Optimizing Multi-Item Lot-Sizing Problem: A Study on Aggregate Service Levels, Piecewise Linear Approximations, and Fix-and-Optimize Heuristics

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Abstract

Optimizing Multi-Item Lot-Sizing Problem: A Study on Aggregate Service Levels, Piecewise Linear Approximations, and Fix-and-Optimize Heuristics

Tanvir Sidhu

In this research thesis, we address the intricate challenges presented by multi-item lot-sizing problems in production environments, considering stochastic demand, capacity constraints, and inventory limitations. Our objective is to formulate an optimized production schedule, drawing inspiration from existing literature models. We introduce two mathematical models for the lot-sizing problem, incorporating aggregate service levels β and γ, and employ novel piece-wise linear approximations to address and extend existing formulations. Our research presents an iterative optimization-based solution approach for the piecewise linear approximation of the stochastic lot-sizing problem. This process involves breaking down the overall planning horizon into smaller intervals, creating a more manageable planning horizon, and iteratively addressing a series of sub-problems. Extensive computational experiments explore the implications of aggregate service levels, comparing the solution quality of the actual piecewise linear approximation model and the Fix-and-Optimize heuristic using four different interval lengths. Results highlight the nuanced relationship between interval lengths and computational efficiency, emphasizing the strategic importance of selecting intervals aligned with operational objectives. For instance, solving the piecewise linear approximation model for the β service level with a higher interval length (9) reduces computational time by 60% on average, with a corresponding average increase of 4.5% in the relative gap (cost). Similarly, for the γ service level, computational time decreases by 38% on average, with an average relative gap increase of 3.7%.

Keywords: Lot sizing, Aggregate service level, Stochastic demand, Mixed integer programming, Fix-and-optimize heuristic

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Chapter 1

Introduction

1.1 Introduction

In contemporary production systems, the complex challenge of allocating shared resources among diverse products is pervasive, especially when confronted with constrained capacities. This problem is especially apparent in various industries where strict regulations, varying production requirements, and diverse product categories necessitate a sophisticated approach to resource allocation *Supply Chain Management and Advanced Planning: Concepts, Models, Software, and Case Studies* (2015). Three distinct examples underscore the complexities inherent in balancing production demands across diverse product lines within shared resource environments. Examples from automotive manufacturing, pharmaceuticals, and the food industry illustrate the intricacies of balancing production demands within shared resource environments. For instance, automotive manufacturers must navigate the production of luxury and compact models, each with unique requirements Kauder and Meyr (2009). In pharmaceuticals, producing generic medicines alongside specialized vaccines adds complexity to resource allocation. The food industry faces challenges concurrently producing staple items and perishable goods Sousa, Liu, Papageorgiou, and Shah (2011). Staple items follow continuous and high-volume production schedules, while perishable goods demand a more flexible and time-sensitive approach, necessitating optimized resource utilization *Supply Chain Management and Advanced Planning: Concepts, Models, Software, and Case Studies* (2015). These examples underscore the need for sophisticated resource allocation strategies to address the nuances of diverse production environments. Balancing these diverse production needs within shared production space highlights the complexity of resource allocation, emphasizing the need for nuanced strategies.

Within this context, we embark on constructing a stochastic model addressing the single-level and multi-product lot-sizing problem, taking into account capacity and inventory constraints. The objective is to formulate an optimized production schedule in response to stochastic demand, with a dual focus on minimizing incurred costs and maintaining aggregate service levels. This modeling approach aligns with the real-world challenges presented in diverse production environments, where resource allocation intricacies necessitate sophisticated strategies for effective and efficient production planning.

In lot sizing problems it is important to build a balance between customer demand satisfaction and overall cost management. The cost related to manufacturing in a supply chain can be defined under different categories such as production setup cost, inventory holding cost, and shortfall penalty cost incurred for the raw material and semi-finished goods procured from suppliers Sereshti, Adulyasak, and Jans (2021a). While Insufficient inventory levels can lead to sale loss and shortages but having an excess inventory of more than the predefined level can lead to an increase in holding cost. Furthermore, in each planning period in which production occurs, a setup cost is incurred which depends on the number of times the production setup has to be performed in a given planning horizon. The objective of the lot sizing problem is to determine an optimal production plan which aims to minimize the cost related to production setup and inventory and satisfy the known demand over a finite and discrete time horizon Pochet and Wolsey (2006a).

The majority of the existing literature on lot sizing primarily focuses on scenarios where all data are known deterministic in advance. Common industrial planning practices often involve utilizing forecasting methods to generate deterministic time series of anticipated future demands. To accommodate uncertainty, a fixed amount of inventory, known as safety stock, is typically reserved. The calculation of this safety stock is frequently based on simple heuristic rules, such as multiplying the standard deviation of demand during the at-risk period by a specific quantile from the standard normal distribution Tempelmeier (2011). However, this approach often falls short of achieving the desired service level. Moreover, the potential influence of lot sizes on risk mitigation tends to be disregarded. For instance, it might be optimal to forgo safety stock entirely when dealing with larger lot sizesTempelmeier and Hilger (2015b). Recognizing the limitations of models that exclude uncertainty, it's clear that their decisions are suboptimal compared to models that explicitly consider uncertainty in their formulation Stadtler and Meistering (2019). Thus, arises the need for methodologies that can effectively manage the risk introduced by uncertainty while concurrently optimizing time-dependent lot sizing and buffer stock decisions within the dynamic lot sizing problem Sereshti, Adulyasak, and Jans (2021b). We will consider a static strategy for our lot sizing problem. Under static uncertainty decisions for all periods are made at the planning horizon and cannot be changed Bookbinder and Tan (1988). The stochastic lot-sizing problem is an extension of the deterministic case and we will determine the production/procurement quantity and schedule under stochastic demand.

The stochastic lot-sizing problem extends beyond the deterministic scenario by involving the determination of production schedules and quantities to meet uncertain demand within a defined planning horizon. In scenarios where maintaining a certain service level is imperative, the central objective revolves around minimizing the overall anticipated cost while adhering to specific demand fulfillment standards Tempelmeier (2007). This is achieved through the incorporation of chance constraints, where the aim is to ensure that the probability of attaining a predetermined service level is greater than or equal to a specified threshold Brahimi, Absi, Dauzère-Pérès, and Nordli (2017). The definition of this service level can be tailored in accordance with the model's objective.

For instance, the β-service level, or fill rate, quantifies the proportion of direct demand fulfillment from existing stock Tempelmeier and Hilger (2015b). Normally, the service levels are applied independently to individual products. Nonetheless, the current research delves into an aggregate service level concept, encompassing multiple products. This approach gains significance in practical scenarios characterized by product diversity Sereshti et al. (2021a). For instance, consider a technology company offering a range of laptop models with varying specifications and configurations. In this scenario, the company might opt to establish an aggregate service level across all laptop models to ensure an overall satisfactory customer experience. Simultaneously, the company could implement individual service levels for each laptop model's distinct configurations, allowing more flexibility. This approach recognizes that some configurations might have higher demand predictability and warrant stricter service levels, while others could have more uncertainty and still meet the aggregate target without sacrificing customer satisfaction. Consider a scenario where a multinational electronics retailer with an extensive product lineup, including smartphones, tablets, laptops, and accessories. In its pursuit of optimizing customer satisfaction while managing costs, the company decides to explore aggregate service levels. Traditionally, ensuring a high aggregate service level like 95% would involve setting the same stringent 95% service level for each product category.

However, the retailer recognizes that different product categories have varying degrees of demand predictability and customer expectations. By adopting a more nuanced approach, the retailer can set an aggregate service level of 95% while strategically assigning distinct individual service levels for each product category. For example, smartphones, which have relatively stable demand patterns, might adhere to a 95% individual service level, maintaining high availability. Meanwhile, less predictable product categories like cutting-edge technology gadgets might be assigned a slightly lower individual service level of 90%, allowing for more flexibility in stock management. This strategy introduces a dynamic balance, wherein certain products adhere to stricter service levels to meet demanding customer expectations, while others, where some level of uncertainty is acceptable, have slightly relaxed individual service levels. As a result, the company achieves the desired overall service level while optimizing inventory costs across its diverse product range.

1.1.1 Contribution

This paper shares close ties with the works of Tempelmeier and Hilger (2015b) and Sereshti et al. (2021b). We adopted their modeling approach to establish production schedules and quantities capable of meeting stochastic demand over a finite planning horizon. Our contribution lies in the proposal of two distinct mathematical models, each catering to different aggregate service levels and employing piece-wise linear approximations. The primary contributions of our research manifest in both the formulation of the models and the solution methodology. From a model formulation perspective, we address the aggregate service level in the stochastic lot sizing problem, presenting mathematical formulations for two distinct service level types (β and γ). Notably, these aggregate service levels provide the flexibility to select individual service levels for each item, allowing them to be used alongside the minimum individual service level constraints found in the existing literature Sereshti et al. (2021b). The piece-wise linear approximation formulations proposed for the β and γ service levels extend existing formulations. Additionally, we introduce a novel iterative optimization heuristic (fix-and-optimize) tailored specifically for these problems Ouhimmou, D'Amours, Beauregard, Ait-Kadi, and Chauhan (2008), offering a comprehensive examination of each formulation's performance. Our paper further contributes through extensive computational experiments, exploring the implications of the aggregate service level in diverse scenarios. A notable strength lies in our adoption of a unified simulation procedure for evaluating the approximation formulations, ensuring a fair comparison of different models and two service levels.

1.1.2 Research project structure

The subsequent sections of this manuscript are structured as follows. Within this chapter, we delve into an extensive review of the academic literature concerning the focal points of this research: Lot sizing and Service levels. In Chapter 2, we detail the mathematical models, their formulations, and the devised methodology for solving these models, specifically tailored for two distinct aggregate service levels. Chapter 3 outlines the proposed iterative optimization-based heuristic (Fix-and-optimize). The details of the computational results are presented in Chapter 4. The thesis is drawn to a close in Chapter 5, which summarizes findings and suggests avenues for future research.

1.2 Literature Review

1.2.1 Lot Sizing

Lot sizing problem, a prevalent challenge in real-life production processes, arises when the initiation of these processes depends on the completion of setup activities involving essential resources, accompanied by associated setup time and costs. This situation necessitates a decision-making process to determine whether it is advantageous to produce future demands in advance, aiming to minimize the frequency of setups *Handbook of Stochastic Models and Analysis of Manufacturing System Operations* (2013). In the literature on lot-sizing problems, the technical structure of existing models can be categorized based on two main dimensions. Firstly, models can be distinguished by whether they account for changes in model parameters over time, leading to classifications such as stationary models (with fixed parameters) and dynamic models (with varying parameters). Secondly, another crucial aspect is whether uncertainty is taken into consideration within the model. This results in classifications such as deterministic models (without uncertainty) and stochastic models (with uncertainty) Glock, Grosse, and Ries (2014).

In the domain of lot-sizing problems, various types of models have been developed to address different characteristics and considerations. The classification of lot-sizing problems can be expanded to include specific types, such as economic order quantity (EOQ) Harris (n.d.), dynamic lot sizing Helber et al. (2013), periodic lot sizing, stochastic lot sizing Sereshti et al. (2021b), and capacitated lot sizing Tavaghof-Gigloo and Minner (2021). These types offer distinct perspectives on the lot-sizing problem, taking into account factors such as demand variability, capacity constraints, and dynamic decision-making. Further exploration of these types can provide valuable insights into the diverse approaches and methodologies employed to tackle lot-sizing challenges in operations research Sethi, Yan, and Zhang (2005).

Capacitated lot-sizing problems with dynamic and stochastic demand have been widely studied in the literature. We can refer to the work of Brandimarte (2006) Tempelmeier and Herpers (2011) Helber et al. (2013), where they have considered uncertain random demand per period of a known probability distribution. As stated by Wemmerlöv (1981), the main reason for such a vast literature on the lot-sizing problem is due to its role in the material requirement planning system.

In the literature, we observe a rapidly increasing amount of papers on stochastic capacitated lot-sizing problems with service level constraints. A detailed discussion of various lot sizing models can be found in Pochet and Wolsey (2006b). However, only a limited amount of researchers have undertaken a static strategy for comparing the output of various types of service levels on both aggregate and individual levels Tempelmeier and Hilger (2015a). The most prevalent method of managing inventory with multiple products is to establish specific service levels or criteria for meeting demand, the idea of defining aggregated service levels has been investigated in the inventory management literature Sereshti et al. (2021a). Kelle (1989) stated that assigning fixed service levels for groups of items with varying demand, cost, and delivery characteristics in order to achieve an overall aggregate service level can be a difficult task. It is important to consider the unique characteristics of each item when determining the appropriate service level and allocation of safety stock. A common approach to deal with a large number of stock-keeping units (SKUs) in the inventory management system is the ABC analysis or classification of SKUs. A number of authors Partovi and Burton (1993); Ramanathan (2006); Zhou and Fan (2007) have considered the use of multiple criteria with the fixed service level in the ABC classification method but Teunter, Babai, and Syntetos (2010) introduced a more efficient approach for ABC classification by using aggregate service level for all SKUs. Recently Stadtler and Meistering (2019) compared the results of a deterministic model for the capacitated lot sizing problem with the results of a rolling schedule strategy addressing a stochastic lot-sizing problem with a given service level but they considered only one service level (beta) for the comparison and they did not consider the setup time for their model. Escalona, Angulo, Weston, Stegmaier, and Kauak (2019) analyzes the impact of two service-level measures on the design of a critical-level policy for fast-moving items. The study uses various service-level constraints to determine the optimal parameters of a continuous review policy with a constant threshold value C to ration the low-priority class. Brandimarte (2006) formulated a stochastic version of the multi-item Capacitated Lot-Sizing Problem (CLSP), where demand uncertainty is explicitly modeled through a scenario tree. The study proposes a plant-location-based model formulation and a heuristic solution approach based on a fix-and-relax strategy. It is important to note that joint and aggregate service levels are different concepts Sereshti et al. (2021b). While both consider multiple products simultaneously, joint service levels refer to situations where service level requirements are imposed on all products simultaneously based on their joint distribution, whereas aggregate service levels refer to constraints imposed on the aggregated values of service levels for individual products. Using aggregate service levels allows for scalability in dealing with practical scenarios where companies need to ensure that a group of products collectively satisfies the service level requirement. Joint service levels result in more strict constraints compared to individual service levels, while aggregate service levels provide more relaxed and flexible constraints. Moreover, joint service level models can be challenging to solve and are not tractable, while aggregate service level models are easier to solve Jiang, Xu, Shen, and Shi (2017). The lot sizing models can be solved using three different strategies: static uncertainty, dynamic uncertainty, and static-dynamic uncertainty as stated by Bookbinder and Tan (1988). The study formulates a stochastic-demand version of the single-stage lot-sizing problem with a service-level constraint on the probability of stock out. The static uncertainty strategy is shown to be the most straightforward to modify and roll along as new demands become known and is computationally simple. The equivalent deterministic problem with time-varying demands for this strategy has optimal or good heuristic solutions. Tarim and Kingsman (2004); Tempelmeier (2007); Tunc, Kilic, Tarim, and Eksioglu (2014); Tunc, Kilic, Tarim, and Rossi (2018) all these studies proposed formulations to solve the multi-period single-item inventory lot-sizing problem with stochastic demands under the "static-dynamic uncertainty" strategy of Bookbinder and Tan (1988). In our research under the static uncertainty strategy, we will use the piecewise approximation method as used by Rossi, Kilic, and Tarim (2015) for the static-dynamic uncertainty strategy. Many studies are done using the static uncertainty strategy and different service levels Helber et al. (2013); Tempelmeier (2011); Tempelmeier and Herpers (2010, 2011); Tempelmeier and Hilger (2015b) and in this study, we evaluate the impact on total cost using aggregate service level on multiple products. As explained by Tunc, Kilic, Tarim, and Eksioglu (2013) when nervousness is low, the static uncertainty strategy is shown to be better in coordinating supply chain inventories compared to dynamic uncertainty and static-dynamic uncertainty strategies. The literature describes nervousness under two categories one is setup-oriented and the other is quantity-oriented, since the production plan with respect to setups and production quantity remains the same, adopting the static strategy is the favorable option Sereshti et al. (2021b).

The literature on inventory management with random demand frequently discusses different service level measures Helber et al. (2013); Tempelmeier (2007). The first is the alpha (α) service level, which ensures that the probability of no stock out during the production or procurement cycle is greater than α but it does not provide any severity of a stockout event. The second measure is the beta (β) service level, also known as the fill rate, which represents the proportion of demand directly filled from stock without backlogging and does not reflect customer waiting time or production quantity determination. The third measure is the gamma (γ) service level, which is the proportion of expected backlog to expected demand. It considers backlog and waiting time, but it can be negative or undefined in periods with small expected demand. Finally, the delta (δ) service level ensures that the proportion of the total expected backlog to the maximum expected backlog is less than or equal to $1 - \delta$ and takes both the backlog and waiting time into consideration Helber et al. (2013); Sereshti et al. (2021b). These measures are calculated for each product separately. Table 1.1 presents a concise summary of the existing literature concerning the lot-sizing problem with service level constraints, along with the associated solving strategies.

1.2.2 Service Levels

There are different service levels that are analyzed and discussed in the literature. In this research, we will investigate two types of service levels β and γ.The rationale behind selecting these two service levels is rooted in their relevance to addressing uncertain demand in the deterministic multi-item lot sizing problem. These service levels offer a thorough understanding of inventory management dynamics, particularly in the context of static strategies, allowing for a comprehensive understanding of decision-making processes in the face of uncertain demand scenarios. Similar to Sereshti et al. (2021b), our research will also focus on the static strategy in multi-item lot-sizing problems, where all decisions are made upfront and cannot be modified once demands are realized. The study assumes independent demand distributions for each product, and in the event of stock-outs, unmet demand is backlogged and fulfilled at the earliest opportunity. These assumptions are applied to address the uncertain demand in the deterministic multi-item lot sizing problem.

Table 1.2 defines the two service levels in separate and aggregate levels. The service levels are

Table 1.1: Overview of literature on lot sizing with aggregate service levels

Year	Authors	Strategy	Service level type
1988	Bookbinder and Tan (1988)	Static Uncertainty	β and γ
2004	Tarim and Kingsman (2004)	Static-Dynamic Uncertainty	α
2007	Tempelmeier (2007)	Static-Dynamic Uncertainty	β and α
2010	Tempelmeier and Herpers (2010)	Static Uncertainty	β
2011	Tempelmeier (2011)	Static Uncertainty	β
2011	Tempelmeier and Herpers (2011)	Static Uncertainty	β
2013	Helber et al. (2013)	Static Uncertainty	δ
2013	Gade and Küçükyavuz (2013)	Deterministic	α and β
2014	Tunc et al. (2014)	Static-dynamic Uncertainty	α
2015	Tempelmeier and Hilger (2015b)	Static Uncertainty	β
2015	Rossi, Tarim, Prestwich, and Hnich (2014)	Static-dynamic Uncertainty	β
2018	Tunc et al. (2018)	Static-dynamic Uncertainty	α and β
2018	Gruson, Cordeau, and Jans (2018)	Deterministic	β and δ
2019	Stadtler and Meistering (2019)	Deterministic	β, α and δ
2020	Tavaghof-Gigloo and Minner (2021)	Static Uncertainty	β
2021	Tunc (2021)	Static Uncertainty	α
2021	Sereshti et al. (2021b)	Static Uncertainty	β, α, δ and γ

defined over the whole planning cycle *Handbook of Stochastic Models and Analysis of Manufacturing System Operations* (2013). In this research, we are not considering service levels that are imposed for each planning period. We assume $\mathbf{K}$ as the set of products and $\mathbf{T}$ as the set of time periods. The first service level mentioned in the Table 1.2 is the β service level, which focuses

Table 1.2: Service levels and their separate and aggregate forms

Service Level (SL)	Separate SL	Aggregate SL
β	$\frac{\sum_{t\in T} E[\overline{BO}_{kt}]}{\sum_{t\in T} E[\overline{D}_{kt}]} \leq 1-\beta \quad \forall k \in K$	$\frac{\sum_{t\in T}\sum_{k\in K} E[\overline{BO}_{kt}]}{\sum_{t\in T}\sum_{k\in K} E[\overline{D}_{kt}]} \leq 1-\beta$
γ	$\frac{\sum_{t\in T} E[\overline{BL}_{kt}]}{\sum_{t\in T} E[\overline{D}_{kt}]} \leq 1-\gamma \quad \forall k \in K$	$\frac{\sum_{t\in T}\sum_{k\in K} E[\overline{BL}_{kt}]}{\sum_{t\in T}\sum_{k\in K} E[\overline{D}_{kt}]} \leq 1-\gamma$

on quantity and considers expected backorders $E[\overline{BO}_{kt}]$ and expected demand $E[\overline{D}_{kt}]$ for a product k in period t. The aggregate service level is calculated as 1 minus the total expected backorders divided by the total average demand across all products and periods Sereshti et al. (2021b). The second service level is the γ service level which reflects backlog quantity and time. It can be calculated as an average over the entire planning horizon and is equal to one minus the total expected backlog divided by total expected demand Helber et al. (2013), where $E[\overline{BL}_{kt}]$ is the expected backlog for product k in period t and $E[\overline{D}_{kt}]$ is the expected mean demand for product k in period t.

Chapter 2

Model Formulation and Solution Methods

We first introduce the mathematical model based on the work of Sereshti et al. (2021b) addressing the stochastic lot-sizing problem, focusing on two distinct service level types β and γ at the aggregate level. The objective is to devise a production plan that minimizes both setup costs and expected inventory carrying costs. Expected inventory level and service level are nonlinear functions of demand and production quantity and therefore the resulting model is a non-linear stochastic capacitated lot-sizing model. To solve this model, a piecewise linear approximation method will be employed Tempelmeier and Hilger (2015b). This model enables the determination of optimal production quantities for each product, aiming to minimize costs and uphold aggregate service levels.

2.1 The non-linear stochastic capacitated lot-sizing problem with aggregate β service level

In this section, we explore the concept of aggregate β-service level, where we set a predefined percentage as a threshold for the total expected backorder divided by the total expected demand. The expected inventory and backorder in each planning period are influenced by the cumulative production, resulting in a non-linear relationship Tempelmeier and Hilger (2015b).

2.1.1 Mathematical model

The notations of related parameters and decision variables are present in Table 2.1. Referring to the model of Sereshti et al. (2021b), We can now state the stochastic capacitated lot sizing problem with aggregate β service level as follows:

Table 2.1: Notation used for the parameters and decision variables of the models with β service level

Indices and index sets	
K	Set of products ($k \in$ 1,.....,K)
T	Set of periods ($t \in$ 1,.....,T)
Deterministic parameters	
β	Target fill rate as an aggregate service level
cap_t	Available production capacity in period *t*
hc_{kt}	Inventory holding cost of product *k* in period *t*
I_{k0}	Initial inventory of product *k*
M	Big number
pt_{kt}	Production time of product *k* in period *t*
sc_{kt}	Setup cost for product *k* in period *t*
st_{kt}	Setup time for product *k* in period *t*
Random variables	
$\overline{BL}_{kt}$	Backlog of product *k* in period *t*
$\overline{BO}_{kt}$	Backorder of product *k* in period *t*
$\overline{D}_{kt}$	Demand of product *k* in period *t*
$\overline{I}_{kt}$	Physical inventory of product *k* in period *t*
Decision variables	
q_{kt}	Production quantity of product *k* in period *t*
y_{kt}	Binary setup variable of product *k* in period *t*, which is equal to 1 if there is a setup, 0 otherwise

$$Min \sum_{t \in T} \sum_{k \in K} (sc_{kt} y_{kt} + hc_{kt} E[\overline{I}_{kt}]) \tag{1}$$

Subject to:

$$\overline{I}_{k,t-1} + q_{kt} + \overline{BL}_{kt} = \overline{I}_{kt} + \overline{D}_{kt} + \overline{BL}_{k,t-1} \quad \forall t \in T, \forall k \in K \tag{2}$$

$$q_{kt} \leq M y_{kt} \quad \forall t \in T, \forall k \in K \tag{3}$$

$$\sum_{k \in K} (st_{kt}\, y_{kt} + pt_{kt}\, q_{kt}) \leq cap_t \quad \forall t \in T \tag{4}$$

$$\begin{aligned} E[\overline{BO}_{kt}] = E[\max\{0, \sum_{j=1}^{t} (\overline{D}_{kj} - q_{kj}) - I_{k0}\}] \\ - E[\max\{0, \sum_{j=1}^{t-1} \overline{D}_{kj} - \sum_{j=1}^{t} q_{kj} - I_{k0}\}] \qquad \forall t \in T, \forall k \in K \end{aligned} \tag{5}$$

$$\frac{\sum_{t \in T} \sum_{k \in K} E[\overline{BO}_{kt}]}{\sum_{t \in T} \sum_{k \in K} E[\overline{D}_{kt}]} \leq 1 - \beta \tag{6}$$

$$y_{kt} \in 0,1 \quad \forall t \in T, \forall k \in K \tag{7}$$

$$q_{kt} \geq 0 \quad \forall t \in T, \forall k \in K \tag{8}$$

$$\overline{I}_{kt} \geq 0 \quad \forall t \in T, \forall k \in K \tag{9}$$

$$\overline{BL}_{kt} \geq 0 \quad \forall t \in T, \forall k \in K \tag{10}$$

The objective function of the model 1 aims to minimize the combined cost which includes the setup costs and the expected inventory holding costs. Constraint 2 ensures that the flow of materials is balanced that is the sum of the opening inventory of a product *k* in period *t* and production quantity of a product *k* in period *t* are equal to the sum of closing inventory of a product in that period, the demand of a product *k* in period *t* and the difference between the backlog of a product *k* in period *t*-1 and the backlog of a product *k* in period *t*. Constraint (3) enforces setup when there is production, ensuring proper sequencing. Constraint (4) incorporates capacity limitations to restrict production quantities which means that the capacity needed for setups and production does not exceed the production capacity in that period. Constraint (5) is used to calculate the expected backorder level for each product in each period. The calculation considers the backlog in the current period and the unsatisfied cumulative demand until the previous period based on the FIFO assumption Sereshti et al. (2021b). Constraint (6) is used to maintain the desired service level which means it ensures an aggregate β service level. Constraint (7) specifies the valid range of decision variables that is it ensures the value of binary setup variables is either 1 or 0. Similarly Constraints (8), (9), and (10) show the valid ranges for the decision variables that is it ensures the non-negativity of the variables.

In this model, since the demand is uncertain, the expected value of inventory level can be calculated using Constraint (11) given below, instead of Constraint (2) Sereshti et al. (2021b).

$$E[\overline{I}_{kt}] = E[\max\{0, I_{k0} + \sum_{j=1}^{t}(x_{kj} - \overline{D}_{kj})\}] \quad \forall t \in T, \forall k \in K \tag{11}$$

2.1.2 A Piecewise linear approximation approach

The expected inventory and expected backorder as shown in Constraints (5) and (11) respectively are non-linear functions of the cumulative production Tempelmeier and Hilger (2015b). A linear approximation of the non-linear function can be done using the piecewise linear approximation method. Based on the extensive study done by Rossi et al. (2014) on the piecewise linear upper and lower bounds for the first-order loss function, we are reformulating the above stochastic lot-sizing model by incorporating piecewise linear approximations of both non-linear functions. The formulation presented here is similar to the work proposed by Sereshti et al. (2021b) and van Pelt and Fransoo (2018). The authors have determined the expected inventory, expected backlog, and expected backorder based on the cumulative demand CD_{kt} up to period t, the cumulative production quantity Q_{kt} up to period t and $\mathcal{L}_{CD_{kt}}Q_{kt}$ the first order loss function of cumulative demand CD_{kt} based on cumulative production quantity Q_{kt}. The cumulative demand CD_{kt} is the sum of t independent and identically normally distributed random variables and hence, our cumulative demand is normal van Pelt and Fransoo (2018).

The inventory balance equation 11 can be rewritten or is equivalent to equation 12.

$$E[\overline{I}_{kt}] = Q_{kt} - E[CD_{kt}] + \mathcal{L}_{CD_{kt}}Q_{kt} \quad \forall t \in T, \forall k \in K \tag{12}$$

Similar to the expected inventory level given by12, we can articulate the expected backlog, commonly referred to as the first-order loss function as equation 13 van Pelt and Fransoo (2018). The expected backorder can be represented as equation 14 Sereshti et al. (2021b) Tempelmeier and Hilger (2015b). It is important to distinguish between backorders and backlog. Backorders are determined periodically, representing the outstanding orders yet to be fulfilled in a specific time period. On the other hand, backlog refers to the cumulative amount of backorders accumulated over time van Pelt (n.d.).

$$E[\overline{BL}_{kt}] = \mathcal{L}_{CD_{kt}} Q_{kt} \quad \forall t \in T, \forall k \in K \tag{13}$$

$$E[\overline{BO}_{kt}] = \mathcal{L}_{CD_{kt}} Q_{kt} - \mathcal{L}_{CD_{k,t-1}} Q_{kt} \quad \forall t \in T, \forall k \in K \tag{14}$$

The derivation of the loss function can be found in the works of Rossi et al. (2014) or van Pelt (n.d.). The steps involved in deriving the expected backlog and expected inventory will be used to linearize both functions and these linearized functions will be used to formulate the model again as a linear optimization problem. The explicit derivation steps are as follows:

The demand for our problem is normally distributed. As presented by van Pelt (n.d.) and Rossi et al. (2014), We assume $f_{CD_{kt}}$ as the probability density function and $F_{CD_{kt}}$ the cumulative density function. Let $\phi(x)$ be the standard normal probability function and $\Phi(x)$ the cumulative distribution function where μ_{CD_k} is the mean demand of product k and σ_{CD_k} is the standard deviation of the product k. The first-order loss function $\mathcal{L}_{CD_{kt}} Q_{kt}$ of cumulative demand CD_{kt} based on cumulative production quantity Q_{kt} can be written as:

$$\begin{aligned}
\mathcal{L}_{CD_{kt}} Q_{kt} &= E[\overline{BL}_{kt}] \\
&= E[\max\{0, CD_{kt} - Q_{kt}\}] \\
&= \int_{-\infty}^{\infty} \max\{0, x - Q_{kt}\} f_{D_{kt}}(x)\, dx \\
&= \int_{Q_{kt}}^{\infty} (x - Q_{kt}) f_{D_{kt}}(x)\, dx \quad \forall t \in T, \forall k \in K
\end{aligned} \tag{15}$$

We can write PDF $f(.)$ of the normal distribution in the form of standard normal PDF, $f(x) = (1/\sigma)\phi(x-\mu)/\sigma$. Where z can be defined as z = $\frac{Q_{kt} - \mu_{CD_k}}{\sigma_{CD_k}}$.

$$\mathcal{L}_{CD_{kt}} Q_{kt} = \int_{-\infty}^{\infty} (D_{kt} - Q_{kt}) \frac{1}{\sigma_{CD_k}} \phi\left(\frac{x - \mu_{CD_k}}{\sigma_{CD_k}}\right) dx$$
$$= \sigma_{CD_k} \int_{Q_{kt}}^{\infty} ((\frac{x - \mu_{CD_k}}{\sigma_{CD_k}}) - z) \frac{1}{\sigma_{CD_k}} \phi\left(\frac{x - \mu_{CD_k}}{\sigma_{CD_k}}\right) dx \quad (16)$$

For solving the equation we will use the substitution rule for integration and integration by parts rule as done by van Pelt (n.d.) given in Appendix A. The equations 13 and 12 can be rewritten as equations 17 and 18 respectively.

$$E[\overline{BL}_{kt}] = \mathcal{L}_{CD_{kt}} Q_{kt} = \sigma_{CD_k}(\phi(z) - z(1 - \Phi(z))) \quad \forall t \in T, \forall k \in K \quad (17)$$

$$E[\overline{I}_{kt}(Q_{kt})] = \sigma_{CD_k}(\phi(z) + z(1 - \Phi(-z))) \quad \forall t \in T, \forall k \in K \quad (18)$$

where

$$z = \frac{Q_{kt} - \mu_{CD_k}}{\sigma_{CD_k}}$$

.

The anticipated backlog and inventory level functions are evidently non-linear as mentioned in equations 17 and 18, requiring us to linearize them for further analysis van Pelt and Fransoo (2018). Consequently, in the next section, we will proceed with a piecewise linear approximation of both functions.

2.1.3 Approximating expected values via piecewise linear functions

It is feasible to substitute the non-linear functions representing the expected backlog and physical inventory in the Stochastic Capacitated Lot-Sizing Problem (SCLSP) with piecewise linear functions. A graphical representation of such piecewise linearization is depicted in Figure 2.1, considering a single period with normally distributed demand. The expected mean demand μ_{CD_k} is assumed to be 100 with a standard deviation σ_{CD_k} of 30. By employing a sufficient number of L line segments, we can approximate both functions with arbitrary precision within the range Rossi et al. (2014).

In the case of a single period, where the period production equals the cumulative production, the graph illustrates the requirement for two specific points to achieve linearization. Firstly, a point representing minimal (zero) production at q = 0 is necessary. Secondly, a point representing the realistic maximum possible production (e.g., q = 300) is essential. As the optimal cumulative production

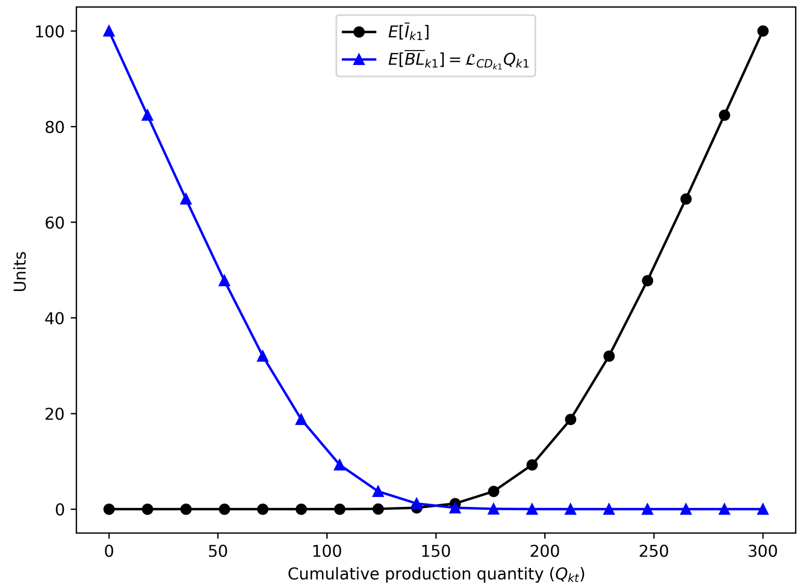

Figure 2.1: Linearize backlog and inventory projections against production Helber et al. (2013)

for any given period is not known in advance, the other supporting points for the linearization are concentrated around the expected demand. This concentration ensures that the deviation from the original non-linear functions is minimal, particularly in the region where the non-linearity of the expected backlog and physical inventory over the production quantity is most pronounced Helber et al. (2013). Figure 2.1 illustrates the relationship between the expected backlog and the production quantity. As the production increases, the expected backlog decreases, while the expected inventory level rises. In order to meet the service level constraint and limit the expected back-orders, it becomes necessary to adjust the production quantity accordingly Tempelmeier and Hilger (2015b). This adjustment directly affects the expected inventory level, which is influenced not only by the service level constraint but also by the trade-off between inventory holding costs and setup costs. Consequently, the determination of the optimal cumulative production quantity Q_{kt} relies on achieving the desired service level at the lowest cost, while considering the capacity constraint van Pelt (n.d.).

Helber et al. (2013) provided a detailed explanation of the approximation/linearization of expected inventory and backlog level. This linearization process needs to be performed for each specific combination of product k and period t. It applies to both the expected physical inventory and the expected backlog, which are expressed as functions of the cumulative production up to period t. An example of this linearization process for a single period is shown in Figure 2.2. It represents the

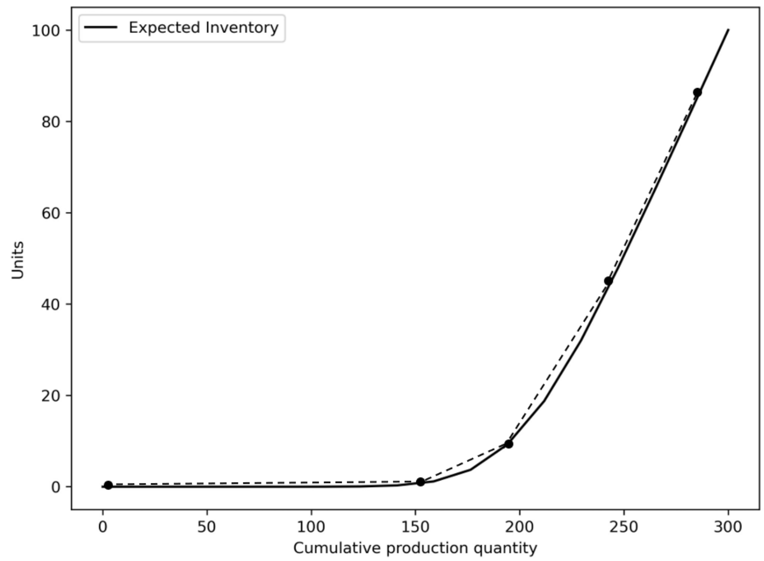

Figure 2.2: Transforming the expected physical inventory function into a linearized form Helber et al. (2013)

approximation of expected inventory by a piecewise linear function with four line segments.

After introducing the expected values, we can use a piecewise linear approximation for both the expected backorder and expected inventory functions van Pelt and Fransoo (2018). The functions are linearized into L line segments with endpoints and interval $[u^0_{kt}, u^l_{kt}]$. The initial point, u^0_{kt}, is set to zero to accommodate the possibility of not needing to produce anything. This decision is based on the assumption of zero initial inventory. On the other hand, the final point, u^l_{kt}, should be sufficiently large to consider the additional production required to meet the service level constraint and find the optimal balance between inventory holding and production setup Helber et al. (2013). Each line segment, represented by the subinterval $[u^{l-1}_{kt}, u^l_{kt}]$, corresponds to a specific segment l within the overall approximation. The slope of each line segment in the expected inventory-on-hand function for item k at time period t can be described as follows,

$$\triangle^l_{I_{kt}} = \frac{(u^l_{kt} - E[CD_{kt}] + \mathcal{L}^1_{CD_{kt}}(u^l_{kt})) - (u^{l-1}_{kt} - E[CD_{kt}] + \mathcal{L}^1_{CD_{kt}}(u^{l-1}_{kt}))}{u^l_{kt} - u^{l-1}_{kt}} \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{19}$$

Additionally, referring previous section and equation 16 it is known that the expected back-orders for period t can be calculated by subtracting $\mathcal{L}^1_{CD_{kt}}(Q_{kt}) - \mathcal{L}^1_{CD_{kt-1}}(Q_{kt})$. Therefore, the expected back-order function can be estimated within the same region, with the slopes determined by this calculation,

$$\triangle^{l}_{BO_{kt}} = \frac{(\mathcal{L}^{1}_{CD_{kt}}(u^{l}_{kt}) - \mathcal{L}^{1}_{CD_{kt-1}}(u^{l}_{kt})) - (\mathcal{L}^{1}_{CD_{kt}}(u^{l-1}_{kt}) - \mathcal{L}^{1}_{CD_{kt-1}}(u^{l-1}_{kt}))}{u^{l}_{kt} - u^{l-1}_{kt}} \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{20}$$

2.1.4 Formulation of the stochastic lot-sizing problem model using piecewise linear approximation

Referring to notations in Table 2.1 and some additional notations in Table 2.2, we can write the piecewise linear approximation of the model as presented in the work of Sereshti et al. (2021b) and van Pelt and Fransoo (2018) as follows:

Table 2.2: Notation used for the parameters and decision variables of the piecewise linear model

Indices and index sets	
L	Set of linearization segments ($l \in$ 1,.....,L)
Deterministic parameters	
u^{0}_{kt}	Lower limit of segment 1 for product k in period t
u^{l}_{kt}	Upper limit of segment l for product k in period t
$\Delta_{I^{l}_{kt}}$	Slope of the inventory function for product k in period t corresponding to segment l
$\Delta_{BO^{l}_{lkt}}$	Slope of the backorder function for product k in period t corresponding to segment l
$\Delta_{I^{0}_{kt}}$	Expected inventory function at point u^{0}_{kt} for product k in period t
$\Delta_{BO^{0}_{lkt}}$	Expected backorder function at point u^{0}_{kt} for product k in period t
Decision variables	
w^{l}_{kt}	Cumulative production quantity associated with segment l for product k in period t
λ^{l}_{kt}	Binary variable which is equal to 1 if w^{l}_{kt} takes a positive value

$$Min \sum_{t\in T}\sum_{k\in K}(sc_{kt}y_{kt} + hc_{kt}[\Delta_{I^{0}_{kt}} + \sum_{l\in L}(\Delta_{I^{l}_{kt}} w^{l}_{kt})]) \tag{21}$$

Subject to:

$$\sum_{k \in K} (st_{kt}\, y_{kt} + pt_{kt}\, q_{kt}) \leq cap_t \quad \forall t \ \in \ T \tag{22}$$

$$w_{kt}^{l-1} \geq (u_{kt}^{l-1} - u_{kt}^{l-2})\lambda_{kt}^{l} \quad \forall t \in T, \forall k \in K, \forall l \in L, l \geq 2 \tag{23}$$

$$w_{kt}^{l} \leq (u_{kt}^{l} - u_{kt}^{l-1})\lambda_{kt}^{l} \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{24}$$

$$\sum_{l \in L} w_{kt}^{l} - \sum_{l \in L} w_{kt-1}^{l} = q_{kt} \quad \forall t \in T, \forall k \in K \tag{25}$$

$$\sum_{l \in L} w_{kt-1}^{l} \leq \sum_{l \in L} w_{kt}^{l} \quad \forall t \in T, \forall k \in K \tag{26}$$

$$\frac{\sum_{t \in T} \sum_{k \in K} (\Delta_{BO_{lkt}^{0}} + \sum_{l \in L} \Delta_{BO_{lkt}^{l}} w_{kt}^{l}}{\sum_{t \in T} \sum_{k \in K} E[\overline{D}_{kt}]} \leq 1 - \beta \tag{27}$$

$$q_{kt} \leq M y_{kt} \quad \forall t \in T, \forall k \in K \tag{28}$$

$$y_{kt} \ \in \ 0, 1 \quad \forall t \in T, \forall k \in K \tag{29}$$

$$q_{kt} \ \geq \ 0 \quad \forall t \in T, \forall k \in K \tag{30}$$

$$w_{kt}^{l} \ \geq \ 0 \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{31}$$

$$\lambda_{kt}^{l} \ \in \ 0, 1 \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{32}$$

The goal of the objective function in equation 21 is to minimize the combined setup cost and an estimated value representing the holding costs. Constraint 22 establishes and upholds the limitations on capacity. Constraint 23 ensures that the value of w_{kt}^{l} falls within the range specified by the interval $[u_{kt}^{l-1}, u_{kt}^{l-2}]$, and Constraint 24 enforces that w_{kt}^{l} is greater than or equal to 0 if and only if there is production associated with line segment l, at which point λ_{kt}^{l} equals one. Constraint 25 calculates the production quantity for each period as the disparity between the cumulative production of successive periods. Constraint 26 along with constraints 23 and 24 ensures the sequential selection of segments van Pelt and Fransoo (2018) to account for the non-convex nature of expected backorder $E[\overline{BO}_{kt}]$ for $t >= 2$. Constraint 27 enforces an aggregate service level requirement, where the total average backorders divided by the total average demand should not exceed 1 - β. Constraint 28 enforces setup when there is a production setup. Constraints 29 to 32 define the feasible ranges for the different variables within the model.

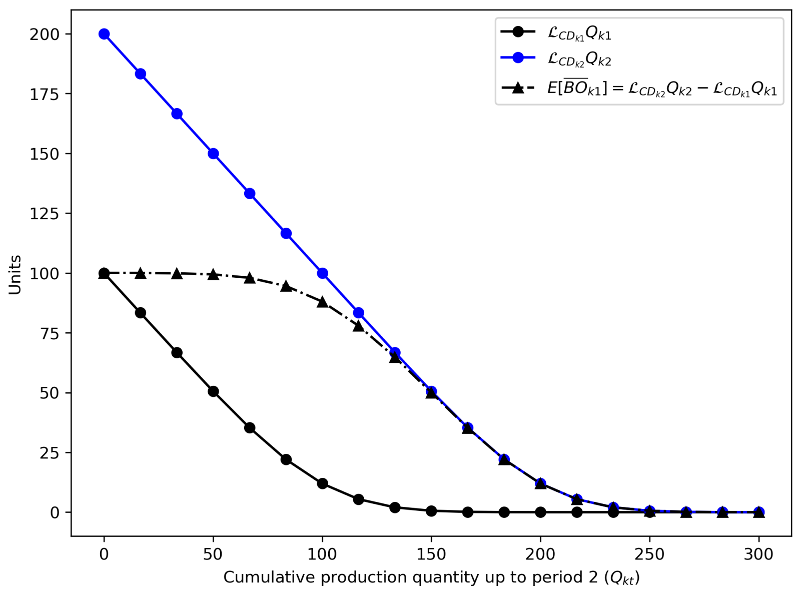

Figure 2.3: First-order loss functions and the expected backorder function for $t = 2$ van Pelt and Fransoo (2018)

The expected backorder $E[\overline{BO}_{kt}]$ illustrated in Figure 2.3 is non-convex in nature, aligning with the findings of van Pelt and Fransoo (2018). Figure 2.3 shows the plot of the first-order loss functions and the expected backorder function. This visualization underscores the idea that a substantial reduction in expected backorders can be achieved while producing fewer units. In other words, fewer w_{kt}^{l}'s need to be filled to their maximum capacity. This is because only those w_{kt}^{l}'s that do not significantly contribute to the reduction of expected backorders remain at zero, while those making the most significant contributions are filled to their maximum capacity van Pelt (n.d.).

2.2 The non-linear stochastic capacitated lot-sizing problem with aggregate γ service level

In this section, we explore the concept of aggregate γ-service level as , which is the time-oriented service level, where we set a predefined percentage as a threshold for the total expected backlog divided by the total expected demand. The expected inventory and backlog in each planning period are influenced by the cumulative production, resulting in a non-linear relationship Helber et al. (2013).

2.2.1 Mathematical Model

The notations of the parameters and decision variables used in this mathematical model are already mentioned in Table 2.1. Referring to the study of Sereshti et al. (2021b), We can now state the stochastic capacitated lot sizing problem with aggregate γ service level as follows:

$$Min \sum_{t \in T} \sum_{k \in K} (sc_{kt} y_{kt} + hc_{kt} E[\overline{I}_{kt}]) \tag{33}$$

Subject to:

$$E[\overline{I}_{kt}] = E[\max\{0, I_{k0} + \sum_{j=1}^{t} (q_{kj} - \overline{D}_{kj})\}] \quad \forall t \in T, \forall k \in K \tag{34}$$

$$q_{kt} \leq M y_{kt} \quad \forall t \ \in \ T, \ \forall k \ \in \ K \tag{35}$$

$$\sum_{k \in K} (st_{kt}\, y_{kt} + pt_{kt}\, q_{kt}) \leq cap_t \quad \forall t \ \in \ T \tag{36}$$

$$y_{kt} \ \in \ 0, 1 \quad \forall t \in T, \forall k \in K \tag{37}$$

$$q_{kt} \ \geq \ 0 \quad \forall t \in T, \forall k \in K \tag{38}$$

$$\overline{I}_{kt} \ \geq \ 0 \quad \forall t \in T, \forall k \in K \tag{39}$$

$$\overline{B}_{kt} \ \geq \ 0 \quad \forall t \in T, \forall k \in K \tag{40}$$

$$E[\overline{BL}_{kt}] = E[\max\{0, \sum_{j=1}^{t} \overline{D}_{kj} - \sum_{j=1}^{t} q_{kj} - I_{k0}\}] \qquad \forall t \in T, \forall k \in K \tag{41}$$

$$\frac{\sum_{t\in T}\sum_{k\in K} E[\overline{BL}_{kt}]}{\sum_{t\in T}\sum_{k\in K} E[\overline{D}_{kt}]} \leq 1-\gamma \tag{42}$$

In this mathematical formulation, $E[BL_{kt}]$ and $E[I_{kt}]$ are two nonlinear functions of the cumulative production.

2.2.2 Piecewise linear approximation

The expected inventory and expected backlog as shown in constraints 34 and 41 are non-linear functions of the cumulative production. Similar to the work proposed by Sereshti et al. (2021b) and van Pelt and Fransoo (2018) we will determine the cumulative demand CD_{kt} up to period *t*, the cumulative production quantity Q_{kt} up to period *t* and $\mathcal{L}_{CD_{kt}}Q_{kt}$ the first order loss function of cumulative demand CD_{kt} based on cumulative production quantity Q_{kt}. We can rewrite the two non-linear functions that are expected inventory and expected backlog mentioned in equations 34 and 41 respectively as equations 12 and 13. That is:

$$E[\overline{I}_{kt}] = Q_{kt} - E[CD_{kt}] + \mathcal{L}_{CD_{kt}}Q_{kt} \quad \forall t \in T, \forall k \in K$$

$$E[\overline{BL}_{kt}] = \mathcal{L}_{CD_{kt}}Q_{kt} \qquad \forall t \in T, \forall k \in K$$

2.2.3 Approximating expected values via piecewise linear functions

Referring to the approximation done for the nonlinear functions in β service level we will employ a sufficient number of line segments L to approximate both expected inventory and expected backlog nonlinear functions with a high degree of precision across the entire range for the γ service level. As explained earlier in section 2.1.3, the figures 2.1 and 2.2 demonstrate the linearization process of both the nonlinear functions.The functions are linearized into L line segments van Pelt and Fransoo (2018) with endpoints and interval $[u^0_{kt}, u^l_{kt}]$. The initial point, u^0_{kt}, is set to zero to accommodate the possibility of not needing to produce anything, and the final point, u^l_{kt}, should be sufficiently large to consider the additional production required to meet the γ service level constraint. The slope of each line segment in the expected inventory-on-hand function for item k at time period t is represented by equation 19. That is:

$$\triangle^{l}_{I_{kt}} = \frac{(u^{l}_{kt} - E[CD_{kt}] + \mathcal{L}^{1}_{CD_{kt}}(u^{l}_{kt})) - (u^{l-1}_{kt} - E[CD_{kt}] + \mathcal{L}^{1}_{CD_{kt}}(u^{l-1}_{kt}))}{u^{l}_{kt} - u^{l-1}_{kt}} \quad \forall t \in T, \forall k \in K, \forall l \in L$$

Additionally, we know from the work of Sereshti et al. (2021b) that equation 13 shows how the expected backlog for the period t can be calculated using $\mathcal{L}_{CD_{kt}} Q_{kt}$ as given in equation 17. Therefore, the expected backlog function can be estimated within the same region, with the slopes determined by the calculation given below:

$$\triangle^{l}_{BL_{kt}} = \frac{(\mathcal{L}^{1}_{CD_{kt}}(u^{l}_{kt}) - \mathcal{L}^{1}_{CD_{kt}}(u^{l-1}_{kt}))}{u^{l}_{kt} - u^{l-1}_{kt}} \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{43}$$

2.2.4 Formulation of the stochastic lot-sizing problem model using piecewise linear approximation

In each planning period, the projected inventory and backlog are influenced by non-linear dependencies on cumulative production. These non-linear relationships are approximated using the linearization of the first-order loss function stemming from the normal distribution, which exhibits convex characteristics concerning cumulative production, as outlined in Rossi et al. (2015). The convex nature of these non-linear relationships for expected inventory and expected backlog obviates the need for the introduction of additional binary decision variables to ensure sequential selection of different segments within piecewise linear functions Sereshti et al. (2021b). This distinction sets this model apart from the one featuring the β service level. For a comprehensive overview of the novel parameters and decision variables introduced by Sereshti et al. (2021b), refer to Tables 2.1, 2.2, and additional notations in Table 2.3.

Table 2.3: Notation used for the parameters and decision variables of the piecewise linear model

Deterministic parameters	
$\Delta_{BL^{l}_{lkt}}$	Slope of the backlog function for product k in period t corresponding to segment l
$\Delta_{BL^{0}_{lkt}}$	Expected backlog function at point u^{0}_{kt} for product k in period t
γ	Target fill rate as an aggregate service level

$$Min \sum_{t \in T} \sum_{k \in K} (sc_{kt} y_{kt} + hc_{kt} E[\overline{I}_{kt}]) \tag{44}$$

subject to constraints: 35, 36, 37, 38, 25 and 31

$$\frac{\sum_{t \in T} \sum_{k \in K} (\Delta_{BL^0_{lkt}} + \sum_{l \in L} \Delta_{BL^l_{lkt}} w^l_{kt}}{\sum_{t \in T} \sum_{k \in K} E[\overline{D}_{kt}]} \leq 1 - \gamma \tag{45}$$

$$w^l_{kt} \leq (u^l_{kt} - u^{l-1}_{kt}) \quad \forall t \in T, \forall k \in K, \forall l \in L \tag{46}$$

The objective function 44 minimizes the total cost which includes the setup cost and the expected inventory holding costs. Constraint 25 calculates the production quantity based on the selected segments. Constraints 46 establish the upper limit for the production quantity that can be assigned to segment l during period t. Constraint 35 ensures the setup in case there is any production in that period. Constraint 36 ensures the production in each period is within the capacity limitation. Constraints 37, 38, and 31 defines the domain of the different variable in the model.

Chapter 3

An iterative optimization-based heuristic

3.1 Fix-and-optimize heuristic

The conventional approach to formulating the stochastic capacitated lot sizing problem (SCLSP), which focuses on production and inventory quantities as detailed in Section 2.1 and 2.2, often results in exceedingly long solution times for all but the smallest problem instances when using a mixed integer programming (MIP) solver. The primary factor influencing the computational effort is the number of binary setup variables, while the number of real-valued variables is of lesser significance Helber and Sahling (2010). Our proposed approach revolves around tackling a series of subproblems, derived from the SCLSP, in an iterative manner. In each iteration of our approach, we adopt a strategy similar to Ouhimmou et al. (2008), where we break down the overall planning horizon into smaller intervals, effectively creating a new, more manageable planning horizon. The central concept behind this heuristic is to tackle a series of successive smaller problems. As shown in Figure 3.1, We assume a sample P_i problem for horizon planning and its starting period is period 1:

In each iteration, we start by solving the problem for the current interval, let's call it P_i during this step, we obtain a solution that includes setting certain binary setup variables to specific values. This solution from P_i is then utilized as a set of new constraints for the subsequent problem, P_{i+1} by doing so, we integrate the current solution's binary setup variable settings into the problem's constraints for the next interval. The key idea here is that this approach narrows down the range of potential solutions for P_{i+1} as it must now adhere to the constraints derived from the previous interval. It's important to note that only binary variables with a value of 1 are fixed, meaning that

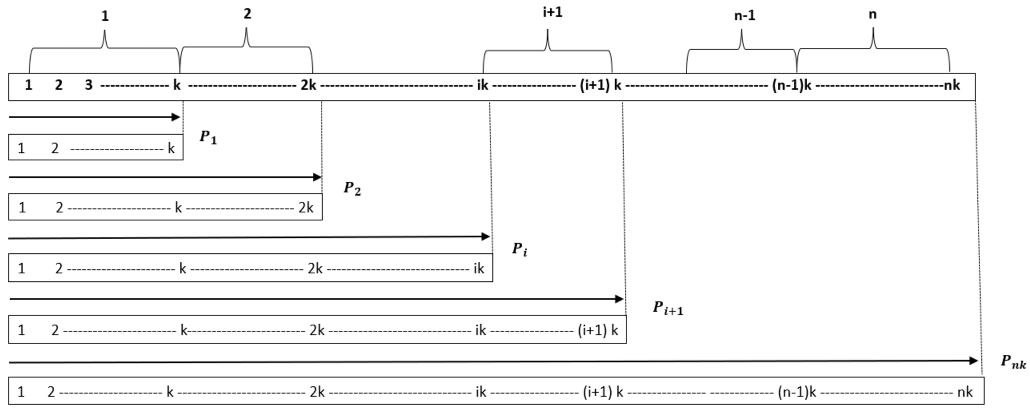

Figure 3.1: Breaking down the planning horizon into smaller and uniform segments

other binary variables with a value of 0 can take on different values in the solution of P_{i+1}. Additionally, continuous variables are not fixed, allowing their values to change in the solution of P_{i+1}. As a result of this process, the number of free binary variables in these subproblems is significantly reduced compared to the original problem, leading to substantially shorter solution times for each subproblem. This iterative approach allows us to efficiently work through the planning horizon by progressively refining solutions, ultimately leading to a more optimized outcome Ouhimmou et al. (2008).

Chapter 4

Computational Experiments

4.1 Computational Experiments

We conducted an extensive computational investigation to assess the efficacy of two service level models, denoted as β and γ, employing the piecewise linear approximation method, as explained in previous sections. Additionally, we compared the outcomes of our two service level models (β and γ) with those obtained from an iterative optimization-based heuristic (Fix-and-optimize) algorithm. Our objective is to scrutinize the performance of each formulation including heuristic with different interval lengths, considering the mean percentage error, CPU time, and relative gap percentage, across varying instance sizes and aggregate service level values for both β and γ service levels. All algorithms were implemented in Python (Jupyter Notebook) and executed on an Intel(R) Xeon(R) Silver 4114 processor at 2.20GHz within a Windows environment. The Gurobi Optimizer version 10.0.2 build v10.0.2rc0 (win64) served as the back-end solver. In the subsequent sections, we analyze the outcomes of our formulation using various values for the number of linear segments in the piecewise linear model of β aggregate service level. After determining the appropriate number of linear segments, we present a summary of our experimental results in terms of model accuracy and heuristic performance for both the aggregate service levels β and γ. Detailed findings are provided in the Appendix.

4.1.1 Benchmark Instances

In this section, we define the data employed to assess the models, categorized into two sets: Set A and Set B. Set A serves the purpose of determining the optimal number of linear segments for our piecewise linear models, featuring two distinct values of β service level—90% and 95%. The mean demand is derived through random sampling from a uniform distribution with a range of 100 to 120. Similarly, the standard deviation is sampled within a range of 2 to 8, while holding costs range from 4 to 8, and setup costs range from 4000 to 6000. The utilization factor for Set A is set at 0.85. The data Set B is utilized to scrutinize the accuracy of our non-linear stochastic capacitated lot-sizing problem for both β and γ service levels, employing piecewise linear approximations both with and without the use of heuristics. Furthermore, we evaluate the performance of our heuristics with different interval lengths. We employed 10 distinct problem sets, generating 10 instances for each set, each with a given value for β and γ aggregate service levels. The mean demand is obtained through random sampling from a uniform distribution with a range of 80 to 240. Similarly, the standard deviation is sampled within the range of 5 to 25, while holding costs range from 2 to 20, and setup costs span from 400 to 1000. The utilization factor for each problem set of Set B is selected between 0.60 and 0.80. A total of 100 instances, encompassing all combinations of the specified parameters, were tested for each service level type (β and γ) and each of the four heuristics within a given interval length. To assess the solutions derived from the approximate formulations for both the data sets A and B, we employ simulation. The results of the models, inclusive of setup decisions and production levels for each product and period, serve as input for this process. Subsequently, 10,000 demand scenarios are generated based on a normal distribution with the same average and variance as the input to the model.

4.1.2 Number of linear line segments

To determine the appropriate number of linear segments for our piecewise linear models, we conducted tests using a model with distinct β service levels. The model was solved with varying numbers of segments for 10 randomly generated problems from data set A, each associated with two different service level values (90% and 95%). Following the methodology outlined in Sereshti et al. (2021b), we assumed the model shares similar characteristics and opted for the β service level for our analysis.

For each number of segments (5, 10, 15, 20, 25, 30, 35, 40, 45, and 50), we solved ten randomly generated problems, considering two aggregate β service level values—90% and 95%. The evaluation was performed using a set of 10,000 demand scenarios. Figure 4.1 illustrates the average solution time and the mean percentage error of the solutions. The mean percentage error represents

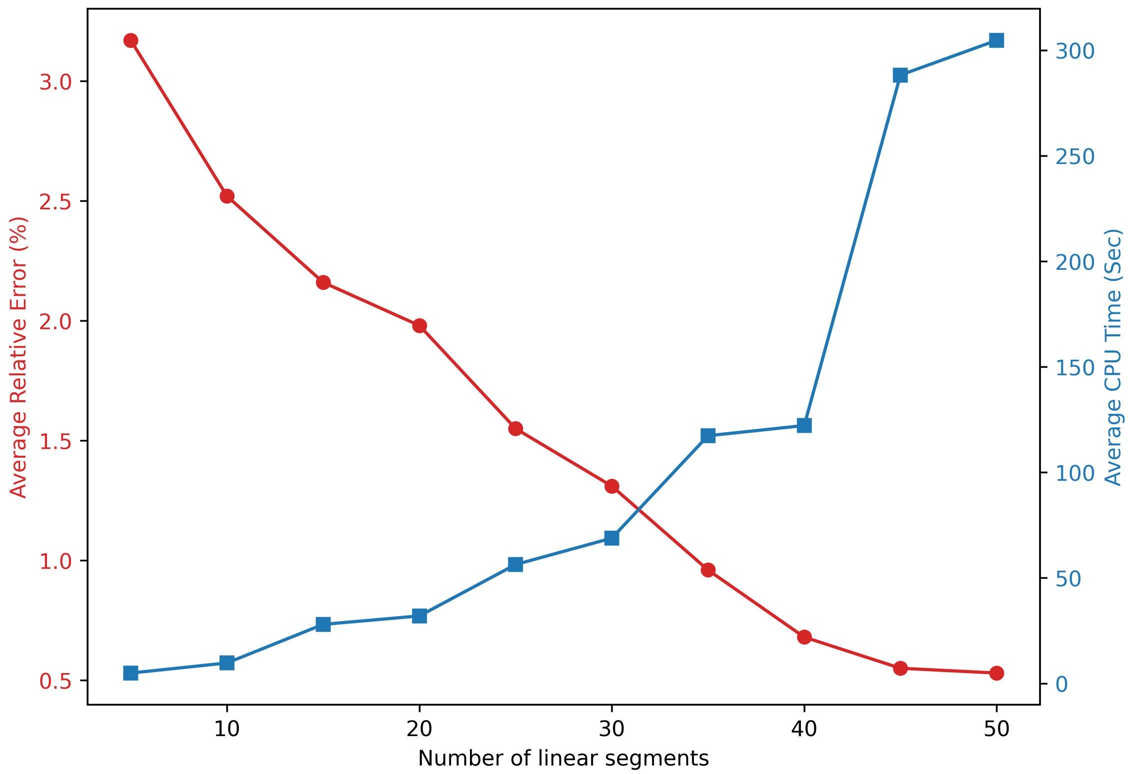

Figure 4.1: Mean Percentage Error % and average CPU time of the piece-wise linear model for the β service level based on the number of linear segments

the difference between the model's objective function and the evaluated objective function (using 10,000 demand scenarios), divided by the evaluated objective function value. Each mean percentage error point was computed using 10 randomly generated problems, and their averages were used for the analysis. After weighing the trade-off between solution time and mean percentage error measures, we pick 40 segments as an appropriate parameter for our piecewise linear approximation. Notably, this result diverges from the findings of Helber et al. (2013) and Sereshti et al. (2021b), who used 18 and 20 segments, respectively, for their piecewise linear models. The detailed findings of the results presented in Figure 4.1 are provided in Appendix B.

Table 4.1 summarizes the results shown in Figure 4.1. The mean percentage error exhibits a decreasing trend with an increasing number of linear segments, indicating that a higher number of segments results in a more accurate cost estimation. The standard deviation of the mean percentage error also decreases with an increasing number of segments. However, the average computation time increases with the segment count.

Table 4.1: Summary Table

L (Number of Segments)	5	10	15	20	25	30	35	40	45	50
Mean Percentage Error (%)	3.17	2.52	2.16	1.98	1.55	1.31	0.96	0.68	0.55	0.53
SD of Mean Percentage Error (%)	1.428	0.817	0.642	0.607	0.503	0.465	0.421	0.254	0.201	0.224
Average CPU Time	4.94	9.72	28.77	32.00	56.35	68.9	117.37	122.21	288.24	304.71

4.1.3 Computational Performance and Discussions of Findings

This section presents the results of experiments conducted on 10 different problem sets derived from data Set B. The primary objectives of these experiments are twofold: to assess the accuracy of the models employing β and γ service levels as shown in Table 4.2 and to evaluate the performance of heuristics under various interval lengths as shown in Table 4.3.

Table 4.2 outlines the results for each problem set, considering 10 different instances (with details in Appendix C). The Error % column indicates the mean percentage error, calculated as the percentage difference between the model's objective function and the cost computed using simulations for 10,000 different demand scenarios. The reported value in the table represents the average across 10 instances. The SD% column represents the standard deviation of errors across these 10 instances, while the Time column denotes the average CPU time in seconds required to solve the model for the 10 instances. The heuristics, denoted as H9, H6, H4, and H3, correspond to time periods (shorter time horizon): 9, 6, 4, and 3 periods respectively. It is essential to calculate the error percentage, standard deviation, and time for several reasons. Firstly, the error percentage provides insights into the accuracy of the model by quantifying the disparity between the model's predicted costs and those obtained through simulation. The standard deviation of errors offers a measure of the consistency or variability in the model's performance across different instances. Lastly, tracking the time required to solve the model is crucial for assessing computational efficiency and scalability, contributing valuable information for practical applications and model deployment.

For the β service level, the Error% for the model solved using a piecewise linear approximation for the full interval length (without heuristic) spans from 0.09 to 2.65. This range indicates that the piecewise linear model generally offers a commendable estimation. Notably, there is a higher Error% for two large problem sets (PS9 and PS10), implying that as the size of our problem sets increases, the error in the objective function and the computed cost tends to rise. This observation underscores the impact of problem set size on model accuracy. The standard deviation across the ten problem sets ranges from 0.23 to 1.18, with an average of 0.65. The standard deviation provides insights into the consistency or variability of the model's performance across different instances. A higher standard deviation suggests greater variability in the model's accuracy across various problem sets. Similarly, for the γ service level, the Error% for the model, solved using a piecewise linear approximation for the full interval length (without heuristic), ranges from 0.03 to 4.81. Two large problem sets (PS9 and PS10) exhibit higher Error%, emphasizing the influence of problem set size on accuracy. The standard deviation for these ten problem sets falls within the range of 0.03 to 1.13, with an average of 0.44. The lower average standard deviation suggests a more consistent performance across different instances compared to the β service level.

In terms of execution time, the γ service level proves to be faster compared to the β service level. This efficiency is attributed to the presence of one extra binary variable in our β service level, contributing to a more computationally demanding process. Understanding these nuances in error percentages, standard deviation, and execution time is crucial for selecting the appropriate service level and model configuration based on computational efficiency and accuracy requirements.

Table 4.2 additionally presents accuracy metrics for heuristic models employing four different interval lengths. The Error% for H9, H6, H4, and H3 heuristics for the β service level ranges from 0.01 to 4.48, 0.13 to 4.92, 0.18 to 4.69, and 0.13 to 4.99, respectively. Notably, there is an increase in the range of error values when utilizing heuristics with varying interval lengths. This variation signifies that the choice of interval length influences the precision of the heuristic models. Examining the average standard deviation across the 10 problem sets for heuristics H9, H6, H4, and H3 reveals an increase when heuristics are introduced. This increase in standard deviation suggests a higher variability in the accuracy of the heuristic models across different problem instances. Simultaneously, the computation time for the heuristics decreases as the interval lengths are reduced, indicating a trade-off between computational efficiency and accuracy. A more comprehensive comparison will be provided in the subsequent section. Similarly, for the γ service level, the Error% for H9, H6, H4, and H3 heuristics ranges from 0.06 to 6.43, 0.09 to 4.54, 0.14 to 4.94, and 0.13 to 5.76, respectively. As observed with the β service level, there is an increase in the range of error values when employing heuristics with different interval lengths. Additionally, the average standard deviation values across the 10 problem sets show an increase, indicating greater variability in the accuracy of the heuristic models. The computation time for the heuristics decreases as the interval lengths are reduced. It's noteworthy that the computational time for both β and γ service levels when using heuristics is relatively similar on average across the four interval lengths. This observation implies that the computational efficiency of the heuristics is comparable for both service levels, regardless of the interval length chosen.

Table 4.2: Model Accuracy

Problem Set						Objective Function			H 9			H 6			H 4			H 3		
PS	**K**	**T**	**L**	**SL %**	**SL Type**	**Error %**	**SD %**	**Time (s)**	**Error %**	**SD %**	**Time (s)**	**Error %**	**SD %**	**Time (s)**	**Error %**	**SD %**	**Time (s)**	**Error %**	**SD %**	**Time (s)**
1	2	18	40	98	Beta	1.47	0.57	5.05	1.70	1.91	3.11	3.14	1.31	3.71	2.86	1.92	2.32	3.31	2.08	2.43
					Gamma	1.98	0.29	1.79	1.93	0.29	0.98	2.03	1.02	1.83	4.66	4.19	1.32	5.80	1.90	1.47
2	3	18	40	98	Beta	1.72	1.28	63.54	0.97	0.93	8.46	1.22	1.58	6.45	1.1	0.65	4.1	1.45	0.92	4.43
					Gamma	2.18	1.02	17.48	3.03	1.58	3.31	4.43	0.68	4.2	4.72	1.80	3.26	5.01	1.66	3.79
3	3	15	40	92	Beta	0.34	0.29	106.89	0.36	0.24	24.17	0.42	0.35	12.91	1.66	2.74	8.09	0.34	0.25	11.14
					Gamma	0.3	0.13	266.46	0.4	0.27	7.81	0.35	0.22	8.71	1.14	2.5	6.41	0.45	0.37	5.8
4	3	15	40	92	Beta	0.76	0.61	85.71	0.69	0.39	17.26	0.75	0.33	8.56	0.72	0.39	7.34	0.63	0.42	6.69
					Gamma	0.3	0.36	18.82	0.38	0.39	6.55	1.64	2.91	7.81	1.83	2.88	5.23	3.91	4.31	5.68
5	3	17	40	94	Beta	0.12	0.23	88.34	0.01	0.34	13.27	0.13	0.07	4.22	0.18	0.10	4.69	0.13	0.23	3.49
					Gamma	0.03	0.03	64.74	0.06	0.12	5.84	0.09	0.18	2.61	0.14	0.16	3.52	0.17	0.20	2.94
6	4	12	40	96	Beta	0.09	0.88	921	0.24	0.59	67.94	0.43	0.76	22.11	0.4	0.91	11.01	0.34	0.50	7.62
					Gamma	0.44	0.38	22.53	1.25	2.81	9.74	0.76	0.47	10.7	0.89	2.00	5.37	3.28	2.95	5.03
7	6	12	40	96	Beta	0.86	0.36	1717.39	0.79	0.25	527.43	2.28	0.94	67.21	1.79	1.20	26.6	0.99	1.47	26.81
					Gamma	1.13	0.34	1678.32	1.04	0.41	748.22	2.32	1.35	71.2	2.06	1.32	11.52	1.66	1.37	11.92
8	4	18	40	98	Beta	1.12	0.82	2095.13	0.94	0.90	151.00	0.56	0.75	51.41	0.71	0.95	26.25	0.60	1.25	24.93
					Gamma	0.94	0.49	141.44	0.92	0.51	47.32	3.70	3.23	24.29	3.13	2.92	17.09	4.68	1.67	18.32
9	6	18	40	94	Beta	2.30	0.58	4500*	4.48	0.77	2600	4.92	1.17	331.9	4.69	1.39	168.46	4.99	1.43	84
					Gamma	4.81	0.27	4500*	6.43	1.66	2813	4.54	0.24	464.34	4.79	1.38	134.35	5.34	1.23	78.49
10	8	12	40	94	Beta	2.65	0.84	5400**	2.92	1.68	4900	3.15	1.70	1346.94	3.75	1.52	255.74	3.22	1.67	79.45
					Gamma	2.27	1.13	5400**	2.43	1.58	4716.63	3.34	1.58	1207.94	4.94	1.33	308.21	3.52	1.27	105.94
Average					Beta	1.14	0.64	1498.30	1.31	0.79	831.26	1.70	0.89	185.54	1.78	1.17	51.46	1.59	0.99	25.09
					Gamma	1.43	0.44	1211.15	1.78	0.96	835.94	2.32	1.18	180.36	2.83	2.04	49.62	3.38	1.73	23.94

* We limit CPU time to 4500 seconds

** We limit CPU time to 5400 seconds

PS: Problem Set

SL: Service Level

Error: Mean Percentage Error

SD: Standard Deviation of Mean Percentage Error

H 9: Heuristic with interval length 9, **H 6**: Heuristic with interval length 6, **H 4**: Heuristic with interval length 4, **H 3**: Heuristic with interval length 3

Table 4.3 illustrates the performance of the four heuristics across four different interval lengths, gauged by the relative gap % and the mean time reduction %. The relative gap % represents the percentage change in cost computed using the heuristic versus the cost computed with the actual interval length based on simulations for 10,000 different demand scenarios. The mean time reduction % indicates the reduction in computational time achieved through the use of the heuristic.

Employing four distinct interval lengths for our heuristic model, we observe that the heuristic with interval length 9 (H9) across the 10 problem sets yields an average reduction in solving time of 59.6 % and 38.2 % for β and γ aggregate service levels, respectively. Simultaneously, the average relative gap is 4.5 % for β and 3.75 % for γ. Likewise, the heuristic with interval length 6 (H6) demonstrates an average reduction in solving time by 82.43 % and 52.63 % for β and γ, with an increase in the average relative gap to 7.38 % and 10.16 % for β and γ, respectively. Continuing, the heuristic with interval length 4 (H4) achieves an average reduction in solving time by 89.66 % and 61.84 % for β and γ, accompanied by an increase in the average relative gap to 15.10 % and 15.36 % for β and γ. Lastly, the heuristic with interval length 3 (H3) results in an average reduction in solving time by 89.61 % and 64.11 % for β and γ, accompanied by an increase in the average relative gap to 19.14 % and 20.10 % for β and γ, respectively.

In the case of Problem Set 1, the full interval length for the γ aggregate service level allows for a rapid solution. Consequently, the heuristics with interval lengths 6, 4, and 3 do not yield significant mean time reduction, coupled with an associated increase in the relative gap. This suggests that for relatively smaller problem sets, the heuristic performance might not justify the additional computational overhead introduced by shorter interval lengths. Conversely, Problem Set 10 poses a substantial challenge, classified as a large problem where the model could not be fully solved within the stipulated time limit of 5400 seconds. Notably, the heuristic with interval length 9 did not exhibit the same level of improvement in terms of the mean time reduction compared to the other nine problem sets. This disparity can be attributed to the fact that even the heuristic with interval length 9 struggled to fully solve the model for a significant portion (6 out of 10 instances) of this dataset.

Upon reviewing the experiments presented in Table 4.3, a crucial recommendation emerges: the choice of an appropriate interval length significantly impacts the model's performance. Opting for a higher interval length, such as 9 in this context, results in better solutions in terms of the mean time reduction, albeit with a moderate increase in the relative gap. Conversely, as the interval length decreases, the model's performance in terms of the mean computational time reduction improves, but at the expense of an increase in the relative gap. Therefore, accurately quantifying the trade-off between the computational mean time reduction and the relative gap is pivotal in selecting the most suitable interval length for the given problem and computational constraints.

Table 4.3: Heuristic Performance

Problem Set						H 9		H 6		H 4		H 3	
PS	K	T	L	Service Level (SL)	SL Type	Relative Gap %	Mean Time Reduction %	Relative Gap %	Mean Time Reduction %	Relative Gap %	Mean Time Reduction %	Relative Gap %	Mean Time Reduction %
1	2	18	40	98	Beta	0.79	37.69	1.92	31.52	4.22	52.78	4.73	50.33
					Gamma	0.37	21.59	16.9	-36.42	14.65	-5.9	24.6	-17.66
2	3	18	40	98	Beta	3.31	86.49	5.63	89.76	11.72	93.48	20.95	92.91
					Gamma	3.86	76.26	12.36	69.84	20.17	75.69	37.48	72.3
2	3	15	40	92	Beta	5.39	63.63	8.93	78.33	14.73	-86.32	20.72	81.17
					Gamma	3.81	32.65	8.61	34.4	8.65	42.55	9.35	47.59
3	3	15	40	92	Beta	5.08	70.25	7.76	86.05	20.43	88.81	25.35	88.96
					Gamma	3.27	16.95	9.15	17.02	10.24	30.3	13.05	37.3
5	3	17	40	94	Beta	5.27	81.62	7.92	93.84	15.55	92.96	19.76	94.72
					Gamma	1.32	22.29	5.95	57.36	6.41	41.96	9.09	53.99
6	4	12	40	96	Beta	4.6	73.72	4.83	95.01	10.04	96.47	13	97.22
					Gamma	5.81	35.81	13.56	52.24	20.74	59.62	28.13	69.98
7	6	12	40	96	Beta	3.62	38.45	14.16	85.72	28.26	95.51	31.66	95.40
					Gamma	3.80	66.55	14.13	84.26	24.66	97.20	28.70	96.88
8	4	18	40	98	Beta	4.68	92.5	4.74	97.41	8.87	98.69	12.44	98.72
					Gamma	7.48	59.59	11.95	80.26	17.5	85.63	21.55	84.37
9	6	18	40	94	Beta	6.74	42.24	8.19	92.63	18.66	96.26	21.95	98.15
					Gamma	3.06	37.51	3.02	89.69	17.67	97.02	18.92	98.26
10	8	12	40	94	Beta	5.51	9.32	9.75	75.08	18.56	95.27	20.82	98.53
					Gamma	4.75	12.74	6.03	77.65	12.86	94.3	10.12	98.04
Average					Beta	4.49	59.59	7.38	82.53	15.10	89.65	19.13	89.61
					Gamma	3.75	38.19	10.16	52.63	15.36	61.84	20.10	64.11

PS: Problem Set
SL: Service Level
Relative Gap %: Percentage difference between heuristic cost and the objective function cost
Mean Time Reduction %: Positive % mean time reduction means that the heuristic method is faster than the actual approximation model
H 9: Heuristic with interval length 9, **H 6**: Heuristic with interval length 6, **H 4**: Heuristic with interval length 4, **H 3**: Heuristic with interval length 3

Chapter 5

Conclusion and Future Research directions

In this comprehensive research endeavor, our primary focus centered on delving into the intricacies of multi-item capacitated lot-sizing problems under the static strategy, specifically addressing two distinct aggregate service levels, denoted as β and γ. We employed the modeling techniques outlined by Tempelmeier and Hilger (2015b) and Sereshti et al. (2021b) to construct a non-linear mathematical model and subsequently solve it by a piecewise linear approximation method.

The incorporation of aggregate service levels provides companies with a powerful tool to flexibly assign service levels across their product range, fostering a nuanced understanding of the computational implications associated with different service level choices. Our research extended beyond model development, conducting extensive numerical experiments to scrutinize the flexibility, computational efforts, and relative cost associated with the β and γ aggregate service levels. Notably, we introduced a Fix and Optimize heuristic, exploring its application across four different interval lengths. Analyzing a set of 10 problem sets derived from experimental data, encompassing up to 8 products and 18 periods for both service levels, our results revealed a nuanced relationship: while heuristics with higher interval lengths outperformed their counterparts in terms of the relative gap, the mean time reduction was less than the heuristic with lower interval lengths. This observation underscores the crucial trade-off between achieving a minimized relative gap and optimizing computational time. For companies seeking to enhance their lot-sizing processes, this highlights the strategic importance of carefully selecting an interval length that aligns with their operational objectives and efficiently balances the dual considerations of gap reduction and computational efficiency.

Comparing our work with existing studies in the multi-item capacitated lot-sizing problems using a static strategy under aggregate service level literature, this research stands out as one of the initial endeavors specifically addressing the importance of computational time reduction through the application of the Fix and Optimize heuristic. However, recognizing the evolving landscape of research possibilities, we acknowledge potential enhancements to our model. Future investigations could explore dynamically increasing the number of linear segments for consecutive periods, especially for larger problem instances. Additionally, extending the analysis to include other service levels available in the literature and conducting sensitivity analyses for different model parameters would provide a more comprehensive understanding of the model's behavior and its applicability in various contexts. To relate these findings to companies and actual practice, organizations can leverage our research insights to optimize their lot-sizing strategies. By understanding the trade-off between computational time reductions and the relative gap, companies can make informed decisions based on their specific operational needs. This research contributes not only to the theoretical foundations of aggregate service level problems but also provides practical implications for companies seeking efficient and effective lot-sizing solutions.

Appendix A

Integration by Substitution:

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_{a}^{b} f(\psi(d))\psi'(d)\,dd$$

Integration by Parts:

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$$

Using two above mentioned rules for integration we can drive $\mathcal{L}_{CD_{kt}}$ as below van Pelt (n.d.);

$$\mathcal{L}_{CD_{kt}} = \sigma_{CD_k} \int_{z}^{\infty} (t-z)\phi(t)\,dt$$

$$= \sigma_{CD_k} \left[\int_{z}^{\infty} t\phi(t)\,dt - \int_{z}^{\infty} z\phi(t)\,dt \right]$$

$$= \sigma_{CD_k} (\left[t\Phi(t) - \int_{z}^{\infty} \Phi(t)\,dt \right] \Big|_{t=z}^{t=\infty} - z(1-\Phi(z)))$$

To express the integrand in the final equation differently, we leverage the property that the antiderivative of the standard Normal Cumulative Distribution Function (CDF) is given by van Pelt (n.d.):

$$\int \Phi(t)\,dt = t\Phi(t) + \phi(t)$$

The resulting expression for $\mathcal{L}_{CD_{kt}}$ becomes:

$$\mathcal{L}_{CD_{kt}} = \sigma_{CD_k} \left[(t\Phi(t) - [t\Phi(t) + \phi(t)]) \Big|_{t=z}^{\infty} - z(1-\Phi(z)) \right]$$

$$= \sigma_{CD_k}([-\phi(t)]\Big|_{t=z}^{\infty} - z(1 - \Phi(z)))$$

$$= \sigma_{CD_k}((-\phi(\infty) + \phi(z)) - z(1 - \Phi(z)))$$

$$= \sigma_{CD_k}(\phi(z) - z(1 - \Phi(z)))$$

where

$$z = \frac{Q_{kt} - \mu_{CD_k}}{\sigma_{CD_k}}$$

Appendix B

In this appendix, we provide a comprehensive summary of the results derived from ten distinct test scenarios performed on line segments of varying lengths (5, 10, 15, 20, 25, 30, 35, 40, 45, and 50). We calculated the relative error by comparing the cost/objective function value with the average computed cost values across 10,000 demand scenarios. The analysis covers two distinct β service levels, as depicted in Figure 4.1.

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost	Scenarios	Mean Percentage Error (%)
4	12	5	90%	51,854.19	3.93	49,457.80		4.85%
				53,396.60	1.64	52,072.55		2.54%
				54,340.21	4.09	52,582.54		3.34%
				54,656.87	3.21	53,249.82		2.64%
				52,149.05	2.08	50,768.29		2.72%
				56,321.52	6.25	54,968.62		2.46%
				54,734.08	13.14	53,549.65		2.21%
				61,798.14	21.59	60,579.00		2.01%
				55,078.88	2.19	54,607.71		0.86%
				58,105.53	1.48	56,818.61		2.26%
4	12	5	95%	73,803.56	3.06	69,744.42		5.82%
				71,600.77	4.28	70,355.12		1.77%
				71,612.27	5.35	68,849.59		4.01%
				71,563.95	2.64	69,430.99		3.07%
				73,922.05	3.08	71,341.27		3.62%
				74,686.50	3.34	72,151.22		3.51%
				73,633.26	4.72	71,511.04		2.97%
				75,295.06	6.49	73,235.50		2.81%
				71,072.37	3.06	66,197.34		7.36%
				67,478.07	3.15	65,825.08		2.51%
			Average		4.94			3.17%

Table B.1: 5 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	10	90%	$59,906.15$	15.31	$58,498.61$	2.41%
				$50,473.11$	6.27	$48,545.78$	3.97%
				$53,037.93$	3.31	$51,760.65$	2.47%
				$53,280.81$	7.85	$52,692.59$	1.12%
				$51,812.42$	7.60	$50,876.06$	1.84%
				$55,845.67$	3.99	$54,967.90$	1.60%
				$53,615.15$	10.56	$52,598.47$	1.93%
				$51,684.14$	12.89	$50,437.95$	2.47%
				$58,265.62$	16.18	$56,646.62$	2.86%
				$53,162.88$	2.97	$52,453.55$	1.35%
4	12	10	95%	$68,703.53$	13.72	$67,099.15$	2.39%
				$67,044.48$	6.50	$66,077.59$	1.46%
				$64,656.59$	11.85	$63,182.38$	2.33%
				$71,818.80$	12.94	$69,805.35$	2.88%
				$71,603.18$	10.97	$69,116.94$	3.60%
				$70,827.16$	11.52	$68,155.79$	3.92%
				$67,795.35$	11.13	$65,746.61$	3.12%
				$74,377.42$	9.21	$71,973.42$	3.34%
				$68,822.32$	7.81	$66,680.43$	3.21%
				$67,977.21$	11.81	$66,583.15$	2.09%
			Average		9.72		2.52%

Table B.2: 10 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	15	90%	$49,152.14$	18.16	$48,327.82$	1.71%
				$62,217.06$	93.43	$61,494.57$	1.17%
				$61,983.60$	29.74	$60,155.99$	3.04%
				$50,305.03$	5.78	$49,532.48$	1.56%
				$53,941.00$	5.66	$52,953.86$	1.86%
				$56,146.01$	27.87	$54,694.30$	2.65%
				$53,541.81$	8.04	$52,338.89$	2.30%
				$50,969.03$	5.97	$49,933.77$	2.07%
				$53,492.70$	23.76	$52,544.39$	1.80%
				$55,074.12$	11.70	$53,652.83$	2.65%
4	12	15	95%	$70,662.20$	29.22	$69,163.25$	2.17%
				$69,929.71$	58.80	$68,949.53$	1.42%
				$72,187.66$	30.96	$70,973.29$	1.71%
				$73,800.71$	28.03	$72,148.82$	2.29%
				$76,256.76$	31.34	$73,447.29$	3.73%
				$75,170.47$	33.52	$74,095.07$	1.45%
				$69,523.27$	41.33	$68,034.85$	2.19%
				$74,815.66$	34.44	$72,858.42$	2.69%
				$70,869.85$	33.08	$68,776.68$	3.04%
				$70,167.44$	24.58	$69,125.91$	1.51%
			Average		28.77		2.16%

Table B.3: 15 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	20	90%	49,152.14	18.26	48,327.82	1.71%
				62,217.06	18.98	61,494.57	1.17%
				61,983.60	30.60	60,155.99	3.04%
				50,305.03	26.38	49,532.48	1.56%
				53,941.00	22.31	52,953.86	1.86%
				56,146.01	18.93	54,694.30	2.65%
				53,541.81	7.26	52,338.89	2.30%
				50,969.03	64.98	49,933.77	2.07%
				53,492.70	9.06	52,544.39	1.80%
				55,074.12	64.60	54,652.83	0.77%
4	12	20	95%	68,265.40	39.30	67,036.75	1.83%
				69,720.34	39.75	68,267.59	2.13%
				67,899.27	29.72	66,997.84	1.35%
				70,534.92	41.78	68,476.96	3.01%
				70,975.08	37.01	69,266.73	2.47%
				59,890.97	38.52	58,937.10	1.62%
				66,308.64	22.28	64,488.94	2.82%
				65,495.20	45.48	64,581.07	1.42%
				68,631.01	26.72	67,000.07	2.43%
				66,543.08	38.08	65,517.16	1.57%
			Average		32		1.98%

Table B.4: 20 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	25	90%	49,350.85	122.29	48,801.71	1.13%
				50,225.67	48.79	49,253.76	1.97%
				54,711.46	10.54	54,293.50	0.77%
				52,340.72	30.21	51,606.19	1.42%
				54,493.17	26.56	53,976.02	0.96%
				52,884.41	28.13	52,182.54	1.35%
				52,230.09	14.99	51,797.26	0.84%
				63,887.58	151.00	63,266.10	0.98%
				57,814.41	23.79	56,996.35	1.44%
				53,788.12	48.81	52,910.88	1.66%
4	12	25	95%	66,521.32	64.54	65,328.79	1.83%
				71,411.02	63.69	70,450.76	1.36%
				72,611.29	72.57	71,270.22	1.88%
				65,203.11	45.92	63,623.36	2.48%
				63,924.99	51.91	62,901.68	1.63%
				63,836.75	51.76	62,809.97	1.63%
				73,143.04	84.95	71,920.95	1.70%
				69,282.31	69.10	67,571.14	2.53%
				66,241.37	48.52	64,760.19	2.29%
				69,631.48	68.84	68,864.18	1.11%
			Average		56.35		1.55%

Table B.5: 25 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	30	90%	57,559.52	37.70	56,859.60	1.23%
				54,418.14	94.75	53,780.72	1.19%
				54,844.67	70.45	54,203.61	1.18%
				52,014.92	36.75	51,284.21	1.42%
				50,236.17	15.90	49,862.64	0.75%
				55,730.87	92.97	54,628.57	2.02%
				56,210.76	39.36	56,017.86	0.34%
				54,728.61	50.26	53,601.26	2.10%
				53,914.02	108.91	53,307.95	1.14%
				50,107.77	41.17	49,458.69	1.31%
4	12	30	95%	65,330.51	96.25	64,874.99	0.70%
				69,336.67	115.12	68,162.72	1.72%
				62,673.39	37.89	62,271.31	0.65%
				74,603.81	107.35	73,555.63	1.43%
				73,516.73	55.43	72,894.63	0.85%
				68,162.88	70.77	66,995.83	1.74%
				68,593.38	66.67	67,516.72	1.59%
				76,837.81	79.46	75,451.54	1.84%
				70,622.94	110.42	69,704.86	1.32%
				61,185.02	50.51	60,168.28	1.69%
			Average		68.90		1.31%

Table B.6: 30 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	35	90%	59,135.35	23.33	58,689.48	0.76%
				53,964.44	94.78	53,512.98	0.84%
				62,937.39	102.29	62,178.85	1.22%
				51,680.20	36.17	51,361.76	0.62%
				53,927.19	51.07	53,275.46	1.22%
				61,740.23	247.53	61,084.37	1.07%
				56,630.98	26.68	56,312.74	0.57%
				55,063.88	61.43	54,776.52	0.52%
				52,042.46	31.40	51,913.46	0.25%
				48,519.12	33.86	48,248.58	0.56%
4	12	35	95%	70,991.91	165.14	70,196.69	1.13%
				64,309.48	40.76	63,851.06	0.72%
				63,338.52	119.27	62,429.01	1.46%
				68,895.25	151.54	68,169.22	1.07%
				66,166.15	84.65	65,582.94	0.89%
				71,395.75	214.24	70,752.50	0.91%
				68,240.96	261.86	67,215.48	1.53%
				72,893.91	180.94	71,387.20	2.11%
				66,199.95	69.04	65,845.10	0.54%
				70,170.62	351.49	69,351.61	1.18%
			Average		117.37		0.96%

Table B.7: 35 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	40	90%	$54,315.98$	95.25	$53,896.40$	0.78%
				$50,593.40$	27.04	$50,296.44$	0.59%
				$53,297.59$	45.00	$53,087.45$	0.40%
				$55,990.27$	47.66	$55,652.84$	0.61%
				$52,832.42$	42.60	$52,396.43$	0.83%
				$51,996.51$	44.58	$51,637.21$	0.70%
				$57,420.30$	48.53	$57,186.77$	0.41%
				$53,899.30$	137.74	$53,490.53$	0.76%
				$57,191.73$	152.66	$56,721.88$	0.83%
				$57,730.03$	54.08	$57,503.87$	0.39%
4	12	40	95%	$67,547.74$	114.34	$66,879.45$	1.00%
				$72,130.42$	153.97	$71,680.52$	0.63%
				$62,014.07$	316.72	$61,318.47$	1.13%
				$66,911.34$	162.30	$66,412.66$	0.75%
				$63,832.11$	74.56	$63,139.08$	1.10%
				$73,012.44$	278.55	$72,677.62$	0.46%
				$65,733.37$	128.96	$65,277.23$	0.70%
				$70,658.93$	194.27	$70,129.72$	0.75%
				$71,544.25$	75.04	$71,533.71$	0.01%
				$66,227.57$	250.39	$65,750.25$	0.73%
			Average		122.21		0.68%

Table B.8: 40 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	45	90%	54,728.15	59.60	54,403.55	0.60%
				52,990.22	25.08	52,719.04	0.51%
				49,902.68	50.21	49,521.92	0.77%
				50,128.69	25.85	49,926.90	0.40%
				54,569.79	341.08	54,408.36	0.30%
				49,067.63	38.26	48,689.85	0.78%
				56,406.50	77.27	56,190.42	0.38%
				60,639.12	319.24	60,171.00	0.78%
				48,906.13	58.92	48,726.58	0.37%
				52,456.40	190.33	52,207.42	0.48%
4	12	45	95%	67,768.15	876.76	67,477.95	0.43%
				64,553.60	247.13	64,499.05	0.08%
				73,827.90	146.47	73,334.89	0.67%
				69,804.72	742.33	69,243.89	0.81%
				68,222.16	232.81	68,008.23	0.31%
				63,319.27	656.58	62,943.06	0.60%
				67,727.95	705.18	67,309.59	0.62%
				64,615.82	306.15	64,081.73	0.83%
				66,701.53	262.29	66,190.15	0.77%
				66,090.55	403.30	65,773.91	0.48%
			Average		288.24		0.55%

Table B.9: 45 - Linear line segments

K	T	L	Service Level (%)	Objective Function Value	CPU Time (s)	Average Cost based on Scenarios	Mean Percentage Error (%)
4	12	50	90%	53,372.39	61.43	53,246.52	0.24%
				55,310.99	480.85	55,016.18	0.54%
				58,401.02	68.44	58,144.10	0.44%
				56,275.01	161.88	56,190.36	0.15%
				54,341.22	37.15	54,261.34	0.15%
				55,050.14	50.67	54,889.76	0.29%
				61,105.36	49.79	60,877.51	0.37%
				59,030.26	392.17	58,866.26	0.28%
				57,503.54	276.44	57,133.18	0.65%
				53,533.64	30.69	53,319.96	0.40%
4	12	50	95%	67,785.51	127.22	67,208.88	0.86%
				71,965.14	288.03	71,580.28	0.54%
				77,075.93	234.31	76,415.15	0.86%
				68,871.34	491.56	68,574.41	0.43%
				69,104.78	803.44	68,612.42	0.72%
				64,335.30	129.25	63,653.01	1.07%
				69,155.96	103.91	68,933.55	0.32%
				70,482.58	1239.10	70,127.79	0.51%
				65,434.19	442.39	64,707.83	1.12%
				73,506.20	525.46	73,068.99	0.60%
			Average		304.71		0.53%

Table B.10: 50 - Linear line segments

Appendix C

In this appendix, we provide a comprehensive summary that includes the results for our objective function as well as outcomes for four heuristic models, each based on different interval lengths. These findings stem from ten diverse test scenarios conducted on ten different problem sets, considering both β and γ aggregate service levels. To gauge performance, we calculated the mean percentage error and its standard deviation by comparing the cost/objective function value with the average computed cost values across 10,000 demand scenarios. The analysis encompasses a range of service levels (92 %, 94 %, 96 %, and 98 %), as outlined in Tables 4.2 and 4.3.

Table C.1: Problem Set - 1

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
2	18	40	98	Beta	1.66	5.80	-1.21	2.97	2.24	4.44	3.94	2.28	6.30	2.60
					1.75	5.92	2.72	4.09	4.75	4.16	5.32	2.25	5.11	2.31
					1.31	5.91	0.36	4.09	4.32	4.15	0.57	2.42	3.86	2.30
					1.81	4.91	5.04	2.66	3.31	3.55	4.64	2.25	2.21	2.38
					-0.01	4.06	3.94	2.72	2.45	3.29	0.79	2.31	0.59	2.21
					1.28	3.94	0.22	2.95	1.42	3.99	1.34	2.49	3.06	3.49
					1.18	5.49	3.05	3.73	3.37	4.33	2.48	2.28	4.76	2.76
					1.76	5.89	0.93	2.63	3.09	3.25	3.43	2.38	3.58	2.13
					2.15	4.14	2.24	2.96	1.11	4.52	0.41	2.33	4.61	1.95
					1.77	4.80	-0.35	3.17	5.31	3.55	5.72	2.29	0.97	2.25
		Average			1.47	5.06	1.70	3.11	3.14	3.73	2.86	2.33	3.31	2.44
2	18	40	98	Gamma	1.98	1.16	1.82	1.02	1.21	1.56	12.90	1.28	10.59	1.55
					2.02	1.38	2.24	0.84	1.78	1.94	3.29	1.34	6.51	1.43
					2.09	0.93	2.44	1.04	4.11	2.04	1.95	1.39	1.5	1.46
					2.00	0.95	1.97	0.88	1.10	1.61	1.94	1.32	4.30	1.42
					2.42	1.28	1.95	0.85	1.58	2.08	3.05	1.25	4.31	1.43
					2.45	1.12	1.71	1.12	3.32	1.70	2.37	1.44	4.58	1.46
					1.89	7.45	2.25	1.01		2.07	2.05	1.24	6.03	1.40
					1.49	1.19	1.78	0.97		2.08	13.04	1.30	6.10	1.56
					2.20	1.15	1.63	1.11	1.61	1.49	2.53	1.35	5.73	1.51
					1.65	1.38	1.46	0.99	1.52	1.83	3.46	1.28	6.46	1.50
		Average			1.99	1.80	1.93	0.98	2.03	1.84	4.66	1.32	5.80	1.47

Table C.2: Problem Set - 2

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
3	18	40	98	Beta	0.06	56.66	1.02	8.74	1.12	7.09	0.74	4.11	1.58	4.76
					1.16	54.85	1.32	8.00	-1.18	4.98	0.82	3.74	0.65	4.36
					1.54	67.69	0.53	9.35	2.68	6.61	1.31	3.78	1.24	3.90
					3.72	66.83	1.66	8.93	2.12	7.36	1.35	4.07	2.71	4.60
					1.68	80.82	1.53	7.21	0.86	6.17	0.46	4.40	0.12	4.15
					1.36	59.48	1.13	9.52	1.70	6.75	2.05	3.24	3.40	3.76
					0.87	65.78	-0.99	8.90	-0.70	6.53	-0.22	4.70	0.75	4.67
					3.20	56.45	2.27	7.12	4.53	6.37	1.92	4.10	1.39	4.75
					3.55	67.14	1.54	8.11	0.15	7.13	1.55	4.30	1.25	4.70
					0.19	59.76	-0.31	8.80	0.89	5.52	1.05	4.59	1.35	4.73
		Average			1.73	63.55	0.97	8.47	1.22	6.45	1.10	4.10	1.45	4.44
3	18	40	98	Gamma	3.28	17.8	4.58	3.02	2.06	4.51	0.59	2.87	4.35	4.29
					0.58	10.6	2.37	3.10	10.58	3.83	3.13	2.87	6.22	3.38
					3.39	12.1	4.28	3.03	10.55	3.58	5.99	3.32	4.46	3.34
					3.18	14.1	2.05	2.83	3.30	4.42	6.96	2.97	4.59	3.95
					1.10	14	1.50	3.18	11.23	3.93	6.46	3.23	2.10	2.80
					1.33	25.7	1.21	5.23	3.36	5.18	4.43	3.54	4.17	
					1.49	11.4	3.76	3.19	2.27	4.73	5.81	3.34	7.12	4.08
					1.81	13	2.02	3.25	3.54	4.39	4.32	3.84	4.70	4.47
					2.17	6.52	6.42	2.8	2.58	3.04	5.72	3.09	8.18	3.28
					3.44	49.6	2.18	3.54	1.63	4.46	3.81	2.91	4.85	4.22
		Average			2.18	17.5	3.03	3.32	5.11	4.21	4.72	3.26	5.01	3.80

Table C.3: Problem Set - 3

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	**T**	**L**	**Service Level (SL)**	**SL Type**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**
3	15	40	92	Beta	0.44	71.94	0.44	13.40	0.33	14.97	0.15	8.77	0.92	23.11
					0.24	60.10	0.22	35.38	0.11	15.57	7.20	10.24	0.29	11.57
					1.11	226.29	0.85	52.37	1.05	14.60	0.59	10.18	0.15	11.36
					0.36	31.27	0.04	34.08	-0.10	14.57	0.11	11.39	-0.03	14.89
					0.22	183.69	0.55	17.13	0.73	7.60	0.38	5.72	0.39	7.11
					0.01	32.91	0.22	9.11	0.09	10.16	0.23	6.58	0.37	7.96
					0.14	57.56	0.04	15.98	0.48	12.93	-0.12	7.29	0.30	6.46
					0.50	64.50	0.35	21.07	0.89	9.80	0.54	6.41	0.56	7.43
					0.17	45.79	0.54	22.74	0.33	18.68	0.43	8.69	0.36	14.17
					0.18	294.87	0.39	20.39	0.28	10.18	7.05	5.60	0.10	7.36
		Average			0.34	106.90	0.36	24.17	0.42	12.91	1.66	8.09	0.34	11.15
3	15	40	92	Gamma	0.15	37.63	0.43	7.10	0.30	10.90	-0.31	6.28	0.48	6.46
					0.42	40.70	0.85	7.58	0.41	7.70	0.34	9.08	0.36	6.93
					0.16	7.09	-0.20	8.84	0.15	8.54	8.61	6.46	0.40	6.41
					0.48	9.67	0.48	10.57	0.27	15.60	0.68	11.65	0.23	9.42
					0.42	2352.25	0.37	11.26	0.56	5.26	0.40	5.53	1.14	4.28
					0.23	131.55	0.28	6.25	0.15	5.47	0.29	4.54	0.65	3.88
					0.23	3.46	0.58	7.86	0.33	10.09	0.31	5.70	0.63	5.19
					0.28	52.65	0.36	6.96	0.06	8.53	0.14	3.85	-0.33	4.65
					0.48	25.21	0.66	5.79	0.38	7.12	0.60	5.40	0.75	5.57
					0.17	4.41	0.14	5.88	0.87	7.83	0.39	5.58	0.20	5.20
		Average			0.30	266.47	0.40	7.81	0.35	8.71	1.14	6.41	0.45	5.80

Table C.4: Problem Set - 4

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	**T**	**L**	**Service Level (SL)**	**SL Type**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**
3	15	40	92	Beta	-0.22	96.96	0.82	12.58	1.05	7.29	1.13	8.91	0.97	7.48
					0.57	34.70	0.92	18.40	0.24	9.07	0.72	8.36	0.90	8.32
					1.05	93.02	0.51	18.78	0.63	12.69	-0.05	10.21	0.15	5.73
					0.43	64.31	0.33	12.72	0.44	8.30	0.12	5.74	0.25	5.90
					0.59	32.42	0.69	19.32	0.82	5.54	0.88	7.31	0.19	9.73
					2.23	132.01	0.58	12.29	0.50	9.60	0.60	8.82	0.26	6.22
					0.79	160.92	1.64	15.19	0.44	8.17	1.05	6.39	0.50	4.99
					0.68	26.70	0.79	18.99	0.87	9.42	1.00	5.37	0.59	5.75
					1.21	124.88	0.35	22.67	1.21	8.77	1.10	7.30	1.02	5.36
					0.28	91.10	0.21	21.72	1.27	6.84	0.62	5.00	1.45	7.46
		Average			0.76	85.70	0.69	17.27	0.75	8.57	0.72	7.34	0.63	6.69
3	15	40	92	Gamma	0.07	7.05	0.04	4.19	0.22	5.28	1.03	4.75	0.48	5.30
					0.06	17.88	0.13	6.12	1.19	7.63	1.33	5.32	9.84	3.22
					0.09	4.99	0.03	5.62	0.59	8.80	1.11	9.35	0.36	4.28
					0.69	4.36	0.09	5.75	0.90	6.25	1.35	5.48	1.24	3.78
					0.06	69.17	0.39	6.81	10.28	8.72	0.65	4.33	0.44	7.03
					0.02	20.47	0.29	8.56	0.12	9.33	1.08	3.83	0.32	7.51
					0.07	46.37	0.64	6.39	0.55	8.15	0.31	6.17	7.99	7.50
					0.82	4.02	1.30	6.68	0.98	8.53	0.58	3.77	8.59	6.21
					0.09	10.12	0.15	8.86	0.14	8.57	0.49	5.63	-0.23	4.46
					1.02	3.78	0.76	6.57	1.42	5.00	10.41	3.77	10.12	7.56
		Average			0.30	18.82	0.38	6.55	1.64	7.81	1.83	5.24	3.91	5.69

Table C.5: Problem Set - 5

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
3	17	40	94	Beta	0.17	229.80	-0.32	22.63	0.09	4.71	0.01	4.44	0.31	4.05
					-0.09	64.05	-0.29	14.30	0.13	3.44	0.40	4.00	0.19	3.26
					0.11	112.89	0.19	10.31	0.04	4.52	0.22	4.54	-0.12	3.19
					0.63	48.15	-0.24	13.67	0.10	3.43	0.06	3.63	0.34	3.63
					-0.09	95.28	0.27	10.12	0.16	4.62	0.30	5.76	0.20	3.59
					0.38	64.66	0.08	13.55	0.03	5.12	0.14	4.36	0.12	3.44
					-0.17	84.70	-0.40	13.50	0.17	5.51	0.20	5.44	-0.41	3.11
					0.10	83.38	0.81	10.48	0.11	4.11	0.17	4.09	0.00	3.31
					-0.05	73.15	-0.13	15.06	0.32	2.59	0.20	5.51	0.27	3.46
					0.22	27.39	0.09	9.16	0.18	4.20	0.11	5.21	0.41	3.87
		Average			0.12	88.34	0.01	13.27	0.13	4.22	0.18	4.70	0.13	3.49
3	17	40	94	Gamma	0.07	2.57	-0.09	2.36	0.03	1.56	0.24	3.65	0.20	2.32
					0.01	8.08	-0.15	6.78	0.45	3.33	0.30	3.53	0.16	3.24
					0.00	3.48	0.19	4.81	0.16	3.18	0.01	3.70	-0.20	2.75
					0.01	6.19	0.05	6.81	-0.05	2.01	0.19	3.37	0.38	2.72
					0.05	53.07	0.06	11.23	0.17	3.22	0.45	3.29	0.35	3.27
					0.06	50.11	0.01	5.29	0.01	2.45	0.19	3.21	0.24	3.38
					0.01	19.25	-0.03	6.21	0.20	2.01	-0.05	3.62	0.40	2.72
					0.08	2.75	0.12	4.95	0.19	2.83	-0.07	3.45	-0.20	2.94
					0.01	500.39	0.27	6.29	0.08	2.62	0.02	4.14	0.25	2.78
					0.03	1.60	0.13	1.72	-0.30	1.21	0.14	1.27	0.15	1.15
		Average			0.03	64.75	0.06	5.85	0.09	2.61	0.14	3.52	0.17	2.95

Table C.6: Problem Set - 6

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
4	12	40	96	Beta	0.42	481.63	1.59	30.16	0.55	15.39	1.79	8.76	0.77	6.4
					0.48	322.45	0.15	65.01	-0.7	24.75	-0.6	9.15	-0.21	6.54
					0.76	351.58	-0.44	66.83	-0.1	33.19	-0.32	12.96	0.32	6.52
					0.61	1648.2	-0.3	64.72	1.6	24.07	1.26	10.4	0.23	5.52
					-1.27	423.77	-0.27	99.31	1.11	15.24	-0.06	8.86	-0.01	9.56
					-0.66	2456.76	-0.19	75.86	-0.35	15.33	1.04	12.66	1.08	9.94
					1.71	275.85	0.11	64.19	1.52	28.13	1.09	9.89	0.24	7.73
					-0.81	979.68	0.33	59.16	-0.35	30.51	1.12	11.59	0.47	9.19
					-0.77	1544.5	0.75	80.51	0.44	23	-1.01	10.61	1.07	6.8
					0.42	734.22	0.65	73.7	0.55	11.58	-0.28	15.2	-0.56	8.03
		Average			0.09	921.87	0.24	67.95	0.43	22.12	0.4	11.01	0.34	7.62
4	12	40	96	Gamma	0.01	15.72	0.7	6.98	1.24	7.45	0.31	5.22	1.29	6.02
					0.73	13.59	0	10.84	1.07	9.18	0.51	5.22	6.96	4.28
					0.47	7.19	0.95	5.96	0.05	29.42	6.7	5.07	5.66	5.15
					0.31	10.89	-0.39	7.11	0.47	7.87	0.53	5.38	7.32	4.53
					-0.49	16.42	-0.14	10.86	0.72	11.46	0.75	5.41	-0.53	4.45
					0.81	37.81	0.76	15.09	0.28	10.63	-0.52	5.78	-0.39	5.12
					0.52	36.79	-0.14	9.76	1.62	9.05	0.7	5.25	5.91	4.12
					0.78	53.33	0.62	10.89	0.46	7.35	0.68	5.58	4.63	5.71
					0.66	28.1	9.59	13.61	0.52	8.5	0	4.92	1.24	4.55
					0.55	5.55	0.58	6.39	1.15	6.18	-0.77	5.95	0.69	6.45
		Average			0.44	22.54	1.25	9.75	0.76	10.71	0.89	5.38	3.28	5.04

Table C.7: Problem Set - 7

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	**T**	**L**	**Service Level (SL)**	**SL Type**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**
6	12	40	96	Beta	1.17	141.03	0.91	141.02	1.29	38.39	2.49	22.75	0.12	21.42
					0.83	6572.76	0.47	155.73	1.17	45	1.15	28.65	0.27	27.97
					0.7	3622.32	0.4	113.92	3.09	50.37	0.63	27.66	2.69	26.22
					1.05	2114.79	0.95	74.29	3.44	58.02	0.5	28.57	4.84	27.69
					0.51	1258.91	0.83	1963.06	2.71	75.96	2.18	27.66	0.21	28.62
					0.57	1155.64	0.8	120.74	1	47.02	4.27	24.53	0.3	27.64
					0.48	136.85	0.39	232.39	3.15	96.79	2.81	27.17	0.32	26.97
					1.71	448.82	1.08	558.69	1.13	107.3	0.68	26.71	0.47	28.72
					0.66	405.49	1.08	298.21	2.7	64.97	2.6	24.5	0.35	26.57
					0.89	141.35	0.95	1616.18	3.07	88.28	0.64	28.37	0.32	26.21
		Average			0.86	1717.39	0.79	527.43	2.28	67.21	1.79	26.6	0.99	26.8
6	12	40	96	Gamma	1.36	2316	1.07	235.15	2.32	15.1	2.63	9.71	4.25	11.65
					1.49	1541.53	0.83	119.22	4.12	54.21	0.5	10.08	2.96	10.22
					0.76	583.14	1.68	420.17	3.95	40.94	2.47	9.22	0.59	10.54
					0.6	123.23	1.28	916.43	3.24	29.73	4.82	11	0.87	11.22
					1.73	1556.38	1.3	69.2	3	25.18	2.35	10.32	2.65	12.18
					1.17	2157.73	1.52	67.83	1.22	29.37	0.49	17.26	0.39	11.96
					1.14	5760.93	0.66	40.13	3.33	27.46	2.33	15.23	3.17	13.84
					1.34	2126.37	0.77	183.13	0.7	368.08	1.07	10.85	0.69	13.25
					0.94	89.7	0.24	5284.99	0.18	84.55	3.21	10.92	0.21	13.28
					0.76	528.23	1.01	145.91	1.11	37.37	0.69	10.62	0.79	11.04
		Average			1.13	1678.32	1.04	748.21	2.32	71.19	2.06	11.52	1.66	11.9

Table C.8: Problem Set - 8

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
4	18	40	98	Beta	1.17	1956.80	1.35	198.40	0.22	40.28	2.07	18.46	-1.19	18.12
					1.29	3092.15	0.54	166.91	-0.77	69.15	2.17	27.48	1.87	18.53
					1.54	2252.93	0.04	119.51	0.45	59.70	0.44	35.11	1.63	32.88
					3.30	2260.31	0.68	215.38	0.43	66.65	1.17	19.11	2.85	37.98
					0.91	1523.58	0.71	160.99	-0.67	55.19	0.97	23.01	-1.14	39.80
					1.10	1776.97	2.66	49.72	1.19	40.88	-0.03	23.04	0.12	20.08
					0.37	2506.77	1.25	120.89	1.61	33.06	1.29	20.15	0.37	15.80
					0.66	2080.83	1.42	169.48	1.14	42.96	-0.08	40.58	1.01	22.44
					0.52	2224.34	-0.82	177.64	1.12	49.48	0.13	31.99	0.98	22.04
					0.36	1276.67	1.57	131.12	0.86	56.78	-1.00	23.62	-0.46	21.66
		Average			1.12	2095.13	0.94	151.00	0.56	51.42	0.71	26.25	0.60	24.93
4	18	40	98	Gamma	1.42	49.60	1.29	40.86	7.30	16.02	1.04	10.41	6.22	10.48
					0.65	118.68	1.69	38.33	-1.02	26.06	0.40	22.96	6.73	32.96
					0.47	82.29	0.80	43.07	7.74	13.55	7.55	11.39	4.11	11.76
					0.70	214.44	0.93	42.78	0.43	17.92	0.98	11.18	4.95	11.95
					0.42	107.34	1.46	50.51	7.11	25.34	7.81	19.54	4.82	27.22
					0.90	99.21	1.14	52.23	6.38	31.02	6.76	23.81	5.12	24.54
					0.96	219.21	0.00	67.49	1.35	38.00	0.90	19.46	4.69	13.95
					1.20	242.53	0.17	61.89	1.56	32.57	3.71	28.37	0.31	27.77
					2.09	203.66	1.10	46.43	0.62	27.61	1.50	11.49	4.06	11.98
					0.58	77.45	0.59	29.69	5.56	14.83	0.69	12.29	5.85	10.64
		Average			0.94	141.44	0.92	47.33	3.70	24.29	3.13	17.09	4.68	18.32

Table C.9: Problem Set - 9

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	**T**	**L**	**Service Level (SL)**	**SL Type**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**	**Error %**	**Time (sec)**
6	18	40	94	Beta	3.73	4501.83	4.99	4517.15	3.76	490.33	5.93	117.63	5.57	128.49
					2.91	4502.03	3.94	3544.72	6.49	237.40	3.17	114.45	3.81	63.85
					1.95	4502.11	4.10	2442.96	6.60	253.50	5.49	184.46	5.06	61.49
					1.74	4503.12	4.05	2235.79	4.05	493.72	3.49	284.35	7.21	110.43
					1.68	4501.61	4.12	3062.47	3.33	332.51	7.40	234.20	3.94	126.51
					2.22	4501.24	4.15	3598.79	6.42	224.24	6.31	206.25	3.55	99.23
					2.08	4501.38	4.56	1407.50	3.74	372.44	3.90	119.73	6.05	47.32
					2.06	4501.48	6.61	2081.97	5.04	453.15	3.75	209.99	3.88	48.49
					2.18	4501.22	4.21	1110.00	5.11	188.06	3.90	115.54	7.39	70.25
					2.42	4501.19	4.09	2000.28	4.65	273.68	3.54	98.01	3.43	78.63
		Average			2.30	4501.72	4.48	2600.16	4.92	331.90	4.69	168.46	4.99	83.47
6	18	40	94	Gamma	4.92	4502.33	8.26	4746.21	4.68	588.41	3.62	277.50	6.50	63.64
					4.52	4501.62	7.78	2472.00	4.24	333.69	3.58	109.90	4.07	126.49
					4.64	4501.43	5.00	1559.08	4.10	211.22	3.92	145.79	4.00	52.58
					4.85	4501.40	3.99	993.94	4.57	413.95	6.95	94.48	6.35	56.31
					5.23	4501.31	5.18	1713.21	4.79	316.32	6.79	110.61	6.63	96.02
					4.47	4501.44	7.54	5181.07	4.68	455.57	4.02	131.48	3.76	77.04
					4.63	4501.56	4.67	3287.74	4.30	493.98	4.03	108.36	4.02	83.31
					4.57	4502.31	7.88	2429.88	4.44	901.66	3.97	103.15	6.83	71.94
					5.19	4501.95	5.27	3352.58	4.74	622.73	4.15	149.66	6.33	93.55
					5.05	4502.68	8.71	2398.29	4.86	305.87	6.90	112.62	4.93	64.00
		Average			4.81	4501.80	6.43	2813.40	4.54	464.34	4.79	134.35	5.34	78.49

Table C.10: Problem Set - 10

Problem Set					Objective Function		H 9		H 6		H 4		H 3	
K	T	L	Service Level (SL)	SL Type	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)	Error %	Time (sec)
8	12	40	94	Beta	2.08	5402.05	4.91	4118.94	2.75	872.44	1.94	283.56	3.62	97.61
					3.62	5406.20	5.39	5501.57	4.50	1205.44	5.62	293.47	1.57	62.92
					2.22	5404.77	1.85	5523.19	5.74	4025.03	5.79	257.33	2.62	123.10
					1.65	5407.27	1.68	5044.52	4.00	1129.48	1.46	293.91	4.58	68.75
					3.08	5403.75	3.34	5471.59	4.93	929.90	2.43	228.48	6.28	46.51
					3.28	5403.35	0.25	5443.00	3.86	937.58	4.14	249.07	2.19	103.57
					4.17	5405.94	3.64	2145.59	2.21	1899.78	2.34	302.76	2.78	59.35
					1.44	5404.07	4.96	4875.02	0.04	742.15	4.05	175.04	5.59	73.23
					2.13	5402.84	1.34	5443.10	0.95	1129.55	5.30	278.78	1.99	78.67
					2.83	5405.29	1.90	5441.29	2.49	598.07	4.39	195.08	0.95	80.87
		Average			2.65	5400.00	2.92	4900.78	3.15	1346.94	3.75	255.75	3.22	79.46
8	12	40	94	Gamma	0.65	5401.93	0.63	5439.58	5.80	989.36	5.87	391.01	2.76	107.18
					2.49	5405.10	1.72	5426.06	2.01	1550.50	3.30	291.56	3.82	54.52
					2.36	5402.63	3.65	5461.43	4.18	1574.90	5.22	393.76	5.66	153.58
					0.83	5407.70	3.44	5465.91	4.83	515.46	6.26	201.48	3.52	115.59
					3.15	5405.88	3.61	2853.85	4.68	1649.56	5.00	228.29	5.11	87.80
					2.84	5403.37	4.70	2435.47	1.03	532.88	4.43	246.20	4.57	90.82
					3.00	5407.74	4.18	5459.48	1.40	1479.49	3.14	363.04	3.10	115.18
					0.79	5407.89	1.58	3723.52	2.21	1469.97	7.21	352.45	3.14	126.14
					2.26	5406.17	0.48	5460.05	4.54	698.83	5.76	371.72	2.36	112.75
					4.30	5404.63	0.28	5441.04	2.70	1618.48	3.22	242.61	1.13	96.18
		Average			2.27	5400.00	2.43	4716.64	3.34	1207.94	4.94	308.21	3.52	105.97

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