Complex Analytic Structure of Stationary Solutions of the Euler Equations

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ABSTRACT

Complex Analytic Structure of Stationary Solutions of the Euler Equations

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This work is devoted to the stationary solutions of the 2D Euler equations describing the time-independent flows of an ideal incompressible fluid. There exists an infinite-dimensional set of such solutions; however, they do not form a smooth manifold in the space of all divergence-free vector fields tangent to the boundary of the flow domain. This circumstance hinders the efforts to understand the structure of the set of stationary flows, and to further study other classes of solutions such as the time-periodic or quasiperiodic flows. The previous authors considered the solutions in the Fréchet space of smooth functions and used powerful methods such as the Nash-Moser-Hamilton implicit function theorem. However, in their approach they overlook a surprising feature of the stationary flows which makes the picture much more transparent, and opens the way to further progress. This is the observation that the particle trajectories in the flow described by arbitrary solutions of the Euler equations in domains with analytic boundary are analytic curves, even if the velocity field has a finite regularity (say, belongs to the Sobolev or Hölder space). In particular, for any stationary solution, the flow lines are analytic curves, despite limited regularity of the velocity field.

To study the stationary flows we change the viewpoint and consider the flow field as a family of analytic flow lines non-analytically depending on parameter. We quantify the analyticity by introducing spaces of functions which have an analytic continuation to some strip containing the real axis such that on the boundary of the strip the function belongs to the Sobolev space. Further, we introduce the class of Sobolev functions of two variables which are analytic (in the above sense) with respect to one variable. Such functions describe the families of flow lines of stationary flows. These partially-analytic functions form a complex Banach space. The stationary solutions satisfy (in the new coordinates) a quasilinear elliptic equation whose local solvability is proved by using the Banach Analytic Implicit Function Theorem (BAIF Theorem). Thus we prove that the set of stationary flows is an analytic

manifold in the complex Banach space of the flows (i.e. families of flow lines).

In our previous work ([9]), we realized this idea in the case of stationary flows in a periodic channel with analytic boundaries. In the present work we study a more complicated case of flows in a domain close to the disc, having one stagnation point. We use polar coordinates centered at the (unknown) stagnation point. This results in an elliptic quasilinear equation in the annulus which is degenerate at one component of the boundary. This makes the analysis more difficult. We introduce function spaces which are adaptations of the Kondratev spaces to the partially-analytic setup, and prove that the problem is Fredholm in those spaces. Further we use the BAIF Theorem, and prove that in our spaces, the set of stationary flows is locally a complex-analytic manifold.

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I also thank Professor Kokotov for aiding in the organization of the defence, the examination committee for their participation and the staff and faculty of the Concordia math department for making it possible.

Finally I thank my family: my father who tutored me from an early age in mathematics and gave me the rawest experience with fluid dynamics on the whitewater rivers of Canada, my mother who knows of every burden without saying, my sister who I would call to bail me out of jail and Tina with whom I finish this chapter of shenanigans with much more than a thesis.

Contents

Chapter 1

Introduction

1.1 Euler Equations

The Euler equations, describing the flow of an incompressible, inviscid fluid of uniform density were first published by Euler in 1757 ([11]). In the absence of external forces, they take the form:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \qquad \nabla \cdot u = 0. \tag{1.1}$$

Here, $u(x,t)$ is a vector field describing the fluid velocity at any moment in time and $p(x,t)$ is a scalar field describing the pressure exerted on the particle occupying position x by the surrounding fluid. The first equation is known as the momentum equation. The second equation is known as the incompressibility condition, which ensures that the volume of any blob of fluid remains constant as it is carried and deformed by the flow. If the fluid occupies a domain with boundary, then additionally, a condition is placed requiring u to be tangent to this boundary.

The standard problem is to find $u(x,t)$ and $p(x,t)$ given an initial velocity $u(x,0)$. Taking the divergence of the momentum equation, one sees that p satisfies some Poisson equation depending on u, and thus up to an additive constant, is uniquely defined by u. For this reason, when one speaks of the solution to the Euler equation, one typically speaks only of u.

Though the Euler equations are valid in any spatial dimension, they are most typically considered in dimensions two and three. The difference in behaviour of the solutions in 2D and 3D is more interesting than a mere reduction in dimension. In fact they display strikingly contrasting behaviour. In 3D turbulence, one typically observes vortices breaking up into smaller vortices, transferring energy from large scales to small scales. This phenomenon is known as the energy cascade. In the 2D case, one typically observes the opposite. Vortices tend to merge with other vortices of like rotation, producing larger vortices, resulting in an inverse energy cascade where energy flows from small scales to large scales.

While truly 2D fluids are unphysical, our interest in them is not solely a mathematical one. There are situations in which the motion of a 3D fluid is inhibited in one direction. For instance, the domain which the fluid occupies may be very thin in one dimension relative to the other two. Stratification and rotation of a fluid also serve to restrict motion in some direction. For example, consider the earth's atmosphere. It is very thin normal to the earth relative to the surface area of the earth. The atmosphere is stratified by the density of air, consisting of layers of decreasing density as we move away from earth. Finally, the earth's rotation induces a rotation on the atmosphere which further inhibits motion normal to the earth. The result is the existence of atmospheric phenomena which are dominated by 2D behaviour, such as hurricanes.

Though written down more than 250 years ago, the Euler equations still contribute a vast pool of unresolved problems in the mathematical and physical sciences. Let us discuss some important accomplishments in their mathematical study.

1.2 Properties of Solutions to the Euler Equations

The local in time existence and uniqueness of classical solutions was proved in the mid 1920s in the works of Lichtenstein ([20]) and Günther ([12]). In two dimension, existence and uniqueness of global in time classical solutions was proved in 1933 by Wolibner ([32]), and later by Kato ([18]) in the 1960s. Yudovich ([33]) extended the result in dimension two to existence and uniqueness of weak solutions with bounded vorticity, i.e. for $\omega_0 = \nabla \times u_0 \in L^\infty$. In modern language, we may state the classical results as follows:

Theorem 1.1.

Suppose $\Omega \in \mathbb{R}^n$ and $u_0(x) \in H^m(\Omega)$, $m > n/2+1$. Then there exists some $T > 0$ depending on u_0 for which the Euler equations 1.1 have a unique solution $u(x,t) \in C\big([0,T), H^m(\Omega)\big)$. If $n = 2$, then $T = \infty$.

In the 60s, Arnold ([3]) presented a geometric formulation of the Euler equations. He interpreted the Euler equations as equations for geodesics on the group of volume preserving diffeomorphisms with respect to the metric given by the energy. Doing so revealed that the Euler equations of fluid dynamics are an infinite dimensional analogue to the Euler equations of rigid body rotation, whose solutions are geodesics on the group of rotations.

Starting in the 90s, a new striking property of the Euler equations was discovered. Serfati ([27]), Shnirelman ([29]), and other authors ([8], [34], [22], [16], [15]) proved that the particle trajectories of solutions to Euler equations are real analytic curves, despite limited regularity of the velocity field. This fact was proved by varying methods, both real and complex. For example, following the work of Lichtenstein, Shnirelman wrote the equation for the trajectories as a Banach space-valued ODE with analytic right-hand side and the result follows from the standard modern theory of such equations. We note, for the time-independent 2D flow, the particle trajectories, flow lines and vorticity lines coincide, so they are analytic curves.

Theorem 1.2.

Under the assumptions of theorem 1.1, the particle trajectories $x_a(t)$, satisfying $x_a(0) = a$, $\frac{\mathrm{d}x_a}{\mathrm{d}t} = u(x_a(t), t)$ are analytic curves.

1.3 Long-time Behaviour of 2D Euler equations

Since in the two dimensional case, the solutions to the Euler equations exist for all time, it is natural to ask what can be said of these solutions as $t \to \infty$? Turning to computer simulations ([26], [28]) one sees the following picture: first there is a brief turbulent period where vortices of like rotation tend to filament under their respective strains and eventually merge to form larger ones. This process ends with the emergence of a stable system of coherent structures locked in some orbital 'dance'. These coherent structures consist of vortices which may have islands inside of them as well as satellites orbiting around them. These islands and satellites may have their own substructures, consisting of subsequent lakes and/or satellites. It is expected that these coherent structures (in the absence of viscosity) can form an infinite hierarchy of systems, subsystems, etc.

In light of this observation, Shnirelman ([28]) conjectured the existence of an attractor for the 2D Euler equations. This attractor is expected to consist of at least stationary, time periodic and time quasi-periodic flows. Given that for stationary flows, the level lines of vorticity $\omega = \nabla \times u$ are analytic curves, it is conjectured the same property holds true for time-periodic and time-quasiperiodic flows, as well as for any other elements of the attractor. Finally, it is conjectured that the components of this attractor are analytic manifolds in the space of divergence-free vector fields.

The conjecture motivates us to initiate a program to describe said attractor. We should start with those flows whose existence is known - the stationary ones. In the preceding work ([9]), we obtained the first result in this direction, where we provided a local description of the set of stationary flows without fixed point in a periodic channel. In a neighbourhood of the constant parallel flow, we showed this set forms an analytic Banach manifold. The next logical step is to provide an analogous local description of stationary flows having a single non-degenerate elliptic fixed point. This thesis is dedicated to accomplishing this task.

Objective. *Our goal is to provide a local description of the 2D stationary flows in a simply connected domain having a single non-degenerate fixed point. Furthermore, we aim to incorporate the analyticity of the flow (vorticity) lines in this description.*

Before we set out to accomplish this, it will be useful to cover some preliminaries relevant to the 2D stationary Euler equations.

1.4 Stationary Flows of the 2D Euler equation

The stationary (time-independent) incompressible Euler equation in a domain Ω is given by

$$u \cdot \nabla u + \nabla p = 0, \qquad \nabla \cdot u = 0, \qquad u \text{ tangent to } \partial\Omega.$$

By taking the curl of the first equation, one can eliminate the pressure term. In two dimensions, this gives $u \cdot \nabla\omega = 0$, where the vorticity $\omega = \nabla \times u$ is a vector normal to the flow and thus taken as a scalar. Observe, this equation says that 2D stationary flows are precisely those vector fields u which point along the level lines of their vorticity. Next, any divergence free vector field u can be written as the curl of some vector potential ψ, unique up to additive constant, known as the stream function. In two dimensions, it is a scalar satisfying $u = \nabla^{\perp}\psi$. It has the property that the flow u points along the level lines of ψ. In other words, the integral curves of u coincide with the level lines of ψ, which we call flow lines. Finally, it is related to the vorticity by the expression $\omega = \Delta\psi$. The equation of stationary flow is equivalent to the statement that $\Delta\psi$ is constant along flow lines $\psi =$ constant. At least locally, where ψ is monotone transversal to its level lines, it must satisfy $\Delta\psi = F(\psi)$. Since the flow is tangent to the boundary, ψ must be constant on each of its components. From here on, when we refer to stationary flows, they are always understood to be 2D, and they satisfy the equation:

$$\Delta\psi = F(\psi) \quad \text{in } \Omega, \qquad \psi = c_i \quad \text{on } \partial\Omega_i. \tag{1.2}$$

where $\partial\Omega_i$ are the components of the boundary of Ω.

Stationary flows can be interpreted another way: they are minimizers of the energy functional on the space of divergence-free vector fields with respect to area preserving diffeomorphisms ([4]). To produce a stationary flow, one can imagine deforming some stream function while preserving the topology of its level lines and area between them, so that it minimizes the Dirichlet energy $\|\nabla\psi\|_{L^2}^2$. One can draw an analogy to a system of elastic bands, each representing a flow line. They will configure themselves in a way to minimize their potential energy. This analogy provides intuition for a number of facts relating to stationary flows. For example, any stationary flow in a parallel channel without fixed point must necessarily be a parallel flow ([14]). Similarly, any stationary flow in a disk having a single fixed point must necessarily be a circular flow with fixed point at the disk centre ([31]). The analyticity of the flow lines can also be intuited through this analogy: any kink present in the system of flow lines stores some energy which can be relaxed if hammered away.

In the work of Šverák & Choffrut ([7]), they produced a smooth local parameterization of the manifold of stationary flows on annular domains. The smoothness of their parameterization necessitates working in the Fréchet space of smooth functions and thus using the Nash-Moser-Hamilton implicit function theorem.

Seeking to incorporate the analyticity of flow lines into the solutions, Shnirelman suggested reformulating the problem by representing a flow as a collection of its flow lines. The central idea he proposed was the coordinate change $\psi(x, y) \to y(x, \psi)$. Differing from the typical picture, the values of the stream function are treated as a variable, and the graphs of its level lines are treated as the unknown. This nonlinear coordinate change was introduced by von Mises in 1927 in his work on boundary layers ([30]), and by Dubreil-Jacotin in her 1934 work on free surface waves ([10]). Since Barron's 1989 ([5]) use of the coordinate change for the numerical study of flows over airfoils, it has seen numerous applications in computational problems, where it is known as the computational von Mises transform (see [13] for a survey). Its success is owed in part to the fact that it converts complicated domains of flow to rectangular 'computational' domains in (x, ψ) coordinates.

Using this idea in [9], we obtained results analogous to those of Šverák & Choffrut with less technical difficulty. We considered stationary flows without fixed point in a periodic channel bounded by flow lines $\psi = 0$ and $\psi = 1$. If these boundary flow lines are the graphs of functions $y = f(x)$ and $y = g(x)$, then the coordinate change $\psi(x, y) \to y = a(x, \psi)$ induces a transformation of equation 1.2 to:

$$\Phi(a) = F(\psi), \qquad a(x, 0) = f(x), \qquad a(x, 1) = g(x), \tag{1.3}$$

where the Laplacian $\Delta\psi$ is given by

$$\Phi(a) = -\frac{1}{a_\psi} a_{xx} + \frac{2a_x}{a_\psi^2} a_{x\psi} - \frac{1 + a_x^2}{a_\psi^3} a_{\psi\psi}$$

and the velocity field in $(\hat{x}, \hat{y})$ coordinates is

$$u(\psi, \theta) = \frac{(1, a_x)}{a_\psi}.$$

Here, $y = a(x, \psi)$ are the family of flow lines parameterized by $(x, \psi) \in \mathbb{T} \times [0, 1]$ and Φ is a second order quasilinear differential operator which is elliptic away from any fixed points of the flow. To incorporate the analyticity of flow lines, each flow line along $x \in \mathbb{T}$ is extended to the complex domain $\mathbb{T}_\sigma = \mathbb{T} \times i(\sigma, \sigma)$. Spaces $X_\sigma^m(\mathbb{T})$ and $Y_\sigma^m(\mathbb{T} \times [0,1])$ of complex analytic flow lines and partially complex analytic families of flow lines respectively, were introduced, with norms:

$$\|a(x)\|_{X_\sigma^m(\mathbb{T})} = \|a(\cdot + i\sigma)\|_{H^m(\mathbb{T})} + \|a(\cdot - i\sigma)\|_{H^m(\mathbb{T})},$$
$$\|a(x,\psi)\|_{Y_\sigma^m(\mathbb{T}\times[0,1])} = \|a(\cdot + i\sigma, \cdot)\|_{H^m(\mathbb{T}\times[0,1])} + \|a(\cdot - i\sigma, \cdot)\|_{H^m(\mathbb{T}\times[0,1])}.$$

By the analytic implicit function theorem in complex Banach spaces, we proved the existence of a local parameterization of solutions near the constant parallel flow:

Theorem 1.3.
Suppose $\|f(x)\|_{X_\sigma^{m-1/2}(\mathbb{T})} < \varepsilon$, $\|g(x) - 1\|_{X_\sigma^{m-1/2}(\mathbb{T})} < \varepsilon$, $\|F(\psi)\|_{H^{m-2}[0,1]} < \varepsilon$ and $\|a(x,\psi) - \psi\|_{Y_\sigma^m(\mathbb{T}\times[0,1])} < \varepsilon$, with ε sufficiently small. Then equation 1.3 has a unique solution $a(x,\psi)$ near the constant parallel flow which depends analytically on parameters (F, f, g).

In this work, the same governing philosophy of viewing a function as a collection of its level lines will be used to generalize the above result to stationary flows having a single elliptic fixed point. We are now ready to formulate how we will do so.

1.5 Stationary Flows with an Elliptic Fixed Point

The prototypical stationary flow having a single, non-degenerate elliptic fixed point is described by stream function $\psi = x^2 + y^2$, our logical starting point. This flow has constant vorticity $F(\psi) = 4$ and describes the motion of a fluid rotating as a rigid body. The flow lines are concentric circles around the origin where the fixed point is located which corresponds to the level set $\psi = 0$. Let us restrict the domain to the unit disk $\mathbb{D}$. Then the boundary flow line is the level set $\psi = 1$.

Suppose there is some suitable perturbation of the domain $\mathbb{D} \to \Omega$ and of the vorticity, generating a new stationary flow ψ with the same topological structure. Suppose that this new stationary flow also satisfies $\psi = 0$ at the fixed point and $\psi = 1$ on the boundary $\partial\Omega$. What can be said of its level sets?

It is known that a stationary flow in a disk having a single fixed point must be circular and therefore the fixed point must be positioned at the disk's centre. This implies that should the perturbation of $\mathbb{D}$ be a mere translation, the fixed point must translate accordingly. We expect this rigidity of the fixed point's orientation to hold for general non-circular flows. So let us introduce the position of the fixed point $p = (p_x, p_y) \in \Omega$ as an unknown in the problem. The remaining flow lines should be close to concentric circles around p. Let (r, θ) be the polar coordinates centred at p. Then the flow lines will be graphs of a family of polar functions $r = a(\psi, \theta)$, parameterized by $\psi \in [0, 1]$ ranging from the fixed point to the boundary. Notice, in order for the level set $\psi = 0$ to define a single point, we must require $r = a(0, \theta) = 0$.

Let us now find an expression for the velocity $u = \nabla^\perp \psi$ in the new coordinates. Inverting the Jacobian $\frac{\partial(r,\theta)}{\partial(\psi,\theta)}$ of the transformation defined by $r = a(\psi, \theta)$ yields relations $\frac{\partial \psi}{\partial r} = \frac{1}{a_\psi}$, $\frac{\partial \psi}{\partial \theta} = -\frac{a_\theta}{a_\psi}$, $\frac{\partial \theta}{\partial r} = 0$, $\frac{\partial \theta}{\partial \theta} = 1$. By the chain rule, we get $\frac{\partial}{\partial r} = \frac{1}{a_\psi}\frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \theta} = -\frac{a_\theta}{a_\psi}\frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta}$. This gives velocity field

$$u(\psi, \theta) = \frac{1}{a_\psi}\big(\frac{a_\theta}{a}, 1\big)$$

in $(\hat{r}, \hat{\theta})$ coordinates. We see that to have a well defined stagnation point at $\psi = 0$, along with condition $a(0, \theta) = 0$, we also require $|a_\psi(\psi, \theta)| \to \infty$ as $\psi \to 0^+$. Finally, $\frac{a_\theta}{a}$ should remain bounded. Observe for example, if $\psi(r, \theta) = r^2$ then $a(\psi, \theta) = \psi^{1/2}$. The critical point of this paraboloid is transformed to a cusp singularity at $\psi = 0$.

Applying the above results, we write $\Delta \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$ in our new coordinates. We define $\Delta \psi = \Xi(a)$ for which we obtain the expression

$$\Xi(a) = -\frac{1}{a_\psi^3}\Big(1 + \frac{a_\theta^2}{a^2}\Big)a_{\psi\psi} + 2\Big(\frac{a_\theta}{a^2 a_\psi^2}\Big)a_{\psi\theta} - \Big(\frac{1}{a^2 a_\psi}\Big)a_{\theta\theta} + \frac{1}{a a_\psi}. \tag{1.4}$$

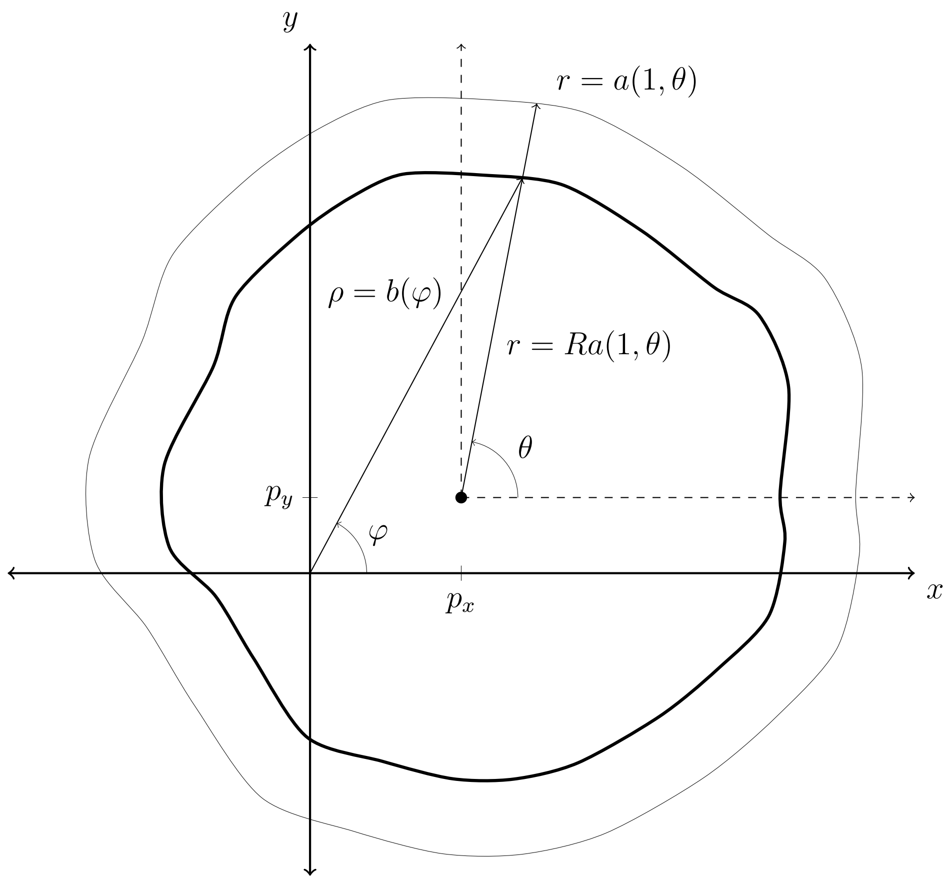
$\Xi(a)$ is a second order quasilinear differential operator of form $Aa_{\psi\psi} + 2Ba_{\psi\theta} + Ca_{\theta\theta} + D$. Such operators are elliptic if $AC - B^2 > 0$. A straightforward calculation shows $AC - B^2 = \frac{1}{a^2 a_\psi^4}$. If we restrict to fixed points that are non-degenerate, then ψ is some deformed paraboloid and we expect $a(\psi, \theta)$ to behave like $\psi^{1/2}$ as $\psi \to 0^+$. In this case $AC - B^2 \to 0$ as $\psi \to 0^+$. We conclude that Ξ is elliptic away from the fixed point, but this ellipticity degenerates as we approach the fixed point.

We now turn to the boundary condition. The boundary $\partial\Omega$ is described by the graph of $r = a(1, \theta)$. This expression is in coordinates (r, θ), which are centred on the fixed point at p, which we have determined is part of the solution, an unknown. To meaningfully treat the domain as a parameter of the problem, we should represent its boundary as the graph of some function in a fixed coordinate frame. Since we are deforming a flow whose fixed point lies at the origin, we expect given a sufficiently small perturbation, that the boundary $\partial\Omega$ can also be described as the graph of a polar function relative to the origin. So let us define (ρ, φ) as the polar coordinates centred at the origin. Then we can treat Ω as a parameter by taking its boundary to be described by the graph of some function $\rho = b(\varphi)$.

Before we can write down the boundary conditions, we must address an additional obstacle. In terms of equation 1.2, we have a peculiar situation. We are trying to solve for ψ with both a boundary condition $\psi|_{\partial\Omega} = 1$, as well as an additional interior point condition $\psi(p) = 0$. From the perspective of solving for the stream function, imposing its value at the fixed point is unnatural and leads to an overdetermined problem. From the perspective of solving for the flow lines, it is rather essential. After all, we are deforming a family of flow lines parameterized by $\psi \in [0, 1]$, and this domain should be a constant if we are to define function spaces for our problem, define operators on these spaces, etc.

To gain some intuition how to overcome this issue, let us consider circular flows around the origin on a disk $\mathbb{D}_R$ of radius R. Additionally, suppose the vorticity $F(\psi) = \omega$ is constant. Then $\psi = \psi(r)$ is radial and direct integration of 1.2 yields a general solution $\psi(r) = \omega r^2/4 + c \ln r + d$. Imposing the condition that $\psi = 0$ at the origin gives $\psi(r) = \omega r^2/4$. Now the boundary condition $\psi(R) = 1$ can only be satisfied for a single choice of ω. Thus imposing ψ both at the fixed point and at the boundary requires some compatibility between the radius of the domain and scaling of vorticity. We expect this requirement to generalize to non-circular flows as well. In other words, given any stationary flow in Ω with values of ψ imposed both at the fixed point and at the boundary, rescaling Ω with respect to the fixed point p yields in our formulation an ill-posed problem. To obtain a well-posed problem then, we should treat only the 'shape' of the domain as a parameter, but not its 'radius' which instead depends on the vorticity.

To work around this, we introduce an additional degree of freedom R to the solution, whose role is to solve the following boundary condition: given a domain Ω, find a solution $a(\psi, \theta)$ which when rescaled radially by R with respect the fixed point p, matches the boundary condition. In other words, the graph of $r = Ra(\psi, \theta)$ describes said boundary. We refer to the following figure of the boundary condition:



The inner deformed circle represents the prescribed boundary flow line, defined by the graph of $\rho = b(\varphi)$. We seek a family of flow lines $a(\psi, \theta)$ about some fixed point p which when rescaled by some R, matches the boundary at $\psi = 1$. The unscaled flow line $r = a(1, \theta)$, depicted by the outer deformed circle, defines a new domain of flow of the same shape as the prescribed one, of a radius compatible with the prescribed vorticity.

We obtain the following equations relating $b(\varphi)$, R and $a(1, \theta)$:

$$b(\varphi)\cos\varphi = p_x + Ra(1,\theta)\cos\theta, \qquad b(\varphi)\sin\varphi = p_x + Ra(1,\theta)\sin\theta.$$

Squaring and summing these equations yields

$$b^2(\varphi) = R^2a^2(1,\theta) + 2Ra(1,\theta)\big(p_x\cos\theta + p_y\sin\theta\big) + p_x^2 + p_y^2.$$

Dividing yields

$$\tan\varphi = \frac{p_y + Ra(1,\theta)\sin\theta}{p_x + Ra(1,\theta)\cos\theta}.$$

Taking the inverse of $\tan\varphi$, we can combine these equations to eliminate φ. To do so, we must be careful. Typically, arctan is a function defined to have values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. To have a meaningful boundary condition, we should instead define $\varphi = \arctan(y, x)$ as the function onto $\mathbb{T}$, whose values are the angle between plane vector (x, y) and the x-axis. We can then define the nonlinear boundary map

$$\begin{aligned} B(b, R, p, a) = -b^2\Big(&\arctan\big(p_y + Ra(1,\theta)\sin\theta, p_x + Ra(1,\theta)\cos\theta\big)\Big) \\ &+ R^2a^2(1,\theta) + 2Ra(1,\theta)\big(p_x\cos\theta + p_y\sin\theta\big) + p_x^2 + p_y^2, \end{aligned} \tag{1.5}$$

and the boundary condition to our problem is given by $B(b, R, p, a) = 0$.

We obtain the following nonlinear boundary value problem for the flow lines of a stationary flow having a single elliptic fixed point:

$$\begin{cases} \Xi(a) = F(\psi) \qquad \text{in } \Pi = (0,1] \times \mathbb{T}, \\ a(0,\theta) = 0, \\ \lim\limits_{\psi\to 0^+} |a_\psi| = \infty, \\ B(b, R, p, a) = 0. \end{cases} \tag{1.6}$$

This equation is to be solved for $R \in \mathbb{R}$, $p \in \mathbb{R}^2$ and function $a(\psi, \theta)$ defined on domain $\Pi = (0,1] \times \mathbb{T}$, given parameters $b(\varphi)$ on $\mathbb{T}$ and $F(\psi)$ on $(0,1]$.

The main tool to solve this problem will be the analytic implicit function theorem in complex Banach spaces, which gives condition under which an operator equation with parameter has a unique local solution.

Theorem 1.4 (Analytic Banach implicit function theorem)**.**
Let X, Y, Z be complex Banach spaces and $f : X \times Y \to Z$ be an analytic map in a neighbourhood of $(x_0, y_0) \in X \times Y$. Suppose $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) : Y \to Z$ is an isomorphism. Then there exists a neighbourhood of $(x_0, y_0, 0) \in X \times Y \times Z$ in which the equation $f(x, y) = 0$ has a unique solution, which is parameterized by an analytic function $y = g(x) : X \to Y$.

We look for solutions near the circular flow with constant vorticity $\psi = r^2$ in the disk, described in our coordinates by $R = 1$, $p = 0$, $a(\psi, \theta) = \psi^{1/2}$, $b(\varphi) = 1$, $F(\psi) = 4$. The bulk of this thesis, split over the next three chapters, is devoted to defining the appropriate function spaces for the problem and proving that in these spaces, the conditions of the analytic implicit function theorem are satisfied. Let us summarize the results which follow.

(i) We start by introducing the Kondratev space of functions $u(\psi,\theta)$ on the strip $\Pi = (0,1] \times \mathbb{T}$, with norm

$$\|u(\psi,\theta)\|^2_{K^m_\gamma(\Pi)} = \sum_{p+q=0}^{m} \left\|\psi^{p-\gamma}\partial^p_\psi\partial^q_\theta u(\psi,\theta)\right\|^2_{L^2(\Pi)} < \infty.$$

While such spaces are the natural setting in which the relevant degenerate operators are Fredholm, their asymptotics as $\psi \to 0^+$ are more flexible than our solutions permit.

We next construct the spaces of functions of fixed asymptotics:

$$J^m_{\lambda,\gamma}(\Pi) = \left\{a(\psi,\theta) = \psi^\lambda v(\theta) + w(\psi,\theta)\colon v(\theta) \in H^m(\mathbb{T}), w(\psi,\theta) \in K^m_{\lambda+\gamma}(\Pi)\right\},$$

whose functions are the sum of a leading term of order ψ^λ and a higher order remainder term taken in the Kondratev space. For $\gamma \geq 1/2$, $J^m_{\lambda,\gamma}(\Pi)$ is a Banach space equivalent to the direct sum $H^m(\mathbb{T}) \oplus K^m_{\lambda+\gamma}(\Pi)$, with norm defined accordingly.

We adapt the Paley-Wiener theorem to give the above functions a partial complex analytic structure. Namely, we consider the subset of above functions which can be analytically continued in θ from $\mathbb{T}$ to the complex strip $\mathbb{T}_\sigma = \mathbb{T} \times i(-\sigma,\sigma)$. We define the space $J^{m,\sigma}_{\lambda,\gamma}(\Pi)$ of such partially-analytic functions with norm

$$\|a(\psi,\theta)\|_{J^{m,\sigma}_{\lambda,\gamma}(\Pi)} = \|a(\cdot,\cdot+i\sigma)\|_{J^m_{\lambda,\gamma}(\Pi)} + \|a(\cdot,\cdot-i\sigma)\|_{J^m_{\lambda,\gamma}(\Pi)}.$$

For $\lambda = 1/2$, $\gamma > 1/2$, $m > 1$, this space appropriately defines the families of complex analytic flow lines for our problem. Its functions are continuous in ψ, analytic in θ and in a sufficiently small neighbourhood of $\psi^{1/2}$, define a unique non-degenerate stagnation point at $\psi = 0$. The position of the stagnation point p and the scaling factor R (both unkowns) are also extended from $\mathbb{R}^2$ to $\mathbb{C}^2$ and $\mathbb{R}$ to $\mathbb{C}$, respectively.

The restriction of functions in $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ to a given flow line at $\psi =$ constant defines the space $X^{m-1/2}_\sigma(\mathbb{T})$ of individual flow lines which are complex analytic in the strip $\mathbb{T}_\sigma$ and Sobolev on the strip boundary. The norm is given by

$$\|a(\theta)\|_{X^m_\sigma(\mathbb{T})} = \|a(\cdot+i\sigma)\|_{H^m(\mathbb{T})} + \|a(\cdot-i\sigma)\|_{H^m(\mathbb{T})}.$$

(ii) We then study the linear problem associated to 1.6. Linearizing with respect to (R, p, a) at solution $(F, b, R, p, a) = (4, 1, 1, 0, \psi^{1/2})$, we obtain maps

$$a \to \frac{\partial \Xi(\psi^{1/2})}{\partial a} a = -8\psi^{-1/2}\Big[\psi^2 \frac{\partial^2}{\partial \psi^2} + 2\psi \frac{\partial}{\partial \psi} + \frac{1}{4}(I + \frac{\partial^2}{\partial \theta^2})\Big] a(\psi, \theta),$$
$$(R, p, a) \to \frac{\partial B(1, 0, 1, \psi^{1/2})}{\partial (R, p, a)}(R, p, a) = 2\Big[R + p_x \cos\theta + p_y \sin\theta + a(1, \theta)\Big].$$

Without loss of generality, we drop the factors -8 and 2 from the above expressions. Writing $\sin\theta$ and $\cos\theta$ in terms of exponentials, we get the degenerate elliptic linear problem

$$\begin{cases} \psi^{-1/2}\Big[\psi^2 \frac{\partial^2}{\partial \psi^2} + 2\psi \frac{\partial}{\partial \psi} + \frac{1}{4}(I + \frac{\partial^2}{\partial \theta^2})\Big] a(\psi, \theta) = f(\psi, \theta) \\ R + (\frac{p_x - ip_y}{2})e^{i\theta} + (\frac{p_x + ip_y}{2})e^{-i\theta} + a(1, \theta) = g(\theta). \end{cases} \tag{1.7}$$

We construct explicit solutions by factoring the above second order operator into the product of two first order degenerate operators, whose inverses are weighted averages. The Hardy inequality is used to establish Fredholmness of 1.7 in the spaces

$$\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to J^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T}).$$

The presence of a two-dimensional cokernel in the above spaces means invertibility can only be established on the codimension-two subspace of the target space, defined by:

$$\widetilde{J}^{m,\sigma}_{0,\gamma}(\Pi) = \left\{ u(\psi, \theta) = v(\theta) + w(\psi, \theta) \in J^{m,\sigma}_{0,\gamma}(\Pi) \colon \int_{\mathbb{T}} v(\theta) e^{\pm 2i\theta}\, \mathrm{d}\theta = 0 \right\}.$$

Namely, for $1/2 < \gamma < 1$, we prove 1.7 defines a Banach space isomorphism in

$$\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T}).$$

(iii) Finally, we study the nonlinear operator $(F, b, R, p, a) \to (\Xi(a) - F, B)$, whose zeroes are the solutions of 1.6.

The analyticity of $a \to \Xi(a) : J^{m,\sigma}_{1/2,\gamma}(\Pi) \to J^{m-2,\sigma}_{0,\gamma}(\Pi)$ is reduced to the study of superposition operators in $J^{m-2,\sigma}_{0,\gamma}(\Pi)$. We show such maps, defined by composition with an analytic function, are analytic in this space when m is sufficiently high. We also show that Ξ actually maps to $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$, a crucial parallel to the linear problem. Taking

$J_{0,\gamma}^{m-2}(0,1]$ to be the space of complex vorticities, we conclude that in a neighbourhood of $a = \psi^{1/2}$, the map 1.4

$$(F, a) \to \Xi(a) - F : J_{0,\gamma}^{m-2}(0,1] \times J_{1/2,\gamma}^{m,\sigma}(\Pi) \to \widetilde{J}_{0,\gamma}^{m-2,\sigma}(\Pi)$$

is complex analytic.

Next we study the nonlinear boundary map 1.5. This map contains a coordinate transformation $\theta \to \varphi$ of the complexified polar angle defining the boundary flow line $r = a(1, \theta)$. This coordinate change is not a self map on the complex strip $\mathbb{T}_\sigma$. Instead, it is some deformation of said strip, depending on the solution itself. For solutions sufficiently close to $R = 1$, $p = 0$, $a = \psi^{1/2}$, the image of every such coordinate change is contained in some slightly larger strip $\mathbb{T}_\tau$, where $\tau > \sigma$. Analyticity of the boundary operator, which follows from results on superposition operators on space $X_\sigma^{m-1/2}(\mathbb{T})$ requires that the boundary flow line $\rho = b(\varphi)$ be holomorphic on this larger strip $\mathbb{T}_\tau$. In particular, it can be taken in any Banach space $\mathrm{H}(\mathbb{T}_\tau)$ of functions analytic in $\mathbb{T}_\tau$. We establish that given any $\tau > \sigma$, there exists a neighbourhood of $R = 1$, $p = 0$, $a = \psi^{1/2}$ for which the boundary map 1.5

$$(b, R, p, a) \to B : \mathrm{H}(\mathbb{T}_\tau) \times \mathbb{C}^3 \times J_{1/2,\gamma}^{m,\sigma}(\Pi) \to X_\sigma^{m-1/2}(\mathbb{T})$$

is complex analytic.

We conclude that for any $\tau > \sigma$ and m sufficiently large, the operator $(F, b, R, p, a) \to (\Xi(a) - F, B)$ defining 1.6 is complex analytic between Banach spaces

$$J_0^{m-2}(0,1] \times \mathrm{H}(\mathbb{T}_\tau) \times \mathbb{C}^3 \times J_{1/2,\gamma}^{m,\sigma}(\Pi) \to \widetilde{J}_{0,\gamma}^{m-2,\sigma}(\Pi) \times X_\sigma^{m-1/2}(\mathbb{T})$$

in a sufficiently small neighbourhood of solution $(F, b, R, p, a) = (4, 1, 1, 0, \psi^{1/2})$. The analytic implicit function theorem thus provides a locally unique solution to 1.6.

Chapter 2

Function Spaces

The theory of elliptic boundary value problems is well established in the standard Sobolev spaces. The Fredholm property of the associated operators and the results of elliptic regularity make these spaces a natural functional setting for posing such problems. The situation changes when the ellipticity of an equation degenerates in some part of the domain. New function spaces must be introduced to obtain results analogous to those of the standard theory. These spaces must reflect the more exotic behaviour of solutions to these equations at the points of degeneracy. The equations of our study (1.6, 1.7) degenerate at the boundary $\{\psi = 0\}$, so we too must look beyond the usual Sobolev spaces.

In this chapter, we develop the appropriate function spaces to formally pose and solve the nonlinear boundary value problem 1.6. We begin by introducing the weighted Sobolev spaces of Kondratev: the natural setting for our linearized equation. Functions in these spaces have asymptotic behaviour as $\psi \to 0^+$ more flexible than our situation permits. We account for this by defining new spaces of functions of a fixed order leading term plus a higher order remainder term, taken in an appropriate Kondratev space. Finally, we will extend these functions to a suitable complex domain to incorporate the partial analytic nature of the solutions. While seemingly exotic, the resulting spaces are the natural and correct setting for our problem. The following chapters will confirm the validity of this claim.

Before continuing, we define for the sake of completeness the Sobolev spaces we will use either explicitly, or that will be relevant to further discussions. Details on their properties may be found in any standard text, for example ([24]).

Definition 2.1 (Integer Sobolev space on the circle)**.**

Let $m \in \mathbb{N}_0$. Define $H^m(\mathbb{T})$ to be the space of measurable functions on $\mathbb{T}$ with norm

$$\|u(\theta)\|^2_{H^m(\mathbb{T})} = \sum_{p=0}^{m} \|D^p u(\theta)\|^2_{L^2(\mathbb{T})} < \infty.$$

Definition 2.2 (Non-integer Sobolev space on the circle)**.**

Let $m \geq 0$ and let $[m]$ be the integer part of m. Define $H^m(\mathbb{T})$ to be the space of measurable functions on $\mathbb{T}$ with norm

$$\|u(\theta)\|^2_{H^m(\mathbb{T})} = \|u(\theta)\|^2_{H^{[m]}(\mathbb{T})} + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\left|D^{[m]}u(\theta) - D^{[m]}u(\theta')\right|^2}{|\theta-\theta'|^{1+2(m-[m])}} \, \mathrm{d}\theta \, \mathrm{d}\theta' < \infty.$$

Remark 2.3. The above spaces (m integer and non-integer) have an equivalent norm in terms of the Fourier series given by

$$\|u(\theta)\|^2_{H^m(\mathbb{T})} = \sum_{k} \left(1+k^2\right)^m |\hat{u}_k|^2 .$$

Definition 2.4 (Integer Sobolev space on the periodic strip)**.**

Let $m \in \mathbb{N}_0$. Define $H^m(\Pi)$ to be the space of measurable functions on $\Pi = (0,1] \times \mathbb{T}$ with norm

$$\|u(\psi,\theta)\|^2_{H^m(\Pi)} = \sum_{p+q=0}^{m} \left\|\partial_\psi^p \partial_\theta^q u(\psi,\theta)\right\|^2_{L^2(\Pi)} < \infty.$$

This space has an equivalent norm in terms of the partial Fourier series given by

$$\|u(\psi,\theta)\|^2_{H^m(\Pi)} = \sum_{p+q=0}^{m} \sum_{k} k^{2q} \|D^p \hat{u}_k(\psi)\|^2_{L^2(0,1]} .$$

2.1 Kondratev Spaces

It is known that the typical elliptic regularity results fail in the presence of singular points in the domain. For example, the solution to the Dirichlet problem for the Poisson equation in a domain with Lipschitz boundary is in general only in $H^{3/2}$, even for smooth right-hand side ([17]). The theory of elliptic equations on domains with conical singularities has been developed since the mid 1960s, starting with the works of Kondratev ([19]). The spaces introduced in his work allow for precise description of the singularities of solutions and their derivatives at the vertex. Furthermore, one can establish shift theorems between these spaces, analogous to those of the standard elliptic regularity.

In fact, elliptic equations on manifolds with singularities are intimately connected with the study of degenerate elliptic equations. See ([23]) for a detailed discussion. To summarize, in practice, one removes the singular point from the manifold and stretches out the resulting open submanifold (imagine a cone, stretched into a cylinder after deleting the vertex). In doing so, the equation on the singular manifold is transferred to a degenerate equation on a regular manifold with boundary. Let us give a particularly relevant example. Consider the surface of a right cone with circular base in $\mathbb{R}^3$. Let r be the distance to the vertex and θ be the polar angle along the circular base. The Laplacian on this surface is given by

$$\Delta = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + c^2\frac{\partial^2}{\partial \theta^2}\right],$$

where c depends on the angle of cone. Notice, the above operator is a polynomial in $r\partial_r$, an order-one degenerate operator.

Let us return to our problem. We have a family of closed level lines parameterized by ψ on the periodic strip Π. This parameterization degenerates at $\{\psi = 0\}$, where the level lines collapse to a point. On this strip we have a degenerate elliptic equation, with, modulo a factor of $\psi^{-1/2}$, degeneracies of type $\psi\partial_\psi$, like those on the surface of the cone. It appears then that our problem is quite analogous to the discussion above. We too have a degenerate equation on a 'stretched' domain with a boundary produced by blowing up a point into a circle. It is not then unreasonable to hope that the function spaces appropriate for posing elliptic boundary value problems on conical domains can also be adapted to pose our own degenerate boundary value problem.

Let us now introduce the Kondratev spaces relevant to our problem; spaces on the interval $(0,1]$ and on the periodic strip $\Pi = (0,1] \times \mathbb{T}$, with degeneracies at $\{0\}$ and $\{0\} \times \mathbb{T}$ respectively.

Definition 2.5 (Kondratev space on the interval)**.**
Let $m \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$. Define $K^m_\gamma(0,1]$ to be the space of measurable functions on $(0,1]$

with norm

$$\|u(\psi)\|^2_{K^m_\gamma(0,1]} = \sum_{p=0}^{m} \left\|\psi^{p-\gamma} D^p u(\psi)\right\|^2_{L^2(0,1]} < \infty.$$

Definition 2.6 (Kondratev space on the periodic strip)**.**
Let $m \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$. Define $K^m_\gamma(\Pi)$ to be the space of measurable functions on Π with norm

$$\|u(\psi,\theta)\|^2_{K^m_\gamma(\Pi)} = \sum_{p+q=0}^{m} \left\|\psi^{p-\gamma} \partial^p_\psi \partial^q_\theta u(\psi,\theta)\right\|^2_{L^2(\Pi)} < \infty.$$

By Parseval's theorem, this space has an equivalent norm in terms of the partial Fourier series given by

$$\|u(\psi,\theta)\|^2_{K^m_\gamma(\Pi)} = \sum_{p+q=0}^{m} \sum_{k} k^{2q} \left\|\psi^{p-\gamma} D^p \hat{u}_k(\psi)\right\|^2_{L^2(0,1]}.$$

These spaces consist of two scales; the usual regularity scale m of integrable weak derivatives, and the scale γ which quantifies the strength of the weight at $\psi = 0$. To get some intuition, suppose for a moment that $m = 0$. If $\gamma > 0$, then the weight blows up at $\psi = 0$, forcing functions in this space to vanish sufficiently rapidly as $\psi \to 0^+$. In contrast, if $\gamma < 0$, the weight vanishes at $\psi = 0$, allowing functions in the space to have some controlled blow up. In general, the smaller γ is, the more singular the functions can be and conversely, the greater γ is, the faster they must decay. Additionally, the weight is homogeneous with respect to differentiation in ψ. That is, the order of the weight increases by one with each derivative in ψ, balancing the corresponding increase in the order of singularity arising from such differentiation. To illustrate the advantage of this feature, consider the function $u = \psi^\lambda$. If $\lambda \notin \mathbb{N}_0$, then this function is either unbounded or eventually its derivatives are. However, the inclusion of this function in the spaces depends only on γ and not on m. If the weight was not homogeneous, then the inclusion would necessarily depend on m. In this regard, the two scales in the above space are uncoupled; m controls the general isotropic regularity, and γ controls the asymptotic behaviour as $\psi \to 0^+$. These scales can be summarized by the inclusion $K^{m_1}_{\gamma_1} \subset K^{m_2}_{\gamma_2}$, for $\gamma_1 \geq \gamma_2$ and $m_1 \geq m_2$.

The first basic property of these spaces worth mentioning is that away from $\psi = 0$, they are equivalent to their unweighted counterparts H^m.

Proposition 2.7. *Away from $\psi = 0$, the K^m_γ and H^m norms are equivalent.*

Proof. Restrict ψ to $\varepsilon \leq \psi \leq 1$, for some $\varepsilon \in (0,1)$. When $p-\gamma \geq 0$, we have $\varepsilon^{p-\gamma} \leq \psi^{p-\gamma} \leq 1$. Similarly, when $p-\gamma \leq 0$, we have $1 \leq \psi^{p-\gamma} \leq \varepsilon^{p-\gamma}$. Thus, each term in the K^m_γ norm has the bound

$$\min\{1, \epsilon^{p-\gamma}\} \|D^p u(\psi)\|_{L^2[\varepsilon,1]} \leq \left\|\psi^{p-\gamma} D^p u(\psi)\right\|_{L^2[\varepsilon,1]} \leq \max\{1, \epsilon^{p-\gamma}\} \|D^p u(\psi)\|_{L^2[\varepsilon,1]}$$

or analogously, on the strip $[\varepsilon, 1] \times \mathbb{T}$. This establishes the equivalence of norms

$$c\|u\|_{H^m} \leq \|u\|_{K^m_\gamma} \leq C\|u\|_{H^m}$$

on either the interval $[\varepsilon, 1]$ or the strip $[\varepsilon, 1] \times \mathbb{T}$. Note, constants c and C depend on ε. $\square$

Remark 2.8. We conclude that the spaces above have the desired property of differing from the standard Sobolev space only in their behaviour as $\psi \to 0^+$. We expect this of any candidate function space, because away from $\psi = 0$, our operators are standard elliptic operators and thus the standard theory should apply.

Let us now discuss how multiplication by powers of ψ and differentiation act on these spaces.

Proposition 2.9.

1. *$u(\psi,\theta) \to \psi^\alpha u(\psi,\theta)$: $K^m_\gamma(\Pi) \to K^m_{\gamma+\alpha}(\Pi)$ defines an isomorphism (for any $\alpha \in \mathbb{R}$).*
2. *$u(\psi,\theta) \to \partial_\psi u(\psi,\theta)$: $K^m_\gamma(\Pi) \to K^{m-1}_{\gamma-1}(\Pi)$ is bounded (for $m \geq 1$).*
3. *$u(\psi,\theta) \to \partial_\theta u(\psi,\theta)$: $K^m_\gamma(\Pi) \to K^{m-1}_{\gamma}(\Pi)$ is bounded (for $m \geq 1$).*

The first two hold analogously for functions $u(\psi)$ on the interval $(0,1]$, replacing ∂_ψ by D.

Proof.
The statements follow from direct computations, with little difference between the case on the interval and the case on the strip. We show them on the strip.

For the first statement, by the product rule, we can write

$$\partial^p_\psi\left(\psi^\alpha u\right) = \sum_{p'=0}^{p} c_{p,p'} \partial_\psi^{p-p'}\left(\psi^\alpha\right) \partial_\psi^{p'}(u) = \sum_{p'=0}^{p} c_{p,p',\alpha} \psi^{\alpha-p+p'} \partial_\psi^{p'} u.$$

Then we get

$$\begin{aligned}
\|\psi^\alpha u\|^2_{K^m_{\gamma+\alpha}(\Pi)} &= \sum_{p+q=0}^{m} \left\|\psi^{p-\gamma-\alpha}\partial^p_\psi\partial^q_\theta\left(\psi^\alpha u\right)\right\|^2_{L^2(\Pi)} \\
&\leq c\sum_{p+q=0}^{m}\sum_{p'=0}^{p}\left\|\psi^{p'-\gamma}\partial^{p'}_\psi\partial^q_\theta u\right\|^2_{L^2(\Pi)} \\
&\leq C\,\|u\|^2_{K^m_\gamma(\Pi)}\,,
\end{aligned}$$

where the last inequality follows because p' in the summation ranges from 0 to $m-q$. This gives the boundedness of map $u \to \psi^\alpha u\colon K^m_\gamma \to K^m_{\gamma+\alpha}$ as well as its inverse $v \to \psi^{-\alpha}v\colon K^m_{\gamma+\alpha} \to K^m_\gamma$, proving the first statement.

For the second statement, we have

$$\begin{aligned}
\|\partial_\psi u\|^2_{K^{m-1}_{\gamma-1}(\Pi)} &= \sum_{p+q=0}^{m-1} \left\|\psi^{p-\gamma+1}\partial^p_\psi\partial^q_\theta\left(\partial_\psi u\right)\right\|^2_{L^2(\Pi)} \\
&= \sum_{p+q=0}^{m-1}\left\|\psi^{p+1-\gamma}\partial^{p+1}_\psi\partial^q_\theta u\right\|^2_{L^2(\Pi)} \\
&\leq \|u\|^2_{K^m_\gamma(\Pi)}\,,
\end{aligned}$$

where the last inequality follows since q ranges from 0 to $m-1$ and $p+1$ ranges from 1 to $m-q$.

The third statement follows similarly. We have

$$\begin{aligned}
\|\partial_\theta u\|^2_{K^{m-1}_{\gamma}(\Pi)} &= \sum_{p+q=0}^{m-1} \left\|\psi^{p-\gamma}\partial^p_\psi\partial^q_\theta\left(\partial_\theta u\right)\right\|^2_{L^2(\Pi)} \\
&= \sum_{p+q=0}^{m-1}\left\|\psi^{p-\gamma}\partial^{p}_\psi\partial^{q+1}_\theta u\right\|^2_{L^2(\Pi)} \\
&\leq \|u\|^2_{K^m_\gamma(\Pi)}\,,
\end{aligned}$$

where this time the last inequality follows since p ranges from 0 to $m-1$ and $q+1$ from 1 to $m-q$. $\square$

Next, we address the relevant traces of functions in these spaces. Namely, restrictions from the strip Π to the circle $\{\psi = \text{constant}\}$. We are particularly interested in restrictions to the boundaries of the strip, that is to $\psi = 0$ and $\psi = 1$. We start with the latter, which follows immediately from the equivalence of K^m_γ and H^m away from $\psi = 0$.

Proposition 2.10. *Let $m > 1/2$ and fix $0 < \psi \leq 1$. Then the restriction to ψ defines the bounded operator*

$$u(\cdot,\cdot) \to u(\psi,\cdot) \colon K^m_\gamma(\Pi) \to H^{m-1/2}(\mathbb{T}).$$

In particular, the restriction to $\psi = 1$ is bounded in these spaces.

Proof. As we saw in 2.7, away from $\psi = 0$, the spaces $K^m_\gamma(\Pi)$ and $H^m(\Pi)$ are equivalent. Then the result follows from the standard trace theorem in Sobolev spaces, where the restriction is bounded from $H^m(\Pi)$ to $H^{m-1/2}(\mathbb{T})$. To be precise, take $0 < \varepsilon < \psi$. Then we have the following inequalities:

$$\|u(\psi,\cdot)\|_{H^{m-1/2}(\mathbb{T})} \leq c\|u\|_{H^m([\varepsilon,1]\times\mathbb{T})} \leq C\|u\|_{K^m_\gamma([\varepsilon,1]\times\mathbb{T})} \leq C\|u\|_{K^m_\gamma(\Pi)}.$$

□

Remark 2.11. Notice the above bound is not uniform in ψ because the constant in the middle inequality depends on ε, which in turn depends on ψ. This is expected, after all, the behaviour as $\psi \to 0^+$ can be singular. The following result improves this estimate to include the dependence on ψ, by exploiting a dilation invariance in the K^m_γ norm.

Proposition 2.12. *There exists $C > 0$ depending on γ, m, k for which*

1. $\left|D^k u(\psi)\right| \leq C\psi^{\gamma-k-1/2}\|u\|_{K^m_\gamma(0,1]}$ *for $m - k > 1/2$*
2. $\left\|\partial^k_\psi u(\psi,\cdot)\right\|_{H^{m-k-1/2}(\mathbb{T})} \leq C\psi^{\gamma-k-1/2}\|u\|_{K^m_\gamma(\Pi)}$ *for $m - k > 1/2$*
3. $\left|\partial^k_\psi u(\psi,\theta)\right| \leq C\psi^{\gamma-k-1/2}\|u\|_{K^m_\gamma(\Pi)}$ *for $m - k > 1$.*

Proof. The proof is adapted directly from [6]. The Sobolev embeddings in continuous functions and the trace theorems tell us:

1. $\left|D^k u(\psi)\right| \leq C\|u\|_{H^m[0,1]}$ for $m - k > 1/2$,
2. $\left\|\partial^k_\psi u(\psi,\cdot)\right\|_{H^{m-k-1/2}(\mathbb{T})} \leq C\|u\|_{H^m([0,1]\times\mathbb{T})}$ for $m - k > 1/2$,
3. $\left|\partial^k_\psi u(\psi,\theta)\right| \leq C\|u\|_{H^m([0,1]\times\mathbb{T})}$ for $m - k > 1$.

In what follows, starting with any of the above estimates will yield the corresponding statement of the proposition, with essentially no difference in the proof. We will pick the second.

Define a new coordinate s by the rescaling $\psi = \lambda s$, for some $\lambda > 0$. Then $u(\psi, \theta) = u(\lambda s, \theta) = v(s, \theta)$. Take, say $s \in (1/2, 1)$. We have

$$\begin{aligned} \left\|\partial_s^k v(s,\cdot)\right\|^2_{H^{m-k-1/2}(\mathbb{T})} &\le C\|v\|^2_{H^m([1/2,1]\times\mathbb{T})} \\ &= C \sum_{p+q=0}^{m} \int_{\frac{1}{2}}^{1} \int_{\mathbb{T}} \left|\partial_s^p \partial_\theta^q v(s,\theta)\right|^2 \mathrm{d}\theta \,\mathrm{d}s \,. \end{aligned}$$

The change of variables gives us: $s = \psi/\lambda$, $\mathrm{d}s = \mathrm{d}\psi\,/\lambda$, $\partial_s^p = \lambda^p \partial_\psi^p$. Applying this to the inequality above gives

$$\left\|\lambda^k \partial_\psi^k u(\psi,\cdot)\right\|^2_{H^{m-k-1/2}(\mathbb{T})} \le C \sum_{p+q=0}^{m} \int_{\frac{\lambda}{2}}^{\lambda} \int_{\mathbb{T}} \left|\lambda^p \partial_\psi^p \partial_\theta^q u(\psi,\theta)\right|^2 \frac{1}{\lambda} \,\mathrm{d}\theta \,\mathrm{d}\psi \,,$$

which we can write as

$$\left\|\partial_\psi^k u(\psi,\cdot)\right\|^2_{H^{m-k-1/2}(\mathbb{T})} \le C \left(\lambda^{\gamma-k-1/2}\right)^2 \sum_{p+q=0}^{m} \left\|\lambda^{p-\gamma} \partial_\psi^p \partial_\theta^q u(\psi,\theta)\right\|^2_{L^2([\lambda/2,\lambda]\times\mathbb{T})} .$$

Since $\lambda/2 \le \psi \le \lambda$, then for any $a \in \mathbb{R}$, we have $\lambda^a \le c\psi^a$ where c depends only on a, not λ. In particular,

$$\lambda^{\gamma-k-1/2} \le c_1 \psi^{\gamma-k-1/2} \quad \text{and} \quad \lambda^{p-k} \le c_2 \psi^{p-k},$$

where c_1 and c_2 depend only on γ, k and p, not λ. We now get

$$\left\|\partial_\psi^k u(\psi,\cdot)\right\|^2_{H^{m-k-1/2}(\mathbb{T})} \le C\psi^{2(\gamma-k-1/2)} \sum_{p+q=0}^{m} \left\|\psi^{p-\gamma} \partial_\psi^p \partial_\theta^q u(\psi,\theta)\right\|^2_{L^2([\lambda/2,\lambda]\times\mathbb{T})}$$

or

$$\left\|\partial_\psi^k u(\psi,\cdot)\right\|_{H^{m-k-1/2}(\mathbb{T})} \le C\psi^{\gamma-k-1/2} \|u\|_{K_\gamma^m([\lambda/2,\lambda]\times\mathbb{T})}.$$

The above estimate is uniform in λ, since the constant C depends on γ, m and k but not on λ. In particular, it holds uniformly for all $\lambda \in (0, 1]$, yielding the statement of the proposition:

$$\left\|\partial_\psi^k u(\psi,\cdot)\right\|_{H^{m-k-1/2}(\mathbb{T})} \le C\psi^{\gamma-k-1/2} \|u\|_{K_\gamma^m(\Pi)}.$$

$\square$

Remark 2.13. The above proposition tells us how the asymptotic behaviour as $\psi \to 0^+$ of functions in K_γ^m depends on γ. We identify the three cases:

- For $\gamma - 1/2 < 0$, functions in K^m_γ can grow unbounded as $\psi \to 0^+$.
- For $\gamma - 1/2 = 0$, functions in K^m_γ remain bounded as $\psi \to 0^+$, though their limit may fail to exist.
- For $\gamma - 1/2 > 0$, functions in K^m_γ necessarily vanish as $\psi \to 0^+$.

We see that only for $\gamma > 1/2$ can we meaningfully define the restriction of $u \in K^m_\gamma$ to the boundary $\psi = 0$ (assuming of course m is sufficiently high), and in this case the restriction is necessarily zero. Additionally, this means that when functions in K^m_γ are continuous away from $\psi = 0$ (by the equivalence with H^m), the continuity extends up to the boundary $\psi = 0$ if $\gamma > 1/2$. We summarize this in the following corollary.

Corollary 2.14. *For $\gamma > 1/2$, we have the embeddings*

- $K^m_\gamma(0,1] \subset C[0,1]$ *when* $m > 1/2$,
- $K^m_\gamma(\Pi) \subset C([0,1] \times \mathbb{T})$ *when* $m > 1$.

Furthermore, under these conditions, functions in K^m_γ vanish as $\psi \to 0^+$.

The previous results can be viewed as an analogue in the Kondratev spaces to the Sobolev embedding into continuous functions. It will also be useful to say something about the Kondratev embeddings into L^p.

Proposition 2.15. $K^1_\gamma(\Pi) \subset L^p(\Pi)$ *when* $\gamma > \frac{1}{2} - \frac{1}{p}$, *with* $\|u\|_{L^p(\Pi)} \leq C\|u\|_{K^1_\gamma(\Pi)}$.

Proof. From 2.12, we have $\|u(\psi,\cdot)\|_{H^{1/2}(\mathbb{T})} \leq C\psi^{\gamma-1/2}\|u\|_{K^m_\gamma(\Pi)}$. By the Sobolev embedding theorem in the critical case, for any $p < \infty$, we have $\|u(\psi,\cdot)\|_{L^p(\mathbb{T})} \leq C\|u(\psi,\cdot)\|_{H^{1/2}(\mathbb{T})}$. This gives

$$\|u(\psi,\cdot)\|^p_{L^p(\mathbb{T})} = \int_{\mathbb{T}} |u(\psi,\cdot)|^p \, \mathrm{d}\theta \leq C\psi^{p(\gamma-1/2)}\|u\|^p_{K^1_\gamma(\Pi)}.$$

Integrating over ψ gives

$$\|u\|^p_{L^p(\Pi)} = \int_0^1 \int_{\mathbb{T}} |u(\psi,\cdot)|^p \, \mathrm{d}\theta \, \mathrm{d}\psi \leq C\|u\|^p_{K^1_\gamma(\Pi)} \int_0^1 \psi^{p(\gamma-1/2)} \, \mathrm{d}\psi \,.$$

The right side is bounded when $p(\gamma - \frac{1}{2}) + 1 > 0$, or $\gamma > \frac{1}{2} - \frac{1}{p}$. $\square$

We include one more estimate that will prove useful.

Proposition 2.16. *Suppose* $u \in K^1_\gamma(\Pi)$*. Then* $I(\theta) = \big(\int_0^1 |\psi^{-\gamma} u(\psi,\theta)|^2 \,\mathrm{d}\psi \big)^{1/2} \in C(\mathbb{T})$ *and* $|I| \le C\|u\|_{K^1_\gamma(\Pi)}$.

Proof. For $u(\psi,\theta) \in K^1_\gamma(\Pi)$, define the vector-valued map $\theta \to \psi^{-\gamma}u(\psi,\theta)$. This map belongs to $H^1(\mathbb{T}, L^2(0,1])$ because

$$\begin{aligned} \|\psi^{-\gamma}u(\psi,\theta)\|^2_{H^1(\mathbb{T},L^2(0,1])} &= \Big\| \|\psi^{-\gamma}u(\psi,\theta)\|_{L^2(0,1]} \Big\|^2_{L^2(\mathbb{T})} + \Big\| \|\psi^{-\gamma}u_\theta(\psi,\theta)\|_{L^2(0,1]} \Big\|^2_{L^2(\mathbb{T})} \\ &\le \|\psi^{-\gamma}u(\psi,\theta)\|^2_{L^2(\Pi)} + \|\psi^{-\gamma}u_\theta(\psi,\theta)\|^2_{L^2(\Pi)} \\ &\le \|u\|^2_{K^1_\gamma(\Pi)}, \end{aligned}$$

which follows from the equivalence $L^2(\mathbb{T}, L^2(0,1]) \cong L^2(\Pi)$.

By the embedding $H^1(\mathbb{T}, L^2(0,1]) \subset C(\mathbb{T}, L^2(0,1])$, the function $I(\theta)$ is a composition of continuous maps $\theta \to \psi^{-\gamma}u(\psi,\theta) : \mathbb{T} \to L^2(0,1]$ and $\|\cdot\|_{L^2(0,1]} : L^2(0,1] \to \mathbb{R}$ and is thus continuous. Finally,

$$|I(\theta)| = \|\psi^{-\gamma}u(\cdot,\theta)\|_{L^2(0,1]} \le \|\psi^{-\gamma}u\|_{C(\mathbb{T},L^2(0,1])} \le C\|\psi^{-\gamma}u\|_{H^1(\mathbb{T},L^2(0,1])} \le C\|u\|_{K^1_\gamma(\Pi)}.$$

$\square$

Having seen a number of properties of the space K^m_γ, we are now ready to discuss its suitability for the study of our problem. Our linear operator 1.7 is modulo a factor of $\psi^{-1/2}$, a polynomial of operators $(\psi\partial_\psi)$ and ∂_θ. Both of these operators are bounded from $K^m_\gamma(\Pi)$ to $K^{m-1}_\gamma(\Pi)$. Additionally, we have a well defined trace in $H^{m-1/2}(\mathbb{T})$ at $\psi = 1$, where our boundary data is defined. This puts us in a good position to pose the linear boundary value problem. In fact, we will see in the following chapter that the linear map is Fredholm on $K^m_\gamma(\Pi) \to K^{m-2}_{\gamma-1/2}(\Pi) \times H^{m-1/2}(\mathbb{T})$ except on a countable set of γ. Furthermore, this space is well suited to include singular functions like $\psi^{1/2}$ (the solution near which we aim to solve the nonlinear problem). Finally, we can guarantee that the functions in this space vanish at $\psi = 0$. After all, to have a meaningful fixed point (level set consisting of a single point), we require that $r = a(\psi,\theta) \to 0$ as $\psi \to 0^+$.

There remains one crucial element we have not accounted for. For our function space to be suitable, the nonlinear operator must be well defined on this space, at least on functions in a sufficiently small neighbourhood of the solution $\psi^{1/2}$. Looking at the operator 1.6, it is clear this can only occur if the functions $r = a(\psi, \theta)$ do not vanish away from $\psi = 0$. We demonstrate how the space K^m_γ fails in satisfying this condition. To see this, let us start by checking for which γ does solution $\psi^{1/2}$ belong to K^m_γ? This is satisfied if $\psi^{1/2-\gamma} \in L^2$, which holds for $\gamma < 1$. So let us fix some $\gamma < 1$. Next, a function ψ^μ belongs to this space if $\mu > \gamma - 1/2$. Since $\gamma < 1$, we can always find a μ satisfying $\gamma - 1/2 < \mu < 1/2$. Fix such a μ and define the function $a = \psi^{1/2} - \varepsilon\psi^\mu$. We see that $\|\psi^{1/2} - a\|_{K^m_\gamma} = \varepsilon\|\psi^\mu\|_{K^m_\gamma}$, which can be made arbitrarily small by control of ε. In other words, for any neighbourhood of $\psi^{1/2}$, we can take ε such that $a = \psi^{1/2} - \varepsilon\psi^\mu$ belongs to this neighbourhood. Finally, notice this function vanishes at $\psi = 0$ as well as $\psi = \varepsilon^{\frac{1}{1/2-\mu}}$. If ε small enough, then the latter condition means that the function $a = \psi^{1/2} - \varepsilon\psi^\mu$ vanishes at some $\psi \in (0, 1]$ and thus the nonlinear operator 1.6 will fail to be well defined there. Let us remind that the value of the function $r = a(\psi, \theta)$ defines the radial coordinate (with respect to the fixed point) of the level line given by ψ. The desired topology of our flows is that of nearly concentric circles around a fixed point. If $r = a(\psi, \theta)$ were to vanish for some $\psi > 0$, we would have some level line at the least pinching the fixed point, if not completely collapsing to it. We must certainly exclude such degeneracies. We rectify this issue in the next section.

2.2 Spaces of fixed asymptotics as $\psi \to 0^+$

The Kondratev space K^m_γ introduced in the previous section fail to be suitable because their functions have asymptotics as $\psi \to 0^+$ that are too flexible. Any neighbourhood containing solution $\psi^{1/2}$ necessarily contains lower order asymptotics, which lead to the breakdown of the topology of the flows we are trying to parameterize. Our candidate solution space requires the asymptotic behaviour of its functions to be firmly capped from below.

We construct such a space as follows. Starting with the solution $\psi^{1/2}$, allowing for some angular dependence gives functions of form $v(\theta)\psi^{1/2}$. If away from $\psi = 0$, we require these functions to be in H^m, then $v(\theta)$ should be taken in $H^m(\mathbb{T})$. Next, we wish to include perturbations by higher order (in ψ) terms, which may include angular dependence as well. These higher order terms can then be taken in $K^m_\gamma(\Pi)$, so long as we ensure to take γ such that it excludes all terms of order $\psi^{1/2}$ and lower. This suggests a space of functions of form $u(\psi,\theta) = \psi^{1/2}v(\theta) + w(\psi,\theta)$, with $v(\theta) \in H^m(\mathbb{T})$ and $w(\psi,\theta) \in K^m_\gamma(\Pi)$, for appropriate γ. It will be useful to define this space in greater generality, namely for leading terms of arbitrary order ψ^λ.

Definition 2.17 (Space of fixed asymptotics on the interval)**.**
Let $m \in \mathbb{N}_0$, and $\lambda, \gamma \in \mathbb{R}$ with $\gamma \geq 1/2$. Define space

$$J^m_{\lambda,\gamma}(0,1] = \left\{u(\psi) = v\psi^\lambda + w(\psi) \colon v \in \mathbb{R}, w(\psi) \in K^m_{\lambda+\gamma}(0,1]\right\}$$

with norm

$$\|u(\psi)\|^2_{J^m_{\lambda,\gamma}(0,1]} = |v|^2 + \|w(\psi)\|^2_{K^m_{\lambda+\gamma}(0,1]}\,.$$

Definition 2.18 (Space of fixed asymptotics on the strip)**.**
Let $m \in \mathbb{N}_0$, and $\lambda, \gamma \in \mathbb{R}$ with $\gamma \geq 1/2$. Define space

$$J^m_{\lambda,\gamma}(\Pi) = \left\{u(\psi,\theta) = v(\theta)\psi^\lambda + w(\psi,\theta) \colon v(\theta) \in H^m(\mathbb{T}), w(\psi,\theta) \in K^m_{\lambda+\gamma}(\Pi)\right\}$$

with norm

$$\|u(\psi,\theta)\|^2_{J^m_{\lambda,\gamma}(\Pi)} = \|v(\theta)\|^2_{H^m(\mathbb{T})} + \|w(\psi,\theta)\|^2_{K^m_{\lambda+\gamma}(\Pi)}\,.$$

Remark 2.19. The space $J^m_{\lambda,\gamma}(0,1]$ is only well defined if its elements $u(\psi)$ uniquely determine v and $w(\psi)$ such that $u(\psi) = v\psi^\lambda + w(\psi)$. In other words, if $\operatorname{span}\left\{\psi^\lambda\right\} \cap K^m_{\lambda+\gamma} = \{0\}$. Similarly, the analogous statement is required of space $J^m_{\lambda,\gamma}(\Pi)$. This is equivalent to the requirement that $\psi^\lambda \notin K^m_{\lambda+\gamma}$, which holds when $\gamma \geq 1/2$. This guarantees that the remainder term w consists only of asymptotics of order greater than ψ^λ. With this condition, the spaces $J^m_{\lambda+\gamma}$ can be identified as the direct sums:

$$J^m_{\lambda,\gamma}(0,1] \approx \mathbb{R} \oplus K^m_{\lambda+\gamma}(0,1],$$

$$J^m_{\lambda,\gamma}(\Pi) \approx H^m(\mathbb{T}) \oplus K^m_{\lambda+\gamma}(\Pi).$$

The parameters λ, γ and m defining $J^m_{\lambda,\gamma}$ can be summarized as follows:

- λ defines the leading order asymptotics as $\psi \to 0^+$.

- $\gamma \geq 1/2$ defines the scale of higher order asymptotics (the greater γ is, the greater the gap between the leading term and higher order terms).

- m is the usual isotropic regularity scale.

Additionally, we have the inclusion $J^{m_1}_{\lambda,\gamma_1} \subset J^{m_2}_{\lambda,\gamma_2}$ for $m_1 \geq m_2$ and $\gamma_1 \geq \gamma_2$, which follows from the inclusions $H^{m_1} \subset H^{m_2}$ and $K^{m_1}_{\lambda+\gamma_1} \subset K^{m_2}_{\lambda+\gamma_2}$.

Next, notice that $J^m_{\lambda,\gamma}$ is equivalent to H^m away from $\psi = 0$. This follows immediately from definition, for $u(\psi,\theta) \in J^m_{\lambda,\gamma}$ is the sum of a term $v(\theta)\psi^\lambda$ (which is in H^m away from $\psi = 0$ if $v(\theta)$ is), and a term in $K^m_{\lambda+\gamma}$ (which we have already seen is equivalent to H^m away from $\psi = 0$). In fact, all the prior properties of K^m_γ hold analogously for the space $J^m_{\lambda,\gamma}$. We only list them, the proofs follow immediately from definition and the results on $K^m_{\lambda+\gamma}$.

Proposition 2.20.

1. *$u(\psi,\theta) \to \psi^\alpha u(\psi,\theta)$: $J^m_{\lambda,\gamma}(\Pi) \to J^m_{\lambda+\alpha,\gamma}(\Pi)$ defines an isomorphism (for any $\alpha \in \mathbb{R}$).*

2. *$u(\psi,\theta) \to \partial_\psi u(\psi,\theta)$: $J^m_{\lambda,\gamma}(\Pi) \to J^{m-1}_{\lambda-1,\gamma}(\Pi)$ is bounded (for $m \geq 1$).*

3. *$u(\psi,\theta) \to \partial_\theta u(\psi,\theta)$: $J^m_{\lambda,\gamma}(\Pi) \to J^{m-1}_{\lambda,\gamma}(\Pi)$ is bounded (for $m \geq 1$).*

The first two hold analogously for functions $u(\psi)$ on the interval $(0,1]$, replacing ∂_ψ by D.

Proposition 2.21. *Let $m > 1/2$. Then the restriction to $\psi = 1$ defines the bounded operator*

$$u(\cdot,\cdot) \to u(1,\cdot)\colon J^m_{\lambda,\gamma}(\Pi) \to H^{m-1/2}(\mathbb{T}).$$

Proposition 2.22. *There exists $C > 0$ depending on λ, γ, m, k for which*

1. *$|D^k u(\psi)| \leq C\psi^{\lambda-k}\left(|v| + \psi^{\gamma-1/2}\|w\|_{\mathcal{K}^m_{\lambda+\gamma}(0,1]}\right)$ for $m - k > 1/2$.*

2. *$\left\|\partial^k_\psi u(\psi,\cdot)\right\|_{H^{m-k-1/2}(\mathbb{T})} \leq C\psi^{\lambda-k}\left(\|v\|_{H^m(\mathbb{T})} + \psi^{\gamma-1/2}\|w\|_{\mathcal{K}^m_{\lambda+\gamma}(\Pi)}\right)$ for $m - k > 1/2$.*

3. *$|\partial^k_\psi u(\psi,\theta)| \leq C\psi^{\lambda-k}\left(\|v\|_{H^m(\mathbb{T})} + \psi^{\gamma-1/2}\|w\|_{\mathcal{K}^m_{\lambda+\gamma}(\Pi)}\right)$ for $m - k > 1$.*

Corollary 2.23. *For $\lambda \geq 0$, $\gamma > 1/2$, we have the embeddings*

- *$J^m_{\lambda,\gamma}(0,1] \subset C[0,1]$ when $m > 1/2$,*

- *$J^m_{\lambda,\gamma}(\Pi) \subset C([0,1]\times\mathbb{T})$ when $m>1$.*

Additionally, under these conditions, we have the following point-wise behaviour.

- *If $\lambda > 0$, then functions in $J^m_{\lambda,\gamma}$ vanish as $\psi \to 0^+$.*

- *If $\lambda = 0$, then $u = v + w \in J^m_{\lambda,\gamma} \to v$ as $\psi \to 0^+$.*

Though we have defined the space $J^m_{\lambda,\gamma}$ for arbitrary λ, in practice we will only use two cases. First, the space of solutions, whose leading asymptotics are of order $1/2$, is defined by $\lambda = 1/2$. Second, the corresponding target space of our differential operators is defined by $\lambda = 0$.

Notice that for $\lambda = 0$ and $\gamma > 1/2$, functions $u(\psi,\theta)$ in $J^m_{0,\gamma}$ have the property that $u(0,\theta) = v(\theta) \in H^m(\mathbb{T})$. Taking $\gamma > 1/2$ guarantees that the contribution of the term $w \in K^m_{0+\gamma}$ is continuous and vanishes at $\psi = 0$. The functions in $J^m_{0,\gamma}$ then have the unusual property that for $\psi \neq 0$, restrictions $u(\psi,\cdot)$ belong to $H^{m-1/2}(\mathbb{T})$, but as $\psi \to 0^+$, this restriction bumps up in regularity to $H^m(\mathbb{T})$.

We will find in the following chapter that the cost of seeking solutions in the space $J^m_{1/2,\gamma}(\Pi)$ is the presence of a two-dimensional cokernel in the linear problem consisting of span $\left\{e^{\pm 2i\theta}\right\}$. To establish our desired isomorphism of the linear boundary value problem 1.7, we are forced to remove this cokernel from the target space. Of course, we will have to ensure this is accounted for in the nonlinear problem 1.6. This leads us to define the additional space:

Definition 2.24.

$$\widetilde{J}^m_{0,\gamma}(\Pi) = \left\{ u(\psi,\theta) = v(\theta) + w(\psi,\theta) \in J^m_{0,\gamma}(\Pi)\colon \int_{\mathbb{T}} v(\theta) e^{\pm 2i\theta}\,\mathrm{d}\theta = 0 \right\}.$$

In this section, we have defined the space $J^m_{\lambda,\gamma}(\Pi)$ of functions with leading asymptotics of order ψ^λ. These functions have well defined restrictions to the boundary $\psi = 1$ lying in $H^{m-1/2}(\mathbb{T})$. For $\lambda \geq 0$ and $\gamma > 1/2$, this space embeds into the continuous functions. We will show in the following chapter that the linear problem 1.7 defines an isomorphism $\mathbb{R}^3 \times J^m_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2}_{0,\gamma}(\Pi) \times H^{m-1/2}(\mathbb{T})$ for some range of γ. Later, we will also demonstrate the well posedness of the nonlinear problem 1.6 in these spaces. So far, our definitions restrict to real valued functions and lack any analytic structure. They will serve as the model spaces which we will extend to incorporate the desired partial analytic structure.

2.3 Spaces of Partially Analytic Functions

In the previous section, we have defined our model space $J^m_{\lambda,\gamma}(\Pi)$ for solutions, interpreted as the graphs of a family of flow lines. We have seen that the restriction to individual flow lines resides in the space $H^{m-1/2}(\mathbb{T})$. The flow lines of solutions to the stationary Euler equation are known to be real analytic curves, that is, functions $a(\psi,\theta)$ in $J^m_{\lambda,\gamma}(\Pi)$ should be additionally analytic in θ. It is our goal in this section to extend (or perhaps better to say refine) $J^m_{\lambda,\gamma}(\Pi)$ to include this partial analytic structure. Similarly, the space of restrictions to individual flow lines, modelled on $H^{m-1/2}(\mathbb{T})$, should be refined to consist of some subset of analytic functions.

Since real analytic functions do not form a Banach space, we can instead consider functions having analytic extensions to some suitable complex domain. In our case, it is natural to extend $\theta \in \mathbb{T}$ to the complex periodic strip $\mathbb{T}_\sigma = \mathbb{T} \times i(\sigma,\sigma)$, where $\sigma > 0$. With this aim, our main tool will be the Paley-Wiener theorem - a group of results relating the decay of a function's Fourier transform with its extension to a complex domain ([25]). They provide a description of spaces of holomorphic (and partially holomorphic) functions, akin to the complex Hardy spaces, defined by an appropriate control of their possible singularities occurring on the boundary of the complex extended domains. This formulation of complexified stationary flows allows us to employ the tools from the theory of complex Banach spaces. We should clarify then, from here on after, all function spaces should be understood to be complex valued (including the real valued spaces defined in the preceding sections).

We now state the following theorem of Paley and Wiener, which characterizes the analytic extensions of $L^2(\mathbb{R})$ functions to the complex strip.

Theorem 2.25 (Paley-Wiener on the complex strip)**.**
Suppose $u(x) \in L^2(\mathbb{R})$ and $\sigma > 0$. Then the following statements are equivalent:

1. $e^{\sigma|\xi|}\hat{u}(\xi) \in L^2(\mathbb{R})$.

2. $u(x)$ extends to $u(z)$ holomorphic in the strip $\{z = x + iy \colon x \in \mathbb{R}, |y| < \sigma\}$ with

$$\sup_{|y|<\sigma} \|u(\cdot + iy)\|_{L^2(\mathbb{R})} < \infty.$$

This theorem enables us to define the space of holomorphic functions on the complex strip whose restrictions to the strip boundaries are $L^2(\mathbb{R})$ functions. With a few changes, we can adapt this theorem to our specific needs. It will be useful to first prove the following lemma.

Lemma 2.26.
Suppose $u(z)$ is holomorphic on the complex periodic strip $\mathbb{T}_\sigma = \{z = \theta + i\tau \colon \theta \in \mathbb{T}, |\tau| < \sigma\}$, with values possibly in a complex Banach space. Then it has the representation:

$$u(z) = \sum_k \hat{u}_k e^{ikz}$$

where

$$\hat{u}_k = \frac{1}{2\pi} \int_{\mathbb{T}} u(\theta) e^{ik\theta} \, \mathrm{d}\theta \, .$$

Proof. We start by showing there is a one-to-one correspondence between holomorphic functions in $\mathbb{T}_\sigma$ and holomorphic functions in the annulus $A_\sigma = \{e^{-\sigma} < |w| < e^\sigma\}$. Consider map $z \to w = e^{iz}$. It defines a holomorphic bijection from $\mathbb{T}_\sigma$ to A_σ. Since $\frac{\mathrm{d}w}{\mathrm{d}z} \neq 0$ everywhere in $\mathbb{T}_\sigma$, then by the inverse function theorem, near any pair (z_0, w_0) with $w_0 = e^{iz_0}$, there exists a holomorphic inverse map $z = F_0(w)$.

Now suppose $g(w)$ is holomorphic in A_σ. Then $u(z) = g(e^{iz})$ is the composition of two holomorphic maps and thus holomorphic in $\mathbb{T}_\sigma$. Conversely, suppose $u(z)$ is holomorphic in $\mathbb{T}_\sigma$. Near any $w_0 \in A_\sigma$ we have some representation $z = F_0(w)$ and so $u(z) = u(F_0(w)) = g(w)$ is analytic near w_0. This holds for all $w_0 \in A_\sigma$, so $g(w)$ is analytic in A_σ. This establishes the one-to-one correspondence between analytic functions $u(z)$ on the strip, and the analytic functions $g(w)$ on the annulus, with $u(z) = g(e^{iz})$.

Given $u(z)$ holomorphic in $\mathbb{T}_\sigma$, take $g(w)$ as above, with $u(z) = g(e^{iz})$. The function $g(w)$ has the Laurent series about $w = 0$ given by

$$g(w) = \sum_k c_k w^k,$$

with

$$c_k = \frac{1}{2\pi i} \oint \frac{g(w)}{w^{k+1}} \, \mathrm{d}w \, .$$

Setting the contour in the above integral to be around the unit circle $w = e^{i\theta}$, we get

$$c_k = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{g(e^{i\theta})}{e^{ik\theta}} \, \mathrm{d}\theta \, .$$

Finally, because $u(z) = g(e^{iz})$, we get

$$c_k = \frac{1}{2\pi} \int_{\mathbb{T}} u(\theta) e^{-ik\theta} \, \mathrm{d}\theta$$

and

$$u(z) = \sum_k c_k e^{ikz}.$$

All the above results hold for complex Banach space valued holomorphic functions, interpreting the integral defining the coefficients c_k as a Bochner integral (see [21]). □

Now, let us adapt the prior Paley-Wiener theorem on complex extensions of $L^2(\mathbb{R})$ functions to the $H^m(\mathbb{T})$ setting.

Theorem 2.27 (Paley-Wiener for Sobolev functions on the complex periodic strip)**.** *Suppose $u(\theta) \in H^m(\mathbb{T})$, where m is a non-negative real number and let $\sigma > 0$. Then the following statements are equivalent:*

1. $\mathcal{F}^{-1}_{k\to\theta} \left\{ e^{\sigma|k|} \hat{u}_k \right\} = \sum_k \hat{u}_k e^{\sigma|k|} e^{ik\theta} \in H^m(\mathbb{T})$.

2. *$u(\theta)$ extends to $u(z)$ holomorphic in the complex periodic strip* $\mathbb{T}_\sigma = \{ z = \theta + i\tau \colon \theta \in \mathbb{T}, |\tau| < \sigma \}$ *with*

$$\sup_{|\tau|<\sigma} \| u(\cdot + i\tau) \|_{H^m(\mathbb{T})} < \infty.$$

Proof. First, we assume the first statement and prove the second. Given $u(\theta) \in H^m(\mathbb{T})$, we have the Fourier series:

$$u(\theta) = \sum_k \hat{u}_k e^{ik\theta}.$$

Next, we extend θ to the complex variable $z = \theta + i\tau$ to get

$$u(z) = \sum_k \hat{u}_k e^{ikz} = \sum_k \hat{u}_k e^{-k\tau} e^{ik\theta}.$$

Now let $\varepsilon > 0$ and $|\tau| < \sigma - \epsilon$. We get

$$\begin{aligned}
|u(z)| &\leq \sum_k |\hat{u}_k e^{-k\tau}| = \|\hat{u}_k e^{-k\tau}\|_{\ell^1} \\
&\leq \sum_k |\hat{u}_k| e^{|k||\tau|} \\
&\leq \sum_k |\hat{u}_k| e^{\sigma|k|} e^{(|\tau|-\sigma)|k|} \\
&\leq \sum_k |\hat{u}_k| e^{\sigma|k|} e^{-\varepsilon|k|} \\
&\leq \|\hat{u}_k e^{\sigma|k|}\|_{\ell^2} \|e^{-\varepsilon|k|}\|_{\ell^2} \\
&< \infty.
\end{aligned}$$

The second last inequality follows from Cauchy-Schwarz. The last inequality follows from the assumption, for if $\mathcal{F}^{-1}_{k\to\theta}\left\{e^{\sigma|k|}\hat{u}_k\right\} \in H^m(\mathbb{T})$, then equivalently $(1+k^2)^{m/2}e^{\sigma|k|}\hat{u}_k \in \ell^2$. Thus, we have shown that $u(z)$ is well defined for any $|\tau| < \sigma - \varepsilon$.

Next, we will show we can also differentiate $u(z)$ under the summation sign, establishing the holomorphy of $u(z)$. Set $f(z,k) = \hat{u}_k e^{ikz}$. From the above inequalities, we saw $f(z,k) \in \ell^1$ for $|\tau| < \sigma - \varepsilon$. Also, $\frac{\partial f}{\partial z} = ik\hat{u}_k e^{ikz}$ certainly exists for each k and all z. Finally we can dominate $\frac{\partial f}{\partial z}$ as follows:

$$\begin{aligned}
|ik\hat{u}_k e^{ikz}| &\leq |k\hat{u}_k| e^{|k||\tau|} \\
&\leq |\hat{u}_k e^{\sigma|k|}||k e^{(|\tau|-\sigma)|k|}| \\
&\leq |\hat{u}_k e^{\sigma|k|}||k e^{-\varepsilon|k|}| \\
&\in \ell^1.
\end{aligned}$$

Thus $\frac{\partial f}{\partial z}$ is dominated by an ℓ^1 function independent of z. By the lemma of differentiation under the integral sign (following from dominated convergence theorem), we conclude that

$$\frac{\mathrm{d}u(z)}{\mathrm{d}z} = \sum_k (ik)\hat{u}_k e^{ikz}$$

for $|\tau| < \sigma - \varepsilon$, and thus u is holomorphic here. Finally, note because this result holds for arbitrary $\varepsilon > 0$, $u(z)$ is in fact holomorphic for $|\tau| < \sigma$.

To conclude the first part of the proof, the norm equivalence of $H^m(\mathbb{T})$ as a weighted ℓ^2 space gives the estimate

$$\|u(\cdot + i\tau)\|_{H^m(\mathbb{T})} \leq \|(1+k^2)^{m/2}\hat{u}_k e^{-k\tau}\|_{\ell^2} \leq \|(1+k^2)^{m/2}\hat{u}_k e^{|k||\tau|}\|_{\ell^2},$$

from which it follows that

$$\sup_{|\tau|<\sigma} \|u(\cdot + i\tau)\|_{H^m(\mathbb{T})} \leq \|(1+k^2)^{m/2}\hat{u}_k e^{\sigma|k|}\|_{\ell^2} < \infty.$$

Now let us assume the second statement and prove the first. Given $u(\cdot + i\tau) \in H^m(\mathbb{T})$, we can write

$$u(\theta + i\tau) = \sum_k \hat{u}_k(\tau) e^{ik\theta}.$$

The key step here is invoking the prior lemma 2.26: because $u(z)$ is holomorphic in $\mathbb{T}_\sigma$, the above expression instead takes the far more rigid form:

$$u(\theta + i\tau) = \sum_k \hat{u}_k e^{-\tau k} e^{ik\theta}.$$

The required estimate then follows again from the equivalence of the space $H^m(\mathbb{T})$ with its weighted ℓ^2 counterpart.

$$\begin{aligned}
\|\mathcal{F}^{-1}_{k\to\theta}\left\{e^{\sigma|k|}\hat{u}_k\right\}\|_{H^m(\mathbb{T})} &= \|(1+k^2)^{m/2}\hat{u}_k e^{\sigma|k|}\|_{\ell^2} \\
&= \sup_{|\tau|<\sigma} \|(1+k^2)^{m/2}\hat{u}_k e^{|\tau||k|}\|_{\ell^2} \\
&\leq \sup_{|\tau|<\sigma} \|(1+k^2)^{m/2}\hat{u}_k \left(e^{\tau k} + e^{-\tau k}\right)\|_{\ell^2} \\
&= \sup_{|\tau|<\sigma} \|u(\cdot - i\tau) + u(\cdot + i\tau)\|_{H^m(\mathbb{T})} \\
&\leq 2 \sup_{|\tau|<\sigma} \|u(\cdot + i\tau)\|_{H^m(\mathbb{T})} \\
&< \infty.
\end{aligned}$$

□

This theorem provides a description of the space of holomorphic functions on the complex periodic strip whose restrictions to the boundary are Sobolev functions. We are now ready to define the space for the individual analytic flow lines.

Definition 2.28. Define $X^m_\sigma(\mathbb{T})$ to be the space of functions satisfying the conditions of theorem 2.27. We have two equivalent characterizations of this space:

1.
$$X^m_\sigma(\mathbb{T}) = \left\{u(\theta) \in H^m(\mathbb{T}) : \mathcal{F}^{-1}_{k\to\theta}\{\hat{u}_k e^{\sigma|k|}\} \in H^m(\mathbb{T})\right\}$$
with norm
$$\|u\|^2_{X^m_\sigma(\mathbb{T})} = \sum_k (1+k^2)^m e^{2\sigma|k|}|\hat{u}_k|^2.$$

2. $X^m_\sigma(\mathbb{T})$ is the space of functions $u(z)$ that are complex analytic in $\mathbb{T}_\sigma$ and satisfy
$$\sup_{|\tau|<\sigma} \|u(\cdot + i\tau)\|_{H^m(\mathbb{T})} < \infty$$
with norm
$$\|u\|^2_{X^m_\sigma(\mathbb{T})} = \sup_{|\tau|<\sigma} \|u(\cdot + i\tau)\|^2_{H^m(\mathbb{T})} \cong \|u(\cdot + i\sigma)\|^2_{H^m(\mathbb{T})} + \|u(\cdot - i\sigma)\|^2_{H^m(\mathbb{T})}.$$

Remark 2.29. It is straightforward to check that the two norms above are indeed equivalent. We can thus interpret $X^m_\sigma(\mathbb{T})$ as the space of analytic flow lines whose possible complex singularities are described by $H^m(\mathbb{T})$ functions on the strip boundaries.

Remark 2.30. The boundedness of derivative map $D\colon X^m_\sigma(\mathbb{T}) \to X^{m-1}_\sigma(\mathbb{T})$ follows immediately from the boundedness of $D\colon H^m(\mathbb{T}) \to H^{m-1}(\mathbb{T})$.

Our next goal is to adapt the Paley-Wiener theorem to characterize the partial analytic nature of our solutions. We must incorporate this anisotropic analyticity while preserving the general structure of the space $J^m_{\lambda,\gamma}(\Pi)$, which we claimed is the appropriate setting for our boundary value problems. To get an idea of how we can do this, let us for a moment consider instead the space $L^2(\mathbb{R}^2)$ of functions $u(x, y)$. How can we define what it means for such functions to be partially analytic, that is analytic in say x, if they are not a priori defined point-wise? Let us consider the restriction to vertical sections $u(x, \cdot)$. If u is only in $L^2(\mathbb{R}^2)$ then of course such a restriction map is not meaningfully defined. However, if we have sufficient additional regularity along x (that is, along the direction perpendicular to the sections we restrict to) then we can define this trace meaningfully as an $L^2(\mathbb{R})$ function. Thus, we can define the partial analyticity by analytic Banach valued maps. Namely, we can say that $u(x,y) \in L^2(\mathbb{R}^2)$ is partially analytic if the map $x \to u(x,\cdot)\colon \mathbb{R} \to L^2(\mathbb{R})$ is analytic. We expect the following extension of the Paley-Wiener theorem:

Theorem 2.31 (Partial Paley-Wiener).
Suppose $u(x,y) \in L^2(\mathbb{R}^2)$ and $\sigma > 0$. Then the following statements are equivalent:

1. $\mathcal{F}^{-1}_{\xi\to x}\left\{e^{\sigma|\xi|}\hat{u}(\xi,y)\right\} \in L^2(\mathbb{R}^2)$.

2. *$u(x,y)$ extends to $u(z,y)$ partially analytic in the sense that the map*

$$z \to u(z,\cdot)\colon\ \{z = x+it\colon |t| < \sigma\} \to L^2(\mathbb{R})$$

   is holomorphic and

$$\sup_{|t|<\sigma} \|u(\cdot + it, \cdot)\|_{L^2(\mathbb{R}^2)} < \infty.$$

We state this because it serves as an example of how to adapt the Paley-Wiener theorem to include a notion of partial analyticity, while preserving the properties of the underlying model space, $L^2(\mathbb{R}^2)$ in the above case. In our case the model space is $J^m_{\lambda,\gamma}(\Pi)$, which recall is the space of functions $u(\psi,\theta) = v(\theta)\psi^\lambda + w(\psi,\theta)$, where $v(\theta) \in H^m(\mathbb{T})$ and $w(\psi,\theta) \in K^m_{\lambda+\gamma}(\Pi)$, and can be identified with $H^m(\mathbb{T}) \oplus K^m_{\lambda+\gamma}(\Pi)$. Since we have already discussed the Paley-Wiener theorem for analytic extensions of $H^m(\mathbb{T})$ to the strip $\mathbb{T}_\sigma$, we need only focus on adapting the above theorem to $K^m_\gamma(\Pi)$ functions. The main point to underline again is that we can define partial analytic functions as holomorphic Banach valued functions, which are valued in the space of restrictions to vertical sections. Where as typically, we expect the trace of a $K^m_\gamma(\Pi)$ function to lose half an order of regularity (as in the Sobolev case), in our case the additional regularity in θ compensates for this. The proof of the following theorem is almost identical to the proof of theorem 2.27, only differing in that $u(\cdot,z)$ is Banach valued.

Theorem 2.32 (Partial Paley-Wiener for Kondratiev functions).
Suppose $u(\psi,\theta) \in K^m_\gamma(\Pi)$ and $\sigma > 0$. Then the following statements are equivalent:

1. $\mathcal{F}^{-1}_{k\to\theta}\left\{e^{\sigma|k|}\hat{u}_k(\psi)\right\} = \sum_k \hat{u}_k(\psi)e^{\sigma|k|}e^{ik\theta} \in K^m_\gamma(\Pi)$.

2. $u(\psi,\theta)$ extends to $u(\psi,z)$ partially analytic in the sense that the map

$$z \to u(\cdot,z)\colon \mathbb{T}_\sigma = \{z=\theta+i\tau\colon |\tau|<\sigma\} \to K_\gamma^m(0,1]$$

is holomorphic and

$$\sup_{|\tau|<\sigma} \|u(\cdot,\cdot+i\tau)\|_{K_\gamma^m(\Pi)} < \infty.$$

Proof. To start, we assume the first statement and prove the second. Given

$$u(\psi,\theta) = \sum_k \hat{u}_k(\psi)e^{ik\theta}$$

we extend to the complex variable $z=\theta+i\tau$ to get

$$u(\psi,z) = \sum_k \hat{u}_k(\psi)e^{ikz} = \sum_k \hat{u}_k(\psi)e^{-k\tau}e^{ik\theta}.$$

We will now show $u(\cdot,z)$ is well defined in $K_\gamma^m(0,1]$. Let $\varepsilon>0$ and $|\tau|<\sigma-\epsilon$. We have

$$\begin{aligned}
\|u(\cdot,z)\|_{K_\gamma^m(0,1]} &\le \sum_k \left\|\hat{u}_k(\psi)e^{-k\tau}\right\|_{K_\gamma^m(0,1]} \\
&\le \sum_k e^{|k||\tau|}\left\|\hat{u}_k(\psi)\right\|_{K_\gamma^m(0,1]} \\
&\le \sum_k e^{\sigma|k|}\left\|\hat{u}_k(\psi)\right\|_{K_\gamma^m(0,1]} e^{(|\tau|-\sigma)|k|} \\
&\le \left\|\left\|\hat{u}_k(\psi)e^{\sigma|k|}\right\|_{K_\gamma^m(0,1]}\right\|_{\ell^2} \left\|e^{-\varepsilon|k|}\right\|_{\ell^2} \\
&\le \left(\sum_k \sum_{p+q=0}^m (k^2)^q \left\|\psi^{p-\gamma}D^p\hat{u}_k(\psi)e^{\sigma|k|}\right\|_{L^2(0,1]}\right)^{1/2} \left\|e^{-\varepsilon|k|}\right\|_{\ell^2} \\
&\le \left\|\mathcal{F}^{-1}_{k\to\theta}\left\{e^{\sigma|k|}\hat{u}_k(\psi)\right\}\right\|_{K_\gamma^m(\Pi)} \left\|e^{-\varepsilon|k|}\right\|_{\ell^2} \\
&< \infty
\end{aligned}$$

where the fourth inequality follows from Cauchy-Schwarz. This shows that $u(\cdot,z)$ is well defined in $K_\gamma^m(0,1]$ for any $|\tau|<\sigma-\varepsilon$.

Next, we will show we can also differentiate $u(\cdot,z)$ under the summation sign, establishing the holomorphy of $u(\cdot,z)$ as a $K_\gamma^m(0,1]$ valued map. Set $f(z,k)=\hat{u}_k(\psi)e^{ikz}$. From the above inequalities, we saw $f(z,k)$ is a $K_\gamma^m(0,1]$ valued ℓ^1 sequence for every $|\tau|<\sigma-\varepsilon$. Also,

$\frac{\partial f}{\partial z} = ik\hat{u}_k(\psi)e^{ikz}$ certainly exists for each k and all z. Finally, this derivative is dominated by

$$\left\| \frac{\partial f}{\partial z}(z,k) \right\|_{K^m_\gamma(0,1]} \leq \left\| e^{\sigma|k|}\hat{u}_k(\psi) \right\|_{K^m_\gamma(0,1]} \left| ke^{-\varepsilon|k|} \right|$$

where the right side is an ℓ^1 sequence. Satisfying the conditions of differentiation under the integral sign we conclude that $z \to u(\cdot, z) \colon \mathbb{T}_{\sigma-\varepsilon} \to K^m_\gamma(0,1]$ is holomorphic. Since $\varepsilon > 0$ is arbitrary, holomorphy holds on $\mathbb{T}_\sigma$.

To conclude the first part of the proof,

$$\|u(\cdot, \cdot + i\tau)\|_{K^m_\gamma(\Pi)} \leq \left\| \mathcal{F}^{-1}_{k\to\theta} \left\{ e^{|\tau||k|}\hat{u}_k(\psi) \right\} \right\|_{K^m_\gamma(\Pi)}$$

from which it follows that

$$\sup_{|\tau|<\sigma} \|u(\cdot, \cdot + i\tau)\|_{K^m_\gamma(\Pi)} \leq \left\| \mathcal{F}^{-1}_{k\to\theta} \left\{ e^{\sigma|k|}\hat{u}_k(\psi) \right\} \right\|_{K^m_\gamma(\Pi)} < \infty.$$

Now to prove the converse let us assume the second statement of the theorem and prove the first. By lemma 2.26 for Banach valued holomorphic functions, $u(\cdot, z)$ takes the form:

$$u(\psi, \theta + i\tau) = \sum_k \hat{u}_k(\psi)e^{-k\tau}e^{ik\theta}.$$

Expressing the $K^m_\gamma(\Pi)$ norm in terms of the partial Fourier series immediately gives

$$\begin{aligned}
\left\| \mathcal{F}^{-1}_{k\to\theta} \left\{ e^{\sigma|k|}\hat{u}_k(\psi) \right\} \right\|^2_{K^m_\gamma(\Pi)} &= \sum_{p+q=0}^{m} \sum_k (k^2)^q e^{2\sigma|k|} \left\| \psi^{p-\gamma} D^p \hat{u}_k(\psi) \right\|^2_{L^2(0,1]} \\
&\leq \sup_{|\tau|<\sigma} \sum_{p+q=0}^{m} \sum_k (k^2)^q \left(e^{-2\tau k} + e^{2\tau k} \right) \left\| \psi^{p-\gamma} D^p \hat{u}_k(\psi) \right\|^2_{L^2(0,1]} \\
&= \sup_{|\tau|<\sigma} \left(\|u(\cdot, \cdot + i\tau)\|^2_{K^m_\gamma(\Pi)} + \|u(\cdot, \cdot - i\tau)\|^2_{K^m_\gamma(\Pi)} \right) \\
&\leq 2 \sup_{|\tau|<\sigma} \|u(\cdot, \cdot + i\tau)\|^2_{K^m_\gamma(\Pi)} \\
&< \infty.
\end{aligned}$$

□

Definition 2.33. Define $K^{m,\sigma}_\gamma(\Pi)$ to be the space of functions satisfying the conditions of theorem 2.32. We have two equivalent characterizations of this space:

1.

$$K_\gamma^{m,\sigma}(\Pi) = \left\{u(\psi,\theta) \in K_\gamma^m(\Pi) : \mathcal{F}^{-1}_{k\to\theta}\{\hat{u}_k(\psi)e^{\sigma|k|}\} \in K_\gamma^m(\Pi)\right\}$$

with norm

$$\|u\|^2_{K_\gamma^{m,\sigma}(\Pi)} = \sum_{p+q=0}^{m} \sum_k k^{2q} e^{2\sigma|k|} \left\|\psi^{p-\gamma} D^p \hat{u}_k(\psi)\right\|^2_{L^2(0,1]}.$$

2. $K_\gamma^{m,\sigma}(\Pi)$ is the space of functions $u(\psi,z)$ that are partially analytic in sense that

$$z \to u(\cdot,z)\colon \mathbb{T}_\sigma = \{z = \theta + i\tau\colon |\tau| < \sigma\} \to K_\gamma^m(0,1]$$

is a holomorphic Banach valued map, and satisfies

$$\sup_{|\tau|<\sigma} \|u(\cdot,\cdot+i\tau)\|_{K_\gamma^m(\Pi)} < \infty,$$

with norm

$$\|u\|^2_{K_\gamma^{m,\sigma}(Pi)} = \sup_{|\tau|<\sigma} \|u(\cdot,\cdot+i\tau)\|^2_{K_\gamma^m(\Pi)} \cong \|u(\cdot,\cdot+i\sigma)\|^2_{K_\gamma^m(\Pi)} + \|u(\cdot,\cdot-i\sigma)\|^2_{K_\gamma^m(\Pi)}.$$

Having defined the spaces $X_\sigma^m(\mathbb{T})$ and $K_\gamma^{m,\sigma}(\Pi)$ it is now straightforward to define the space of partially analytic flow lines modelled on $J_{\lambda,\gamma}^m(\Pi)$, as the direct sum of these spaces.

Definition 2.34. For $\gamma \geq 1/2$, we define

$$\begin{aligned} J_{\lambda,\gamma}^{m,\sigma}(\Pi) &= \left\{u(\psi,\theta) = v(\theta)\psi^\lambda + w(\psi,\theta)\colon v(\theta) \in X_\sigma^m(\mathbb{T}), w(\psi,\theta) \in K_{\lambda+\gamma}^{m,\sigma}(\Pi)\right\} \\ &\approx X_\sigma^m(\mathbb{T}) \oplus K_{\lambda+\gamma}^{m,\sigma}(\Pi) \end{aligned}$$

with norm

$$\|u(\psi,\theta)\|^2_{J_{\lambda,\gamma}^{m,\sigma}(\Pi)} = \|v(\theta)\|^2_{X_\sigma^m(\mathbb{T})} + \|w(\psi,\theta)\|^2_{K_{\lambda+\gamma}^{m,\sigma}(\Pi)}.$$

Additionally, we define the co-dimension two subspace:

Definition 2.35.

$$\widetilde{J}_{0,\gamma}^{m,\sigma}(\Pi) = \left\{u(\psi,\theta) = v(\theta) + w(\psi,\theta) \in J_{0,\gamma}^{m,\sigma}(\Pi)\colon \int_{\mathbb{T}} v(\theta)e^{\pm 2i\theta}\,\mathrm{d}\theta = 0\right\}.$$

Remark 2.36. All the properties of propositions 2.20, 2.21, 2.22 and corollary 2.23 hold analogously in the space $J_{\lambda,\gamma}^{m,\sigma}(\Pi)$. This follows immediately from the fact that if $u \in J_{\lambda,\gamma}^{m,\sigma}(\Pi)$, then on every periodic strip $\{|\tau| \leq \sigma$ is fixed.$\}$, $u(\cdot,\cdot+i\tau)$ belongs to $J_{\lambda,\gamma}^m(\Pi)$.

Let us now summarize the results of this chapter by giving some context in terms of the equation of study 1.6. We have defined the space $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ of partially complex analytic flow lines which will serve as the space for solutions $r = a(\psi, \theta)$. For $m > 1$, $\gamma > 1/2$, these functions are continuous. If $a(\psi, \theta) = \psi^{1/2} v(\theta) + w(\psi, \theta)$ and $v(\theta)$ is never zero, then $a(\psi, \theta)$ vanishes like $\psi^{1/2}$ as $\psi \to 0^+$ and blows up like $\psi^{-1/2}$. Such functions thus have the required behaviour at $\psi = 0$ to properly describe the stagnation point of the associated velocity field $u(\psi, \theta)$. These functions have well defined restrictions at the boundary $\{\psi = 1\}$, which reside in $X^{m-1/2}_{\sigma}(\mathbb{T})$. The space $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ serves as the target space of our equations. For $m - 2 > 1$, $\gamma > 1/2$, its functions are continuous and have the property that $u(\psi, \theta) = v(\theta) + w(\psi, \theta) \to v(\theta)$ as $\psi \to 0^+$. Finally, the space $J^{m-2}_{0,\gamma}(0, 1]$ embeds naturally into $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ and serves as the space of complex vorticities $F(\psi)$. Functions in this space are continuous for $\gamma > 1/2$, and consist of a leading constant term plus a higher order perturbation. Now that we have defined the relevant function spaces, we can proceed with establishing the well-posedness of the linear and nonlinear problem in these spaces.

Chapter 3

Linearized Problem

The goal of this chapter is to prove that the linearization 1.7 of the nonlinear boundary value problem at solution $a(\psi,\theta) = \psi^{1/2}$ defines an isomorphism between our defined function spaces. We remind this linearized problem is given by:

Definition 3.1 (Linear Problem)**.**

$$\begin{cases} \psi^{-1/2}\Big[\psi^2\frac{\partial^2}{\partial\psi^2} + 2\psi\frac{\partial}{\partial\psi} + \frac{1}{4}(I + \frac{\partial^2}{\partial\theta^2})\Big]u(\psi,\theta) = f(\psi,\theta) \\ R + (\frac{p_x - ip_y}{2})e^{i\theta} + (\frac{p_x+ip_y}{2})e^{-i\theta} + u(1,\theta) = g(\theta), \end{cases}$$

to be solved for $R \in \mathbb{C}$, $p = (p_x, p_y) \in \mathbb{C}^2$ and $u(\psi,\theta) \in J^{m,\sigma}_{1/2,\gamma}(\Pi)$, given parameters $f(\psi,\theta) = \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ and $g(\theta) \in X^{m-\frac{1}{2}}_{\sigma}(\mathbb{T})$.

By remark 2.36 and proposition 2.20, multiplication by $\psi^{1/2}$ defines an isomorphism from $J^{m,\sigma}_{0,\gamma}(\Pi)$ to $J^{m,\sigma}_{1/2,\gamma}(\Pi)$. It is then equivalent to consider instead the problem

$$\begin{cases} Lu(\psi,\theta) = \Big[\psi^2\frac{\partial^2}{\partial\psi^2} + 2\psi\frac{\partial}{\partial\psi} + \frac{1}{4}(I + \frac{\partial^2}{\partial\theta^2})\Big]u(\psi,\theta) = f(\psi,\theta) \\ R + (\frac{p_x - ip_y}{2})e^{i\theta} + (\frac{p_x+ip_y}{2})e^{-i\theta} + u(1,\theta) = g(\theta), \end{cases} \tag{3.1}$$

where $f(\psi,\theta)$ is now to be taken in $\widetilde{J}^{m-2,\sigma}_{1/2,\gamma}(\Pi)$.

We break down the proof into four parts. First, the boundedness of the linear map in the above spaces follows immediately from the work done in the previous chapter. Second, we solve the homogeneous problem when $f = 0$ and bound the solution by the boundary data. This requires a restriction on the permissible values of γ. Finally, we solve the inhomogeneous problem on the component spaces that make up $J^{m,\sigma}_{1/2,\gamma}(\Pi)$. The linear problem on the leading term component reduces to an ODE in θ, solved by Fourier inversion. The bulk of the work is then dedicated to solving the linear problem on the remainder component term in $K^{m,\sigma}_{1/2+\gamma}(\Pi)$ and establishing the required bounds. Expanding to a Fourier series in θ yields a sequence of second order ODEs in ψ. Each corresponding second order differential operator can be factored into the product of two first order operators. Their inverses, which can be computed explicitly, are operators taking weighted averages. The main tool to establish their boundedness is the Hardy inequality ([6]), which bounds the L^2 norm of the weighted average of a function by the L^2 norm of said function:

Theorem 3.2 (Hardy Inequality)**.**

- *If* $\alpha < 1/2$,
$$\left\| y^{\alpha-1}\int_0^y x^{-\alpha} f(x)\,\mathrm{d}x \right\|_{L^2[0,1]} \leq \frac{1}{\frac{1}{2}-\alpha}\|f\|_{L^2[0,1]}.$$
- *If* $\alpha > 1/2$,
$$\left\| y^{\alpha-1}\int_y^1 x^{-\alpha} f(x)\,\mathrm{d}x \right\|_{L^2[0,1]} \leq \frac{1}{\alpha-\frac{1}{2}}\|f\|_{L^2[0,1]}.$$

Let us now proceed, starting with the boundedness of the linear operator.

Proposition 3.3.

The linearized problem 3.1 is bounded from $\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi)$ *to* $J^{m-2,\sigma}_{1/2,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T})$

Proof. This result follows immediately from the results of the previous chapter. The operators $\psi^2\partial^2_\psi$, $\psi\partial_\psi$ and ∂^2_θ are bounded from $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ to $J^{m-2,\sigma}_{1/2,\gamma}(\Pi)$. This follows from proposition 2.20, remark 2.36, and the boundedness of the identity map from $J^{m-1,\sigma}_{1/2,\gamma}(\Pi)$ to $J^{m-2,\sigma}_{1/2,\gamma}(\Pi)$. Thus L is bounded in these spaces. Next, by proposition 2.21 and remark 2.36, the trace map $u(\cdot,\cdot) \to u(1,\theta)$ is bounded from $J^{m,\sigma}_{1/2,\gamma}$ to $X^{m-1/2}_{\sigma}(\mathbb{T})$. Finally, using the Fourier series representation of the norm of $X^{m-1/2}_{\sigma}(\mathbb{T})$, we get $\|R\|_{X^{m-1/2}_{\sigma}(\mathbb{T})} = |R|$ and $\|(p_x \mp ip_y)e^{\pm i\theta}\|_{X^{m-1/2}_{\sigma}(\mathbb{T})} \leq C\left(|p_x| + |p_x|\right)$. Putting it all together gives the bound

$$\|f\|_{J^{m-2,\sigma}_{1/2,\gamma}(\Pi)} + \|g\|_{X^{m-1/2}_{\sigma}(\mathbb{T})} \leq |R| + |p| + \|u\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)}.$$

$\square$

Next, we tackle the homogeneous linear problem.

Proposition 3.4.

Let $1/2 \leq \gamma < 1$. Then the homogeneous problem obtained from 3.1 by setting $f = 0$ is invertible and its solution has bound

$$|R| + |p| + \|u\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} \leq C\|g\|_{X^{m-1/2}_\sigma(\mathbb{T})}.$$

Proof. Expanding $u(\psi, \theta)$ in a Fourier series in θ gives the family of 2nd order Cauchy-Euler equations

$$\left(\psi^2 D^2 + 2\psi D + \frac{1-k^2}{4}\right)\hat{u}_k(\psi) = 0.$$

Solving gives the general homogeneous solution

$$u(\psi, \theta) = c_0\psi^{-\frac{1}{2}} + d_0\psi^{-\frac{1}{2}}\ln(\psi) + \sum_{k\neq 0}\left(c_k\psi^{\frac{-1+|k|}{2}} + d_k\psi^{\frac{-1-|k|}{2}}\right)e^{ik\theta}.$$

The space $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ is the direct sum of a leading term of order $\psi^{1/2}$ and a remainder in $K^{m,\sigma}_{1/2+\gamma}(\Pi)$ of higher order terns. Thus we must discard all terms from the homogeneous solution whose order is less than $\psi^{1/2}$, namely, we must set c_0, $c_{\pm 1}$ and all d_k terms to zero. This gives us the homogeneous solution

$$u(\psi, \theta) = \sum_{|k|\geq 2} c_k\psi^{\frac{-1+|k|}{2}}e^{ik\theta}.$$

Observe that the 0th and 1st order modes are entirely absent from this solution. Their absence is accounted for by the extra degrees of freedom R and p, provided in the solution. We split up the solution as follows:

$$u(\psi, \theta) = \psi^{\frac{1}{2}}\left(c_2 e^{2i\theta} + c_{-2}e^{-2i\theta}\right) + \sum_{|k|\geq 3} c_k\psi^{\frac{-1+|k|}{2}}e^{ik\theta}.$$

The first term is of order $\psi^{1/2}$, and its angular contribution is entire, thus certainly in $X^m_\sigma(\mathbb{T})$. We should now guarantee that the remaining sum belongs to $K^{m,\sigma}_{1/2+\gamma}(\Pi)$ without having to discard any additional modes. We need thus ensure that the lowest order term of the remainder, that is ψ, is in $K^{m,\sigma}_{1/2+\gamma}(\Pi)$. This is satisfied when $\gamma < 1$. If we were to allow $\gamma \geq 1$, we would have to drop sufficient additional low order modes from the remainder term, which would render the boundary value problem non-surjective. Also, recall that $J^{m,\sigma}_{1/2,\gamma}(\Pi)$

is only well defined if $\gamma \geq 1/2$. Taking the Fourier series of $g(\theta)$, we can now match the boundary condition. We get:

$$\begin{cases} R = \hat{g}_0 \\ p_x = \hat{g}_1 + \hat{g}_{-1} \\ p_y = i(\hat{g}_1 - \hat{g}_{-1}) \\ c_k = \hat{g}_k \text{ for } |k| \geq 2. \end{cases}$$

Finally, we show the bound on this solution. Substituting the above and from definition of $J^{m,\sigma}_{1/2,\gamma}(\Pi)$, we get

$$\begin{aligned} |R|^2 + |p_x|^2 + |p_y|^2 + \|u(\psi,\theta)\|^2_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} &= |\hat{g}_0|^2 + |\hat{g}_1 + \hat{g}_{-1}|^2 + |\hat{g}_1 - \hat{g}_{-1}|^2 \\ &+ \big\|\hat{g}_2 e^{2i\theta} + \hat{g}_{-2} e^{-2i\theta}\big\|^2_{X^m_\sigma(\mathbb{T})} + \Big\|\sum_{|k|\geq 3} \hat{g}_k \psi^{\frac{-1+|k|}{2}} e^{ik\theta}\Big\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}. \end{aligned}$$

The last term is bounded as follows:

$$\begin{aligned} \Big\|\sum_{|k|\geq 3} \hat{g}_k \psi^{\frac{-1+|k|}{2}} e^{ik\theta}\Big\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} &= \sum_{p+q=0}^{m} \sum_{|k|\geq 3} (k^2)^q e^{2\sigma|k|} |\hat{g}_k|^2 \big\|\psi^{p-\frac{1}{2}-\gamma} D^p \psi^{\frac{-1+|k|}{2}}\big\|^2_{L^2(0,1]} \\ &= \sum_{p+q=0}^{m} \sum_{|k|\geq 3} (k^2)^q e^{2\sigma|k|} |\hat{g}_k|^2 c_{p,k} \big\|\psi^{-1-\gamma+\frac{|k|}{2}}\big\|^2_{L^2(0,1]} \\ &= \sum_{p+q=0}^{m} \sum_{|k|\geq 3} C_{p,k} \frac{(k^2)^q}{-1-2\gamma+|k|} e^{2\sigma|k|} |\hat{g}_k|^2 \\ &\leq C \sum_{|k|\geq 3} (1+k^2)^{m-1/2} e^{2\sigma|k|} |\hat{g}_k|^2 \end{aligned}$$

where the third equality follows so long as $\psi^{-1-\gamma+\frac{|k|}{2}} \in L^2(0,1]$, which is satisfied given $\gamma < 1$ and $|k| \geq 3$. We thus get the required bound

$$|R|^2 + |p_x|^2 + |p_y|^2 + \|u(\psi,\theta)\|^2_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} \leq C \sum_k (1+k^2)^{m-1/2} e^{2\sigma|k|} |\hat{g}_k|^2 = C\|g(\theta)\|_{X^{m-1/2}_\sigma(\mathbb{T})}.$$

$\square$

Let us now tackle the inhomogeneous problem. We write $u(\psi,\theta) = v(\theta)\psi^{1/2} + w(\psi,\theta)$ and $f(\psi,\theta) = \xi(\theta)\psi^{1/2} + \eta(\psi,\theta)$, where $v \in X^m_\sigma(\mathbb{T})$, $w \in K^{m,\sigma}_{1/2+\gamma}(\Pi)$, $\xi \in X^{m-2}_\sigma(\mathbb{T})$ and $\eta \in K^{m-2,\sigma}_{1/2+\gamma}(\Pi)$. We can consider the linear problem on components $v(\theta)\psi^{1/2}$ and $w(\psi,\theta)$

separately, so long as we are careful to distribute the boundary condition modes carefully. Let us start with the first component. We have just seen that v can only account for the $|k| = 2$ modes of the boundary condition. Let us then consider the problem

$$\begin{cases} L\left(v(\theta)\psi^{1/2}\right) = \xi(\theta)\psi^{1/2} \\ \hat{v}_{\pm 2} = 0. \end{cases} \tag{3.2}$$

Proposition 3.5.

The linear problem 3.2 is invertible between $v(\theta) \in X^m_\sigma(\mathbb{T})$ *and* $\xi(\theta) \in \widetilde{X}^{m-2}_\sigma(\mathbb{T}) = \left\{\xi \in X^{m-2}_\sigma(\mathbb{T})\colon \int_{\mathbb{T}} \xi(\theta) e^{\pm 2i\theta} = 0\right\}$ *and its solution has bound* $\|v\|_{X^m_\sigma(\mathbb{T})} \leq C\|\xi\|_{X^{m-2}_\sigma(\mathbb{T})}$.

Proof. A direct computation shows $L(v\psi^{1/2}) = \psi^{1/2}\left(v(\theta) + \frac{D^2 v(\theta)}{4}\right)$. Taking the Fourier series, we get the family of algebraic equations $(1 - k^2/4)\,\hat{v}_k = \hat{\xi}_k$. For $|k| = 2$, the left side vanishes. We get the solution

$$\begin{cases} \hat{v}_k = (1 - k^2/4)^{-1}\hat{\xi}_k, \text{ for } |k| \neq 2, \\ \hat{v}_{\pm 2} = 0 \end{cases}$$

and deduce L is not surjective onto $X^{m-2}_\sigma(\mathbb{T})$ but rather onto $\widetilde{X}^{m-2}_\sigma(\mathbb{T})$. To establish the boundedness of this inverse, we have

$$\begin{aligned} \|v\|^2_{X^m_\sigma(\mathbb{T})} &= \sum_k (1+k^2)^m e^{2\sigma|k|}|\hat{v}_k|^2 \\ &= \sum_{|k|\neq 2} (1+k^2)^m e^{2\sigma|k|}\frac{|\hat{\xi}_k|^2}{(1-k^2/4)^2} \\ &\leq C\sum_k (1+k^2)^{m-2} e^{2\sigma|k|}|\hat{\xi}_k|^2 \\ &\leq C\|\xi\|^2_{X^{m-2}_\sigma(\mathbb{T})}. \end{aligned}$$

□

It remains to solve the inhomogeneous problem on the second component. We must be careful to distribute the boundary conditions correctly. We have seen that the $k = 0$ mode of the boundary condition is controlled for by R. Next, the $|k| = 1$ modes are controlled for by

p. Finally, the $|k| = 2$ modes are controlled for by the first component of u. Thus we should expect that only the remaining $|k| \geq 3$ modes are controlled by the second component. The inhomogeneous problem then is: given $\eta(\psi, \theta) \in K^{m-2,\sigma}_{1/2,\gamma}(\Pi)$, solve the following equation for $w(\psi, \theta) \in K^{m,\sigma}_{1/2,\gamma}(\Pi)$:

$$\begin{cases} Lw(\psi, \theta) = \eta(\psi, \theta) \\ \hat{w}_k(1) = \int_{\mathbb{T}} w(1, \theta) e^{-ik\theta} \, \mathrm{d}\theta = 0 \text{ for } |k| \geq 3. \end{cases} \tag{3.3}$$

Proposition 3.6.

Let $1/2 < \gamma < 1$. The inhomogeneous problem 3.3 is invertible with bound

$$\|w\|_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} \leq C \|\eta\|_{K^{m-2,\sigma}_{1/2+\gamma}(\Pi)}.$$

Proof. Expanding in a Fourier series gives the family of ODEs

$$\begin{cases} L_k \hat{w}_k(\psi) = \hat{\eta}_k(\psi) \\ \hat{w}_k(1) = 0 \text{ for } |k| \geq 3, \end{cases}$$

where

$$L_k = \psi^2 D^2 + 2\psi D + \frac{1-k^2}{4} I.$$

We can factor L_k into the product of two 1st order operators as follows:

$$L_k = \left(\psi D + \frac{1+|k|}{2} I \right) \left(\psi D + \frac{1-|k|}{2} I \right) = L_k^+ \cdot L_k^-.$$

Next we can rewrite these operators as

$$\begin{aligned} L_k^+ &= \psi D + \frac{1+|k|}{2} I = \psi^{1-\frac{1+|k|}{2}} D \left(\psi^{\frac{1+|k|}{2}} I \right), \\ L_k^- &= \psi D + \frac{1-|k|}{2} I = \psi^{1-\frac{1-|k|}{2}} D \left(\psi^{\frac{1-|k|}{2}} I \right). \end{aligned}$$

We can accordingly define a factorization of L by

$$L = L_+ \cdot L_-,$$

where

$$L_\pm w(\psi, \theta) = \sum_k L_k^\pm \hat{w}_k(\psi) e^{ik\theta}.$$

Notice each of these operators is bounded from $K^{m,\sigma}_{1/2+\gamma}(\Pi)$ to $K^{m-1,\sigma}_{1/2+\gamma}(\Pi)$. Using the boundary conditions, we can explicitly invert $L^{\pm}_k$ to construct $L^{-1}_{\pm}$ and show that it is bounded from $K^{m-1,\sigma}_{1/2+\gamma}(\Pi)$ to $K^{m,\sigma}_{1/2+\gamma}(\Pi)$. Since L^{-1} is the composition of L^{-1}_{-} and L^{-1}_{+}, it follows that it is bounded from $K^{m-2,\sigma}_{1/2+\gamma}(\Pi)$ to $K^{m,\sigma}_{1/2+\gamma}(\Pi)$.

Let us proceed now with inverting L^{+}_k and L^{-}_k. They take the general form of operator

$$A_{\lambda_k} = \psi D + \lambda_k I = \psi^{1-\lambda_k} D\left(\psi^{\lambda_k} I\right),$$

where in our case $\lambda_k = \frac{1\pm|k|}{2}$. Writing the equation $A_{\lambda_k} w_k(\psi) = \psi^{1-\lambda_k} D\left(\psi^{\lambda_k} w_k(\psi)\right) = \eta_k(\psi)$, we can solve by direct integration to get

$$w_k(\psi) = A^{-1}_{\lambda_k}\eta_k(\psi) = \psi^{-\lambda_k}\int_0^{\psi} t^{\lambda_k - 1}\eta_k(t)\,\mathrm{d}t + c\psi^{-\lambda_k},$$

or equivalently

$$w_k(\psi) = A^{-1}_{\lambda_k}\eta_k(\psi) = -\psi^{-\lambda_k}\int_{\psi}^{1} t^{\lambda_k - 1}\eta_k(t)\,\mathrm{d}t + d\psi^{-\lambda_k}.$$

Next, we should check for which $\lambda_k = \frac{1\pm|k|}{2}$ does the $\psi^{-\lambda_k}$ term belong to $K^{m,\sigma}_{1/2+\gamma}$, keeping in mind we have the restriction $1/2 \leq \gamma < 1$ from the homogeneous problem. This occurs only for L^{-}_k when $|k| \geq 3$, that is, when $\lambda_k = \frac{1-|k|}{2}$ and $|k| \geq 3$. In this case, we need the boundary condition to find the inverse. Otherwise, we must set constants c or d to zero and no boundary condition is available. Let us then write the inverses as follows.

$$\left(L^{-}_k\right)^{-1}\hat{\eta}_k(\psi) = \begin{cases} \psi^{-\frac{1-|k|}{2}} \int_0^{\psi} t^{\frac{-1-|k|}{2}}\hat{\eta}_k(t)\,\mathrm{d}t & \text{for } |k| < 3 \\ -\psi^{-\frac{1-|k|}{2}} \int_{\psi}^{1} t^{\frac{-1-|k|}{2}}\hat{\eta}_k(t)\,\mathrm{d}t & \text{for } |k| \geq 3 \end{cases} \tag{3.4}$$

$$\left(L^{+}_k\right)^{-1}\hat{\eta}_k(\psi) = \psi^{-\frac{1+|k|}{2}}\int_0^{\psi} t^{\frac{-1+|k|}{2}}\hat{\eta}_k(t)\,\mathrm{d}t \qquad \text{for all } k. \tag{3.5}$$

Notice that the inverses above are operators that take the weighted average of a Fourier mode of η from 0 to ψ or from ψ to 1. The choice we made is in anticipation of using the Hardy inequality 3.2. Also notice the boundary conditions available were each used precisely once. We have thus constructed explicitly the inverse of L, which can be written as

$$L^{-1}\eta(\psi,\theta) = \sum_k \left(L^{-}_k\right)^{-1}\cdot\left(L^{+}_k\right)^{-1}\hat{\eta}_k(\psi)e^{ik\theta}$$

and can be described as the Fourier series of the composition of two varying weighted averages of the Fourier modes of η.

Now, we must demonstrate the boundedness from $K^{m-1,\sigma}_{1/2,\gamma}(\Pi)$ to $K^{m,\sigma}_{1/2,\gamma}(\Pi)$ of operators

$$L_{\pm}^{-1}\eta(\psi,\theta) = \sum_k \left(L_k^{\pm}\right)^{-1}\hat{\eta}_k(\psi)e^{ik\theta}.$$

That is, we are looking to establish the bound $\|L_{\pm}^{-1}\eta(\psi,\theta)\|_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} \leq C\|\eta(\psi,\theta)\|_{K^{m-1,\sigma}_{1/2+\gamma}(\Pi)}$ given norm

$$\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} = \sum_k \sum_{p+q=0}^{m} (k^2)^q e^{2\sigma|k|}\left\|\psi^{p-1/2-\gamma}D^p\hat{w}_k(\psi)\right\|^2_{L^2(0,1]}. \tag{3.6}$$

Having already constructed $\hat{w}_k(\psi)$ in 3.4 and 3.5, we should next find an expression for its derivatives $D^p\hat{w}_k(\psi)$. Let us again write

$$A_{\lambda_k}\hat{w}_k(\psi) = (\psi D + \lambda_k I)\,\hat{w}_k(\psi) = \hat{\eta}_k(\psi).$$

Rearranging, gives

$$D\hat{w}_k(\psi) = \frac{1}{\psi}\hat{\eta}_k(\psi) - \frac{\lambda_k}{\psi}\hat{w}_k(\psi).$$

Continued differentiation and substitution yields the expression

$$D^p\hat{w}_k(\psi) = \sum_{n=1}^{p}(-1)^{n+1}\frac{(\lambda_k+p-1)!}{(\lambda_k+p-n)!}\cdot\frac{1}{\psi^n}\cdot D^{p-n}\hat{\eta}_k(\psi) + (-1)^p(\lambda_k+p-1)!\frac{\hat{w}_k(\psi)}{\psi^p}, \tag{3.7}$$

where we use the factorial sign ! in a modified sense to mean

$$\begin{cases} (\lambda+l)! = (\lambda+l)(\lambda+l-1)\cdots(\lambda+1)(\lambda) & \text{for } l\in\mathbb{N}_0, \\ (\lambda-1)! = 1. \end{cases} \tag{3.8}$$

Substituting this expression into 3.6 and using triangle inequality for the summations, we get

$$\begin{aligned}\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}} \leq C\sum_k \sum_{\substack{p+q=1\\ p\geq 1}}^{m}(k^2)^q e^{2\sigma|k|}\sum_{n=1}^{p}\left(\frac{(\lambda_k+p-1)!}{(\lambda_k+p-n)!}\right)^2\left\|\psi^{p-n-\frac{1}{2}-\gamma}D^{p-n}\hat{\eta}_k(\psi)\right\|^2_{L^2(0,1]} \\ + C\sum_k\sum_{p+q=0}^{m}(k^2)^q e^{2\sigma|k|}\big((\lambda_k+p-1)!\big)^2\left\|\psi^{-\frac{1}{2}-\gamma}\hat{w}_k(\psi)\right\|^2_{L^2(0,1]}.\end{aligned}$$

From our modified factorial 3.8 and that $\lambda_k = \frac{1 \pm |k|}{2}$, we have that

$$(\lambda_k + p - 1)! \sim |k|^p \text{ as } |k| \to \pm\infty$$

and

$$\frac{(\lambda_k + p - 1)!}{(\lambda_k + p - n)!} \sim |k|^{n-1} \text{ as } |k| \to \pm\infty.$$

Applying this to the inequality above gives

$$\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}} \le C \sum_k \sum_{\substack{p+q=1\\ p\ge 1}}^{m} \sum_{n=1}^{p} (k^2)^{n-1+q} e^{2\sigma|k|} \left\|\psi^{p-n-\frac{1}{2}-\gamma} D^{p-n} \hat{\eta}_k(\psi)\right\|^2_{L^2(0,1]}$$
$$+ C \sum_k \sum_{p+q=0}^{m} (k^2)^{p+q} e^{2\sigma|k|} \left\|\psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi)\right\|^2_{L^2(0,1]}.$$

Let us split the right hand side into two terms, with

$$A = \sum_k \sum_{\substack{p+q=1\\ p\ge 1}}^{m} \sum_{n=1}^{p} (k^2)^{n-1+q} e^{2\sigma|k|} \left\|\psi^{p-n-\frac{1}{2}-\gamma} D^{p-n} \hat{\eta}_k(\psi)\right\|^2_{L^2(0,1]}$$

and

$$B = \sum_k \sum_{p+q=0}^{m} (k^2)^{p+q} e^{2\sigma|k|} \left\|\psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi)\right\|^2_{L^2(0,1]}.$$

In term A, setting $q' = n - 1 + q$ and $p' = p - n$, then $p' + q' = p + q - 1$ ranges from 0 to $m - 1$. We immediately get

$$A \le C \sum_k \sum_{p'+q'=0}^{m-1} (k^2)^{q'} e^{2\sigma|k|} \left\|\psi^{p'-\frac{1}{2}-\gamma} D^{p'} \hat{\eta}_k(\psi)\right\|^2_{L^2(0,1]} = C\|\eta(\psi,\theta)\|_{K^{m-1,\sigma}_{1/2+\gamma}(\Pi)}.$$

Now we turn to bounding B, starting with the factor $\left\|\psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi)\right\|_{L^2(0,1]}$ in the summation. To achieve this we will use now the Hardy inequality. Recall, $\hat{w}_k(\psi)$ is given by 3.5 and 3.4, depending on if we are inverting L_k^+ or L_k^-. This gives us three separate cases of expressions for $\hat{w}_k(\psi)$.

Let us start with the case when $\hat{w}_k(\psi) = \left(L_k^+\right)^{-1} \hat{\eta}_k(\psi)$, and k is arbitrary. In this case,

$$\hat{w}_k(\psi) = \psi^{-\frac{1+|k|}{2}} \int_0^\psi t^{\frac{-1+|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t\,.$$

Setting $\alpha - 1 = -\frac{1}{2} - \gamma - \frac{1+|k|}{2}$, we get $\alpha = -\gamma - \frac{|k|}{2} \leq -\gamma < 1/2$ for all k, as long as $\gamma > -1/2$. Also set $t^{\frac{-1+|k|}{2}} \hat{\eta}_k(t) = t^{-\alpha} \zeta_k(t)$. Applying Hardy's inequality for $\alpha < 1/2$, we get

$$\begin{aligned}
\left\| \psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi) \right\|_{L^2(0,1]} &= \left\| \psi^{-\frac{1}{2}-\gamma-\frac{1+|k|}{2}} \int_0^\psi t^{\frac{-1+|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t \right\|_{L^2(0,1]} \\
&= \left\| \psi^{\alpha-1} \int_0^\psi t^{-\alpha} \zeta_k(t)\, \mathrm{d}t \right\|_{L^2(0,1]} \\
&\leq \frac{1}{\frac{1}{2} - \alpha} \left\| \zeta_k(\psi) \right\|_{L^2(0,1]} \\
&\leq \frac{1}{\frac{1}{2} + \gamma + \frac{|k|}{2}} \left\| \psi^{-\gamma-\frac{1}{2}} \hat{\eta}_k(\psi) \right\|_{L^2(0,1]}.
\end{aligned}$$

Next, we consider the case when $\hat{w}_k(\psi) = \left(L_k^-\right)^{-1} \hat{\eta}_k(\psi)$ and $|k| < 3$. In this case,

$$\hat{w}_k(\psi) = \psi^{-\frac{1-|k|}{2}} \int_0^\psi t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t\,.$$

Setting $\alpha - 1 = -\frac{1}{2} - \gamma - \frac{1-|k|}{2}$, we get $\alpha = -\gamma + \frac{|k|}{2} < 1/2$ for $|k| \leq 2$, only as long as $\gamma > 1/2$. Also set $t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t) = t^{-\alpha} \zeta_k(t)$. Applying Hardy's inequality for $\alpha < 1/2$, we get

$$\begin{aligned}
\left\| \psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi) \right\|_{L^2(0,1]} &= \left\| \psi^{-\frac{1}{2}-\gamma-\frac{1-|k|}{2}} \int_0^\psi t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t \right\|_{L^2(0,1]} \\
&= \left\| \psi^{\alpha-1} \int_0^\psi t^{-\alpha} \zeta_k(t)\, \mathrm{d}t \right\|_{L^2(0,1]} \\
&\leq \frac{1}{\frac{1}{2} - \alpha} \left\| \zeta_k(\psi) \right\|_{L^2(0,1]} \\
&\leq \frac{1}{\frac{1}{2} + \gamma - \frac{|k|}{2}} \left\| \psi^{-\gamma-\frac{1}{2}} \hat{\eta}_k(\psi) \right\|_{L^2(0,1]}.
\end{aligned}$$

The third case occurs when $\hat{w}_k(\psi) = \left(L_k^-\right)^{-1} \hat{\eta}_k(\psi)$ and $|k| \geq 3$. In this case,

$$\hat{w}_k(\psi) = -\psi^{-\frac{1-|k|}{2}} \int_\psi^1 t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t\,.$$

Setting $\alpha - 1 = -\frac{1}{2} - \gamma - \frac{1-|k|}{2}$, we get $\alpha = -\gamma + \frac{|k|}{2} > 1/2$ for $|k| \geq 3$, as long as $\gamma < 1$. Also set $t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t) = t^{-\alpha} \zeta_k(t)$. Applying Hardy's inequality now for $\alpha > 1/2$, we get

$$\begin{aligned}
\left\| \psi^{-\frac{1}{2}-\gamma} \hat{w}_k(\psi) \right\|_{L^2(0,1]} &= \left\| \psi^{-\frac{1}{2}-\gamma-\frac{1-|k|}{2}} \int_\psi^1 t^{\frac{-1-|k|}{2}} \hat{\eta}_k(t)\, \mathrm{d}t \right\|_{L^2(0,1]} \\
&= \left\| \psi^{\alpha-1} \int_0^\psi t^{-\alpha} \zeta_k(t)\, \mathrm{d}t \ \right\|_{L^2(0,1]} \\
&\leq \frac{1}{\alpha - \frac{1}{2}} \left\| \zeta_k(\psi) \right\|_{L^2(0,1]} \\
&\leq \frac{1}{-\frac{1}{2} - \gamma + \frac{|k|}{2}} \left\| \psi^{-\gamma-\frac{1}{2}} \hat{\eta}_k(\psi) \right\|_{L^2(0,1]}.
\end{aligned}$$

We have thus found that in each of the three cases, we get the bound

$$\left\|\psi^{-\frac{1}{2}-\gamma}\hat{w}_k(\psi)\right\|_{L^2(0,1]} \leq C_{\gamma,k}\left\|\psi^{-\frac{1}{2}-\gamma}\hat{\eta}_k(\psi)\right\|_{L^2(0,1]}.$$

The crucial detail is the additional strict restriction to $\gamma > 1/2$, which avoids the critical case of the Hardy inequality when $\alpha = 1/2$. This ensures the constant $C_{\gamma,k}$ above is bounded for all k. Note this constant decays like $2/|k|$.

Returning to B and applying the above, we get the bound

$$\begin{aligned} B &= \sum_k \sum_{p+q=0}^{m} (k^2)^{p+q} e^{2\sigma|k|} \left\|\psi^{-\frac{1}{2}-\gamma}\hat{w}_k(\psi)\right\|^2_{L^2(0,1]} \\ &\leq C \sum_k \sum_{q'=0}^{m-1} (k^2)^{q'} e^{2\sigma|k|} \left\|\psi^{-\frac{1}{2}-\gamma}\hat{\eta}_k(\psi)\right\|^2_{L^2(0,1]} \\ &\leq C \left\|\eta(\psi,\theta)\right\|_{K^{m-1,\sigma}_{1/2+\gamma}(\Pi)}. \end{aligned}$$

From A and B, we get

$$\left\|L_{\pm}^{-1}\eta(\psi,\theta)\right\|_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} \leq C \left\|\eta(\psi,\theta)\right\|_{K^{m-1,\sigma}_{1/2+\gamma}(\Pi)}$$

and thus

$$\left\|w(\psi,\theta)\right\|_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} = \left\|L_{-}^{-1}\cdot L_{+}^{-1}\eta(\psi,\theta)\right\|_{K^{m,\sigma}_{1/2+\gamma}(\Pi)} \leq C \left\|\eta(\psi,\theta)\right\|_{K^{m-2,\sigma}_{1/2+\gamma}(\Pi)}.$$

□

We now have the ingredients to prove the main result of this chapter.

Theorem 3.7. *For $1/2 < \gamma < 1$, the linear problem 3.1 defines an isomorphism*

$$\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T}).$$

Proof. By proposition 3.3 and the fact that multiplication by $\psi^{-1/2}$ defines an isomorphism from $J^{m-2,\sigma}_{1/2,\gamma}(\Pi)$ to $J^{m-2,\sigma}_{0,\gamma}(\Pi)$, the linear map is bounded in the above spaces.

To construct and bound the inverse, we must be careful to match the boundary conditions correctly. Let $u_g(\psi,\theta) = \psi^{1/2}v(\theta) + w(\psi,\theta)$, where $v(\theta)$ is solution to 3.2 and $w(\psi,\theta)$

is solution to 3.3. Let $u_h(\psi,\theta) = \sum_{|k|\geq 2} c_k \psi^{\frac{-1+|k|}{2}} e^{ik\theta}$ be the homogeneous solution with coefficients c_k to be determined. Then the full solution is

$$\begin{aligned} u(\psi,\theta) &= u_g(\psi,\theta) + u_h(\psi,\theta) \\ &= \psi^{\frac{1}{2}} \left(\hat{v}(\theta) + c_2 e^{2i\theta} + c_{-2} e^{-2i\theta}\right) + w(\psi,\theta) + \sum_{|k|\geq 3} c_k \psi^{\frac{-1+|k|}{2}} e^{ik\theta} \\ &= \psi^{\frac{1}{2}} \left(\sum_k \hat{v}_k e^{ik\theta} + \sum_{|k|=2} c_k e^{ik\theta}\right) + \sum_k \hat{w}_k(\psi) e^{ik\theta} + \sum_{|k|\geq 3} c_k \psi^{\frac{-1+|k|}{2}} e^{ik\theta}. \end{aligned}$$

Now we must match the boundary condition

$$R + (\frac{p_x - ip_y}{2})e^{i\theta} + (\frac{p_x + ip_y}{2})e^{-i\theta} + u(1,\theta) = g(\theta).$$

Keeping in mind that $\hat{v}_{\pm 2} = 0$, and $\hat{w}_k(1) = 0$ for $|k| \geq 3$, the boundary condition yields

$$R + (\frac{p_x - ip_y}{2})e^{i\theta} + (\frac{p_x + ip_y}{2})e^{-i\theta} + \sum_{|k|\neq 2} \hat{v}_k e^{ik\theta} + \sum_{|k|\leq 2} \hat{w}_k(1) e^{ik\theta} + \sum_{|k|\geq 2} c_k e^{ik\theta} = \sum_k \hat{g}_k e^{ik\theta}.$$

This gives us the following set of equations on the Fourier modes

$$\begin{cases} R + \hat{v}_0 + \hat{w}_0(1) = \hat{g}_0 \\ \frac{p_x \mp ip_y}{2} + \hat{v}_{\pm 1} + \hat{w}_{\pm 1}(1) = \hat{g}_{\pm 1} \\ \hat{w}_{\pm 2}(1) + c_{\pm 2} = \hat{g}_{\pm 2} \\ \hat{v}_k + c_k = \hat{g}_k, \text{ for } |k| \geq 3. \end{cases}$$

Solving for R, p_x, p_y and c_k, we get

$$\begin{cases} R = \hat{g}_0 - \hat{v}_0 - \hat{w}_0(1) \\ p_x = \left(\hat{g}_1 + \hat{g}_{-1}\right) - \left(\hat{v}_1 + \hat{v}_{-1}\right) - \left(\hat{w}_1(1) + \hat{w}_{-1}(1)\right) \\ -ip_y = \left(\hat{g}_1 - \hat{g}_{-1}\right) - \left(\hat{v}_1 - \hat{v}_{-1}\right) - \left(\hat{w}_1(1) - \hat{w}_{-1}(1)\right) \\ c_{\pm 2} = \hat{g}_{\pm 2} - \hat{w}_{\pm 2}(1) \\ c_k = \hat{g}_k - \hat{v}_k, \text{ for } |k| \geq 3. \end{cases}$$

Having now constructed the solution to the boundary value 3.1, we can now establish its boundedness. We start with

$$
\begin{aligned}
|R|^2 + |p_x|^2 + |p_y|^2 &\le C \sum_{|k|\le 1} \left(|\hat{g}_k|^2 + |\hat{v}_k|^2 + |\hat{w}_k(1)|^2\right) \\
&\le C \left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + \|w(1,\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}\right) \\
&\le C \left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + \|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}\right) \\
&\le C \left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|\xi(\theta)\|^2_{X^{m-2}_\sigma(\mathbb{T})} + \|\eta(\psi,\theta)\|^2_{K^{m-2,\sigma}_{1/2+\gamma}(\Pi)}\right) \\
&= C \left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|f(\psi,\theta)\|^2_{J^{m-2,\sigma}_{1/2,\gamma}(\Pi)}\right),
\end{aligned}
$$

where the third inequality follows from the second by boundedness of restriction to $\psi = 1$ from $K^{m,\sigma}_{1/2+\gamma}(\Pi)$ to $X^{m-1/2}_\sigma(\mathbb{T})$, and the fourth inequality follows from the third by propositions 3.5, 3.6.

Next we bound the leading term of the solution

$$
\begin{aligned}
\Big\|v(\theta) + \sum_{|k|=2} c_k e^{ik\theta}\Big\|^2_{X^m_\sigma(\mathbb{T})} &\le C\|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + C \sum_{|k|=2} (1+k^2)^m e^{2\sigma|k|} \left(|\hat{g}_k|^2 + |\hat{w}_k(1)|^2\right) \\
&\le C\|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + C \sum_{|k|=2} (1+k^2)^{m-1/2} e^{2\sigma|k|} \left(|\hat{g}_k|^2 + |\hat{w}_k(1)|^2\right) \\
&\le C \left(\|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + \|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|w(1,\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}\right) \\
&\le C \left(\|v(\theta)\|^2_{X^m_\sigma(\mathbb{T})} + \|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}\right) \\
&\le C \left(\|\xi(\theta)\|^2_{X^{m-2}_\sigma(\mathbb{T})} + \|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|\eta(\psi,\theta)\|^2_{K^{m-2,\sigma}_{1/2+\gamma}(\Pi)}\right) \\
&\le C \left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})} + \|f(\psi,\theta)\|^2_{J^{m-2,\sigma}_{1/2,\gamma}(\Pi)}\right),
\end{aligned}
$$

where again we have used the boundedness of the restriction to $\psi = 1$ and propositions 3.5, 3.6.

Finally we bound the remainder term of the solution

$$
\begin{aligned}
&\Big\|w(\psi,\theta)+\sum_{|k|\geq 3} c_k\psi^{\frac{-1+|k|}{2}}e^{ik\theta}\Big\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}\\
&\qquad\leq C\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}+C\Big\|\sum_{|k|\geq 3}(\hat g_k-\hat v_k)\psi^{\frac{-1+|k|}{2}}e^{ik\theta}\Big\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}\\
&\qquad\leq C\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}+\|g(\theta)-v(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}\\
&\qquad\leq C\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}+\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}+\|v(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}\\
&\qquad\leq C\|w(\psi,\theta)\|^2_{K^{m,\sigma}_{1/2+\gamma}(\Pi)}+\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}+\|v(\theta)\|^2_{X^{m}_\sigma(\mathbb{T})}\\
&\qquad\leq C\|\eta(\psi,\theta)\|^2_{K^{m-2,\sigma}_{1/2+\gamma}(\Pi)}+\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}+\|\xi(\theta)\|^2_{X^{m-2}_\sigma(\mathbb{T})}\\
&\qquad= C\left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}+\|f(\psi,\theta)\|^2_{J^{m-2,\sigma}_{1/2,\gamma}(\Pi)}\right)
\end{aligned}
$$

where we have used propositions 3.4, 3.5 and 3.6.

We have thus established the bound

$$
|R|^2+|p|^2+\|u(\psi,\theta)\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)}\leq C\left(\|g(\theta)\|^2_{X^{m-1/2}_\sigma(\mathbb{T})}+\|f(\psi,\theta)\|^2_{J^{m-2,\sigma}_{1/2,\gamma}(\Pi)}\right)
$$

for the inverse to 3.1.

Finally, since multiplication by $\psi^{-1/2}$ is an isomorphism from $J^{m-2,\sigma}_{1/2,\gamma}(\Pi)$ to $J^{m-2,\sigma}_{0,\gamma}(\Pi)$, we conclude that the linear problem 3.1 defines an isomorphism

$$
\mathbb{C}^3\times J^{m,\sigma}_{1/2,\gamma}(\Pi)\to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)\times X^{m-1/2}_\sigma(\mathbb{T}) \tag{3.9}
$$

for $1/2<\gamma<1$. □

Remark 3.8. Let us comment on how the above result compares with the standard boundary value problem associated to operator 3.1 in the Kondratev space. That is, let's consider

$$
\begin{cases} Lu(\psi,\theta)=f(\psi,\theta)\\ u(1,\theta)=g(\theta)\end{cases}
$$

in spaces $K^m_\gamma(\Pi)\to K^{m-2}_\gamma(\Pi)\times H^{m-1/2}(\mathbb{T})$.

To start, we address the presence of the cokernel given by $\operatorname{span}\{e^{\pm 2i\theta}\}$ in 3.5. Solving $Lu=\psi^{\frac12}e^{\pm 2i\theta}$ explicitly yields solution $u=\frac12\psi^{\frac12}\ln(\psi)e^{\pm 2i\theta}$. If we take γ such that the right hand side is in $K_\gamma^{m-2}(\Pi)$, then the above solution is necessarily in $K_\gamma^m(\Pi)$. The presence of this cokernel is thus a direct consequence of solving in $J_{1/2,\gamma}^m(\Pi)$, which excludes the above lower order solutions. On the other hand, the space $K_\gamma^m(\Pi)$ has enough flexibility in the asymptotics that it contrastingly includes such solutions.

Next, following the proof of 3.6 and applying the Hardy inequality to invert L_k, one finds that whenever $\gamma = k/2$, the Hardy Inequality fails to apply as we fall into the critical case $\alpha = 1/2$. To avoid this, we must remove $\gamma = 0, \pm 1/2, \pm 1, \pm 3/2, \text{etc}$ from the pool of allowable γ values.

Finally, recall that the solution to the homogeneous problem $Lu = 0$ is given by

$$u(\psi,\theta) = c_0\psi^{-\frac{1}{2}} + d_0\psi^{-\frac{1}{2}}\ln(\psi) + \sum_{k\neq 0}\left(c_k\psi^{\frac{-1+|k|}{2}} + d_k\psi^{\frac{-1-|k|}{2}}\right)e^{ik\theta}.$$

The homogeneous boundary value problem can only be surjective onto $H^{m-1/2}(\mathbb{T})$ if all Fourier modes are represented in the above expression. That is, all c_k terms should be in the span of solutions. This necessarily implies that the map is not injective, for if γ is such that $c_0\psi^{-\frac{1}{2}}$ is in $K_\gamma^m(\Pi)$, then $d_0\psi^{-\frac{1}{2}}\ln(\psi)$ is also in $K_\gamma^m(\Pi)$, and thus $\psi^{-\frac{1}{2}}\ln(\psi)$ belongs to the kernel of the map. We can summarize with the following observation. The greater γ is, fewer low order modes of u span the solution, and thus the greater the dimension of the cokernel in the space of boundary value functions. Conversely, the smaller γ is, the greater the dimension of the kernel is, as more d_k terms are permitted to exist. The details are due to the specific coupling defined by L of Fourier modes k and asymptotics ψ^λ in the homogeneous solution.

We find in particular that the above linear problem is Fredholm for $\gamma \neq \frac{n}{2}$, $n \in \mathbb{Z}$. Furthermore, given $\frac{n}{2} < \gamma < \frac{n+1}{2}$, the index of the map is equal to $-2n-1$. We see then, that without imposing additional conditions or adding additional degrees of freedom to the solution (depending on γ), the problem is not invertible in these spaces. This parallels our main result, which requires R and p as part of the solution to achieve surjectivity.

To conclude this chapter, let us summarize our findings. We have shown that the linearization to the nonlinear problem 1.6 at solution $\psi^{1/2}$ defines an isomorphism in our spaces when $1/2 < \gamma < 1$. Recall that $J_{\lambda+\gamma}^{m,\sigma}(\Pi)$ is well defined for $\gamma \geq 1/2$ and that γ quantifies the gap between the leading term asymptotics of order ψ^λ and the order of asymptotics of the remainder term. The restriction $\gamma < 1$ ensures that this gap is small enough that we have sufficient low order terms to match the boundary condition. The restriction $\gamma > 1/2$ is needed to apply the Hardy inequality to establish the boundedness of the inverse. Recall that though $J^{m,\sigma}_{\lambda+\gamma}(\Pi)$ is well defined for $\gamma \geq 1/2$, it is precisely when $\gamma > 1/2$ that this space embeds into continuous functions. Since continuity is certainly not an unreasonable expectation of our stationary flows, we need not view this restriction on the lower bound of γ as a limitation of our result. Finally, we have seen how the additional degrees of freedom in the solution provided by R and p are crucial for the surjectivity of the linear problem. They accommodate for the $|k| \leq 1$ Fourier modes of the boundary perturbation, corresponding to dilations and translations of the solution.

Chapter 4

Nonlinear Results

In the preceding chapters, we defined the function spaces relevant to our problem and established the invertibility of the linearization in these spaces. Our task in this chapter is to prove that the nonlinear map defining the equation of stationary flow is an analytic operator, at least in a neighbourhood of our reference solution $a(\psi,\theta) = \psi^{1/2}$.

To be precise, given $\varepsilon > 0$ sufficiently small and $|R-1| < \varepsilon$, $|p| < \varepsilon$, $\|a(\psi,\theta) - \psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} < \varepsilon$, we want to prove that the operators

$$\big(F(\psi), a(\psi,\theta)\big) \to \Xi\big(a(\psi,\theta)\big) - F(\psi)$$
$$\big(b(\varphi), R, p, a(\psi,\theta)\big) \to B\big(b(\varphi), R, p, a(1,\theta)\big)$$

are well defined into $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ and $X^{m-1/2}_{\sigma}(\mathbb{T})$ respectively, and that these maps are complex analytic. The choice of function spaces for parameters $b(\varphi)$ and $F(\psi)$ are flexible in the sense that they are not explicitly determined by the linear problem. They should be chosen to be as large as possible (while satisfying the above criteria), so that the resulting parameterization of solutions given by the implicit function theorem is maximal in the defined solution space $\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi)$.

We split the result between two sections: one for the differential operator and one for the boundary operator. We will see that the former can be reduced to a study of superposition operators on space $J^{m,\sigma}_{0,\gamma}(\Pi)$. The latter involves a study of superposition operators on $X^{m-1/2}_{\sigma}(\mathbb{T})$.

Operators $u(x) \to f\big(x, u(x)\big)$, called superposition operators, form an extensive field of interest ([2]) given the pool of nonlinear problems they generate . A typical question is to find conditions on f such that $u(x) \to f(x, u(x))$ maps between two given function spaces. In our case, the functions f defining the superposition operators are determined by our problem, and we must verify they are well behaved in our spaces. Let us start with a best case warm-up: operators of form $u(x) \to f\big(u(x)\big)$ acting on $H^m(\mathbb{T}) \subset C(\mathbb{T})$ when $m > 1/2$ is an integer.

Proposition 4.1.

Let $f \in C^m$ on a domain containing image of u. Then $u \to f(u) : H^m(\mathbb{T}) \to H^m(\mathbb{T})$ is well defined and continuous for integer $m > 1/2$. Furthermore, if $f \in C^{m+1}$, then this map is C^1.

Proof. Let $u \in H^m(\mathbb{T})$ and $f \in C^m$ on a domain containing image of u. Then we can define the composition $f(u(x))$. To show this superposition is in $H^m(\mathbb{T})$, we must estimate the $L^2(\mathbb{T})$ norm of its derivatives. The p-th derivative in x of function $f\big(u(x)\big)$ has form

$$D^p f(u) = \sum_{j=1}^{p} \sum_{\substack{\alpha_1+\cdots+\alpha_j=p \\ \alpha_i \geq 1}} C_{\alpha_1,\ldots,\alpha_j} f^{(j)}(u)(D^{\alpha_1}u)\cdots(D^{\alpha_j}u).$$

If $m > 1/2$, then $H^m(\mathbb{T}) \subset C(\mathbb{T})$ and so $f^{(j)}(u)$ is the composition of continuous functions and thus continuous. Next, $D^{\alpha_i}u \in H^{m-\alpha_i}(\mathbb{T}) \in C(\mathbb{T})$, unless $\alpha_i = m$. First suppose each $\alpha_i < m$. Then

$$\|f^{(j)}(u)(D^{\alpha_1}u)\cdots(D^{\alpha_j}u)\|_{L^2(\mathbb{T})} \leq \|f^j\|_\infty \|D^{\alpha_1}u\|_\infty \cdots \|D^{\alpha_j}u\|_\infty \leq C\|f\|_{C^m}\|u\|^j_{H^m(\mathbb{T})}.$$

Next, if some $\alpha_i = m$, then because $\alpha_1 + \cdots + \alpha_j = p = m$ and $\alpha_i \geq 1$, then we know $j = 1$. In such a case, we have

$$\|f'(u)D^m u\|_{L^2(\mathbb{T})} \leq \|f'(u)\|_\infty \|D^m u\|_{L^2(\mathbb{T})} \leq \|f\|_{C^m}\|u\|_{H^m(\mathbb{T})}.$$

We conclude that $f(u) \in H^m(\mathbb{T})$ and obtain the estimate

$$\|f(u)\|_{H^m(\mathbb{T})} \leq \|f\|_{C^m} \sum_{j=0}^{m} C_j \|u\|^j_{H^m(\mathbb{T})} \leq C\|f\|_{C^m}\big(1 + \|u\|^m_{H^m(\mathbb{T})}\big).$$

To see this map is continuous, let $u_n \to u \in H^m(\mathbb{T})$ and consider $f(u_n) - f(u)$. The p-th derivative of this difference is a sum of terms of form

$$f^{(j)}(u_n)(D^{\alpha_1}u_n)\cdots(D^{\alpha_j}u_n) - f^{(j)}(u)(D^{\alpha_1}u)\cdots(D^{\alpha_j}u).$$

Let us write $A_i(u) = D^{\alpha_i}u$ and $f^{(j)}(u) = A_{j+1}(u)$. We can add and subtract to the above expression a term as follows to obtain

$$A_1(u_n)\cdots A_{j+1}(u_n) - A_1(u)\cdots A_{j+1}(u) = A_1(u_n)\cdots A_j(u_n)\Big(A_{j+1}(u_n) - A_{j+1}(u)\Big) \\ + \Big(A_1(u_n)\cdots A_j(u_n) - A_1(u)\cdots A_j(u)\Big)A_{j+1}(u_n).$$

Continuing in this fashion on the second term, we get a sum of terms of form

$$A_1\cdots A_{i-1}\Big(A_i(u_n) - A_i(u)\Big)A_{i+1}\cdots A_{j+1},$$

where for $k \neq i$, A_k indicates either $A_k(u_n)$ or $A_k(u)$. Now applying the same bounds as earlier, we get that the $L^2(\mathbb{T})$ norm of the above expression vanishes as $n \to \infty$ because the bounding factor $\|u_n - u\|_{H^m(\mathbb{T})}$ for $i < j+1$ or $\|f^{(j)}(u_n) - f^{(j)}(u)\|_\infty$ for $i = j+1$ vanishes, while the other factors remain bounded. This proves the continuity of map $u \to f(u) : H^m(\mathbb{T}) \to H^m(\mathbb{T})$.

Next, the Gâteaux derivative of $f(u)$ in direction v is given by $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(u+tv) = f'(u)v$. If $f \in C^{m+1}$, then $f'(u)$ is a well defined superposition operator on $H^m(\mathbb{T})$. Since this space is an algebra for $m > 1/2$, then the multiplication map $v \to f'(u)v : H^m(\mathbb{T}) \to H^m(\mathbb{T})$ is a well defined linear map with $|||f'(u)||| \leq C\|f'(u)\|_{H^m(\mathbb{T})}$. If a map has a linear Gâteaux derivative that is continuous in operator norm, then it is continuously Fréchet differentiable, (see [1]). So suppose $\|u_n - u\|_{H^m(\mathbb{T})} \to 0$ as $n \to \infty$. Then $|||f'(u_n) - f'(u)||| \leq C\|f'(u_n) - f'(u)\|_{H^m(\mathbb{T})} \to 0$. The latter follows since $u \to f'(u) : H^m(\mathbb{T}) \to H^m(\mathbb{T})$ is continuous. Thus we have shown that if $f \in C^{m+1}$, then $u \to f(u)$ is a C^1 map from $H^m(\mathbb{T})$ to itself. □

4.1 Differential Operator

In this section we study the differential operator $\big(F(\psi), a(\psi,\theta)\big) \to \Xi\big(a(\psi,\theta)\big) - F(\psi)$, mapping into space $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$. Since the invertibility of the linearized problem is contingent on the restriction to $1/2 < \gamma < 1$, we need not concern ourselves with γ outside this range for the nonlinear problem.

We start with the observation that the identity map $F(\psi,\theta) = F(\psi)$ naturally embeds any function $F(\psi) \in J^{m-2}_{0,\gamma}(0,1]$ into $J^{m-2,\sigma}_{0,\gamma}(\Pi)$. By construction, such functions are constant along θ, so they in fact embed into $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$. Since the above embedding is bounded and linear from $J^{m-2,\sigma}_{0,\gamma}(0,1]$ to $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$, it is analytic with respect to F. Thus $J^{m-2}_{0,\gamma}(0,1]$ serves as the natural parameter space of vorticities in our problem.

It remains to show $a \to \Xi(a) : J^{m,\sigma}_{1/2+\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ is analytic near $a = \psi^{1/2}$, where operator Ξ is given by the expression:

$$\Xi(a) = -\frac{1}{a_\psi^3}\Big(1 + \frac{a_\theta^2}{a^2}\Big)a_{\psi\psi} + 2\Big(\frac{a_\theta}{a^2 a_\psi^2}\Big)a_{\psi\theta} - \Big(\frac{1}{a^2 a_\psi}\Big)a_{\theta\theta} + \frac{1}{a a_\psi}. \tag{4.1}$$

Notice, this map is a rational function of derivatives of $a(\psi,\theta)$, in other words a superposition map $a \to f(a, a_\psi, a_\theta, a_{\psi\psi}, a_{\psi\theta}, a_{\theta\theta})$, defined by a rational function f. The trouble is that these derivatives have distinct leading term asymptotics ψ^λ, defined by different weights of $J^{m,\sigma}_{\lambda,\gamma}(\Pi)$. It is hopeless to expect any general results of superposition operators on such spaces, regardless of m. To see this, it is enough to compare the difference between how maps $u \to u^2$ and $u \to 1/u$ act on $u = \psi^{1/2}$. There is one exception to this observation, the case when $\lambda = 0$, and thus the leading term is of order $\psi^0 = 1$. In this case, squaring or taking the reciprocal will still yield a function of leading order 1 (assuming the function does not vanish). This should remain true for other superposition maps on $J^{m,\sigma}_{0,\gamma}(\Pi)$. Given that we require Ξ to map to $J^{m-2,\sigma}_{0,\gamma}(\Pi)$, we are motivated to rewrite Ξ as an operator on $J^{m-2,\sigma}_{0,\gamma}(\Pi)$.

To do this, we exploit the first part of 2.20, which tells us $a \to \psi^\alpha a : J^{m,\sigma}_{1/2+\gamma}(\Pi) \to J^{m,\sigma}_{1/2+\alpha,\gamma}(\Pi)$ is an isomorphism. In particular we observe, if $a \in J^{m,\sigma}_{1/2,\gamma}(\Pi)$, then each of the following functions belongs to $J^{m-2,\sigma}_{0,\gamma}(\Pi)$:

$$[\psi^{-1/2}a]\ ,\ [\psi^{1/2}a_\psi]\ ,\ [\psi^{-1/2}a_\theta]\ ,\ [\psi^{-1/2}a_{\theta\theta}]\ ,\ [\psi^{1/2}a_{\psi\theta}]\ ,\ [\psi^{3/2}a_{\psi\psi}].$$

Writing, $a = \psi^{1/2}[\psi^{-1/2}a]$, $a_\psi = \psi^{-1/2}[\psi^{1/2}a_\psi]$, etc, and substituting into $\Xi(a)$, we find

$$\begin{aligned}\Xi(a) = &-\frac{1}{[\psi^{1/2}a_\psi]^3}\Big(1 + \frac{[\psi^{-1/2}a_\theta]^2}{[\psi^{-1/2}a]^2}\Big)[\psi^{3/2}a_{\psi\psi}] + 2\frac{[\psi^{-1/2}a_\theta][\psi^{1/2}a_{\psi\theta}]}{[\psi^{-1/2}a]^2[\psi^{1/2}a_\psi]^2}\\ &- \frac{[\psi^{-1/2}a_{\theta\theta}]}{[\psi^{-1/2}a]^2[\psi^{1/2}a_\psi]} + \frac{1}{[\psi^{-1/2}a][\psi^{1/2}a_\psi]}.\end{aligned}$$

Notice that all of the ψ^α terms outside of the square brackets have cancelled. What remains is a rational function of only square brackets. Each of the square brackets is a multiplication and derivative of $a(\psi,\theta)$ lying in $J^{m-2,\sigma}_{0,\gamma}(\Pi)$, thus analytically depends on $a(\psi,\theta) \in J^{m,\sigma}_{1/2,\gamma}(\Pi)$.

The rational function defining Ξ as written above is analytic so long as the denominator is never zero, that is so long as $\psi^{-1/2}a \neq 0$ and $\psi^{1/2}a_\psi \neq 0$. Since $\Xi(a)$ is now a sum of reciprocals and products of functions in $J^{m-2,\sigma}_{0,\gamma}(\Pi)$, it is enough to prove that the maps

$$(u,v) \to uv : J^{m,\sigma}_{0,\gamma}(\Pi) \times J^{m,\sigma}_{0,\gamma}(\Pi) \to J^{m,\sigma}_{0,\gamma}(\Pi),$$
$$u \to \frac{1}{u} : J^{m,\sigma}_{0,\gamma}(\Pi) \to J^{m,\sigma}_{0,\gamma}(\Pi)$$

are well defined and analytic. We start with the following result:

Proposition 4.2.
For $m > 1$ and $\gamma > 1/2$, $K^m_\gamma(\Pi)$ is an algebra with $\|uv\|_{K^m_\gamma(\Pi)} \leq C\|u\|_{K^m_\gamma(\Pi)}\|v\|_{K^m_\gamma(\Pi)}$.

Proof. Suppose $m > 1$ and $\gamma > 1/2$. Then $K^m_\gamma(\Pi) \subset C(\overline{\Pi})$ and these functions vanish at $\psi = 0$. Take $u, v \in K^m_\gamma(\Pi)$. We must show $uv \in K^m_\gamma(\Pi)$. By the product rule, we can write

$$\partial^p_\psi \partial^q_\theta (uv) = \sum_{p'=0}^{p}\sum_{q'=0}^{q} C^{p',q'}_{p,q}\partial^{p-p'}_\psi \partial^{q-q'}_\theta (u)\partial^{p'}_\psi \partial^{q'}_\theta (v).$$

We thus get

$$\begin{aligned}\|uv\|^2_{K^m_\gamma(\Pi)} &= \sum_{p+q=0}^{m} \|\psi^{p-\gamma}\partial^p_\psi\partial^q_\theta(uv)\|^2_{L^2(\Pi)} \\ &\leq C\sum_{p+q=0}^{m}\sum_{p'=0}^{p}\sum_{q'=0}^{q}\|\psi^{p-\gamma}\partial^{p-p'}_\psi \partial^{q-q'}_\theta (u)\partial^{p'}_\psi \partial^{q'}_\theta (v)\|^2_{L^2(\Pi)}.\end{aligned}$$

To bound each of the terms above, we make use of the following estimate (from 2.12)

$$|\partial^p_\psi\partial^q_\theta u| \leq C\psi^{\gamma-1/2-p}\|u\|_{K^m_\gamma(\Pi)} \leq C\psi^{-p}\|u\|_{K^m_\gamma(\Pi)},$$

which holds for $m - (p+q) > 1$, $\gamma > 1/2$.

First we consider the case when $m - (p - p' + q - q') > 1$. Then the previous estimate yields $|\partial^{p-p'}_\psi \partial^{q-q'}_\theta u| \leq C\psi^{p'-p}\|u\|_{K^m_\gamma(\Pi)}$ which then gives

$$\|\psi^{p-\gamma}\partial^{p-p'}_\psi \partial^{q-q'}_\theta (u)\partial^{p'}_\psi \partial^{q'}_\theta (v)\|_{L^2(\Pi)} \leq C\|u\|_{K^m_\gamma(\Pi)}\|\psi^{p'-\gamma}\partial^{p'}_\psi \partial^{q'}_\theta (v)\|_{L^2(\Pi)} \leq C\|u\|_{K^m_\gamma(\Pi)}\|v\|_{K^m_\gamma(\Pi)}.$$

The analogous argument holds if $m-(p'+q') > 1$. It thus remains to consider the case when both $m - (p - p' + q - q') \leq 1$ and $m - (p' + q') \leq 1$. If we sum these two inequalities we get $2m-(p+q)\leq 2$. Rearranging gives $2m-2\leq p+q$. But $p+q\leq m$. We conclude that $2m-2\leq m$ and so $m\leq 2$. Since we must have $m>1$, this leaves only $m=2$. Thus for $m>2$, the statement is proven. In the case when $m=2$, the only terms where the above arguments don't apply are

$$\|\psi^{2-\gamma}(\partial_\psi u)(\partial_\psi v)\|_{L^2(\Pi)},\quad \|\psi^{1-\gamma}(\partial_\psi u)(\partial_\theta v)\|_{L^2(\Pi)},\quad \|\psi^{-\gamma}(\partial_\theta u)(\partial_\theta v)\|_{L^2(\Pi)}.$$

We now apply Hölder's inequality to get the following three inequalities:

$$\begin{aligned}
\|\psi^{2-\gamma}(\partial_\psi u)(\partial_\psi v)\|_{L^2(\Pi)} &= \|(\psi^{1-\frac{\gamma}{2}}\partial_\psi u)(\psi^{1-\frac{\gamma}{2}}\partial_\psi v)\|_{L^2(\Pi)} \leq \|\psi^{1-\frac{\gamma}{2}}\partial_\psi u\|_{L^4(\Pi)}\|\psi^{1-\frac{\gamma}{2}}\partial_\psi v\|_{L^4(\Pi)},\\
\|\psi^{1-\gamma}(\partial_\psi u)(\partial_\theta v)\|_{L^2(\Pi)} &= \|(\psi^{1-\frac{\gamma}{2}}\partial_\psi u)(\psi^{-\frac{\gamma}{2}}\partial_\theta v)\|_{L^2(\Pi)} \leq \|\psi^{1-\frac{\gamma}{2}}\partial_\psi u\|_{L^4(\Pi)}\|\psi^{-\frac{\gamma}{2}}\partial_\theta v\|_{L^4(\Pi)},\\
\|\psi^{-\gamma}(\partial_\theta u)(\partial_\theta v)\|_{L^2(\Pi)} &= \|(\psi^{-\frac{\gamma}{2}}\partial_\theta u)(\psi^{-\frac{\gamma}{2}}\partial_\theta v)\|_{L^2(\Pi)} \leq \|\psi^{-\frac{\gamma}{2}}\partial_\theta u\|_{L^4(\Pi)}\|\psi^{-\frac{\gamma}{2}}\partial_\theta v\|_{L^4(\Pi)}.
\end{aligned}$$

Notice, function $\psi^{1-\gamma/2}\partial_\psi u$ and $\psi^{-\gamma/2}\partial_\theta u$ belong to $K^1_{\gamma/2}(\Pi)$. By 2.15, these functions belong to $L^4(\Pi)$ when $\frac{\gamma}{2}>\frac{1}{2}-\frac{1}{4}$, which is precisely when $\gamma>1/2$. We thus get the estimates:

$$\begin{aligned}
\|\psi^{2-\gamma}(\partial_\psi u)(\partial_\psi v)\|_{L^2(\Pi)} &\leq C\|u\|_{K^2_\gamma(\Pi)}\|v\|_{K^2_\gamma(\Pi)},\\
\|\psi^{1-\gamma}(\partial_\psi u)(\partial_\theta v)\|_{L^2(\Pi)} &\leq C\|u\|_{K^2_\gamma(\Pi)}\|v\|_{K^2_\gamma(\Pi)},\\
\|\psi^{-\gamma}(\partial_\theta u)(\partial_\theta v)\|_{L^2(\Pi)} &\leq C\|u\|_{K^2_\gamma(\Pi)}\|v\|_{K^2_\gamma(\Pi)}.
\end{aligned}$$

This proves the $m=2$ case and thus concludes the proof of the proposition and establishes the bound $\|uv\|_{K^m_\gamma(\Pi)}\leq C\|u\|_{K^m_\gamma(\Pi)}\|v\|_{K^m_\gamma(\Pi)}$. □

Proposition 4.3.

Let $m>1/2$. Given $\xi(\theta)\in H^m(\mathbb{T})$ and $u(\psi,\theta)\in K^m_\gamma(\Pi)$, then $\xi u\in K^m_\gamma(\Pi)$ with $\|\xi u\|_{K^m_\gamma(\Pi)}\leq C\|\xi\|_{H^m(\mathbb{T})}\|u\|_{K^m_\gamma(\Pi)}$.

Proof. By product rule, we have

$$\|\xi u\|^2_{K^m_\gamma(\Pi)} = \sum_{p+q=0}^{m}\|\psi^{p-\gamma}\partial^p_\psi\partial^q_\theta(\xi u)\|^2_{L^2(\Pi)} \leq C\sum_{p+q=0}^{m}\sum_{q'=0}^{q}\|\psi^{p-\gamma}\big(D^{q-q'}\xi\big)\big(\partial^p_\psi\partial^{q'}_\theta u\big)\|^2_{L^2(\Pi)}.$$

For $m>1/2$, because $\xi(\theta)\in H^m(\mathbb{T})$, we have $|D^{q-q'}\xi|\leq C\|\xi\|_{H^m(\mathbb{T})}$ when $q-q'<m$. In this case, we immediately get

$$\|\psi^{p-\gamma}\big(D^{q-q'}\xi\big)\big(\partial^p_\psi\partial^{q'}_\theta u\big)\|_{L^2(\Pi)} \leq C\|D^{q-q'}\xi\|_\infty\|\psi^{p-\gamma}\partial^p_\psi\partial^{q'}_\theta u\|_{L^2(\Pi)} \leq C\|\xi\|_{H^m(\mathbb{T})}\|u\|_{K^m_\gamma(\Pi)}.$$

In the case when $q - q' = m$, that is $p = 0$, $q = m$, $q' = 0$, by 2.16 (since $m \geq 1$), we get

$$\begin{aligned}\|\psi^{-\gamma}(D^m\xi)u\|_{L^2(\Pi)} &= \Big(\int_{\mathbb{T}} |D^m\xi|^2 \big(\int_0^1 |\psi^{-\gamma}u|^2 \, \mathrm{d}\psi \big) \, \mathrm{d}\theta \Big)^{1/2} \\ &\leq \|I\|_\infty \|D^m\xi\|_{L^2(\mathbb{T})} \\ &\leq C\|\xi\|_{H^m(\mathbb{T})}\|u\|_{K^m_\gamma(\Pi)}.\end{aligned}$$

Thus we have shown $\|\xi u\|_{K^m_\gamma(\Pi)} \leq C\|\xi\|_{H^m(\mathbb{T})}\|u\|_{K^m_\gamma(\Pi)}$. □

Corollary 4.4.

For $m > 1$ and $\gamma > 1/2$, $J^{m,\sigma}_{0,\gamma}(\Pi)$ is an algebra with $\|uv\|_{J^{m,\sigma}_{0,\gamma}(\Pi)} \leq C\|u\|_{J^m_{0,\gamma}(\Pi)}\|v\|_{J^{m,\sigma}_{0,\gamma}(\Pi)}$.

Proof. First, let $u = v(\theta) + w(\psi, \theta)$, $\zeta = \xi(\theta) + \eta(\psi, \theta) \in J^m_{0,\gamma}(\Pi)$. This means $v, \xi \in H^m(\mathbb{T})$ and $w, \eta \in K^m_\gamma(\Pi)$. Multiplying, we get $u\zeta = v\xi + v\eta + \xi w + w\eta$. The leading term $v\xi$ is in $H^m(\mathbb{T})$ because this space is an algebra. The remaining terms belong to $K^m_\gamma(\Pi)$, by the preceding two propositions. Furthermore, from the bounds established previously and the definition of norm of $J^m_{0,\gamma}(\Pi)$, we have the bound

$$\begin{aligned}&\|u\zeta\|_{J^m_{0,\gamma}(\Pi)} = \|v\xi\|^2_{H^m(\mathbb{T})} + \|v\eta + \xi w + w\eta\|_{K^m_\gamma(\Pi)} \\ &\leq C\big(\|v\|_{H^m(\mathbb{T})}\|\xi\|_{H^m(\mathbb{T})} + \|v\|_{H^m(\mathbb{T})}\|\eta\|_{K^m_\gamma(\mathbb{T})} + \|\xi\|_{H^m(\mathbb{T})}\|w\|_{K^m_\gamma(\mathbb{T})} + \|w\|_{K^m_\gamma(\mathbb{T})}\|\eta\|_{K^m_\gamma(\mathbb{T})}\big) \\ &\leq C\|u\|_{J^m_{0,\gamma}(\Pi)}\|\zeta\|_{J^m_{0,\gamma}(\Pi)}.\end{aligned}$$

This confirms $J^m_{0,\gamma}(\Pi)$ is an algebra. In the case when $u, \zeta \in J^{m,\sigma}_{0,\gamma}(\Pi)$, we have $v, \xi \in X^m_\sigma(\mathbb{T})$ and $w, \eta \in K^{m,\sigma}_\gamma(\Pi)$. Then $v\xi \in X^m_\sigma(\mathbb{T})$ because it is the product of two holomorphic functions in $\mathbb{T}_\sigma$ and so holomorphic itself, and since $H^m(\mathbb{T})$ is an algebra, we get

$$\begin{aligned}&\|v\xi\|_{X^m_\sigma(\mathbb{T})} = \|v(\cdot + i\sigma)\xi(\cdot + i\sigma)\|_{H^m(\mathbb{T})} + \|v(\cdot - i\sigma)\xi(\cdot - i\sigma)\|_{H^m(\mathbb{T})} \\ &\leq C\|v(\cdot + i\sigma)\|_{H^m(\mathbb{T})}\|\xi(\cdot + i\sigma)\|_{H^m(\mathbb{T})} + C\|v(\cdot - i\sigma)\|_{H^m(\mathbb{T})}\|\xi(\cdot - i\sigma)\|_{H^m(\mathbb{T})} \\ &\leq C\|v\|_{X^m_\sigma(\mathbb{T})}\|\xi\|_{X^m_\sigma(\mathbb{T})}.\end{aligned}$$

Next, if $\xi(\theta) \in X^m_\sigma(\mathbb{T})$ and $w \in K^{m,\sigma}_\gamma(\Pi)$, then the map $\theta \to \xi(\theta)w(\cdot, \theta)$ is holomorphic as a map from $\mathbb{T}_\sigma$ to $K^m_\gamma(0, 1]$, with bound $\|\xi w\|_{K^{m,\sigma}_\gamma(\Pi)} \leq C\|\xi\|_{X^m_\sigma(\mathbb{T})}\|w\|_{K^{m,\sigma}_\gamma(\Pi)}$, analogously obtained as above. Finally, the same argument holds for the product of $w, \eta \in K^{m,\sigma}_\gamma(\Pi)$. It is the product of holomorphic functions $\mathbb{T}_\sigma \to K^m_\gamma(0, 1]$, the latter of which is an algebra and so this product is also holomorphic as a Banach valued map. The similar bound applies, namely that $\|w\eta\|_{K^{m,\sigma}_{\gamma}(\Pi)} \le C\|w\|_{K^{m,\sigma}_{\gamma}(\Pi)}\|\eta\|_{K^{m,\sigma}_{\gamma}(\Pi)}$. Putting it all together, we conclude that $J^{m}_{0,\gamma}(\Pi)$ is an algebra with $\|u\zeta\|_{J^{m,\sigma}_{0,\gamma}(\Pi)} \le C\|u\|_{J^{m,\sigma}_{0,\gamma}(\Pi)}\|\zeta\|_{J^{m,\sigma}_{0,\gamma}(\Pi)}$. □

The next step is to prove that multiplication of functions in $J^{m,\sigma}_{0,\gamma}(\Pi)$ is an analytic operator.

Proposition 4.5.
The map $(u, v) \to uv : J^{m,\sigma}_{0,\gamma}(\Pi) \times J^{m,\sigma}_{0,\gamma}(\Pi) \to J^{m,\sigma}_{0,\gamma}(\Pi)$ *is analytic for* $m > 1$, $\gamma > 1/2$.

Proof. Let $X = J^{m,\sigma}_{0,\gamma}(\Pi)$, and write $(x_1, x_2) \to B(x_1, x_2) = x_1x_2 : X \times X \to X$. Such a map is bilinear, and we have just seen it satisfies $\|B(x_1, x_2)\|_X \le C\|x_1\|_X\|x_2\|_X$. The statement follows almost immediately from these two facts. From bilinearity, we have

$$B(x_1 + h_1, x_2 + h_2) = B(x_1, x_2) + B(h_1, x_2) + B(x_1, h_2) + B(h_1, h_2).$$

Next, write $A(x)h = B(h_1, x_2) + B(x_1, h_2)$. From the bilinearity and boundedness of B, we get that $A(x)$ is a bounded linear operator on X, thus a candidate for the Fréchet derivative of B. We can thus write

$$B(x_1 + h_1, x_2 + h_2) - B(x_1, x_2) - A(x)h = B(h_1, h_2).$$

Since $\|B(h_1, h_2)\|_X \le C\|h_1\|_X\|h_2\|_X \le C\|h\|^2_{X\times X}$, we get that

$$\frac{\|B(h_1, h_2)\|_X}{\|h\|_{X\times X}} \le C\|h\|_{X\times X} \to 0 \quad \text{as} \quad \|h\|_{X\times X} \to 0.$$

Thus by definition (see [1]), the map $B : X \times X \to X$ is Fréchet differentiable with derivative $DB(x)h = A(x)h = B(h_1, x_2) + B(x_1, h_2)$. By the theory of holomorphy in complex Banach spaces (see [21]), since $X = J^{m,\sigma}_{0,\gamma}(\Pi)$ is a complex Banach space and B is complex Fréchet differentiable, the map is analytic. □

Now that we have determined multiplication in $J^{m,\sigma}_{0,\gamma}(\Pi)$ defines an analytic operator, we turn to the operator $u \to \frac{1}{u}$ on $J^{m,\sigma}_{0,\gamma}(\Pi)$. In fact, we will consider the more general problem, of superposition operator $u \to f(u)$.

Theorem 4.6.

Let $f \in C^{m+1}$ on a domain containing image of u. Then $u \to f(u) : J^m_{0,\gamma}(\Pi) \to J^m_{0,\gamma}(\Pi)$ is well defined and continuous for $m > 1$, $\gamma > 1/2$. Furthermore, if $f \in C^{m+2}(\Omega)$ then this map is C^1.

Proof. Given $u(\psi, \theta) \in J^m_{0,\gamma}(\Pi)$, we write $u(\psi, \theta) = \xi(\theta) + v(\psi, \theta)$, where $\xi \in H^m(\mathbb{T})$ and $v \in K^m_\gamma(\Pi)$. Recall that for $m > 1$, $\gamma > 1/2$, u is continuous and $v \to 0$ as $\psi \to 0^+$. In other words, ξ defines the behaviour of u along $\psi = 0$, so we write $u(0, \theta) = \xi(\theta)$. By continuity of f, we have $f(\xi + v) \to f(\xi)$ as $\psi \to 0^+$. So the behaviour of $f(u)$ at $\psi = 0$ is defined by $f(\xi)$. We thus have the decomposition $f(\xi + v) = f(\xi) + f(\xi + v) - f(\xi)$. Since $\xi \in H^m(\mathbb{T})$, then by 4.1, $f(\xi) \in H^m(\mathbb{T})$. This forms the leading term of $f(u) \in J^m_{0,\gamma}(\Pi)$. The main task then is to prove that the remainder term, $f(\xi + v) - f(\xi)$, belongs to $K^m_\gamma(\Pi)$. Intuitively, this means that this term vanishes as $\psi \to 0^+$ at the same rate as v does. We must bound

$$\|f(\xi + v) - f(\xi)\|^2_{K^m_\gamma(\Pi)} = \sum_{p+q=0}^{m} \|\psi^{p-\gamma} \partial^p_\psi \partial^q_\theta \big(f(\xi + v) - f(\xi)\big)\|^2_{L^2(\Pi)}.$$

To start, given a composition $f\big(u(\psi, \theta)\big)$, we have the following expressions for its partial derivatives:

$$\partial^p_\psi \partial^q_\theta f(u) = \sum_{j=1}^{p+q} \sum_{\substack{\alpha_1+\cdots+\alpha_j=p \\ \beta_1+\cdots+\beta_j=q \\ \alpha_i+\beta_i \geq 1}} C^{\beta_1,\ldots,\beta_j}_{\alpha_1,\ldots,\alpha_j} \big(\partial^{\alpha_1}_\psi \partial^{\beta_1}_\theta u\big) \cdots \big(\partial^{\alpha_j}_\psi \partial^{\beta_j}_\theta u\big) f^{(j)}(u).$$

If $p \geq 1$, then $\partial^p_\psi \partial^q_\theta f(\xi) = 0$. In this case, with $u = \xi(\theta) + v(\psi, \theta)$, we obtain:

$$\partial^p_\psi \partial^q_\theta f(\xi + v) = \sum_{j=1}^{p+q} \sum_{\substack{\alpha_1+\cdots+\alpha_j=p \\ \beta_1+\cdots+\beta_j=q \\ \alpha_i+\beta_i \geq 1}} \sum_{\text{or}} C^{\beta_1,\ldots,\beta_j}_{\alpha_1,\ldots,\alpha_j} \big(\partial^{\alpha_1}_\psi \partial^{\beta_1}_\theta \xi \text{ or } v\big) \cdots \big(\partial^{\alpha_j}_\psi \partial^{\beta_j}_\theta \xi \text{ or } v\big) f^{(j)}(\xi + v),$$

where the summation over 'or' indicates we sum over all choices of ξ or v in the above factors. Note though, since $p \geq 1$, at least some $\alpha_i \neq 0$ and thus the case when all factors choose ξ vanishes. If on the other hand $p = 0$, then we instead obtain the expression

$$\partial^q_\theta \Big(f(\xi + v) - f(\xi)\Big) = \sum_{j=1}^{q} \sum_{\substack{\alpha_1+\cdots+\alpha_j=q \\ \alpha_i \geq 1}} C_{\alpha_1,\ldots,\alpha_j} \Bigg[\Big(f^{(j)}(\xi + v) - f^{(j)}(\xi)\Big) (D^{\alpha_1}\xi) \cdots (D^{\alpha_j}\xi)$$
$$+ \sum_{\text{or}} f^{(j)}(\xi + v) (D^{\alpha_1}\xi \text{ or } v) \cdots (D^{\alpha_j}\xi \text{ or } v)\Bigg]$$

where summation over 'or' excludes case when all factors choose ξ.

Let us now start with bounds for the case $p \geq 1$. Namely, for $1 \leq j \leq p+q \leq m$, $\alpha_1 + \cdots + \alpha_j = p$, $\beta_1 + \cdots + \beta_j = q$, $\alpha_i + \beta_i \geq 1$, we must bound

$$A = \|\psi^{p-\gamma}\big(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}\xi \text{ or } v\big)\cdots\big(\partial_\psi^{\alpha_j}\partial_\theta^{\beta_j}\xi \text{ or } v\big)f^{(j)}(\xi+v)\|_{L^2(\Pi)}.$$

Since $f \in C^{m+1}(\Omega)$, we immediately get

$$A \leq \|f^{(j)}\|_\infty \|\psi^{p-\gamma}\big(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}\xi \text{ or } v\big)\cdots\big(\partial_\psi^{\alpha_j}\partial_\theta^{\beta_j}\xi \text{ or } v\big)\|_{L^2(\Pi)}.$$

Next, $\xi \in H^m(\mathbb{T})$ and thus $\partial_\theta^{\beta_i}\xi \in C(\mathbb{T})$ unless $\beta_i = m$. This occurs only if $q = m$ and thus $p = 0$, which is outside of the current case $p \geq 1$. Thus we can assume all $\beta_i \leq m$ and so each factor $\partial_\theta^{\beta_i}\xi$ is continuous, and thus can be factored out of the norm. There are j factors of form $\partial_\psi^{\alpha_j}\partial_\theta^{\beta_j}(\xi \text{ or } v)$, but at most $j-1$ of them choose ξ. We thus get

$$A \leq \|f^{(j)}\|_\infty \|\xi\|_{H^m(\mathbb{T})}^\lambda \|\psi^{p-\gamma}\underbrace{\big(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}v\big)\cdots\big(\partial_\psi^{\alpha_j}\partial_\theta^{\beta_j}v\big)}_{\lambda \text{ terms missing}}\|_{L^2(\Pi)},$$

where $0 \leq \lambda \leq j-1$. Next, if $m - (\alpha_i + \beta_i) > 1$, then

$$|\partial_\psi^{\alpha_i}\partial_\theta^{\beta_i}v| \leq C\psi^{\gamma-1/2-\alpha_i}\|v\|_{K_\gamma^m(\Pi)} \leq C\psi^{-\alpha_i}\|v\|_{K_\gamma^m(\Pi)},$$

since $\gamma > 1/2$. This condition is not satisfied only when $\alpha_i + \beta_i = m-1$ or m. First suppose $\alpha_i + \beta_i = m$. Then $j = 1$ and $\alpha_1 = p$ and immediately we get

$$A \leq \|f^{(j)}\|_\infty \|\psi^{p-\gamma}\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}v\|_{L^2(\Pi)} \leq \|f^{(j)}\|_\infty\|v\|_{K_\gamma^m(\Pi)} \leq \|f^{(j)}\|_\infty\|u\|_{J_{0,\gamma}^m(\Pi)}$$

Next, assume without loss of generality that $\alpha_1 + \beta_1 = m-1$. Then either $j = 1$, and the same estimate as above holds, or $j = 2$. Either $\lambda = 1$ and immediately

$$A \leq \|f^{(j)}\|_\infty\|\xi\|_{H^m(\mathbb{T})}\|\psi^{p-\gamma}\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}v\|_{L^2(\Pi)} \leq \|f^{(j)}\|_\infty\|\xi\|_{H^m(\mathbb{T})}\|v\|_{K_\gamma^m(\Pi)} \leq \|f^{(j)}\|_\infty\|u\|_{J_{0,\gamma}^m(\Pi)}^2,$$

or $\lambda = 0$. Necessarily $(\alpha_2, \beta_2) = (1,0)$ or $(0,1)$. If $m - (\alpha_2 + \beta_2) = m - 1 > 1$, then

$$|\partial_\psi^{\alpha_2}\partial_\theta^{\beta_2}v| \leq C\psi^{\gamma-1/2-\alpha_2}\|v\|_{K_\gamma^m(\Pi)} \leq C\psi^{-\alpha_2}\|v\|_{K_\gamma^m(\Pi)}$$

and so

$$\begin{aligned}
A &\leq \|f^{(j)}\|_\infty\|\psi^{\alpha_1+\alpha_2-\gamma}\big(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}v\big)\big(\partial_\psi^{\alpha_2}\partial_\theta^{\beta_2}v\big)\|_{L^2(\Pi)}\\
&\leq \|f^{(j)}\|_\infty\|v\|_{K_\gamma^m(\Pi)}\|\psi^{\alpha_1-\gamma}\big(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1}v\big)\|_{L^2(\Pi)}\\
&\leq \|f^{(j)}\|_\infty\|u\|_{J_{0,\gamma}^m(\Pi)}^2.
\end{aligned}$$

If on the other hand $m - \alpha_2 + \beta_2 = m - 1 \leq 1$, then $m \leq 2$, so $m = 2$, and we have $A = \|\psi^{2-\gamma}(\partial_\psi v)(\partial_\psi v) f^{(j)}(u)\|_{L^2(\Pi)}$ or $A = \|\psi^{1-\gamma}(\partial_\psi v)(\partial_\theta v) f^{(j)}(u)\|_{L^2(\Pi)}$. We have seen in 4.2, how to bound this expression using Hölder's inequality and embeddings of $\psi^{-\frac{\gamma}{2}}\partial_\theta v$, $\psi^{1-\frac{\gamma}{2}}\partial_\psi v$ into $L^4(\Pi)$ bu 2.15. This gives $A \leq \|f^{(j)}\|_\infty \|u\|^2_{J^m_{0,\gamma}(\Pi)}$.

Finally, in the case when $\alpha_i + \beta_i < m - 1$, applying the point-wise estimate $|\partial_\psi^{\alpha_i}\partial_\theta^{\beta_i} v| \leq C\psi^{-\alpha_i}\|v\|_{K^m_\gamma(\Pi)}$ to all but one factor in the expression

$$
\begin{aligned}
A &\leq \|f^{(j)}\|_\infty \|\xi\|^\lambda_{H^m(\mathbb{T})} \|\psi^{p-\gamma} \underbrace{\left(\partial_\psi^{\alpha_1}\partial_\theta^{\beta_1} v\right) \cdots \left(\partial_\psi^{\alpha_j}\partial_\theta^{\beta_j} v\right)}_{\lambda \text{ terms missing}} \|_{L^2(\Pi)} \\
&\leq C\|f^{(j)}\|_\infty \|\xi\|^\lambda_{H^m(\mathbb{T})} \|v\|^{j-\lambda-1}_{K^m_\gamma(\Pi)} \|\psi^{\alpha_i - \gamma}\partial_\psi^{\alpha_i}\partial_\theta^{\beta_i} v\|_{L^2(\Pi)} \\
&\leq C\|f^{(j)}\|_\infty \|u\|^j_{J^m_{0,\gamma}(\Pi)}.
\end{aligned}
$$

This concludes the case when $p \geq 1$, where we have found each term of $\|\psi^{p-\gamma}\partial_\psi^p\partial_\theta^q\big(f(\xi + v) - f(\xi)\big)\|_{L^2(\Pi)}$ is bounded by $C\|f^{(j)}\|_\infty \|u\|^j_{J^m_{0,\gamma}(\Pi)}$, for $1 \leq j \leq p + q \leq m$. Thus for $p \geq 1$ we can write

$$
\|\psi^{p-\gamma}\partial_\psi^p\partial_\theta^q\big(f(\xi + v) - f(\xi)\big)\|_{L^2(\Pi)} \leq C\|f\|_{C^m}\big(\|u\|_{J^m_{0,\gamma}(\Pi)} + \|u\|^m_{J^m_{0,\gamma}(\Pi)}\big).
$$

Now we consider the case when $p = 0$. Recall, $\partial_\theta^q\big(f(\xi + v) - f(\xi)\big)$ is a sum of

$$
\sum_{j=1}^{q} \sum_{\substack{\alpha_1 + \cdots + \alpha_j = q \\ \alpha_i \geq 1}} C_{\alpha_1,\ldots,\alpha_j} \big(f^{(j)}(\xi + v) - f^{(j)}(\xi)\big) \left(D^{\alpha_1}\xi\right) \cdots \left(D^{\alpha_j}\xi\right)
$$

and

$$
\sum_{j=1}^{q} \sum_{\substack{\alpha_1 + \cdots + \alpha_j = q \\ \alpha_i \geq 1}} C_{\alpha_1,\ldots,\alpha_j} \sum_{\text{or}} f^j(\xi + v) \left(D^{\alpha_1}\xi \text{ or } v\right) \cdots \left(D^{\alpha_j}\xi \text{ or } v\right).
$$

Bounding the latter is identical to the previous case of $p \geq 1$. So we have only the first part to bound. That is, for $1 \leq j \leq q \leq m$, $\alpha_1 + \cdots + \alpha_j = q$, $\alpha_i \geq 1$, we must bound

$$
B = \|\psi^{-\gamma}\big(f^{(j)}(\xi + v) - f^{(j)}(\xi)\big) \left(D^{\alpha_1}\xi\right) \cdots \left(D^{\alpha_j}\xi\right)\|_{L^2(\Pi)}.
$$

We use the fundamental theorem of calculus to write

$$
f^{(j)}(\xi + v) - f^{(j)}(\xi) = \int_0^\psi f^{(j+1)}(u)\partial_t v(t, \theta)\,\mathrm{d}t\,.
$$

Now we apply Hardy's inequality. Set $-\gamma = \alpha - 1$. Then $\alpha = 1-\gamma < 1/2$ since $\gamma > 1/2$. Also set $f^{(j+1)}(u)\partial_t v(t,\theta) = t^{-\alpha} g(t,\theta)$. We get

$$B \leq C\|\psi^{1-\gamma} f^{(j+1)}(u)\,(\partial_\psi v)\,(D^{\alpha_1}\xi)\cdots(D^{\alpha_j}\xi)\|_{L^2(\Pi)}$$

Now suppose each $\alpha_i < m$. Then each factor $D^{\alpha_i}\xi$ is continuous and thus

$$B \leq C\|f^{(j+1)}\|_\infty \|\xi\|^j_{H^m(\mathbb{T})}\|\psi^{1-\gamma}\,(\partial_\psi v)\|_{L^2(\Pi)} \leq C\|f^{(j+1)}\|_\infty \|u\|^{j+1}_{J^m_{0,\gamma}(\Pi)}.$$

If on the other hand $\alpha_1 = m$, then necessarily $j=1$. We have $\psi v_\psi \in K^{m-1}_\gamma(\Pi)$ and $m-1 \geq 1$, thus by 2.16, $\|\psi^{1-\gamma}\partial_\psi v(\psi,\theta)\|_{L^2(0,1]}$ is continuous with respect to θ. We thus get

$$\begin{aligned}
B &\leq \|f^{(j+1)}\|_\infty \|\psi^{1-\gamma}\,(\partial_\psi v)\,(D^m\xi)\|_{L^2(\Pi)} \\
&\leq \|f^{(j+1)}\|_\infty \Big(\int_{\mathbb{T}} |D^m\xi|^2 \int_0^1 |\psi^{1-\gamma}\partial_\psi v|^2 \,\mathrm{d}\psi\,\mathrm{d}\theta\Big)^{1/2} \\
&\leq C\|f^{(j+1)}\|_\infty \|v\|_{K^m_\gamma(\Pi)}\|D^m\xi\|_{L^2(\mathbb{T})} \\
&\leq C\|f^{(j+1)}\|_\infty \|u\|^2_{J^m_{0,\gamma}(\Pi)}.
\end{aligned}$$

We have thus established the bounds for $p=0$ case. Together with the prior $p \geq 1$ case, we get

$$\|f(\xi+v) - f(\xi)\|_{K^m_\gamma(\Pi)} \leq C\|f\|_{C^{m+1}}\left(\|u\|_{J^m_{0,\gamma}(\Pi)} + \|u\|^{m+1}_{J^m_{0,\gamma}(\Pi)}\right).$$

Combining this with bound

$$\|f(\xi)\|_{H^m(\mathbb{T})} \leq C\|f\|_{C^m}\left(1 + \|\xi\|^m_{H^m(\mathbb{T})}\right)$$

from 4.1, we find our desired bound

$$\begin{aligned}
\|f(u)\|_{J^m_{0,\gamma}(\Pi)} &= \|f(\xi)\|_{H^m(\mathbb{T})} + \|f(\xi+v) - f(\xi)\|_{K^m_\gamma(\Pi)} \\
&\leq C\|f\|_{C^{m+1}}\left(1 + \|u\|^{m+1}_{J^m_{0,\gamma}(\Pi)}\right).
\end{aligned}$$

The continuity and Fréchet differentiability follow analogously to the proof of 4.1. □

Corollary 4.7.

Suppose f is complex analytic on a domain containing image of u. Then $u \to f(u) : J^{m,\sigma}_{0,\gamma}(\Pi) \to J^{m,\sigma}_{0,\gamma}(\Pi)$ is complex analytic for $m > 1$, $\gamma > 1/2$.

Proof. Let $u \in J^{m,\sigma}_{0,\gamma}(\Pi)$. Write $u = \xi + v$, with $\xi \in X^m_\sigma(\mathbb{T})$, $v \in K^{m,\sigma}_\gamma(\Pi)$. Now, as in the proof of the prior theorem, write $f(u) = f(\xi) + f(\xi+v) - f(\xi)$. By definition, $\xi(\cdot + it) \in H^m(\mathbb{T})$ for all $|t| \le \sigma$. By 4.1, $f(\xi(\cdot + it)) \in H^m(\mathbb{T})$ for $|t| \le \sigma$. Since $f(\xi)$ is the composition of analytic functions, it is itself analytic in $\mathbb{T}_\sigma$, and so $f(\xi) \in X^m_\sigma(\mathbb{T})$.

Next, by definition $u(\cdot, \cdot + it) \in J^m_{0,\gamma}(\Pi)$ for all $|t| \le \sigma$. From the previous theorem, $g(u) = f(\xi + v) - f(\xi) \in K^m_\gamma(\Pi)$ for each fixed $|t| \le \sigma$. Now fix $z \in \mathbb{T}_\sigma$, then $u(\cdot, z) \in \mathbb{C} \times K^m_\gamma(0,1]$. All of the prior results in this chapter on $K^m_\gamma(\Pi)$ and $J^m_{0,\gamma}(\Pi)$ likewise apply to $K^m_\gamma(0,1]$ and $J^m_{0,\gamma}(0,1]$. Namely, these spaces are algebras and superposition maps are well defined them. Thus $g(z) = f\big(\xi(z) + v(\cdot, z)\big) - f\big(\xi(z)\big)$ is a $K^m_\gamma(0,1]$ valued map of z. Differentiating gives

$$\begin{aligned} g'(z) &= f'\big(\xi(z) + v(\cdot,z)\big)\big(\xi'(z) + \partial_z v(\cdot,z)\big) - f'\big(\xi(z)\big)\xi'(z) \\ &= \Big(f'\big(\xi(z) + v(\cdot,z)\big) - f'\big(\xi(z)\big)\Big)\xi'(z) + f'(\xi)\partial_z v(\cdot,z) \\ &+ \Big(f'\big(\xi(z) + v(\cdot,z)\big) - f'\big(\xi(z)\Big)\partial_z v(\cdot,z). \end{aligned}$$

The first two terms are products of a $K^m_\gamma(0,1]$ function and scalar, the last term is the product of two $K^m_\gamma(0,1]$, which is itself in $K^m_\gamma(0,1]$, since this space is an algebra. Thus we have showed that $z \to f\big(\xi(z) + v(\cdot,z)\big) - f\big(\xi(z)\big)$ is well defined and complex differentiable, thus analytic. This means $f(\xi + v) - f(\xi) \in K^{m,\sigma}_\gamma(\Pi)$. We conclude that $f(u) = f(\xi) + f(\xi + v) - f(\xi) \in J^{m,\sigma}_{0,\gamma}(\Pi)$. ☐

We now state the main result of this section.

Theorem 4.8.
For $m > 3$, $\gamma > 1/2$, the map $a \to \Xi(a) : J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ is analytic in a neighbourhood of $a = \psi^{1/2}$.

Proof. Earlier in this chapter, we showed we can write

$$\begin{aligned} \Xi(a) = &-\frac{1}{[\psi^{1/2}a_\psi]^3}\Big(1 + \frac{[\psi^{-1/2}a_\theta]^2}{[\psi^{-1/2}a]^2}\Big)[\psi^{3/2}a_{\psi\psi}] + 2\frac{[\psi^{-1/2}a_\theta][\psi^{1/2}a_{\psi\theta}]}{[\psi^{-1/2}a]^2[\psi^{1/2}a_\psi]^2} \\ &- \frac{[\psi^{-1/2}a_{\theta\theta}]}{[\psi^{-1/2}a]^2[\psi^{1/2}a_\psi]} + \frac{1}{[\psi^{-1/2}a][\psi^{1/2}a_\psi]}. \end{aligned}$$

Thus Ξ is a composition of maps $a \to [\cdots] : J^{m,\sigma}_{1/2,\gamma}(\Pi) \to J^{m-2,\sigma}_{0,\gamma}(\Pi)$, which are linear and thus analytic, and a rational function of the square brackets. Given that the square brackets are valued in $J^{m-2}_{0,\gamma}(\Pi)$, then by 4.5 and 4.7, this rational function is an analytic map $J^{m-2}_{0,\gamma}(\Pi) \times \cdots \times J^{m-2}_{0,\gamma}(\Pi) \to J^{m-2}_{0,\gamma}(\Pi)$ when $m-2>1$, so long as the denominator does not vanish. To see it does not, suppose $\|a-\psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} < \varepsilon$. By boundedness of multiplication by $\psi^{-1/2}$, we get

$$\|\psi^{-1/2}a-1\|_{J^{m,\sigma}_{0,\gamma}(\Pi)} \le C\|a-\psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} \le C\varepsilon.$$

By 2.22,

$$|\psi^{-1/2}a-1| \le D\|\psi^{-1/2}a-1\|_{J^{m,\sigma}_{0,\gamma}(\Pi)} \le CD\varepsilon.$$

Taking ε small enough, we can ensure $\psi^{-1/2}a$ is close enough to 1 in $\mathbb{C}$ that it is never zero. Similarly, by boundedness of ∂_ψ,

$$\|a_\psi - 1/2\psi^{-1/2}\|_{J^{m-1,\sigma}_{-1/2,\gamma}(\Pi)} \le C\|a-\psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} \le C\varepsilon.$$

By boundedness of multiplication by $\psi^{1/2}$,

$$\|\psi^{1/2}a_\psi - 1/2\|_{J^{m-1,\sigma}_{0,\gamma}(\Pi)} \le D\|a_\psi - 1/2\psi^{-1/2}\|_{J^{m-1,\sigma}_{-1/2,\gamma}(\Pi)}.$$

By 2.22

$$|\psi^{1/2}a_\psi - 1/2| \le E\|\psi^{1/2}a_\psi - 1/2\|_{J^{m-1,\sigma}_{0,\gamma}(\Pi)} \le CDE\varepsilon.$$

Again, taking ε small enough, we can ensure $\psi^{1/2}a_\psi$ is close enough to $1/2$ in $\mathbb{C}$ it is never zero. Thus we have shown $\Xi : U \to J^{m-2,\sigma}_{0,\gamma}(\Pi)$ is an analytic map on a neighbourhood $U \subset J^{m,\sigma}_{1/2,\gamma}(\Pi)$ of $\psi^{1/2}$.

It remains to show that Ξ is in fact $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ valued. This means the leading $H^m(\mathbb{T})$ term of $\Xi(a)$ has zero second-order Fourier coefficients. We have seen that for $\gamma > 1/2$, the leading term of $\Xi(a)$ depends only on the leading term of the square brackets, which in turn depend only on the leading term of $a(\psi,\theta)$, that is, depend only on $\psi^{1/2}\xi(\theta)$. Thus we must show $\int_{\mathbb{T}} \Xi\big(\psi^{1/2}\xi(\theta)\big)e^{\pm 2i\theta}\,\mathrm{d}\theta = 0$. We find

$$\Xi(\psi^{1/2}\xi) = \frac{4}{\xi^2} + \frac{6(D\xi)^2}{\xi^4} - \frac{2D^2\xi}{\xi^3}.$$

We find

$$\begin{aligned}\int_{\mathbb{T}} \Xi(\psi^{1/2}\xi)e^{\pm 2i\theta}\,\mathrm{d}\theta &= 2\int_{\mathbb{T}}\Big(\frac{2}{\xi^2}+\frac{3(D\xi)^2}{\xi^4}-\frac{D^2\xi}{\xi^3}\Big)e^{\pm 2i\theta}\,\mathrm{d}\theta\\ &= 2\int_{\mathbb{T}}\Big(\frac{2}{\xi^2}\pm 2i\frac{D\xi}{\xi^3}\Big)e^{\pm 2i\theta}\,\mathrm{d}\theta\\ &= 2\int_{\mathbb{T}}\Big(\frac{2}{\xi^2}\mp iD\big(\frac{1}{\xi^2}\big)\Big)e^{\pm 2i\theta}\,\mathrm{d}\theta\\ &= 0\end{aligned}$$

where we have integrated by parts the last term on the first line, and again the last term on the third line. We conclude that $\Xi(a) \in \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$, thus the statement of the theorem is proved. □

Remark 4.9. We introduced the space $\widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ to establish the bijection of the linearization, which kills these second-order Fourier modes of the leading term. The above result demonstrates that this property is in fact inherited from the nonlinear problem. This fact is crucial for the successful application of the implicit function theorem.

Corollary 4.10.
For $m > 3$, $\gamma > 1/2$, the map $(F, a) \to \Xi(a) - F : J^{m-2}_{0,\gamma}(0, 1] \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi)$ is analytic in a neighbourhood of $F(\psi) = 4$, $a(\psi, \theta) = \psi^{1/2}$.

To conclude this section, we comment on the structure of the stagnation point of the flows defined by our function space $J^{m,\sigma}_{1/2,\gamma}(\Pi)$. In a sufficiently small neighbourhood of $a(\psi, \theta) = \psi^{1/2}$, the function $r = (\psi, \theta)$ is never vanishing except as $\psi \to 0^+$, where it behaves likes $\psi^{1/2}$. This property confirms that the flow lines do collapse to a single point at $\psi = 0$. Next, a_ψ behaves like $\psi^{-1/2}$ and thus blows up as we approach $\psi = 0$. This means that the associated velocity field $u(\psi, \theta) = \frac{1}{a_\psi}\big(\frac{a_\theta}{a}, 1\big)$ of the fluid does in fact vanish at $\psi = 0$ (a_θ and a have the same asymptotics and thus their ratio remains bounded), defining a true stagnation point of the fluid. Finally, since $a(\psi, \theta)$ behaves like $\psi^{1/2}$ asymptotically, then the associated stream function $\psi(r, \theta)$ locally near $r = 0$ resembles the paraboloid $\psi = r^2$, which defines a non-degenerate stagnation point. To confirm this, the Hessian matrix of stream function ψ

in polar coordinates is given by

$$H_\psi(r,\theta) = \begin{pmatrix} \psi_{rr} & \frac{1}{r}\psi_{r\theta} - \frac{1}{r^2}\psi_\theta \\ \frac{1}{r}\psi_{r\theta} - \frac{1}{r^2}\psi_\theta & \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} \end{pmatrix}.$$

The coordinate change $\psi(r,\theta) \to r = a(\psi,\theta)$ gives

$$H_a(\psi,\theta) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$\begin{aligned} H_{11} &= -\frac{1}{a_\psi^3} a_{\psi\psi}, \\ H_{12} = H_{21} &= \frac{a_\theta}{a a_\psi^3} a_{\psi\psi} - \frac{1}{a a_\psi^2} a_{\psi\theta} + \frac{a_\theta}{a^2 a_\psi}, \\ H_{22} &= -\frac{a_\theta^2}{a^2 a_\psi^3} a_{\psi\psi} + 2\frac{a_\theta}{a^2 a_\psi^2} a_{\psi\theta} - \frac{1}{a^2 a_\psi} a_{\theta\theta} + \frac{1}{a a_\psi}. \end{aligned}$$

A similar argument used to prove 4.6 shows that for $a(\psi,\theta) \sim \psi^{1/2}$, we also have $H_{11} \sim 2$, $H_{12} = H_{21} \sim 0$, $H_{22} \sim 2$ and thus $|H_a(\psi,\theta)| \sim 4$, confirming that the stagnation points generated by functions in $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ near $\psi^{1/2}$ are indeed non-degenerate.

4.2 Boundary Operator

Having established the analyticity of the nonlinear differential map defining the equation of stationary flow, we now turn our attention to the nonlinear boundary map. Namely, we should prove that in a neighbourhood of $b(\varphi) = 1$, $R = 1$, $p = 0$, $a(\psi,\theta) = \psi^{1/2}$, the map $B(b,R,p,a)$ is well defined and analytic into $X^{m-1/2}_\sigma(\mathbb{T})$. The boundary map B is defined by

$$\begin{aligned} B(b,R,p,a) = -b^2\Big(&\arctan\big(p_y + Ra(1,\theta)\sin\theta, p_x + Ra(1,\theta)\cos\theta\big)\Big) \\ &+ R^2a^2(1,\theta) + 2Ra(1,\theta)(p_x\cos\theta + p_y\sin\theta) + p_x^2 + p_y^2. \end{aligned}$$

We start by generalizing the previous section's results to superposition maps acting on fractional Sobolev spaces.

Proposition 4.11.

Let $f \in C^{m+1}$ on a domain containing image of u. Then $u \to f(u) : H^{m+1/2}(\mathbb{T}) \to H^{m+1/2}(\mathbb{T})$ is well defined and continuous for integer $m \geq 1$. Furthermore, if $f \in C^{m+2}$ then this map is C^1.

Proof. First, recall the fractional order Sobolev norm:

$$\|u\|^2_{H^{m+1/2}(\mathbb{T})} = \|u\|^2_{H^m(\mathbb{T})} + \|D^m u\|^2_{H^{1/2}(\mathbb{T})}$$

where

$$\|D^m u\|^2_{H^{1/2}(\mathbb{T})} = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|D^m u(x) - D^m u(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

We have already seen in 4.1 that the statement is true for integer order $H^m(\mathbb{T})$. To show this map is well defined in the fractional order space, it remains to bound the fractional order part of the norm. We can write

$$\begin{aligned} D^m f\big(u(x)\big) - D^m f\big(u(y)\big) = \sum_{j=1}^{m} \sum_{\substack{\alpha_1+\cdots+\alpha_j=m \\ \alpha_i \geq 1}} C_{\alpha_1,\cdots,\alpha_j} \Big(& f^{(j)}(u(x)) D^{\alpha_1} u(x) \cdots D^{\alpha_j} u(x) \\ & - f^{(j)}(u(y)) D^{\alpha_1} u(y) \cdots D^{\alpha_j} u(y) \Big) \end{aligned}$$

and so

$$\begin{aligned} |D^m f\big(u(x)\big) - D^m f\big(u(y)\big)|^2 \leq C \sum_{j=1}^{m} \sum_{\substack{\alpha_1+\cdots+\alpha_j=m \\ \alpha_i \geq 1}} \Big| & f^{(j)}(u(x)) D^{\alpha_1} u(x) \cdots D^{\alpha_j} u(x) \\ & - f^{(j)}(u(y)) D^{\alpha_1} u(y) \cdots D^{\alpha_j} u(y) \Big|^2. \end{aligned}$$

First consider the case when each $\alpha_i < m-1$. Then necessarily $m > 2$. Each $D^{\alpha_i} u \in C^1(\mathbb{T})$ and so $f^{(j)}(u) D^{\alpha_1} u \cdots D^{\alpha_j} u \in C^1(\mathbb{T})$. By mean value theorem, we have

$$\begin{aligned} \Big| f^{(j)}(u(x)) D^{\alpha_1} u(x) \cdots D^{\alpha_j} u(x) - f^{(j)}(u(y)) D^{\alpha_1} u(y) \cdots D^{\alpha_j} u(y) \Big| & \\ \leq \|D\big(f^{(j)}(u) D^{\alpha_1} u \cdots D^{\alpha_j} u\big)\|_\infty |x-y| & \\ \leq C\|f\|_{C^{j+1}}\big(\|u\|^j_{H^m} + \|u\|^{j+1}_{H^m}\big). & \end{aligned}$$

This gives

$$\|f^{(j)}(u) D^{\alpha_1} u \cdots D^{\alpha_j} u\|_{H^{1/2}(\mathbb{T})} \leq C\|f\|_{C^{m+1}}\big(\|u\|^3_{H^m(\mathbb{T})} + \|u\|^{m+1}_{H^m(\mathbb{T})}\big).$$

Next, we consider the case when $\alpha_i = m-1$ for some i. Then necessarily $j=2$ and $m \geq 2$. Without loss of generality, suppose $\alpha_1 = m-1$ and $\alpha_2 = 1$. We have

$$\begin{aligned}
&\Big| f''(u(x))D^{m-1}u(x)Du(x) - f''(u(y))D^{m-1}u(y)Du(y)\Big| \\
&\quad \leq |f''(u(x))Du(x)||D^{m-1}u(x) - D^{m-1}u(y)| + |D^{m-1}u(y)||f''(u(x))||Du(x) - Du(y)| \\
&\qquad\qquad + |D^{m-1}u(y)||Du(y)||f''(u(x)) - f''(u(y))| \\
&\quad \leq \|f''\|_\infty \|Du\|_\infty |D^{m-1}u(x) - D^{m-1}u(y)| + \|f''\|_\infty \|D^{m-1}u\|_\infty |Du(x) - Du(y)| \\
&\qquad\qquad + \|f'''\|_\infty \|Du\|_\infty^2 \|D^{m-1}u\|_\infty |x-y|,
\end{aligned}$$

where we have used continuity of of Du, $D^{m-1}u$, and mean value theorem on $f''(u)$. This gives

$$\|f''(u)D^{m-1}uDu\|_{H^{1/2}(\mathbb{T})} \leq C\|f\|_{C^{m+1}}\big(\|u\|^2_{H^m(\mathbb{T})} + \|u\|^3_{H^m(\mathbb{T})}\big).$$

Finally, consider the case when $\alpha_1 = m$, thus $j=1$ and $m \geq 1$. We have

$$\begin{aligned}
&|f'(u(x))D^m u(x) - f'(u(y))D^m u(y)| \\
&\qquad\qquad \leq \|f'\|_\infty |D^m u(x) - D^m u(y)| + |D^m u(y)||f'(u(x)) - f'(u(y))|
\end{aligned}$$

and

$$|f'(u(x)) - f'(u(y))| \leq \int_y^x |f''(u(t))Du(t)|\,\mathrm{d}t \leq \|f''\|_\infty \int_y^x |Du(t)|\,\mathrm{d}t\,.$$

Next, $Du \in H^{m-1/2} \subset H^{1/2}$ for $m \geq 1$. By the critical Sobolev embedding, we have $Du \in L^q$ for any $q \in (1,\infty)$. Let $1/p + 1/q = 1$. By Hölder's inequality,

$$\int_y^x |Du(t)|\,\mathrm{d}t \leq |x-y|^{1/p}\|Du\|_{L^q}.$$

Finally, for $p < 2$ (and thus $q > 2$), we have

$$\int_{\mathbb{T}} \frac{1}{|x-y|^{2-2/p}}\,\mathrm{d}x = \frac{(2\pi - y)^{2/p-1} + y^{2/p-1}}{2/p - 1}$$

where the right-hand side is bounded on $y \in \mathbb{T}$. We thus get the bound

$$\|f'(u)D^m u\|_{H^{1/2}(\mathbb{T})} \leq C\|f\|_{C^{m+1}}\big(\|u\|_{H^{m+1/2}(\mathbb{T})} + \|u\|^2_{H^{m+1/2}(\mathbb{T})}\big).$$

Putting the three cases together with the results of 4.1, we have shown that for $m \geq 1$,

$$\|f(u)\|_{H^{m+1/2}(\mathbb{T})} \leq C\|f\|_{C^{m+1}}\big(1 + \|u\|^{m+1}_{H^{m+1/2}(\mathbb{T})}\big).$$

The continuity and Fréchet differentiability of $u \to f(u) : H^{m+1/2}(\mathbb{T}) \to H^{m+1/2}(\mathbb{T})$ follow analogously as in 4.1. $\square$

Remark 4.12. We have thus shown $u \to f(u) : H^{m-1/2}(\mathbb{T}) \to H^{m-1/2}(\mathbb{T})$ is C^1 when f is C^{m+1} and $m \geq 2$.

Corollary 4.13.

Suppose f is complex analytic on a domain containing image of u. Then $u \to f(u) : X_\sigma^{m-1/2}(\mathbb{T}) \to X_\sigma^{m-1/2}(\mathbb{T})$ is complex analytic for $m \geq 2$.

Proof. If $u \in X_\sigma^{m-1/2}(\mathbb{T})$ and f complex analytic on a domain containing image of u, then $f(u)$ is a composition of holomorphic functions and thus holomorphic in $\mathbb{T}_\sigma$. Furthermore, $u(\cdot + it) \in H^{m-1/2}(\mathbb{T})$ for every $|t| \leq \sigma$ and thus by the previous result, $f\big(u(\cdot + it)\big) \in H^{m-1/2}(\mathbb{T})$ for every $|t| \leq \sigma$. Thus $f(u) \in X_\sigma^{m-1/2}(\mathbb{T})$. Additionally, this map is complex differentiable and thus analytic. $\square$

To apply the above result to our boundary map, we define the superposition operators:

$$(R, p, a) \to X(\theta) = p_x + Ra(1,\theta)\cos\theta, \tag{4.2}$$

$$(R, p, a) \to Y(\theta) = p_y + Ra(1,\theta)\sin\theta, \tag{4.3}$$

$$(R, p, a) \to f(\theta) = R^2a^2(1,\theta) + 2Ra(1,\theta)(p_x\cos\theta + p_y\sin\theta) + p_x^2 + p_y^2, \tag{4.4}$$

$$(R, p, a) \to \varphi(\theta) = \arctan\big(p_y + Ra(1,\theta)\sin\theta, p_x + Ra(1,\theta)\cos\theta\big). \tag{4.5}$$

Then the boundary map can be written $(R, p, a) \to B(\theta) = -b^2(\varphi(\theta)) + f(\theta)$. First we establish analyticity of superposition maps X, Y and f. Second we will address the map φ and the composition $b^2(\varphi)$.

Corollary 4.14.

The maps

$$(R, p, a) \to X, Y, f : \mathbb{C}^3 \times J_{1/2,\gamma}^{m,\sigma}(\Pi) \to X_\sigma^{m-1/2}(\mathbb{T})$$

defined in 4.2 are complex analytic for $m \geq 2$.

Proof. First, the restriction map $a(\psi, \theta) \to a(1, \theta) : J_{1/2,\gamma}^{m,\sigma}(\Pi) \to X_\sigma^{m-1/2}(\mathbb{T})$ is linear and thus analytic. Multiplication of functions by $\cos\theta$ and $\sin\theta$ is a linear map, well defined into

$X_\sigma^{m-1/2}(\mathbb{T})$ and thus also analytic. In particular, the maps $(R, p, a) \to X, Y, f$ can be viewed as compositions of the linear map

$$(R, p_x, p_y, a(\psi, \theta)) \to (R, p_x, p_y, p_x \cos\theta, p_y \sin\theta, a(1, \theta))$$
$$\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \underbrace{X_\sigma^{m-1/2}(\mathbb{T}) \times \cdots \times X_\sigma^{m-1/2}(\mathbb{T})}_{6 \text{ times}},$$

and a polynomial on $\mathbb{C}^6$. By 4.13, these maps are thus analytic $\mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X_\sigma^{m-1/2}(\mathbb{T})$, so long as $m \geq 2$. □

Now we must consider the map $(R, p, a) \to \varphi(\theta)$. This maps takes the graph of polar function $r = Ra(1, \theta)$ centered at p and returns the corresponding angle coordinate of this graph in (ρ, φ) coordinates centered at the origin. So long as p is close to 0 and the graph $a(1, \theta)$ is close to a circle so that it corresponds to the graph of a polar function in both coordinates, then this nonlinear coordinate change will be well defined. In the real case, it will be some diffeomorphism of $\mathbb{T}$. In the complex case, we expect a biholomorphism from $\mathbb{T}_\sigma$ to a slightly deformed complex periodic strip $\varphi\{T_\sigma\}$. Since this deformation should be continuous with respect to (R, p, a), then for any $\tau > \sigma > 0$, we can take (R, p, a) close enough to $(1, 0, \psi^{1/2})$ that we get $\varphi\{\mathbb{T}_\sigma\} \subset \mathbb{T}_\tau$.

Proposition 4.15.
Let $m \geq 2$. For any $\tau > \sigma > 0$, there exists $\varepsilon > 0$ small enough such that if $|R - 1| < \varepsilon$, $|p| < \varepsilon$ and $\|a - \psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} < \varepsilon$, then the map

$$(R, p, a) \to \varphi(\theta) : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X_\sigma^{m-1/2}(\mathbb{T})$$

is analytic and the image $\varphi\{\mathbb{T}_\sigma\}$ is contained in $\mathbb{T}_\tau$.

Proof.

In the real case, $(x, y) \in \mathbb{R}^2 \setminus \{0\} \to \varphi = \text{atan}(y, x)$ is a $\mathbb{T}$-valued function giving the angle between the plane vector (x, y) and the x-axis. Equivalently, one can think of $\text{atan}(y, x)$ as a (helicoidal) multivalued function with the property that if $\text{atan}(y, x) = \varphi$, then also $\text{atan}(y, x) = \varphi + 2\pi k$ for any $k \in \mathbb{Z}$.

Let us define the usual one-argument arctangent by $\operatorname{atan}(y/x) \in (-\pi/2, \pi/2)$ for $x > 0$. Using the one-argument arctangent, we can for example define the following four charts of the multivalued arctangent:

$$\operatorname{atan}(y,x) = \begin{cases} \operatorname{atan}(y/x) & \text{if } x > 0 \\ \pi/2 - \operatorname{atan}(x/y) & \text{if } y > 0 \\ \pi - \operatorname{atan}(y/x) & \text{if } x < 0 \\ 3\pi/2 - \operatorname{atan}(x/y) & \text{if } y < 0. \end{cases}$$

These charts correspond to values of φ in $(-\pi/2, \pi/2)$, $(0, \pi)$, $(\pi/2, 3\pi/2)$ and $(\pi, 2\pi)$ respectively. Adding $2\pi k$ with $k \in \mathbb{Z}$ defines the remaining charts of the full helicoid.

Next, consider the complexifications $x \to X = x + i\xi$ and $y \to Y = y + i\eta$. Analogous charts of $\operatorname{atan}(Y, X)$ for $x > 0$, $y > 0$, $x < 0$ and $y < 0$ are defined by use of the complex one-argument function $\operatorname{atan}(z)$, with $z = X/Y$ or $z = Y/X$. The function $\operatorname{atan}(z)$, which is the complex extension of the real one-argument arctangent with values in $(-\pi/2, \pi/2)$, is analytic except at $\{z : \operatorname{Re}\{z\} = 0, |\operatorname{Im}\{z\}| \geq 1\}$.

Now, treating R, p, a, θ as real, define the complex extensions: $p_x \to p_x + i\pi_x$, $p_y \to p_y + i\pi_y$, $Ra(1,\theta) \to \alpha + i\beta$ and $\theta \to \theta + it$. The last extension gives identities

$$\cos(\theta + it) = \cos\theta \cosh t - i \sin\theta \sinh t \ , \quad \sin(\theta + it) = \sin\theta \cosh t + i \cos\theta \sinh t.$$

These induce complexifications of $p_x + Ra(1,\theta)\cos\theta$ and $p_y + Ra(1,\theta)\sin\theta$, given by

$$X = (p_x + \alpha\cos\theta\cosh t + \beta\sin\theta\sinh t) + i(\pi_x - \alpha\sin\theta\sinh t + \beta\cos\theta\cosh t) = x + i\xi,$$
$$Y = (p_y + \alpha\sin\theta\cosh t - \beta\cos\theta\sinh t) + i(\pi_y + \alpha\cos\theta\sinh t + \beta\sin\theta\cosh t) = y + i\eta.$$

First suppose $x > 0$. We have

$$z = \frac{Y}{X} = \frac{(xy + \xi\eta) + i(x\eta - y\xi)}{x^2 + \xi^2}.$$

If $\operatorname{Re}\{z\} \neq 0$, then $\operatorname{atan}(z)$ is analytic. If on the other hand $\operatorname{Re}\{z\} = 0$, then $\operatorname{atan}(z)$ is analytic when $|\operatorname{Im}\{z\}| < 1$. So suppose $\operatorname{Re}\{z\} = 0$. Then $xy + \xi\eta = 0$. Since $x > 0$, we have $y = -\xi\eta/x$. From this we find that $\operatorname{Im}\{z\} = \eta/x$. Thus for $\operatorname{atan}(z)$ to be analytic, we require that $|\eta/x| < 1$, or equivalently, that $|\eta| < |x|$.

Substituting expressions for x, y, ξ, η into condition $xy + \xi\eta = 0$ and using identity $\cosh^2 t - \sinh^2 t = 1$, we get

$$\begin{aligned}
&p_x p_y + \pi_x \pi_y + (\alpha p_x + \beta \pi_x)\sin\theta\cosh t + (\alpha\pi_x - \beta p_x)\cos\theta\sinh t \\
&\qquad + (\alpha p_y + \beta\pi_y)\cos\theta\cosh t - (\alpha\pi_y - \beta p_y)\sin\theta\sinh t + (\alpha^2 + \beta^2)\sin\theta\cos\theta = 0.
\end{aligned}$$

From the statement of the theorem, we have $p_x, p_y, \pi_x, \pi_y, \beta \sim \varepsilon$ and $\alpha \sim 1$. The above equality implies that the last term of the left hand side is of the same order as the other terms, thus we deduce $\sin\theta\cos\theta \sim \varepsilon(\sin\theta + \cos\theta)(\cosh t + \sinh t) \sim \varepsilon$, since $|t| < \sigma$ and σ is fixed. If ε is small enough, then $\sin\theta\cos\theta \sim \varepsilon$ implies either $\sin\theta \sim \varepsilon$ or $\cos\theta \sim \varepsilon$. Since we work on the chart $x > 0$, we can assume without loss of generality that $\sin\theta \sim \theta \sim \varepsilon$ and thus $\cos\theta \sim 1$. The other case can be handled by charts $y > 0$ and $y < 0$.

Returning to the desired estimate $|\eta| < |x|$, observe $\eta = \pi_y + \alpha\cos\theta\sinh t + \beta\sin\theta\cosh t$ and $x = p_x + \alpha\cos\theta\cosh t + \beta\sin\theta\sinh t$. We have

$$\begin{aligned}
|\eta| &= |\pi_y + \alpha\cos\theta\sinh t + \beta\sin\theta\cosh t| \\
&\le |\pi_y| + \alpha\cos\theta|\sinh t| + |\beta\sin\theta|\cosh t \\
&= |\pi_y| + |p_x| - |p_x| + (\alpha\cos\theta - |\beta\sin\theta|)(|\sinh t| - \cosh t) - |\beta\sin\theta\sinh t| \\
&\quad + 2|\beta\sin\theta\sinh t| + \alpha\cos\theta\cosh t \\
&< |\pi_y| + |p_x| + 2|\beta\sin\theta\sinh\sigma| + (\alpha\cos\theta - |\beta\sin\theta|)(\sinh\sigma - \cosh\sigma) \\
&\quad + |p_x + \alpha\cos\theta\cosh t + \beta\sin\theta\sinh t|.
\end{aligned}$$

Here we have used the fact that $|t| < \sigma$. Finally, since p_x, π_y, β, $\sin\theta \sim \varepsilon$ and $\alpha\cos\theta - |\beta\sin\theta| \sim 1$, for any σ, we can take ε small enough such that $|\pi_y| + |p_x| + 2|\beta\sin\theta\sinh\sigma| + (\alpha\cos\theta - |\beta\sin\theta|)(\sinh\sigma - \cosh\sigma) < 0$. We thus get $|\eta| < |x|$.

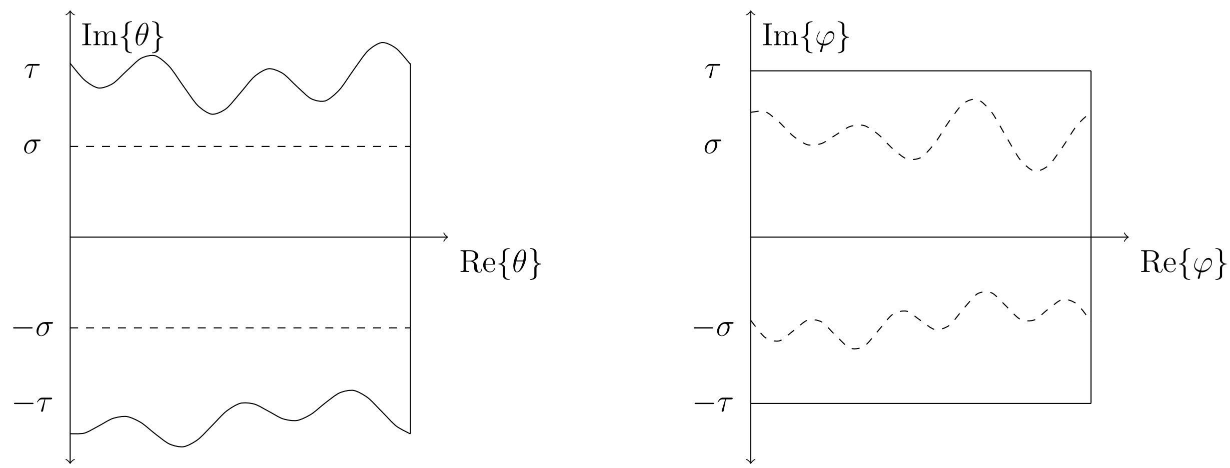
Analogous arguments hold for the other charts (with $z = X/Y$ for $y > 0$ and $y < 0$). Returning to our standard notation where R, p and $a(\psi,\theta)$ are $\mathbb{C}$-valued, we conclude that for any $\sigma > 0$, there exists $\varepsilon > 0$ small enough such that for $|R - 1| < \varepsilon$, $|p| < \varepsilon$ and $\|a - \psi^{1/2}\|_{J^{m,\sigma}_{1/2,\gamma}(\Pi)} < \varepsilon$, the multivalued function $\varphi = \operatorname{atan}(Y, X)$ is analytic on the image of $X(\theta) = p_x + Ra(1,\theta)\cos\theta$, $Y(\theta) = p_y + Ra(1,\theta)\sin\theta$. By 4.13, for $m \ge 2$,

$$(R, p, a) \to (X, Y) \to \operatorname{atan}(X, Y) : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X^{m-1/2}_\sigma(\mathbb{T}) \times X^{m-1/2}_\sigma(\mathbb{T}) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is the composition of analytic maps and is thus analytic.

Observe that $\varphi = \operatorname{atan}\big(p_y + Ra(1,\theta)\sin\theta, p_x + Ra(1,\theta)\cos\theta\big)$ defines a conformal map of θ in the periodic strip $\mathbb{T}_\sigma$ which is conformal to an annulus. Thus its image is some deformed periodic strip of equal modulus of annulus. By consequence of the above result, $(R,p,a) \to \varphi : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X^{m-1/2}_\sigma(\mathbb{T})$ is continuous at $(R,p,a) = (1,0,\psi^{1/2})$, where we have $\varphi(1,0,\psi^{1/2}) = \operatorname{atan}(\sin\theta,\cos\theta) = \theta$. By continuity of this map and the embedding $X^{m-1/2}_\sigma(\mathbb{T}) \in C(\overline{\mathbb{T}}_\sigma)$, we can make this deformation arbitrary small. In particular, for any $\tau > \sigma$, we can find ε small enough such that $\varphi\{\mathbb{T}_\sigma\} \subset \mathbb{T}_\tau$.

$\square$



The graph of a polar function can be represented in coordinates (r,θ) and (ρ,φ). The map $\theta \to \varphi$ takes $\mathbb{T}_\sigma$ to some deformed strip (enclosed by the dashed curves on the right side) which is contained in $\mathbb{T}_\tau$. Conversely, $\varphi \to \theta$ takes $\mathbb{T}_\tau$ to some deformed strip (enclosed by the solid curves on the left side) which contains $\mathbb{T}_\sigma$.

Remark 4.16. The point is that the nonlinear coordinate change $\theta \leftrightarrow \varphi$ between domains of analyticity is not a self map on $\mathbb{T}_\sigma$. From a reverse perspective, given a prescribed boundary function $\rho = b(\varphi)$ analytic on some domain, the domain of analyticity of $r = Ra(1,\theta)$ will depend on the solution itself (namely the position p). Since we require the pool of solutions to be taken from the same Banach space, we must fix the domain of solutions. To work around this, we enlarge the domain of analyticity of prescribed boundary functions $\rho = b(\varphi)$ to $\mathbb{T}_\tau$ with $\tau > \sigma$, so that in a sufficiently small neighbourhood of solution $(R,p,a) = (1,0,\psi^{1/2})$, all solutions map $\theta \to \varphi : \mathbb{T}_\sigma \to \mathbb{T}_\tau$. That is, we prescribe an analytic boundary function $\rho = b(\varphi)$ whose complex singularities are restricted to $|\mathrm{Im}\{\varphi\}| \geq \tau$. Then, we describe our solutions on domain $\theta \in \mathbb{T}_\sigma$ with $|\operatorname{Im}\{\varphi(\theta)\}| < \tau$ so that they do not include the prescribed singularities. For this reason, these prescribed singularities can be of any strength and the boundary functions $\rho = b(\varphi)$ can be taken in any Banach space $\mathrm{H}(\mathbb{T}_\tau)$ of functions holomorphic in $\mathbb{T}_\tau$.

We thus arrive at the main result of this section.

Theorem 4.17.

Let $m \geq 2$. For any $\tau > \sigma > 0$, there exists a neighbourhood of solution $R = 1$, $p = 0$ and $a(\psi, \theta) = \psi^{1/2}$ on which the boundary map

$$(b, R, p, a) \to B : \mathrm{H}(\mathbb{T}_\tau) \times \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is analytic, for any Banach space $\mathrm{H}(\mathbb{T}_\tau)$ of functions holomorphic in $\mathbb{T}_\tau$.

Proof.

We saw that the composition

$$(R, p, a) \to (X, Y) \to \varphi : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X^{m-1/2}_\sigma(\mathbb{T}) \times X^{m-1/2}_\sigma(\mathbb{T}) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is analytic and the image of $\varphi(\theta)$ is contained in $\mathbb{T}_\tau$. By 4.13, the map

$$(b, \varphi) \to b \circ \varphi : \mathrm{H}(\mathbb{T}_\tau) \times X^{m-1/2}_\sigma(\mathbb{T}) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is well defined and analytic in φ. Also it is linear and thus analytic in b. Thus it is analytic in the product space $\mathrm{H}(\mathbb{T}_\tau) \times X^{m-1/2}_\sigma(\mathbb{T})$. Again by 4.13, the map

$$b \circ \varphi \to (b \circ \varphi)^2 : X^{m-1/2}_\sigma(\mathbb{T}) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is analytic. Finally, we saw also that the map

$$(R, p, a) \to f : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to X^{m-1/2}_\sigma(\mathbb{T})$$

is analytic.Thus, $(b, R, p, a) \to B = -b^2(\varphi) + f$ is the composition and sum of analytic maps and thus analytic.

□

Combining with the main result of the previous section, we have proved:

Theorem 4.18. *Let $m > 3$ and $\gamma > 1/2$. For any $\tau > \sigma > 0$, there exists a neighbourhood of $(F, b, R, p, a) = (4, 1, 1, 0, \psi^{1/2})$ in which the map*

$$(F, b, R, p, a) \to \big(\Xi(a) - F, B\big)$$
$$J^{m-2}_{0,\gamma}(0, 1] \times \mathrm{H}(\mathbb{T}_\tau) \times \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T})$$

is complex analytic.

Chapter 5

Conclusion

5.1 Main Result

We now have all of the required components to prove our main result. The principle driving our work is the representation of a flow as a collection of its flow lines. We have introduced function spaces which describe families of topologically circular flow lines around a single non-degenerate elliptic fixed point. A partial complex analytic structure on these function spaces incorporates the flow line analyticity. In our formulation, stationary flows are governed by a nonlinear degenerate elliptic boundary value problem, which can be expressed as an analytic operator equation in the defined function spaces.

Theorem 5.1 (Main Result)**.**
Let $m > 3$, $1/2 < \gamma < 1$ and $\tau > \sigma > 0$. There exists a neighbourhood of $F(\psi) = 4$ in $J^{m-2}_{0,\gamma}(0,1]$, $b(\varphi) = 1$ in $\mathrm{H}(\mathbb{T}_\tau)$, $R = 1$ in $\mathbb{C}$, $p = 0$ in $\mathbb{C}^2$ and $a(\psi,\theta) = \psi^{1/2}$ in $J^{m,\sigma}_{1/2,\gamma}(\Pi)$, in which 1.6 has a unique solution that is parameterized by analytic map $(F, b) \to (R, p, a) : J^{m-2}_{0,\gamma}(0,1] \times \mathrm{H}(\mathbb{T}_\tau) \to \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi)$.

Proof. Equation 1.6 can be written as an operator equation

$$(F, b, R, p, a) \to \big(\Xi(a) - F, B\big) = 0$$

between complex Banach spaces

$$J^{m-2}_{0,\gamma}(0,1] \times \mathrm{H}(\mathbb{T}_\tau) \times \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T}).$$

This equation has a solution at $(F, b, R, p, a) = (4, 1, 1, 0, \psi^{1/2})$ and in a neighbourhood of this solution, the above operator is analytic. The linearization

$$\frac{\partial\big(\Xi(a) - F, B\big)}{\partial(R, p, a)} : \mathbb{C}^3 \times J^{m,\sigma}_{1/2,\gamma}(\Pi) \to \widetilde{J}^{m-2,\sigma}_{0,\gamma}(\Pi) \times X^{m-1/2}_{\sigma}(\mathbb{T})$$

at this solution defines a Banach isomorphism. By the analytic implicit function theorem in complex Banach spaces, the result follows. $\square$

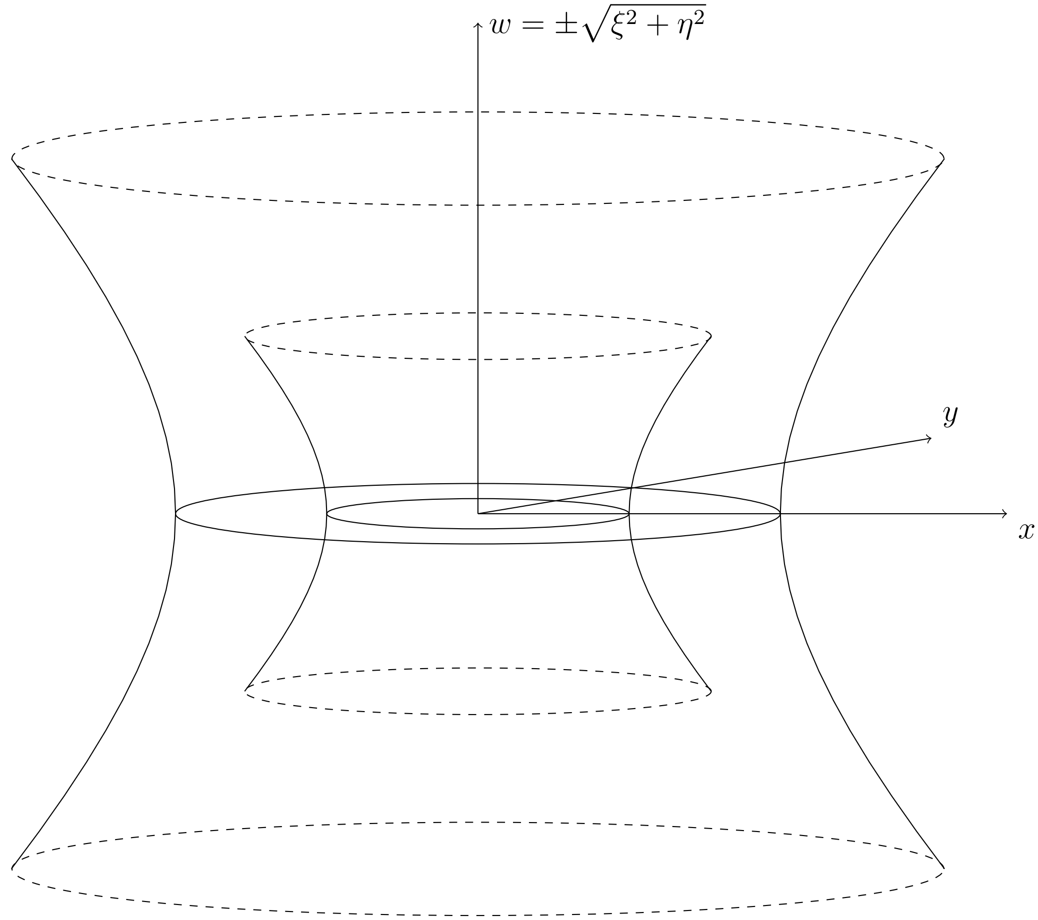
Recall that the unknown R was introduced into the solution as an extra degree of freedom to accommodate the fact that specifying ψ at the fixed point (as we have done) yields an overdetermined problem. Under such circumstances, only the vorticity and the 'shape' of domain should be treated as parameters, where as the 'radius' of domain depends on vorticity. In our construction, the solutions for which $R \neq 1$ are fictitious in that they are produced by incompatible choices of vorticity and domain. Taking the pre-image of solutions with $R = 1$ defines a codimension-one submanifold of the parameter space, consisting of precisely the compatible parameters.

Theorem 5.2.
Under the conditions of 5.1, in a neighbourhood of the circular flow of constant vorticity, the set of stationary flows having a single, non-degenerate elliptic fixed point form a complex Banach manifold in $J^{m,\sigma}_{1/2,\gamma}(\Pi)$ parameterized by a codimension-one submanifold of $J^{m-2}_{0,\gamma}(0, 1] \times \mathrm{H}(\mathbb{T}_\tau)$.

Now let us say something about the analytic structure of these solutions. In the real picture, our flow lines $r = a(\psi, \theta)$ are parameterized in the plane by concentric circles $(x, y) = \psi^{1/2}(\cos\theta, \sin\theta)$, i.e. the level sets of $\psi = x^2 + y^2$. Complexifying the circle to the periodic strip $\theta \to \theta + it : \mathbb{T} \to \mathbb{T}_\sigma$ induces complexifications $x \to x + i\xi$, $y \to y + i\eta$ and we get the following domain in $\mathbb{C}^2$ parameterizing each flow line:

$$(x, y) + i(\xi, \eta) = \psi^{1/2} \cosh t(\cos\theta, \sin\theta) + i\psi^{1/2} \sinh t(-\sin\theta, \cosh\theta).$$

Thus, θ sweeps out circles in the real plane, as well as the complex plane. As $\psi \to 0^+$, the parameter domain collapses to a point.



Given coordinates $(x + i\xi, y + i\eta) \in C^2$, complex flow lines are analytic deformations of the level sets of $\psi = (x + i\xi)^2 + (y + i\eta)^2$. Passing to coordinates $(x, y, \pm\sqrt{\xi^2 + \eta^2})$, these flow lines can be visualized as deformations of a family of nested hyperboloids.

To further aid in visualizing these parameterizing domains, the level sets of $(x + i\xi)^2 + (y + i\eta)^2 = \psi \in \mathbb{R}$ are defined by two equations:

$$x^2 + y^2 - \xi^2 - \eta^2 = \psi \quad \text{and} \quad x\xi + y\eta = 0.$$

Define coordinate $w = \pm\sqrt{\xi^2 + \eta^2}$. Then the first equation $x^2 + y^2 - w^2 = \psi$ defines a family of hyperboloids in $\mathbb{R}^3$. In light of the prior observations, these hyperboloids decrease in height $|w|$ and collapse to a point as $\psi \to 0^+$.

The complex flow lines defining our stationary flow are analytic deformations of the parameter sets in $\mathbb{C}^2$ described above, with at worst, weak singularities on their boundary of Sobolev type.

Comparing to our previous work [9] in which we obtained an analytic parameterization of stationary flows in a periodic strip without fixed point, our result here is a touch weaker. The parameterization of the prior work includes in its description the prescribed singularities of the boundary flow lines (which may occur along $\partial\mathbb{T}_\sigma$). In the parameterization provided here, the prescribed singularities are explicitly avoided from the description. The limitation seems only of a technical nature resulting from the coordinate changes induced by translations of the fixed point of the flow. In the case of solutions for which the fixed point does not deviate from the origin, this coordinate change does not occur. We then expect the following strengthening of our main result:

Theorem 5.3.
Suppose $(b, F) \in X_\tau^{m-1/2}(\mathbb{T}) \times J_{0,\gamma}^{m-2}(0, 1]$ *are such that solution* $a(\psi, \theta) \in J_{1/2,\gamma}^{m,\sigma}(\Pi)$ *has fixed point at* $p = 0$*. Then in fact* $a(\psi, \theta) \in J_{1/2,\gamma}^{m,\tau}(\Pi)$*.*

It remains to be seen how to show this improvement. Doing so would bring our result in exact analogy with the prior work on the periodic strip.

5.2 Considerations and Future Problems

Let us underline the spirit which drove our success: viewing a function as a collection of its level sets, or specific to our case, a flow as a family of flow lines. First, this formulation allows us to directly incorporate the analyticity of the flow lines. Second, such constructions allow us to conveniently single out particular topologies of flow lines we wish to study. Such a loss in generality in effect transfers to the strength of the results as well as the ease in acquiring them. In [9], these were flow lines without fixed point in a periodic channel. In the work presented, these are flow lines with a single non-degenerate elliptic fixed point in a simply connected domain. The next logical step is to apply these principles to describe flows with a non-degenerate hyperbolic fixed point.

The prototypical example of such stationary flows is described by $\psi = xy$, whose level curves are hyperbolas in each quadrant pinching on the axes. It seems natural to use a system of orthogonal hyperbolas to describe their perturbations. The immediate trouble we face is that such a coordinate system couples opposing quadrants; level lines of $\psi = xy \neq 0$ consist of two curves in opposite quadrants. This introduces an undesirable restriction in the generality of the problem. To circumvent this, perhaps one can introduce a system of hyperbolic coordinates, each covering a single half-plane. By patching these coordinate systems along each quadrant and writing the equations of stationary flow, we expect the stationary flows to be governed by a system of degenerate elliptic equations.

Additional trouble arises because such flow lines are unbounded. It would be more appropriate to work with some analogue of the case above, whose flow lines are compact. For example, one can imagine flows containing a figure-eight. Despite the additional obstacles, it at least seems reasonable to expect that the general ideas behind this work can be extended to solve this problem.

Once this is accomplished, we have local descriptions of the set of stationary flows near three prototypes of distinct topology. From here, more difficult problems are abundant : providing local descriptions near arbitrary solutions having a single fixed point or none, patching such solutions together to produce local descriptions of stationary flows with an arbitrary number of fixed points, properties of transition maps between charts on the prospective manifold of solutions and eventually questions regarding the global structure of the set of stationary flows.

Finally, we remind that the conjectured attractor of the 2D Euler equations consists of not just stationary flows, but time-periodic and time-quasiperiodic flows as well. Their existence should be proven. Can we use a similar construction to the ones presented here, based on the flow lines or perhaps vorticity lines? How do we let them 'breath'?

As we see, subsequent questions are plentiful. How far can we push the principle philosophy behind this work to answer them?

Chapter 6

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