)) \to H^i(X, \mathcal{O}_X(K_X + (m + k)L))$$

naturally defined via D are injective for all $i \geq 0$ and $m > 0$.

Theorem 2.2.5. *Let $f : X \to Y$ be a surjective morphism of projective varieties with X smooth. Then:*

i $R^j f_* \mathcal{O}_X(K_X)$ *is torsion-free for all* j.

ii $R^j f_* \mathcal{O}_X(K_X) = 0$ *for* $j > \dim X - \dim Y$.

iii For any ample divisor A on Y and every $j \in \mathbb{N}$,

$$H^i(Y, R^j f_* \mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(A)) = 0, \ i > 0.$$

2.3 Fujita Theorem

In this section is presented a very important result, due to Fujita, about the positivity of the direct image of the relative canonical bundle of a fibration over a curve. The Theorem first appeared in [9] and is, there, proved through analytic means. Due to the very algebraic nature of this discussion we chose not to present such proof, but instead we present the Theorem as stated by Fujita and prove an analogous result as presented by Lazarsfeld in [18] and, in the end, spend some words on some generalizations. The main references are [18] and [30].

Definition 2.3.1. Given $f : X \to Y$ an algebraic fibre space, the *relative canonical bundle* of f is defined as

$$\omega_{X/Y} = \omega_X \otimes f^* \omega_Y^\vee$$

Theorem 2.3.2. *Let X be a smooth projective variety of dimension n and suppose $f : X \to C$ is a surjective morphism with connected fibres to a smooth projective curve. Then $f_* \omega_{X/C}$ is a nef vector bundle on C.*

Proof. Omitted. See [9]. □

In the case of a smooth fibration, the proof is much easier and can be generalized to the case of a target of arbitrary dimension. Before stating the result and proving it, we need an auxiliary result.

Theorem 2.3.3. *Let $f : X \to Y$ a morphism of smooth projective varieties, $\mathcal{F}$ a coherent sheaf over X, Y' another smooth projective variety and $u : Y' \to Y$ a smooth morphism. Moreover let $X' = X \times_Y Y'$ and v the induced morphism such that the following diagram is commutative*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

Then,

$$u^* R^i f_* \mathcal{F} \cong R^i g_* v^* \mathcal{F},$$

canonically, for every $i > 0$. Furthermore if $y \in Y$, X_y is the fibre through f and $\mathcal{F}_y$ the induced sheaf on the fibre and $\underline{\mathbb{C}(y)}$ be constant sheaf at y, for every $i \in \mathbb{N}$

$$H^i(X_y, \mathcal{F}_y) \cong H^i(X, \mathcal{F} \otimes \underline{\mathbb{C}(y)})$$

and the isomorphism is canonical.

Proof. Omitted. See [12] Chapter III, Proposition 9.3 and Corollary 9.4. □

Proposition 2.3.4. *Let $f : X \to Y$ be a smooth surjective morphism between smooth projective varieties. Then $f_* \omega_{X/Y}$ is nef.*

Proof. Fix $s > 0$ and consider the fibre product

$$f^{(s)} : X^{(s)} = \underbrace{X \times_Y \cdots \times_Y X}_{s \text{ times}} \to Y.$$

By the smoothness of f also $X^{(s)}$ is smooth. Moreover, by the Base Change Formula (Theorem 2.3.3) it holds that

$$f^{(s)}_* \omega_{X^{(s)}/Y} = (f_* \omega_{X/Y})^{\otimes s}.$$

If $\dim Y = d$ and B is a very ample line on Y such that $B \otimes \omega_Y^\vee$ is ample. Consider

$$H^i(Y, (f_* \omega_{X/Y})^{\otimes s} \otimes B^{\otimes (d+1-i)}) = H^i(Y, f^{(s)}_* \omega_{X^{(s)}/Y} \otimes B^{\otimes (d+1-i)}).$$

Now, by

$$f^{(s)}_* \omega_{X^{(s)}/Y} = f^{(s)}_* (\omega_{X^{(s)}} \otimes f^{(s)*} \omega_Y^\vee)$$

and the Projection Formula, then

$$H^i(Y, f^{(s)}_* \omega_{X^{(s)}/Y} \otimes B^{\otimes (d+1-i)}) = H^i(Y, f^{(s)}_* \omega_{X^{(s)}} \otimes \omega_Y^\vee \otimes B^{\otimes (d+1-i)})$$

Lastly by Kollár Vanishing Theorem (Theorem 2.2.5), this last cohomology group is zero. But then $f_*^{(s)}\omega_{X^{(s)}/Y}$ is $(d+1)$-regular with respect to B, hence

$$f_*^{(s)}\omega_{X^{(s)}/Y} \otimes B^{\otimes(d+1)} = (f_*\omega X/Y)^{\otimes s} \otimes B^{\otimes(d+1)}$$

is globally generated by the Castelnuovo-Mumford Regularity Theorem (Theorem 2.1.8).
This holds for every $s > 0$. In particular, for every $s > 0$ also

$$S^s f_*\omega_{X/Y} \otimes B^{\otimes(d+1)}$$

is globally generated, thus, by Theorem 1.5.8, $f_*\omega_{X/Y}$ is nef. □

When considering the most general case, which is of an arbitrary morphism onto a variety of arbitrary dimension, Fujita's Theorem yields a weaker result: the direct image of the relative canonical bundle has still some positivity properties, but, in full generality, cannot be nef. In order to give the precise statement we need to give some definitions.

Definition 2.3.5. A *quasi-projective variety* is an open subvariety of a projective variety.

Definition 2.3.6. Fixed a reduced quasi-projective variety Y, $Y_0 \subseteq Y$ an open dense subvariety of Y and $\mathcal{G}$ a locally free sheaf over Y of finite constant rank. $\mathcal{G}$ is said to be *weakly positive* over Y_0 if for every ample invertible sheaf $\mathcal{H}$ on Y and every $\alpha > 0$ there exists a $\beta > 0$ such that

$$S^{\alpha\beta}\mathcal{G} \otimes \mathcal{H}^{\beta}$$

is globally generated over Y_0.

Theorem 2.3.7. *Let $f : X \to Y$ be a surjective morphism of smooth projective complex varieties and $Y_0 \subseteq Y$ be the largest open subvariety such that*

$$f_{|_{f^{-1}(Y_0)}} : f^{-1}(Y_0) \to Y_0$$

is smooth. Then, $f_*\omega_{X/Y}$ is weakly positive over Y_0

Proof. Omitted. See [30] Theorem 2.41 and following results. □

Remark 2.3.8. The proof we cited is completely algebraic.

2.4 Mourougane Theorem

Positivity results like Fujita Theorem are key results in the study of the known cases of the $C_{n,m}$ conjecture. We will later prove how Fujita Theorem directly implies $C_{n,1}$ and more recently, in an expository paper by Hacon, Popa and Schnell study the positivity properties of

$$f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$$

where L is a nef and relatively big line bundle, h a singular hermitian metric with semi-positive curvature on L and $\mathcal{I}(h)$ is the multiplier ideal sheaf of h (see Definition 3.4.10), in order to deduce the $C_{n,m}$ in the case of a base space being an abelian variety. We will not go into much detail about this proof, but the everything can be found either in the paper by Hacon-Popa-Schnell [10] or in the original article by Cao and Păun [5].
Here we present with its proof an intermidiate result of Mourougane. The main reference is [21]. Before that we will give some definitions and proofless results that will be useful throughout the proof.

Definition 2.4.1. Given a fibre space $f : X \to Y$ an algebraic fibre space, a vector bundle E on X is *relatively big* or *f-big* is its restriction to a general fibre of f is a big vector bundle on the fibre.

Theorem 2.4.2. *Let $f : X \to Y$ be smooth morphism of smooth projective varieties, $\mathcal{F}$ a coherent sheaf over X which is flat over Y. If furthermore there exists $i \in \mathbb{N}$ such that*

$$y \longmapsto h^i(y, \mathcal{F}) = \dim_{\mathbb{C}(y)} H^i(X_y, \mathcal{F}_y)$$

is constant on Y, then $R^i f_* \mathcal{F}$ is locally free on Y.

Proof. Omitted. See [12] Chapter III, Corollary 12.9. □

Theorem 2.4.3. *Let $f : X \to Y$ a smooth surjective morphism between two smooth projective varieties. Let L be a nef and f-big line bundle over X. Then*

$$f_*(\omega_{X/Y} \otimes L)$$

is locally free and nef.

Proof. Consider for every $i > 0$ the following direct image sheaf:

$$R^i f_*(\omega_{X/Y} \otimes L) = R^i f_*(\omega_X \otimes L \otimes f^*\omega_Y^\vee);$$

then by Projection Formula, the latter is equal to

$$R^i f_*(\omega_X \otimes L) \otimes \omega_Y^\vee.$$

By Kawamata-Viehweg Vanishing Theorem (Theorem 2.2.3), as L is nef and big on the fibres, we have that

$$H^i(X, \omega_X \otimes L) = 0.$$

In particular, this holds on the fibres, hence,

$$R^i f_*(\omega_X \otimes L) = 0.$$

By smoothness of f the map

$$\varphi : y \longmapsto \dim_{\mathbb{C}(y)} H^0(F_y, (\omega_{X/Y} \otimes L)_{|F_y})$$

is locally constant in Y: as f is smooth, then it is flat. So we can apply Theorem 2.3.3. By the second statement of this theorem, if F_y is the fibre over a fixed $y \in Y$

$$H^0(F_y, (\omega_{X/Y} \otimes L)_{|F_y}) = H^0(X, \omega_{X/Y} \otimes L \otimes \underline{\mathbb{C}(y)}).$$

Observe now that, as Y is smooth, each point is smooth and the Krull dimension of $\mathcal{O}_{Y,y}$ is locally constant, therefore also the transcendence degree of $\mathbb{C}(y)$ is locally constant for $y \in Y$.
And since $\omega_{X/Y} \otimes L$ is locally free over X the map φ is indeed locally constant. By Grauert's Theorem (Theorem 2.4.2) we conclude that

$$f_*(\omega_{X/Y} \otimes L)$$

is locally free.
Now we want to prove that $f_*(\omega_{X/Y} \otimes L)$ is globally generated. Fix a very ample line bundle B on Y and let $d = \dim Y$.

$$H^i(Y, f_*(\omega_{X/Y} \otimes L) \otimes \omega_Y \otimes B^{\otimes(d+1-i)}) = H^i(Y, f_*(\omega_X \otimes L) \otimes B^{\otimes(d+1-i)})$$

and the equality holds by the Projection Formula. Again by the Projection Formula

$$H^i(Y, f_*(\omega_X \otimes L) \otimes B^{\otimes(d+1-i)}) = H^i(Y, f_*(\omega_X \otimes L \otimes f^*B^{\otimes(d+1-i)})).$$

And as the higher direct images of $\omega_X \otimes L$ vanish we can write

$$H^i(Y, f_*(\omega_X \otimes L \otimes f^*B^{\otimes(d+1-i)})) = H^i(X, \omega_X \otimes L \otimes f^*B^{\otimes(d+1-i)}).$$

As L is nef and B is very ample, $L \otimes f^*B^{\otimes(d+1-i)}$ is nef. Now, for every i, if $n = \dim X$

$$\int_X c_1(L \otimes f^*B^{d+1-i})^n = \int_X (c_1(L) + c_1(f^*B^{d+1-i}))^n =$$

$$= \int_X \sum_{k=0}^{n} \binom{n}{k} c_1(L)^{n-k} c_1(f^*B^{\otimes(d+1-i)})^k$$

Now, by linearity of the Intersection Product, Naturality of the Chern classes and the Projection Formula for the integration along the fibres

$$\int_X c_1(L \otimes f^*B^{d+1-i})^n = \sum_{k=0}^{n} \binom{n}{k} \int_Y f_* c_1(L)^{n-k} c_1(B^{\otimes(d+1-i)})^k.$$

Notice that, by the very-ampleness of B and the relative bigness of L all the summands are non-negative and, in particular, the last is positive. This implies that $L \otimes f^*B^{\otimes(d+1-i)}$ is big for every i.
By Kawamata-Viehweg, the cohomology group

$$H^i(X, \omega_X \otimes L \otimes f^*B^{\otimes(d+1-i)})$$

vanishes for every $1 \leq i \leq d$. Therefore, $f_*(\omega_{X/Y} \otimes L) \otimes \omega_Y$ is $d+1$-regular with respect to $B^{\otimes(d+1-i)}$. And therefore $f_*(\omega_{X/Y} \otimes L)$ is globally generated. Fix $s > 0$ and let $X^{(s)}$ be the fibre product of s copies of X over Y. Let π_i the projection on the i-th factor. And let $f^{(s)}$ the resulting map to Y. As f is smooth, so are $X^{(s)}$ and $f^{(s)}$. Now, let

$$L^{(s)} = \bigotimes_{i=1}^{s} \pi_i^* L$$

Moreover, by the Base Change Formula,

$$f_*^{(s)}(\omega_{X^{(s)}/Y} \otimes L^{(s)}) = (f_*(\omega_{X/Y} \otimes L))^{\otimes s}$$

Now, the very same proof we used to prove that $f_*(\omega_{X/Y} \otimes L)$ is globally generated can be redone for $f^{(s)}$ and therefore, one proves that for every $s \in \mathbb{N}$:

$$(f_*(\omega_{X/Y} \otimes L))^{\otimes s}$$

is globally generated, but this implies that for every $s \in \mathbb{N}$

$$S^s f_*(\omega_{X/Y} \otimes L)$$

is globally generated as well, hence, $f_*(\omega_{X/Y} \otimes L)$ is nef. $\square$

Chapter 3

Kodaira Dimension of Algebraic Fibre Spaces

In this last chapter we finally present the most general statement of the Iitaka Conjecture for smooth projective varieties. Then we state and prove some basic facts on the Kodaira dimension under some particular hypothesis. We then prove the Conjecture in the known case of the base space being a curve and present some ideas from a paper by Markus Wessler about an alternative proof of $C_{2,1}$ (see [32]). Lastly, we present some ideas about the proofs of the case in which the base space is either a variety of general type or an abelian variety.

3.1 Iitaka Conjecture

In this section we will present some ideas that are deeply linked to the Iitaka Fibration Theorem and have lead to the statement of the Iitaka Conjecture. The main reference for this section is Fujino's book [7].

In his paper [13], Shigeru Iitaka proved the Iitaka Fibration Theorem (in this discussion Theorem 1.4.6), linking some birational properties of a complex smooth projective variety to a fixed fibration whose domain is birationally equivalent to the variety one started with.

By the proof of Iitaka Theorem and by the Fujita-like results, it is clear that, among all varieties and line bundles that one can consider, a main role is played by those varieties X that satisfy the following

$$\kappa(X) \in \{-\infty, 0, \dim X\}$$

respectively called, by Iitaka, of *elliptic, parabolic and hyperbolic type*. In particular, the Iitaka Fibration Theorem implies that in order to study the birational classification, one need to concetrate on either this kind of varieties or to fibrations whose general fibre is of parabolic type.
The study of such fibrations lead Iitaka to pose, in [14], the following

Conjecture 3.1.1. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties X of dimension n and Y of dimension m, with connected fibres. Then*

$$\kappa(X) \geq \kappa(X_y) + \kappa(Y)$$

for a sufficiently general fibre X_y of f.

3.2 General Results

Here we present some general results about the Iitaka dimension and the Kodaira. The main reference are some notes by Popa [25]. Before giving proofs, we need the following:

Definition 3.2.1. Let Y be non-singular closed subvariety of the non-singular variety X. Let $\mathcal{I}$ be its sheaf ideal. We define the normal bundle of Y in X as

$$\mathcal{N}_{Y/X} = (\frac{\mathcal{I}}{\mathcal{I}^2})^{\vee}.$$

Theorem 3.2.2. *Let $f : X \to Y$ be an algebraic fibre space of smooth projective varieties. Then*

$$\kappa(X) \leq \kappa(F) + \dim Y$$

where F is a general fibre of f.

Proof. In $\kappa(X) = -\infty$ the statement is clear.
We will, at first, prove the more general statement: let $f : X \to Y$ be an algebraic fibre space of smooth projective varieties and let L be a line bundle on X. Then

$$\kappa(X, L) \leq \kappa(F, L_{|_F}) + \dim Y.$$

Assume $\kappa(X, L) \geq 0$. By Iitaka Fibration Theorem there exists m sufficiently large and divisible such that the canonical rational map φ_m, associated to $|L^{\otimes m}|$ is actually the Iitaka Fibration. Thus, denoting $Z = \overline{\varphi_m(X)}$, it holds

that $\dim Z = \kappa(X, L)$.
Moreover, m is such that, for the general fibre F,

$$\dim \overline{\varphi_m(F)} \leq \kappa(F, L_{|_F}).$$

Indeed, the restriction map

$$H^0(X, L^{\otimes m}) \to H^0(F, L_{|_F}^{\otimes m}),$$

induces a rational map

$$\Phi : \mathbb{P}(H^0(F, L_{|_F}^{\otimes m})) \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

such that the following diagram is commutative

$$\begin{array}{ccc} F & \overset{\psi_m}{\dashrightarrow} & \mathbb{P}(H^0(F, L_{|_F}^{\otimes m})) \\ \downarrow & & \vdots\ \Phi \\ X & \underset{\varphi_m}{\longrightarrow} & \mathbb{P}(H^0(X, L^{\otimes m})) \end{array}$$

with obvious definition of ψ_m. Therefore

$$\dim \overline{\varphi_m(F)} = \dim \overline{\Phi \circ \psi_m(F)} \leq$$
$$\leq \dim \overline{\psi_m(F)} \leq \kappa(F, L_{|_F}).$$

Now, let

$$W = \overline{(f, \varphi_m)(X)} \subseteq Y \times Z.$$

As the canonical projection on the second factor

$$\pi_2 : W \to Z$$

is surjective,

$$\kappa(X) = \dim Z \leq \dim W;$$

on the other hand, W is the union of all $\pi_1^{-1}(y), y \in Y$. These sets have, for $y \in Y$ general, the same dimension and therefore,

$$\dim W = \dim Y + \dim \pi_1^{-1}(Y);$$

but

$$\dim \pi_1^{-1}(y) \leq \dim \overline{\varphi_m(F)} \leq \kappa(F, L_{|_F})$$

and by putting everything together one obtains the statement.
Now, fix $y \in Y$ general and fix a local holomorphic coordinate system

$$(\eta_1, \dots, \eta_n)$$

for Y around y. Consider also the normal bundle $N = \mathcal{N}_{F/X}$ on F the fibre of y. The dual of N is globally trivialized by

$$f^*(d\eta_1 \wedge \dots \wedge d\eta_n);$$

therefore it is trivial, and so is N. By the Adjuction Formula (See Chapter II, Proposition 8.20 of [12] as a reference):

$$\omega_F = \omega_X \otimes \bigwedge^{\dim X - \dim F} N = \omega_X \otimes \bigwedge^{\dim X - \dim F} \mathcal{O}_F = \omega_X \otimes \mathcal{O}_F = \omega_{X_{|_F}}.$$

Specializing the general statement to ω_X:

$$\kappa(X) = \kappa(X, \omega_X) \leq \kappa(F, \omega_{X_{|_F}}) + \dim Y = \kappa(F, \omega_F) + \dim Y = \kappa(F) + \dim Y.$$

□

Before proving the next result we need to recall a couple of results from Hartshorne [12]. For the proofs, one can check the Propositions 8.10 and 8.11 of the second chapter of [12].

Proposition 3.2.3. *Let $f : X \to Y, g : Y' \to Y$ be morphisms of smooth projective varieties and $f' : X' = X \times_Y Y' \to Y'$ be the base change morphism. Then*

$$\Omega_{X'/Y'} = \pi_1^* \Omega_{X/Y}$$

where π_1 is the projection on the first factor and the Ω's are the sheaves of relative differentials.

Proposition 3.2.4. *Let $f : X \to Y, g : Y \to Z$ be morphisms of smooth projective varieties. Then there exists an exact sequence of sheaves on X*

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Theorem 3.2.5. *Let X, Y be smooth projective varieties. Then*

$$\kappa(X \times Y) = \kappa(X) + \kappa(Y).$$

Proof. In order to apply the Base Change Formula for the sheaf of differentials, just notice that our varieties are all schemes over $\mathbb{C}$. So, we have that if Ω_X, Ω_Y are the sheaves of differentials relative to $\mathbb{C}$ on X and Y respectively, then, by the Base Change Formula

$$\Omega_{X\times Y/X} = \pi_1^*\Omega_X,\ \Omega_{X\times Y/Y} = \pi_2^*\Omega_Y$$

where π_i is the projection on the i-th factor for $i \in \{1, 2\}$.
Then we have exact sequences

$$\pi_1^*\Omega_X \xrightarrow{\varphi_1} \Omega_{X\times Y} \xrightarrow{\varphi_2} \pi_2^*\Omega_Y \longrightarrow 0$$

$$\pi_2^*\Omega_Y \xrightarrow{\psi_1} \Omega_{X\times Y} \xrightarrow{\psi_2} \pi_1^*\Omega_X \longrightarrow 0$$

We now want to prove that ψ_1 is a right inverse to φ_2 and that φ_1 is a right inverse for ψ_2. Hence the short exact sequences

$$0 \longrightarrow \frac{\pi_1^*\Omega_X}{\ker(\varphi)} \xrightarrow{\varphi} \Omega_{X\times Y} \longrightarrow \pi_2^*\Omega_Y \longrightarrow 0$$

$$0 \longrightarrow \frac{\pi_2^*\Omega_Y}{\ker(\psi)} \xrightarrow{\psi} \Omega_{X\times Y} \longrightarrow \pi_1^*\Omega_X \longrightarrow 0$$

are split. We therefore have isomorphisms

$$\pi_1^*\Omega_X \oplus \frac{\pi_2^*\Omega_Y}{\ker(\psi)} \cong \Omega_{X\times Y} \cong \frac{\pi_1^*\Omega_X}{\ker(\varphi)} \oplus \pi_2^*\Omega_Y.$$

And this implies that

$$\begin{cases} \ker(\varphi) = \ker(\psi) = 0 \\ \Omega_{X\times Y} \cong \pi_1^*\Omega_X \oplus \pi_2^*\Omega_Y \end{cases}.$$

The proof of these facts is very scheme-theoretic, so we will regard the varieties as schemes and assume as known all the definitions and constructions related

to the sheaves of differentials. For more details one can check [12] Chapter II, Section 8.
We need to prove that a certain map of sheaves is the identity and, since this is a local property at each point, we can restrict to an arbitrary open affine cover of $X \times Y$ and prove it here. So take $\mathrm{Spec}(A) \subseteq X$ and $\mathrm{Spec}(B) \subseteq Y$ be open affines, then $\mathrm{Spec}(A \otimes_{\mathbb{C}} B)$ is an open affine in $X \times Y$ and varying A, B on all the rings that define open affines of X and Y respectively, we get an open affine cover of $X \times Y$.
By how the first exact sequence is defined, and using again the base change in the other direction, we know that the sequence of sheaves derives from the following exact sequence of $A \otimes_{\mathbb{C}} B$-modules

$$\Omega_{A/\mathbb{C}} \otimes_A (A \otimes_{\mathbb{C}} B) \xrightarrow{\widetilde{\varphi_1}} \Omega_{A \otimes_{\mathbb{C}} B/\mathbb{C}} \xrightarrow{\widetilde{\varphi_2}} \Omega_{B/\mathbb{C}} \otimes_B (A \otimes_{\mathbb{C}} B) \longrightarrow 0$$

where the $\Omega_{\cdot/\mathbb{C}}$'s are the modules of $\mathbb{C}$-differentials on the ring.
The map $\widetilde{\varphi_1}$ is defined as follows on the simple tensors

$$\widetilde{\varphi_1} : (a \otimes b) \otimes d\alpha \longmapsto (a \otimes b) d(\alpha \otimes 1)$$

and

$$\widetilde{\varphi_2} : d(a \otimes b) \longmapsto (a \otimes 1) \otimes db$$

In the same way the second exact sequence derives from

$$\Omega_{B/\mathbb{C}} \otimes_B (A \otimes_{\mathbb{C}} B) \xrightarrow{\widetilde{\psi_1}} \Omega_{A \otimes_{\mathbb{C}} B/\mathbb{C}} \xrightarrow{\widetilde{\psi_2}} \Omega_{A/\mathbb{C}} \otimes_A (A \otimes_{\mathbb{C}} B) \longrightarrow 0$$

with definitions,

$$\widetilde{\psi_1} : (a \otimes b) \otimes d\beta \longmapsto (a \otimes b) d(1 \otimes \beta)$$

and

$$\widetilde{\psi_2} : d(a \otimes b) \longmapsto (1 \otimes b) \otimes da$$

By direct computations

$$\widetilde{\varphi_2} \circ \widetilde{\psi_1}((a \otimes b) \otimes d\beta) = \widetilde{\varphi_2}((a \otimes b) d(1 \otimes \beta)) = (a \otimes b)(1 \otimes 1) \otimes d\beta = (a \otimes b) \otimes d\beta;$$

and, on the other hand,

$$\widetilde{\psi_2} \circ \widetilde{\varphi_1}((a \otimes b) \otimes d\alpha) = \widetilde{\psi_2}((a \otimes b)d(\alpha \otimes 1)) = (a \otimes b)(1 \otimes 1) \otimes d\alpha = (a \otimes b) \otimes d\alpha.$$

Therefore the sequences split.
Now, by the functoriality of the exterior power and by the linearity of the pull-back

$$\bigwedge^{\dim X + \dim Y} \Omega_{X \times Y} = \pi_1^* \bigwedge^{\dim X} \Omega_X \otimes \pi_2^* \bigwedge^{\dim Y} \Omega_Y = \pi_1^* \omega_X \otimes \pi_2^* \omega_Y.$$

Hence, for every $m \in \mathbb{N}$

$$H^0(X \times Y, \omega_{X \times Y}^{\otimes m}) = H^0(X, \omega_X^{\otimes m}) \otimes H^0(Y, \omega_Y^{\otimes m});$$

therefore

$$h^0(X \times Y, \omega_{X \times Y}^{\otimes m}) = h^0(X, \omega_X^{\otimes m}) \cdot h^0(Y, \omega_Y^{\otimes m})$$

and by the logarithmic definition of the Kodaira dimension we deduce the formula. □

3.3 Fibre Spaces over Curves

In this section we focus the attention on fibrations over curves; more precisely we will prove $C_{n,1}$ in the case of a curve of general type as base space and the additional hypothesis that the geometric genus of the general fibre is positive. The main reference is [18]. Later we will also analize the case of $C_{2,1}$ for which Markus Wessler's presented a proof that completely uses methods from Algebraic Geometry in [32].

3.3.1 Curves of General Type

A variety is said *of general type* if its Kodaira dimension and its topological dimension coincide.
In this section we will omit the cases of the base space being an elliptic curve and the case of the general fibre having zero global sections for the canonical bundle.
Notice that, by the computation of the Kodaira dimension for curves that $\kappa(\mathbb{P}^1) = -\infty$, so if we are studying a fibration

$$f : X \to \mathbb{P}^1,$$

then the Iitaka Conjecture is trivially satisfied.

Theorem 3.3.1. *In the setting of Fujita Theorem (Theorem 2.3.2), assume that C is of general type and the general fibre F of the fibration has positive geometric genus $p_g(F) = h^0(F, \omega_F)$. Then*

$$\kappa(X) = \kappa(F) + 1.$$

Proof. By the Easy Addition Formula, one has

$$\kappa(X) \leq \kappa(F) + \dim C = \kappa(F) + 1$$

for every general fibre F. Fix one. By the assumption of positive geometric genus for F, we have that $\omega_{X/C} \neq 0$ and as C is of general type, ω_C is ample. Observe that

$$\begin{gathered} f_*\omega_{X/C} = f_*(\omega_X \otimes f^*\omega_C^\vee) = f_*\omega_X \otimes \omega_C^\vee \\ \implies f_*\omega_X = f_*\omega_{X/C} \otimes \omega_C. \end{gathered}$$

By Fujita Theorem, $f_*\omega_{X/C}$ is nef, thus $f_*\omega_X$ is ample.
Fix a very ample divisor H on C, then by the characterization of ample vector bundles, for all m sufficiently large

$$S^m f_*\omega_X \otimes \mathcal{O}_C(-H)$$

is globally generated, fix such an m. Moreover notice that

$$H^0(X, \omega_X^{\otimes m} \otimes f^*\mathcal{O}_C(-H)) = H^0(C, f_*(\omega_X^{\otimes m} \otimes f^*\mathcal{O}_C(-H))) = H^0(C, f_*\omega_X^{\otimes m} \otimes \mathcal{O}_C(-H)).$$

As $S^m f_*\omega_X$ can be realized as a subsheaf of $f_*\omega_X^{\otimes m}$ the same holds for $S^m f_*\omega_X \otimes \mathcal{O}_C(-H)$ with respect to $f_*\omega_X^{\otimes m} \otimes \mathcal{O}_C(-H)$. As the first one is globally generated its global sections have positive dimension and therefore

$$H^0(C, f_*\omega_X^{\otimes m} \otimes \mathcal{O}_C(-H)) \neq 0.$$

Take a global section of $\omega_X^{\otimes m} \otimes f^*\mathcal{O}_C(-H)$, then it induces an injective map

$$\alpha : H^0(X, f^*\mathcal{O}_C(H)) \to H^0(X, \omega_X^{\otimes m})$$

But,

$$H^0(X, f^*\mathcal{O}_C(H)) = H^0(C, \mathcal{O}_C(H)) = H^0(X, \mathcal{O}_X(f^*H)).$$

Hence, $H^0(X, \mathcal{O}_X(f^*H))$ is a subspace of $H^0(X, \omega_X^{\otimes m})$.
Now we check that $\kappa(X) > 0$. By definition

$$\kappa(X) = \max_{n \geq m} \dim \varphi_n(X)$$

where φ_n is the rational map canonically induced by the linear system $\left|\omega_X^{\otimes n}\right|$. Notice that even though the usual definition is without the lower bound m, by what we proved in Lemma 1.2.12, there are infinitely many n's that realize the Kodaira dimension, thus taking only $n \geq m$ does not change the definition. By contraddiction assume $\kappa(X) = 0$, then, φ_n is a constant map for every $n \in \mathbb{N}$. On the other hand, though, for all $n \geq m$

$$0 \neq H^0(C, \mathcal{O}_C(H)) = H^0(X, \mathcal{O}_C(f^*H)) \subseteq H^0(X, \omega_X^{\otimes n}).$$

Hence

$$C \subseteq \mathbb{P}(H^0(X, \omega_X^{\otimes n}))$$

So there exists an $n \in \mathbb{N}$ such that φ_n is non-constant. This is a contraddiction. We deduce that $\kappa(X)$ is positive. We can therefore apply the Iitaka Fibration Theorem. Let

$$X_\infty \to V$$

be the Iitaka fibration, but as X is birationally equivalent to X_∞ we can consider the Iitaka Fibration as a morphism

$$\rho : X \to V$$

Consider then the factorization of f

$$X \to V \dashrightarrow C$$

The fact that the fibration f factorizes through the Iitaka Fibration is obvious by the construction in the Theorem of the Iitaka Fibration.
Take $v \in V$ a general point such that v is not in the indeterminacy locus of the rational map

$$\varphi : V \dashrightarrow C$$

and let $G = \rho^{-1}(v)$, $c = \varphi(v)$, $F = \varphi^{-1}(c)$.
Let $x \in G$, then $\varphi(\rho(x)) = c$; by the factorization above

$$f = \varphi \circ \rho,$$

so $x \in f^{-1}(c) = F$. We showed that every general fibre of the Iitaka fibration is contained in a general fibre of f. Therefore, by the Iitaka Fibration Theorem

$$\kappa(G, \omega_{F_{|G}}) = \kappa(G, \omega_{X_{|F_{|G}}}) = \kappa(G, \omega_{X_{|G}}) = 0.$$

As ρ is the Iitaka Fibration, $\dim V = \kappa(X)$ and

$$\kappa(X) - 1 = \dim V - \dim C$$

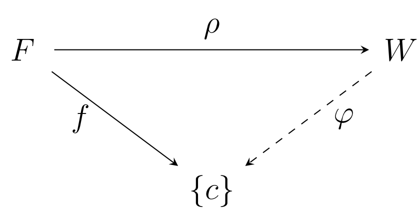
is the topological dimension of the general fibre of φ. On the other hand we also have that

$$\dim F = \dim X - 1,\ \dim G = \dim X - \kappa(X).$$

Substracting the second from the first we find that

$$\dim F - \dim G = \kappa(X) - 1.$$

By denoting $W = \overline{\rho(F)}$ we have a commutative triangle where the morphisms are trivially fibre spaces:



And by applying the Preliminary Result to the Easy Addition Formula to

$$F \to W$$

we have that

$$\kappa(F) \leq \kappa(G, \omega_{F_{|G}}) + \dim W = 0 + (\kappa(X) - 1) = \kappa(X) - 1.$$

By reordering one finds the inequality we sought. $\square$

3.3.2 The Case of a Surface

In the case of a fibration

$$f : S \to C$$

where S is a smooth projective surface and C is a smooth projective curve, one can prove the Iitaka inequality using only methods from Algebraic Geometry: indeed the proof we presented uses either Fujita Theorem whose proof deeply relies on Analytic methods or, even in the case of a smooth fibration, Kollár Vanishing Theorem that is also proved via analytic means.
The special case of $C_{2,1}$, instead, due to the classical knowledge of the birational geometry of the surfaces and thanks to the developments by Viehweg and Mumford can be therefore proved via only methods from Algebraic Geometry. This was shown by Wessler in [32]. We present here the main ideas citing the references for the most classical statements.
Before going into detail with Wessler's paper, we need to recall a definition and a result form [1].

Discriminant of a morphism and Branched Covering Trick

In order to give the definition of the discriminant of a finite locally free morphism, we need to work in the setting of schemes.
Let $\pi : X \to Y$ be a finite locally free morphism of schemes and consider the canonical $\mathcal{O}_Y$-linear trace map

$$\mathrm{Tr}_\pi : \pi_*\mathcal{O}_X \to \mathcal{O}_Y$$

sending a local section f of $\pi_*\mathcal{O}_X$ to the trace of the multiplication by f on $\pi_*\mathcal{O}_X$. The composition

$$\mathcal{O}_Y \to \pi_*\mathcal{O}_X \to \mathcal{O}_Y$$

is the multiplication by the degree of π, which is a locally constant map on Y. Therefore one can define a trace pairing

$$Q_\pi : \pi_*\mathcal{O}_X \times \pi_*\mathcal{O}_X \to \mathcal{O}_Y$$

associating to every couple (f, g) the trace map of the product. One can think Q_π as a map from $\pi_*\mathcal{O}_X$ to its dual. These two sheaves have equal rank r. Therefore, we have a determinant

$$\det Q_\pi : \bigwedge^r \pi_*\mathcal{O}_X \to \bigwedge^r (\pi_*\mathcal{O}_X)^\vee.$$

We can consider $\det Q_\pi$ as a global section of $\bigwedge^r(\pi_*\mathcal{O}_X)^{\otimes -2}$. The discriminant of π is the divisor cut out by this section.

Theorem 3.3.2. *Let $p : B \to X$ be an holomorphic $\mathbb{P}^1$-bundle over the smooth complex variety X. If S is an irreducible divisor on B meeting the general fibre in n points, then there exists a smooth complex variety Y, a generically finite surjective morphism $f : Y \to X$ and n effective divisors $S_1, \dots, S_n$ on $B' = B \times_X Y$ all meeting the general fibre in one point, such that*

$$g^*S = S_1 + \cdots + S_n$$

where $g : B' \to B$ is the projection.

Proof. Omitted. See [1] Chapter I, Theorem 18.2. □

Wessler's proof

We introduce some notations: let S be a smooth projective surface and C a smooth projective curve both defined over $\mathbb{C}$. Let

$$f : S \to C$$

be an algebraic fibre space; let g be the genus of the general fibre and let δ be the number of singular fibres.
If $g = 0$, then the inequality is trivially verified, so we can assume that $g > 0$. Furthermore we may assume, up to blow-down all the (-1)-curves (curves of self intersection equals to -1) in the fibres, that f is a relatively minimal model.
We can also assume that C has positive genus otherwise the inequality is, again, trivial.

Definition 3.3.3. A fibraton is said to be *isotrivial* if all the general fibres are isomorphic.

Theorem 3.3.4. *In the notation above, if f is isotrivial with general fibre isomorphic to a curve A, then there exisits a smooth curve B and a finite group G acting algebraically and faithfully on A and B such that the following diagram commutes*

$$\begin{array}{ccc} S & \overset{\sim}{\dashrightarrow} & \frac{A}{G} \times \frac{B}{G} \\ {\scriptstyle f}\downarrow & & \downarrow \\ C & \overset{\sim}{\longrightarrow} & \frac{B}{G} \end{array}$$

the horizontal maps are birational and the projection

$$B \to \frac{B}{G}$$

is étale.

Proof. Omitted. See [26]. □

Consider now the base change

$$S' = S \times_{\frac{B}{G}} B.$$

The second projection gives a fibration and the first projection is étale: it is a fibre product along two smooth morphisms, thus it is smooth and as B is finite over $\frac{B}{G}$ so is S' over S. Furthermore,

$$S' \cong A \times B;$$

therefore,

$$\kappa(S) = \kappa(S') = \kappa(A) + \kappa(B) = \kappa(A) + \kappa(\frac{B}{G}) = \kappa(A) + \kappa(C);$$

where we used Theorem 3.2.5 and the following:

Proposition 3.3.5. *Let $f : X \to Y$ be a generically finite surjective morphism between smooth projective varieties. Then*

i $\kappa(Y) \leq \kappa(X)$*;*

ii if f is étale, then we have an equality.

Proof. Omitted. See [25] Example 1.3.9. □

Thus we may also assume that f is non-isotrivial.
By Theorem 2.3.7 we know that the direct image of the relative canonical bundle is weakly-positive. This notion, on a curve, coincides with the nefness. It holds the following

Corollary 3.3.6. *In the same notations as above, the relative canonical bundle $\omega_{S/C}$ is nef and is such that $c_1(\omega_{S/C})^2 \geq 0$.*

Proof. Let $A \subseteq S$ be an irreducible curve. If A is a fibre, then by the fact that f is relatively minimal

$$c_1(\omega_{S/C}) \cdot A \geq 0.$$

Otherwise, the natural map

$$f^* f_* \omega_{S/C} \to \omega_{S/C}$$

is surjective when restricted to C. Hence, $\omega_{S/C}$, when restricted to C, is quotient of the pull-back of a nef line bundle. Thus

$$c_1(\omega_{S/C}) \cdot A \geq 0$$

and therefore is nef and the relation $c_1(\omega_{S/C})^2 \geq 0$ is obviously satisfied. □

What we want to prove is that under some further assumptions, it has positive degree. We will also treat the cases excluded by the assumptions.

Definition 3.3.7. A morphism

$$f : X \to Y$$

between a smooth surface and a smooth curve is said to be a *semi-stable model* if for all varieties $V \subseteq W$ all the irreducible components of $f^{-1}(V)$ are reduced and smooth and if P is in the intersection of some of these components $Y_1, \ldots, Y_r$ with local equations $f_1, \ldots, f_r$ respectively, then the images of the f_i's are linearly independent in $\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}$. Such a divisor in X is called *normal crossing divisor*.

We want to prove that we can assume that f is a semi-stable model. Up to consider a finite covering

$$\tau = C' \to C,$$

and taking the base change

$$f' : S' = S \times_C C' \to C',$$

and a resolution of singularities

$$d : S'' \to S',$$

we have a semi-stable morphism. Notice that the existence of such a resolution is guaranteed by the following

Theorem 3.3.8. *Let Y be any curve on the surface X. Then there exists a finite sequence of blow-ups at a point*

$$X' = X_n \to \cdots \to X_0 = X$$

such that $f^{-1}(Y)$ is a normal crossing divisor.

Proof. Omitted. See [12] Chapter V, Theorem 3.9. □

Consider now the trace map

$$d_* \omega_{S''} \to \omega_{S'} :$$

this map is defined as follows (with abuse of notation we confuse the bundles and their associated sheaves): at first, we have a canonical map

$$d_* d^{-1} \omega_{S'} \to \omega_{S'}$$

On the other hand, for $U \subseteq S''$ sufficiently small, both U and $d(U)$ are in some open subsets of S'' and S' that trivialize $\omega_{S''}$ and $\omega_{S'}$ simultaneously. Therefore for each these U's we have a canonical map

$$\omega_{S''_{|U}} = \mathcal{O}_U \to d^{-1}\mathcal{O}_{S'_{|d(U)}} = (d^{-1}\omega_{S'})_U.$$

We can obviuosly glue these morphisms and the U's we've considered determine an open covering of S'', hence we have a canonically defined map

$$\omega_{S''} \to d^{-1}\omega_{S'}$$

and therefore the trace map is the composition

$$d_*\omega_{S''} \to d_* d^{-1}\omega_{S'} \to \omega_{S'}.$$

Moreover, as d is, on an open dense subset of S an isomorphism, on the stalks at the points of such an open, the trace map is injective. It induces an injective map

$$d_*\omega_{S''/C'} \to \omega_{S'/C'} = \tau'^*\omega_{S/C}.$$

By flat base change (Theorem 2.3.3) we have an injective map

$$f''_*\omega_{S''/C'} = f'_* d_*\omega_{S''/C'} \to f'_*\tau'^*\omega_{S/C} = \tau^* f_*\omega_{S/C}.$$

Thus, if $f''_*\omega_{S''/C'}$ has positive degree, then also $f_*\omega_{S/C}$ does.
This shows that we can consider f to be semi-stable.

Lemma 3.3.9. *Under the assumptions above and in the same notations, if λ_n is the degree of*

$$f_*\omega_{S/C}^{\otimes n}$$

for every $n \geq 1$, then if $g > 1$ we have that

$$\lambda_n = \binom{n}{2}(12\lambda_1 - \delta) + \lambda_1 = \binom{n}{2} c_1(\omega_{S/C})^2 + \lambda_1$$

and for $g = 1$

$$12\lambda_n = n\delta = 12n\lambda_1.$$

Proof. Omitted. See [22] Theorem 5.10. □

By Mumford Lemma, we have that

$$12\lambda_1 - \delta \geq 0 \iff 12\lambda_1 \geq \delta$$

If $\delta > 0$ we have proved the claim on the positivity of λ_1. For the case of a smooth fibration things can be solved relatively esaily.
Moreover, for $\delta = 0$ and $g > 1$, we have that $\lambda_1 > 0$ if and only if $\lambda_n > 0$ for every $n \in \mathbb{N}$ if and only if $\lambda_n > 0$ for some $n \in \mathbb{N}$. Thus, in order to prove $\lambda_1 > 0$, it suffice to show $\lambda_n > 0$ for some $n \in \mathbb{N}$.
If the general fibre is an elliptic curve ($g = 1$) then, consider the map

$$c \longmapsto j(f^{-1}(c)) :$$

this defines, by the properties of the j-invariant a morphism

$$j : C \to \mathbb{A}^1(\mathbb{C}).$$

But C is projective, hence complete and $\mathbb{A}^1(\mathbb{C})$ is affine. Therefore j is a constant morphism and the fibration is isotrivial.

Definition 3.3.10. A curve X is called *hyperelliptic* if $g \geq 2$ and there exists a finite morphism of degree 2

$$X \to \mathbb{P}^1$$

Notice that, as any morphism to a curve is either constant or dominant, any hyperelliptic curve is a double covering of $\mathbb{P}^1$. We have that

Lemma 3.3.11. *In the notations and assumptions as above, if all the fibres are hyperelliptic curves, the fibration is isotrivial*

Proof. Consider the projective bundle

$$\pi : \mathbb{P}(f_*\omega_{S/C}) \to C$$

and the morphism induced by the canonical surjection

$$f^*f_*\omega_{S/C} \to \omega_{S/C}.$$

Denote this morphism with

$$\varphi : S \to \mathbb{P}(f_*\omega_{S/C}).$$

As, by definition, we have that

$$f = \pi \circ \varphi$$

and all the fibres of f are smooth hyperelliptic, also all fibres of φ are smooth hyperelliptic, hence, they are double coverings of $\mathbb{P}^1$. So if $\mathbb{P}$ is the image of φ, then $\mathbb{P}$ is a ruled surface.
Let Δ be the discriminant of φ. Then Δ intersects the general fibre in $\deg\varphi = 2g+2$ points. By the Branched Covering Trick (Theorem 3.3.2) there exists an étale covering

$$\gamma : C' \to C$$

such that, if

$$\mathbb{P}' = \mathbb{P} \times_C C',$$

then the pull-back of Δ to $\mathbb{P}'$ has $2{+}2g$ disjoint components. These components correspond to $2+2g$ disjoint global sections of $\mathbb{P}'$, thus a linear combination of these sections yields a never-vanishing global section of $\mathbb{P}'$. Therefore it is trivial.
As the fibration f' factors through a trivial fibre bundle, it is isotrivial. Therefore f itself is isotrivial. □

From now on, assume that not all fibres are hyperelliptic.
Furthermore, we recall

Proposition 3.3.12. *Let X be a curve of genus $g \geq 2$. Then $|\omega_X|$ is very ample if and only if X is non-hyperelliptic.*

Proof. Omitted. See [12] Chapter IIII, Proposition 5.2. □

Lemma 3.3.13. *In the same notations and assumptions, for $n \in \mathbb{N}$ sufficiently large, the multiplication map*

$$\mu_n : S^n f_*\omega_{S/C} \to f_*\omega_{S/C}^{\otimes n}$$

is surjective outside the hyperelliptic fibres.

Proof. Let S_c be a non-hyperelliptic fibre of f, thus ω_{S_c} is very ample. Consider the closed embedding

$$S_c \to \mathbb{P}^{g-1}$$

given by the global sections of ω_{S_c}.
We can identify these sections with those of $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ and therefore there is an isomorphism

$$S^n H^0(S_c, \omega_{S_c}) \to H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n))$$

for every $n \in \mathbb{N}$. Let I_c be the sheaf of ideals associated to S_c, then we have a short exact sequence

$$0 \longrightarrow I_c \longrightarrow \mathcal{O}_{\mathbb{P}^{g-1}}(1) \longrightarrow \omega_{S_c} \longrightarrow 0$$

Twist by $n{-}1$. Now, remember that, through the closed immersion canonically associated to ω_{S_c}, namely λ, it holds that

$$\omega_{S_c} = \lambda^* \mathcal{O}_{P^{g-1}}(1);$$

therefore

$$\omega_{S_c} \otimes \mathcal{O}_{P^{g-1}}(n-1) = \omega_{S_c} \otimes \lambda^* \mathcal{O}_{P^{g-1}}(n-1) = \omega_{S_c}^{\otimes n}$$

and thus, for n sufficiently large, by Serre's Vanishing Theorem, the map

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{P^{g-1}}(n)) \to H^0(S_c, \omega_{S_c})$$

is surjective. Thus the composition

$$S^n H^0(S_c, \omega_{S_c}) \to H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(S_c, \omega_{S_c})$$

is surjective for n sufficiently large. By base change we conclude. □

At this point, if some of the fibres are hyperelliptic, then consider, for $n \in \mathbb{N}$ sufficiently large, the image of μ_n: as $f_*\omega_{S/C}$ is weakly-positive, then also $S^n f_*\omega_{S/C}$ is weakly-positive, then it has non-negative degree as well as its image, via μ_n. Furthermore, this image is a proper subsheaf of $f_*\omega_{S/C}^{\otimes n}$, thus

$$\lambda_n > \deg \mu_n(S^n f_*\omega_{S/C}) \geq 0$$

hence, $\lambda_1 > 0$.
We now prove the following.

Theorem 3.3.14. *Let S be a smooth projective complex surface and C a smooth projective complex curve. Moreover let $f : S \to C$ be an algebraic fibre space such that: f is a relatively minimal semi-stable model, it is non-isotrivial and all the fibres are smooth non-hyperelliptic curves of general type. Then $\lambda_n > 0$ for some $n \in \mathbb{N}$.*

Proof. At first we will describe the so-called "method of the universal basis" introduced by Viehweg.
Let $\mathcal{E}$ be a locally free sheaf of rank m over C and let

$$\pi : \mathbb{P} \to C$$

be the projective bundle associated to $(\mathcal{E}^\vee)^{\oplus m}$. Then we canonically have a surjective map

$$\pi^*(\mathcal{E}^\vee)^{\oplus m} \to \mathcal{O}_\mathbb{P}(1),$$

and by dualizing we obtain an injective map

$$\mathcal{O}_\mathbb{P}(-1) \to \pi^* \mathcal{E}^{\oplus m}$$

defined on each local section by

$$\ell \longmapsto (s_1, \ldots, s_m).$$

This induces another (injective) map

$$s : \mathcal{O}_{\mathbb{P}}(-1)^{\oplus m} \to \pi^* \mathcal{E}$$
$$(f_1 \ell, \ldots, f_m \ell) \longmapsto \sum_{i=1}^{m} f_i s_i.$$

This map is called the universal basis of $\mathcal{E}$.
Consider s the universal basis of $f_* \omega_{S/C}$

$$\mathcal{O}_{\mathbb{P}}(-1)^{\oplus m} \to \pi^* f_* \omega_{S/C}$$

where $\mathbb{P}$ is the projective bundle of $(f_* \omega_{S/C}^{\vee})^{\oplus m}$. Taking the n-th symmetric power we have a map

$$S^n \mathcal{O}_{\mathbb{P}}(-1)^{\oplus m} \to S^n \pi^* f_* \omega_{S/C} = \pi^* S^n f_* \omega_{S/C}.$$

On the other hand by pull-back (which preserves surjections) we also have a morphism

$$\pi^* S^n f_* \omega_{S/C} \to \pi^* f_* \omega_{S/C}^{\otimes n}.$$

Then, composing we get another morphism

$$S^n \mathcal{O}_{\mathbb{P}}(-1)^{\oplus m} \to \pi^* f_* \omega_{S/C}^{\otimes n}$$

that is surjective outside of D the zero divisor of $\det s$ for n sufficiently large.
Let $\mathcal{B}$ be the image sheaf of such morphism.
Consider a blow-up

$$\tau : \mathbb{P}' \to \mathbb{P}$$

with center D. We may assume that $\mathcal{B}' = \tau^* \mathcal{B}$ is, modulo torsion, locally free.
Moreover denote $\mathcal{O}_{\mathbb{P}'}(k) = \tau^* \mathcal{O}_{\mathbb{P}}(k)$ for every integer k and

$$\pi' : \mathbb{P}' \to C$$

the composed morphism. By pulling-back we obtain a surjection

$$S^n \mathcal{O}_{\mathbb{P}'}(-1)^{\oplus m} \to \mathcal{B}'.$$

Let now G be the Grassmanian manifold parametrizing the r-dimensional quotients of $S^n\mathbb{C}^m$ where r is the rank of $f_*\omega_{S/C}^{\otimes n}$ and let $V = \bigwedge^r S^n\mathbb{C}^m$. By the properties of the projective bundle, the induced quotient map

$$\underline{V} \otimes \mathcal{O}_{\mathbb{P}'} = \bigwedge^r S^n\mathcal{O}_{\mathbb{P}'}^{\oplus m} \to \bigwedge^r(\mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(n)) = \bigwedge^r \mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(nr),$$

where $\underline{V}$ is the sheaf of locally constant continuous V-functions on $\mathbb{P}'$, corresponds to a morphism of varieties

$$\Phi' : \mathbb{P}' \to \mathbb{P}(V)$$

that factors through G.
Take $P \in \mathbb{P}' - \tau^*D$. Then P identifies uniquely a fibre S_c of f and a basis of $H^0(S_c, \omega_{S_c})$ up to multiplication by a non zero scalar. Of course the fibre we refer is the fbre of $\pi'(P)$ through f and the basis is given by the fact that the map

$$S^n\mathcal{O}_{\mathbb{P}}(-1)^{\oplus m} \to \pi^* f_*\omega_{S/C}^{\otimes n}$$

is, by definition, non-zero on P and by reverse engineering the proof of Lemma 3.3.13.
The choice of P, therefore, determines an isomorphism

$$\mathbb{P}^{g-1} \cong \mathbb{P}(H^0(S_c, \omega_{S_c}))$$

and consequently, as ω_{S_c} is very ample, an embedding of S_c into $\mathbb{P}^{g-1}$.
But if $\Phi'(P) \in G$ is then given by the quotient map

$$\mu_n : S^n H^0(S_c, \omega_{S_c}) \to H^0(S_c, \omega_{S_c}^{\otimes n})$$

that is the multiplication map. By Lemma 3.3.13, for n sufficiently large its kernel determines both S_c and its embedding in the suitable projective space. As the genus of S_c is at least 2, by a Hurwitz's result, its automorphism group is finite. Thus the subgroup of $\mathrm{PGL}_g(\mathbb{C})$ that leaves $S_c \subseteq \mathbb{P}^{g-1}$ invariant has to be finite.
So, if $A \subseteq \mathbb{P}' - \tau^*D$ is a curve mapping to a point in G, we obtain that A cannot be contained in a fibre of π'. Thus the restriction map

$$A \to C$$

is surjective. This implies that for every point in A the kernel of the associated multiplication map is the same, otherwise it would not be mapped into a point of G. Thus

$$X \times_C A$$

is trivial, but this contraddicts the non-isotriviality of f. Thus there cannot be curves in the fibres of $\Phi'_{|_{\mathbb{P}'-\tau^* D}}$.
As a consequence,

$$\det \mathcal{B} \otimes \mathcal{O}_{\mathbb{P}'}(nr) = \Phi'^* \mathcal{O}_{\mathbb{P}(V)}(1)$$

is ample and the inclusion

$$\det \mathcal{B} \otimes \mathcal{O}_{\mathbb{P}'}(nr) \subseteq \pi'^* \det f_* \omega_{S/C}^{\otimes n} \otimes \mathcal{O}_{\mathbb{P}'}(nr) = \mathcal{L}$$

implies that $\mathcal{L}$ is big.
Take now F a general fibre of π', for every $h \in \mathbb{N}$, consider the short exact sequence

$$0 \longrightarrow \mathcal{L}^{\otimes h}(-F) \longrightarrow \mathcal{L}^{\otimes h} \longrightarrow \mathcal{L}^{\otimes h}_{|_F} \longrightarrow 0$$

As $\mathcal{L}$ is big, for h sufficiently large, there exists a non-trivial section.
For P in general position, the projection formula implies

$$\mathcal{O}_C \to \Lambda_n^{\otimes h} \otimes \mathcal{O}_{\mathbb{P}'}(hnr) = \Lambda_n^{\otimes h} \otimes (S^{hnr} f_* \omega_{S/C}^{\oplus m})^{\vee}$$

where $\Lambda_n = \det f_* \omega_{S/C}^{\otimes n}$ and the last equality holds by the properties of the projective bundle. Dualizing we obtain a non trivial map

$$S^{hnr} f_* \omega_{S/C}^{\oplus m} \to \Lambda_n^{\otimes h}(-P)$$

Since $f_* \omega_{S/C}$ is weakly-positive, the degree of $\Lambda_n^{\otimes h}(-P)$ is non negative, hence $\lambda_n > 0$. □

We now conclude the proof with this result.

Theorem 3.3.15. *In the same assumptions and notations as above*

$$\kappa(S) \geq \kappa(C) + \kappa(S_c)$$

Proof. By the previous discussion we may assume that the general fibre is of general type and the base curve is not a rational curve.
By the previous Theorem we have that the line bundle

$$f_*\omega_{S/C}$$

has positive degree on C, therefore it is ample. In particular

$$f_*\omega_S = f_*\omega_{S/C} \otimes \omega_C$$

is ample because, by the assumption on C, ω_C is, at least, nef and the tensor product of a nef line bundle with an ample one is ample.
Moreover, by the observation we made right after stating Mumford Lemma, all the tensor powers of $f_*\omega_{S/C}$ have positive degree.
Thus, for every $m \in \mathbb{N}, n \neq 0$

$$h^0(S, \omega_S^{\otimes m}) = h^0(C, f_*\omega_S^{\otimes m}) > 1.$$

This implies that the $\kappa(X) \geq 1$. If C is an elliptic curve there is nothing else to prove.
So assume C is of general type. By the relative minimality of f, and the fact that the Kodaira dimension is positive then ω_S is nef. In particular, for a canonical divisor K_S,

$$K_S^2 \geq 0.$$

If the inequality is strict, in particular, K_S^2 is ample and the Kodaira dimension of X is maximal, hence the Iitaka inequality is verified as an equality.
If the square is 0, then, $\omega_S^{\otimes 2} = \mathcal{O}_S$ and therefore

$$f_*\omega_{S/C}^{\otimes 2} = f_*(\omega_S^{\otimes 2} \otimes (f^*\omega_C^\vee)^{\otimes 2}) = f_*(\mathcal{O}_S \otimes (f^*\omega_C^\vee)^{\otimes 2}) = (\omega_C^\vee)^{\otimes 2}$$

which is not nef. This is a contraddiction, so if both S_c and C are of general type, so must be S. □

Remark 3.3.16. We specialized this proof in the case of $\mathbb{C}$, but it actually holds over every algebraically closed field k. Therefore substituting everywhere $\mathbb{C}$ with k, yields a proof of $C_{2,1}$ over any field.

3.4 Fibres Spaces over Abelian Varieties

In this last section, we conclude by presenting the main ideas in the expository paper by Hacon-Popa-Schnell [10] about the proof that Cao and Păun gave of the Iitaka Conjecture in the case of the base space being an abelian variety in their paper [5].
The main theorem they proved is the Iitaka inequality in the case of a variety of maximal Albanese dimension. Anyway, this case follows quite easily from the assumption that the domain of the fibration has Kodaira dimension 0 and the base space being an abelian variety.
We intend to present the main ideas, therefore most of the proofs will be omitted. Anyway, after stating all the necessary results we will prove the inequality in the restricted case and then in the general case. All proofs can be found in [10] unless otherwise stated.

3.4.1 Preliminaries

In this subsection we will assume to always be (unless otherwise stated) in the following situation:

$$f : X \to A$$

is an algebraic fibre space of smooth projective varieties such that $\kappa(X) = 0$ and A is an abelian variety. Furthermore let $m \in \mathbb{N}$ be such that $P_m(X) = 1$ and consider the following sheaf

$$\mathcal{F}_m = f_*\omega_X^{\otimes m}.$$

This is a torsion-free coherent sheaf over A whose rank is equal to $P_m(F)$ at each point.

Abelian Varieties and Albanese Variety

Fix A a smooth projective abelian variety over $\mathbb{C}$ of dimension n and let e be its identity element. Consider the Zariski tangent space of A at e, namely, if $\mathfrak{m}_e \subseteq \mathcal{O}_{A,e}$ are the maximal ideal and the local ring of A at e

$$T_eA = \frac{\mathfrak{m}_e}{\mathfrak{m}_e^2}$$

As e is smooth in A, this is an n-dimensional $\mathbb{C}$-vector space. Moreover, the action of the group via translations, give isomorphisms between all the

tangent spaces. This implies, by the biholomorphicity of the translations, that the tangent bundle of A, needs to be free. In particular its dual is free as well, and therefore the canonical bundle is trivial.
This implies that in order to prove Iitaka inequality, we just need to show that for the general fibre F, $\kappa(F) = 0$ and therefore it suffices to show that the rank of $\mathcal{F}_m$ is at each point 1 for at least one m.

Definition 3.4.1. Let V be a smooth projective variety. The Albanese variety of V is an abelian variety A together with a morphism

$$\alpha : V \to A$$

such that for every abelian variety B and every morphism

$$\beta : V \to B$$

there exists a unique

$$f : A \to B$$

such that $\beta = f \circ \alpha$.

Theorem 3.4.2. *Let V a smooth projective variety. Then there exists an Albanese variety (A, α) for V. A is uniquely determined up to a birational isomorphism and f is uniquely determined up to a translation.*

Proof. Omitted. See [17], Chapter II, Section 3, Theorem 11. □

Generic Vanishing and Unipotency

Consider the following

$$V^0(A, \mathcal{F}_m) = \{P \in \mathrm{Pic}^0(A) : H^0(A, \mathcal{F}_m \otimes P) \neq 0\} =$$
$$= \{P \in \mathrm{Pic}^0(A) : H^0(X, \omega_X^{\otimes m} \otimes f^*P) \neq 0\} \subseteq \mathrm{Pic}^0(A).$$

We have that

Theorem 3.4.3. *Let X be a smooth projective variety. Then, for every $m \in \mathbb{N}$ the locus*

$$\{P \in Pic^0(X) : H^0(X, \omega_X^{\otimes m} \otimes P) \neq 0\} \subseteq Pic^0(X)$$

is a finite union of abelian subvarieties translated by points of finite order.

Remark 3.4.4. Theorem 3.4.3 implies that also $V^0(A, \mathcal{F}_m)$ is also a finite union of abelian subvarieties translated by points of finite order. Indeed, by Example 1.2.17, as f is an algebraic fibre space,

$$f^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$$

is injective.
More precisely we can prove that

$$V^0(A, \mathcal{F}_m) = \{\mathcal{O}_A\}.$$

Indeed, as we assume $P_m(X) = 1$, then $\mathcal{O}_A \in V^0(A, \mathcal{F}_m)$. Take $s_0 \in H^0(X, \omega_X^{\otimes m})$ any non-zero section and let $P \in V^0(A, \mathcal{F}_m)$ a non-trivial line bundle. By the previous Theorem we can assume P has finite order $d \neq 1$. Take

$$s_1 \in H^0(A, \mathcal{F}_m \otimes P) = H^0(X, \omega_X^{\otimes m} \otimes f^*P) \neq 0.$$

Then, $s_0^{\otimes d}, s_1^{\otimes d}$ are linearly independent global sections of $\omega_X^{\otimes dm}$. This contraddicts the fact that $P_{dm}(X) \leq 1$.

Definition 3.4.5. A coherent sheaf $\mathcal{F}$ on A is called a *GV-sheaf* if for every $i \in \mathbb{N}$

$$V^i(A, \mathcal{F}) = \{P \in \mathrm{Pic}^0(A) : H^i(A, \mathcal{F} \otimes P) \neq 0\}$$

is such that $\mathrm{codim} V^i(A, \mathcal{F}) \geq i$.

Theorem 3.4.6. *Let $f : X \to A$ be an algebraic fibre space from a smooth projective variety to an abelian variety. Then for every $m \in \mathbb{N}$, the sheaf $\mathcal{F}_m$ is a GV-sheaf.*

Lemma 3.4.7. *Let X be a smooth projective variety and $\mathcal{F}$ a GV-sheaf on X. Then $\mathcal{F}$ is zero if and only if $V^0(X, \mathcal{F}) = \emptyset$.*

Definition 3.4.8. A vector bundle E on A is called *unipotent* if it has a filtration

$$0 = E_0 \subseteq \cdots \subseteq E_n = E$$

such that $\frac{E_i}{E_{i-1}} \cong \mathcal{O}_A$ for every $1 \leq i \leq n$. It is called *homogeneous* if it has a filtration

$$0 = E_0 \subseteq \cdots \subseteq E_n = E$$

such that $\frac{E_i}{E_{i-1}} \in \mathrm{Pic}^0(A)$. Is *decomposable* if there are vector bundles E_1, E_2 on A such that

$$E \cong E_1 \oplus E_2$$

indecomposable if it is not decomposable.

Theorem 3.4.9. *It holds that:*

i if $\mathcal{F}_m \neq 0$ for some $m \in \mathbb{N}$, then it is homogeneous and indecomposable.

ii if $H^0(X, \omega_X^{\otimes m}) \neq 0$ for some $m \in \mathbb{N}$, then $\mathcal{F}_m$ is indecomposable and unipotent.

Singular Hermitian Metrics on Pushforwards of Pluricanonical Bundles

We now state the main analytic results that are necessary to conclude the proof. We assume all the basic definitions.

Definition 3.4.10. Given a singular hermitian metric h on a line bundle L, its *multiplier ideal sheaf* is the sheaf of the functions that are locally square-integrable with respect to h and it is denoted as $\mathcal{I}(h)$.

Theorem 3.4.11. *Let $f : X \to Y$ be a projective morphism of smooth varieties, and let (L, h) be a line bundle on X together with a singular Hermitian metric of semi-positive curvature. Then the torsion-free sheaf $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ has a canonical singular Hermitian metric with semi-positive curvature.*

Theorem 3.4.12. *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties. Let (L, h) be a line bundle on X with a singular Hermitian metric with semi-positive curvature and define $\mathcal{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$. Then*

i if $c_1(\det \mathcal{F}) = 0 \in H^2(Y, \mathbb{R})$ then the torsion-free sheaf $\mathcal{F}$ is locally free and the singular Hermitian metric of Theorem 3.4.11 is smooth and flat.

ii Every non-zero morphism

$$\mathcal{F} \to \mathcal{O}_Y$$

is split surjective.

Now, in the setting of an algebraic fibre space these results are particularly interesting. Let $f : X \to Y$ be an algebraic fibre space and F be a general fibre of the fibration. Then, for every $m \in \mathbb{N}$ such that $P_m(F) \neq 0$, the space of m-canonical forms on the smooth fibres of f induce in a canonical way a singular hermitian metric with semi-positive curvature on $\omega_{X/Y}$, called the *m-th Narashiman-Simha metric*. Denote with h the singular Hermitian metric induced on $L = \omega_{X/Y}^{\otimes m}$. By construction, the inclusion

$$f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)) \subseteq f_*(\omega_{X/Y} \otimes L) = f_*\omega_{X/Y}^{\otimes m}$$

is generically an isomorphism, so we can apply the Theorems 3.4.11 and 3.4.12.

Corollary 3.4.13. *Let $f : X \to Y$ an algebraic fibre space.*

i For any $m \in \mathbb{N}$, the torsion-free sheaf $f_*\omega_{X/Y}^{\otimes m}$ has a canonical singular Hermitian metric with semi-positive curvature.

ii If $c_1(\det f_*\omega_{X/Y}^{\otimes m}) = 0 \in H^2(Y, \mathbb{R})$, then $f_*\omega_{X/Y}^{\otimes m}$ is locally-free and the singular Hermitian metric is smooth and flat.

iii Every non-zero morphism

$$f_*\omega_{X/Y}^{\otimes m} \to \mathcal{O}_Y$$

is split surjective.

3.4.2 Kawamata's Results

In this subsection we present, mainly without proofs, some results shown by Yujiro Kawamata in [15] that will be crucial in the proof of the general case. The main reference is [15].

Theorem 3.4.14. *Let $f : X \to A$ be a finite morphism from a smooth projective variety to an abelian variety. Then $\kappa(X) \geq 0$ and there are an abelian subvariety B of A, étale coverings $\widetilde{X}, \widetilde{B}$ of X and B respectively and a smooth projective variety $\widetilde{Y}$ such that*

i $\widetilde{Y}$ is finite over $\frac{A}{B}$.

ii $\widetilde{X}$ is isomorphic to $\widetilde{B} \times \widetilde{Y}$.

iii $\kappa(\widetilde{Y}) = \dim \widetilde{Y} = \kappa(X)$.

Proof. Omitted. See [15] Theorem 13. ∎

Theorem 3.4.15. *Let $f : X \to Y$ an algebraic fibre space such that $\kappa(X) \geq 0$ and Y is of general type. Then*

$$\kappa(X) = \kappa(Y) + \kappa(F)$$

where F is a general fibre of f.

Proof. Omitted. See [15] Theorem 3. □

Theorem 3.4.16. *Let X be a smooth projective variety such that $\kappa(X) = 0$. Then the Albanese map of X*

$$\alpha : X \to A(X)$$

is an algebraic fibre space.

Proof. Let Z be the image of the Albanese map, and let

$$X \to Y \to Z$$

be its Stein factorization. Thus the following diagram is commutative

$$\begin{array}{ccccc} X & & \xrightarrow{\ \alpha\ } & & Z & \xrightarrow{\ i\ } & A(X) \\ & \searrow^{f} & & \nearrow_{g} & & & \\ & & Y & & & & \end{array}$$

By Theorem 3.4.14 there exists an étale covering $\widetilde{Y}$ of Y such that

$$\widetilde{Y} = B \times W$$

where B is an abelian variety and W is a variety of general type. Consider $\widetilde{X} = X \times_Y \widetilde{Y}$. As also the projection

$$\widetilde{X} \to X$$

is étale, by Proposition 3.3.5 we conclude that $\kappa(\widetilde{X}) = \kappa(X) = 0$. Obviously the projections

$$\widetilde{Y} \to W,\ \widetilde{X} \to \widetilde{Y}$$

are algebraic fibre spaces, so is their composition. By Theorem 3.4.15 the following relations hold

$$0 = \kappa(\widetilde{X}) = \kappa(F) + \kappa(W) \geq \kappa(W) = \dim W \geq 0$$

where F is a general fibre of the composition. The above inequalities show that W is a point, therefore $\widetilde{Y}$ is an abelian variety. Thus Y itself is abelian. By the Universal Property of the Albanese map, f factors through $A(X)$, thus there exists

$$\varphi : A(X) \to Y$$

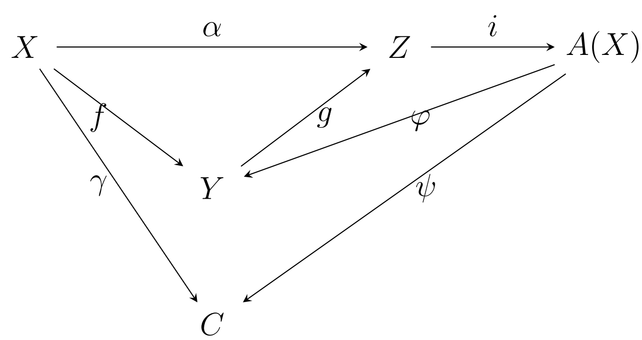
such that

$$\varphi \circ \alpha = f.$$

Consider then C another abelian variety, and

$$\gamma : X \to C$$

a morphism. The morphism γ factors through α, so we have a commutative diagram



But now, the map $\psi \circ i \circ g = \eta$. Then

$$\gamma = \eta \circ f$$

hence f is an Albanese map for X. This implies that Y and $A(X)$ are isomorphic. As f is a fibre space, so is α. □

3.4.3 The Main Theorems

Theorem 3.4.17. *Let $f : X \to A$ be an algebraic fibre space with A an abelian variety and $\kappa(X) = 0$. Then*

$$\mathcal{F}_m = f_* \omega_X^{\otimes m} \cong \mathcal{O}_A$$

for every $m \in \mathbb{N}$ such that $P_m(X) \neq 0$.

Proof. By Theorem 3.4.9 $\mathcal{F}_m$ is an indecomposable unipotent vector bundle on A. In particular its determinant is trivial.
By Corollary 3.4.13 it also has a smooth and flat singular Hermitian metric. All of this implies that $\mathcal{F}_m$ is a successive extension of the trivial bundle $\mathcal{O}_A$ and can be split into direct summands through the metric. Thus

$$\mathcal{F}_m \cong \mathcal{O}_A^{\oplus r}$$

where $r \geq 1$ is the rank of $\mathcal{F}_m$, but by indecomposability, $r = 1$. □

Definition 3.4.18. A variety Y is said to have *maximal Albanese dimension* if its Albanese map is generically finite.

Theorem 3.4.19. *Let $f : X \to Y$ be an algebraic fibre space such that Y has maximal Albanese dimension. Then*

$$\kappa(X) \geq \kappa(Y) + \kappa(F)$$

where F is a general fibre of the fibration.

Proof. Assume that $\kappa(X) = -\infty$. If, by contraddiction $\kappa(F) \neq -\infty$, there exists $m > 0$ such that $P_m(F) > 0$ and therefore, $f_* \omega_X^{\otimes m} \neq 0$.
Let $Y \to A$ be the Albanese map of Y and let $g : X \to A$ the composed morphism. As, F is an irreducible component of the general fibre of g, it also holds that $g_* \omega_X^{\otimes m} \neq 0$.
By Theorem 3.4.6 this is a GV-sheaf and by Lemma 3.4.7 the set

$$V^0(A, g_* \omega_X^{\otimes m}) = \{P \in \mathrm{Pic}^0(A) : H^0(A, g_* \omega_X^{\otimes m} \otimes P) \neq 0\}$$

is non-empty.
By Theorem 3.4.3 and Remark 3.4.4 there exists a torsion point $P \in V^0(A, g_* \omega_X^{\otimes m})$. Fix such a point P of order k. In particular

$$h^0(A, g_* \omega_X^{\otimes m}) = h^0(A, g_* \omega_X^{\otimes m} \otimes P) \neq 0;$$

thus

$$h^0(X, \omega_X^{\otimes km}) = h^0(X, (\omega_X^{\otimes m} \otimes g^*P)^{\otimes k}) = h^0(A, (g_*\omega_X^{\otimes m} \otimes P)^{\otimes k}) \neq 0$$

and this contraddicts the fact that $\kappa(X) = -\infty$.
Now assume $\kappa(X) \geq 0$ and $\kappa(Y) = 0$. By Theorem 3.4.16, since Y is of maximal Albanese dimension, it is birational to its Albanese variety. So, birationally, we can assume that Y is abelian.
Consider now

$$h : X \to Z$$

the Iitaka fibration associated to ω_X. We may assume that Z is smooth. Take G a general fibre of h; then $\kappa(G) = 0$ and again by Theorem 3.4.16, the Albanese map of G is surjective. This implies that $f(B)$ is an abelian variety as well. Indeed, if α_G is an Albanese map for G, then the restriction of f factors through α_G, hence there exists a morphism of varieties

$$\varphi : A(G) \to f(B)$$

such that

$$f_{|G} = \varphi \circ \alpha_G$$

By a known fact about abelian varieties, φ is, up to a translation, a group homomorphism as well. Thus, up to choose another Albanese map, we may assume that φ is also a group homomorphism and therefore, $f(B)$ is an abelian subvariety of Y.
Take the Stein factorization

$$G \to B' \to B$$

of the restriction of f. As

$$B' \to B$$

is étale over B, B' is abelian as well. So the fibre space

$$G \to B'$$

is onto an abelian variety and $\kappa(G) = 0$. Its general fibre is $H = F \cap G$: indeed it is contained in both G and F and the converse inclusion is as well trivial. By Theorem 3.4.17 $\kappa(H) = 0$.

On the other hand, H is also a connected component of the general fibre of the restriction of h to F. Then, by the Easy Addition Formula

$$\kappa(F) = \kappa(H) + \dim h(F) = \dim h_{|F}(F) \leq \dim Z = \kappa(X).$$

Notice that in order to apply the Easy Addition Formula we need to assume that $h(F)$ is smooth, but up to a desingularization (which is a birational map) we can make this assumption.
In the end consider the general case. Since Y has maximal Albanese dimension, its Albanese map is a generically finite morphism. Up to consider the finite part of the Stein Factorization of such map, we can assume that it is already finite.
By Theorem 3.4.14 there exists an étale covering $\widetilde{Y}$ of Y, an abelian subvariety K of $A(Y)$, an étale covering $\widetilde{K}$ of K and a general type variety Z such that

$$\widetilde{Y} = Z \times \widetilde{K}.$$

So, up to consider this étale covering (that does not change the Kodaira dimension of any of the involved varieties), and for X a desingularization of the fibre product that this étale covering would induce, we can assume that $Y = Z \times K$ for a general type variety Z and an abelian variety K.
It holds that

$$\kappa(Y) = \kappa(Z \times K) = \kappa(Z) + \kappa(K) = \kappa(Z) = \dim Z.$$

The first projection induces a morphism

$$X \to Z.$$

Let E be a general fibre of such morphism, then there exists an induced morphism

$$E \to K.$$

The general fibre of this last morphism is F, indeed, for $k \in K$ general, the fibre in E is sent through f to a fixed general point, thus the fibre is F.
By the case $\kappa(Y) = 0$, we have $\kappa(E) \geq \kappa(F)$, thus

$$\kappa(X) = \kappa(E) + \kappa(Z) \geq \kappa(F) + \kappa(Z) = \kappa(F) + \kappa(Y).$$

And the first equality comes from Theorem 3.4.15. $\square$

Conclusion

As of today, thanks to Mori, the main approach to the classification problem relies on the Minimal Model Program.
The main ideas in MMP can be summarized as follows: fix X a projective variety. Then we distinguish two cases.
If the Kodaira dimension of X is negative, then we want to find projective varieties X', Y with X' birational to X, and a fibration

$$f : X' \to Y$$

such that, if F is a general fibre of f, the anticanonical bundle of F is ample. This means that F is a *Fano variety*.
Otherwise if the Kodaira dimension of X is non-negative, we look for a projective variety X' that is birational to X and has nef canonical bundle. If such X' exists, is said to be a *minimal model* for X.
In [20] Mori proved, under some additional hypotheses, the existence of minimal models for 3-folds. In general, the existence of minimal models is still a widely open problem. In particular, there are many conjectures that are still open about these objects.
One of the most important is the Generalized Abundance Conjecture (for a reference, see [7] Conjecture 2.3.59). Still in [7], it also proved that, the Generalized Abundance Conjecture implies the Iitaka Conjecture (Theorem 1.2.3 of [7]).
Therefore, it is very natural, in order to solve Iitaka Conjecture, to study the Generalized Abundance Conjecture and the Minimal Model Program. Nowadays, in fact, this is one of the most successful approaches to $C_{n,m}$; a good instance of this is Birkar's paper [2].

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