Triple Product p-adic L-functions for Finite Slope Families and A p-adic Gross-Zagier Formula

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Abstract

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In this thesis, we generalize the p-adic Gross-Zagier formula of Darmon-Rotger on triple product p-adic L-functions to finite slope families. First, we recall the construction of triple product p-adic L-functions for finite slope families developed by Andreatta-Iovita. Then we proceed to compute explicitly the p-adic Abel-Jacobi image of the generalized diagonal cycle. We also established a theory of finite polynomial cohomology with coefficients for varieties with good reduction. It simplifies the computation of the p-adic Abel-Jacobi map and has the potential to be applied to more general settings. Finally, we show by q-expansion principle that the special value of the L-function is equal to the Abel-Jacobi image. Hence, we conclude the formula.

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We can forgive a man for making a useful thing as long as he does not admire it. The only excuse for making a useless thing is that one admires it intensely.

Oscar Wilde, *The Picture of Dorian Gray*

Contents

1 Introduction

The theory of L-functions has long been a central topic in Number Theory due to its various arithmetic applications. Some notable ones are the Birch and Swinnerton-Dyer (BSD) conjecture and its generalizations, mostly attributed to Bloch and Kato.

Let E be an elliptic curve defined over $\mathbb{Q}$. One has the Hasse–Weil L-function $L(E,s)$ and the Mordell–Weil group $E(\mathbb{Q})$ associated with E. The BSD conjecture predicts the relation

$$\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s)$$

between the rank of the abelian group $E(\mathbb{Q})$ and the vanishing order of $L(E,s)$.

Let

$$\rho : \mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_L(V_\rho) \cong \mathrm{GL}_n(L)$$

be an Artin representation factoring through a finite Galois extension $H/\mathbb{Q}$, where V_ρ is an n-dimensional L-vector space with $L \subset \mathbb{C}$ being a finite extension of $\mathbb{Q}$. One then has the twisted L-function $L(E,\rho,s)$. Let $E(H)^\rho := \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes_{\mathbb{Z}} L)$ be the ρ-isotypic component of the Mordell–Weil group. The analytic and algebraic rank of E twisted by ρ are defined as

$$r_{\mathrm{an}}(E,\rho) := \mathrm{ord}_{s=1} L(E,\rho,s), \quad r_{\mathrm{alg}}(E,\rho) := \dim_L(E(H)^\rho).$$

The (Galois-)equivariant BSD conjecture predicts that $r_{\mathrm{an}}(E,\rho) = r_{\mathrm{alg}}(E,\rho)$.

Besides BSD-type conjectures, there are many other studies on special values of L-functions. For example, Katz's p-adic Kronecker limit formula [Kat76] relates special values of a two-variable p-adic L-function, which interpolates central critical values of the complex L-function attached to a p-adic family of Eisenstein series twisted by a family of algebraic Hecke characters, to p-adic logarithms of elliptic units. Another example is due to Gross and Zagier [GZ86], usually known as the Gross–Zagier formula, which expresses the value of the first derivative of the complex L-function attached to an elliptic curve as the height of a certain rational point on the elliptic curve. Since the Gross–Zagier formula serves as an inspiration of its p-adic version, which is the main subject of this thesis, we will sketch it below.

Let E be an elliptic curve over $\mathbb{Q}$ with conductor N and with complex multiplication by an imaginary quadratic field K. When all the prime divisors of N split in K (usually called the Heegner assumption), the L-function $L(E/K,s)$ necessarily vanishes at $s=1$. Hence one shifts his attention to the first derivative $L'(E/K,1)$. On the other hand, the Heegner assumption provides points (called *Heegner points*) with CM by K on the modular curve $X_0(N)(H_K)$, where H_K is the Hilbert class field of K. By the modular parametrization, one can construct a point $y_K \in E(H_K)$, which is actually defined over K. The Gross–Zagier formula then states that $L'(E/K,1)$ is equal, up to an explicit constant, to the *height* $h_E(y_K)$ of the point y_K.

In the work of Bertolini–Darmon–Prasanna [BDP13], they generalized Katz's p-adic L-functions to cusp forms, and then related the central critical values of the p-adic L-function to the p-adic Abel–Jacobi images of *generalized Heegner cycles*. The result was considered as a p-adic analogue of the Gross–Zagier formula.

Later in [DR14], Darmon and Rotger constructed a Garret–Rankin triple product p-adic L-function attached to three Hida families of modular forms. Similarly, they proved a p-adic Gross–Zagier formula which relates

- the special values of the triple product p-adic L-function at classical points lying outside the region of interpolation, to
- the p-adic Abel–Jacobi images of *generalized diagonal cycles* in the product of three Kugo–Sato varieties, evaluated at certain differentials.

It has been shown, in [Ber+14] and [DR17] for example, that these two p-adic Gross–Zagier formulae have applications to Euler systems and the equivariant BSD conjecture.

The main focus of this article is to generalize the p-adic Gross–Zagier formula in [DR14] to the case of finite slope families of modular forms.

In order to state our main result, we need to introduce several notations. Let

$$\begin{aligned} f &= \sum a_n(f)q^n \in S_k(N_f, \chi_f), \\ g &= \sum a_n(g)q^n \in S_\ell(N_g, \chi_g), \\ h &= \sum a_n(h)q^n \in S_m(N_h, \chi_h) \end{aligned}$$

be a triple of normalized primitive cuspidal eigenforms of weights $k, \ell, m \geq 2$, levels $N_f, N_g, N_h \geq 1$, and nebentypus characters χ_f, χ_g, χ_h, respectively. Set $N = \mathrm{lcm}(N_f, N_g, N_h)$ and assume that

$$\chi_f \cdot \chi_g \cdot \chi_h = 1,$$

which implies that $k + \ell + m$ is even.

The triple (k, ℓ, m) is said to be *balanced* if the largest number is strictly less than the sum of the other two. A triple which is not balanced is called *unbalanced*, and the largest number in an unbalanced triple is called the *dominant weight*.

In §2.3, we will recall the definition of the (complex) Garrett–Rankin L-function $L(f, g, h; s)$. It can be viewed as the L-function attached to the tensor product

$$V(f, g, h) = V(f) \otimes V(g) \otimes V(h)$$

of the (compatible systems of) p-adic Galois representations attached to f, g, and h. This L-function admits a functional equation relating values at s and $k + \ell + m - 2 - s$. As a consequence, the order of vanishing of $L(f, g, h; s)$ at the central point $c = \frac{k+\ell+m-2}{2}$ is determined by the root number $\varepsilon \in \{\pm 1\}$ of the functional equation. The root number ε can be further written as a product $\prod_{v|N\infty} \varepsilon_v$ of local root numbers $\varepsilon_v \in \{\pm 1\}$. We will assume the following assumption throughout this thesis.

Assumption H: The local root numbers $\varepsilon_v = +1$ for all finite primes $v \mid N$.

As indicated in [DR14], this assumption holds in a broad collection of settings. For instance, it is satisfied when

- $\gcd(N_f, N_g, N_h) = 1$, or,
- $N = N_f = N_g = N_h$ is square-free and $a_v(f)a_v(g)a_v(h) = -1$ for all primes $v \mid N$.

Under Assumption H, $\varepsilon = \varepsilon_\infty$, which in turns depends only on whether the triple of weights (k, ℓ, m) is balanced or not:

$$\varepsilon_\infty = \begin{cases} -1 & \text{if } (k, \ell, m) \text{ is balanced;} \\ +1 & \text{if } (k, \ell, m) \text{ is unbalanced.} \end{cases}$$

In particular, the L-function necessarily vanishes at the central point c when (k, ℓ, m) is balanced. In this situation, instead of studying the values of $L(f, g, h; s)$, one naturally studies the values of its first derivative $L'(f, g, h; s)$ and expects it to encode arithmetic information.

Let $E \to X = X_1(N)$ be the universal elliptic curve over the modular curve. For any $n \geq 0$, we let W_n be the n-th Kuga–Sato variety over $X_1(N)$. It is an $(n+1)$-dimensional variety obtained by desingularizing the n-fold fiber product E^n over X (c.f. [BDP13, Apeendix]). Then the Galois representation $V(f, g, h)$ appears in the middle cohomology of the triple product

$$W^* := W_{r_1} \times W_{r_2} \times W_{r_3},$$

where the notation $(r_1, r_2, r_3) = (k-2, \ell-2, m-2)$ will be commonly used in this article.

When (k, ℓ, m) is balanced and Assumption H is satisfied, the conjectures of Bloch–Kato and Beilinson–Bloch predict that the vanishing of $L(f, g, h; c)$ implies the existence of a non-trivial cycle $\Delta_{f,g,h}$ in the Chow group $\mathbb{Q} \otimes \mathrm{CH}^c(W^*)_0$ of rational equivalence classes of null-homologous cycles of codimension c in W^*.

Such a cycle is defined in [DR14], whose construction will be recalled in §5.8. We here only give a brief description. Set $r := \frac{r_1+r_2+r_3}{2}$. There exists an essentially unique way to closedly embed W^r into W^*. Its image, under suitable modifications, gives rise to a homologically trivial cycle

$$\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W^*)_0 := \ker(\mathrm{CH}^{r+2}(W^*) \xrightarrow{\mathrm{cl}} H^{2r+4}_{\mathrm{dR}}(W^*/\mathbb{C})).$$

The cycle $\Delta_{f,g,h}$ can be then defined as the (f,g,h)-isotypic component of $\Delta_{k,\ell,m}$ with respect to the Hecke actions.

In the archimedean setting, the height $h(\Delta_{f,g,h})$ of the cycle in the sense of Beilinson and Bloch is conjectured to be well-defined and related to the first derivative of $L(f,g,h;s)$ at $s=c$. That is, one expects the relation

$$h(\Delta_{f,g,h}) \stackrel{?}{=} (\text{Explicit non-zero factors}) \times L'(f,g,h,c).$$

In this article, we will focus on the p-adic analogue. To be more precise, we aim to relate the image of $\Delta_{k,\ell,m}$ under the p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W^*)_0(\mathbb{C}_p) \to [\mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W^*/\mathbb{C}_p)]^\vee$$

to the special value of the triple product p-adic L-function attached to three finite slope families of modular forms. We will now describe the setup in detail.

Choose a prime $p \nmid N$ such that f,g,h are of finite slope at p. Then we may find Coleman families $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ passing through f,g,h, defined over certain weight spaces $\Omega_f, \Omega_g, \Omega_h$, respectively. In the language of rigid analytic geometry, the space $\Omega_\bullet$ for $\bullet \in \{f,g,h\}$ is a finite rigid analytic cover over a subset of the weight space $\mathcal{W} := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ which always contains the integers $\mathbb{Z}$ via the identification $k \mapsto (\gamma \mapsto \gamma^k)$.

When $\mathbf{f}$ is of slope $a \in \mathbb{Q}_{\geq 0}$, a point $x \in \Omega_f$ is said to be classical if its image in Ω, denoted by $\kappa(x)$, belongs to $\mathbb{Z}_{>a+1}$. We will denote the set of classical points in Ω_f by $\Omega_{f,\mathrm{cl}}$ and usually identify $x \in \Omega_{f,\mathrm{cl}}$ with $\kappa(x) \in \mathbb{Z}$ by an abuse of notations. For almost all $x \in \Omega_{f,\mathrm{cl}}$, the specialization of $\mathbf{f}$ at $x \in \Omega_{f,\mathrm{cl}}$, denoted by $\mathbf{f}_x$ (or simply f_x), is a normalized eigenform of weight $\kappa(x)$ on $\Gamma_1(N,p) := \Gamma_1(N) \cap \Gamma_0(p)$. For all but finitely many such x, $\mathbf{f}_x$ is the (finite slope) p-stabilization of a normalized eigenform of the same weight on $\Gamma_1(N)$, denoted by f_x^0.

When $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ are Hida families, one can view them as maps

$$\mathbf{f} : \Omega_f \to \mathbb{C}_p[\![q]\!], \ \mathbf{g} : \Omega_g \to \mathbb{C}_p[\![q]\!], \ \mathbf{h} : \Omega_h \to \mathbb{C}_p[\![q]\!].$$

The triple product p-adic L-function for Hida families is defined to be

$$\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h}) := \frac{(\mathbf{f}^*, e_{\mathrm{ord}}(\theta^\bullet \mathbf{g}^{[p]} \times \mathbf{h}))}{(\mathbf{f}^*, \mathbf{f}^*)},$$

where $\mathbf{f}^*$ is the Atkin–Lehner involution of $\mathbf{f}$, e_{ord} is Hida's ordinary projector, $\theta = q\frac{d}{dq}$ is Serre's operator, and $\mathbf{g}^{[p]} := (1-VU)\mathbf{g}$ is the p-depletion of $\mathbf{g}$. The philosophy behind this definition will be explained in §2.3.

For finite slope families, one needs a different approach. A construction of triple product p-adic L-functions for finite slope families is developed by F. Andreatta and A. Iovita in [AI21]. Sections §3 and §4 are dedicated to describe their construction.

Roughly speaking, one wants to define the triple product p-adic L-function as the ratio

$$\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h}) := \frac{(\mathbf{f}^*, \nabla^\bullet \mathbf{g}^{[p]} \times \mathbf{h})}{(\mathbf{f}^*, \mathbf{f}^*)},$$

where ∇ is the Gauss–Manin connection (c.f. §2.1). In order for this expression to make sense, one needs to p-adically iterate the connection ∇. When $\mathbf{f}, \mathbf{g}, \mathbf{h}$ are Hida families, as we saw in above definition, one may replace ∇ by $\theta = q\frac{d}{dq}$ and works only on q-expansions. Then there is a straightforward way to iterate θ to

p-adic powers, provided that the form it acts on lies in the kernel of U (e.g. $\mathbf{g}^{[p]}$). The argument for ∇ is more complicated and will be recalled in §4.8.

The triple product p-adic L-function $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ can be viewed as a three variable function

$$\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) : \Omega_f \times \Omega_g \times \Omega_h \to \mathbb{C}_p,$$

whose value at $(x, y, z) \in \Omega_f \times \Omega_g \times \Omega_h$ is

$$\frac{(\mathbf{f}_x^*, \nabla^{-t}\mathbf{g}_y^{[p]} \times \mathbf{h}_z)}{(\mathbf{f}_x^*, \mathbf{f}_x^*)},$$

where $t \in \Omega$ satisfies $\kappa(x) = \kappa(y) + \kappa(z) - 2t$. As one easily observes, the form $\mathbf{f}$ (or $\mathbf{f}^*$) plays a distinct role in $\mathscr{L}_p^f$. What is not so obvious is that $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is symmetric, up to a $\pm$ sign, in the last two components. Similarly, one can define $\mathscr{L}_p^g = \mathscr{L}_p^g(\mathbf{g}, \mathbf{f}, \mathbf{h})$ and $\mathscr{L}_p^h$.

Set $\Sigma = \Omega_f \times \Omega_g \times \Omega_h$ and $\Sigma_{\mathrm{cl}} = \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$. The set Σ_{cl} can be naturally divided into four disjoint subsets:

$$\begin{aligned}
\Sigma_{\mathrm{bal}} &:= \{(x, y, z) \in \Sigma_{\mathrm{cl}} \mid (\kappa(x), \kappa(y), \kappa(z)) \text{ is balanced}\};\\
\Sigma_f &:= \{(x, y, z) \in \Sigma_{\mathrm{cl}} \mid \kappa(x) \geq \kappa(y) + \kappa(z)\};\\
\Sigma_g &:= \{(x, y, z) \in \Sigma_{\mathrm{cl}} \mid \kappa(y) \geq \kappa(x) + \kappa(z)\};\\
\Sigma_h &:= \{(x, y, z) \in \Sigma_{\mathrm{cl}} \mid \kappa(z) \geq \kappa(x) + \kappa(y)\}.
\end{aligned}$$

The p-adic L-function $\mathscr{L}_p^f$ on Σ_f interpolates the *square roots* of *central critical values* of the classical L-functions $L(f_x^0, g_y^0, h_z^0; s)$, which is a combined result of the Ichino formula (c.f. Theorem 2.24), [DR14, Theorem 4.7] and [AI21, Corollary 5.13].

Points in Σ_{bal}, on the other hand, lie outside this region of interpolation. By definition, the value of $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at a balanced point (x, y, z) involves a "negative" power ∇^{-t}, which is originally defined as a p-adic limit. On the other hand, the interpolating property implies that ∇^{-t} satisfies $\nabla^s \circ \nabla^{-t} = \nabla^{-t+s}$. In particular, this negative power also serves as the t-th anti-derivative with respect to ∇. Hence, we may expect to express these special values via an integration theory. A suitable candidate is Coleman p-adic integration or its generalization, Besser's finite polynomial cohomology. In particular, in [Bes00a], Besser showed that one can interpret the p-adic Abel–Jacobi map as a p-adic integration, which is in analogue to the classical case.

Now fix a balanced weight $(x, y, z) \in \Sigma_{\mathrm{bal}}$ and write $(x, y, z) = (r_1 + 2, r_2 + 2, r_3 + 2)$ as before. We let $r := \frac{1}{2}(r_1 + r_2 + r_3)$. Assume that the specializations $\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z$ are p-stabilizations of f_x^0, g_y^0, h_z^0, respectively. For simplicity, we will further assume that they are newforms of the same level N in this introduction.

To any classical newform ϕ of weight $k = s + 2$ on $\Gamma_1(N)$, there is a cohomology class

$$\omega_\phi \in \mathrm{Fil}^{s+1} H_{\mathrm{dR}}^{s+1}(W_s/\overline{\mathbb{Q}}) \subset H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p).$$

The ϕ-isotypic part $H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi]$ is two-dimensional and contains ω_ϕ. If ϕ is ordinary at p, then there is a one-dimensional unit root subspace $H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi]^{\mathrm{u\text{-}r}}$, on which the Frobenius endomorphism acts as multiplication by a p-adic unit. Moreover, one has a direct sum decomposition

$$H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi] = \mathrm{Fil}^{s+1} H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi] \oplus H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi]^{\mathrm{u\text{-}r}},$$

which is well known as the unit root splitting. When ϕ has finite slope $a \in \mathbb{Q}$, a similar result holds if $a < k - 1 = s + 1$. To be more precise, one has a direct sum decomposition

$$H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi] = \mathrm{Fil}^{s+1} H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi] \oplus H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi]^a$$

where $H_{\mathrm{dR}}^{s+1}(W_s/\mathbb{C}_p)[\phi]^a$ is the one-dimensional subspace on which the Frobenius acts as multiplication by an element of p-adic valuation a. As a result of Poincaré duality, there is a unique element $\eta_\phi^a \in$

$H^{s+1}_{\mathrm{dR}}(W_s/\mathbb{C}_p)[\phi]^a$ such that for any cusp form ω on $\Gamma_1(N)$ of weight k, we have

$$\langle \eta^a_\phi, \omega\rangle := \frac{(\phi^*, \omega)}{(\phi^*, \phi^*)},$$

where $\langle\ ,\ \rangle$ denotes the non-degenerate Poincaré pairing on $H^{s+1}_{\mathrm{dR}}(W_s/\mathbb{C}_p)$, ϕ^* is the Atkin–Lehner involution of ϕ, and $(\ ,\ )$ denotes the Petersson inner product of level $\Gamma_1(N)$ and weight k (c.f. 2.1).

Suppose that the family $\mathbf{f}$ is of slope a and $a < x-1$. Then we construct an element (associated with (f^0_x, g^0_y, h^0_z))

$$\begin{aligned}\eta^a_f \otimes \omega_g \otimes \omega_h &\in H^{r_1+1}_{\mathrm{dR}}(W_{r_1}) \otimes \mathrm{Fil}^{r_2+1} H^{r_2+1}_{\mathrm{dR}}(W_{r_2}) \otimes \mathrm{Fil}^{r_3+1} H^{r_3+1}_{\mathrm{dR}}(W_{r_3})\\ &\subset \mathrm{Fil}^{r+2}(H^{r_1+1}_{\mathrm{dR}}(W_{r_1}) \otimes H^{r_2+1}_{\mathrm{dR}}(W_{r_2}) \otimes H^{r_3+1}_{\mathrm{dR}}(W_{r_3}))\\ &\subset \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W^*/\mathbb{C}_p),\end{aligned}$$

where the first inclusion is by the balancedness assumption, and the second one is from the Künneth decomposition of $W^* = W_{r_1} \times W_{r_2} \times W_{r_3}$. In particular, $\eta^a_f \otimes \omega_g \otimes \omega_h$ lies inside the domain of $\mathrm{AJ}_p(\Delta_{x,y,z})$.

Lastly, we need to introduce several Euler factors. For any $\phi \in S_k(\Gamma_1(N), \chi)$, we shall write the Hecke polynomial

$$x^2 - a_p(\phi)x + \chi(p)p^{k-1} = (x-\alpha_\phi)(x-\beta_\phi)$$

with $\mathrm{ord}_p(\alpha_\phi) \le \mathrm{ord}_p(\beta_\phi)$. For the modular form f^0_x associated with f_x, we also assume that $a = \mathrm{ord}_p(\alpha_{f^0_x})$. For simplicity, we will write α_f for $\alpha_{f^0_x}$ and similarly for other modular forms.

The main result of this thesis is the theorem below.

Theorem 1.1. *Given $(x,y,z) \in \Sigma_{\mathrm{bal}}$. Let $c := (x+y+z-2)/2$ and write $x = y+z-2t$ with $t > 0$. Then*

$$\mathscr{L}^f_p(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = (-1)^{t-1}\frac{\mathscr{E}(f^0_x, g^0_y, h^0_z)}{(t-1)!\mathscr{E}_0(f^0_x)\mathscr{E}_1(f^0_x)}\mathrm{AJ}_p(\Delta_{x,y,z})(\eta^a_f \otimes \omega_g \otimes \omega_h), \tag{1}$$

where the Euler factors are given by

$$\begin{aligned}\mathscr{E}_0(f^0_x) &:= 1 - \beta_f^2\chi_f^{-1}(p)p^{1-x},\\ \mathscr{E}_1(f^0_x) &:= 1 - \beta_f^2\chi_f^{-1}(p)p^{-x},\\ \mathscr{E}(f^0_x, g^0_y, h^0_z) &:= (1-\beta_f\alpha_g\alpha_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c}).\end{aligned} \tag{2}$$

2 Preliminaries

As the theory of modular forms is a vital ingredient in this thesis, we will set up the notations and recall several facts for future use. Some notable results are the relations between modular forms and cohomology groups, overconvergent modular forms, and Hida families.

Then we proceed to recall the work of H. Darmon and V. Rotger on the p-adic Gross–Zagier formula for triple product p-adic L-functions associated with Hida families. Few proofs will be given in this part since our results for the finite slope cases will naturally cover the ordinary case. Nevertheless, we think that giving a picture of the proof for the p-adic Gross–Zagier formula first will help readers to understand this thesis better.

2.1 Modular forms and cohomology

Classical modular forms. Fix an integer $N \geq 4$ and a prime $p \nmid N$. Let $Y = Y_1(N)$ be the affine modular curve of level $\Gamma_1(N)$ defined over $\mathbb{Q}_p$ and $X = X_1(N)$ be the compactified modular curve. We let $C := X \backslash Y$ be the subscheme of cusps. The modular curve X is proper smooth over $\mathbb{Q}_p$, and it has a smooth model $\mathcal{X}$ defined over $\mathbb{Z}_p$.

Let $\pi : E \to Y$ be the universal elliptic curve and $\omega_E := \pi_* \Omega^1_{E/Y}$ denote the invertible sheaf (line bundle) on Y. The sheaf ω_E extends canonically to a sheaf over X, which is still denoted by ω_E. To be more precise, around a cusp, we have

$$H^0(\operatorname{Spec} \mathbb{Q}_p[[q]], \omega_E) = \mathbb{Q}_p[[q]] \cdot \omega_{\mathrm{can}}$$

where $\omega_{\mathrm{can}} := \frac{dt}{t}$ is the canonical differential on the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$.

Let $\mathbb{Q}_p \subset K \subset \mathbb{C}_p$ be a tower of field extensions. A modular form f on X of weight $k \geq 0$ with Fourier coefficients in K then corresponds to a global section ω_f of the sheaf ω_E^k over X_K (c.f. [Kat73]). If f is a cusp form of weight $k \geq 2$, then ω_f can be further identified as a section of $\omega_E^{k-2} \otimes \Omega^1_X$. The notation $k = r + 2$ will be frequently used.

Let

$$\mathcal{H} := \mathbb{R}^1 \pi_* \Omega^\bullet_{E/Y}$$

be the sheaf of relative de Rham cohomology on Y, which also extends canonically to a coherent sheaf of rank 2 on X and will be denoted by the same notation. Alternatively, the extensions of ω_E and $\mathcal{H}$ to X can be described as

$$\begin{aligned} \omega_E &= \pi_* \Omega^1_E(\log \pi^{-1}(C)), \\ \mathcal{H} &= \mathbb{R}^1 \pi_* \Omega^1_E(\log \pi^{-1}(C)). \end{aligned}$$

The sheaf $\mathcal{H}$ is equipped with a Hodge filtration

$$0 \to \omega_E \to \mathcal{H} \to \omega_E^{-1} \to 0 \tag{3}$$

and a Gauss–Manin connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_X(\log C).$$

For any $r \in \mathbb{N}$, the r-th symmetric power $\mathcal{H}^r := \operatorname{Sym}^r \mathcal{H}$ is of rank $r + 1$ and is equipped with the induced Hodge filtration and Gauss–Manin connection. Locally around a cusp, we have

$$H^0(\operatorname{Spec} \mathbb{Q}_p[[q]], \mathcal{H}^r) = \mathbb{Q}_p[[q]]\omega^r_{\mathrm{can}} + \mathbb{Q}_p[[q]]\omega^{r-1}_{\mathrm{can}}\eta_{\mathrm{can}} + \cdots + \mathbb{Q}_p[[q]]\eta^r_{\mathrm{can}}$$

where $\eta_{\mathrm{can}} := \nabla(q\frac{d}{dq})(\omega_{\mathrm{can}})$ and the connection ∇ is given by

$$\nabla \omega_{\mathrm{can}} = \eta_{\mathrm{can}} \otimes \frac{dq}{q}, \quad \nabla \eta_{\mathrm{can}} = 0$$

together with the Leibniz's rule.

The Hodge filtration on $\mathcal{H}^r$ is decreasing with successive quotients

$$\mathcal{H}^r / \operatorname{Fil}^1 \mathcal{H}^r \cong \omega_E^{-r}, \operatorname{Fil}^1 \mathcal{H}^r / \operatorname{Fil}^2 \mathcal{H}^r \cong \omega_E^{-r+2}, \dots, \operatorname{Fil}^r \mathcal{H}^r \cong \omega_E^r.$$

The connection ∇ satisfies Griffiths transversality

$$\nabla(\operatorname{Fil}^i \mathcal{H}^r) \subset \operatorname{Fil}^{i-1} \mathcal{H}^r \otimes \Omega^1_X(\log C)$$

and further induces isomorphisms

$$\nabla : \frac{\operatorname{Fil}^i \mathcal{H}^r}{\operatorname{Fil}^{i+1} \mathcal{H}^r} \cong \frac{\operatorname{Fil}^{i-1} \mathcal{H}^r}{\operatorname{Fil}^i \mathcal{H}^r} \otimes \Omega^1_X(\log C) \tag{4}$$

of $\mathcal{O}_X$-modules. In particular, when $r = 1$ and $i = 1$, it gives rise to the Kodaira–Spencer isomorphism

$$\mathrm{KS} : \omega_E^2 \cong \Omega^1_X(\log C).$$

Next, we would like study the cohomology of $\mathcal{H}^r$. However, instead of using the complex $\mathcal{H}^r \to \mathcal{H}^r \otimes \Omega^1_X(\log C)$, we need a modified version of it.

Definition 2.1. ([Sch85, § 2]) The parabolic complex $(\mathcal{H}^r \otimes \Omega^\bullet_X)_{\mathrm{par}}$ is a subcomplex of

$$0 \to \mathcal{H}^r \to \mathcal{H}^r \otimes \Omega^1_X(\log C) \to 0$$

defined by

$$\begin{aligned}
(\mathcal{H}^r \otimes \Omega^0_X)_{\mathrm{par}} &:= \mathcal{H}^r, \\
(\mathcal{H}^r \otimes \Omega^1_X)_{\mathrm{par}} &:= \nabla(\mathcal{H}^r) + \mathcal{H}^r \otimes \Omega^1_X.
\end{aligned}$$

Locally around a cusp, one has

$$H^0(\operatorname{Spec} \mathbb{Q}_p[[q]], (\mathcal{H}^r \otimes \Omega^1_X)_{\mathrm{par}}) = \left(\mathbb{Q}_p[[q]]\omega^r_{\mathrm{can}} + \mathbb{Q}_p[[q]]\omega^{r-1}_{\mathrm{can}}\eta_{\mathrm{can}} + \cdots + q\mathbb{Q}_p[[q]]\eta^r_{\mathrm{can}}\right) \frac{dq}{q}.$$

The hypercohomology of $(\mathcal{H}^r \otimes \Omega^\bullet_X)_{\mathrm{par}}$ on X_K will be denoted by $H^i_{\mathrm{par}}(X_K, \mathcal{H}^r)$.

Remark 2.2. By examining the graded pieces and using the isomorphism (4), we have an isomorphism (c.f. [Sch85, § 2])

$$H^0(X_K, \omega_E^r \otimes \Omega^1_X) \cong \frac{H^0(X_K, (\mathcal{H}^r \otimes \Omega^1_X)_{\mathrm{par}})}{\nabla H^0(X_K, \mathcal{H}^r)}. \tag{5}$$

The parabolic cohomology $H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)$ is equipped with a short exact sequence

$$0 \to H^0(X_K, \omega_E^r \otimes \Omega^1_X) \to H^1_{\mathrm{par}}(X_K, \mathcal{H}^r) \to H^1(X_K, \omega_E^{-r}) \to 0, \tag{6}$$

and the Hodge filtration on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$ is given by

$$\begin{aligned}
&\operatorname{Fil}^0 = H^1_{\mathrm{par}}(X_K, \mathcal{H}^r), \\
&\operatorname{Fil}^1 = \operatorname{Fil}^2 = \cdots = \operatorname{Fil}^{r+1} = H^0(X_K, \omega_E^r \otimes \Omega^1_X), \\
&\operatorname{Fil}^{r+2} = 0.
\end{aligned}$$

As we will see in §5, there is an action of Frobenius ϕ on $H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)$. This makes the vector space $H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)$ into a filtered Frobenius module.

We also remark that there is a non-degenerate Poincaré pairing

$$\langle\ ,\ \rangle : H^1_{\mathrm{par}}(X_K, \mathcal{H}^r) \times H^1_{\mathrm{par}}(X_K, \mathcal{H}^r) \to K(-1-r) \tag{7}$$

where $(-1-r)$ denotes the Tate twist of a filtered Frobenius module (c.f. §5).

p-adic and overconvergent modular forms. In this part, we will assume some basic knowledge of rigid analytic geometry. A beginner friendly reference would be [FP03].

Let $P_1, \dots, P_s \in \bar{X}(\mathbb{F}_{p^2})$ denote the supersingular points of the special fiber of $\mathcal{X}$ (also called the reduction of X). These points are the zeros of the Hasse invariant $\overline{\mathrm{Ha}}$, which is a mod p modular form of weight $p-1$. We may choose lifts $\tilde{P}_1, \dots \tilde{P}_s \in \mathcal{X}(\mathbb{Z}_{p^2})$ of the supersingular points. For example, when $p \geq 5$, one may simply take the zeros of the Eisenstein series E_{p-1}. We let $\mathcal{X}' := \mathcal{X} - \{\tilde{P}_1, \dots \tilde{P}_s\}$ be the resulting affine curve over $\mathbb{Z}_p$, whose generic fiber and special fiber will be denoted X' and $\bar{X}'$ respectively.

Now view $X(\mathbb{C}_p)$ as a rigid analytic space X^{an} and let $\mathrm{red} : X^{\mathrm{an}} \to \bar{X}$ be the reduction map. Then $\mathcal{A} = \mathcal{A}_{\mathrm{ord}} := \mathrm{red}^{-1}(\bar{X}')$ is a connected affinoid, called the ordinary locus. Let Ha be a lift of the Hasse invariant (e.g. E_{p-1} when $p \geq 5$). One has the following description of the ordinary locus

$$\mathcal{A} = \{x \in X(\mathbb{C}_p) \mid \mathrm{ord}_p \mathrm{Ha}(x) = 0\}.$$

For a real number $\epsilon > 0$, we define

$$W_\epsilon := \{x \in X(\mathbb{C}_p) \mid \mathrm{ord}_p \mathrm{Ha}(x) < \epsilon\}.$$

When $0 < \epsilon < 1$, the region W_ϵ is independent of the choice of lifts of the Hasse invariant. Such a W_ϵ is called a strict neighborhood or a wide open neighborhood of $\mathcal{A}$ in X^{an}. If $\mathrm{ord}_p \mathrm{Ha}(x) < \frac{p}{p+1}$, then the elliptic curve A_x for which x represents admits a canonical subgroup Z_x. This allows us to define a map Φ on W_ϵ for $\epsilon < \frac{p}{p+1}$, whose restriction to $\mathcal{A}$ is a lift of the Frobenius on $\bar{X}'$ (c.f. [Kat73] or [DR14, § 2.1]).

By abuse of notations, we will let Ω^1_X denote its corresponding rigid analytic sheaf on X^{an} and similarly for $\omega_E^r, \mathcal{H}^r$. Let K be a complete subfield of $\mathbb{C}_p$, we let

$$S_2^p(N; K) := H^0(\mathcal{A}/K, \Omega^1_X), \quad S_2^{\mathrm{oc}}(N; K) := \varinjlim_{\epsilon > 0} H^0(W_\epsilon/K, \Omega^1_X)$$

be the spaces of p-adic cusp forms and overconvergent cusp forms of weight two with coefficients in K. For $k > 2$, we let

$$S_k^p(N; K) := H^0(\mathcal{A}/K, \omega_E^{k-2} \otimes \Omega^1_X), \quad S_k^{\mathrm{oc}}(N; K) := \varinjlim_{\epsilon > 0} H^0(W_\epsilon/K, \omega_E^{k-2} \otimes \Omega^1_X).$$

When $K = \mathbb{C}_p$, we will simply write $S_k^p(N)$ and $S_k^{\mathrm{oc}}(N)$. Notice that by restriction, we may identify $S_k^{\mathrm{oc}}(N; K) \subset S_k^p(N; K)$ as a subspace.

For any $f \in S_k^p(N; K)$, we have the q-expansion

$$f(q) = \sum_{n \geq 1} a_n q^n$$

of f, which uniquely determines f (c.f. [Kat73]). We have two non-commuting operators U and V on $f \in S_k^p(N; K)$, defined via q-exapnsions by

$$(Uf)(q) = \sum_{n \geq 1} a_{pn} q^n, \quad (Vf)(q) = \sum_{n \geq 1} a_n q^{pn}.$$

Obviously, these operators satisfy

$$UVf = f, \quad VUf(q) = \sum_{n \geq 1} a_{pn} q^{pn}.$$

We define the p-depletion of f to be

$$f^{[p]} := (1 - VU)f, \text{ with } f^{[p]}(q) = \sum_{p \nmid n} a_n q^n.$$

The operator U restricted to the p-adic Banach space $S_k^{\mathrm{oc}}(N)$ is a compact (or completely continuous), which gives rise to a slope decomposition of the infinite-dimensional vector space $S_k^{\mathrm{oc}}(N)$ (c.f. §7).

Definition 2.3. An overconvergent modular form is said to be ordinary if it belongs to the slope 0 subspace for U. We let $S_k^{\mathrm{ord}}(N)$ denote the subspace of ordinary modular forms. We also recall Hida's ordinary projector

$$e_{\mathrm{ord}} := \lim_{n\to\infty} U^{n!},$$

which gives a Hecke equivariant projection from $S_k^p(N)$ tp $S_k^{\mathrm{ord}}(N)$.

Nearly holomorphic and nearly overconvergent modular forms. All the modular forms we mentioned before are sections of ω_E^k and $\omega_E^r \otimes \Omega_X^1$. In this part, we will shift our attention to sections of $\mathcal{H}^r$ and introduce some differential operators. A more complete study can be found in [Urb14].

We first start with the complex case $K = \mathbb{C}$. Hodge theory provides a canonical real analytic, but non-holomorphic splitting

$$\mathrm{Spl}_{\mathrm{Hdg}} : \mathcal{H} \to \omega_E$$

of the exact sequence (3) over X, or equivalently, a decomposition $\mathcal{H} = \omega_E \oplus \overline{\omega}_E$. We also let the same symbol denote the induced map $\mathcal{H}^r \to \omega_E^r$ as well as the following map

$$\mathrm{Spl}_{\mathrm{Hdg}} : H^0(X_{\mathbb{C}}, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \to H^0(X(\mathbb{C})_{\mathrm{an}}, (\omega_E^r \otimes \Omega_X^1)_{\mathrm{par}}) \tag{8}$$

where by $X(\mathbb{C})_{\mathrm{an}}$ we means the real analytic structure of $X(\mathbb{C})$.

Definition 2.4. The image of $\mathrm{Spl}_{\mathrm{Hdg}}$ in (8) is called the space of nearly holomorphic cusp forms of weight $k = r+2$ on $\Gamma_1(N)$ and will be denoted by $S_k^{\mathrm{nh}}(N;\mathbb{C})$.

We will recall, mostly without proof, the following facts about nearly holomorphic forms.

1. The map $\mathrm{Spl}_{\mathrm{Hdg}}$ in (8) is injective, and hence induces an isomorphism of complex vector spaces

   $$H^0(X_{\mathbb{C}}, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \cong S_k^{\mathrm{nh}}(N;\mathbb{C}).$$

   As a result, we will also call elements in $H^0(X_{\mathbb{C}}, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ nearly holomorphic cusp forms.

2. For any subfield K of $\mathbb{C}$, the image of $H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ under $\mathrm{Spl}_{\mathrm{Hdg}}$ yields a natural K-structure on $S_k^{\mathrm{nh}}(N;\mathbb{C})$ and is denoted by $S_k^{\mathrm{nh}}(N;K)$.

3. Let $\phi \in H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$, then equation (5) allows us to write

   $$\phi = \Pi^{\mathrm{holo}}(\phi) + \nabla s, \quad \text{with } \begin{cases} \Pi^{\mathrm{holo}}(\phi) \in H^0(X_K, \omega_E^r \otimes \Omega_X^1) = S_k(N;K), \\ s \in H^0(X_K, \mathcal{H}^r). \end{cases} \tag{9}$$

   The modular form $\Pi^{\mathrm{holo}}(\phi)$ is called the holomorphic projection of ϕ.

4. By composing with the inverse of Kodaira–Spencer isomorphism and the natural morphism $\omega_E^2 \otimes \mathcal{H}^r \hookrightarrow \mathcal{H}^2 \otimes \mathcal{H}^r \to \mathcal{H}^{r+2}$, one may view the Gauss–Manin connection as a map

   $$\tilde{\nabla} : H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \to H^0(X_K, (\mathcal{H}^{r+2} \otimes \Omega_X^1)_{\mathrm{par}}).$$

   This map is related to the weight k Shimura–Maass differential operator $\delta_k = \frac{1}{2\pi i}(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}})$ via the following commutative diagram

   $$\begin{tikzcd} H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\mathrm{Hdg}}"] \arrow[d, "\tilde{\nabla}"'] & S_k^{\mathrm{nh}}(N;K) \arrow[d, "\delta_k"] \\ H^0(X_K, (\mathcal{H}^{r+2} \otimes \Omega_X^1)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\mathrm{Hdg}}"] & S_{k+2}^{\mathrm{nh}}(N;K). \end{tikzcd}$$

This follows from a direct computation. Recall that in terms of the standard coordinates $\tau \in \mathbb{H}$ and $z \in \mathbb{C}/\langle 1, \tau \rangle$ (c.f. [Urb14, § 2.2.1]), we have

$$\nabla(2\pi i dz) = 2\pi i \left(\frac{dz - d\bar{z}}{\tau - \bar{\tau}} \right) \otimes d\tau, \quad \nabla(d\bar{z}) = 0, \quad \mathrm{KS}((2\pi i dz)^{\otimes 2}) = 2\pi i d\tau.$$

One can then take the successive composition $\delta_k^t := \delta_{k+2t-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k$ for all $t \in \mathbb{N}$, which sends $S_k^{\mathrm{nh}}(N;K)$ to $S_{k+2t}^{\mathrm{nh}}(N;K)$. In particular, when $g \in S_\ell(N;K)$ and $h \in S_m(N;K)$, the product $\delta_\ell^t g \times h$ belongs to $S_{\ell+2t+m}^{\mathrm{nh}}(N;K)$. More precisely, it is the image of the element

$$\tilde{\nabla}^t(\omega_g) \otimes \omega_h \in H^0(X_K, (\mathcal{H}^{\ell+2t+m} \otimes \Omega_X^1)_{\mathrm{par}})$$

under the splitting $\mathrm{Spl}_{\mathrm{Hdg}}$.

5. Hodge theory also gives a canonical splitting of the exact sequence (6):

$$H^1_{\mathrm{par}}(X_{\mathbb{C}}, \mathcal{H}^r) = H^0(X_{\mathbb{C}}, \omega_E^r \otimes \Omega_X^1) \oplus \overline{H^0(X_{\mathbb{C}}, \omega_E^r \otimes \Omega_X^1)}.$$

One may then define the Petersson scalar product $(\ ,\ )_N$ on $S_k^{\mathrm{nh}}(N;K)$ by the rather unusual rule

$$(f_1, f_2)_N := \int_{\Gamma_1(N)\backslash\mathbb{H}} \overline{f_1(\tau)} f_2(\tau) y^k \frac{dxdy}{y^2} \tag{10}$$

where $\tau := x + iy$. Notice that we follow the definition in [DR14]. This product is Hermitian-linear in the first component and $\mathbb{C}$-linear in the second component.

Lemma 2.5. *For all $\eta \in S_k(N;\mathbb{C})$ and $\phi \in S_k^{\mathrm{nh}}(N;\mathbb{C})$, we have*

$$(\eta, \phi)_N = (\eta, \Pi^{\mathrm{holo}}(\phi))_N.$$

This equation may serve as an alternative definition of the holomorphic projection.

Proof. As $X_{\mathbb{C}}$ is a compact manifold, the class of $\nabla(s)$ is zero in $H^1_{\mathrm{par}}(X_{\mathbb{C}}, \mathcal{H}^r)$ for all $s \in H^0(X_{\mathbb{C}}, \mathcal{H}^r)$. The result then follows from the relation between the Petersson product and the Poincaré pairing. ☐

We now consider the p-adic case. Recall that we have the ordinary locus $\mathcal{A}$ and a system of strict neighborhoods $W_\epsilon \supset \mathcal{A}$. Since W_ϵ is a Stein space, the hyper-cohomology $\mathbb{H}^i(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^\bullet)_{\mathrm{par}})$ (with respect to the rigid analytic topology) can be computed by the complex of global sections. In other words, we have

$$H^1_{\mathrm{rig,par}}(W_\epsilon, \mathcal{H}^r) := \mathbb{H}^1(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^\bullet)_{\mathrm{par}}) = \frac{H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})}{\nabla H^0(W_\epsilon, \mathcal{H}^r)}. \tag{11}$$

The comparison theorem between rigid cohomology and de Rham cohomology then implies that

$$H^1_{\mathrm{rig,par}}(W_\epsilon, \mathcal{H}^r) \cong H^1_{\mathrm{par}}(X'_{\mathbb{C}_p}, \mathcal{H}^r)$$

for all $0 < \epsilon < 1$. In particular, the space $H^1_{\mathrm{rig,par}}(W_\epsilon, \mathcal{H}^r)$ is independent of ϵ.

It is known (c.f. [Col95]) that for an overconvergent modular form $s \in H^0(W_\epsilon, \omega_E^{-r})$ of weight $-r$, it admits a unique lift $\tilde{s} \in H^0(W_\epsilon, \mathcal{H}^r)$ under the projection $H^0(W_\epsilon, \mathcal{H}^r) \to H^0(W_\epsilon, \omega_E^{-r})$ satisfying $\nabla \tilde{s} \in H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1)$. Moreover, the section $\nabla\tilde{s}$ corresponds to the overconvergent form $\theta^{r+1}s$ where $\theta = q\frac{d}{dq}$ is Serre's operator. Notice that θ sends p-adic modular forms of weight k to p-adic modular forms of weight $k+2$ and in general does not preserve overconvergence.

Again by (4), any rigid analytic section $\phi \in H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ can be written as

$$\phi = \phi_0 + \nabla s, \text{ with } \phi_0 \in H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1), \quad s \in H^0(W_\epsilon, \mathcal{H}^r). \tag{12}$$

Hence one may rewrite (11) as

$$H^1_{\mathrm{rig,par}}(W_\epsilon, \mathcal{H}^r) = \frac{H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1)}{\nabla H^0(W_\epsilon, \mathcal{H}^r) \cap H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1)} = \frac{S^{\mathrm{oc}}_{r+2}(N)}{\theta^{r+1} S^{\mathrm{oc}}_{-r}(N)}. \tag{13}$$

Over $\mathcal{A}$, the slope decomposition with respect to the action of Frobenius gives the unit-root splitting

$$\mathrm{Spl}_{\text{u-r}} : \mathcal{H} \to \omega_E$$

of the exact sequence (3). We will use the same notation to denote the associated map $\mathrm{Spl}_{\text{u-r}} : \mathcal{H}^r \to \omega_E^r$ and

$$\mathrm{Spl}_{\text{u-r}} : \varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \to H^0(\mathcal{A}, \omega_E^r \otimes \Omega_X^1) \tag{14}$$

via composing with the restriction map.

Definition 2.6. The image of the map $\mathrm{Spl}_{\text{u-r}}$ in (14), denoted by $S_k^{\text{n-oc}}(N; \mathbb{C}_p)$, is called the space of nearly overconvergent modular forms of weight k on $\Gamma_1(N)$.

Remark 2.7. The space $S_k^{\text{n-oc}}(N; \mathbb{C}_p)$ is contained in the space of p-adic modular forms by definition. It contains the space of overconvergent modular forms by the inclusion $H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1) \subset H^0(W_\epsilon, \mathcal{H}^r \otimes \Omega_X^1)$. But the two spaces are not equal, since the unit root splitting does not extend to any strict neighborhood W_ϵ (c.f. [Urb14, Proposition 3.1.3]).

We recall the following important facts about nearly overconvergent modular forms, which are analogous to those in the complex setting:

1. The map $\mathrm{Spl}_{\text{u-r}}$ in (14) is injective and induces an isomorphism of p-adic Fréchet spaces (c.f. [CGJ95]):

$$\mathrm{Spl}_{\text{u-r}} : \varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \xrightarrow{\sim} S_k^{\text{n-oc}}(N; \mathbb{C}_p).$$

   As a results, we will also call elements in $\varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ nearly overconvergent modular forms.

2. If K is any subfield of $\mathbb{C}_p$, the image of $\varinjlim_{\epsilon>0} H^0(W_\epsilon/K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ under $\mathrm{Spl}_{\text{u-r}}$ yields a natural K-vector subspace $S_k^{\text{n-oc}}(N; K) \subset S_k^{\text{n-oc}}(N; \mathbb{C}_p)$.

3. Let $\phi \in \varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$. As we already mentioned, one can write

$$\phi = \phi_0 + \nabla s, \text{ with } \phi_0 \in H^0(W_\epsilon, \omega_E^r \otimes \Omega_X^1), \quad s \in H^0(W_\epsilon, \mathcal{H}^r).$$

   The overconvergent modular forms $\Pi^{\mathrm{oc}}(\phi) := \phi_0$ is called the overconvergent projection of the nearly overconvergent modular form ϕ. Note that $\Pi^{\mathrm{oc}}(\phi)$ is only well-defined modulo $\theta^{r+1}(S^{\mathrm{oc}}_{-r}(N))$.

4. The map $\tilde{\nabla}$ corresponds under $\mathrm{Spl}_{\text{u-r}}$ to the operator $\theta = q\frac{d}{dq}$. That is, we have the commutative diagram

$$\begin{tikzcd}
\varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\text{u-r}}"] \arrow[d, "\tilde{\nabla}"'] & S_k^{\text{n-oc}}(N; \mathbb{C}_p) \arrow[d, "\theta"] \\
\varinjlim_{\epsilon>0} H^0(W_\epsilon, (\mathcal{H}^{r+2} \otimes \Omega_X^1)_{\mathrm{par}}) \arrow[r, "\mathrm{Spl}_{\text{u-r}}"] & S_{k+2}^{\text{n-oc}}(N; \mathbb{C}_p).
\end{tikzcd}$$

5. Suppose K is equipped with embeddings into $\mathbb{C}$ and $\mathbb{C}_p$. Then there are natural maps

$$S_k^{\mathrm{nh}}(N; K) \xleftarrow{\mathrm{Spl}_{\mathrm{Hdg}}} H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}}) \xrightarrow{\mathrm{Spl}_{\text{u-r}}} S_k^{\text{n-oc}}(N; K).$$

   By definition, the images of $H^0(X_K, (\mathcal{H}^r \otimes \Omega_X^1)_{\mathrm{par}})$ under the holomorphic and overconvergent projections both take values in $S_k(N, K)$ and are equal.

The following lemma studies the relation between Hida's ordinary projection and the overconvergent projection.

Lemma 2.8. *Let ϕ be a nearly overconvergent modular form on $\Gamma_1(N)$ of weight $k \geq 2$. Its image under e_{ord} is overconvergent, and is thus classical on $\Gamma_1(N) \cap \Gamma_0(p)$ (c.f. [Col95]). Moreover, we have*

$$e_{\mathrm{ord}}\phi = e_{\mathrm{ord}}\Pi_N^{\mathrm{oc}}(\phi). \tag{15}$$

Proof. We first write $\phi = \phi_0 + \nabla s$ as in the definition of the overconvergent projection. Then we observe that e_{ord} annihilate ∇s by looking at the q-expansion. More precisely, e_{ord} annihilates $\theta(\mathbb{C}_p \otimes \mathcal{O}_{\mathbb{C}_p}[[q]])$. The result hence follows. □

Let $g \in S_\ell(N;K)$ and $h \in S_m(N;K)$ be classical cusp forms defined over K, with fixed embeddings of K into $\mathbb{C}$ and $\mathbb{C}_p$. The forms g, h can be regarded as complex and overconvergent forms simultaneously. The proposition below relates $\theta^t g \times h$ to $\delta_\ell^t g \times h$.

Proposition 2.9 ([DR14, Proposition 2.8])**.** *For all $t \geq 0$, the modular form $\theta^t g \times h$ belongs to $S_{\ell+2t+m}^{n\text{-}oc}(N;K)$ and*

$$e_{\mathrm{ord}}(\theta^t g \times h) = e_{\mathrm{ord}}\Pi^{\mathrm{holo}}(\delta_\ell^t g \times h).$$

Proof. Observe that both modular forms come from the section $\tilde{\nabla}\omega_g \otimes \omega_h$. The result then follows from a direct computation together with the previous lemma. □

The situation becomes more intriguing when one wants to take negative power of θ. In order to do so, one needs to replace g by its p-depletion $g^{[p]} \in S_k^{\mathrm{oc}}(N;\mathbb{C}_p)$. The form $\theta^{-t} g^{[p]} \times h$ (with $t > 0$) is still a p-adic modular form of weight $k := \ell + m - 2t$. The following proposition shows that it is nearly overconvergent in certain cases.

Proposition 2.10. *Assume that $1 \leq t \leq \ell - 1$ so that $k := \ell + m - 2t \geq 2$. Then the p-adic modular form $\theta^{-t} g^{[p]} \times h$ belongs to $S_k^{n\text{-}oc}(N;\mathbb{C}_p)$, and in particular*

$$e_{\mathrm{ord}}(\theta^{-t} g^{[p]} \times h) \in S_k^{\mathrm{ord}}(N;\mathbb{C}_p) \subset S_k^{\mathrm{ord}}(\Gamma_1(N) \cap \Gamma_0(p);\mathbb{C}_p).$$

Proof. For a proof in the ordinary case, we refer it to [DR14, Proposition 2.9]. However, we will give a more general (but implicit) proof in Section 6. □

Periods of modular forms. Let $f \in S_k(N_f, \chi_f; K_f) \subset S_k(N_f; K_f)$ be an newform of nebentypus χ_f. It generates an automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. For any multiple N of N_f and any field extension $K_f \subset K \subset \mathbb{C}$, we let $S_k(N;K)[f] := S_k(N;K)[\pi_f]$ denote the f-isotypic subspace of $S_k(N;K)$, which consists of modular forms in $S_k(N;K)$ on which the Hecke operators T_ℓ for $(\ell, N_f) = 1$ and the diamond operators $\langle d \rangle$ for $d \in (\mathbb{N}/N\mathbb{N})^\times$ act with the same eigenvalues as those on f. The space $S_k(N;K)[f]$ is finite dimensional over K with dimension equal to $\sigma_0(N/N_f) := \sum_{0<d|(N/N_f)} 1$. A basis is given by $\{f(d \cdot \tau) = f(q^d)\}_{0<d|(N/N_f)}$. Similarly, one can define $H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)[f]$.

Definition 2.11. Suppose f is a cusp form of weight k, level N. We define the Atkin-Lehner involution of f to be

$$f^*(\tau) = (w_N f)(\tau) := i^k N^{-\frac{k}{2}} \tau^{-k} f(-1/N\tau).$$

It can be easily check that w_N is indeed an involution and the Hecke eigenvalues of $w_N(f)$ outside N are equal to those of f twisted by χ_f^{-1}. For this reason, we will often write $f \otimes \chi_f^{-1}$ to denote f^*. Moreover, w_N is self-adjoint. As a consequence, we have $(f,f)_N = (f^*, f^*)_N$ (c.f. [DS10, § 5.5, § 5.10]).

Lemma 2.12 ([DR14, Lemma 2.21])**.** *For any $\breve{f} \in S_k(N;K)[f]$ and all $\phi \in S_k(N;K)$, the Petersson scalar product $(\breve{f}^*, \phi)_N$ is a K-multiple of $(f^*, f^*)_N$.*

We then define the period to be

$$J(\breve{f}^*, \phi) := \frac{(\breve{f}^*, \phi)_N}{(f^*, f^*)_N} \in K.$$

Let

$$\eta_{\breve{f}}^{\text{a-h}} \in \overline{H^0(X_{\mathbb{C}}, \omega_E^r \otimes \Omega_X^1)} \subset H^1_{\text{par}}(X_{\mathbb{C}}, \mathcal{H}^r)$$

denote the class of $\frac{1}{\langle \omega_{\breve{f}}, \overline{\omega_{\breve{f}}} \rangle} \cdot \overline{\omega_{\breve{f}}}$ and $\overline{\eta_{\breve{f}}^{\text{a-h}}}$ be its natural image in $H^1(X_{\mathbb{C}}, \omega_E^{-r})$ under the projection in (6).

Corollary 2.13 ([DR14, Corollary 2.13])**.** *The class $\overline{\eta_{\breve{f}}^{a\text{-}h}}$ belongs to $H^1(X_{K_f}, \omega_E^{-r})$.*

We now give an alternative descriptions of $J(\breve{f}^*, \phi)$, following [BSV22, § 2]. The Poincaré pairing (7) descends to a perfect pairing

$$H^1_{\text{par}}(X_K, \mathcal{H}^r)[\breve{f}]/\operatorname{Fil}^{r+1} \times S_k(N; K)[\breve{f}^*] \to K.$$

The K-linear functional $J(\breve{f}^*, \cdot)$ on $S_k(N;K)[\breve{f}^*]$ then corresponds to a unique element

$$\overline{\eta_{\breve{f}}} \in H^1_{\text{par}}(X_K, \mathcal{H}^r)[\breve{f}]/\operatorname{Fil}^{r+1}.$$

Under the identification $H^1_{\text{par}}(X_K, \mathcal{H}^r)[\breve{f}]/\operatorname{Fil}^{r+1} \cong H^1(X_K, \omega_E^{-r})$, the class $\overline{\eta_{\breve{f}}}$ is sent to $\overline{\eta_{\breve{f}}^{\text{a-h}}}$.

Remark 2.14. Notice that $J(\breve{f}^*, \cdot)$ is K-linear because of our unusual choice of the Petersson scalar product. If one wished to use the usual Petersson scalar product, then one should exchange the positions of $\breve{f}^*$ and ϕ in the definition of J.

2.2 A quick recall on Hida families

Hida families. Let $f \in S_k(N_f, \chi_f; K_f) \subset S_k(N_f; K_f)$ be an eigenform of nebentypus χ_f. We factor the Hecke polynomial at a prime $p \nmid N_f$ as

$$T^2 - a_p(f)T + \chi_f(p)p^{x-1} = (T - \alpha_f)(T - \beta_f)$$

with $a := \operatorname{ord}_p(\alpha_f)$. In the ordinary setting, we will assume $a = 0$, *i.e.*, α_f is a p-adic unit. The ordinary p-stabilization of f is defined to be

$$f^{(p)} := (1 - \beta_f V_p)f,$$

on which the U_p acts with the eigenvalue α_f. We will use similar notations for $g \in S_\ell(N_g, \chi_g; K_g)$ and $h \in S_m(N_h, \chi_h; K_h)$. Suppose that K_f, K_g, K_h are contained in a field K with a fixed p-adic embedding $K \hookrightarrow \mathbb{C}_p$, and let $\mathcal{O}$ denote the ring of integers of the closure of K in $\mathbb{C}_p$.

Set $\Gamma = 1 + p\mathbb{Z}_p$ and $\Lambda = \mathcal{O}[[\Gamma]]$ be the completed group algebra. The weight space is defined to be

$$\Omega = \operatorname{Spf}(\Lambda)(\mathcal{O}) = \operatorname{Hom}_{\mathcal{O}\text{-alg}}(\Lambda, \mathcal{O}),$$

which can be naturally identified as the set $\operatorname{Hom}_{\text{cts}}(\Gamma, \mathcal{O}^\times)$ of continuous homomorphisms. The subset of classical weights is defined to be

$$\Omega_{\text{cl}} = \{\chi_k := \gamma \mapsto \gamma^k \mid k \in \mathbb{Z}_{\geq 2}\}.$$

For any finite flat extension Λ_f of Λ, let

$$\Omega_f := \operatorname{Spf}(\Lambda_f)(\mathcal{O}),$$

this space is endowed with a natural projection $\kappa : \Omega_f \to \Omega$ induced by the inclusion $\Lambda \subset \Lambda_f$. A point $x \in \Omega_f$ for which $\kappa(x) \in \Omega_{\text{cl}}$ is called a classical point. By abuse of notations, we will usually identify a classical point x with the integer $\kappa(x) \in \mathbb{Z}$ even though there may be multiple points sent to the same classical weight $\kappa(x)$.

We now introduce the notion of Hida families. We follow the definitions used in [DR14], which are slightly restrictive compared to definitions in other literature.

Definition 2.15. Let $N_f \geq 1$ be an integer and p be a prime not dividing N_f. A Hida family of tame level N_f is a quadruple $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$, where

1. Λ_f is a finite flat extension of Λ;
2. Ω_f is a non-empty open subset of $X_f := \mathrm{Hom}(\Lambda_f, \mathbb{C}_p)$ and $\Omega_{f,\mathrm{cl}}$ is a p-adically dense subset of Ω_f consisting of classical points;
3. $\mathbf{f} = \sum \mathbf{a}_n(\mathbf{f}) q^n \in \Lambda_f[[q]]$ is a formal power series with coefficients in Λ_f such that, for any $x \in \Omega_{f,\mathrm{cl}}$, the power series

$$\mathbf{f}_x(q) := \sum_{n=1}^{\infty} \mathbf{a}_n(\mathbf{f})_x q^n \in \mathbb{C}_p[[q]]$$

is the q-expansion of the ordinary p-stabilization of a normalized newform of weight $\kappa(x)$ on $\Gamma_1(N_f)$.

Remark 2.16. In order to lighten the typing load, the notation f_x will be frequently used to denote the specialization of $\mathbf{f}$ at x throughout this thesis. The corresponding normalized newform on $\Gamma_1(N_f)$ will then be denoted by f_x^0.

Theorem 2.17 (Hida)**.** *Let $f \in S_k(N_f; K)$ be a newform with ordinary p-stabilization $f^{(p)}$. Then there exists a Hida family $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$ of tame level N_f and a classical point $x_0 \in \Omega_{f,\mathrm{cl}}$ satisfying*

$$\kappa(x_0) = k, \quad \mathbf{f}_{x_0} = f^{(p)}.$$

It will be convenient to introduce a more general definition of families of modular forms.

Definition 2.18. A Λ-adic modular form of tame level N is a quadruple $(R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}, \boldsymbol{\phi})$, where

1. R is a complete, finitely generated, flat extension of Λ;
2. Ω_ϕ is a non-empty open subset of $\mathrm{Hom}(R, \mathbb{C}_p)$ and $\Omega_{\phi,\mathrm{cl}}$ is a p-adically dense subset of Ω_ϕ consisting of classical points;
3. $\boldsymbol{\phi} = \sum \mathbf{a}_n(\boldsymbol{\phi}) q^n \in R[[q]]$ is a formal power series with coefficients in R such that, for any $x \in \Omega_{\phi,\mathrm{cl}}$, the power series

$$\boldsymbol{\phi}_x(q) := \sum_{n=1}^{\infty} \mathbf{a}_n(\boldsymbol{\phi})_x q^n \in \mathbb{C}_p[[q]]$$

is the q-expansion of a classical ordinary cusp form in $S_{\kappa(x)}(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{C}_p)$.

One can define families of old forms as below. Let $f \in S_k(N_f; K_f)$ be a new form and N a multiple of N_f with $p \nmid N$. Let $\breve{f} \in S_k(N; K_f)[\pi_f]$ be an old form. Then there are unique scalars $\lambda_d \in K_f$ for $d \mid (N/N_f)$ such that

$$\breve{f}(q) = \sum_{d \mid (N/N_f)} \lambda_d \cdot f(q^d).$$

The p-stabilization of $f(q^d)$ is the weight k specialization of the formal power series (Λ-adic form)

$$\mathbf{f}(q^d) := \sum_n \mathbf{a}_n(\mathbf{f}) q^{dn},$$

where $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$ is the Hida family of tame level N_f attached to f. Hence we can set

$$\breve{\mathbf{f}} := \sum_{d \mid (N/N_f)} \lambda_d \mathbf{f}(q^d).$$

The quadruple $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \breve{\mathbf{f}})$ is a Λ-adic modular form which specializes to the ordinary p-stabilization of $\breve{f}$ at the weight k point $x_0 \in \Omega_{f,\mathrm{cl}}$ in Theorem 2.17.

Remark 2.19. Given a Hida family $\breve{\mathbf{f}}$ of nebentypus χ_f, there is the Atkin–Lehner involution $\breve{\mathbf{f}}^* := \breve{\mathbf{f}} \otimes \chi_f^{-1}$, such that for all $x \in \Omega_{f,\mathrm{cl}}$, $\breve{\mathbf{f}}_x^* = (\breve{\mathbf{f}}_x)^*$.

Products of families. Let $(\Lambda_g, \Omega_g, \Omega_{g,\mathrm{cl}}, \mathbf{g}), (\Lambda_h, \Omega_h, \Omega_{h,\mathrm{cl}}, \mathbf{h})$ be two Λ-adic modular forms of tame level N. Let $\Lambda_{gh} := \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$ be the finitely generated Λ-algebra with the natural diagonal embedding $\Lambda \to \Lambda_g \otimes \Lambda_h$ given by sending the group-like element $[a]$ to $[a] \otimes [a]$. We set

$$\Omega_{gh} := \Omega_g \times \Omega_h, \quad \Omega_{gh,\mathrm{cl}} := \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}.$$

The naive product $\mathbf{g} \times \mathbf{h} \in \Lambda_{gh}[[q]]$ may not be an ordinary family. Nevertheless, we can take its ordinary projection $e_{\mathrm{ord}}(\mathbf{g} \times \mathbf{h})$. Then the quadruple $(\Lambda_{gh}, \Omega_{gh}, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(\mathbf{g} \times \mathbf{h}))$ is a Λ-adic modular forms.

Operators on families. Recall that we have the operators U, V and $\theta = q\frac{d}{dq}$ on modular forms, defined primarily via q-expansions. We will extend these operators to families of modular forms. For U and V, the generalizations are straightforward. In particular, we can define the p-depletion of $\mathbf{g}$ to be

$$\mathbf{g}^{[p]} := (1 - VU)\mathbf{g} = \sum_{p \nmid n} \mathbf{a}_n(\mathbf{g}) q^n.$$

The specialization $g_y^{[p]}$ of $\mathbf{g}^{[p]}$ at a classical weight y can either be viewed as a p-adic modular form of tame level N or a classical modular form of level Np^2. Notice that $\mathbf{g}^{[p]}$ is far from being ordinary. In fact, it is annihilated by U, hence by e_{ord}.

For any $n \in \mathbb{Z}$ with $p \nmid n$, we can view it as the group-like element $[n] \in \Lambda$. The specialization $[n]_k$ of $[n]$ at a point $k \in \Omega$ is simply $n^k := k(n) \in \mathcal{O}$. Now we consider the q-expansion

$$\theta^{\bullet}\mathbf{g}^{[p]} := \sum_{p \nmid n} [n] \otimes \mathbf{a}_n(\mathbf{g}) q^n, \tag{16}$$

viewed as an element in $\Lambda \otimes_{\mathcal{O}} \Lambda_g[[q]]$. It is a "two-variable" family, whose specialization at $(t, y) \in \Omega_{\mathrm{cl}} \times \Omega_{g,\mathrm{cl}}$ is the p-adic modular form

$$\theta^t g_y^{[p]}.$$

Define $R_{gh} := \Lambda \otimes_{\mathcal{O}} \Lambda_g \otimes_{\mathcal{O}} \Lambda_h$, regarded as a Λ-algebra by the embedding of group-like elements $[a] \mapsto [a^2] \otimes [a] \otimes [a]$. Then the map from $\mathrm{Hom}(R_{gh}, \mathbb{C}_p) = \Omega \times \Omega_g \times \Omega_h$ to the weight space Ω sends the classical point (t, y, z) to $\kappa(y) + \kappa(z) + 2t \in \Omega_{\mathrm{cl}}$. Hence, the product $e_{\mathrm{ord}}(\theta^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h})$ is a Λ-adic form with coefficients in R_{gh}, whose specialization at (t, y, z) is equal to the p-adic modular form $e_{\mathrm{ord}}(\theta^t g_y^{[p]} \times h_z)$ of weight $y + z + 2t$.

We here give a lemma which is easy to prove but is extremely useful.

Lemma 2.20. *Let g and h be two p-adic modular forms of tame level N, then U annihilate $g^{[p]} \times Vh$, and in particular*

$$e_{\mathrm{ord}}(g^{[p]} \times Vh) = 0.$$

The same result also holds for families of modular forms.

Proof. This is obvious from the q-expansion, where one observes that $a_n(g^{[p]} \times Vh) = 0$ whenever $p|n$. $\square$

We conclude the discussion on products of families with the following proposition.

Proposition 2.21. *Let*

$$\Omega_{gh,\mathrm{cl}} := \{(t, y, z) \in \Omega_{\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}} \mid t > \max(1 - \kappa(y), 1 - \kappa(z))\}.$$

The quadruple

$$e_{\mathrm{ord}}(\theta^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}) = (R_{gh}, \Omega \times \Omega_g \times \Omega_h, \Omega_{gh,\mathrm{cl}}, e_{\mathrm{ord}}(\theta^{\bullet}\mathbf{g}^{[p]} \times \mathbf{h}))$$

is a Λ-adic modular form of tame level N. Moreover, its specialization at $(t, y, z) \in \Omega_{gh,\mathrm{cl}}$ is a the classical modular form

$$e_{\mathrm{ord}}(g_y^{[p]} \times h_z) \in S_x(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{C}_p)$$

where $x = y + z + 2t$.

Periods of families. Write $\mathbf{S}^{\mathrm{ord}}(N;R)$ for the space of Λ-adic modular forms with coefficients in a Λ-algebra R. A Hida family $(\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \mathbf{f})$ then gives rise to a subspace

$$\mathbf{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_{\mathbf{f}}] := \{\breve{\mathbf{f}} \in \mathbf{S}^{\mathrm{ord}}(N;\Lambda_f) \mid T_n\breve{\mathbf{f}} = \mathbf{a}_n(\mathbf{f})\breve{\mathbf{f}} \quad \forall (n,N)=1\}.$$

Let $\boldsymbol{\phi} = (R, \Omega_\phi, \Omega_{\phi,\mathrm{cl}}\boldsymbol{\phi})$ be another Λ-adic modular form and consider a pair $(x,y) \in \Omega_{f,\mathrm{cl}} \times_{\Omega_{\mathrm{cl}}} \Omega_{\phi,\mathrm{cl}}$, *i.e.*, $\kappa(x) = \kappa(y)$. The specialization ϕ_y needs not be the p-stabilization of a classical modular form, but its projection $\phi_{x,y} := e_{f_x}\phi_y$ to the f_x-isotypic part is the p-stabilization of a classical modular form, which will be denoted by $\phi^0_{x,y}$.

Lemma 2.22. *For all $\breve{\mathbf{f}} \in \mathbf{S}^{\mathrm{ord}}(N;\Lambda_f)[\pi_{\mathbf{f}}]$ and $\boldsymbol{\phi} \in \mathbf{S}^{\mathrm{ord}}(N;R)$, there exists a unique $J(\breve{\mathbf{f}},\boldsymbol{\phi}) \in \mathrm{Frac}(\Lambda_f)\otimes_\Lambda R$ such that, for all classical points $(x,y) \in \Omega_{f,\mathrm{cl}} \times_{\Omega_{\mathrm{cl}}} \Omega_{\phi,\mathrm{cl}}$,*

$$J(\breve{\mathbf{f}},\boldsymbol{\phi})(x,y) = \frac{(\breve{f}_x, e_{f_x}\phi_y)_{N,p}}{(f_x,f_x)_{N,p}} = \frac{(\breve{f}^0_x, \phi^0_{x,y})_N}{(f^0_x, f^0_x)_N} = \langle \eta_{f^0_x}, \phi^0_{x,y}\rangle_{\mathrm{dR}} \tag{17}$$

where $(\ ,\ )_{N,p}$ stands for the Petersson scalar product on $S_x(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{C})$, and the last pairing is the Poincaré pairing on $H^1_{\mathrm{par}}(X_{\mathbb{C}_p}, \mathcal{H}^{x-2})$.

Proof. We would like to leave it to [DR14, Lemma 2.19]. □

2.3 The p-adic Gross–Zagier formula of Darmon–Rotger

Classical triple product L-functions. Let $f \in S_k(N_f,\chi_f;K_f)$, $g \in S_\ell(N_g,\chi_g;K_g)$, and $h \in S_m(N_h,\chi_h;K_h)$ be a triple of normalized primitive cusp forms with coefficients in $K_\bullet \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ for $\bullet \in \{f,g,h\}$. We set $N = \mathrm{lcm}(N_f,N_g,N_h)$ and $K = K_{f,g,h} := K_f \cdot K_g \cdot K_h$. We assume that $\chi_f \cdot \chi_g \cdot \chi_h = 1$, which implies $k+\ell+m$ is even.

The Garrett–Rankin triple product L-function $L(f,g,h;s)$ is defined by an Euler product

$$L(f,g,h;s) = \prod_p L^p(f,g,h;p^{-s})^{-1},$$

where for $p \nmid N$, the local factor is the degree 8 polynomial

$$\begin{aligned} L^p(f,g,h;T) = &(1-\alpha_{f,p}\alpha_{g,p}\alpha_{h,p}T)\times(1-\alpha_{f,p}\alpha_{g,p}\beta_{h,p}T)\\ &\times(1-\alpha_{f,p}\beta_{g,p}\alpha_{h,p}T)\times(1-\beta_{f,p}\alpha_{g,p}\alpha_{h,p}T)\\ &\times(1-\alpha_{f,p}\beta_{g,p}\beta_{h,p}T)\times(1-\beta_{f,p}\alpha_{g,p}\beta_{h,p}T)\\ &\times(1-\beta_{f,p}\beta_{g,p}\alpha_{h,p}T)\times(1-\beta_{f,p}\beta_{g,p}\beta_{h,p}T). \end{aligned} \tag{18}$$

Piatetski-Shapiro and Rallis gave a precise recipe in [PR87] for the local factors at primes $p|N$ as well as the archimedean factor $L^\infty(f,g,h;s)$. They also showed that the completed L-function

$$\Lambda(f,g,h;s) := L^\infty(f,g,h;s)\cdot L(f,g,h;s)$$

admits a functional equation

$$\Lambda(f,g,h;s) = \varepsilon(f,g,h)\Lambda(f,g,h;k+\ell+m-2-s)$$

where $\varepsilon(f,g,h) \in \{\pm1\}$. The sign of $\varepsilon(f,g,h)$ then determine the parity of the order of vanishing of $L(f,g,h;s)$ at the central point

$$c = c_{f,g,h} := \frac{k+\ell+m-2}{2},$$

at which there is no pole (c.f. [PR87, Theorem 5.2]). The root number $\varepsilon(f,g,h)$ can be expressed as a product

$$\varepsilon(f,g,h) = \prod_q \varepsilon_q(f,g,h)$$

where $q \leq \infty$ runs through all the places of $\mathbb{Q}$ and each local root number $\varepsilon_q(f,g,h) \in \{\pm 1\}$.

Throughout this article, we will assume that all root numbers at finite primes are $+1$ (c.f. [DR14, § 1]). Then $\varepsilon(f,g,h)$ is determined by the local root number at infinity, which depends only on the weights of (f,g,h). Precisely, we have

$$\varepsilon(f,g,h) = \varepsilon_\infty(f,g,h) = \begin{cases} -1 & \text{if } (k,\ell,m) \text{ is balanced;} \\ +1 & \text{if } (k,\ell,m) \text{ is unbalanced.} \end{cases} \tag{19}$$

We recall the definitions of balanced and unbalanced triples below.

Definition 2.23. A triple $(x,y,z) \in \mathbb{N}^3$ is said to be balanced if the largest number is strictly less than the sum of the other two; it is called unbalanced if otherwise.

In the unbalanced case, we recall the following result of M. Harris and S. Kulda [HK91], refined by A. Ichino [Ich08] and T. C. Watson [Wat02]. The theorem is usually known as the Ichino formula.

Theorem 2.24. *Let f,g,h be as above and assume that the weights (k,ℓ,m) is unbalanced with $k = \ell+m+2t$ for some $t \in \mathbb{N}$. Then there exist modular forms (called test vectors)*

$$\breve{f} \in S_k(N, K_{f,g,h})[f],\ \ \breve{g} \in S_\ell(N, K_{f,g,h})[g],\ \ \breve{h} \in S_m(N, K_{f,g,h})[h]$$

and constants $C_q \in K$ depending only on the local components of $\breve{f},\breve{g},\breve{h}$ at all $q \mid N\infty$ such that

$$\frac{\prod_q C_q}{\pi^{2k}} L\left(\breve{f},\breve{g},\breve{h},\frac{k+l+m-2}{2}\right) = |I(\breve{f},\breve{g},\breve{h})|^2,$$

where

$$I(\breve{f},\breve{g},\breve{h}) := (\breve{f}^*, \delta_\ell^t \breve{g} \times \breve{h})_N.$$

Moreover, there exists a choice of $\breve{f},\breve{g},\breve{h}$ such that $C_q \neq 0$ for all q.

Triple product p-adic L-functions for Hida families. Let the triple (f,g,h) be as before, and fix an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Also let $(\breve{f},\breve{g},\breve{h})$ be test vectors as in Theorem 2.24. Assume further that $(\breve{f},\breve{g},\breve{h})$ are ordinary with respect to ι_p and let

$$\breve{\mathbf{f}} = (\Lambda_f, \Omega_f, \Omega_{f,\mathrm{cl}}, \breve{\mathbf{f}}), \quad \breve{\mathbf{g}} = (\Lambda_g, \Omega_g, \Omega_{g,\mathrm{cl}}, \breve{\mathbf{g}}), \quad \breve{\mathbf{h}} = (\Lambda_h, \Omega_h, \Omega_{h,\mathrm{cl}}, \breve{\mathbf{h}})$$

be the Hida families of tame level N interpolating the ordinary p-stabilizations of $(\breve{f},\breve{g},\breve{h})$ respectively. We write

$$\Sigma := \{(x,y,z) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}\},$$

and

$$\begin{aligned} \Sigma_f &:= \{(x,y,z) \in \Sigma \mid x-y-z \in 2\mathbb{N}\}, \\ \Sigma_{\mathrm{bal}} &:= \{(x,y,z) \in \Sigma \mid (x,y,z) \text{ is balanced and } x+y+z \in 2\mathbb{N}\}. \end{aligned}$$

In light of Theorem 2.24, one would like to have a p-adic L-function associated with $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ that interpolates values related to $I(\breve{f},\breve{g},\breve{h}) := (\breve{f}^*, \delta_\ell^t \breve{g} \times \breve{h})_N$. This motivates the following definition in [DR14, § 4].

Definition 2.25. The Garrett–Rankin triple product p-adic L-function attached to the families $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ is the element

$$\mathscr{L}_p^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}) := J(\breve{\mathbf{f}}^*, e_{\mathrm{ord}}(\theta^\bullet \breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}})) \in \operatorname{Frac}\Lambda_f \otimes_\Lambda (\Lambda \otimes \Lambda_g \otimes \Lambda_h) \tag{20}$$

where the family $e_{\mathrm{ord}}(\theta^\bullet \breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}})$ and the period J are defined as in previous sections.

Notice that $\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ has poles only for finitely many $x \in \Omega_f$. By definition, its value at $(x, y, z) \in \Sigma_f$, after setting $x = y + z + 2t$, is

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x, y, z) = \frac{(\breve{f}_x^*, e_{\mathrm{ord}}(\theta^t \breve{g}_y^{[p]} \times \breve{h}_z))_{N,p}}{(\breve{f}_x, \breve{f}_x)_{N,p}}. \tag{21}$$

In particular, the value is algebraic. Furthermore, in virtue of the relation between θ and δ, one has the following interpolation formula.

Theorem 2.26 ([DR14, Theorem 4.7]). *Let $(x, y, z) \in \Sigma_f$ and f_x^0, g_y^0, h_z^0 be classical modular forms of level N whose ordinary p-stabilizations are $\breve{f}_x, \breve{g}_y, \breve{h}_z$ respectively. Then*

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x, y, z) = \frac{\mathscr{E}(f_x^0, g_y^0, h_z^0)}{\mathscr{E}_0(f_x^0)\mathscr{E}_1(f_x^0)} \times \frac{I(f_x^0, g_y^0, h_z^0)}{(f_x^0, f_x^0)_N} \tag{22}$$

where the Euler factors $\mathscr{E}(f_x^0, g_y^0, h_z^0), \mathscr{E}_0(f_x^0), \mathscr{E}_1(f_x^0)$ are defined as in the introduction.

Evaluations at balanced weights. It is more than natural to ask what happens when we specialize $\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ to a classical point $(x, y, z) \in \Sigma_{\mathrm{bal}}$. These points lie outside the range of interpolations in the sense that Ichino formula does not apply to them.

One sees from the definition of $\mathscr{L}_p^f$ that the value at a point $(x, y, z) \in \Sigma_{\mathrm{bal}}$ involves a negative power of θ. Let $t \in \mathbb{Z}_{>0}$, the negative power $\theta^{-t} g^{[p]}$ is originally defined as a p-adic limit of modular forms. However, it does satisfy the identity $\theta^t(\theta^{-t} g^{[p]}) = g^{[p]}$. In other words, $\theta^{-t} g^{[p]}$ may be viewed as the t-th anti-derivative of $g^{[p]}$ with respect to the differential operator θ. Hence, one may expect the values on Σ_{bal} can be expressed as certain integrals. This is the motivation of the p-adic Gross–Zagier formula. In this case, the special values can be further expressed as p-adic Abel–Jacobi images, which are indeed certain p-adic integrals, as we will show in later sections.

We here recall the p-adic Gross–Zagier formula for Hida families without fully defining every ingredient.

Theorem 2.27 ([DR14, Theorem 1.3]). *Let notations be as in Theorem 2.26, except that $(x, y, z) \in \Sigma_{\mathrm{bal}}$ now. We write $x = y + z - 2t$ with $t \in \mathbb{Z}_{>0}$. Then*

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(x, y, z) = (-1)^{t-1} \frac{\mathscr{E}(f_x^0, g_y^0, h_z^0)}{(t-1)!\mathscr{E}_0(f_x^0)\mathscr{E}_1(f_x^0)} \times \mathrm{AJ}_p(\Delta_{x,y,z})(\eta_f^{u\text{-}r} \otimes \omega_g \otimes \omega_h) \tag{23}$$

where $\mathrm{AJ}_p(\Delta_{x,y,z})$ is the p-adic Abel–Jacobi image of the diagonal cycle associated with the weight (x, y, z), and $\eta_f^{u\text{-}r}, \omega_g, \omega_h$ are elements in the cohomology groups of various Kuga–Sato varieties associated with the classical modular forms f_x^0, g_y^0, h_z^0 (c.f. §5).

Remark 2.28. The formula obviously has two sides, the L-function part and the Abel–Jacobi map part. To generalize the formula to finite slope families, one needs to generalize these two factors. Sections 3 and 4 will recall the construction of triple product p-adic L-functions for finite slope families in [AI21], while Section 5 will focus on the p-adic Abel–Jacobi map. As our results for finite slope families will cover the ordinary case, we would like to omit the proof of Theorem 2.27.

3 Vector bundles with marked sections

In this section, we will study vector bundles with marked sections (abbr. VBMS). It serves as an important tool in Section 4 for constructing both the modular and de Rham sheaves. The results in this section all come from [AI21, § 2] and I wish to claim no originality of the proofs.

3.1 Formal vector bundles with marked sections

Let S be a formal scheme with ideal of definition $\mathcal{I}$ which is invertible, and let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank n. Let $s_1, s_2, \ldots, s_m \in H^0(S, \mathcal{E}/\mathcal{I}\mathcal{E})$ with $m \leq n$ be a collection of sections such that they generate a direct summand of $\bar{\mathcal{E}} := \mathcal{E}/\mathcal{I}\mathcal{E}$. Such a data $(\mathcal{E}, s_1, \ldots, s_m)$ is called a locally free sheaf with marked sections. The goal is to construct a vector bundle associated with $(\mathcal{E}, s_1, \ldots, s_m)$.

Definition 3.1. A formal vector bundle of rank n over S is a formal vector group scheme $f : X \to S$ such that locally on S it is isomorphic to the n-fold product of the additive group $\mathbb{G}^n_{a,S}$.

Theorem 3.2 ([AI21, § 2.2])**.** *Let $(\mathcal{E}, s_1, \ldots, s_m)$ be as above. We have the following results.*

i. The functor that sends a morphism of formal schemes $t : T \to S$ (with the ideal of definition of T being $t^*(\mathcal{I})$) to the set

$$\mathbb{V}(\mathcal{E})(t : T \to S) := \operatorname{Hom}_{\mathcal{O}_T}(t^*\mathcal{E}, \mathcal{O}_T)$$

is represented by the formal vector bundle $\mathbb{V}(\mathcal{E}) := \operatorname{Spf}(\widehat{\operatorname{Sym}}(\mathcal{E}))$, where $\widehat{\operatorname{Sym}}(\mathcal{E})$ is the $\mathcal{I}$-adic completion of the symmetric algebra $\operatorname{Sym}(\mathcal{E}) := \oplus_{i\in\mathbb{N}} \operatorname{Sym}^i_{\mathcal{O}_S}(\mathcal{E})$.

ii. The subfunctor $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ of $\mathbb{V}(\mathcal{E})$, which sends $t : T \to S$ as above to the set

$$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(t : T \to S) := \{h \in \mathbb{V}(\mathcal{E})(t : T \to S) \mid h(\text{mod } t^*(\mathcal{I}))(t^*(s_j)) = 1\ \forall j\},$$

is represented by an open formal subscheme $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ in a formal admissible blow-up of $\mathbb{V}$.

Proof. (of Theorem 3.2) The first part is obvious, see [AI21, Lemma 2.2]. We here give a detailed proof for the second part since the local description is crucial in later sections.

The sections $s_1, \ldots, s_m$ define a subsheaf of $\mathcal{O}_{\bar{S}}$-module in $\bar{\mathcal{E}}$. By assumption, the quotient Q is a locally free $\mathcal{O}_{\bar{S}}$-module of rank $n - m$. Consider the quotient map

$$\operatorname{Sym}(\bar{\mathcal{E}}) := \oplus_{i\in\mathbb{N}} \operatorname{Sym}^i_{\mathcal{O}_{\bar{S}}}(\bar{\mathcal{E}}) \to \operatorname{Sym}(Q)$$

which has kernel $\bar{\mathcal{J}} := (s_1 - 1, \ldots, s_m - 1)$. Taking the induced map on spectra (relative to $\bar{S}$), it defines a closed subscheme in $\operatorname{Spec}(\operatorname{Sym}(\bar{\mathcal{E}}))$. Let $\mathcal{J} \subset \mathcal{O}_{\mathbb{V}(\mathcal{E})}$ be the inverse image of $\bar{\mathcal{J}}$.

Now consider the $\mathcal{I}$-adic completion of the open formal subscheme of the blow-up of $\mathbb{V}(\mathcal{E})$ with respect to the ideal $\mathcal{J}$, defined by the requirement that the ideal generated by the inverse image of $\mathcal{J}$ coincides with the inverse image of $\mathcal{I}$. We claim that such an open formal subscheme, denoted by $\mathbb{B}$, represents the desired functor.

In local coordinates, let $U = \operatorname{Spf}(R) \subset S$ be an open formal subscheme such that $\mathcal{I}$ is generated by α, $\mathcal{E}|_U$ is free with basis $e_1, \ldots, e_n$ such that $e_i \equiv s_i$ modulo α for $i = 1, \ldots, m$, and $e_{m+1}, \ldots, e_n$ define a basis of Q. Then $\mathbb{V}(\mathcal{E})|_U$ is the formal scheme associated with $R\langle X_1, \ldots, X_n\rangle$ where X_i corresponds to e_i and $\mathcal{J}|_U$ is the ideal $(\alpha, X_1 - 1, \ldots X_m - 1)$. In particular, one sees that

$$\mathbb{B}|_U = \operatorname{Spf} R\langle Z_1, \ldots, Z_m, X_{m+1}, \ldots, X_n\rangle$$

with morphism $\mathbb{B}|_U \to \mathbb{V}(\mathcal{E})|_U$ given by the map $X_i \mapsto X_i$ for $i = m + 1, \ldots, n$ and $X_i \mapsto 1 + \alpha Z_i$ for $i = 1, \ldots, m$.

For every formal scheme T over U, a section $\rho \in \mathbb{V}(\mathcal{E})(T) = \operatorname{Hom}_{\mathcal{O}_T}(t^*\mathcal{E}, \mathcal{O}_T)$ is defined by the images $a_i := \rho(t^*(e_i))$ of X_i's. Then $\rho \in \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(T)$ if and only if $a_i \equiv 1 \bmod \alpha$ for $i = 1, \ldots, m$. Hence, ρ

determines uniquely a T-point of $\mathbb{B}|_U$ given by $X_i \mapsto a_i$ for $i = m+1, \ldots, n$ and $Z_i \mapsto \frac{a_i - 1}{\alpha}$ for $i = 1, \ldots, m$. Conversely, any T-point of $\mathbb{B}|_U$ defines a section $\rho \in \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(T)$.

One can then verify that the isomorphism $\mathbb{B}|_U \cong \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)|_U$ glues to an isomorphism $\mathbb{B} \cong \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ as formal schemes over $\mathbb{V}(\mathcal{E})$. The functoriality is immediately checked by construction. □

3.2 Filtrations on vector bundles with marked sections

Let S and $\mathcal{E}$ be as before. Suppose $\mathcal{F} \subset \mathcal{E}$ is a subsheaf, locally free of rank m, such that $\mathcal{E}/\mathcal{F}$ is locally free of rank $n - m$. Suppose also that the sections $s_1, \ldots, s_m$ of $\bar{\mathcal{E}}$ define an $\mathcal{O}_{\bar{S}}$-basis of $\bar{\mathcal{F}}$. By the functoriality, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) & \longrightarrow & \mathbb{V}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathbb{V}_0(\mathcal{F}, s_1, \ldots, s_m) & \longrightarrow & \mathbb{V}(\mathcal{F}). \end{array}$$

We let $f : \mathbb{V}(\mathcal{E}) \to S,\ g : \mathbb{V}(\mathcal{F}) \to S$ and $f_0 : \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \to S,\ g_0 : \mathbb{V}_0(\mathcal{F}, s_1, \ldots, s_m) \to S$ denote the structural morphisms.

Lemma 3.3 ([AI21, Lemma 2.5])**.** *The above diagram is Cartesian. In particular, the vertical morphisms are principal homogeneous spaces under the formal vector group scheme $\mathbb{V}(\mathcal{E}/\mathcal{F})$.*

Corollary 3.4 ([AI21, Corollary 2.6])**.** *Let notations be as above. The sheaf $f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}$ is endowed with an increasing filtration $\mathrm{Fil}_\bullet\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}$ with graded pieces*

$$\mathrm{Gr}_h\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)} \cong g_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{F}, s_1, \ldots, s_m)} \otimes_{\mathcal{O}_S} \mathrm{Sym}^h(\mathcal{E}/\mathcal{F}).$$

Remark 3.5. The proofs of above two results can be derived from the following local description. Locally, let $U = \mathrm{Spf}(R) \subset S$ is an open formal affine subscheme such that $\mathcal{F}, \mathcal{E}$ are free with basis $\{e_1, \ldots, e_m\}$ (resp. $\{e_1, \ldots, e_m, f_{m+1}, \ldots, f_n\}$). We can write

$$\begin{aligned} \mathbb{V}_0(\mathcal{F}, s_1, \ldots, s_m)|_U &\cong \mathrm{Spf}(R\langle Z_1, \ldots, Z_m\rangle), \\ \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)|_U &\cong \mathrm{Spf}(R\langle Z_1, \ldots, Z_m, X_{m+1}, \ldots, X_n\rangle) \end{aligned}$$

as in previous section. In particular, the filtration $\mathrm{Fil}_h\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}(U)$ consists of polynomials of degree at most h in the variables $X_{m+1}, \ldots, X_n$ with coefficients in $R\langle Z_1, \ldots, Z_m\rangle$.

3.3 Connections on vector bundles with marked sections

Fix a $\mathbb{Z}_p$-algebra A and an element $\alpha \in A$ such that A is α-adically complete and separated. Suppose that S is a formal scheme locally of finite type over $\mathrm{Spf}(A)$ and the topology of S is the α-adic topology. We let $\Omega^1_{S/A}$ be the $\mathcal{O}_S$-module of continuous Kähler differentials.

Grothendieck's description of integrable connections. Let $\mathcal{P} = \mathcal{P}_S := S \times_A S$ and $\Delta : S \to \mathcal{P}_S$ be the diagonal embedding. It is a locally closed immersion, and we let $\mathcal{P}^{(1)}$ be the first infinitesimal neighborhood of Δ: if locally on $\mathcal{P}$ the morphism Δ is defined by an ideal $\mathscr{I}$, then $\mathcal{P}^{(1)}$ is defined by $\mathscr{I}^2$. There are two natural projections $j_1, j_2 : \mathcal{P}^{(1)} \to S$.

Then, giving an integrable connection $\nabla : M \to M \otimes_{\mathcal{O}_S} \Omega^1_{S/A}$ on a locally free $\mathcal{O}_S$-module of finite rank is equivalent to giving an isomorphism of $\mathcal{O}_{\mathcal{P}^{(1)}}$-modules

$$\epsilon : j_2^*(M) = \mathcal{O}_{\mathcal{P}^{(1)}} \otimes_{\mathcal{O}_S} M \cong j_1^*(M) = M \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathcal{P}^{(1)}}$$

such that $\Delta^*(\epsilon) = \mathrm{Id}$ on M and ϵ satisfies a suitable cocycle condition with respect to the pullbacks of three possible embeddings from $S \times_A S$ to $S \times_A S \times_A S$.

Remark 3.6. This is a well-known fact (c.f. [BO78, § 2]), but we will recall it for later use. First, one observes that the sheaf $\mathcal{O}_S \otimes \mathcal{O}_S$ has two $\mathcal{O}_S$-module structures. Namely, the one given by $\mathcal{O}_S \otimes 1$ and the other given by $1 \otimes \mathcal{O}_S$. They give rise to two $\mathcal{O}_S$-structures on $\mathcal{O}_{\mathcal{P}^{(1)}}$ which correspond to the structures induced by j_1^* and j_2^*. Also, we have a natural map $\mathcal{O}_S \otimes \mathcal{O}_S / \mathscr{I} \to \mathcal{O}_S$ and hence a map $\mathcal{O}_{\mathcal{P}^{(1)}}/(\mathscr{I}/\mathscr{I}^2) \to \mathcal{O}_S$.

Now suppose we have an isomorphism ϵ as above. The condition $\Delta^*(\epsilon) = \mathrm{Id}$ is equivalent to saying that $\epsilon = \mathrm{Id}$ on M modulo $\mathscr{I}/\mathscr{I}^2$. We also remark that the $\mathcal{O}_S$-structure on $\mathcal{O}_{\mathcal{P}^{(1)}}$ used for the tensor product $\mathcal{O}_{\mathcal{P}^{(1)}} \otimes_{\mathcal{O}_S} M$ is the one given by $\mathcal{O}_S \to 1 \otimes \mathcal{O}_S$. One defines $\theta := \epsilon \circ j_2^* : M \to \mathcal{O}_{\mathcal{P}^{(1)}} \otimes_{\mathcal{O}_S} M$ and

$$\nabla(m) := \theta(m) - m \otimes 1.$$

By the assumption of ϵ, $\nabla(m) \in M \otimes \mathscr{I}/\mathscr{I}^2 \cong M \otimes \Omega_S^1$. For the Leibniz rule, one computes, for $s \in \mathcal{O}_S$ and $m \in M$, that

$$\begin{aligned}
\nabla(sm) &= \theta(sm) - sm \otimes 1 = (1 \otimes s)\theta(m) - (s \otimes 1)(m \otimes 1) \\
&= (1 \otimes s)\theta(m) - (1 \otimes s)(m \otimes 1) + (1 \otimes s)(m \otimes 1) - (s \otimes 1)(m \otimes 1) \\
&= (1 \otimes s)\nabla(m) + ds \cdot (m \otimes 1) = s\nabla(m) + m \otimes ds
\end{aligned}$$

where we identify $1 \otimes s - s \otimes 1 = ds$ in $\mathscr{I}/\mathscr{I}^2 \cong \Omega_S^1$.

Conversely, given a connection ∇, one defines

$$\epsilon(1 \otimes m) = m \otimes 1 + \nabla(m).$$

One can then reverse the above computation to verify that ϵ has the desired properties.

Lastly, we remark that this formulation allows us to define connections on an arbitrary quasi-coherent $\mathcal{O}_S$-module M (not necessarily locally free of finite rank).

Suppose now that $\mathcal{E}$ is a locally free $\mathcal{O}_S$-module with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_S^1$ and s_i's are marked sections of $\bar{\mathcal{E}}$ that are horizontal for ∇ modulo $\mathcal{I}$. This means that the reduction of the associated isomorphism $\epsilon : j_2^*\mathcal{E} \to j_1^*\mathcal{E}$ satisfies $\bar{\epsilon}(j_2^*(s_i)) = j_1^*(s_i)$ for all i. Let $f : \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \to S$ be the structural morphism. We would like to give a connection on the sheaf $f_*\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}$.

First, by functoriality, we have the commutative diagram of formal schemes over S

$$\begin{tikzcd}
\mathcal{P}^{(1)}_{S/A} \times_S \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \arrow[r, "\epsilon_0"] \arrow[d] & \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \times_S \mathcal{P}^{(1)}_{S/A} \arrow[d] \\
\mathcal{P}^{(1)}_{S/A} \times_S \mathbb{V}(\mathcal{E}) \arrow[r, "\epsilon'"] & \mathbb{V}(\mathcal{E}) \times_S \mathcal{P}^{(1)}_{S/A}
\end{tikzcd}$$

such that $\Delta^*(\epsilon_0) = \mathrm{Id}$ and $\Delta^*(\epsilon') = \mathrm{Id}$. Passing to functions, we obtain compatible isomorphisms

$$\begin{tikzcd}
j_2^*(f_*\mathcal{O}_{\mathbb{V}(\mathcal{E})}) \arrow[r, "\epsilon'^{,*}"] \arrow[d] & j_1^*(f_*\mathcal{O}_{\mathbb{V}(\mathcal{E})}) \arrow[d] \\
j_2^*(f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}) \arrow[r, "\epsilon_0^*"] & j_1^*(f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)})
\end{tikzcd}$$

such that $\Delta^*(\epsilon_0^*) = \mathrm{Id}$ and $\Delta^*(\epsilon'^{,*}) = \mathrm{Id}$. By construction, when restricted to the submodule $\mathcal{E} \subset f_*\mathcal{O}_{\mathbb{V}(\mathcal{E})}$, $\epsilon'^{,*}$ coincides with ϵ. Hence, via Grothedieck's correspondence, there are compatible integrable connections:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\nabla"] \arrow[d] & \mathcal{E} \otimes_{\mathcal{O}_S} \Omega^1_{S/A} \arrow[d] \\
f_*\mathcal{O}_{\mathbb{V}(\mathcal{E})} \arrow[r, "\nabla'"] \arrow[d] & f_*\mathcal{O}_{\mathbb{V}(\mathcal{E})} \hat{\otimes}_{\mathcal{O}_S} \Omega^1_{S/A} \arrow[d] \\
f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)} \arrow[r, "\nabla_0"] & f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)} \hat{\otimes}_{\mathcal{O}_S} \Omega^1_{S/A}.
\end{tikzcd}$$

Assume further that we have a direct summand $\mathcal{F} \subset \mathcal{E}$ as in the setting of Section 3.2. Recall that there is a filtration $\mathrm{Fil}_\bullet\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)}$ on $f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)}$.

Lemma 3.7 ([AI21, Lemma 2.9])**.** *The connection ∇_0 satisfies Griffiths transversality with respect to the filtration $\mathrm{Fil}_\bullet\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)}$. That is, for every integer h, we have*

$$\nabla(\mathrm{Fil}_h\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)}) \subset \mathrm{Fil}_{h+1}\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)} \hat{\otimes}_{\mathcal{O}_S} \Omega^1_{S/A}.$$

Proof. The statement can be checked locally. Let $U = \mathrm{Spf}(R) \subset S$ be an open formal affine subscheme such that $\mathcal{I}$ is generated by $u \in R$. Suppose that $\mathcal{F}, \mathcal{E}$ are free with basis $\{e_1, \dots, e_m\}$ (resp. $\{e_1, \dots, e_m, f_{m+1}, \dots, f_n\}$). Write

$$\begin{aligned}
\mathbb{V}_0(\mathcal{E})|_U &\cong \mathrm{Spf}(R\langle X_1, \dots, X_n\rangle),\\
\mathbb{V}_0(\mathcal{F}, s_1, \dots, s_m)|_U &\cong \mathrm{Spf}(R\langle Z_1, \dots, Z_m\rangle),\\
\mathbb{V}_0(\mathcal{E}, s_1, \dots, s_m)|_U &\cong \mathrm{Spf}(R\langle Z_1, \dots, Z_m, X_{m+1}, \dots, X_n\rangle)
\end{aligned}$$

as before.

By assumption, for $1 \le s \le m$, we can write $\nabla(e_s) = \sum_{i=1}^m ue_i \otimes \alpha_{s,i} + \sum_{j=m+1}^n uf_j \otimes \beta_{s,j}$ where $\alpha_{s,i}, \beta_{s,j} \in \Omega^1_U$. Hence,

$$\nabla'(X_s) = \sum_{i=1}^m uX_i \otimes \alpha_{s,i} + \sum_{j=m+1}^n uX_j \otimes \beta_{s,j}.$$

Since $X_s = 1 + uZ_s$, we deduce that

$$\nabla_0(Z_s) = \sum_{i=1}^m (X_i \otimes \alpha_{s,i} - Z_i \otimes du) + \sum_{j=m+1}^n X_j \otimes \beta_{s,j}$$

for $1 \le s \le m$. Recall that the filtration $\mathrm{Fil}_h\, f_{0,*}\mathcal{O}_{\mathbb{V}_0(\mathcal{E},s_1,\dots,s_m)}(U)$ consists of polynomials of degree at most h in the variables $X_{m+1}, \dots, X_n$ with coefficients in $R\langle Z_1, \dots, Z_m\rangle$. One then verifies Griffiths transversality by Leibniz's rule. $\square$

4 Overconvergent modular and de Rham sheaves

In this part, we will recall the constructions of overconvergent sheaves by using vector bundles with marked sections. The arguments mainly follow [AI21, § 3], with some inputs from [Kaz22]. Again, we do not claim any originality of the proofs.

Throughout this section, we will let $N \geq 4$ be an integer, p be a prime not dividing N and let $q = 4$ if $p = 2$ and $q = p$ if otherwise.

4.1 The weight space

We let $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[(\mathbb{Z}/q\mathbb{Z})^\times][\![T]\!]$ be the Iwasawa algebra. Here the isomorphism is given by $\exp(q) \mapsto 1 + T$. We also let $\Lambda^0 = \mathbb{Z}_p[\![T]\!]$. Then we have two formal schemes

$$\mathfrak{W} := \mathrm{Spf}(\Lambda) \text{ and } \mathfrak{W}^0 := \mathrm{Spf}(\Lambda^0)$$

which are called the weight spaces. Recall that for any p-adically complete $\mathbb{Z}_p$-algebra R,

$$\mathrm{Hom}_{\mathrm{cont},\mathbb{Z}_p}(\Lambda, R) \cong \mathrm{Hom}_{\mathrm{cont},\mathbb{Z}}(\mathbb{Z}_p^\times, R^\times).$$

Then $\mathfrak{W}^0$ is the connected component of the trivial character in $\mathfrak{W}$. We define $\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}}$ to be the analytic adic space associated with Λ, and similarly $\mathcal{W}^0 := \mathrm{Spa}(\Lambda^0, \Lambda^0)^{\mathrm{an}}$.

For any interval $I = [p^a, p^b] \subset [0, \infty]$ with $a \in \mathbb{N} \cup \{-\infty\}$ and $b \in \mathbb{N} \cup \{\infty\}$, we let

$$\mathcal{W}_I^0 = \{x \in \mathcal{W}^0 \mid |p|_x \leq |T^{p^a}|_x \neq 0, |T^{p^b}|_x \leq |p|_x \neq 0\}.$$

We will mainly focus on the following two cases of I:

1. $I = [0, p^b]$ for some $b \in \mathbb{N}$;
2. $I = [p^a, p^b]$ with $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$.

We then have the descriptions of $\mathcal{W}_I^0$ for these two cases:

$$\mathcal{W}^0_{[0,p^b]} = \mathrm{Spa}\left(\Lambda^0 \left\langle \frac{T^{p^b}}{p} \right\rangle \left[\frac{1}{p}\right], \Lambda^0 \left\langle \frac{T^{p^b}}{p} \right\rangle\right);$$

$$\mathcal{W}^0_{[p^a,p^b]} = \mathrm{Spa}\left(\Lambda^0 \left\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \right\rangle \left[\frac{1}{p}\right], \Lambda^0 \left\langle \frac{p}{T^{p^a}}, \frac{T^{p^b}}{p} \right\rangle\right).$$

For I as above, we can similarly define $\mathcal{W}_I$, which is the componentwise union of $\mathcal{W}_I^0$. We let $\Lambda_I^0 := \Gamma(\mathcal{W}_I^0, \mathcal{O}^+_{\mathcal{W}_I^0})$, $\Lambda_I := \Gamma(\mathcal{W}_I, \mathcal{O}^+_{\mathcal{W}_I})$ and $\mathfrak{W}_I^0 := \mathrm{Spf}\,\Lambda_I^0$, $\mathfrak{W}_I := \mathrm{Spf}\,\Lambda_I$. For $I = [0, p^b]$, we choose a pseudo-uniformizer $\alpha = p$ and for $I = [p^a, p^b]$, we take $\alpha = T$.

Remark 4.1. Throughout this thesis, we will be mostly dealing with the case $I = [0, 1]$ and $\alpha = p$.

The universal character.

Definition 4.2. Given a p-adically complete and separated ring R, we say that a continuous homomorphism $k : \mathbb{Z}_p^\times \to R^\times$ is an analytic weight of radius p^a for $a \in \mathbb{N}$ if there exists an element $u \in R[\frac{1}{p}]$ such that $k(t) = \exp(u \log t)$ for every $t \in 1 + p^a\mathbb{Z}_p$.

Let $k^{\mathrm{univ}} : \mathbb{Z}_p^\times \to \Lambda$ be the universal character. We denote by $k^0 := k^{\mathrm{univ},0} : \mathbb{Z}_p^\times \to \Lambda^0$ the character obtained by k^{univ} via the projection $\Lambda \to \Lambda^0$. We also let $k_I^0 : \mathbb{Z}_p^\times \to \Lambda_I^0$ be the restriction of k^0 to $\mathcal{W}_I^0$.

Lemma 4.3. *For $I \subset [0, q^{-1}p^n]$, the restriction of k_I^0 to $1 + qp^{n-1}\mathbb{Z}_p$ is analytic. In particular, it extends to a character*

$$k_I^0 : \mathcal{W}_I^0 \times \mathbb{Z}_p^\times(1 + qp^{n-1}\mathbb{G}_a^+) \to \mathbb{G}_m^+$$

which further restricts to

$$k_I^0 : \mathcal{W}_I^0 \times (1 + qp^{n+m-1}\mathbb{G}_a^+) \to 1 + qp^m\mathbb{G}_a^+$$

for all $m \geq 0$.

Proof. ([AIP18, Proposition 2.1]) For simplicity, we let $I = [0, q^{-1}p^n]$. One first observes that $k_I^0(\exp(qp^{n-1})) - 1 = (1+T)^{p^{n-1}} - 1 \in (T^{p^{n-1}}, pT^{p^{n-2}}, p^{n-1}T)$. Since $T^{p^{n-1}}$ is divisible by q in $\Lambda_I^0 = \Lambda^0\langle T^{q^{-1}p^n}/p\rangle$, We can write $(1+T)^{p^{n-1}} = 1 + qh(T)$ for some $h \in \Lambda_I^0$. In particular, the element $u' = \log((1+T)^{p^{n-1}})$ is well-defined in Λ_I^0. As $1 + qp^{n-1}\mathbb{Z}_p$ is isomorphic to $qp^{n-1}\mathbb{Z}_p$ via the exponential and the logarithm, for any $r \in \mathbb{Z}_p$, we have

$$k_I^0(\exp(qp^{n-1}r)) = \exp\left(r\log((1+T)^{p^{n-1}})\right) = \exp\left(qp^{n-1}r \cdot \frac{1}{qp^{n-1}}\log((1+T)^{p^{n-1}})\right).$$

Hence k_I^0 is analytic with $u = \frac{1}{qp^{n-1}}\log((1+T)^{p^{n-1}})$. The rest of the statements then follow from direct computations. ☐

4.2 Modular curves and Igusa towers

Let $Y = Y_1(N)/\mathbb{Z}_p$ be the moduli scheme of elliptic curves with $\Gamma_1(N)$-level structure. Let $X = X_1(N)$ be its smooth proper compactification of Y that classifies generalized elliptic curves with $\Gamma_1(N)$-level structure. We let $\mathfrak{X}$, $\mathfrak{Y}$ be the formal completions of X, Y along their special fibers respectively. Lastly, we let $\pi : E \to \mathfrak{X}$ be the universal semi-abelian scheme.

Similarly as in §2.1, the two sheaves $\pi_*(\Omega^1_{E_{\mathfrak{Y}}/\mathfrak{Y}})$ and $\mathbb{R}^1\pi_*(\Omega^\bullet_{E_{\mathfrak{Y}}/\mathfrak{Y}})$ have canonical extensions to $\mathfrak{X}$, which will again be denoted by ω_E and $\mathcal{H}$ respectively. The sheaf ω_E is locally free of rank 1 and $\mathcal{H}$ is locally free of rank 2 with a Hodge filtration and a Gauss–Manin connection.

The Hodge ideal, denoted by Hdg, is the ideal of $\mathcal{O}_\mathfrak{X}$ defined locally by: on an open affine $U = \mathrm{Spf}(R)$ of $\mathfrak{X}$ such that $\omega_E|_U$ is a free R-module of rank 1, Hdg is generated by the value $\widetilde{\mathrm{Ha}}(E/R, \omega)$, where $\widetilde{\mathrm{Ha}}$ is a lift of the Hasse invariant and ω is any R-generator of $\omega_{E|_U}$. Recall that when $p \geq 5$ case we can simply take $\widetilde{\mathrm{Ha}} = E_{p-1}$, the Eisenstein series of weight $p-1$. By abuse of notations, we will also write Hdg for a local generator of the ideal Hdg.

The Partial Igusa tower. Fix an interval $I = [p^a, p^b]$. Let $\mathfrak{X}_I := \mathfrak{X} \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathfrak{W}_I^0$. By an abuse of notation, we also let Hdg be the Hodge ideal inside $\mathcal{O}_{\mathfrak{X}_I}/(\alpha)$ where recall that α is the chosen pseudo-uniformizer depending on I. For any $r \geq 1$, consider the inverse image of $\mathrm{Hdg}^{p^{r+1}}$ under the natural map $\mathcal{O}_{\mathfrak{X}_I} \to \mathcal{O}_{\mathfrak{X}_I}/(\alpha)$ and call this ideal Hdg_r. Locally over $\mathrm{Spf}(R) \subset \mathfrak{X}_I$, Hdg_r is equal to $(\alpha, \mathrm{Hdg}^{p^{r+1}})$.

Let $\mathfrak{X}_{r,I} \to \mathfrak{X}_I$ be the open in the admissible blow-up of $\mathfrak{X}_I$ with respect to the ideal Hdg_r, defined such that the inverse image of Hdg_r is locally generated by $\mathrm{Hdg}^{p^{r+1}}$.

In case 1, that is, $I = [0, p^b]$, then $\frac{p}{\mathrm{Hdg}^{p^{r+1}}} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$. If $I = [p^a, p^b]$ as in case 2, then $\frac{p}{\mathrm{Hdg}^{p^{a+r+1}}} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$. We let n be an integer with $1 \leq n \leq r$ in case 1, and $1 \leq n \leq a + r$ in case 2.

Proposition 4.4. *For I, r, n as above, the semi-abelian scheme $E \to \mathfrak{X}_{r,I}$ admits a canonical subgroup H_n. To be more precise, we have the following properties:*

1. *H_n lifts* $\ker(\mathrm{Frob})$ *modulo p/λ, where* Frob *is the Frobenius and $\lambda = \mathrm{Hdg}^{\frac{p^n-1}{p-1}}$,*
2. *For any α-adically complete admissible Λ_I^0-algebra R with a morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{r,I}$,*

$$H_n(R) = \{s \in E[p^n](R) \mid s \bmod p/\lambda \in \ker(\mathrm{Frob})\},$$

3. *Let $L_n = E[p^n]/H_n$. Then ω_{L_n} is killed by λ and we have $\omega_{L_n} \cong \omega_E/\lambda\omega_E$,*

4. $E[p^n]/H_n \cong H_n^\vee$ *through the Weil pairing.*

Proof. The proof of these facts about the canonical subgroup can be found in [AIP18, Appendice A]. $\square$

Definition 4.5. Let I, r, n be as above. We let $\mathcal{IG}_{n,rI} \to \mathcal{X}_{r,I}$ be the adic space which classifies trivializations $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} H_n^\vee$, and define $g_n : \mathfrak{IG}_{n,rI} \to \mathfrak{X}_{r,I}$ to be the normalization of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,rI}$.

Fact. (c.f. [AI21, § 3.1]) The morphism $\mathcal{IG}_{n,rI} \to \mathcal{X}_{r,I}$ is finite étale and Galois with the Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. The morphism $g_n : \mathfrak{IG}_{n,rI} \to \mathfrak{X}_{r,I}$ is finite and is endowed with an action of $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

4.3 Heuristics on vector bundles with marked sections and overconvergent sheaves

In this subsection, we will briefly explain the idea of using vector bundles with marked sections to construct overconvergent sheaves. We figure it would be helpful to give a picture of the theory before going into detailed constructions. We also set up several notations that will be fixed during the following subsections.

Fix a positive integer n, then we take $I = [p^a, p^b]$ such that k_I^0 is analytic on $1 + p^{n-1}\mathbb{Z}_p$ and r such that H_n is defined over $\mathfrak{X}_{r,I}$. For example,

1. $I = [0, 1]$: For $p \neq 2$, take $2 \leq r$ and $2 \leq n \leq r$. For $p = 2$, take $r \geq 4$ and $4 \leq n \leq r$.
2. $I = [p^a, p^b]$ with $a, b \in \mathbb{N}$: For $p \neq 2$, take $b + 2 \leq a + r$ and $b + 2 \leq n \leq a + r$. For $p = 2$, $2 \leq r$, take $b + 4 \leq a + r$ and $b + 4 \leq n \leq a + r$.

The idea is fairly simple. First, one would like to construct a $(1 + p^{n-1}\mathbb{Z}_p)$-torsor $\tau : \mathfrak{T} \to \mathfrak{X}_{r,I}$. Then for an analytic weight k_I, we consider the sheaf $\tau_*\mathcal{O}_{\mathfrak{T}}[k_I^0]$ on which $1 + p^{n-1}\mathbb{Z}_p$ acts by k_I^0. The sections of such a sheaf should be close to a family of modular forms of weight k_I^0. Vector bundles with marked sections, especially the marked sections (say, modulo p^{n-1}), provide a desired $1 + p^{n-1}\mathbb{Z}_p$-action. Following this idea, we can construct overconvergent modular and de Rham sheaves, denoted by $\mathfrak{w}_{k_I}^0$ and $\mathbb{W}_{k_I}^0$ in §4.4 and §4.5.

On the other hand, let $k_I^f : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{Z}_p^\times \to \Lambda_I$ be the torsion part of the universal character k_I. For $p \neq 2$, we consider the morphism $h : \mathfrak{IG}_{1,r,I} \to \mathfrak{X}_{r,I}$ and for $p = 2$ we consider $h : \mathfrak{IG}_{2,r,I} \to \mathfrak{X}_{r,I}$. Set $\mathfrak{w}_{k_I}^f := h_*\mathcal{O}_{\mathfrak{IG}_{i,r,I}}[k_I^f]$, which takes care of the torsion part action. We then let $\mathfrak{w}_{k_I} := \mathfrak{w}_{k_I}^0 \hat{\otimes} \mathfrak{w}_{k_I}^f$ and $\mathbb{W}_{k_I} := \mathbb{W}_{k_I}^0 \hat{\otimes} \mathfrak{w}_{k_I}^f$. These are sheaves that interpolate $\mathrm{Sym}^\ell \omega_E$ and $\mathrm{Sym}^\ell \mathcal{H}$ respectively for all classical weight $\ell \in \mathbb{Z}$.

Remark 4.6. According to A. Kazi's observation in [Kaz22, § 1.3], the idea of using vector bundles with marked sections can be explained in the following way. As ω_E is a line bundle, its isomorphism class $[\omega_E]$ can be viewed as an element in $\check{H}^1(\mathfrak{X}, \mathbb{G}_m)$, where the cohomology is the Čech cohomology. For a classical weight $\ell \in \mathbb{Z}$, ω_E^ℓ corresponds to the image of $[\omega_E]$ under the map $\check{H}^1(\mathfrak{X}, \mathbb{G}_m) \xrightarrow{\ell} \check{H}^1(\mathfrak{X}, \mathbb{G}_m)$ induced by the ℓ-th power map on $\mathbb{G}_m$.

Now one would like to extend the same process to a p-adic weight. To do so, one needs the analyticity of k. Suppose k is analytic on an open subgroup $1 + p^{n-1}\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. Then we consider the map $k : 1 + p^{n-1}\mathbb{G}_a \to \widehat{\mathbb{G}_m}$. It is not hard to see that the group $\check{H}^1(\mathfrak{X}, 1 + p^{n-1}\mathbb{G}_a)$ classifies line bundles $\mathscr{L}$ on $\mathfrak{X}$ together with an isomorphism $\mathcal{O}_{\mathfrak{X}}/p^{n-1}\mathcal{O}_{\mathfrak{X}} \cong \mathscr{L}/p^{n-1}\mathscr{L}$. In other words, the sheaf $\mathscr{L}$ comes with a marked section corresponding to the isomorphism. This gives a more intuitive explanation of the use of vector bundles with marked sections.

4.4 The overconvergent modular sheaf $\mathfrak{w}_k$

Let n, r, I be fixed as in previous sections. Then the trivialization of $H_n^\vee$ over $\mathcal{IG}_{n,r,I}$ induces an equality

$$\mathbb{Z}/p^n\mathbb{Z} \cong H_n^\vee(\mathfrak{IG}_{n,r,I}) = H_n^\vee(\mathcal{IG}_{n,r,I}).$$

Let P^{univ} denote the image of $\bar{1} \in \mathbb{Z}/p^n\mathbb{Z}$ in $H_n^\vee(\mathfrak{IG}_{n,r,I})$. We have a diagram of $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-linear maps (c.f. [AI21, A.1])

$$\begin{tikzcd} & \omega_E \arrow[d] & \\ H_n^\vee(\mathfrak{IG}_{n,r,I}) \arrow[r, "d\log"] & \omega_{H_n} \arrow[r] & \omega_E/p^n \operatorname{Hdg}^{-\frac{p^n-1}{p-1}} \omega_E. \end{tikzcd}$$

Recall that a point $P \in H^\vee(R)$ gives a group homomorphism $\gamma_P : H_{n,R} \to \mathbb{G}_{m,R}$, which allows as to defined $d\log(P) := \gamma_P^*(dt/t)$ where dt/t is the canonical differential on $\mathbb{G}_m$.

Set $\beta_n := p^n \operatorname{Hdg}^{\frac{-p^n}{p-1}}$ and view $s := d\log(P^{\mathrm{univ}}))$ as a section of $\omega_E/\beta_n\omega_E$ by a further quotient of the above diagram.

One may naively expect (ω_E, s) to be a locally free sheaf with a marked section and then construct the modular sheaf via $\mathbb{V}_0(\omega_E, s)$. Unfortunately, the data (ω_E, s) does not satisfy the condition in Section 3 since the cokernel of the inclusion

$$s \cdot (\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/\beta_n\mathcal{O}_{\mathfrak{IG}_{n,r,I}}) \hookrightarrow \omega_E/\beta_n\omega_E$$

is annihilated precisely by $\delta := \operatorname{Hdg}^{\frac{1}{p-1}}$. Hence one needs to modify either ω_E or s.

Definition 4.7. Define $\Omega_E \subset \omega_E$ to be the inverse image of $d\log(P^{\mathrm{univ}})$ under the map $\omega_E \to \omega_{H_n}$. We call Ω_E the modified modular sheaf.

Then we have the following properties (c.f. [AI21, A.1]):

1. Ω_E is a locally free $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-module of rank 1.
2. The map $d\log$ induces an isomorphism

$$H^\vee(\mathfrak{IG}_{n,r,I}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/\beta_n \cong \Omega_E/\beta_n\Omega_E.$$

In particular, the pair $(\Omega_E.s)$ is a locally free sheaf with a marked section. Also, we have $\Omega_E = \delta\omega_E$, which implies that Ω_E is actually a free $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-module for $p \geq 5$.

Now, we have the formal scheme $\mathbb{V}_0(\Omega_E, s)$ with morphisms

$$u : \mathbb{V}_0(\Omega_E, s) \xrightarrow{u_0} \mathfrak{IG}_{n,r,I} \xrightarrow{g_n} \mathfrak{X}_{r,I}.$$

The morphism $u_0 : \mathbb{V}_0(\Omega_E, s) \to \mathfrak{IG}_{n,r,I}$ carries an action of the formal group scheme $\mathfrak{I} := 1+\beta\mathbb{G}_a$, which realizes $\mathbb{V}_0(\Omega_E, s)$ as a torsor. The action can be described explicitly on points. Let $\rho : \operatorname{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ be a morphism such that $\rho^*(\omega_E)$ is free of rank 1. By abuse of notations, we let β_n and δ be a generator of $\rho^*(\beta_n)$ and $\rho^*(\delta)$.

An R-point $(\rho, f) \in \mathbb{V}_0(\Omega_E, s)(R)$ consists of a morphism ρ as above and an element $f \in \operatorname{Hom}_R(\rho^*\Omega_E, R)$ such that $f(\rho^*s) \pmod{\beta_n} = 1$. Then for any $t \in 1 + \beta_n R$, the action is defined by $t * (\rho, f) := (\rho, tf)$. It is clear that (ρ, tf) is still an element of $\mathbb{V}_0(\Omega_E, s)(R)$.

One can also describe it in local coordinates. Recall that we can take an R-basis e of $\rho^*\Omega_E$ such that $e \pmod{\beta_n} = \rho^*(s)$. Then we have $\mathbb{V}_0(\Omega_E, s)(R) = \operatorname{Spf} R\langle Z\rangle \to \mathbb{V}_0(\Omega_E)(R) = \operatorname{Spf} R\langle X\rangle$ induced by $X \to 1 + \beta_n Z$, where X corresponds to the chosen basis e.

On the other hand,

$$\mathbb{V}_0(\Omega_E, s)(R) = \{f : \rho^*\Omega_E \to R \mid f(\rho^*s) \mod \beta_n = 1\} = (1 + \beta_n R)e^\vee$$

where $e^\vee$ is the dual basis to e. In particular, an element $(1 + \beta_n r)e^\vee$ correspond to the map $R\langle Z\rangle \to R$ sending $Z \to r = \frac{(1+\beta_n r)-1}{\beta_n}$.

One can then define the action of $1 + \beta_n R$ on $R\langle Z\rangle$ by letting $t * Z$ be the element in $R\langle Z\rangle$ such that $(t * f)(Z) = f(t * Z)$ for all $f \in \mathbb{V}_0(\Omega_E, s)(R)$ and $t \in 1 + \beta_n R$.

Suppose that $t = 1 + \beta_n b$, $f = (1 + \beta_n a)e^\vee$ and $t * Z = \sum_{m=0}^\infty c_m Z^m$. By definition, $f(t * Z) = \sum c_m a^m$, and $t*f = (1+\beta_n b)(1+\beta_n a)e^\vee = (1+\beta_n(a+b+\beta_n ab))e^\vee$. We see that $(t*f)(Z) = a+b+\beta_n ab = b+(1+\beta_n b)a$, and hence $t * Z = b + (1 + \beta_n b)Z = \frac{t-1}{\beta_n} + tZ$.

Since there is a $(\mathbb{Z}/p^n\mathbb{Z})^\times$-action on $g_n : \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$, the action of $\mathfrak{I}$ can be extended to an action of $\mathfrak{I}^{\text{ext}} := \mathbb{Z}_p^\times(1 + \beta_n \mathbb{G}_a)$ on $u : \mathbb{V}_0(\Omega_E, s) \to \mathfrak{X}_{r,I}$. We here describe this action.

Let $\lambda \in \mathbb{Z}_p^\times$ and $\bar\lambda$ be its image in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Then $\bar\lambda$ may be viewed as an isomorphism $\bar\lambda : \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$. Hence it induces a natural isomorphism $\gamma_\lambda : \Omega_E \cong \bar\lambda^*\Omega_E$ characterized by $\gamma_\lambda(s) \equiv \bar\lambda^{-1} \cdot s \pmod{p^n}$. The action of $\mathbb{Z}_p^\times$ is defined by $\lambda * (\rho, f) := (\bar\lambda \circ \rho, f \circ \gamma_\lambda^{-1})$

Definition 4.8. Let $k_I : \mathbb{Z}_p^\times \to \Lambda_I$ be the universal character on $\mathcal{W}_I$. Then we define $\mathfrak{w}_{k_I} := u_*(\mathcal{O}_{\mathbb{V}_0(\Omega_E,s)})[k_I]$. That is, the subsheaf on which $\mathfrak{I}^{\text{ext}}$ acts by k_I.

Local description of $\mathfrak{w}^0_{k_I}$. For simplicity, we write $k = k_I$. Let $\mathfrak{w}^0_k : u_{0*}(\mathcal{O}_{\mathbb{V}_0(\Omega_E,s)})[k]$, which is a sheaf on $\mathfrak{IG}_{n,r,I}$.

Lemma 4.9. *Let $\rho : \mathrm{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ be as before. Then we have the local description of $\mathfrak{w}^0_k$.*

1. $\rho^*(\mathfrak{w}^0_k) = R\langle Z\rangle[k] = R \cdot k(1 + \beta_n Z)$.
2. *$\mathfrak{w}^0_k$ is a locally free sheaf of rank* 1.

Proof. 1. First note that by the analyticity of k, $k(1 + \beta_n Z) \in R\langle Z\rangle$ is a well-defined element. Moreover, for $t \in 1 + \beta_n R$, we have

$$t * (1 + \beta_n Z) = t \cdot (1 + \beta_n Z).$$

This implies that $t * k(1 + \beta_n Z) = k(t) \cdot k(1 + \beta_n Z)$, i.e. $R \cdot k(1 + \beta_n Z) \subset R\langle Z\rangle[k]$. For the inverse inclusion, we refer to [AI21, Lemma 3.9].

2. It follows directly from part 1. □

Remark 4.10. The element $k(1 + \beta_n Z)$ should be viewed as the "k-th power" of $1 + \beta_n Z$. For this reason, we will also write $(1 + \beta_n Z)^k$ for $k(1 + \beta_n Z)$.

Remark 4.11. If we wish to describe $\mathfrak{w}_k$, we need to consider the $\mathbb{Z}_p^\times$-action on $\mathfrak{w}^0_k$ and descend from $\mathfrak{IG}_{n,r,I}$ to $\mathfrak{X}_{r,I}$. In particular, one sees that $g_{n*}(\mathfrak{w}^0_k)$ decomposes into $|(\mathbb{Z}/p^n\mathbb{Z})^\times|$-pieces according to the residual action.

4.5 The overconvergent de Rham sheave $\mathbb{W}_k$

Similar to the modular sheaf, the pair $(\mathcal{H}, s)$ is not a locally free sheaf with a marked section either. So again, we need to modify it.

Definition 4.12. We define $\mathcal{H}^\sharp := \Omega_E + \delta^p\mathcal{H}$ to be the modified de Rham sheaf. It is the push-out in the category of coherent sheaves on $\mathfrak{X}_{r,I}$ of the diagram

$$\begin{array}{ccc} \delta^p\omega_E & \longrightarrow & \delta^p\mathcal{H} \\ \cap & & \\ \Omega_E & & \end{array} \quad .$$

As δ is locally free, $\mathcal{H}^\sharp$ is locally free of rank 2. Moreover, it sits inside the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \delta^p\omega_E \arrow[r] \arrow[d, phantom, "\cap"] & \delta^p\mathcal{H} \arrow[r] \arrow[d] & \delta^p\omega_E^{-1} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
0 \arrow[r] & \Omega_E \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{H}^\sharp \arrow[r] \arrow[d] & \delta^p\omega_E^{-1} \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
0 \arrow[r] & \omega_E \arrow[r] & \mathcal{H} \arrow[r] & \omega_E^{-1} \arrow[r] & 0.
\end{tikzcd}$$

It follows that $(\mathcal{H}^\sharp, s)$ is a locally free sheaf with a marked section, where recall that $s = d\log(P^{\mathrm{univ}})$ is viewed as a section of $\Omega_E/\beta_n\Omega_E \subset \mathcal{H}^\sharp/\beta_n\mathcal{H}^\sharp$. Hence we have $\mathbb{V}_0(\mathcal{H}^\sharp, s)$, together with a morphism $v_0 : \mathbb{V}_0(\mathcal{H}^\sharp, s) \to \mathfrak{IG}_{n,r,I}$ that factors through u_0. We let $v := g_n \circ v_0 : \mathbb{V}_0(\mathcal{H}^\sharp, s) \to \mathfrak{X}_{r,I}$.

Recall that $\Omega_E \subset \mathcal{H}^\sharp$ gives a filtration of locally free sheaves with a marked section $(\Omega_E, s) \subset (\mathcal{H}^\sharp, s)$. Therefore, the sheaf $v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})$ is endowed with a filtration $\mathrm{Fil}_\bullet(v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)}))$.

Similarly, there is an action of $\mathfrak{I}$ on $\mathbb{V}_0(\mathcal{H}^\sharp, s)$ over $\mathfrak{IG}_{n,r,I}$ and an action of $\mathfrak{I}^{\mathrm{ext}}$ on $\mathbb{V}_0(\mathcal{H}^\sharp, s)$ over $\mathfrak{X}_{r,I}$.

Definition 4.13. Let $k_I : \mathbb{Z}_p^\times \to \Lambda_I$ be the universal character on $\mathcal{W}_I$. Then we define $\mathbb{W}_{k_I} := v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})[k_I]$.

Theorem 4.14 ([AI21, Theroem 3.11])**.** *The action of $\mathfrak{I}^{\mathrm{ext}}$ preserves the filtration on $v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})$. For any $h \in \mathbb{N}$, define $\mathrm{Fil}_h \mathbb{W}_{k_I} := \mathrm{Fil}_h v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})[k_I]$. Then,*

1. *Each $\mathrm{Fil}_h \mathbb{W}_{k_I}$ is a coherent $\mathcal{O}_{\mathfrak{X}_{r,I}}$-module of finite rank.*
2. *$\mathbb{W}_{k_I}$ is the α-adic completion of $\lim_{h\to\infty} \mathrm{Fil}_h \mathbb{W}_{k_I}$.*
3. *$\mathrm{Fil}_0 \mathbb{W}_{k_I} = \mathfrak{w}_{k_I}$ and $\mathrm{Gr}_h \mathbb{W}_{k_I} \cong \mathfrak{w}_{k_I} \otimes_{\mathcal{O}_{\mathfrak{X}_{r,I}}} \mathrm{Hdg}^h \omega_E^{-2h}$.*

When we specialize k_I to a classical weight $m \in \mathbb{N}$, viewed as a point in the weight space, and write $\mathbb{W}_m$ for the corresponding base-change of $\mathbb{W}_{k_I}$, then we have the identification

$$\mathrm{Sym}^m(\mathcal{H})[1/p] = \mathrm{Fil}_m(W_m)[1/p]$$

over the adic fiber $\mathcal{X}_{r,I}$. Moreover, this identification is compatible with the natural Hodge filtration on $\mathrm{Sym}^m(\mathcal{H})$.

Local description of $\mathbb{W}_k^0$ For simplicity, we write $k = k_I$ and $\mathbb{W}_k^0 := v_{0*}\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp)}[k]$. We first give descriptions of the $\mathfrak{I}$-action.

Let $\rho : \mathrm{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ be a morphism such that $\rho^*(\omega_E)$ is a free R-module of rank one and β_n, δ be as before. We fix an R-basis (e, e') of $\rho^*(\mathcal{H}^\sharp)$ such that $e \pmod{\beta_n} = \rho^*(s)$. We denote by $(e^\vee, e'^\vee)$ the dual basis. Then we have

$$\mathbb{V}_0(\mathcal{H}^\sharp, s)(R) = \{f = ae^\vee + be'^\vee \mid a \in 1 + \beta_n R, b \in R\}.$$

Also, we have $\mathbb{V}_0(\mathcal{H}^\sharp, s) \times_{\mathfrak{IG}_{n,r,I}} \mathrm{Spf}\, R = \mathrm{Spf}(R\langle Z, Y\rangle)$, where a point $ae^\vee + be'^\vee$ corresponds to the R-algebra homomorphism $R\langle Z, Y\rangle \to R$ given by $Z \mapsto \frac{a-1}{\beta_n}$ and $Y \mapsto b$. As before, the action of $t \in 1 + \beta_n R$ on f is simply $t * f = tf$. In terms of coordinates Z and Y, we have $t * Z = \frac{t-1}{\beta_n} + tZ$, $t * Y = tY$.

Lemma 4.15. *Let $\rho : \mathrm{Spf}(R) \to \mathfrak{IG}_{n,r,I}$ be as above. Then we have*

$$\rho^*(v_{0*}(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})[k]) = R\langle Z, Y\rangle[k] = \{\sum_{m=0}^{\infty} a_m(1 + \beta_n Z)^k \frac{Y^m}{(1 + \beta_n Z)^m}\},$$

where $a_m \in R$ for all m and $a_m \to 0$ as $m \to \infty$. Moreover, the filtration is given by

$$\rho^*(\mathrm{Fil}_h v_{0*}(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp,s)})[k]) = \{\sum_{m=0}^{h} a_m(1 + \beta_n Z)^k \frac{Y^m}{(1 + \beta_n Z)^m}\}.$$

One can see that this description is in accordance with the fact that $\mathrm{Fil}_0 \mathbb{W}_k^0 = \mathfrak{w}_k^0$.

Proof. (c.f. [AI21, Lemma 3.13]) The proof is similar to Lemma 4.9, so we adopt the same notations. Clearly, $(1+\beta_n Z)^{k-m}Y^m \in R\langle Z, Y\rangle$. In addition, for $t \in 1+\beta_n R$, one can check that

$$t * (1+\beta_n Z)^{k-m}Y^m = k(t)\cdot(1+\beta_n Z)^{k-m}Y^m$$

for all $m \geq 0$.

For the converse inclusion, one needs to observe that the $(1+\beta_n R)$-invariant subspace of $R\langle Z, Y\rangle$ is $R\langle V\rangle$ where $V := \frac{Y}{1+\beta_n Z}$ (c.f. proof of [AI21, Lemma 3.13]). If $f(Z,Y) \in R\langle Z, Y\rangle[k]$, then

$$\frac{f(Y,Z)}{(1+\beta_n Z)^k} \in R\langle Z, Y\rangle^{1+\beta_n R} = R\langle V\rangle.$$

Hence, $f(Y,Z) \in (1+\beta_n Z)^k R\langle V\rangle$, which proves the lemma. $\square$

4.6 The q-expansions

For any morphism $\mathfrak{S} \to \mathfrak{X}$, we let $\mathfrak{S}^{\mathrm{ord}}$ be the the inverse image of the ordinary locus $\mathfrak{X}^{\mathrm{ord}}$. Over $\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}$, we have $\mathcal{H}^\sharp = \mathcal{H} = \omega_E \oplus \omega_E^{-1}$ via the unit-root splitting. In particular, the splitting induces an isomorphism

$$\mathbb{V}_0(\mathcal{H}^\sharp, s)^{\mathrm{ord}} \to \mathbb{V}_0(\omega_E, s)^{\mathrm{ord}} \times_{\mathfrak{IG}^{\mathrm{ord}}_{n,r,I}} \mathbb{V}(\omega_E^{-1}).$$

Assume that the weight k is analytic on $1+p^n\mathbb{Z}_p$. Let $u^{\mathrm{ord}} : \mathbb{V}_0(\omega_E, s)^{\mathrm{ord}} \to \mathfrak{X}^{\mathrm{ord}}$ be the canonical projection, then the global sections of $\omega^{\mathrm{ord}}_{k^0} := u^{\mathrm{ord}}_*(\mathcal{O}_{\mathbb{V}_0(\omega_E,s)^{\mathrm{ord}}})[k^0]$ over $\mathfrak{X}^{\mathrm{ord}}$ coincide with Katz's p-adic modular forms of weight k^0. Similarly, we can define $\omega^{\mathrm{ord}}_k := \omega^{\mathrm{ord}}_{k^0} \otimes \omega_{k_f}|_{\mathfrak{X}^{\mathrm{ord}}}$. Then we have canonical decompositions

$$\mathbb{W}^0_k|_{\mathfrak{X}^{\mathrm{ord}}} \cong \omega^{\mathrm{ord}}_{k^0} \hat{\otimes} \operatorname{Sym}(\omega_E^{-2}),$$
$$\mathbb{W}_k|_{\mathfrak{X}^{\mathrm{ord}}} \cong \omega^{\mathrm{ord}}_{k} \hat{\otimes} \operatorname{Sym}(\omega_E^{-2}).$$

So there are natural splittings

$$\mathbb{W}^0_k|_{\mathfrak{X}^{\mathrm{ord}}} \to \omega^{\mathrm{ord}}_{k^0}, \quad \mathbb{W}_k|_{\mathfrak{X}^{\mathrm{ord}}} \to \omega^{\mathrm{ord}}_{k}$$

of the filtrations $\operatorname{Fil}_0 \mathbb{W}^0_k$ and $\operatorname{Fil}_0 \mathbb{W}_k$.

Definition 4.16. Using the q-expansion map for Katz's p-adic modular forms, we obtain the following map

$$H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k) \xrightarrow{res} H^0(\mathfrak{X}^{\mathrm{ord}}, \mathbb{W}_k|_{\mathrm{ord}}) \to H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^{\mathrm{ord}}_k) \to \Lambda_I((q)),$$

which we will call the q-expansion map.

The q-expansion map can be made more explicit by using the Tate curve. Consider the Tate curve $E = \mathrm{Tate}(q^N)$ over $\mathrm{Spf}(R)$ with $R = \Lambda^0_I((q))$. We fix a basis $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}} := \nabla(\theta)(\omega_{\mathrm{can}})\}$ of $\mathcal{H}$, where θ is the derivation dual to $\frac{dq}{q} = \mathrm{KS}(\omega^2_{\mathrm{can}})$. Let $\mathbb{W}^0_k(q)$ be the module of $\mathbb{W}^0_k$ over the Tate curve E. Then we have the local description $\mathbb{W}^0_k(q) = R\langle V\rangle(1+p^nZ)^k$ where $V := \frac{Y}{1+p^nZ}$. If we set $V_{k,i}(q) = Y^i(1+p^nZ)^{k-i}$, then $\operatorname{Fil}_h \mathbb{W}^0_k(q) = \sum_{i=0}^h R\cdot V_{k,i}$. The q-expansion map correspond to the projection $\mathbb{W}^0_k(q) \to R$ given by $\sum_{i\geq 0} a_i V_{k,i}(q) \mapsto a_0$. The q-expansion map for $\mathbb{W}_k$ can be described similarly by replacing $\Lambda^0_I((q))$ with $\Lambda_I((q))$.

Remark 4.17. As the local description is constructed by using the canonical basis $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}}\}$ of $\mathcal{H}$ on the Tate curve, when specialized to a classical weight $\ell \in \mathbb{N}$, the basis $V_{\ell,i}$ is specialized to the element $\omega^{\ell-i}_{\mathrm{can}}\eta^i_{\mathrm{can}}$ for all $i \leq \ell$.

4.7 The operators U and V

Similar to the classical case, we can define the operators U and V by using Hecke correspondences, as recalled below.

Consider the two morphism $p_1, p_2 : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$ defined on the universal elliptic curve by $E \mapsto E$ and $E \mapsto E' := E/H_1$ respectively. Over $\mathfrak{IG}_{n,r+1,I}$, we have the isogeny $\lambda : E' \to E$, which is dual to the natural map $\pi : E \to E'$.

Proposition 4.18 ([AI21, Proposition 3.24]). *The isogeny λ induces a morphism of $\mathcal{O}_{\mathfrak{X}_{r,I}}$-modules*

$$\mathcal{U} : p_{2*}p_1^*\mathbb{W}_k \to p_{2*}p_2^*\mathbb{W}_k$$

which commutes with the Gauss–Manin connection and preserves the filtrations.

Further, as the map p_2 is finite flat of degree p, there are well-defined trace map $\mathrm{tr} : p_{2*}\mathcal{O}_{\mathfrak{X}_{r+1,I}} \to \mathcal{O}_{\mathfrak{X}_{r,I}}$ and $\mathrm{tr} : p_{2*}p_2^*\mathbb{W}_k \to \mathbb{W}_k$. We then define the operator U as

$$U : H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k) \xrightarrow{\mathcal{U}\circ p_{2*}p_1^*} H^0(\mathfrak{X}_{r,I}, p_{2*}p_2^*\mathbb{W}_k) \xrightarrow{\frac{1}{p}\,\mathrm{tr}} H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)[p^{-1}].$$

Proposition 4.19 ([AI21, Proposition 3.25]). *Assume that $I \subset [0,1]$ and let $\alpha = p$ or assume that $I \subset [1,\infty]$ and $\alpha = T$. Then $U(H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)) \subset \frac{1}{\alpha}H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)$, and the induced map on $H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k/\operatorname{Fil}_h \mathbb{W}_k)$ is 0 modulo $\alpha^{[h/p]-1}$ for $h \geq p$. Moreover, if $m \in \mathbb{N}$ is an integral weight, then the identification $\operatorname{Fil}_m \mathbb{W}_m[p^{-1}]|_{\mathcal{X}_r} = \operatorname{Sym}^m \mathcal{H}[p^{-1}]|_{\mathcal{X}_r}$ on the adic fiber is compatible with the U-operators defined on the two sides.*

Passing to the adic fiber $\mathcal{X}_{r,I}$, we have the following result.

Corollary 4.20 ([AI21, Corollary 3.26]). *The operator U on $H^0(\mathcal{X}_{r,I}, \mathbb{W}_k)$ admits a Fredholm determinant $P_I(k, X) \in \Lambda_I[\![X]\!]$, and for any non-negative $a \in \mathbb{Q}$, the space $H^0(\mathcal{X}_{r,I}, \mathbb{W}_k)$ admits a slope a-decomposition. Similarly, for any $h \in \mathbb{N}$, we also have the Fredholm determinant $P_I^h(k, X)$ on $H^0(\mathcal{X}_{r,I}, \operatorname{Fil}_h \mathbb{W}_k)$ and a slope a-decomposition. Finally, for a fixed $a \in \mathbb{Q}$, the inclusion $H^0(\mathcal{X}_{r,I}, \operatorname{Fil}_h \mathbb{W}_k)^{\leq a} \subset H^0(\mathcal{X}_{r,I}, \mathbb{W}_k)^{\leq a}$ is an equality for h large enough.*

For the operator V, we first observe that the map $\Phi : \mathfrak{X}_{r+1,I} \to \mathfrak{X}_{r,I}$, previously denoted by p_2, lifts naturally to $\Phi : \mathfrak{IG}_{n+1,r+1,I} \to \mathfrak{IG}_{n,r,I}$. By [AI21, § A2] the dual isogeny λ induces an isomorphism $\lambda^* : \Omega_E \to \Phi^*(\Omega_E)$ over $\mathfrak{IG}_{n+1,r+1,I}$ and hence a morphism $\mathbb{W}_k \to \Phi^*(\mathbb{W}_k)$ which provides an isomorphism $\lambda^* : \mathfrak{w}_k \to \Phi^*(\mathfrak{w}_k)$ on the 0-th filtrations. We then define

$$V : H^0(\mathfrak{X}_{r,I}, \mathfrak{w}_k) \to H^0(\mathfrak{X}_{r+1,I}, \mathfrak{w}_k), \quad V(\gamma) := (\lambda^*)^{-1}\Phi^*(\gamma).$$

Note that on the q-expansion $\sum_{n\geq 0} a_n q^n$ of an element in $H^0(\mathfrak{X}_{r,I}, \mathfrak{w}_k)$, we have the familiar formulae

$$U\left(\sum a_n q^n\right) = \sum a_{np} q^n,$$
$$V\left(\sum a_n q^n\right) = \sum a_n q^{np}.$$

In particular, $U \circ V = \mathrm{Id}$.

Definition 4.21. Let $f \in H^0(\mathfrak{X}_{r,I}, \mathfrak{w}_k)$. The p-depletion of f, denoted by $f^{[p]}$, is defined as $f^{[p]} := (1-VU)f$.

Remark 4.22. Let $\sum_{n\geq 0} a_n q^n$ be the q-expansion of f, then the q-expansion of $f^{[p]}$ is $\sum_{n\geq 0, p\nmid n} a_n q^n$. Consequently, one easily sees that $U(f^{[p]}) = 0$.

4.8 p-adic iterations of the Gauss–Manin connection

We first study the explicit description of the connection on $\mathbb{W}_k$. Let $\rho : S = \operatorname{Spf}(R) \to \mathfrak{IG}'_{n,r,I}$ be a morphism of formal scheme over $\operatorname{Spf}(\Lambda_I^0)$. Assume that the composite of ρ and the projection to the modular curve $\mathfrak{X}$ factors through $\zeta : S \to U$ where $U \subset \mathfrak{X}$ is an open affine over which $\mathcal{H}$ is free with basis $\{\omega, \eta\}$, and ω

spans ω_E. Let the notations δ, β_n be as before. By definition, the R-modules $\zeta^*\mathcal{H}$ and $\rho^*\mathcal{H}^\sharp$ are free of rank 2 with basis $\{\zeta^*\omega, \zeta^*\eta\}$ and $\{e' = \delta\omega, e = \delta^p\eta\}$ respectively.

To describe the connection, we utilize Grothendieck's formalism. Let $\mathcal{P}^{(1)} \subset \mathrm{Spf}(R\hat{\otimes}_{\Lambda_I^0} R)$ be the first infinitesimal neighborhood. The R-module $\rho^*(\mathcal{H}^\sharp)$ admits an integrable connection $\nabla^\sharp$, which can be expressed as an isomorphism $\epsilon^\sharp : j_2^*(\rho^*(\mathcal{H}^\sharp)) \cong j_1^*(\rho^*(\mathcal{H}^\sharp))$, where $j_i : S \times S \to S$ is the projection to the i-th component. Let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}^{(1)}})$$

be the the matrix of $\epsilon^\sharp$ with respect to the basis $\{1 \otimes e', 1 \otimes e\}$ and $\{e' \otimes 1, e \otimes 1\}$.

Lemma 4.23 ([AI21, Lemma 3.20])**.** *We have*

1. $a = 1 + a_0$, $d = 1 + d_0$ *with* $a_0, b, c, d_0 \in I(\Delta)$. *So* $a_0^2 = b^2 = c^2 = d_0^2 = 0$ *in* $\mathcal{O}_{\mathcal{P}^{(1)}}$.
2. *If we interpret* $a_0, b, c, d_0 \in I(\Delta)/I(\Delta)^2$ *as elements in* Ω^1_{R/Λ_I^0}, *then* $a_0, b, c, d_0 \in \frac{1}{\mathrm{Hdg}} \cdot \zeta^*(\Omega^1_{\mathfrak{X}})$ *and* $\mathrm{Hdg} \cdot c$ *is the Kodaira–Spencer differential* $\mathrm{KS}(\omega, \eta)$ *associated to the local basis* $\{\omega, \eta\}$ *of* $\mathcal{H}$.

Proof. The first statement comes from the fact that $A \equiv \mathrm{Id} \pmod{I(\Delta)}$, and $I(\Delta)^2 = 0$ in $\mathcal{O}_{\mathcal{P}^{(1)}}$.

For the second statement, recall that $\nabla^\sharp$ is uniquely determined by the connection on $\rho^*\mathcal{H}^\sharp \subset \zeta^*\mathcal{H}$. Note that we have $\delta^{p-1} = \mathrm{Hdg}$ by definition, and in this situation $\delta^{p-1} = \zeta^*(u)$ for a section $u \in H^0(U, \mathcal{O}_U)$. So we have

$$d\zeta^*(u) = d\delta^{p-1} = (p-1)\delta^{p-2}d\delta = (p-1)\delta^{p-1}d\log(\delta).$$

This implies that $d\log(\delta) = (p-1)^{-1}d\log(\zeta^*(u)) \in \frac{1}{\mathrm{Hdg}}\zeta^*\Omega^1_{\mathfrak{X}/\mathbb{Z}_p}$. The Kodaira–Spencer isomorphism

$$\mathrm{KS} : \omega_E \to \omega_E^{-1} \otimes \Omega^1_{\mathfrak{X}/\mathbb{Z}_p},$$

provides a basis element $\Theta := \mathrm{KS}(\omega, \eta)$ of $\Omega^1_{\mathfrak{X}/\mathbb{Z}_p}$ over U characterized by the property $\mathrm{KS}(\omega) = \eta \otimes \Theta$. We may now write the connection as

$$\begin{aligned} \nabla(\omega) &= m\omega \otimes \Theta + \eta \otimes \Theta \\ \nabla(\eta) &= q\omega \otimes \Theta + r\eta \otimes \Theta, \end{aligned}$$

where $m, q, r \in H^0(U, \mathcal{O}_U)$. Hence, omitting the notation ζ^* for simplicity, we have

$$\begin{aligned} \nabla^\sharp(e') &= \nabla(\delta\omega) = \delta\nabla(\omega) + \delta\omega \otimes d\log(\delta) \\ &= me' \otimes \Theta + e' \otimes \frac{du}{(p-1)u} + \frac{1}{\delta^{p-1}} e \otimes \Theta, \\ \nabla^\sharp(e) &= \nabla(\delta^p\eta) = \delta^p\nabla(\eta) + p\delta^p\eta \otimes d\log(\delta) \\ &= \delta^{p-1}qe' \otimes \Theta + re \otimes \Theta + pe \otimes \frac{du}{(p-1)u}. \end{aligned}$$

Carefully writing down the correspondence between $\nabla^\sharp$ and the isomorphism $\epsilon^\sharp$, we see that the entry c of the matrix A is $\frac{1}{\delta^{p-1}}\Theta$. ☐

Now we wish to understand the induced connection on the vector bundle with a marked section. Let

$$\zeta^*(\mathbb{W}_k^0) = \{\sum_{m=0}^{\infty} a_m V^m (1 + \beta_n Z)^k \mid a_m \in R,\ a_m \to 0,\ V = \frac{Y}{1 + \beta_n Z}\}$$

be the local description of $\mathbb{W}_k^0$. Similarly, we have

$$j_i^*(\zeta^*(\mathbb{W}_k^0)) = \{\sum_{m=0}^{\infty} a_m V^m (1 + \beta_n Z)^k \mid a_m \in j_i^*(R),\ a_m \to 0,\ V = \frac{Y}{1 + \beta_n Z}\}$$

To describe the connection ∇_k on $\mathbb{W}_k$, it suffices to describe its corresponding isomorphism $\epsilon_k : j_2^*(\zeta^*(\mathbb{W}_k^0)) \to j_1^*(\zeta^*(\mathbb{W}_k^0))$. This isomorphism is induced by the matrix A. As $X = 1 + \beta_n Z$ corresponds to the basis e' and Y corresponds to the basis e, X is sent to $aX + cY$ and Y is sent to $bX + dY$. Hence, one gets

$$\begin{aligned}\epsilon_k \circ j_2^*(V^m(1+\beta_n Z)^k) &= j_1^*[(b+dV)^m(a+cV)^{-m}(a+cV)^k(1+\beta_n Z)^k] \\ &= j_1^*[(a+cV)^{k-m}(b+dV)^m(1+\beta_n Z)^k].\end{aligned}$$

Use the fact that k is analytic with $k(t) = \exp(u\log(t))$ for some element $u \in p^{1-n}\Lambda_I^0$, we can write $(a+cV)^{k-m} = \exp((u-m)\log(1+a_0+cV)) = 1 + (u-m)(a_0+cV)$. On the other hand, $(b+dV)^m = (V+b+d_0V)^m = V^m + mV^{m-1}(b+d_0V)$. Therefore,

$$\epsilon_k \circ j_2^*(V^m(1+\beta_n Z)^k) = j_1^*\left[\left((1+md_0+(u-m)a_0)V^m + mbV^{m-1} + (u-m)cV^{m+1}\right)(1+\beta_n Z)^k\right].$$

We can now recover ∇_k by using the relation $\nabla_k(V^m(1+\beta_n Z)^k) = \epsilon_k \circ j_2^*(V^m(1+\beta_n Z)^k) - j_1^*(V^m(1+\beta_n Z)^k)$. To be more precise, we have

$$\begin{aligned}&\nabla_k(V^m(1+\beta_n Z)^k) \\ &\quad = \left(mV^m \otimes d_0 + (u-m)V^m \otimes a_0 + mV^{m-1} \otimes b + (u-m)V^{m+1} \otimes c\right) \times ((1+\beta_n Z)\otimes 1)^k. \qquad (24)\end{aligned}$$

Let $E = \mathrm{Tate}(q^N)$ be the Tate curve over $\mathrm{Spf}(R)$ with $R = \Lambda_I((q))$ as before. Recall that we have a basis $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}} := \nabla(\theta)(\omega_{\mathrm{can}})\}$ with $\theta = q\frac{d}{dq}$. With respect to this basis, the matrix of the Gauss–Manin connection ∇ is given by

$$\begin{pmatrix} 0 & 0 \\ \frac{dq}{q} & 0 \end{pmatrix}$$

By equation (24), on $\mathbb{W}_k(q)$ we have

$$\nabla(aV_{k,h}) = \theta(a)V_{k+2,h} + a(u_k - h)V_{k+2,h+1}.$$

Note that when specialized to a classical weight $\ell \in \mathbb{Z}$, this is the familiar formula for ∇ on $\mathrm{Sym}^\ell \mathcal{H}$ as $u_k \mapsto \ell$ under the specialization.

Iterating this formula, we have the following lemma.

Lemma 4.24. *Let $g(q) \in R$ and $s \in \mathbb{N}$. Then we have the formula*

$$\nabla^s(g(q)V_{k,h}) = \sum_{j=0}^{s} a_{s,k,h,j}\theta^{s-j}g(q)\cdot V_{k+2s,h+j}$$

where $a_{s,k,h,j} \in R$ are given by $a_{s,k,h,0} = 1$ and

$$\begin{aligned}a_{s,k,h,j} &= \binom{s}{j}\frac{(u_k-h+s-1)\cdots(u_k-h+1)(u_k-h)}{(u_k-h+s-1-j)\cdots(u_k-h+1)(u_k-h)} \\ &= \binom{s}{j}\prod_{i=0}^{j-1}(u_k-h+s-1-i).\end{aligned}$$

One might naively expect the same formula to hold for an arbitrary analytic p-adic weight $s : \mathbb{Z}_p^\times \to \Lambda_{I_s}$. However, one needs to worry about the convergence of the operator. One should also be aware of that the norm we are considering is not the naively one given by the Gauss norm on q-expansions, but rather the sup-norm of functions on a strict neighborhood.

We will list below a series of results on (integral) iterations of the Gauss–Manin connection, which in the end leads to the convergence of certain p-adic powers of ∇ (Theorem 4.30). We prefer to omit the proofs, since they are highly technical and lengthy. Detailed computations can be found in [AI21, § 3.10 & § 4]. We hope that by providing these statements, the reader can at least grasp the idea of this construction.

Proposition 4.25 ([AI21, Proposition 3.41 & Claim 3.42]). *Assume that $g(q) \in R[\![q]\!]$ is the q-expansion of an overconvergent modular form g of weight k and $U(g(q)) = 0$. For every positive integer N, we may write*

$$\left(\frac{(\nabla^{p-1} - \mathrm{Id})^{Np}}{p^N}\right) g(q)V_{k,0} = \sum_{r=0}^{(p-1)pN} \sum_{h=0}^{\infty} p^{N-2r-h} \frac{((1+pZ)^{2(p-1)} - 1)^{hp}}{p^h} g_{r,h}^{(N)} V_{k,r}$$

where $g_{r,h}^{(N)} \in R^{U=0}[1+pZ]$ is a polynomial in $1+pZ$ with coefficients in $R^{U=0}$. If we further assume that $u_k \in p\Lambda_I$, then $p^{N-2r-h} g_{r,h}^{(N)} \in R^{U=0}[1+pZ]$ for every r and h. In particular, p^{2r+h-N} divides $g_{r,h}^{(N)}$ whenever $2r + h - N \geq 0$.

Write $\mathfrak{w}_k(q)$ for the evaluation of the sheaf $\mathfrak{w}_k$ at the Tate curve. It can be viewed as a submodule of $\mathbb{W}_k(q)$ via the identification $\mathfrak{w}_k(q) = \mathrm{Fil}_0 \mathbb{W}_k(q)$.

Corollary 4.26 ([AI21, Corollary 3.43]). *Suppose that we have $g(q) \in \mathfrak{w}_k(q)$ with $U(g(q)) = 0$. Then for every positive integer N, we have*

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(g(q)) \in \sum_{n=0}^{(p-1)pN} p^{2N-2n} \mathfrak{w}_k(q)[Z] V_{k,n}.$$

Moreover, if $u_k \in p\Lambda_I$, then

$$(\nabla^{p-1} - \mathrm{Id})^{Np}(g(q)) \in p^N \cdot \left(\sum_{n=0}^{(p-1)pN} \mathfrak{w}_k(q)[Z] V_{k,n}\right).$$

Assumption 4.27. Assume that the weights k and s satisfy the condition: $k = \chi \cdot k_0 \cdot v$ and $s = \chi' \cdot s_0 \cdot w$, where

1. χ, χ' are finite-order character on $\mathbb{Z}_p^\times$ and χ is even.
2. k_0 and s_0 are integral weights such that k_0 is even modulo p. That is, there are integers a, b with a even modulo p such that $k_0(t) = t^a$, $s_0(t) = t^b$ for all $t \in \mathbb{Z}_p^\times$.
3. v, w are analytic weights such that there exist $u_v \in p\Lambda_I$, $u_w \in q\Lambda_{I_s}$ satisfying $v(t) = \exp(u_v \log(t))$ and $w(t) = \exp(u_w \log(t))$ for all $t \in \mathbb{Z}_p^\times$.

Suppose that $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)^{U=0}$ and k, s satisfy Assumption 4.27. The idea is to set

$$\nabla_k^s(g) := \exp\left(\frac{u_s}{p-1} \log(\nabla_k^{p-1})\right)(g)$$

and show that this expression makes sense and gives rise to a section of $\mathbb{W}_{k+2s}$.

Recall that we have fixed integers r and n as in §4.3. Set $\mathbb{W} := v_*(\mathcal{O}_{\mathbb{V}_0(\mathcal{H}^\sharp, s)})$ and $\mathbb{W}'_k \subset \mathbb{W}$ be the subsheaf defined by $\mathbb{W}'_k := \sum_{i \in \mathbb{Z}} \mathbb{W}_{k+2i}$.

Proposition 4.28 ([AI21, Propopsition 4.11]). *Let s be a non-negative integer. Then there exists a positive integer $b \geq r$, depending on r, n and s such that for every section $g \in \mathrm{Hdg}^{-s} H^0(\mathfrak{X}_{r,I}, \mathbb{W})$ with $g|_{\mathfrak{X}_{r,I}^{\mathrm{ord}}} \in H^0(\mathfrak{X}_{r,I}^{\mathrm{ord}}, p^j \mathbb{W})$ for some $j \in \mathbb{N}$, we have $g \in H^0(\mathfrak{X}_{b,I}, p^{[j/2]} \mathbb{W})$. In other words, one can retain some information on the divisibility by restricting to the ordinary locus.*

Proposition 4.29 ([AI21, Corollary 4.12]). *There exist integers b and c_n depending on r and n such that, for every $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)^{U=0}$ and every positive integer N, we have*

$$\mathrm{Hdg}^{c_n(p-1)^2} (\nabla^{p-1} - \mathrm{Id})^N (g) \subset p^{[N/2p]} H^0(\mathfrak{X}_{b,I}, \mathbb{W}) \cap H^0(\mathfrak{X}_{b,I}, \mathbb{W}'_k).$$

At last, we have the following main theorem.

Theorem 4.30 ([AI21, Theorem 4.3]). *Let k, s be two weights that satisfy Assumption 4.27, and let $g \in H^0(\mathfrak{X}_{r,I}, \mathbb{W}_k)^{U=0}$. Then there exists positive integers γ and b depending on r, n, p and a unique element $\nabla_k^s(g)$ of $\mathrm{Hdg}^{-\gamma} H^0(\mathfrak{X}_{b,I}, \mathbb{W}_{k+2s})$ such that on the q-expansions, if $g(q) = \sum_h g_h(q) V_{k,h}$, then*

$$\nabla_k^s(g)(q) = \sum_h \sum_{j=0}^{\infty} \binom{u_s}{j} \prod_{i=0}^{j-1} (u_k + u_s - h - 1 - i) \theta^{s-j}(g_h(q)) V_{k+2s,h+j},$$

where $\binom{u_s}{j} = \frac{u_s(u_s-1)\cdots(u_s-j+1)}{j!}$, and

$$\theta^{s-j}(g_h(q)) = \sum s(n) n^{-j} a_{h,n} q^n$$

if we write $g_h(q) = \sum_{n, p \nmid n} a_{h,n} q^n$.

Remark 4.31. In [Kaz22], a more refined tool called vector bundles with marked sections and marked splitting is considered. With this modification, one can get a better control on the convergence and loosen Assumption 4.27.

Remark 4.32. In our application later, we will be particularly interested in the specialization of $\nabla_k^s(g)$ to a pair of classical weights $(k, s) \mapsto (y, -t)$ where $y \geq 2$ and $y > t > 0$. Let g_y be the specialization of g at y, viewed as a section of $\omega_E^{y-2} \otimes \Omega_X^1$ over a strict neighborhood W_ϵ. The condition $U(g) = 0$ implies the class $[g_y]$ in $H^1_{\mathrm{rig,par}}(W_\epsilon, \mathcal{H}^{y-2})$ is 0. As W_ϵ is a Stein space, this means that there is a section G_y of $\mathcal{H}^{y-2}$ over W_ϵ such that $\nabla(G_y) = g_y$. In fact, one can choose G_y such that it is equal to $\nabla_k^s(g)$ specialized at $(y, -1)$. The same argument also holds for $t > 1$.

4.9 The overconvergent projection

In this section we will introduce the overconvergent projection for families. We first explain the idea in the following paragraph.

Let g be a section of ω_E^r over some strict neighborhood $W = W_\epsilon$. For any positive integer N, we have the nearly overconvergent form $\nabla^{N+1}(g)$, viewed as a section of $\mathcal{H}^{r+2N} \otimes \Omega_W^1$. In practice, one usually wants to deal with overconvergent modular forms instead of nearly overconvergent modular forms. So the natural question is: can we assign an element $H^\dagger(f) \in H^0(W, \omega_E^r \otimes \Omega_W^1)$ for any $f \in H^0(W, \mathcal{H}^r \otimes \Omega_W^1)$ in a canonical way? As we have seen in §2.1, a natural choice is to take the overconvergent projection $\Pi^{\mathrm{oc}}(f) \in H^0(W, \omega_E^r \otimes \Omega_W^1)$ of f. The overconvergent projection $\Pi^{\mathrm{oc}}(f)$ satisfies the property that the two classes $[\Pi^{\mathrm{oc}}(f)], [f] \in H^1_{\mathrm{rig,par}}(W, \mathcal{H}^r)$ coincide. One should note that the overconvergent projection is only well-defined up to $\mathrm{Im}\,\nabla \cap H^0(W, \omega_E^r \otimes \Omega_W^1)$.

The goal now is to extend the above method to sections of $\mathbb{W}_{k_I}$ for $I \subset [0, \infty]$. Throughout this section, we will work with the adic space $\mathcal{X}_{r,I}$ and write $k = k_I$ and $u = u_k$ for simplicity.

Recall that we can view the connection ∇_k as a complex of sheaves $\mathbb{W}_k^\bullet : \mathbb{W}_k \to \mathbb{W}_{k+2}$ on the adic space $\mathcal{X}_{r,I}$. We let $H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}_k^\bullet)$ be the i-th hypercohomology group of this complex. As the connection satisfies Griffiths transversality, we have the following diagram of sheaves on $\mathcal{X}_{r,I}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}_n(\mathbb{W}_k) & \longrightarrow & \mathbb{W}_k & \longrightarrow & \mathbb{W}_k / \mathrm{Fil}_n \mathbb{W}_k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nabla_k} & & \downarrow{\scriptstyle \nabla_k} & & \downarrow{\scriptstyle \nabla_k} & & \\ 0 & \longrightarrow & \mathrm{Fil}_{n+1}(\mathbb{W}_{k+2}) & \longrightarrow & \mathbb{W}_{k+2} & \longrightarrow & \mathbb{W}_{k+2} / \mathrm{Fil}_{n+1} \mathbb{W}_{k+2} & \longrightarrow & 0 \end{array}$$

The rows are exact by definition, and we denote by $\mathrm{Fil}_n^\bullet(\mathbb{W}_k)$ and $(\mathbb{W}_k / \mathrm{Fil}_n \mathbb{W}_k)^\bullet$ the first and last column of the above diagram respectively. With this notation, we can rewrite the above diagram as the following exact sequence of complexes

$$0 \to \mathrm{Fil}_n^\bullet \mathbb{W}_k \to \mathbb{W}_k^\bullet \to (\mathbb{W}_k / \mathrm{Fil}_n \mathbb{W}_k)^\bullet \to 0, \tag{25}$$

which gives rise to the long exact sequence

$$\begin{aligned}0 &\to H^0_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k) \to H^0_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k) \to H^0_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathbb{W}_k/\mathrm{Fil}_n \mathbb{W}_k)^\bullet) \\ &\to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k) \to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k) \to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathbb{W}_k/\mathrm{Fil}_n \mathbb{W}_k)^\bullet) \to \cdots \end{aligned} \tag{26}$$

We recall that the sheaves $\mathrm{Fil}_n \mathbb{W}_k$ and $\mathrm{Fil}_{n+1} \mathbb{W}_k$ are coherent, while the rest are not. As $\mathcal{X}_{r,I}$ is a Stein space, the hypercohomology of the complex $\mathrm{Fil}^\bullet_n \mathbb{W}_k$ can be computed by the cohomology of the complex of global sections. That is, for $i \geq 0$, we have

$$H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k) = H^i\left(H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_n \mathbb{W}_k) \xrightarrow{\nabla_k} H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_{n+1} \mathbb{W}_k)\right).$$

Lemma 4.33 ([AI21, Lemma 3.32])**.** *We have an exact sequence, with equivariant morphisms for the action of U,*

$$0 \to H^0(\mathcal{X}_{r,I}, \mathfrak{w}_{k+2}) \to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k) \to \bigoplus_{i=0}^{n} H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{-i}) \to 0,$$

where the first map is induced by the inclusion $\mathfrak{w}_{k+2} = \mathrm{Fil}_0 \mathbb{W}_{k+2} \subset \mathbb{W}_{k+2}$, j_i is the closed immersion $\mathcal{X}_{r,I} \times_{\mathcal{W}_I} \mathbb{Q}_p \subset \mathcal{X}_{r,I}$ defined by the $\mathbb{Q}_p$-valued point $k = i$ of $\mathcal{W}_I$, and the action of U on $H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{-i})$ is divided by p^{i+1}. Moreover, if we let $H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k)^{\mathrm{tf}}$ be the Λ_I-torsion-free part, we have the following U-equivariant exact sequence

$$0 \to H^0(\mathcal{X}_{r,I}, \mathfrak{w}_{k+2}) \to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k)^{\mathrm{tf}} \to \bigoplus_{i=0}^{n} \theta^{i+1}(H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{-i})) \to 0,$$

where $\theta^i : H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{-i}) \to H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{i+2})$ is the theta operator defined in [Col95, Proposition 4.3].

Proof. We here only give the proof of the first part, and leave the second part to [AI21, Lemma 3.32].

Recall that we have the identification $\mathrm{Gr}_i \mathbb{W}_k = \mathfrak{w}_{k-2i}$ over $\mathcal{X}_{r,I}$. The connection $\nabla_k : \mathrm{Fil}_n \mathbb{W}_k \to \mathrm{Fil}_{n+1} \mathbb{W}_{k+2}$ then induces the map

$$\mathfrak{w}_{k-2n} \cong \mathrm{Gr}_n(\mathbb{W}_k) \to \mathrm{Gr}_{n+1}(\mathbb{W}_{k+2}) \cong \mathfrak{w}_{k-2n}$$

which is an isomorphism times the multiplication by $u - n$ map. This map is injective, and the cokernel is identified with $\mathfrak{w}_{k-2n}/(u-n)\mathfrak{w}_{k-2n} \cong \omega_E^{-n}$. The first statement then follows by induction on n, where the case for $n = 0$ one use the identification $\mathfrak{w}_{k+2} = \mathrm{Fil}_0 \mathbb{W}_{k+2}$. ☐

Notice that the U operators on $H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k)$ and $H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathbb{W}_k/\mathrm{Fil}_n \mathbb{W}_k)^\bullet)$ are also compact. So we also have slope decompositions on these spaces.

Lemma 4.34. *Fix a rational $a \geq 0$, then for n large enough, the exact sequence (25) induces isomorphisms*

$$H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k)^{\leq a} \cong H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k)^{\leq a}$$

for all $i \geq 0$.

Proof. This follows from a similar description for the U operator on $\mathbb{W}_k/\mathrm{Fil}_n \mathbb{W}_k$ as in Proposition 4.19. In particular,

$$H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, (\mathbb{W}_k/\mathrm{Fil}_n \mathbb{W}_k)^\bullet)^{\leq a} = 0$$

for n large enough. Therefore, the long exact sequence (26) implies the claim. ☐

Summarizing the above results, we have the following theorem.

Theorem 4.35. *Given a finite slope $a \geq 0$, locally over the weight space, the spaces $H^i_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k)$ admits slope a-decomposition. Moreover, for some n large enough, we have an exact sequence*

$$0 \to H^0(\mathcal{X}_{r,I}, \mathfrak{w}_{k+2})^{\leq a} \to H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k)^{\leq a} \to \bigoplus_{i=0}^{n} H^0(\mathcal{X}_{r,I}, j_{i*}(\omega_E)^{-i})^{\leq \frac{a}{p^{i+1}}} \to 0.$$

Definition 4.36. We define

$$H^\dagger_n : H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathrm{Fil}_n(\mathbb{W}_k)^\bullet) \otimes_{\Lambda_I} \Lambda_I \left[\prod_{i=0}^{n}(u_k - i)^{-1}\right] \cong H^0(\mathcal{X}_{r,I}, \mathfrak{w}_{k+2}) \otimes_{\Lambda_I} \Lambda_I \left[\prod_{i=0}^{n}(u_k - i)^{-1}\right]$$

and

$$H^\dagger : H^1_{\mathrm{dR}}(\mathcal{X}_{r,I}, \mathbb{W}^\bullet_k)^{\leq a} \otimes_{\Lambda_I} \Lambda_I \left[\prod_{i=0}^{n_a}(u_k - i)^{-1}\right] \cong H^0(\mathcal{X}_{r,I}, \mathfrak{w}_{k+2})^{\leq a} \otimes_{\Lambda_I} \Lambda_I \left[\prod_{i=0}^{n_a}(u_k - i)^{-1}\right]$$

to be the isomorphisms given by Lemma 4.33 and Theorem 4.35. These maps are called overconvergent projections in families. We will often drop the subscript n of $H^\dagger_n$ for simplicity.

We now proceed to describe the overconvergent projection in terms of q-expansions. Recall that we have the identity

$$\nabla(aV_{k,h}) = \theta(a)V_{k+2,h} + a(u_k - h)V_{k+2,h+1}.$$

on $\mathbb{W}_k(q)$.

Given a section $\gamma \in H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_{n+1}\mathbb{W}_{k+2})$ with its q-expansion $\sum_{i=0}^{n+1}\gamma_i(q)V_{k+2,i}$, the overconvergent projection of the class $[\gamma]$ is the element $H^\dagger(\gamma) = \gamma^\dagger(q)V_{k+2,0}$ such that $\gamma \equiv H^\dagger(\gamma)$ modulo the image of ∇. Hence, after shifting the indices, we have the following result.

Proposition 4.37. *Let $\gamma \in H^0(\mathcal{X}_{r,I}, \mathrm{Fil}_n\mathbb{W}_k)$ and $\gamma(q) = \sum_{i=0}^{n}\gamma_i(q)V_{k,i}$ be its q-expansion. Then the q-expansion of $H^\dagger([\gamma])$ is*

$$\sum_{i=0}^{n}(-1)^i \frac{\theta^i\gamma_i(q)}{(u_{k-2} - i + 1)(u_{k-2} - i + 2)\cdots u_{k-2}} V_{k,0}.$$

Remark 4.38. It is clear from the above formula that one needs to invert some elements in Λ_I for the overconvergent projection in families. Also, when specialized to a classical weight $m > n$, one can see that $H^\dagger_n$ coincides with the definitions of overconvergent projections in [Urb14, Lemma 3.3.4] or [DR14, § 2.4].

4.10 The triple product p-adic L-functions for finite slope families

With all the results in the previous sections, we are now able to define triple product p-adic L-functions for finite slope families.

Let $f \in S_k(N_f, \chi_f)$, $g \in S_\ell(N_g, \chi_g)$, $h \in S_m(N_h, \chi_h)$ be a triple of normalized primitive cuspidal eigenforms such that f has slope $a > 0$ and $\chi_f \cdot \chi_g \cdot \chi_h = 1$. We further assume that and $a < 1$ if $k = 2$, and $2a < k - 1$ if $k > 2$ (see Remark 4.41).

Let $N = \mathrm{lcm}(N_f, N_g, N_h)$, and let $\breve{f}$, $\breve{g}$, $\breve{h}$ be as in Theorem 2.24. We denote by K a finite extension of $\mathbb{Q}_p$ that contains all the Fourier coefficients of f, g, h as well as the values of χ_f, χ_g, χ_h and denote by $\mathcal{O}_K$ the ring of integers in K.

Let $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be overconvergent families of modular forms deforming the p-stabilizations of f, g, h and similarly $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$ for $\breve{f}, \breve{g}, \breve{h}$. We denote by

$$k_f : \mathbb{Z}_p^* \to \Lambda_f, \quad k_g : \mathbb{Z}_p^* \to \Lambda_g, \quad k_h : \mathbb{Z}_p^* \to \Lambda_h$$

the analytic weights of these families respectively. After base-change to $\mathcal{O}_K$, we may further assume that $\Lambda_f, \Lambda_g, \Lambda_h$ are $\mathcal{O}_K$-algebras. Then there exists some $n \in \mathbb{N}$ such that we have families

$$\begin{aligned} \mathbf{f}, \breve{\mathbf{f}} &\in H^0(\mathcal{X}_n, \mathfrak{w}_{k_f}), \\ \mathbf{g}, \breve{\mathbf{g}} &\in H^0(\mathcal{X}_n, \mathfrak{w}_{k_g}), \\ \mathbf{h}, \breve{\mathbf{h}} &\in H^0(\mathcal{X}_n, \mathfrak{w}_{k_h}). \end{aligned}$$

As we want to apply p-adic powers of ∇ to the p-depletion $\breve{\mathbf{g}}^{[p]}$ later, we need the following assumption on the weights.

Assumption 4.39. Suppose the weight $k_f - k_g - k_h$ is even, i.e., there is an analytic weight $\nu : \mathbb{Z}_p^* \to \Lambda_f \hat{\otimes}_{\mathcal{O}_K} \Lambda_g \hat{\otimes}_{\mathcal{O}_K} \Lambda_h$ such that $k_f - k_g - k_h = 2\nu$. Moreover, we require that k_g and ν satisfy Assumption 4.27 in this order.

With this assumption, we have

$$\nabla_{k_g}^{\nu} \breve{\mathbf{g}}^{[p]} \in H^0(\mathcal{X}_{n'}, \mathbb{W}_{k_g+2\nu})$$

for some $n' \geq n$. Therefore, $(\nabla_{k_g})^{\nu} \breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}} \in H^0(\mathcal{X}_{n'}, \mathbb{W}_{k_f})$ and we may consider its class in $H^1_{\mathrm{dR}}(\mathcal{X}_{n'}, \mathbb{W}_{k_f-2})$. After base change to $\mathfrak{K}_f$, we obtain a section in $H^1_{\mathrm{dR}}(\mathcal{X}_{n'}, \mathbb{W}_{k_f-2}) \otimes_{\Lambda_f} \mathfrak{K}_f$, where $\mathfrak{K}_f$ is obtained from Λ_f by inverting the elements $\{u_{k_f} - n \mid n \in \mathbb{N}\}$ (or one may simply take it to be $\mathrm{Frac}(\Lambda_f)$). We then consider its overconvergent projection

$$H^{\dagger, \leq a}(\nabla_{k_g}^{\nu} \breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}}) \in H^0(\mathcal{X}_{n'}, \mathfrak{w}_{k_f})^{\leq a} \otimes \mathfrak{K}_f.$$

Definition 4.40. The Garrett-Rankin triple product p-adic L-function attached to the triple $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of finite slope families of modular forms is

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) := \frac{(\breve{\mathbf{f}}^*, H^{\dagger, \leq a}(\nabla_{k_g}^{\nu} \breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}}))}{(\breve{\mathbf{f}}^*, \breve{\mathbf{f}}^*)} \in \mathfrak{K}_f \hat{\otimes} \Lambda_g \hat{\otimes} \Lambda_h$$

where $\breve{\mathbf{f}}^*$ is the the Atkin-Lehner involution of $\breve{\mathbf{f}}$ (c.f. [AI21, Definition 5.2]) and $(\ , \ )$ is the Petersson product defined in [AI21, § 5.2.1]. When f, g, h are ordinary at p, one can see that this definition coincides with the one given in [DR14, § 4.2].

Remark 4.41. As explained in [AI21, § 5.2.1], when specialized to a classical weight $x > 2$ with $2a < x-1$ or $x = 2$ with $a < 1$, the Petersson product interpolates the classical Petessons inner product on $S_x(\Gamma_1(N, p))$, up to a constant multiple.

In fact, given $\mathbf{f}$, what is really required for the weight x is the condition $\alpha(\mathbf{f}_x)^2 \neq p^{x-1} b(\mathbf{f}_x)$, where $\alpha(\mathbf{f}_x), b(\mathbf{f}_x)$ are the eigenvalues of U_p and $\langle p \rangle$ acting on $\mathbf{f}_x$, respectively. In other words, if f_x^0 is the classical modular form on $\Gamma_1(N)$ with one of its p-stabilizations being $\mathbf{f}_x$, it means that the two roots α, β of the Hecke polynomial of f_x^0 at p are different. Equivalently in a fancier language, one wants to restrict to a subset of the weight space over which the eigencurve giving rise to the family $\mathbf{f}$ is étale.

5 Finite polynomial cohomology and the p-adic Abel–Jacobi map

In this section, we aim to formulate a theory of finite polynomial cohomology with coefficients in a special case, which generalizes Besser's results in [Bes00a].

Technically speaking, this machine is not necessary in our application since one can use the so-called Liebermann's trick to avoid dealing with coefficients, as H. Darmon and V. Rotger did in [DR14]. Still, our theory with coefficients has its own advantages. The most obvious one is that it may be applied to objects without universal abelian varieties. The second one is that it makes the computation of Abel-Jacobi maps simpler.

The language of rigid geometry and Coleman integration will be used heavily throughout this section. For those who are not familiar with these topics, Appendix §8 provides a rather short but sufficient introduction.

Notation. We fix a prime p and let K be a finite unramified extension of $\mathbb{Q}_p$. We denote by $V = \mathcal{O}_K$ the ring of integers in K and κ the residue field with $|\kappa| = q = p^r$.

We also recall the category of filtered Frobenius modules (abbr. ffm) over K.

Definition 5.1. Let $\sigma : K \to K = \mathrm{Frac}(W(\kappa))$ be the Frobenius induced from the Frobenius on the Witt ring $W(\kappa)$. By a filtered Frobenius module (over K), we means a finite dimensional K-vector space M with the following data:

1. An exhaustive and separated decreasing filtration $\mathrm{Fil}^i M$ of K-vector spaces.
2. A σ-linear automorphism $\phi : M \to M$.

We consider K as a filtered Frobenius module with $\mathrm{Fil}^i = K$ for $i \leq 0$, $\mathrm{Fil}^i = 0$ for $i > 0$ and $\phi = \mathrm{Id}$. For any $n \in \mathbb{Z}$, we let $K(n)$ be the filtered Frobenius module whose underlying space is just K, with filtration $\mathrm{Fil}^i K(n) = \mathrm{Fil}^{i+n} K$ and $\phi = q^{-n}\,\mathrm{Id}$. For any filtered Frobenius module M, we let $M(n) := M \otimes_K K(n)$. The dual $M^\vee$ of a filtered Frobenius module M is again a filtered Frobenius module. Its underlying space is $\mathrm{Hom}_K(M, K)$, with filtration $\mathrm{Fil}^i M^\vee := (\mathrm{Fil}^{-i+1} M)^\perp$ and Frobenius $\phi(f) = f \circ \phi^{-1}$. The collection of filtered Frobenius modules over K forms a category with the obvious definition of morphisms.

5.1 Finite polynomial cohomology without coefficients and applications

In this section, we briefly recall the definition of finite polynomial cohomology without coefficients (c.f. [Bes00a]). Then we will compute the Abel–Jacobi map in the special case of weight $(2, 2, 2)$, which follows closely [Bes16].

Let X be a proper, irreducible and smooth V-scheme of relative dimension d. We denote by X_K its general fiber and X_κ its special fiber. Then, by [Bes00b, § 4, 5], we have the following functorial complexes of K-vector spaces

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K),\ \mathbb{R}\Gamma_{\mathrm{dR}}(X_K),\ \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K)$$

which compute the rigid cohomology $H^i_{\mathrm{rig}}(X_\kappa/K), H^i_{\mathrm{dR}}(X_K/K)$ and $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K)$ respectively. We will drop the subscripts κ and K inside $\mathbb{R}\Gamma_\bullet$ and $H^i_\bullet$ as the context will make it clear what we are referring. We note that there is a canonical map $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X)$ and a natural quasi-isomorphism $\mathbb{R}\Gamma_{\mathrm{dR}}(X) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X)$.

On $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$, there is an K-linear action of the Frobenius ϕ (of degree q). By [CLS98, Théorèm 1.2], its eigenvalues on $H^i_{\mathrm{rig}}(X)$ are of pure Weil weight i, *i.e.*, are algebraic of complex absolute value $p^{\frac{i}{2}}$. By using the comparison $H^i_{\mathrm{rig}}(X) \cong H^i_{\mathrm{dR}}(X)$ induced by the above quasi-isomorphism, we may translate the action of ϕ to $H^i_{\mathrm{dR}}(X)$.

Remark 5.2. To obtain the functorial complexes $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K)$ and $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K)$, one can use Godement resolutions (in the Zariski topology). For the rigid complex $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa)$, this method does not work in rigid analytic topology. However, if one passes to adic spaces, which are genuine topological spaces, the same trick would provide a desired functorial complex. As explained in [Bes00b, § 7], the property of being a complex is used only for constructing Chern classes (hence the cycle class maps) and comparing it to Chern classes in de Rham cohomology. Outside this situation, one can just work with the derived category of complexes.

Definition 5.3. Let $\mathfrak{P} \subset K[T]$ be the monoid of polynomials with constant term 1 and algebraic roots. For any $s \in \mathbb{Z}$, we let $\mathfrak{P}_s \subset \mathfrak{P}$ be the submonoid of polynomials with constant term 1 and whose roots are of Weil weight s.

Definition 5.4. For any polynomial $P \in \mathfrak{P}$ and $n \in \mathbb{N}$, we define the syntomic P-complex $\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n)$ to be

$$\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n) := \mathrm{Cone}\left(\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X)\right)[-1].$$

Here Cone means the mapping cone and the map $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X)$ should be understood as the composite

$$\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X)$$

where the first map is the natural map and the second one is the quasi-isomorphism mentioned above. The i-the cohomology of $\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n)$ will be denoted by $H^i_{\mathrm{syn},P}(X,n)$.

Remark 5.5. We would like to recall our convention for the mapping cone. Let $A^\bullet, B^\bullet$ be two complexes with a map $f : A^\bullet \to B^\bullet$. The complex $\mathrm{Cone}(A^\bullet \xrightarrow{f} B^\bullet)[-1]$ is called the mapping fiber of f in some literature and is denoted by $\mathrm{MF}(f : A^\bullet \to B^\bullet)$ or simply $\mathrm{MF}(f)$. At degree i, it is given by $A^i \oplus B^{i-1}$ with differential $d(a,b) = (da, f(a) - db)$.

Definition 5.6. For any $s \in \mathbb{Z}$, we define

$$\mathbb{R}\Gamma_{\mathrm{fp},s}(X,n) := \varinjlim_{P \in \mathfrak{P}_s} \mathbb{R}\Gamma_{\mathrm{syn},P}(X,n)$$

where the connecting map is induced by the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \arrow[r, "P(\phi)"] \arrow[d, "="] & \mathbb{R}\Gamma_{\mathrm{rig}}(X) \arrow[d, "Q(\phi)"] \\
\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \arrow[r, "PQ(\phi)"] & \mathbb{R}\Gamma_{\mathrm{rig}}(X).
\end{tikzcd}$$

The i-th cohomology of $\mathbb{R}\Gamma_{\mathrm{fp},s}(X,n)$ will be denoted by $H^i_{\mathrm{fp},s}(X,n)$. If $i = s$, we will abbreviate it as $H^i_{\mathrm{fp}}(X,n)$.

Proposition 5.7 ([Bes00a, Proposition 2.5])**.** *The space $H^i_{\mathrm{fp}}(X,n)$ satisfies the following properties:*

1. *There is a short exact sequence*

$$0 \to H^{i-1}_{\mathrm{dR}}(X)/\mathrm{Fil}^n \xrightarrow{\mathrm{i}_{\mathrm{fp}}} H^i_{\mathrm{fp}}(X,n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \mathrm{Fil}^n H^i_{\mathrm{dR}}(X) \to 0,$$

   which will be called as the fundamental exact sequence.

2. *There is a cup-product $\cup : H^i_{\mathrm{fp}}(X,n) \times H^j_{\mathrm{fp}}(X,m) \to H^{i+j}_{\mathrm{fp}}(X,n+m)$ that is compatible with the cup product on de Rham cohomology via the fundamental exact sequence. In particular, $\langle x, \mathrm{i}_{\mathrm{fp}}(y)\rangle_{\mathrm{fp}} = \langle \mathrm{pr}_{\mathrm{fp}}(x), y\rangle_{\mathrm{dR}}$ for any $x \in H^i_{\mathrm{fp}}(X,n)$ and $y \in H^j_{\mathrm{dR}}(X)/\mathrm{Fil}^m H^j_{\mathrm{dR}}(X)$.*

3. *The map $\mathrm{i}_{\mathrm{fp}} : H^{2d}_{\mathrm{dR}}(X)/\mathrm{Fil}^{d+1} \to H^{2d+1}_{\mathrm{fp}}(X, d+1)$ is an isomorphism and induces the following trace map*

$$\mathrm{tr}_{\mathrm{fp}} : H^{2d+1}_{\mathrm{fp}}(X,d+1) \xrightarrow{\mathrm{i}_{\mathrm{fp}}^{-1}} H^{2d}_{\mathrm{dR}}(X)/\mathrm{Fil}^{d+1} H^{2d}_{\mathrm{dR}}(X) = H^{2d}_{\mathrm{dR}}(X) \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} K.$$

   Moreover, the pairing

$$\langle\ ,\ \rangle_{\mathrm{fp}} : H^i_{\mathrm{fp}}(X,n) \times H^{2d+1-i}_{\mathrm{fp}}(X, d+1-n) \xrightarrow{\cup} H^{2d+1}_{\mathrm{fp}}(X,d+1) \xrightarrow{\mathrm{tr}_{\mathrm{fp}}} K$$

   is a perfect pairing.

4. *Suppose $\iota_Z : Z \to X$ is a smooth irreducible closed subscheme of codimension c. Then we have a pushforward map*
$$\iota_{Z,*} : H^i_{\mathrm{fp}}(Z, n) \to H^{i+2c}_{\mathrm{fp}}(X, n + c)$$
which is adjoint to the pullback ι_Z^* with respect to the pairing $\langle\ ,\ \rangle_{\mathrm{fp}}$.

Proof. Let $P(T) \in \mathfrak{P}$ be a polynomial such that $P(\phi)$ annihilates $H^i_{\mathrm{rig}}(X)$ but acts bijectively on $H^{i-1}_{\mathrm{rig}}(X)$. We set $C^\bullet_P := \mathrm{Cone}(\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X))[-1]$. By the long exact sequence of mapping cones, we have the following long exact sequence

$$\cdots \longrightarrow \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X) \xrightarrow{P(\phi)} H^{i-1}_{\mathrm{rig}}(X) \longrightarrow H^i(C^\bullet_P)$$

$$\longrightarrow \mathrm{Fil}^n H^i_{\mathrm{dR}}(X) \xrightarrow{P(\phi)} H^i_{\mathrm{rig}}(X) \longrightarrow \cdots .$$

The choice of the polynomial P, together with the comparison $H^i_{\mathrm{rig}}(X) \cong H^i_{\mathrm{dR}}(X)$, then imply the short exact sequence

$$0 \to H^{i-1}_{\mathrm{dR}}(X)/P(\phi)\,\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X) \xrightarrow{\mathrm{i}'_{\mathrm{fp}}} H^i_{\mathrm{syn},P}(X, n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \mathrm{Fil}^n H^i_{\mathrm{dR}}(X) \to 0.$$

As $P(\phi)$ is an isomorphism on $H^{i-1}_{\mathrm{dR}}(X)$, we have $H^{i-1}_{\mathrm{dR}}(X)/\mathrm{Fil}^n \cong H^{i-1}_{\mathrm{dR}}(X)/P(\phi)\,\mathrm{Fil}^n$. Hence, we get

$$0 \to H^{i-1}_{\mathrm{dR}}(X)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X) \xrightarrow{\mathrm{i}_{\mathrm{fp}}} H^i_{\mathrm{syn},P}(X, n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \mathrm{Fil}^n H^i_{\mathrm{dR}}(X) \to 0,$$

where the first map is twisted by $P(\phi)$. This twist is important in later computations and should not be forgotten. The fundamental exact sequence then follows from taking limit in the direct system. This proves 1.

For 2 and 3, we would like to refer to [Bes00a, § 3, 4]. The construction of the cup product and the compatibility with de Rham cohomology rely on an alternative description of the cone $C^\bullet_P$ (see §5.2 below). The trace map $\mathrm{tr}_{\mathrm{fp}}$, on the other hand, is a direct result of 1.

For 4, one first observes that the complexes $\mathbb{R}\Gamma_{\mathrm{syn},P}(X, n)$ and $\mathbb{R}\Gamma_{\mathrm{fp}}(X, n)$ are functorial since both $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}$ and $\mathbb{R}\Gamma_{\mathrm{rig}}$ are. Hence, the pullback $\iota_Z^* : H^j_{\mathrm{fp}}(X, m) \to H^j_{\mathrm{fp}}(Z, m)$ is well defined. The pushforward $\iota_{Z,*}$ can be defined directly as the dual of ι_Z^* with respect to the pairing $\langle\ ,\ \rangle_{\mathrm{fp}}$. One then works out the indices to get the statement in 4. ☐

Remark 5.8. If one is only interested in the functoriality of finite polynomial cohomology between two fixed proper smooth schemes $Y \to X$, one does not really need to deal with the direct limit. In fact, most of the explicit computations are done at a "finite level," *i.e.*, at a fixed polynomial P. To be more precise, one can find a polynomial $P \in \mathfrak{P}_i$ such that $P(\phi)$ simultaneously annihilates $H^i_{\mathrm{dR}}(X), H^i_{\mathrm{dR}}(Y)$ and acts bijectively on $H^{i-1}_{\mathrm{dR}}(X), H^{i-1}_{\mathrm{dR}}(Y)$. Then one has a natural pullback map $H^i_{\mathrm{syn},P}(X, n) \to H^i_{\mathrm{syn},P}(Y, n)$. The process of taking direct limit over all polynomials in $\mathfrak{P}_i$ is to secure the functoriality between all proper smooth schemes. It is important to note that one can take polynomials in $\mathfrak{P}_i$ because of the Weil conjecture for rigid cohomology of proper smooth schemes. In order to replace the structure sheaf by some overconvergent F-isocrystal $\mathcal{E}$, one would like to have a similar condition on the rigid cohomology groups for all proper smooth pullback of $\mathcal{E}$, or at least for the schemes one wants to study.

Abel–Jacobi maps via finite polynomial cohomology. Suppose $Z = \sum a_i Z_i \in A^c(X)$ is a cycle where each Z_i is closed smooth irreducible of codimension c in X. The cycle Z is said to be de Rham (co)homologous to zero or (co)homologically trivial if Z is sent to $0 \in H^{2c}_{\mathrm{dR}}(X)$ under the class map cl_X. We denote by $A^c(X)_0$ the subgroup of homologically trivial cycles.

This can be re-interpreted by finite polynomial cohomology in the following way. For each i, there is a canonical element $1_{Z_i} \in H^0_{\mathrm{fp}}(Z, 0) \cong H^0_{\mathrm{dR}}(Z_i) \cong K$. The pushforward of 1_{Z_i} by $\iota_{Z_i} : Z_i \to X$ is the cycle class $\mathrm{cl}_{\mathrm{fp}}(Z_i) \in H^{2c}_{\mathrm{fp}}(X,c)$ that satisfies $\langle \mathrm{cl}_{\mathrm{fp}}(Z_i), y\rangle_{\mathrm{fp}} = \mathrm{tr}_{Z_i,\mathrm{fp}}(\iota^*_{Z_i} y)$ for all $y \in H^{2(d-c)+1}_{\mathrm{fp}}(X, d-c+1)$ where we recall that $d-c$ is the relative dimension of Z_i.

It is showed in [Bes00a, § 5] that if $Z = \sum a_i Z_i$ is de Rham homologous to zero, then we have $\mathrm{pr}_{\mathrm{fp}}(\sum a_i \,\mathrm{cl}_{\mathrm{fp}}(Z_i)) = 0 \in H^{2c}_{\mathrm{dR}}(X)$. Hence $\sum a_i \,\mathrm{cl}_{\mathrm{fp}}(Z_i) \in \ker \mathrm{pr}_{\mathrm{fp}} = H^{2c-1}_{\mathrm{dR}}(X)/\operatorname{Fil}^c$. By further applying Poincaré duality, we may identify $H^{2c-1}_{\mathrm{dR}}(X)/\operatorname{Fil}^c$ with $[\operatorname{Fil}^{d+1-c} H^{2(d-c)+1}_{\mathrm{dR}}(X)]^\vee$. The resulting map

$$\mathrm{AJ}_p : A^c(X)_0 \to [\operatorname{Fil}^{d+1-c} H^{2(d-c)+1}_{\mathrm{dR}}(X)]^\vee$$

is called the (p-adic) syntomic Abel–Jacobi map and it coincides with the étale Abel–Jacobi map under the p-adic comparison theory (c.f. [Bes00b, Remark 9.11]). To be more precise, given $Z \in A^c(X)_0$ and $\omega \in \operatorname{Fil}^{d+1-c} H^{2(d-c)+1}_{\mathrm{dR}}(X)$, we have $\mathrm{AJ}_p(Z)(\omega) = \langle \mathrm{cl}_{\mathrm{fp}}(Z), \tilde{\omega}\rangle_{\mathrm{fp}}$, where $\tilde{\omega} \in H^{2(d-c)+1}_{\mathrm{fp}}(X, d+1-c)$ is a lift of ω under $\mathrm{pr}_{\mathrm{fp}}$.

Two important variations. In this paragraph, we will briefly introduce two variations of the finite polynomial cohomology. Some definitions will be omitted as they are lengthy. However, we will provide examples that will suffice our future use.

First, notice that the constructions of $\mathbb{R}\Gamma_{\mathrm{syn},P}$ and $\mathbb{R}\Gamma_{\mathrm{fp},s}$ can be extended to a smooth affine scheme X', though some statements in Proposition 5.7 may fail due to the fact that $H^i_{\mathrm{rig}}(X')$ may no longer be of pure weight. Nevertheless, an affine scheme has the advantage that the rigid cohomology is easier to compute, as illustrated below (c.f. §8.1).

Example. Let $X' \subset X$ be an smooth affine subscheme of a smooth proper scheme X over V. The space $\mathcal{A} :=]X'_\kappa[\subset X^{\mathrm{rig}}_K$ is an affinoid. We define the dagger differential complex of X'_κ to be

$$\Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa) := \varinjlim_{W} \Omega^{\bullet}_{\mathrm{rig}}(W)$$

where W runs through all strict neighborhoods of $\mathcal{A}$ in X^{rig}_K and $\Omega^\bullet_{\mathrm{rig}}$ is the rigid differential complex. Then the rigid cohomology $H^i_{\mathrm{rig}}(X'_\kappa)$ can be computed simply as the i-th cohomology of the complex $\Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa)$.

Second, we introduce the Gros style finite polynomial cohomology $\tilde{H}^i_{\mathrm{fp}}(X,n)$. The idea is to replace the de Rham complex $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X)$ with a rigid complex $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X)$. That is,

$$\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn},P}(X,n) := \operatorname{Cone}\left(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X)\right)[-1].$$

For the exact definition of $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X)$, we leave it to [Bes00b, § 9]. We remark that when X is proper, $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X)$ is canonically quasi-isomorphic to $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X)$. The Gros style cohomology has the advantage that it is more convenient for comparing to Coleman's p-adic integration theory, especially in the affine case where we can use the dagger differential complex.

Example. Let X be a smooth proper curve over V, and $X' \subset X$ be an affine subscheme. We would like to describe the cohomology $\tilde{H}^i_{\mathrm{fp}}(X',n)$. In this setting, the complexes $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X')$ and $\mathbb{R}\Gamma_{\mathrm{rig}}(X')$ can be represented by

$$\operatorname{Fil}^n \Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa) \text{ and } \Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa),$$

where $\operatorname{Fil}^n \Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa)$ is the stupid filtration $\tau^{\geq n}\Omega^{\dagger;\bullet}_{\mathrm{rig}}(X'_\kappa)$. As a result, an element in $\tilde{H}^i_{\mathrm{syn},P}(X',n)$ can be represented by a pair (ω, G) where $\omega \in \operatorname{Fil}^n \Omega^{\dagger;i}_{\mathrm{rig}}(X'_\kappa)$ is closed, $G \in \Omega^{\dagger;i-1}_{\mathrm{rig}}(X'_\kappa)$, and $dG = P(\phi)\omega$. In particular, if we restrict to the natural image of $\tilde{H}^1_{\mathrm{syn},P}(X,1)$ in $\tilde{H}^1_{\mathrm{syn},P}(X',1)$, an element can be represented by a pair (ω, G) where $\omega \in \Omega^1_{\mathrm{rig}}(X^{\mathrm{rig}}_K)$, $G \in \mathcal{O}(W)$ for some strict neighborhood W with $dG = P(\phi)\omega$ on a possibly smaller neighborhood W'. It should be clear now how to relate the class (ω, G) in $\tilde{H}^1_{\mathrm{syn},P}(X,1)$ to a Coleman integral of ω (c.f. §8.1 or [Bes00a, § 9]).

5.2 Explicit cup product formula

In this section, we will explicitly describe the cup product for finite polynomial cohomology. This is a crucial step if one wants to compute the syntomic Abel–Jacobi map. As a byproduct, we illustrate (partial) proofs for statements 2. and 3. in Proposition 5.7. Most of the results are from [Bes00a, § 3, 4] and we will skip some of the proofs about homological algebra. The reason for recalling these constructions is that the same arguments can be applied to the case with coefficients with little modifications.

We start with an alternative description of the complex $\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n)$. Given a polynomial $P \in \mathfrak{P}$, we let $\mathfrak{V}_P$ be the cyclic $K[T]$-module whose generator has characteristic polynomial $P(T)$. In other words, let $V_P^\bullet$ be the complex

$$V_P^\bullet : K[T] \xrightarrow{\cdot P(T)} K[T]$$

which is concentrated at degree -1 and 0. Then $\mathfrak{V}_P$ is the 0-th cohomology of $V_P^\bullet$ and $V_P^\bullet$ serves as a projective resolution of $\mathfrak{V}_P$.

Remark 5.9. In [Bes00a], Besser only used $\mathbb{Q}[T]$, which is not ideal for most of the applications. Indeed, since the eigenvalues of the Frobenius on rigid cohomology are algebraic, we can always find a "large" enough polynomial that belongs to $\mathbb{Q}[T]$. However, in practice, one prefers a simpler polynomial that is suitable for computation. For example, the Hecke polynomial of a cusp form usually does not have coefficients in $\mathbb{Q}$.

Let $C^\bullet$ be a complex of K-vector spaces with a K-linear endomorphism ϕ. We can then view $C^\bullet$ as a complex of $K[T]$-modules where T acts as ϕ.

Lemma 5.10 ([Bes00a, Lemma 3.1])**.** *Let $C^\bullet$ and P be as above. There is a canonical isomorphism*

$$\mathrm{Hom}(V_P^\bullet, C^\bullet) \cong \mathrm{MF}(P(\phi) : C^\bullet \to C^\bullet),$$

where Hom *is the total complex of the double complex of homomorphisms between different components of $V_P^\bullet$ and $C^\bullet$.*

Proof. The statement can be verified straightforward once we recall the notation of Hom. For two complexes $A^\bullet$ and $B^\bullet$, the (p,q)-degree of the Hom double complex is defined to be $\mathrm{Hom}(A^{-p}, B^q)$. In particular, we identify $\mathrm{Hom}(K[T], C^i) \cong C^i$ in the natural way, and pullback by $P(T)$ on $K[T]$ transforms to the action of $P(\phi)$ on C^i. By definition, the degree i part of $\mathrm{Hom}(V_P^\bullet, C^\bullet)$ is

$$\mathrm{Hom}(V_P^0, C^i) \oplus \mathrm{Hom}(V_P^{-1}, C^{i-1}) \cong C^i \oplus C^{i-1}.$$

One can then check the differential on the total complex coincides with the differential on the mapping fiber $\mathrm{MF}(P(\phi))$. ☐

Remark 5.11. There is a canonical map $K[T] \to V_P^\bullet$ given by the identity at degree 0. The induced map $\mathrm{Hom}(V_P^\bullet, C^\bullet) \to \mathrm{Hom}(K[T], C^\bullet) \cong C^\bullet$ can be then identified with the natural map $\mathrm{MF}(P(\phi)) \to C^\bullet$.

Definition 5.12. Suppose we have three complexes $X^\bullet, Y^\bullet, Z^\bullet$ with two morphisms $f : X^\bullet \to Z^\bullet$ and $g : Y^\bullet \to Z^\bullet$. One can define the fiber product $X^\bullet \times_{Z^\bullet} Y^\bullet$ simply by componentwise fiber products. On the other hand, there is the quasi-fiber product

$$X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet := \mathrm{MF}(f - g : X^\bullet \oplus Y^\bullet \to Z^\bullet).$$

One can check that the canonical map $X^\bullet \times_{Z^\bullet} Y^\bullet \to X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet$ taking (x,y) to $((x,y),0)$ is a quasi-isomorphism if $f - g$ is surjective. The advantage of the quasi-fiber product is that there is a natural sequence of complexes

$$Z^\bullet[-1] \to X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet \to X^\bullet \oplus Y^\bullet.$$

Another advantage of the quasi-fiber product concerns the cup product formula, which is illustrated in the below lemma.

Lemma 5.13 ([Bes00b, Lemma 3.2]). *Suppose we have complexes $X_i^\bullet, Y_i^\bullet, Z_i^\bullet$ with maps $f_i : X_i^\bullet \to Z_i^\bullet$ and $g_i : Y_i^\bullet \to Z_i^\bullet$ as above for $i = 1, 2, 3$. Suppose further that there are maps of complexes (cup products) $\cup : X_1^\bullet \otimes X_2^\bullet \to X_3^\bullet$ and similarly for Y and Z, which are compatible with the maps f_i and g_i in the obvious way. Then one has an induced map*

$$(X_1^\bullet \tilde{\times}_{Z_1^\bullet} Y_1^\bullet) \otimes (X_2^\bullet \tilde{\times}_{Z_2^\bullet} Y_2^\bullet) \xrightarrow{\cup} X_3^\bullet \tilde{\times}_{Z_3^\bullet} Y_3^\bullet.$$

More precisely, it can be described by the following formula

$$\begin{aligned}(x_1, y_1, z_1) \cup (x_2, y_2, z_2) = (&x_1 \cup x_2, y_1 \cup y_2, \\ &z_1 \cup (\gamma f_2(x_2) + (1-\gamma) g_2(y_2)) + (-1)^{\deg x_1}((1-\gamma) f_1(x_1) + \gamma g_1(y_1)) \cup z_2)\end{aligned}$$

for any arbitrary choice of $\gamma \in K$.

Following the definitions and carefully writing down the maps in various complexes, we have the following result on mapping fibers.

Lemma 5.14 ([Bes00a, Lemma 3.2 & Lemma 3.3]). *Suppose there is another complex $D^\bullet$ with $\alpha : D^\bullet \to C^\bullet$. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{MF}(P(\phi) \circ \alpha : D^\bullet \to C^\bullet) \arrow[r, "\sim"] \arrow[d] & D^\bullet \times_{C^\bullet} \mathrm{Hom}(V_P^\bullet, C^\bullet) \arrow[d] \\ D^\bullet \tilde{\times}_{C^\bullet} \mathrm{MF}(P(\phi) : C^\bullet \to C^\bullet) \arrow[r, "\sim"] & D^\bullet \tilde{\times}_{C^\bullet} \mathrm{Hom}(V_P^\bullet, C^\bullet). \end{tikzcd}$$

In particular, given $P, Q \in \mathfrak{P}$ and consider the natural map $\mathfrak{V}_{PQ} \to \mathfrak{V}_P$ by further reduction mod $P(T)$. This map lifts to the map on their projective resolutions given by

$$\begin{tikzcd} V_{PQ}^\bullet : \arrow[d] & K[T] \arrow[r, "PQ(T)"] \arrow[d, "Q(T)"'] & K[T] \arrow[d, equal] \\ V_P^\bullet : & K[T] \arrow[r, "P(T)"] & K[T]. \end{tikzcd}$$

Then the maps of mapping fibers induced by

$$\begin{tikzcd} D^\bullet \arrow[r, "P(\phi)\circ\alpha"] \arrow[d, equal] & C^\bullet \arrow[d, "Q(T)"] \\ D^\bullet \arrow[r, "PQ(\phi)\circ\alpha"] & C^\bullet \end{tikzcd}$$

can be identified as the (quasi-)fiber product of $D^\bullet$ with the map $\mathrm{Hom}(V_P^\bullet, C^\bullet) \to \mathrm{Hom}(V_{PQ}^\bullet, C^\bullet)$ induced by the map $V_{PQ}^\bullet \to V_P^\bullet$.

Finally, we obtain the alternative descriptions of i_{fp} and $\mathrm{pr}_{\mathrm{fp}}$ in the fundamental exact sequence.

Proposition 5.15 ([Bes00a, Proposition 3.4]). *1. Let $P \in \mathfrak{P}$. There are canonical maps*

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{syn},P}(X, n) &\to \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \times_{\mathbb{R}\Gamma_{\mathrm{rig}}(X)} \mathrm{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \\ &\to \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{rig}}(X)} \mathrm{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)),\end{aligned}$$

where the first map is an isomorphism and the second one is a quasi-isomorphism. Moreover, after taking $\varinjlim$ over $P \in \mathfrak{P}_s$, one has similarly

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{fp},s}(X, n) &\to \varinjlim_{P \in \mathfrak{P}_s} \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \times_{\mathbb{R}\Gamma_{\mathrm{rig}}(X)} \mathrm{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \\ &\to \varinjlim_{P \in \mathfrak{P}_s} \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{rig}}(X)} \mathrm{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)),\end{aligned}$$

where again the first map is an isomorphism and the second one is a quasi-isomorphism. The transition maps in the direct limit are induced by $V_{PQ}^\bullet \to V_P^\bullet$ described above.

2. *Fix a polynomial $P \in \mathfrak{P}_i$, then the maps* i_{fp} *and* $\mathrm{pr}_{\mathrm{fp}}$ *in the fundamental exact sequence are induced, respectively, by the first and second maps of the diagram*

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{rig}}(X)[-1] &\to \mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \oplus \operatorname{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X)\big)\\ &\to \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X)\end{aligned} \tag{27}$$

at degree i, where the map in the mapping fiber is the difference of the two natural maps from the two components to $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$.

Proof. The first statement is a direct results from previous lemmas. For the second statement, we will describe the two maps in the sequence.

First, one can identify $\operatorname{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X))$ with $\mathrm{MF}(P(\phi))$. Then an element of degree i in the mapping fiber $\mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \oplus \operatorname{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X)\big)$ can be written as $(x \oplus (y, z), w)$ where x is of degree i in $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X)$ and y, z, w are in $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$ of degrees $i, i-1, i-1$ respectively. The second map is defined by sending $(x \oplus (y, z), w)$ to x. Given a closed element w of degree $i-1$ in $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$, the first map takes w to $(0 \oplus (0, 0), w)$. In particular, one sees that the two maps are compatible with the cup products on the rigid and de Rham cohomology.

The quasi-isomorphism between $\mathbb{R}\Gamma_{\mathrm{syn},P}$ and the mapping fiber appearing in (27) is a direct result of the first statement. However, one still needs to check that the two maps in (27) agree with i_{fp} and $\mathrm{pr}_{\mathrm{fp}}$. For the second map, it is clearly that it coincides with $\mathrm{pr}_{\mathrm{fp}}$. For the first map, recall that the identification

$$\mathbb{R}\Gamma_{\mathrm{syn},P} \to \mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X) \oplus \operatorname{Hom}(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X)\big)$$

sends a representative (x, z) of $\mathbb{R}\Gamma_{\mathrm{syn},P}$ to $(x \oplus (x, z), 0)$. One then observes that the two elements $(0 \oplus (0, 0), w)$ and $(0 \oplus (0, P(\phi)w), 0)$ differ by a coboundary. Hence they induce the same map i_{fp}.

□

Definition 5.16. Let P, Q be two polynomials in $\mathfrak{P}$ which factor as $P(T) = \prod(1 - \alpha_i T)$ and $Q(T) = \prod(1 - \beta_j T)$. We define their star product to be

$$P * Q(T) := \prod_{i,j}(1 - \alpha_i \beta_j T).$$

We now aim to construct a cup product

$$H^i_{\mathrm{syn},P}(X, n) \times H^j_{\mathrm{syn},Q}(X, m) \to H^{i+j}_{\mathrm{syn},P*Q}(X, n+m).$$

First, we need the following lemma.

Lemma 5.17 ([Bes00a, Lemma 4.2]). 1. *If P and Q are the characteristic polynomials of operators T and S, respectively, on some finite dimensional vector spaces. Then $P * Q$ is the characteristic polynomial of $T \otimes_K S$.*

2. *There is a canonical map $\mathfrak{V}_{P*Q} \to \mathfrak{V}_P \otimes \mathfrak{V}_Q$ sending the generator 1 to $1 \otimes 1$. Moreover, this map lifts, uniquely up to homotopy, to a map of complexes*

$$V^\bullet_{P*Q} \to V^\bullet_P \otimes_K V^\bullet_Q.$$

3. *In the polynomial rings $K[T_1, T_2]$ one can find polynomials $a(T_1, T_2)$ and $b(T_1, T_2)$ such that*

$$(P * Q)(T_1 T_2) = a(T_1, T_2) P(T_1) + b(T_1, T_2) Q(T_2).$$

We now sketch the construction of the cup product. One first observes that the cup product on $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$ and the map $V^\bullet_{P*Q} \to V^\bullet_P \otimes V^\bullet_Q$ induce a map

$$\mathrm{Hom}(V^\bullet_P, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \otimes \mathrm{Hom}(V^\bullet_Q, \mathbb{R}\Gamma_{\mathrm{rig}}(X)) \to \mathrm{Hom}(V^\bullet_{P*Q}, \mathbb{R}\Gamma_{\mathrm{rig}}(X)).$$

Hence, one obtains a map

$$\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n) \otimes \mathbb{R}\Gamma_{\mathrm{syn},Q}(X,m) \to \mathbb{R}\Gamma_{\mathrm{syn},P*Q}(X,n+m).$$

One also needs to check its compatibility with the cup product on $\mathbb{R}\Gamma_{\mathrm{dR}}(X)$ via i_{fp} and $\mathrm{pr}_{\mathrm{fp}}$. This essentially comes from the proof of statement 2. in Proposition 5.15.

Remark 5.18. To explicitly describe the cup product, we endow $\mathbb{R}\Gamma_{\mathrm{rig}}(X) \otimes \mathbb{R}\Gamma_{\mathrm{rig}}(X)$ with an action of $K[T_1, T_2]$ by letting T_1 act as $\phi \otimes 1$ and T_2 as $1 \otimes \phi$. We also represent an element of $\mathbb{R}\Gamma_{\mathrm{syn},P}(X,n)$ by a pair (x,y) with $x \in \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X)$ and $y \in \mathbb{R}\Gamma_{\mathrm{rig}}(X)$ of degrees $i, i-1$ respectively. Then by Lemma 5.13, the cup product is given by the formula

$$(x_1, y_1) \cup (x_2, y_2) = \Big(x_1 \cup x_2, (-1)^{\deg x_1} \bigcup b(T_1,T_2)(x_1 \otimes y_2) + \bigcup a(T_1,T_2)(y_1 \otimes x_2)\Big)$$

where by an abuse of notation we let x_i denote its natural image in $\mathbb{R}\Gamma_{\mathrm{rig}}(X)$ in the second coordinate.

One should notice that the polynomials $a(T_1,T_2), b(T_1,T_2)$ are not unique. However, for two different pairs of polynomials, one can show that the resulting maps on the complexes are homotopic to each other.

5.3 Applications to modular forms of weight 2

Let $X = X_1(N)$ be the modular curve defined over V. We let $X' \subset X$ be the affine subscheme obtained by removing lifts of all supersingular points. The notation W will now be reserved to denote a suitable strict neighborhood of the ordinary locus in X_K^{an}. In particular, we will represent an element in $\tilde{H}^1_{\mathrm{syn},P}(X,1) \subset \tilde{H}^1_{\mathrm{syn},P}(X',1)$ by a pair (ω, G) as in last section. We also recall that the Hecke operator $U = U_p$ acts on the dagger complex $\Omega^{\dagger,\bullet}_{\mathrm{rig}}(X'_\kappa)$, which can be described by the usual rule on q-expansions.

Since X is proper, the Gros style finite polynomial cohomology $\tilde{H}^i_{\mathrm{fp}}(X,n)$ is canonically isomorphic to the non-tilde version $H^i_{\mathrm{fp}}(X,n)$. Hence we will make no difference between the two versions in the following discussions.

Proposition 5.19. *For $i = 2,3$ let $P_i(T) = (1-\alpha_i T)$ and suppose that we have two classes $\tilde{\omega}_i := (\omega_i, G_i) \in H^1_{\mathrm{syn},P_i}(X,1)$. Then*

$$\tilde{\omega}_2 \cup \tilde{\omega}_3 = (0, G_2\omega_3 - \alpha_2 G_3 \phi\omega_2) \in H^2_{\mathrm{syn},P_2*P_3}(X,2).$$

Proof. This follows directly from the explicit cup product formula, where we choose $a(T_1,T_2) = 1$ and $b(T_1,T_2) = \alpha_2 T_1$. ☐

Theorem 5.20. *Suppose that we have two classes $\tilde{\omega}_i = (\omega_i, G_i) \in H^1_{\mathrm{syn},1-\alpha_i T}(X,1)$ with $UG_i = 0$ for $i = 2,3$. Let $P(T) = (1-\alpha_2\alpha_3 T)$ and consider the cup product $\tilde{\omega}_2 \cup \tilde{\omega}_3 \in H^2_{\mathrm{syn},P}(X,2)$. Let $\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3) \in H^1_{\mathrm{dR}}(X)$ be the inverse and recall that the map $\mathrm{i}_{\mathrm{fp}} : H^1_{\mathrm{dR}}(X) \to H^2_{\mathrm{syn},P}(X,2) \cong H^1_{\mathrm{dR}}(X)$ is twisted by $P(\phi)$. If $\eta \in H^1_{\mathrm{dR}}(X)$ is an eigenvector of ϕ with eigenvalue α and $\mathrm{ord}_p(\alpha) = a \in \mathbb{Q}$, then*

$$\langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle_{\mathrm{dR}} = (1-\beta\alpha_2\alpha_3)^{-1}\langle \eta, e^{\leq a}(G_2\omega_3)\rangle_{\mathrm{dR}},$$

where $\beta = p/\alpha$ and $e^{\leq a}$ is the slope a-projection with respect to U. We here abuse the notation by writing $e^{\leq a}(G_2\omega_3)$ for its class in $H^1_{\mathrm{dR}}(X)$.

Proof. We first recall several facts. First, the Poincaré pairing satisfies $\langle \phi\eta_1, \phi\eta_2\rangle_{\mathrm{dR}} = p\langle \eta_1, \eta_2\rangle_{\mathrm{dR}}$. Second, as we assume that $UG_3 = 0$, we have $U(G_3\phi\omega_2) = 0$ by examining the q-expansion. The fact that $e^{\leq a}$ can be written as a power series in U with no constant term further implies that and $e^{\leq a}(G_3\phi\omega_2) = 0$. Lastly, recall that $\tilde{\omega}_2 \cup \tilde{\omega}_3 = (0, G_2\omega_3 - \alpha_2 G_3\phi\omega_2)$. The inverse $\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)$ is then an element in $H^1_{\mathrm{dR}}(X)$ such that $P(\phi)(\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)) = G_2\omega_3 - \alpha_2 G_3\phi\omega_2$.

Therefore, after applying Lemma 7.3, we have

$$\begin{aligned}\langle \eta, e^{\leq a}(G_2\omega_3 - \alpha_1 G_3\phi\omega_2)\rangle &= \langle \eta, G_2\omega_3 - \alpha_2 G_3\phi\omega_2\rangle \\ &= \langle \eta, P(\phi)\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle \\ &= \langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle - \alpha_2\alpha_3\langle \eta, \phi(\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3))\rangle \\ &= \langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle - \alpha_2\alpha_3 p\langle \phi^{-1}\eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle \\ &= \langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle - \alpha_2\alpha_3 p\langle \alpha^{-1}\eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)\rangle.\end{aligned}$$

The result then follows. □

We now set up some notations for modular forms. Let f be a cusp form of weight 2 of level $\Gamma_1(N)$, which we also view as a global section ω_f of Ω^1_X. We write the Hecke polynomial of f as

$$T^2 - a_p(f)T + \chi_f(p)p = (T - \alpha_f)(T - \beta_f)$$

and assume $a = \mathrm{ord}_p(\alpha_f) \neq \mathrm{ord}_p(\beta_f)$. We let

$$P_f(T) = 1 - a_p(f)p^{-1}T + \chi_f(p)p^{-1}T^2 = (1 - \alpha_f p^{-1}T)(1 - \beta_f p^{-1}T)$$

and write $\alpha'_f := \alpha_f p^{-1}$, $\beta'_f := \beta_f p^{-1}$ for simplicity. The polynomial $P_f(T)$ is defined such that $P_f(\phi)$ annihilates the class of ω_f in $H^1_{\mathrm{dR}}(X)$. More precisely, $P_f(\phi)\omega_f = \omega_{f^{[p]}}$ as section over W and there is a unique section $F^{[p]} = \theta^{-1}f^{[p]}$ such that $dF^{[p]} = \omega_{f^{[p]}}$.

Remark 5.21. As the f-isotypic part $H^1_{\mathrm{dR}}(X)[f]$ is defined such that $\omega_f \in H^1_{\mathrm{dR}}(X)[f]$, the above result implies that the characteristic polynomial of ϕ on $H^1_{\mathrm{dR}}(X)[f]$ is (a power of) $P_f(T)$. In particular, the eigenvalues of ϕ are $\frac{p}{\alpha_f} = \beta_f\chi_f(p)^{-1} = \beta_{f^*}$ and $\frac{p}{\beta_f} = \alpha_f\chi_f(p)^{-1} = \alpha_{f^*}$. We remind that our notations are different than those in [BSV22, § 2.5].

Corollary 5.22. *Let g, h be two cusp forms on $\Gamma_1(N)$ of weight 2 which are eigenforms for the Hecke operator T_p and ω_g, ω_h be the associated differential form on X. Let $\tilde{\omega}_g, \tilde{\omega}_h \in H^1_{\mathrm{fp}}(X, 1)$ be the lifts of ω_g, ω_h such that the associated Coleman integrals vanish at the ∞-cusp. Suppose that $\eta \in H^1_{\mathrm{dR}}(X)$ is an eigenvector of ϕ with eigenvalue α and $\mathrm{ord}_p(\alpha) = a \in \mathbb{Q}$. Let $\beta = p/\alpha$. Then we have*

$$\begin{aligned}\langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h)\rangle &= (1 - \beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h) \\ &\quad \times \left((1 - \beta\alpha'_g\alpha'_h)(1 - \beta\alpha'_g\beta'_h)(1 - \beta\beta'_g\alpha'_h)(1 - \beta\beta'_g\beta'_h)\right)^{-1} \\ &\quad \times \langle \eta, e^{\leq a}(G^{[p]}\omega_h)\rangle.\end{aligned}$$

Proof. By the choice of the polynomial $P_g(T)$, we have the element $(\omega_g, G^{[p]}) \in H^1_{\mathrm{syn},P_g}(X, 1)$. In order to apply Theorem 5.20, one needs to modify P_g into a degree one polynomial. This can be done by considering the two p-stablizations of g. Namely, we let

$$\begin{aligned}\omega_{g,\alpha} &:= (1 - \beta'_g\phi)\omega_g, \quad \omega_{g,\beta} := (1 - \alpha'_g\phi)\omega_g \\ \omega_{h,\alpha} &:= (1 - \beta'_h\phi)\omega_h, \quad \omega_{h,\beta} := (1 - \alpha'_h\phi)\omega_h.\end{aligned}$$

Then they may be lifted to classes

$$\begin{aligned}\tilde{\omega}_{g,\alpha} &= (\omega_{g,\alpha}, G^{[p]}) \in H^1_{\mathrm{syn},1-\alpha'_g T}(X, 1) \subset H^1_{\mathrm{fp}}(X, 1) \\ \tilde{\omega}_{g,\beta} &= (\omega_{g,\beta}, G^{[p]}) \in H^1_{\mathrm{syn},1-\beta'_g T}(X, 1) \subset H^1_{\mathrm{fp}}(X, 1)\end{aligned}$$

and similarly for h. The obvious relation $\omega_g = (\alpha'_g - \beta'_g)^{-1}(\alpha'_g\omega_{g,\alpha} - \beta'_g\omega_{g,\beta})$ implies the same relation

$$\tilde{\omega}_g = (\alpha'_g - \beta'_g)^{-1}(\alpha'_g\tilde{\omega}_{g,\alpha} - \beta'_g\tilde{\omega}_{g,\beta})$$

in $H^1_{\mathrm{fp}}(X,1)$. Hence, the pairing $\langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde\omega_g \cup \tilde\omega_h)\rangle$ decomposes into 4 terms

$$(\alpha'_g-\beta'_g)^{-1}(\alpha'_h-\beta'_h)^{-1}\left[\langle \alpha'_g\tilde\omega_{g,\alpha}, \alpha'_h\tilde\omega_{h,\alpha}\rangle - \langle \alpha'_g\tilde\omega_{g,\alpha}, \beta'_h\tilde\omega_{h,\beta}\rangle - \langle \beta'_g\tilde\omega_{g,\beta}, \alpha'_h\tilde\omega_{h,\alpha}\rangle + \langle \beta'_g\tilde\omega_{g,\beta}, \beta'_h\tilde\omega_{h,\beta}\rangle\right]$$

and each term satisfies the assumption of Theorem 5.20. For example, the first cup product in the bracket is

$$\begin{aligned}\alpha'_g\alpha'_h\langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde\omega_{g,\alpha}\cup\tilde\omega_{h,\alpha})\rangle &= \alpha'_g\alpha'_h(1-\beta\alpha'_g\alpha'_h)^{-1}\langle \eta, e^{\leq a}G^{[p]}\omega_{h,\alpha}\rangle\\ &= \alpha'_g\alpha'_h(1-\beta\alpha'_g\alpha'_h)^{-1}\langle \eta, e^{\leq a}G^{[p]}\omega_{h}\rangle\end{aligned}$$

where we again use the fact that $U(G^{[p]}(1-\beta'_h\phi)\omega_h) = U(G^{[p]}\omega_h) - \beta'_h U(G^{[p]}\phi\omega_h) = U(G^{[p]}\omega_h)$. The rest is then reduced to the following computation

$$\begin{aligned}&(\alpha'_g-\beta'_g)^{-1}(\alpha'_h-\beta'_h)^{-1}\left(\frac{\alpha'_g\alpha'_h}{1-\beta\alpha'_g\alpha'_h} - \frac{\beta'_g\alpha'_h}{1-\beta\beta'_g\alpha'_h} - \frac{\alpha'_g\beta'_h}{1-\beta\alpha'_g\beta'_h} + \frac{\beta'_g\beta'_h}{1-\beta\beta'_g\beta'_h}\right)\\ &= (1-\beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h)\times\big((1-\beta\alpha'_g\alpha'_h)(1-\beta\alpha'_g\beta'_h)(1-\beta\beta'_g\alpha'_h)(1-\beta\beta'_g\beta'_h)\big)^{-1}.\end{aligned}$$

□

The remaining task is to relate the above result to the p-adic syntomic Abel–Jacobi image of the diagonal cycle $\Delta_{2,2,2}$. Consider the triple product X^3 and let $\pi_i : X^3 \to X$ be the projection of the i-th component. Let $o \in X$ be the ∞-cusp and consider the following embeddings of X into X^3

$$\begin{aligned}\iota_{123} :&X \xrightarrow{\Delta} X\times X\times X,\\ \iota_{12} :&X \xrightarrow{\Delta} X\times X \to X\times X\times\{o\} \hookrightarrow X^3,\\ \iota_{13} :&X \xrightarrow{\Delta} X\times X \to X\times\{o\}\times X \hookrightarrow X^3,\\ \iota_{23} :&X \xrightarrow{\Delta} X\times X \to \{o\}\times X\times X \hookrightarrow X^3,\\ \iota_{1} :&X \to X\times\{o\}\times\{o\} \hookrightarrow X^3,\\ \iota_{2} :&X \to \{o\}\times X\times\{o\} \hookrightarrow X^3,\\ \iota_{3} :&X \to \{o\}\times\{o\}\times X \hookrightarrow X^3.\end{aligned}$$

Recall that the diagonal cycle $\Delta_{2,2,2} \in A^2(X^3)_0$ is defined as

$$\Delta_{2,2,2} = \iota_{123}(X) - \iota_{12}(X) - \iota_{23}(X) - \iota_{13}(X) + \iota_1(X) + \iota_2(X) + \iota_3(X).$$

For simplicity, we will write X_I for $\iota_I(X)$ for $\emptyset \neq I \subset \{1,2,3\}$.

Now we want to evaluate $\mathrm{AJ}_p(\Delta_{2,2,2})$ at $\eta\otimes\omega_g\otimes\omega_h$, where η, ω_g, ω_h are as in Corollary 5.22. We first take a lift of $\eta\otimes\omega_g\otimes\omega_h \in \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3)$ in $H^3_{\mathrm{fp}}(X^3, 2)$. Such a lift can be taken to be $\pi_1^*\tilde\eta \cup \pi_2^*\tilde\omega_g \cup \pi_3^*\tilde\omega_h$ where $\tilde\eta$ is the preimage of η under the isomorphism $H^1_{\mathrm{fp}}(X,0) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} H^1_{\mathrm{dR}}(X)$, while $\tilde\omega_g, \tilde\omega_h$ are lifts such that the associated Coleman integrals vanish at the ∞-cusp.

Then, by definition,

$$\begin{aligned}\mathrm{AJ}_p(\Delta_{2,2,2})(\eta\otimes\omega_g\otimes\omega_h) &= -\sum_{\emptyset\neq I\subset\{1,2,3\}}(-1)^{|I|}\langle \mathrm{cl}_{\mathrm{fp}}(X_I), \pi_1^*\tilde\eta\cup\pi_2^*\tilde\omega_g\cup\pi_3^*\tilde\omega_h\rangle_{\mathrm{fp}}\\ &= -\sum_{\emptyset\neq I\subset\{1,2,3\}}(-1)^{|I|}\,\mathrm{tr}_{\mathrm{fp},X}(\iota_I^*(\pi_1^*\tilde\eta\cup\pi_2^*\tilde\omega_g\cup\pi_3^*\tilde\omega_h)).\end{aligned}$$

This allows us to compute everything over X instead of over X^3. Since $\pi_i \circ \iota_I$ is the identity map if $i \in I$ and is the constant map o otherwise, we see that $\mathrm{AJ}(\Delta_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h)$ is the trace of

$$\begin{aligned}
&\tilde{\eta} \cup \tilde{\omega}_g \cup \tilde{\omega}_h \\
&\quad - \tilde{\eta} \cup o^*\tilde{\omega}_g \cup \tilde{\omega}_h - \tilde{\eta} \cup \tilde{\omega}_g \cup o^*\tilde{\omega}_h \\
&\quad - o^*\tilde{\eta} \cup \tilde{\omega}_g \cup \tilde{\omega}_h \\
&\quad + \tilde{\eta} \cup o^*(\tilde{\omega}_g \cup \tilde{\omega}_h) \\
&\quad + o^*\tilde{\eta} \cup o^*\tilde{\omega}_g \cup \tilde{\omega}_h + o^*\tilde{\eta} \cup \tilde{\omega}_g \cup o^*\tilde{\omega}_h.
\end{aligned}$$

As $o^*\tilde{\eta} \in H^1_{\mathrm{fp}}(\operatorname{Spec} \mathcal{O}_K, 0) = 0$, the third and the fifth lines are 0. The fourth line is also zero since $H^2_{\mathrm{fp}}(\operatorname{Spec} \mathcal{O}_K, 2) = 0$. By [Bes00a, Theorem 1.1], $o^*\tilde{\omega}_g \in K$ is the evaluation of the associated Coleman integration at the point o. Hence $o^*\tilde{\omega}_g$ and $o^*\tilde{\omega}_h$ are both 0 by our choices.

In conclusion, we have $\mathrm{AJ}(\Delta_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h) = \tilde{\eta} \cup \tilde{\omega}_g \cup \tilde{\omega}_h = \langle \tilde{\eta}, \tilde{\omega}_g \cup \tilde{\omega}_h \rangle_{\mathrm{fp}}$. Moreover, since $\mathrm{i}_{\mathrm{fp}} : H^1_{\mathrm{dR}}(X) \to H^2_{\mathrm{fp}}(X, 2)$ is an isomorphism, the compatibility of cup products implies that $\langle \tilde{\eta}, \tilde{\omega}_g \cup \tilde{\omega}_h \rangle_{\mathrm{fp}} = \langle \mathrm{pr}_{\mathrm{fp}}(\tilde{\eta}), \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h) \rangle_{\mathrm{dR}} = \langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h) \rangle_{\mathrm{dR}}$.

In conclusion, we have the following modified version of [DR14, Theorem 3.14].

Theorem 5.23. *Let η, g, h be as in Corollary 5.22 and $\Delta_{2,2,2}$ be the diagonal cycle defined above. Then we have*

$$\begin{aligned}
\mathrm{AJ}(\Delta_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h) = &(1 - \beta^2 \alpha'_g \alpha'_h \beta'_g \beta'_h) \\
&\times \left((1 - \beta \alpha'_g \alpha'_h)(1 - \beta \alpha'_g \beta'_h)(1 - \beta \beta'_g \alpha'_h)(1 - \beta \beta'_g \beta'_h) \right)^{-1} \\
&\times \langle \eta, e^{\leq a}(G^{[p]} \omega_h) \rangle_{\mathrm{dR}}.
\end{aligned}$$

5.4 Finite polynomial cohomology with coefficients

Let X be proper smooth V-scheme of relative dimension $d = d_X$ and fix a overconvergent F-isocrystal (E, ϕ) on X which is of pure weight w. Again, following the idea in [Bes00b], one has canonical functorial complexes of K-vector spaces

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa, (E, \phi)),\ \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (E, \nabla)),\ \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, (E, \nabla))$$

computing cohomology groups of their namesakes respectively (c.f. §8.2). For simplicity, we will drop the notations κ, K, ϕ and write $\mathbb{R}\Gamma_{\mathrm{rig}}(X, E), \mathbb{R}\Gamma_{\mathrm{dR}}(X, E), \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X, E)$ when no confusion might be caused. As before, we also have natural quasi-isomorphsim $\mathbb{R}\Gamma_{\mathrm{dR}}(X, E) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X, E)$, which induces the comparison $H^i_{\mathrm{dR}}(X, E) \cong H^i_{\mathrm{rig}}(X, E)$.

Remark 5.24. As mentioned in remark 5.2, these functorial complexes can be constructed by Godement resolutions. For the rigid complex, one needs to pass to adic spaces so that the construction of Godement resolution works. On the other hand, since our construction of the cycle classes map with coefficients requires no input from K-theory, we will simply view them as objects in the derived category of K-complexes. The existences and functorialities are provided by their respective cohomology theory. In addition, the language of mapping cones also applies to derived complexes.

Definition 5.25. For any polynomial $P(T) \in \mathfrak{P}$ and $n \in \mathbb{Z}$, we define the syntomic P-complex $\mathbb{R}\Gamma_{\mathrm{syn},P}(X, E, n)$ by

$$\mathbb{R}\Gamma_{\mathrm{syn},P}(X, E, n) := \mathrm{Cone}\left(\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X, E) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X, E) \right)[-1].$$

Its i-th cohomology will be denoted by $H^i_{\mathrm{syn},P}(X, E, n)$.

For any $s \in \mathbb{Z}$, we define

$$\mathbb{R}\Gamma_{\mathrm{fp},s}(X, E, n) := \varinjlim_{P \in \mathfrak{P}_s} \mathbb{R}\Gamma_{\mathrm{syn},P}(X, E, n)$$

where the connecting map is similar to the one in Definition 5.4. Its cohomology will be denoted by $H^i_{\mathrm{fp},s}(X, E, n)$.

We will be particularly interested in cohomology $H^i_{\mathrm{fp},w+i}(X,E,n)$, which will be denoted simply by $H^i_{\mathrm{fp}}(X,E,n)$. The reason is encoded in the following proposition analogous to Proposition 5.7.

Proposition 5.26. *The group $H^i_{\mathrm{fp}}(X,E,n)$ satisfies the following properties:*

1. *We have the fundamental exact sequence*

$$0 \to H^{i-1}_{\mathrm{dR}}(X,E)/\operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X,E) \xrightarrow{\mathrm{i}_{\mathrm{fp}}} H^i_{\mathrm{fp}}(X,E,n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \operatorname{Fil}^n H^i_{\mathrm{dR}}(X,E) \to 0.$$

2. *Let $E^\vee$ be the dual of E such that Poincaré pairing reads*

$$H^i_{\mathrm{dR}}(X,E) \times H^{2d-i}_{\mathrm{dR}}(X,E^\vee) \to H^{2d}_{\mathrm{dR}}(X) \cong K(-d).$$

   Then we have a perfect pairing $\langle\ ,\ \rangle_{\mathrm{fp}}$ induced by the cup product

$$H^i_{\mathrm{fp}}(X,E,n) \times H^{2d+1-i}_{\mathrm{fp}}(X,E^\vee,d+1-n) \xrightarrow{\cup} H^{2d+1}_{\mathrm{fp}}(X,d+1) \cong K$$

   which is compatible with the Poincaré pairing on the de Rham cohomology via the fundamental exact sequence.

3. *Suppose that $\iota_Z : Z \to X$ is a smooth irreducible closed subscheme of codimension c. Then we have a pushforward map*

$$\iota_{Z,*} : H^i_{\mathrm{fp}}(Z,\iota_Z^*E,n) \to H^{i+2c}_{\mathrm{fp}}(X,E,n+c)$$

   which can be described as the adjoint of the pullback ι_Z^* under the above cup product pairing.

Proof. The proof is highly similar to the proof of Proposition 5.7. In fact, everything in §5.1 and §5.2 can be translated by replacing

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa),\ \mathbb{R}\Gamma_{\mathrm{dR}}(X_K),\ \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K)$$

with

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa,(E,\phi)),\ \mathbb{R}\Gamma_{\mathrm{dR}}(X_K,(E,\nabla)),\ \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K,(E,\nabla)),$$

respectively. As before, we will work with a fixed polynomial. The results will then follow by taking inductive limit.

Let $P \in \mathfrak{P}$. We set $C^\bullet_P := \operatorname{Cone}(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(X,E))[-1]$. Assume that $P(\phi)$ annihilates $H^i_{\mathrm{rig}}(X,E)$ but acts bijectively on $H^{i-1}_{\mathrm{rig}}(X,E)$. Notice that the existence of such a polynomial P is from our pure weight assumption on E. In particular, one may further assume that $P \in \mathfrak{P}_{i+w}$.

Then we have the following long exact sequence

$$\cdots \longrightarrow \operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X,E) \xrightarrow{P(\phi)} H^{i-1}_{\mathrm{rig}}(X,E) \longrightarrow H^i(C^\bullet_P)$$

$$\longrightarrow \operatorname{Fil}^n H^i_{\mathrm{dR}}(X,E) \xrightarrow{P(\phi)} H^i_{\mathrm{rig}}(X,E) \longrightarrow \cdots.$$

By the choice of P and the comparison $H^i_{\mathrm{rig}}(X,E) \cong H^i_{\mathrm{dR}}(X,E)$, one obtains the following short exact sequence

$$0 \to H^{i-1}_{\mathrm{dR}}(X,E)/P(\phi)\operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X,E) \xrightarrow{\mathrm{i}'_{\mathrm{fp}}} H^i_{\mathrm{syn},P}(X,E,n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \operatorname{Fil}^n H^i_{\mathrm{dR}}(X,E) \to 0.$$

As $P(\phi)$ is an isomorphism on $H^{i-1}_{\mathrm{dR}}(X,E)$, we have $H^{i-1}_{\mathrm{dR}}(X,E)/\operatorname{Fil}^n \cong H^{i-1}_{\mathrm{dR}}(X,E)/P(\phi)\operatorname{Fil}^n$. Hence we can rewrite the above sequence as

$$0 \to H^{i-1}_{\mathrm{dR}}(X,E)/\operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X,E) \xrightarrow{\mathrm{i}_{\mathrm{fp}}} H^i_{\mathrm{syn},P}(X,E,n) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \operatorname{Fil}^n H^i_{\mathrm{dR}}(X,E) \to 0,$$

where the first map is now twisted by $P(\phi)$.

For 2 and 3, we will adopt the notations used in §5.2. First, we remark that for two overconvergent F-isocrystals E, E' on X, we have the maps

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X,E)\otimes_K \mathbb{R}\Gamma_{\mathrm{rig}}(X,E')\to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E\otimes E') \text{ and}$$
$$\mathbb{R}\Gamma_{\mathrm{dR}}(X,E)\otimes_K \mathbb{R}\Gamma_{\mathrm{dR}}(X,E')\to \mathbb{R}\Gamma_{\mathrm{dR}}(X,E\otimes E')$$

that induce the cup products on the cohomology groups. When $E'=E^\vee$, one gets a perfect pairing

$$H^i_\bullet(X,E)\otimes_K H^{2d-i}_\bullet(X,E^\vee)\to H^{2d}_\bullet(X,E\otimes E^\vee)\xrightarrow{\mathrm{ev}} H^{2d}_\bullet(X)\cong K(-d)$$

where $\mathrm{ev}: E\otimes E^\vee\to \mathcal{O}_X$ is the evaluation map and $\bullet\in\{\mathrm{dR},\mathrm{rig}\}$. For de Rham cohomology, the proof can be found in [AB01]. For rigid cohomology, it is proved in [Ked06]. Alternatively, since X is proper, one can apply rigid GAGA to reduce to the de Rham case. As a consequence, when E if of pure weight w, $E^\vee$ is of pure weight $-w$.

As in Proposition 5.15, one can compute $\mathbb{R}\Gamma_{\mathrm{syn},P}(X,E,n)$ by the complex

$$\mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E)\oplus \mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E))\to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E)\big),$$

and the maps i_{fp} and $\mathrm{pr}_{\mathrm{fp}}$ can be interpreted as the first and second maps in the following sequence

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{rig}}(X,E)[-1]&\to \mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E)\oplus \mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E))\to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E)\big)\\ &\to \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E),\end{aligned}$$

respectively.

Consider now the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{i-1}_{\mathrm{dR}}(X,E)/\operatorname{Fil}^n \arrow[r, "\mathrm{i}_{\mathrm{fp}}"] \arrow[d, "\sim"] & H^i_{\mathrm{fp}}(X,E,n) \arrow[r, "\mathrm{pr}_{\mathrm{fp}}"] \arrow[d] & \operatorname{Fil}^n H^i_{\mathrm{dR}}(X,E) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \Big(\operatorname{Fil}^{d+1-n} H^{2d+1-i}_{\mathrm{dR}}(X,E^\vee)\Big)^\vee \arrow[r, "\mathrm{pr}_{\mathrm{fp}}^\vee"] & H^{2d+1-i}_{\mathrm{fp}}(X,E^\vee,d+1-n)^\vee \arrow[r, "\mathrm{i}_{\mathrm{fp}}^\vee"] & \Big(H^{2d-i}_{\mathrm{dR}}(X,E^\vee)/\operatorname{Fil}^{d+1-n}\Big)^\vee \arrow[r] & 0.
\end{tikzcd}$$

The first and the third vertical maps come from Poincaré duality of de Rham cohomology. For the middle one, recall from §5.2 that we can construct a map

$$\mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E))\otimes \mathrm{Hom}(V_Q^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E^\vee))\to \mathrm{Hom}(V_{P*Q}^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X)),$$

where $P\in\mathfrak{P}_{i+w}$ and $Q\in\mathfrak{P}_{2d+1-i-w}$ are such that $P(\phi)$ annihilates $H^i_{\mathrm{rig}}(X,E)$ and acts invertibly on $H^{i-1}_{\mathrm{rig}}(X,E)$, while $Q(\phi)$ annihilates $H^{2d+1-i}_{\mathrm{rig}}(X,E^\vee)$ and acts invertibly on $H^{2d-i}_{\mathrm{rig}}(X,E^\vee)$. This induces a product between

$$\mathrm{MF}\big(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E)\oplus \mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E))\to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E)\big)$$

and

$$\mathrm{MF}\big(\operatorname{Fil}^{d+1-n} \mathbb{R}\Gamma_{\mathrm{dR}}(X,E^\vee)\oplus \mathrm{Hom}(V_Q^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X,E^\vee))\to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E^\vee)\big)$$

to the mapping fiber $\mathrm{MF}\big(\operatorname{Fil}^{d+1}\mathbb{R}\Gamma_{\mathrm{dR}}(X)\oplus \mathrm{Hom}(V_{P*Q}^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X))\to \mathbb{R}\Gamma_{\mathrm{rig}}(X)\big)$. Taking cohomology, we get a pairing

$$H^i_{\mathrm{syn},P}(X,E,n)\times H^{2d+1-i}_{\mathrm{syn},Q}(X,E^\vee,d+1-n)\to H^{2d+1}_{\mathrm{syn},P*Q}(X,d+1)\cong K \tag{28}$$

which gives rise to the middle map. Notice that since $P*Q\in\mathfrak{P}_{2d+1}$, one can identify $H^{2d+1}_{\mathrm{syn},P*Q}(X,d+1)$ with K. In general, one has a cup product

$$H^i_{\mathrm{syn},P}(X,E,n)\times H^j_{\mathrm{syn},Q}(X,E^\vee,m)\to H^{i+j}_{\mathrm{syn},P*Q}(X,n+m).$$

Moreover, it can be explicitly described as in Remark 5.18.

We also remark that commutativity of the diagram again comes from the descriptions of i_{fp} and $\mathrm{pr}_{\mathrm{fp}}$ via $\mathrm{MF}\big(\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X,E) \oplus \mathrm{Hom}(V_P^{\bullet}, \mathbb{R}\Gamma_{\mathrm{rig}}(X,E)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X,E)\big)$. Finally, the snake lemma concludes that equation (28) is a perfect pairing.

Given a proper smooth morphism $f : Y \to X$, we have the natural maps $f^* : \mathbb{R}\Gamma_{\mathrm{syn},P}(X,E,n) \to \mathbb{R}\Gamma_{\mathrm{syn},P}(Y,f^*E,n)$ and $f^* : \mathbb{R}\Gamma_{\mathrm{fp},s}(X,E,n) \to \mathbb{R}\Gamma_{\mathrm{fp},s}(Y,f^*E,n)$, since both $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}$ and $\mathbb{R}\Gamma_{\mathrm{rig}}$ are functorial. However, in order for the pullback map f^* to be compatible with the fundamental exact sequences on Y and X, we need the condition in Definition 8.13. To be more precise, this condition ensures that we can take a polynomial $P \in \mathfrak{P}_{w+i}$ such that $P(\phi)$ annihilates $H^i_{\mathrm{rig}}(X,E)$ and $H^i_{\mathrm{rig}}(Y,f^*E)$ simultaneously, while acts invertibly on $H^{i-1}_{\mathrm{rig}}(X,E)$ and $H^{i-1}_{\mathrm{rig}}(Y,f^*E)$. In particular, let $\iota_Z : Z \to X$ be as in the statement 3, the pushforward $\iota_{Z,*}$ is defined as the dual of ι_Z^* with respect to the pairing $\langle\ ,\ \rangle_{\mathrm{fp}}$.

□

5.5 Abel–Jacobi maps with coefficients

We begin with a description of Abel–Jacobi maps in the trivial coefficient case.

Recall that if $Z \subset X$ is a proper smooth irreducible subvariety of codimension c defined over K, we have an isomorphism $H^0_{\mathrm{dR}}(Z) \cong K$ and the Gysin sequence

$$0 \to H^{2c-1}_{\mathrm{dR}}(X) \to H^{2c-1}_{\mathrm{dR}}(X\backslash Z) \to H^{2c}_{\mathrm{dR},Z}(X)(-c) \cong H^0_{\mathrm{dR}}(Z)(-c) \xrightarrow{\mathrm{cl}_X} H^{2c}_{\mathrm{dR}}(X) \to \cdots .$$

Suppose now $Z = \sum a_j Z_j \in A^c(X)_0$ with each Z_j smooth irreducible, the condition of being cohomologically trivial implies that the sum $\sum a_j \mathrm{cl}_X(1_{Z_j})$ is zero in $H^{2c}_{\mathrm{dR}}(X)$. The coefficients a_j's define a unique K-line in $H^0_{\mathrm{dR}}(Z)$. Hence one gets an extension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(X) & \longrightarrow & D & \longrightarrow & K(-c) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(X) & \longrightarrow & H^{2c-1}_{\mathrm{dR}}(X\backslash Z) & \longrightarrow & H^0_{\mathrm{dR}}(Z)(-c) & &
\end{array}$$

via pullback.

Remark 5.27. One can also define the Abel–Jacobi map under the framework of Galois representations and étale cohomology (c.f. [Nek00]), which is essentially based on the Hochschild–Serre spectral sequence. In the case of varieties with good reduction, it translates back to the above extension (of filtered Frobenius modules) via p-adic comparison theory.

The isomorphism class of D gives an element in $\mathrm{Ext}^1_{\mathrm{ffm}}(K(-c), H^{2c-1}_{\mathrm{dR}}(X))$ where ffm stands for the category of filtered Frobenius modules. This extension class, denoted by $\mathrm{AJ}(D)$, is the Abel–Jacobi image of D. Via the isomorphism (c.f. [BDP13, Proposition 3.5] or [AI19, Lemma 7.2])

$$\mathrm{Ext}^1_{\mathrm{ffm}}(K(-c), H^{2c-1}_{\mathrm{dR}}(X)) \cong H^{2c-1}_{\mathrm{dR}}(X)/\,\mathrm{Fil}^c,$$

one can then view $\mathrm{AJ}(D)$ as an element in $H^{2c-1}_{\mathrm{dR}}(X)/\,\mathrm{Fil}^c$. By further identifying $H^{2c-1}_{\mathrm{dR}}(X)/\,\mathrm{Fil}^c$ with $[\mathrm{Fil}^{d+1-c} H^{2d-2c+1}_{\mathrm{dR}}(X)]^\vee$ under Poincaré duality, this coincides with the Abel–Jacobi map described in §5.1.

Now turn to non-trivial coefficients E. Inspired by the trivial coefficient case, the question reduces to finding a subspace in $H^0_{\mathrm{dR}}(Z,E)$ isomorphic to $K(j)$ for some $j \in \mathbb{Z}$. This is not easy in general. However, when the sheaf E comes from a universal object W over X, one has the following approach. For simplicity, we will restrict to the case where $\dim X = 1$.

Suppose we have a commutative diagram of smooth irreducible K-schemes with good reductions

$$\begin{array}{ccccc}
W & \longleftarrow & W_Z & \longleftarrow & Y \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow & & \\
X & \longleftarrow & Z & &
\end{array}$$

where

- $\dim W = d_W$, $\dim X = 1$, and $\dim Z = 0$.
- $\pi : W \to X$ is proper, smooth, and the square is Cartesian.
- $Y = \sum a_i Y_i$ is a cycle of codimension n in W, hence of codimension $n-1$ in W_Z. Moreover, we require that $Y \in A^n(W)_0$ but $Y \in A^{n-1}(W_Z)$ is not cohomologically trivial.

Consider

$$E := \epsilon \mathbb{R}^w \pi_* \Omega^{\bullet}_{W/X}$$

for some $w \in \mathbb{Z}$ and an auxiliary idempotent $\epsilon \in \mathbb{Q}[\mathrm{Aut}(W/X)]$. By the relative Leray spectral sequence (c.f. [Kat70, (3.3.0)])

$$E_2^{ij} := H^i_{\mathrm{dR}}(X, \mathbb{R}^j \pi_* \Omega^{\bullet}_{W/X}) \Rightarrow H^{i+j}_{\mathrm{dR}}(W),$$

we can view $H^1_{\mathrm{dR}}(X, E)$ as a piece of $H^{w+1}_{\mathrm{dR}}(W)$ (or more precisely, $\epsilon H^{w+1}_{\mathrm{dR}}(E)$). Hence one sees that E has pure weight w.

By setting $W' = W \backslash W_Z$, we have the Gysin sequence

$$\cdots \to H^{2n-3}_{\mathrm{dR}}(W_Z)(-1) \to H^{2n-1}_{\mathrm{dR}}(W) \to H^{2n-1}_{\mathrm{dR}}(W') \to H^{2n-2}_{\mathrm{dR}}(W_Z)(-1) \to H^{2n}_{\mathrm{dR}}(W) \to \cdots .$$

Note that each component Y_i of Y provides a subspace $K(-n+1)$ in $H^{2n-2}_{\mathrm{dR}}(W_Z)$ by the pushforward map

$$H^0_{\mathrm{dR}}(Y_i)(-n+1) \cong K(-n+1) \to H^{2n-2}_{\mathrm{dR}}(W_Z).$$

Since we want to relate $H^1_{\mathrm{dR}}(X, E)$ to $H^{2n-1}_{\mathrm{dR}}(W)$, we will take $w = 2n-2$ and assume the idempotent ϵ is chosen so that:

1. the cycle Y is stable under ϵ;
2. it annihilates $H^j_{\mathrm{dR}}(W)$ for $j \neq 2n-1$;
3. it gives rise to an isomorphism $\epsilon H^{2n-1}_{\mathrm{dR}}(W) \cong H^1_{\mathrm{dR}}(X, E)$ via the Leray spectral sequence.

Consequently, one obtains a diagram

$$\begin{tikzcd}
0 \arrow[r] & \epsilon H^{2n-1}_{\mathrm{dR}}(W) \arrow[r] & \epsilon H^{2n-1}_{\mathrm{dR}}(W') \arrow[r] & \epsilon H^{2n-2}_{\mathrm{dR}}(W_Z)(-1) \arrow[r] & \cdots . \\
& H^1_{\mathrm{dR}}(X, E) \arrow[u, "\cong"'] & & H^0_{\mathrm{dR}}(Z, E)(-n) \arrow[u, hook] & \\
& & & H^0_{\mathrm{dR}}(Y)(-n) \arrow[u, hook] &
\end{tikzcd}$$

The inclusion $H^0_{\mathrm{dR}}(Y)(-n) \hookrightarrow \epsilon H^{2n-2}_{\mathrm{dR}}(W_Z)(-1)$ provides a unique $K(-n)$-line determined by the coefficients a_i's. In summary, we obtain by pullback an extension

$$0 \to H^1_{\mathrm{dR}}(X, E) \to D \to K(-n) \to 0.$$

Notice that D now gives a class in $\mathrm{Ext}^1_{\mathrm{ffm}}(K(-n), H^1_{\mathrm{dR}}(X, E)) \cong H^1_{\mathrm{dR}}(X, E)/\,\mathrm{Fil}^n$, where $n = \frac{w}{2} + 1$ is different from the trivial coefficient case.

Remark 5.28. In general, let E be an arbitrary overconvergent F-isocrystal of pure weight w on X. The above discussion suggests that in order to define Abel–Jacobi maps with the coefficient E, the weight w must be even. This restriction can also be found in [BDP13] and [DR14]. Furthermore, for a cohomologically trivial cycle Z of codimension c with coefficient E, a correct Abel–Jacobi map should send Z into $H^{2c-1}_{\mathrm{dR}}(X, E)/\,\mathrm{Fil}^{\frac{w}{2}+c}$. The philosophy is that we still treat E as if it came from $\mathbb{R}^w \pi_* \Omega^{\bullet}_{W/X}$ for some morphism $\pi : W \to X$ that may not even exist.

Now let X be a proper smooth irreducible V-scheme of relative dimension $d = d_X$ as before and E be a convergent F-isocrystal on X_κ of pure weight $w \in 2\mathbb{Z}$. We first need to define the cycle class group with coefficient in E.

In the trivial coefficient case, the target of the cycle class map $\mathrm{cl}_X : A^c(X) \to H^{2c}_{\mathrm{dR}}(X)$ is a K-vector space. Hence we may extend scalar to $A^c(X) \otimes_{\mathbb{Z}} K$. If we let $Z^c(X)$ be the collection of smooth irreducible closed subscheme Z in X such that they form a base of $A^c(X)$. Then we may write $A^c(X) \otimes_{\mathbb{Z}} K = \oplus_{Z \in Z^c(X)} K = \oplus_{Z \in Z^c(X)} H^0_{\mathrm{dR}}(Z)$. Inspired by this observation, we have the following definition.

Definition 5.29. The cycle group with coefficients in E of codimension c, denoted by $A^c(X, E)$, is defined to be

$$A^c(X,E) := \bigoplus_{Z \in Z^i(X)} H^0_{\mathrm{fp}}(Z, E, \frac{w}{2}) = \bigoplus_{Z \in Z^i(X)} \mathrm{Fil}^{\frac{w}{2}} H^0_{\mathrm{dR}}(Z, E).$$

The finite polynomial class map $\mathrm{cl}_{\mathrm{fp}}$ is simply defined as the sum of the pushforward maps

$$\mathrm{cl}_{\mathrm{fp}} := \bigoplus_{Z \in Z^i(X)} \iota_{Z,*} : A^c(X,E) \to H^{2c}_{\mathrm{fp}}(X, E, \frac{w}{2} + c).$$

We then define $A^c(X,E)_0 = \ker(\mathrm{pr}_{\mathrm{fp}} \circ \mathrm{cl}_{\mathrm{fp}})$ to be the subgroup of cohomologically trivial cycles with coefficients in E. By the fundamental exact sequence

$$0 \to H^{2c-1}_{\mathrm{dR}}(X,E)/\mathrm{Fil}^{\frac{w}{2}+c} \xrightarrow{\mathrm{i}_{\mathrm{fp}}} H^{2c}_{\mathrm{fp}}(X, E, \frac{w}{2} + c) \xrightarrow{\mathrm{pr}_{\mathrm{fp}}} \mathrm{Fil}^{\frac{w}{2}+c} H^{2c}_{\mathrm{dR}}(X,E) \to 0,$$

we define the finite polynomial Abel–Jacobi map with coefficients in E

$$\mathrm{AJ}_{\mathrm{fp}} : A^c(X,E)_0 \to H^{2c-1}_{\mathrm{dR}}(X,E)/\mathrm{Fil}^{\frac{w}{2}+c}$$

simply by the $\mathrm{i}_{\mathrm{fp}}^{-1} \circ \mathrm{cl}_{\mathrm{fp}}$. One can further identify the target with $[\mathrm{Fil}^{d+1-\frac{w}{2}-c} H^{2d-2c+1}_{\mathrm{dR}}(X, E^\vee)]^\vee$ under our notation of duality.

5.6 The special case $\mathcal{H}^r$

In this section, we will deal with the sheaf $\mathcal{H}^r$. Some modifications are needed because we are interested in its parabolic cohomology instead of the usual de Rham cohomology.

We first recall several definitions that already appeared in §2. Let $X = X_1(N)$ be as in the previous section, $E \xrightarrow{\pi} X$ be the universal generalized elliptic curve, and $Y \subset X$ be the affine modular curve. Recall that $\mathcal{H}$ is the canonical extension of $\mathbb{R}^1\pi_*\Omega^1_{E/X}$ on Y to X. It comes with an exact sequence

$$0 \to \underline{\omega} \to \mathcal{H} \to \underline{\omega}^{-1} \to 0$$

which defines the Hodge filtration on $\mathcal{H}$ and a Gauss–Manin connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X$.

For any positive integer r, we let $\mathcal{H}^r := \mathrm{Sym}^r \mathcal{H}$ be the r-th symmetric power of $\mathcal{H}$, together with the induced Hodge filtration and Gauss–Manin connection.

The parabolic complex $(\mathcal{H}^r \otimes \Omega^\bullet_X)_{\mathrm{par}}$ is a subcomplex of

$$0 \to \mathcal{H}^r \to \mathcal{H}^r \otimes \Omega^1_X(\log C) \to 0$$

defined by

$$(\mathcal{H}^r \otimes \Omega^0_X)_{\mathrm{par}} := \mathcal{H}^r,$$
$$(\mathcal{H}^r \otimes \Omega^1_X)_{\mathrm{par}} := \nabla(\mathcal{H}^r) + \mathcal{H}^r \otimes \Omega^1_X.$$

The hypercohomology of $(\mathcal{H}^r \otimes \Omega^\bullet_X)_{\mathrm{par}}$ will be denoted by $H^i_{\mathrm{par}}(X, \mathcal{H}^r)$. One also has its rigid analogue, the rigid parabolic cohomology $H^i_{\mathrm{rig,par}}(X, \mathcal{H}^r)$.

The parabolic cohomology $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$ is equipped with a short exact sequence

$$0 \to H^0(X, \underline{\omega}^r \otimes \Omega^1_X) \to H^1_{\mathrm{par}}(X, \mathcal{H}^r) \to H^1(X, \underline{\omega}^{-r}) \to 0.$$

However, the Hodge filtration on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$ is not the naive 2-step filtration. As explained in [DR14, Lemma 2.2], the parabolic cohomology can be viewed as the image of $H^{r+1}_{\mathrm{dR}}(W_r)$ under a certain idempotent $\epsilon_r \in \mathbb{Q}[\mathrm{Aut}(W_r/X)]$. Here W_r is the Kuga–Sato variety appearing in the introduction (also see §5.8). In particular, the Hodge filtration on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$ is given by the Hodge filtration on $H^{r+1}_{\mathrm{dR}}(W_r)$. That is,

$$\begin{aligned}
&\mathrm{Fil}^0 = H^1_{\mathrm{par}}(X, \mathcal{H}^r),\\
&\mathrm{Fil}^1 = \mathrm{Fil}^2 = \cdots = \mathrm{Fil}^{r+1} = H^0(X, \underline{\omega}^r \otimes \Omega^1_X),\\
&\mathrm{Fil}^{r+2} = 0.
\end{aligned}$$

Also, Poincaré duality on the fibers induces a duality $\mathcal{H}^r \times \mathcal{H}^r \to \mathcal{O}_X(-r)$ which allows us to identify $(\mathcal{H}^r)^\vee$ with $\mathcal{H}^r(r)$.

We then have the derived complexes

$$\mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r),\ \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r),\ \mathbb{R}\Gamma_{\mathrm{rig,par}}(X, \mathcal{H}^r)$$

which compute $H^i_{\mathrm{par}}(X, \mathcal{H}^r), \mathrm{Fil}^n H^i_{\mathrm{par}}(X, \mathcal{H}^r)$ and $H^i_{\mathrm{rig,par}}(X, \mathcal{H}^r)$ respectively. Moreover, there are a natural map $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r) \to \mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r)$ and a quasi-isomorphism $\mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r) \to \mathbb{R}\Gamma_{\mathrm{rig,par}}(X, \mathcal{H}^r)$. The Frobenius ϕ on $\mathbb{R}\Gamma_{\mathrm{rig,par}}(X, \mathcal{H}^r)$ then induces a Frobenius action ϕ on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$.

Since W_r is of good reduction, the Frobenius action ϕ on $H^{r+1}_{\mathrm{dR}}(W_r)$ is pure of Weil weight $r+1$. As a consequence, all the eigenvalues of the Frobenius ϕ on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)$ are also of Weil weight $r+1$.

Replace everything with its parabolic version, we have the following definition.

Definition 5.30. For any polynomial $P(T) \in \mathfrak{P}$ and $n \in \mathbb{N}$, we define the syntomic P-complex $\mathbb{R}\Gamma_{\mathrm{syn},P}(X, \mathcal{H}^r, n)$ by

$$\mathbb{R}\Gamma_{\mathrm{syn},P}(X, \mathcal{H}^r, n) := \mathrm{Cone}\left(\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{par}}(X, \mathcal{H}^r) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig,par}}(X, \mathcal{H}^r)\right)[-1].$$

Its i-th cohomology will be denoted by $H^i_{\mathrm{syn},P}(X, \mathcal{H}^r, n)$.

For any $s \in \mathbb{Z}$, we define

$$\mathbb{R}\Gamma_{\mathrm{fp},s}(X, \mathcal{H}^r, n) := \varinjlim_{P \in \mathfrak{P}_s} \mathbb{R}\Gamma_{\mathrm{syn},P}(X, \mathcal{H}^r, n)$$

where the connecting map is similar to the one in Definition 5.4. Its cohomology will be denoted by $H^i_{\mathrm{fp},s}(X, \mathcal{H}^r, n)$.

Remark 5.31. Technically speaking, we should use the notation $H^i_{\mathrm{syn,par},P}$ and $H^i_{\mathrm{fp,par},s}$. However, we prefer to drop the subscript in order to keep notations clean. Any cohomology that involves $\mathcal{H}^r$ should be understood as the parabolic version in this article.

As before, we are particularly interested in the group $H^1_{\mathrm{fp}}(X, \mathcal{H}^r, n) := H^1_{\mathrm{fp},r+1}(X, \mathcal{H}^r, n)$.

Proposition 5.32 (c.f. Proposition 5.26)**.** *The group $H^1_{\mathrm{fp}}(X, \mathcal{H}^r, n)$ satisfies the following properties:*

1. *We have the exact sequence*

$$0 \to H^0_{\mathrm{par}}(X, \mathcal{H}^r)/\mathrm{Fil}^n H^0_{\mathrm{par}}(X, \mathcal{H}^r) \xrightarrow{\mathrm{i_{fp}}} H^1_{\mathrm{fp}}(X, \mathcal{H}^r, n) \xrightarrow{\mathrm{pr_{fp}}} \mathrm{Fil}^n H^1_{\mathrm{par}}(X, \mathcal{H}^r) \to 0.$$

Moreover, the map $\mathrm{pr_{fp}}$ is in fact an isomorphism.

2. *Let $(\mathcal{H}^r)^\vee$ be the dual of $\mathcal{H}^r$ such that the Poincaré pairing reads*

$$H^1_{\mathrm{par}}(X, \mathcal{H}^r) \times H^1_{\mathrm{par}}(X, (\mathcal{H}^r)^\vee) \to K(-1).$$

Then we have a perfect pairing induced by the cup product

$$H^i_{\mathrm{fp}}(X, \mathcal{H}^r, n) \times H^{3-i}_{\mathrm{fp}}(X, (\mathcal{H}^r)^\vee, 2-n) \xrightarrow{\cup} H^3_{\mathrm{fp}}(X, 2) \cong K$$

which is compatible with the Poincaré pairing on H^1_{par} via the fundamental exact sequence.

Definition 5.33. Similarly to Definition 5.29, when r is even we have the finite polynomial Abel–Jacobi map

$$\mathrm{AJ}_{\mathrm{fp}} : A^1(X, \mathcal{H}^r)_0 \to H^1_{\mathrm{par}}(X, \mathcal{H}^r)/\operatorname{Fil}^{\frac{r}{2}+1} H^1_{\mathrm{par}}(X, \mathcal{H}^r) \cong [\operatorname{Fil}^{1-\frac{r}{2}} H^1_{\mathrm{par}}(X, (\mathcal{H}^r)^\vee)]^\vee.$$

Remark 5.34. Recall that we have a perfect pairing

$$H^1_{\mathrm{par}}(X, \mathcal{H}^r) \times H^1_{\mathrm{par}}(X, \mathcal{H}^r) \to K(-r-1).$$

This provides the identification $H^1_{\mathrm{par}}(X, (\mathcal{H}^r)^\vee) \cong H^1_{\mathrm{par}}(X, \mathcal{H}^r)(r)$. Hence, as K-vector spaces,

$$\operatorname{Fil}^{1-\frac{r}{2}} H^1_{\mathrm{par}}(X, (\mathcal{H}^r)^\vee) = \operatorname{Fil}^{\frac{r}{2}+1} H^1_{\mathrm{par}}(X, \mathcal{H}^r).$$

One can check that this numerology is compatible with the Abel–Jacobi map in [BDP13, § 3]. Further explanation can be found in [HW22, § 6.4].

In next section, we will also consider the sheaf $\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3} := \pi_1^*\mathcal{H}^{r_1} \otimes \pi_2^*\mathcal{H}^{r_2} \otimes \pi_3^*\mathcal{H}^{r_3}$ on X^3. By Künneth decomposition, the only interesting (parabolic) cohomology is at middle degree 3, with

$$H^3_{\mathrm{par}}(X^3, \mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3}) \cong H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}) \otimes H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2}) \otimes H^1_{\mathrm{par}}(X, \mathcal{H}^{r_3}).$$

Hence, the Frobenius acts on $H^3_{\mathrm{par}}(X^3, \mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})$ with eigenvalues of Weil weight $r_1 + r_2 + r_3 + 3$, and $\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3}$ is of pure weight $r_1 + r_2 + r_3$ (for the parabolic cohomology). One can then define $H^i_{\mathrm{fp}}(X^3, \mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})$ and $H^i_{\mathrm{fp}}(X^3, (\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})^\vee)$ similarly, where $(\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})^\vee$ is of pure weight $-(r_1 + r_2 + r_3)$. In particular, the finite polynomial Abel–Jacobi map we will use later is of the form

$$\mathrm{AJ}_{\mathrm{fp}} : A^2(X^3, (\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})^\vee)_0 \to [\operatorname{Fil}^{\frac{r_1+r_2+r_3}{2}+2} H^3_{\mathrm{par}}(X^3, \mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3})]^\vee.$$

5.7 Applications to modular forms of weight > 2

We will show how to compute the Abel–Jacobi map appearing in Section 6.2 by using finite polynomial cohomology with coefficients. The ideas and computations are almost identical to those in Section 5.1, with only some minor modifications.

Let $(x, y, z) \in \mathbb{Z}^3_{>2}$ be a triple of balanced weights such that $x = y + z - 2t$ with $t \in \mathbb{Z}_{>0}$. Write $x = r_1 + 2, y = r_2 + 2, z = r_3 + 2$ and $r = \frac{1}{2}(r_1 + r_2 + r_3)$ as before. Then $r_1 = r_2 + r_3 - 2(t-1)$ and $r \leq r_2 + r_3$.

Let $\iota = \iota_{123} : X \to X^3$ be the diagonal embedding. We set $Z := \iota(X) \subset X^3$ to be the smooth closed subscheme of codimension 2 and will often identify it with X in the following arguments. When restricted to Z (pull-back via ι), the sheaf $\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3}$ is identified with $\mathcal{H}^{r_1} \otimes \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3}$ on X. For simplicity, we will denote $\mathcal{H}^{r_1} \boxtimes \mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3}$ by $\mathcal{H}^{\boxtimes}$ and $\mathcal{H}^{r_1} \otimes \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3}$ by $\mathcal{H}^{\otimes}$.

Recall that we want to evaluate the Abel–Jacobi map at an element of the form $\eta \otimes \omega_2 \otimes \omega_3$ with

$$\eta \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}),$$
$$\omega_2 \in \operatorname{Fil}^{r_2+1} H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2}),$$
$$\omega_3 \in \operatorname{Fil}^{r_3+1} H^1_{\mathrm{par}}(X, \mathcal{H}^{r_3}).$$

Note that $\eta \otimes \omega_2 \otimes \omega_3$ can be viewed as an element in $\mathrm{Fil}^{r_2+r_3+2} H^3_{\mathrm{par}}(X^3, \mathcal{H}^{\boxtimes}) \subset \mathrm{Fil}^{r+2} H^3_{\mathrm{par}}(X^3, \mathcal{H}^{\boxtimes})$. We then take lifts

$$\begin{aligned}
\tilde{\eta} &\in H^1_{\mathrm{fp}}(X, \mathcal{H}^{r_1}, 0),\\
\tilde{\omega}_2 &\in H^1_{\mathrm{fp}}(X, \mathcal{H}^{r_2}, r_2+1),\\
\tilde{\omega}_3 &\in H^1_{\mathrm{fp}}(X, \mathcal{H}^{r_3}, r_3+1)
\end{aligned}$$

of η, ω_2, ω_3 under the isomorphism $\mathrm{pr}_{\mathrm{fp}}$ respectively. A lift of $\eta \otimes \omega_2 \otimes \omega_3$ can then be taken to be $\pi_1^*\tilde{\eta} \cup \pi_2^*\tilde{\omega}_2 \cup \pi_3^*\tilde{\omega}_3$.

We would like to define the diagonal cycle $\Delta \in A^2(X^3, (\mathcal{H}^{\boxtimes})^\vee)$ such that it is supported only on the diagonal Z. In other words, we want $\Delta = \delta \cdot Z$ with $\delta \in H^0_{\mathrm{fp}}(Z, (\mathcal{H}^{\boxtimes})^\vee, -r) = H^0_{\mathrm{dR}}(Z, (\mathcal{H}^{\boxtimes})^\vee)$. The job now reduces to find the correct section δ. By definition, the Abel–Jacobi map $\mathrm{AJ}_{\mathrm{fp}}(\Delta)(\eta \otimes \omega_2 \otimes \omega_3)$ computes the cup product

$$\langle \delta, \iota^*(\pi_1^*\tilde{\eta} \cup \pi_2^*\tilde{\omega}_2 \cup \pi_3^*\tilde{\omega}_3)\rangle_{\mathrm{fp}}$$

on $Z \cong X$.

Observe that by the assumption on r_i's, there is a canonical direct sum decomposition

$$\mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \cong \oplus_{j=0}^{\min\{r_2,r_3\}} \mathcal{H}^{r_2+r_3-2j}(-j).$$

In particular, there is a projection $\mathrm{pr}_{r_1} : \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \to \mathcal{H}^{r_1}(1-t)$. We may then consider

$$\mathrm{pr}_{r_1}(\tilde{\omega}_2 \cup \tilde{\omega}_3) \in H^2_{\mathrm{fp}}(X, \mathcal{H}^{r_1}(1-t), r_2+r_3+2)$$

and the cup product $\tilde{\eta} \cup \mathrm{pr}_{r_1}(\tilde{\omega}_2 \cup \tilde{\omega}_3) \in H^3_{\mathrm{fp}}(X, \mathcal{H}^{r_1} \otimes \mathcal{H}^{r_1}(1-t), r_2+r_3+2)$. Now apply Poincaré duality $\mathcal{H}^{r_1} \otimes \mathcal{H}^{r_1} \to \mathcal{O}_X(-r_1)$, we end up with an element in $H^3_{\mathrm{fp}}(X, \mathcal{O}_X(-r_1+1-t), r_2+r_3+2)$.

Since $-r_1+1-t+r_2+r_3 = 0$, we have, as vector spaces,

$$H^3_{\mathrm{fp}}(X, \mathcal{O}_X(-r_1+1-t), r_2+r_3+2) = H^3_{\mathrm{fp}}(X, \mathcal{O}_X, 2) \cong K$$

where the last isomorphism is the trace map $\mathrm{tr}_{\mathrm{fp}}$ for the trivial coefficient. In summary, we showed that there is a non-trivial map $\tau : H^3_{\mathrm{fp}}(X, \mathcal{H}^{\otimes}, r_2+r_3+2) \to K$. In particular $H^3_{\mathrm{fp}}(X, \mathcal{H}^{\otimes}, r_2+r_3+2)$ is non-empty.

By the fact that we have a perfect pairing

$$\langle\ ,\ \rangle_{\mathrm{fp}} : H^0_{\mathrm{fp}}(X, (\mathcal{H}^{\otimes})^\vee, n) \times H^3_{\mathrm{fp}}(X, \mathcal{H}^{\otimes}, 2-n) \to H^3_{\mathrm{fp}}(X, 2) \cong K,$$

there is an element $\mathbb{1}_Z \in H^0_{\mathrm{fp}}(X, (\mathcal{H}^{\otimes})^\vee, -r_2-r_3) = H^0_{\mathrm{par}}(X, (\mathcal{H}^{\otimes})^\vee)$ such that

$$\langle \mathbb{1}_Z, \tilde{\eta} \cup \tilde{\omega}_2 \cup \tilde{\omega}_3\rangle_{\mathrm{fp},X} = \tilde{\eta} \cup \mathrm{pr}_{r_1}(\tilde{\omega}_2 \cup \tilde{\omega}_3) = \tau(\tilde{\eta} \cup \tilde{\omega}_2 \cup \tilde{\omega}_3) \in K.$$

In fact, we can take $\mathbb{1}_Z \in H^0_{\mathrm{fp}}(X, (\mathcal{H}^{\otimes})^\vee, -r)$ since $-r + (r_2+r_3+2) \geq 2$ and $H^3_{\mathrm{fp}}(X, m) \cong K$ for all $m \geq 2$. By doing so, the r_i's now play equal roles in the sense that the Abel–Jacobi map of the cycle defined below can also be evaluated at elements of the form $\omega_1 \otimes \eta_2 \otimes \omega_3$ or $\omega_1 \otimes \omega_2 \otimes \eta_3$.

Definition 5.35. Let $\mathbb{1}_Z$ be as above. We define $\Delta^{x,y,z}_{2,2,2} := \mathbb{1}_Z \cdot Z \in A^2(X^3, (\mathcal{H}^{\boxtimes})^\vee)$. The superscript is to denote that we are interested in modular forms of weights x, y and z (or equivalently, the sheaves $\mathcal{H}^{r_1}, \mathcal{H}^{r_2}$ and $\mathcal{H}^{r_3}$). While the subscript is to remind us that it is supported on closed subscheme Z of X^3, as in the weight $(2,2,2)$ case.

One can check that the push-forward $\iota_*(\mathbb{1}_Z) \in H^4_{\mathrm{fp}}(X^3, (\mathcal{H}^{\boxtimes})^\vee, -r+2)$ in fact lies in

$$\ker(\mathrm{pr}_{\mathrm{fp}}) = H^3_{\mathrm{par}}(X^3, (\mathcal{H}^{\boxtimes})^\vee) / \mathrm{Fil}^{-r+2} H^3_{\mathrm{par}}(X^3, (\mathcal{H}^{\boxtimes})^\vee) \cong [\mathrm{Fil}^{r+2} H^3_{\mathrm{par}}(X^3, \mathcal{H}^{\boxtimes})]^\vee.$$

This justifies the evaluation of $\mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})$ at $\eta \otimes \omega_2 \otimes \omega_3$.

Let (f, g, h) be three cusp forms on X of weight (x, y, z), respectively. Recall that we write the Hecke polynomial of f as

$$T^2 - a_p(f)T + \chi_f(p)p^{x-1} = (T - \alpha_f)(T - \beta_f)$$

with $a = \mathrm{ord}_p(\alpha_f)$, and similarly for g, h. We also write

$$\begin{aligned} P_g(T) &= 1 - a_p(g)p^{1-y}T + \chi_g(p)p^{1-y}T^2 = (1 - \alpha_g p^{1-y}T)(1 - \beta_g p^{1-y}T), \\ P_h(T) &= 1 - a_p(h)p^{1-z}T + \chi_h(p)p^{1-z}T^2 = (1 - \alpha_h p^{1-z}T)(1 - \beta_h p^{1-z}T) \end{aligned}$$

and let $\alpha'_g := \alpha_g p^{1-y}$, $\beta'_g := \beta_g p^{1-y}$, $\alpha'_h := \alpha_h p^{1-z}$ and $\beta'_h := \beta_h p^{1-z}$. The polynomial $P_g(T)$ is defined such that $P_g(\phi)$ annihilates the class of ω_g, where ϕ is the Frobenius on the cohomology $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2})$. In particular, $P_g(\phi)\omega_g = \omega_{g^{[p]}}$ as sections over some strict neighborhood W of the ordinary locus.

Similarly as before, we may represent a class in $H^1_{\mathrm{syn},P}(X, \mathcal{H}^{r_i}, r_i + 1)$ by a pair (ω, G) where $\omega \in H^0(X, \mathcal{H}^r \otimes \Omega^1_X)$ and $G \in H^0(W, \mathcal{H}^r)$ are such that $P(\phi)(\omega) = \nabla G$. Then we have the following analogues of Theorem 5.20 and Corollary 5.22, whose proofs will be omitted.

Theorem 5.36. *Suppose that we have two classes $\tilde{\omega}_i = (\omega_i, G_i) \in H^1_{\mathrm{syn},1-\alpha_i T}(X, \mathcal{H}^{r_i}, r_i + 1)$ with $UG_i = 0$ for $i = 2, 3$. Let $P(T) = (1 - \alpha_2\alpha_3 T)$ and consider the cup product*

$$\tilde{\omega}_2 \cup \tilde{\omega}_3 \in H^1_{\mathrm{syn},P}(X, \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3}, r_2 + r_3 + 2).$$

Let $i_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3) \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3})$ be the inverse and apply the projection pr_{r_1} on it to obtain the element $\mathrm{pr}_{r_1}(i_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)) \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1 - t))$. If $\eta \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1})$ is an eigenvector of ϕ with eigenvalue α and $\mathrm{ord}_p(\alpha) = a \in \mathbb{Q}$, then

$$\langle \eta, \mathrm{pr}_{r_1}(\mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_2 \cup \tilde{\omega}_3)) \rangle_{\mathrm{dR}} = (1 - \beta\alpha_2\alpha_3)^{-1} \langle \eta, e^{\leq a}\, \mathrm{pr}_{r_1}(G_2\omega_3) \rangle_{\mathrm{dR}}$$

where $\beta = p^{x-1+t-1}/\alpha$ and $e^{\leq a}$ is the slope a-projection with respect to U.

Remark 5.37. One needs to be careful about the twisting. The above pairing $\langle\ ,\ \rangle_{\mathrm{dR}}$ is the Poincaré pairing

$$H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}) \times H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1 - t)) \to K(-r_1 - 1 + 1 - t) = K(1 - x + 1 - t).$$

Hence we have $\beta = p^{x-1+t-1}/\alpha$, instead of $\beta = p/\alpha$ in the weight $(2, 2, 2)$ case.

Corollary 5.38. *Let g, h be as above and suppose they are eigenfroms form the Hecke operator T_p. Let $\tilde{\omega}_g, \tilde{\omega}_h \in H^1_{\mathrm{fp}}(X, 1)$ be the lifts of ω_g, ω_h. Suppose that $\eta \in H^1_{\mathrm{dR}}(X)$ is an eigenvector of ϕ with eigenvalue α and $\mathrm{ord}_p(\alpha) = a \in \mathbb{Q}$. Let $\beta = p^{x-1+t-1}/\alpha$. Then we have*

$$\begin{aligned} \langle \eta, \mathrm{i}_{\mathrm{fp}}^{-1}(\tilde{\omega}_g \cup \tilde{\omega}_h) \rangle_{\mathrm{dR}} &= (1 - \beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h) \\ &\quad \times \big((1 - \beta\alpha'_g\alpha'_h)(1 - \beta\alpha'_g\beta'_h)(1 - \beta\beta'_g\alpha'_h)(1 - \beta\beta'_g\beta'_h)\big)^{-1} \\ &\quad \times \langle \eta, e^{\leq a}\, \mathrm{pr}_{r_1}(G^{[p]}\omega_h) \rangle_{\mathrm{dR}}. \end{aligned}$$

Together with the definition of $\mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h)$, we have the following theorem.

Theorem 5.39. *Let notations be as before. Then we have*

$$\begin{aligned} \mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h) &= (1 - \beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h) \\ &\quad \times \big((1 - \beta\alpha'_g\alpha'_h)(1 - \beta\alpha'_g\beta'_h)(1 - \beta\beta'_g\alpha'_h)(1 - \beta\beta'_g\beta'_h)\big)^{-1} \\ &\quad \times \langle \eta, e^{\leq a}\, \mathrm{pr}_{r_1}(G^{[p]}\omega_h) \rangle_{\mathrm{dR}}. \end{aligned}$$

Remark 5.40. In the applications later, the eigenvalue α will be $\alpha_{f^*} = \chi_f(p)^{-1}\alpha_f$. Then $\beta = p^{x-1+t-1}/\alpha_{f^*} = p^{t-1}\beta_f$. One can re-write

$$\begin{aligned}
1 - \beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h &= 1 - p^{2t-2}\beta_f^2\alpha_g\beta_g\alpha_h\beta_h p^{2-2y}p^{2-2z} \\
&= 1 - p^{2t-2}\beta_f^2\chi_g(p)p^{y-1}\chi_h(p)p^{z-1}p^{2-2y}p^{2-2z} \\
&= 1 - p^{2t-2}\beta_f^2\chi_f^{-1}(p)p^{1-y}p^{1-z} \\
&= 1 - \beta_f^2\chi_f^{-1}(p)p^{2t-y-z} \\
&= 1 - \beta_f^2\chi_f^{-1}(p)p^{-x},
\end{aligned}$$

which is equal to the Euler factor $\mathscr{E}_1(f)$ defined in the introduction. Similarly, one can re-write

$$\begin{aligned}
&(1-\beta\alpha'_g\alpha'_h)(1-\beta\alpha'_g\beta'_h)(1-\beta\beta'_g\alpha'_h)(1-\beta\beta'_g\beta'_h) \\
&= (1-\beta_f\alpha_g\alpha_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c})
\end{aligned}$$

with $c := \frac{x+y+z-2}{2}$. It is then equal to the Euler factor $\mathscr{E}(f,g,h)$. Hence, we will use the following notations

$$\begin{aligned}
\mathscr{E}_1(\eta) &:= (1-\beta^2\alpha'_g\alpha'_h\beta'_g\beta'_h), \\
\mathscr{E}(\eta,g,h) &:= (1-\beta\alpha'_g\alpha'_h)(1-\beta\alpha'_g\beta'_h)(1-\beta\beta'_g\alpha'_h)(1-\beta\beta'_g\beta'_h).
\end{aligned}$$

5.8 Comparison to the generalized diagonal cycle without coefficient

In this section, we recall the generalized diagonal cycle (without coefficient) defined in [DR14] and its Abel–Jacobi image. In the end, we will see that their generalized diagonal cycle $\Delta_{x,y,z}$ is essentially equal to $\Delta_{2,2,2}^{x,y,z}$ we defined in the last section.

Kuga–Sato varieties and the generalized diagonal cycles. For any $r \geq 0$, we have the Kuga–Sato variety W_r, which is the desingularization (c.f. [BDP13, Appendix]) of the r-fold fiber product

$$W'_r := \mathcal{E} \times_X \mathcal{E} \cdots \times_X \mathcal{E}.$$

Then one may see the parabolic cohomology $H^1_{\mathrm{par}}(X,\mathcal{H}^r)$ as a subspace in a correct degree of the de Rham cohomology of W_r, which is illustrated in the following lemma.

Lemma 5.41 ([BDP13, Lemma 2.2]). *Assume that $r \geq 1$. Then there is an idempotent $\epsilon_r \in \mathbb{Q}[\mathrm{Aut}(W_r/X)]$, defined in [DR14, §3.1], such that we have*

$$\epsilon_r H^j_{\mathrm{dR}}(W_r/K) = \begin{cases} 0 & j \neq r+1 \\ H^1_{\mathrm{par}}(X,\mathcal{H}^r) & j = r+1 \end{cases}.$$

Suppose now we have a triple of balanced weights $(x,y,z) = (r_1+2, r_2+2, r_3+2)$. We further assume that $r_1 > 0, r_2 > 0, r_3 > 0$ and $r := \frac{r_1+r_2+r_3}{2} \in \mathbb{N}$.

Set $W^* = W_{r_1} \times W_{r_2} \times W_{r_3}$, which is of dimension $2r+3$ over the base field. We now briefly recall the definition of the generalized diagonal cycle $\Delta_{x,y,z} \in \mathrm{CH}^{r+2}(W^*)$ (c.f. [DR14, Definition 3.3]).

Choose three subsets

$$A = \{a_1,\ldots,a_{r_1}\}, \quad B = \{b_1,\ldots,b_{r_2}\}, \quad C = \{c_1,\ldots,c_{r_3}\}$$

of $\{1,\ldots,r\}$ such that $A \cap B \cap C = \emptyset$ and $A \cup B = B \cup C = A \cup C = \{1,\ldots,r\}$. One can see that the balancedness assumption guarantees the existence of such sets. Then we consider the map

$$\begin{aligned}
\varphi_{ABC} : W_r &\to W_{r_1} \times W_{r_2} \times W_{r_3} \\
(x; P_1,\ldots,P_r) &\mapsto ((x; P_{a_1},\ldots,P_{a_{r_1}}), (x; P_{b_1},\ldots,P_{b_{r_2}}), (x; P_{c_1},\ldots,P_{c_{r_3}})),
\end{aligned}$$

which is a closed embedding of W_r into W^*.

Definition 5.42. The generalized diagonal cycle is defined by

$$\Delta_{x,y,z} = (\epsilon_{r_1}, \epsilon_{r_2}, \epsilon_{r_3})\varphi_{ABC}(W_r) \in \mathrm{CH}^{r+2}(W^*).$$

Remark 5.43. For the case where some $r_i = 0$, we refer the definitions to [DR14, § 3.1].

By using the Künneth decomposition of $H^{2r+4}_{\mathrm{dR}}(W^*)$ and examining the image of the idempotents ϵ_{r_i}'s, it follows that the cycle $\Delta_{x,y,z}$ is homologous to zero. That is,

$$\Delta_{x,y,z} \in \mathrm{CH}^{r+2}(W^*)_0 := \ker(\mathrm{cl} : \mathrm{CH}^{r+2}(W^*) \to H^{2r+4}_{\mathrm{dR}}(W^*)).$$

Fix a prime $p \nmid N$, we have the p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W^*)_0 \to \mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W^*)^\vee$$

as introduced in [Nek00]. We recall the following result by Darmon–Rotger.

Theorem 5.44 ([DR14, § 3.4])**.** *Let η, ω_g, ω_h be as in Corollary 5.38. Then*

$$\mathrm{AJ}_p(\Delta_{x,y,z})(\eta \otimes \omega_g \otimes \omega_h) = \frac{\mathscr{E}_1(\eta)}{\mathscr{E}(\eta, g, h)} \cdot \langle \eta, e^{\leq a} \operatorname{pr}_{r_1}(G^{[p]} \times \omega_h)\rangle.$$

As a result, we see that the two definitions of cycles are essentially the same, as least when we evaluate their Abel–Jacobi images at $\eta \otimes \omega_g \otimes \omega_h$.

Theorem 5.45. *Let η, ω_g, ω_h be as before. Then*

$$\mathrm{AJ}_p(\Delta_{x,y,z})(\eta \otimes \omega_g \otimes \omega_h) = \mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})(\eta \otimes \omega_g \otimes \omega_h) = \frac{\mathscr{E}_1(\eta)}{\mathscr{E}(\eta, g, h)} \cdot \langle \eta, e^{\leq a} \operatorname{pr}_{r_1}(G^{[p]} \times \omega_h)\rangle.$$

Notice that in the first Abel–Jacobi map, we view $\eta \otimes \omega_g \otimes \omega_h$ as elements in the de Rham cohomology $\mathrm{Fil}^{r+2} H^{2r+3}_{\mathrm{dR}}(W^*)$. Whereas in the second one, it is viewed as an element in $\mathrm{Fil}^{r+2} H^3_{\mathrm{par}}(X^3, \mathcal{H}^{\boxtimes})$.

Lastly, we would like to make an expedition to the proof of [DR14, Theorem 3.8] for $(k, \ell, m) > (2, 2, 2)$. We briefly recall the setting.

Given ω_g, ω_h as before, one obtain a class

$$\omega_g \otimes \omega_h \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2}) \otimes H^1_{\mathrm{par}}(X, \mathcal{H}^{r_3}).$$

Let $\Phi := \phi \otimes \phi$. As explained in [DR14, § 3.4], one can find a polynomial $P(T)$ such that $P(\Phi)$ annihilates the class $\omega_g \otimes \omega_h$ and none of its roots is of Weil weight $r_2 + r_3 + 1$.

Then, for a suitable strict neighborhood $W = W_\epsilon$, there is an rigid analytic 1-form $\rho(P, \omega_2, \omega_h)$ on $W \times W$ such that

$$\nabla\rho(P, \omega_g, \omega_h) = P(\Phi)(\omega_g \otimes \omega_h).$$

Recall that we have a map $\operatorname{pr}_{r_1} : \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \to \mathcal{H}^{r_1}(1-t)$. Composed with the pullback by the diagonal embedding $W \to W \times W$, we get a map $\mathcal{H}^{r_2} \boxtimes \mathcal{H}^{r_3} \to \mathcal{H}^{r_1}(1-t)$, which will still be denoted by pr_{r_1}.

Set $\xi(P, \omega_g, \omega_h) :=$ the class of $\operatorname{pr}_{r_1} \rho(P, \omega_g, \omega_h) \in H^1_{\mathrm{rig}}(W, \mathcal{H}^{r_1}(1-t))$. If one further requires that $P(\phi)$ composed with the residue map to each supersingular residue disk is zero for the sheaf $\mathcal{H}^{r_1}(1-t)$ (these eigenvalues are of Weil weight $2r+2$), then $\xi(P, \omega_g, \omega_h)$ actually lies in $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1-t))$.

As the eigenvalues of ϕ on $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1-t))$ are of Weil weight $r_1 + 1 + 2t - 2 = r_2 + r_3 + 1$, $P(\phi)$ is an isomorphism on $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1-t))$. One then set

$$\xi(\omega_g, \omega_h) : P(\phi)^{-1}\xi(P, \omega_g, \omega_h) \in H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1}(1-t)).$$

It can be checked that this class is independent of the choice of the polynomial as long as it satisfies the conditions above.

Proposition 5.46 ([DR14, theorem 3.8]). *Let notations be as before. Then*

$$\mathrm{AJ}_p(\Delta_{x,y,z})(\eta\otimes\omega_g\otimes\omega_h)=\mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})(\eta\otimes\omega_g\otimes\omega_h)=\langle\eta,\xi(\omega_g,\omega_h)\rangle_{\mathrm{dR}}.$$

We now illustrate the proof by using finite polynomial cohomology with coefficients. In particular, we show how it simplifies the argument in [DR14, Lemma 3.11]. One should then see that this approach is more coherent to the one for weight $(2,2,2)$ case in [DR14, § 3.3].

By this time, one might already see that the pair $(\omega_g\otimes\omega_h,\rho(P,\omega_g,\omega_h))$ gives rise to a class

$$\widetilde{\omega_g\otimes\omega_h}\in H^2_{\mathrm{syn},P}(X^2,\mathcal{H}^{r_2}\boxtimes\mathcal{H}^{r_3},r_2+r_3+2).$$

Consider now the following diagram (we temporarily set $r'=r_2+r_3+2$ to make the diagram smaller):

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{par}}(X^2,\mathcal{H}^{r_2}\boxtimes\mathcal{H}^{r_3}) \arrow[r, "\mathrm{i_{fp}}"] \arrow[d, "\mathrm{pr}_{r_1}"] & H^2_{\mathrm{syn},P}(X^2,\mathcal{H}^{r_2}\boxtimes\mathcal{H}^{r_3},r') \arrow[r, "\mathrm{pr_{fp}}"] \arrow[d, "\mathrm{pr}_{r_1}"] & \mathrm{Fil}^{r'}H^2_{\mathrm{par}}(X^2,\mathcal{H}^{r_2}\boxtimes\mathcal{H}^{r_3}) \arrow[r] \arrow[d, "\mathrm{pr}_{r_1}"] & 0 \\
0 \arrow[r] & H^1_{\mathrm{par}}(X,\mathcal{H}^{r_1}(1-t)) \arrow[r, "\mathrm{i_{fp}}"] & H^2_{\mathrm{syn},P}(X,\mathcal{H}^{r_1}(1-t)),r') \arrow[r, "\mathrm{pr_{fp}}"] & \mathrm{Fil}^{r'}H^2_{\mathrm{par}}(X,\mathcal{H}^{r_1}(1-t)) \arrow[r] & 0.
\end{tikzcd}$$

Notice that $\mathrm{Fil}^{r_2+r_3+2}H^2_{\mathrm{par}}(X,\mathcal{H}^{r_1}(1-t))$ is 0 since the sheaf $\mathcal{H}^{r_1}(1-t)$ has no non-trivial global horizontal section. So the map $\mathrm{i_{fp}}$ in the bottom row is an isomorphism. Also recall that the map $\mathrm{i_{fp}}$ on the bottom row sends a class ζ to $(P(\phi)\zeta,0)$ (since we are on a curve, any representative of the class ζ is necessarily closed). As a consequence, the class $\xi(\omega_g,\omega_h)$ is nothing but the preimage of $\mathrm{pr}_{r_1}(\widetilde{\omega_g\otimes\omega_h})$ under $\mathrm{i_{fp}}$. The result then follows from our definition of $\mathrm{AJ}_{\mathrm{fp}}(\Delta^{x,y,z}_{2,2,2})(\eta\otimes\omega_g\otimes\omega_h)$ in §5.7 and the compatibility between cup products on finite polynomial cohomology and de Rham cohomology.

6 Special values of triple product p-adic L-functions

In this section, we will focus on the specializations of the triple product p-adic L-functions at certain balanced classical weights, and try to relate the values to p-adic Abel-Jacobi images.

Recall that the triple product p-adic L-function is defined as

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) := \frac{(\breve{\mathbf{f}}^*, H^{\dagger,\leq a}(\nabla_{k_g}^{\nu}\breve{\mathbf{g}}^{[p]} \times \breve{\mathbf{h}}))}{(\breve{\mathbf{f}}^*, \breve{\mathbf{f}}^*)} \in \mathfrak{K}_f \hat{\otimes} \Lambda_g \hat{\otimes} \Lambda_h$$

From now on, we will simply write $\mathbf{f}^*$ for $\breve{\mathbf{f}}^*$ and similarly for $\breve{\mathbf{g}}$ and $\breve{\mathbf{h}}$.

6.1 Special values at balanced classical weights

We first specify at which classical points we want to study the values of $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

Suppose $\mathbf{f}$ is a family of finite slope $a = a_f$, and $\mathbf{g}, \mathbf{h}$ are of slopes a_g, a_h respectively. We are interested in the values of $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at classical weights $(x, y, z) \in \mathbb{Z}^3$ such that

1. The specializations of $\mathbf{f}, \mathbf{g}, \mathbf{h}$ at (x, y, z) are p-old, *i.e.*, $\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z$ are p-stabilizations of some classical modular forms f_x^0, g_y^0, h_z^0 on $X_1(N)$;
2. (x, y, z) is balanced and $x = y + z - 2t$ for some $t \in \mathbb{Z}_{>0}$;
3. $x > 2a + 1$, $y > 2a_g + 1$ and $z > 2a_h + 1$.

We denote the set consisting of all such weights by $\Sigma_{\mathbf{f},\mathbf{g},\mathbf{h}}$. By abuse of notations, we will write $f_x := \mathbf{f}_x$, $g_y := \mathbf{g}_y$, $h_z := \mathbf{h}_z$ and denote the classical forms f_x^0, g_y^0, h_z^0 simply by f, g, h. Notice that they are different from the modular forms f, g, h (of weight k, ℓ, m, resp.) which the families $\mathbf{f}, \mathbf{g}, \mathbf{h}$ deform.

Remark 6.1. The third condition is explained in Remark 4.41. We can modify it into the following condition:
$3'$. $x - 1 > a, y - 1 > a_g, z - 1 > a_h$ and $\alpha_{f_x^0} \neq \beta_{f_x^0}$, $\alpha_{g_y^0} \neq \beta_{g_y^0}$, $\alpha_{h_z^0} \neq \beta_{h_z^0}$.

As we will proceed our proofs by studying the q-expansions of various forms, we here recall several definitions. Around the Tate curve, one has a basis $\{\omega_{\text{can}}, \eta_{\text{can}}\}$ of $\mathcal{H}$ that satisfy

$$\nabla(\omega_{\text{can}}) = \eta_{\text{can}} \otimes \frac{dq}{q}, \quad \nabla(\eta_{\text{can}}) = 0.$$

As we mentioned before, the basis $V_{r,i}$ of the sheaf $\mathbb{W}_r$ at the Tate curve corresponds to the basis $\omega_{\text{can}}^{r-i}\eta_{\text{can}}^i$ of $\mathcal{H}^r$ at the Tate curve when r is a positive integer and $i \leq r$. From now on, we will make no difference between these two notations and use them interchangeably.

We now examine the q-expansion of $H^\dagger(\nabla_{k_g}^{\nu}(\mathbf{g}^{[p]}) \times \mathbf{h})$ at (y, z). Let

$$g_y^{[p]}(q)V_{y,0} \text{ and } h_z(q)V_{z,0}$$

be the q-expansions of the specializations of $\mathbf{g}^{[p]}$ and $\mathbf{h}$ at y and z respectively.

Since we assume the triple (x, y, z) to be balanced, $b := y - t$ is a positive integer. Apply the formula in Theorem 4.30, we see that

$$\begin{aligned}\nabla^{-t}(g_y^{[p]}V_{y,0}) &= \sum_{j}^{\infty} \binom{-t}{j} \prod_{i=0}^{j-1}(y - t - 1 - i)\theta^{-t-j} g_y^{[p]}(q)V_{y-2t,j} \\ &= \sum_{j=0}^{b-1} \binom{-t}{j} \prod_{i=0}^{j-1}(y - t - 1 - i)\theta^{-t-j} g_y^{[p]}(q)V_{y-2t,j}\end{aligned}$$

is in fact a finite sum. We simplify the above formula as

$$\nabla^{-t}(g_y^{[p]}V_{y,0}) = \sum_{j=0}^{b-1} C_j \cdot \theta^{-t-j} g_y^{[p]}(q) V_{y-2t,j}$$

by letting $C_j := \binom{-t}{j} \prod_{i=0}^{j-1}(y-t-1-i)$.

The q-expansion of the product $\nabla^{-t} g_y^{[p]}(q)V_{y,0} \times h_z(q)V_{z,0}$ can now be expressed as

$$\sum_{j=0}^{b-1} C_j \cdot \theta^{-t-j} g_y^{[p]}(q) V_{y-2t,j} \times h_z(q)V_{z,0} = \sum_{j=0}^{b-1} C_j \cdot \theta^{-t-j} g_y^{[p]}(q) \times h_z(q) V_{x,j}.$$

We here remark that $b-1 \le x$. Since if $b-1 = y-t-1 > x$, by using the balancedness assumption $x+z-y>0$, we have $z > t+1$. But this would imply $x = y-t+z-t > y-t+1 > y-t-1$, which is a contradiction. As a result, one has the identification

$$\sum_{j=0}^{b-1} C_j \cdot \theta^{-t-j} g_y^{[p]}(q) \times h_z(q) V_{x,j} = \sum_{j=0}^{b-1} C_j \cdot \theta^{-t-j} g_y^{[p]}(q) \times h_z(q) \omega_{\mathrm{can}}^{x-j} \eta_{\mathrm{can}}^{j}.$$

Applying the overconvergent projection formula in Prop. 4.37 with

$$\gamma_j(q) = C_j \cdot \theta^{-t-j} g_y^{[p]}(q) \times h_z(q),$$

we get

$$H^{\dagger}(\nabla^{-t}(g_y^{[p]}V_{y,0}) \times h_z V_{z,0})(q) = \sum_{j=0}^{b-1} (-1)^j C_j \cdot \frac{\theta^j(\theta^{-t-j} g_y^{[p]}(q) \times h_z(q))}{(x-2-j+1)(x-2-j+2)\cdots(x-2)} V_{x,0}.$$

6.2 The p-adic Abel–Jacobi images

In this section, we will review the computation of the Abel–Jacobi map. In particular, we will define an element η_f^a attached to a modular form f of finite slope $\le a$ and compute $\mathrm{AJ}(\Delta_{2,2,2}^{x,y,z})(\eta_f^a \otimes \omega_g \otimes \omega_h)$.

Let f, g, h be a triple of cusp forms on $X_1(N)$ of balanced weights ($x = r_1+2,\ y = r_2+2,\ z = r_3+2$) such that $x = y+z-2t$ as before. The cusp form f then corresponds to a section $\omega_f \in H^0(X, \underline{\omega}^{r_1} \otimes \Omega_X^1)$ and similar for g and h. We also fix a strict neighborhood W of the ordinary locus in X and sometimes view ω_g, ω_h as sections over W via restriction.

As explained in §2.1, we have a class $\overline{\eta_f} \in H^1_{\mathrm{par}}(X, \mathcal{H}^r)[f]/\operatorname{Fil}^{r+1}$ such that for any cusp form $\omega \in S_x(\Gamma_1(N))$, we have

$$\langle \overline{\eta_f}, \omega \rangle_{\mathrm{dR}} = \frac{(f^*, \omega)_N}{(f^*, f^*)_N}.$$

Notice that on $H^1_{\mathrm{par}}(X, \mathcal{H}^r)[f]$, the Frobenius ϕ acts with eigenvalues α_f^* and β_f^*. We want to find a lifting $\eta_f = \eta_f^a \in H^1_{\mathrm{par}}(X, \mathcal{H}^r)[f]$ such that the Frobenius ϕ acts with eigenvalue $\alpha_{f^*} = \alpha_f \chi_f(p)^{-1}$, which also has p-adic valuation a. In order to do so, we need the decomposition

$$H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1})[f] = H^0(X, \underline{\omega}^{r_1} \otimes \Omega_X^1)[f] \oplus H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1})[f]^{\phi = \alpha_{f^*}}.$$

The existence of the decomposition is unknown in general. However, it is true under the non-critical assumption $\mathrm{ord}_p(\alpha_{f^*}) < x-1$ (c.f. [BC09, § 2.4.3]). We then define η_f^a to be the unique lift of $\overline{\eta_f}$ in $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1})[f]^{\phi = \alpha_{f^*}}$.

Then we have the following special case of Theorem 5.39.

Theorem 6.2. *Let $\Delta_{2,2,2}^{x,y,z}$ be the general diagonal cycle with coefficients defined in §5 and $\eta_f^a, \omega_g, \omega_h$ be as above. Then we have*

$$\mathrm{AJ}_{\mathrm{fp}}(\Delta_{2,2,2}^{x,y,z})(\eta_f^a \otimes \omega_g \otimes \omega_h) = \frac{\mathscr{E}_1(f)}{\mathscr{E}(f,g,h)} \cdot \langle \eta_f^a, e^{\leq a} \operatorname{pr}_{r_1}(G^{[p]} \times \omega_h) \rangle.$$

Now we shift our attention to the term $\operatorname{pr}_{r_1}(G^{[p]} \times \omega_h)$. As $1 - VU$ annihilates the cohomology $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_2})$, the form $\omega_g^{[p]}$ over W is ∇-exact. A primitive $G^{[p]}$ over W can be chosen (uniquely) such that its polynomial q-expansion takes the form

$$G^{[p]}(q) = \sum_{i=0}^{r_2} (-1)^i i! \binom{r_2}{i} \theta^{-i-1} g^{[p]}(q) \omega_{\mathrm{can}}^{r_2 - i} \eta_{\mathrm{can}}^i.$$

The product $G^{[p]} \times \omega_h$ can be viewed as a section of $\mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \otimes \Omega^1_X$. By assumption on the weights (x, y, z), it follows that there is a projection $\operatorname{pr}_{r_1} : \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \to \mathcal{H}^{r_1}(1-t)$ (c.f. [DR14, Prop. 2.9]).

Remark 6.3. We here explain the projection pr_{r_1} in detail, for it is crucial in later computations.

Set $r := r_2 + r_3 - (t-1)$, and let $\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(r)}$ be r-copies of $\mathcal{H}$ and $\mathcal{H}^{\otimes r} := \mathcal{H}^{(1)} \otimes \cdots \mathcal{H}^{(r)}$. We then have a natural embedding of $\mathcal{H}^r := \operatorname{Sym}^r \mathcal{H}$ into $\mathcal{H}^{\otimes r}$.

We choose subsets $B \subset \{1, 2, \dots, r\}$ of cardinality r_2 and $C \subset \{1, 2, \dots, r\}$ of cardinality r_3 such that $B \cup C = \{1, 2, \dots, r\}$ as before. Notice that we automatically have $\#(B \cap C)^c = r_1$. We may fix a simple choice $B = \{1, 2, \dots, r_2\}$ and $C = \{r - r_3 + 1 = r_2 - t + 2, \dots, r\}$.

We may embed $\mathcal{H}^{r_2}$ canonically into $\mathcal{H}^{\otimes r_2}$, then embed it into $\mathcal{H}^{\otimes r}$ via the set B, and embed $\mathcal{H}^{r_3}$ into $\mathcal{H}^{\otimes r}$ via C similarly. In terms of the basis $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}}\}$, the element $1 \cdot \omega_{\mathrm{can}}^{r_2}$ of $\mathcal{H}^{r_2}$ is sent to $\omega_{\mathrm{can}}^{(r_2)} = 1 \cdot \omega_{\mathrm{can}} \otimes \omega_{\mathrm{can}} \cdots \otimes \omega_{\mathrm{can}}$ of $\mathcal{H}^{\otimes r_2}$. On the other hand, $1 \cdot \omega_{\mathrm{can}}^{r_2 - 1} \eta_{\mathrm{can}}$ is sent to

$$\frac{1}{\binom{r_2}{1}} \sum_{j=1}^{r_2} \eta_{\mathrm{can}}^{(j)} \otimes \omega_{\mathrm{can}}^{(\hat{j})}$$

where $\eta_{\mathrm{can}}^{(j)} = 1 \otimes \cdots \otimes 1 \otimes \eta_{\mathrm{can}} \otimes 1 \otimes \cdots \otimes 1$ with only one η_{can} at the j-th component, and $\omega_{\mathrm{can}}^{(\hat{j})} = \omega_{\mathrm{can}} \otimes \cdots \otimes \omega_{\mathrm{can}} \otimes 1 \otimes \omega_{\mathrm{can}} \otimes \cdots \otimes \omega_{\mathrm{can}}$ with only one 1 at the j-th component. One should be able to work out the general cases explicitly.

Now apply the Poincaré pairing $\mathcal{H} \times \mathcal{H} \to \mathcal{O}_X(-1)$ component-wise on the images of $\mathcal{H}^{r_2}$ and $\mathcal{H}^{r_3}$. Since there are $(t-1)$-many overlapping components corresponding to $B \cap C$, after symmetrization, we may identify the resulting sheaf as $\mathcal{H}^{r_1}(1-t)$. The symmetrization sends, for example, $\sum_{j=1}^{r_2} \alpha_j \eta_{\mathrm{can}}^{(j)} \otimes \omega_{\mathrm{can}}^{(\hat{j})}$ to $(\sum_{j=1}^{r_2} \alpha_j) \cdot \omega_{\mathrm{can}}^{r_2 - 1} \eta_{\mathrm{can}}$.

Notice that one may view the projection pr_{r_1} as a generalization of the decomposition

$$\operatorname{Sym}^{r_2} V \otimes \operatorname{Sym}^{r_3} V \cong \bigoplus_{j=0}^{\min\{r_2, r_3\}} \operatorname{Sym}^{r_2 + r_3 - 2j} V$$

for a two dimensional vector space V (or viewed as the standard representation of SL_2).

6.3 The p-adic Gross–Zagier formula and its proof

After studying the two ingredients, we are now able to prove the p-adic Gross–Zagier formula. Before doing so, we first fix some notations.

Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be three finite slope families and $(x, y, x) \in \Sigma_{\mathbf{f},\mathbf{g},\mathbf{h}}$ be a triple of balanced weights as in Section 6.1. The specializations f_x, g_y, h_z are classical modular forms on $X_1(N, p)$ and will be viewed as sections over the strict neighborhood W in X. Furthermore, they are the p-stabilizations of modular forms f, g, h on $X_1(N)$ and $f_x = (1 - \beta_f V) f$ by our assumption. We also have the elements $\eta_f = \eta_f^a, \omega_g, \omega_h$ associated with f, g, h in their respective cohomology groups as defined in Section 6.2.

Theorem 6.4 (p-adic Gross–Zagier formula). *Let notations be as above. We have*

$$\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = (-1)^{t-1}\frac{\mathscr{E}(f,g,h)}{(t-1)!\mathscr{E}_0(f)\mathscr{E}_1(f)} \times \mathrm{AJ}_p(\Delta_{x,y,z})(\eta_f\otimes\omega_g\otimes\omega_h),$$

where the Euler factors $\mathscr{E}(f,g,h)$ and $\mathscr{E}_1(f)$ are as before, and

$$\mathscr{E}_0(f) = (1-\beta_f^2\chi_f^{-1}(p)p^{1-x}).$$

The strategy of the proof is the following: from the two overconvergent forms g_y, h_z, there are two ways to construct an overconvergent form of weight x (or an element in $H^1_{\mathrm{par}}(X,\mathcal{H}^{r_1})$).

One is $H^\dagger(\nabla^{-t}(g_y^{[p]}V_{y,0})\times h_zV_{z,0})$, which relates to the triple product L-function. The other one is $H^\dagger(\mathrm{pr}_{r_1}(G^{[p]}\times\omega_h))$, which relates to the Abel-Jacobi image. Hence, proving the p-adic Gross–Zagier formula is essentially equal to relating the two overconvergent forms (or cohomological classes) mentioned above.

The crucial observation is the following lemma, which should be considered the main technical novelty of this paper.

Lemma 6.5. *Let notations be as before, then we have*

$$(-1)^{t-1}(t-1)!\cdot H^\dagger(\nabla^{-t}(g_y^{[p]}V_{y,0})\times h_zV_{z,0}) = H^\dagger(\mathrm{pr}_{r_1}(G^{[p]}\times h_z))V_{x,0}.$$

Proof. We prove this equality by examining their q-expansions and using the q-expansion principle.

First, we observe that the two forms g and $g_y = (1-\beta_gV)g$ has the same p-depletion. So we may replace $g_y^{[p]}$ with $g^{[p]}$.

Let $b := y-t$ be as before, recall that

$$H^\dagger(\nabla^{-t}(g^{[p]}V_{y,0})\times hV_{z,0}) = \sum_{j=0}^{b-1}(-1)^j\binom{-t}{j}\prod_{i=0}^{j-1}(y-t-1-i)\frac{\theta^j(\theta^{-t-j}g^{[p]}\times h)}{(x-2-j+1)\cdots(x-2)}V_{x,0}. \tag{29}$$

On the other hand, we have

$$G^{[p]} = \sum_{s=0}^{r_2}(-1)^s s!\binom{r_2}{s}\theta^{-s-1}g^{[p]}(q)\omega_{\mathrm{can}}^{r_2-s}\eta_{\mathrm{can}}^s$$

where $G^{[p]}$ is the primitive of the overconvergent form $g^{[p]}$ with respect to the Gauss–Manin connection ∇. We now view h_z as a section of $\mathcal{H}^{r_3}\otimes\Omega^1_X$ over $\mathcal{W}$ and write its q-expansion as $h_z(q)\omega_{\mathrm{can}}^{r_3}\cdot\frac{dq}{q}$.

Following the recipe in Remark 6.3, we have the polynomial q-expansion

$$\mathrm{pr}_{r_1}(G^{[p]}\times h_z) = \sum_{s=t-1}^{r_2}(-1)^s s!\binom{r_2-(t-1)}{s-(t-1)}\theta^{-s-1}g^{[p]}\times h_z\ \omega_{\mathrm{can}}^{r_1-(s-(t-1))}\eta_{\mathrm{can}}^{s-(t-1)}\cdot\frac{dq}{q}.$$

After the change of variable $\alpha := s-(t-1)$ and noticing that $r_2-(t-1) = b-1$, it can be written as

$$\mathrm{pr}_{r_1}(G^{[p]}\times h_z) = \sum_{\alpha=0}^{b-1}(-1)^{\alpha+(t-1)}(\alpha+t-1)!\binom{b-1}{\alpha}\theta^{-t-\alpha}g^{[p]}\times h_z\ \omega_{\mathrm{can}}^{r_1-\alpha}\eta_{\mathrm{can}}^{\alpha}\cdot\frac{dq}{q}.$$

The overconvergent projection $H^\dagger(\mathrm{pr}_{r_1}(G^{[p]}\times h_z))$ then takes the form

$$\sum_{\alpha=0}^{b-1}(-1)^{2\alpha+(t-1)}(\alpha+t-1)!\binom{b-1}{\alpha}\frac{\theta^\alpha(\theta^{-t-\alpha}g^{[p]}\times h_z)}{(x-2-\alpha+1)\cdots(x-2)}\ \omega_{\mathrm{can}}^{r_1}\cdot\frac{dq}{q}. \tag{30}$$

The lemma then reduces to comparing the numbers

$$(-1)^j \binom{-t}{j} \prod_{i=0}^{j-1} (b-1-i)$$

and

$$(-1)^{t-1}(j+t-1)! \binom{b-1}{j}.$$

Observe that

$$\begin{aligned}
(-1)^j \binom{-t}{j} \prod_{i=0}^{j-1} (b-1-i) &= (-1)^j \frac{(-t)(-t-1)\cdots(-t-j+1)}{j!} \prod_{i=0}^{j-1} (b-1-i) \\
&= (-1)^{2j} \cdot t(t+1)\cdots(t+j-1) \cdot \frac{1}{j!} \cdot \prod_{i=0}^{j-1} (b-1-i) \\
&= t(t+1)\cdots(t+j-1)\frac{1}{j!} \cdot j! \binom{b-1}{j} \\
&= t(t+1)\cdots(t+j-1) \cdot \binom{b-1}{j}.
\end{aligned}$$

Hence we have

$$(-1)^{t-1}(t-1)! \cdot \left((-1)^j \binom{-t}{j} \prod_{i=0}^{j-1} (b-1-i) \right) = (-1)^{t-1}(j+t-1)! \binom{b-1}{j}$$

for all $0 \le j \le b-1$. The lemma then follows. In fact, one sees that the equality holds even without applying the overconvergent projection. ☐

Remark 6.6. In the ordinary case (c.f. [DR14, Prop 2.9]), the proof is easier as one may replace the overconvergent projection with the unit root splitting. That is, one only needs to prove the equality between the first terms of the two polynomial q-expansions.

Corollary 6.7. *With notations as before, we have*

$$\begin{aligned}
\mathrm{AJ}(\Delta_{x,y,z})(\eta_f \otimes \omega_g \otimes \omega_h) &= \frac{\mathscr{E}_1(f)}{\mathscr{E}(f,g,h)} \cdot \langle \eta_f, e^{\le a}(\mathrm{pr}_{r_1}(G^{[p]} \times h)\rangle \\
&= \frac{\mathscr{E}_1(f)}{\mathscr{E}(f,g,h)} \cdot \langle \eta_f, e^{\le a}(\mathrm{pr}_{r_1}(G^{[p]} \times h_z)\rangle \\
&= (-1)^{t-1}(t-1)! \frac{\mathscr{E}_1(f)}{\mathscr{E}(f,g,h)} \cdot \langle \eta_f, e^{\le a} H^\dagger(\nabla^{-t}(g^{[p]}) \times h_z)\rangle.
\end{aligned}$$

Proof. It only suffices to prove the second equation. As $U(G^{[p]} \cdot Vh) = 0$, and $e^{\le a}$ has no constant term as a power series in U (c.f. Remark 7.23), $e^{\le a}$ annihilates $\mathrm{pr}_{r_1}(G^{[p]} \times Vh)$. Hence we may replace h by one of its p-stabilization h_z and the second equality follows. ☐

Now back to the triple product p-adic L-function $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$. By definition, its value at the point (x,y,z) is

$$\frac{(f_x^*, H^{\dagger, \le a}(\nabla^{-t}(g_y^{[p]}) \times h_z))_{N,p}}{(f_x^*, f_x^*)_{N,p}}$$

where the Petersson product is on $X_1(N,p)$.

We need to translate this Petersson product on $X_1(N,p)$ back to one on $X_1(N)$. Recall that f_x is the p-stabilization of f. In other words, as classes in $H^1_{\mathrm{par}}(X, \mathcal{H}^{r_1})$, we have $f_x = (1 - \beta'_f \phi) f = (1 - \beta_f V) f$ and $f_x^* = (1 - \beta_{f^*} V) f^*$.

For this we recall the formula in Lemma 2.22. Let ω be any modular form on $X_1(N,p)$ of weight x, we have

$$\frac{(f_x^*,\omega)_{N,p}}{(f_x^*,f_x^*)_{N,p}}=\frac{(f_x^*,e_{f_x^*}\omega)_{N,p}}{(f_x^*,f_x^*)_{N,p}}=\frac{(f^*,\omega^0)_N}{(f^*,f^*)_N}=\langle\eta_f,\omega^0\rangle_{\mathrm{dR}}.$$

Here $e_{f_x^*}$ is the projection to the f_x^*-isotypic component, and ω^0 is a modular form on $X_1(N)$ such that

$$e_{f^*}\omega=(1-\beta'_{f^*}\phi)\omega^0=(1-\beta_{f^*}V)\omega^0.$$

Note that here we also write ω^0 for its (unique) corresponding element in $\mathrm{Fil}^{r_1+1}H^1_{\mathrm{par}}(X,\mathcal{H}^{r_1})$ by an abuse of notations, rather than using the awkward ω_{ω^0}. Now we let ω be the modular form $H^{\dagger,\leq a}(\nabla^{-t}(g_y^{[p]})\times h_z)$ on $X_1(N,p)$, viewed as a class in $H^1_{\mathrm{dR}}(X,\mathcal{H}^{r_1})$. Then,

$$\begin{aligned}\langle\eta_f,\omega\rangle=\langle\eta_f,e_{f^*}\omega\rangle&=\langle\eta_f,(1-\beta_{f^*}V)\omega^0\rangle\\&=\langle\eta_f,\omega^0\rangle-\beta_{f^*}\langle\eta_f,V\omega^0\rangle\\&=\langle\eta_f,\omega^0\rangle-\beta_{f^*}\langle\phi^{-1}\eta_f,\omega^0\rangle\\&=(1-\frac{\beta_{f^*}}{\alpha_{f^*}})\langle\eta_f,\omega^0\rangle\\&=\mathscr{E}_0(f)\langle\eta_f,\omega^0\rangle\end{aligned}$$

where $\mathscr{E}_0(f):=(1-\frac{\beta_{f^*}}{\alpha_{f^*}})=(1-\frac{\beta_f}{\alpha_f})=(1-\beta_f^2\chi_f^{-1}(p)p^{1-x})$. As a result,

$$\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z)=\frac{1}{\mathscr{E}_0(f)}\langle\eta_f,H^{\dagger,\leq a}(\nabla^{-t}(g_y^{[p]})\times h_z)\rangle_{\mathrm{dR}}.$$

This equality, together with Corollary 6.7, then imply

$$\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z)=(-1)^{t-1}\frac{\mathscr{E}(f,g,h)}{(t-1)!\mathscr{E}_0(f)\mathscr{E}_1(f)}\times\mathrm{AJ}_p(\Delta_{x,y,z})(\eta_f\otimes\omega_g\otimes\omega_h).$$

6.4 Comparison to the result of Bertolini–Seveso–Venerucci

Lastly, we would like to make some remarks on the result in [BSV20]. They computed the Bloch–Kato logarithm of a certain cohomology class $\kappa(f,g,h)$, evaluated at $\eta_f\otimes\omega_g\otimes\omega_h$, and related this value to a certain period attached to the triple (f,g,h). We here recall their theorem without fully explain every ingredient. Those who are familiar with the relation between Bloch–Kato logarithm and syntomic Abel–Jacobi map will immediately see its connection to Theorem 6.4.

Theorem 6.8 ([BSV20, Theorem A])**.** *Let (f,g,h) be as before, and let $\kappa(f,g,h)$ be the diagonal class defined in [BSV20, § 2]. Then,*

$$\log_p(\kappa(f,g,h))(\eta_f\otimes\omega_g\otimes\omega_h)=(-1)^{t-1}N^{c-2}\frac{\mathscr{E}(f,g,h)}{(t-1)!\mathscr{E}_0(f)\mathscr{E}_1(f)}\times I_p(f,g,h),\tag{31}$$

where

$$I_p(f,g,h):=\frac{(f^*,e_{f^*}(\theta^{-t}g^{[p]}\times h))_{Np}}{(f^*,f^*)_{Np}}.$$

Here, the Petersson product is taken over level $\Gamma_1(Np)$.

Remark 6.9. One notices that they are using the p-adic modular form $\theta^{-t}g^{[p]}\times h$ instead of overconvergent modular forms. This difference is explained in [BSV20, § 4.4], where they utilized the result of [Nik10]. Roughly speaking, it says that one can study certain overconvergent forms by restricting to the ordinary locus. This is coherent to [Urb14, Proposition 3.2.4], where the q-expansion map is proved to be injective on the space of overconvergent modular forms. The same technique, however, does not work for finite slope families of overconvergent forms. The main reason is that one now enters into the realm of infinitely dimensional p-adic Banach spaces. Whereas every norm on a finite dimensional vector space is equivalent, the same statement is not true for infinitely dimensional vector spaces.

7 Appendix. A lemma on pairings of p-adic Banach spaces

In this appendix, we make a detour to the theory of p-adic Banach spaces. Our goal is to explore more on the relation between finite slope projector $e^{\leq a}$ and the Poincaré pariring. To be more precise, we wish to establish a result analogous to [DR14, Prop 2.3] and [DR14, Prop 2.11]. A combined statement as well as its proof will be recalled below.

Let K be a finite extension of $\mathbb{Q}_p$ and $X = X_1(N)$ be the modular curve defined over $\mathbb{Z}_p$ as before.

Lemma 7.1. *Let $\eta \in H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)^{\mathrm{u-r}}$ be a class in the unit root part, i.e., the Frobenius ϕ acts on η as multiplying by a p-adic unit. Suppose g is a nearly overconvergent modular form of weight $k = r+2 \geq 2$ on $\Gamma_1(N)$ with vanishing residues at all the supersigular annuli. We let $\omega = \omega_g$ in $H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)$ be the class given by g. Then we have*

$$\langle \eta, \omega \rangle = \langle \eta, e_{\mathrm{ord}}\omega \rangle$$

where $\langle\ ,\ \rangle$ is the Poincaré pairing.

Proof. ([Bes16, [Prop. 3.2]) First, recall that we have the operators U and V, which are inverse to each other on the level of cohomology. We also have the Frobenius morphism ϕ that satisfies $\langle \phi\eta, \phi\omega \rangle = p^{k-1}\langle \eta, \omega \rangle$ and $\phi = p^{k-1}V = p^{k-1}U^{-1}$ on the level of cohomology. Let β be the eigenvalue of ϕ acting on η, which is a p-adic unit. Then,

$$\begin{aligned}
\langle \eta, U\omega \rangle &= p^{1-k}\langle \phi\eta, \phi U\omega \rangle \\
&= \langle \phi\eta, p^{1-k}\phi U\omega \rangle \\
&= \langle \phi\eta, \omega \rangle \\
&= \beta\langle \eta, \omega \rangle.
\end{aligned}$$

Taking limit, we have

$$\langle \eta, e_{\mathrm{ord}}\omega \rangle = \langle \eta, \lim_{n\to\infty} U^{n!}\omega \rangle = \lim_{n\to\infty} \beta^{n!}\langle \eta, \omega \rangle = \langle \eta, \omega \rangle$$

as desired. ☐

Remark 7.2. In [DR14], it is proved in another way. First, one uses the relation $\langle \phi\eta, \phi\omega \rangle = p^{k-1}\langle \eta, \omega \rangle$ to deduce that the Poincaré pairing descends to

$$\langle\ ,\ \rangle : H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)^{\mathrm{u-r}} \times H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)^{\phi, k-1} \to K(1-r)$$

where the subscript $\phi, k-1$ denotes the slope $= k-1$ subspace for ϕ. Then one identifies $S^{\mathrm{ord}}_k(N)$, the space of ordinary overconvergent modular forms of weight k, with $H^1_{\mathrm{par}}(X_K, \mathcal{H}^r)^{\phi, k-1}$.

The main result in this section is the following lemma:

Lemma 7.3 (General form)**.** *Let M be a p-adic Banach space over a p-adic field K and u be a compact operator on M. Suppose there is a bilinear pairing $\langle\ ,\ \rangle : M \times M \to K$ and another operator ϕ on M that satisfies*

$$\langle m_1, um_2 \rangle = \langle \phi m_1, m_2 \rangle$$

for any $m_1, m_2 \in M$.

Let $\eta \in M$ be an eigenvector for ϕ with eigenvalue α, whose p-adic valuation is $a \in \mathbb{Q}$. Let $e^{\leq a}$ be the projector onto the slope $\leq a$ subspace of M for the operator u. Then we have

$$\langle \eta, m \rangle = \langle \eta, e^{\leq a}m \rangle$$

for all $m \in M$.

In our application, it takes the following form:

Lemma 7.4. *Suppose $\eta \in H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)$ is an element for which the Frobenius ϕ acts with eigenvalue α such that $v_p(\alpha) \le a$. Then for any $\omega \in H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)$, we have*

$$\langle \eta, \omega \rangle = \langle \eta, e^{\le a} \omega \rangle$$

where $e^{\le a}$ is the projector to the slope $\le a$ part for the U_p operator and the pairing is the Poincaré pairing.

Remark 7.5. As we will eventually restrict to classical forms, we are dealing with the finite dimensional vector space $H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)$. If we understand the action of U (or its characteristic polynomial) on $H^1_{\mathrm{dR}}(X_K, \mathcal{H}^r)$, the projector e_{ord} or $e^{\le a}$ can be expressed as polynomials in U. So technically speaking, we do not really need to deal with (infinite dimensional) p-adic Banach spaces and compact operators. However, we wish to provide a more general statement that may be applied to non-classical forms.

We first recall some facts about p-adic Banach spaces and compact operators. The main references are [Ser62], [Col97] and [Buz07].

Let K be a complete non-archimedean valuation field, A be its ring of valuation, $\mathfrak{m}$ be its maximal ideal, and $k = A/\mathfrak{m}$ be the residue field. We also let $G \subset \mathbb{R}^\times$ denote the image of $K^\times$ under the valuation map.

Definition 7.6. A Banach space E over K is a complete normed vector space over K that satisfies

$$|a + b| \le \max\{|a|, |b|\}$$

for all $a, b \in E$. For such a space E, we let $E_0 = \{x \in E \mid |x| \le 1\}$. It is clear that E_0 is a A-module, and the topology of E is determined by E_0.

We will have to consider the following property (N): for all $x \in E$, $|x| \in \overline{G}$, the closure of G.

Example. Let I be a set and let $c(I)$ be the set of families $x = (x_i)_{i \in I}, x_i \in K$ such that x_i tends to 0 on the complement of an increasing filtration of finite subsets of I (denoted by $x_i \to 0$ when $i \to \infty$). The norm is given by $|x| = \sup\{|x_i|\}$.

The space $c(I)$ will be our main object of interest, and it turns out that most of the Banach spaces are of this type.

Proposition 7.7. *Suppose the valuation on K is discrete. Then any Banach space E/K satisfying property (N) is isomorphic to $c(I)$ for some set I.*

Definition 7.8. An orthonormal basis of a Banach space E is a family $(e_i)_{i \in I}, e_i \in E$, such that any $x \in E$ can be written uniquely as

$$x = \sum x_i e_i, \ x_i \in K$$

with $|x_i| \to 0$ and $|x| = \sup_i |x_i|$. A space E is called orthonormizable if it admits an orthonormal basis.

Suppose E, F are two Banach spaces. Let $\mathcal{L}(E, F)$ be the vector space of continuous linear maps from E to F. The space $\mathcal{L}(E, F)$ is equipped with the norm

$$|u| := \sup_{x \neq 0} \frac{|ux|}{|x|}$$

which is equivalent to $\sup_{0 \neq |x| \le 1} |ux|$. Under this norm, $\mathcal{L}(E, F)$ is a Banach space. If E, F satisfy (N), then so does $\mathcal{L}(E, F)$.

Now suppose $E = c(I)$ is orthonormizable with basis $(e_i)_{i \in I}$, and $F = c(J)$ with basis $(e'_j)_{j \in J}$. Then for any $u \in \mathcal{L}(E, F)$, the element $u_i := f(e_i)$ takes the form $(n_{ij})_{j \in J}$ where $|n_{ij}| \to 0$ when $j \to \infty$ and i is fixed. We have $|u| = \sup_{i,j} |n_{ij}|$, and if $x = (x_i) \in E$, then $u(x) = (y_j)_{j \in J}$ where $y_j = \sum_i n_{ij} x_i$. We call (n_{ij}) the matrix of u with respect to the orthonormal basis of E, F. On the other hand, we may write $u(x) = (w_j x)_{j \in J}$ where w_j is in the dual E^* of E. Then $|u| = \sup_j |w_j|$.

Definition 7.9. A map $u \in \mathcal{L}(E, F)$ is called completely continuous (or compact) if it lies in the closure of the subspace of finite rank maps. We denote this set by $\mathcal{C}(E, F)$.

It is clear by definition that the composition $v \circ u$ is completely continuous if either v or u is. In particular, $\mathcal{C}(E, E)$ is a two-sided ideal in $\mathcal{L}(E, E)$.

Proposition 7.10. *Suppose $F = c(J)$, then the above identification defines an isomorphism from $\mathcal{C}(E, F)$ to $c_{E^*}(J)$, the space of collections $(w_j)_{j\in J}$ of elements in E^* tending to 0, with the norm defined by* $\sup|w_j|$.

Fredholm determinant. From now on, we will always assume the Banach spaces E, F are orthonormizable.

It is tempting to define a certain characteristic power series for $u \in \mathcal{L}(E, F)$ analogous to what we do in finite dimensional linear algebra. However, as a Banach space can be infinite dimensional, one needs to worry about the convergence. It turns out that we are able to define $\det(1 - Tu)$ for completely continuous map u, which can be essentially computed by using the matrix coefficients (n_{ij}) of u.

We briefly recall some definitions. For any finite subset S of I, define

$$c_S = \sum_\tau \operatorname{sgn}(\tau) \prod_{i\in S} n_{i,\tau(i)}$$

and $c_n = (-1)^n \sum_{|S|=n} c_S$, where the sum is taken over all finite subsets S of index n.

Proposition 7.11 ([Ser62, § 5])**.** *Let u be an element in $\mathcal{C}(E, E)$ with matrix (n_{ij}). We can define the Fredholm determinant* $\det(1 - Tu) \in K[[T]]$ *that satisfies the following properties:*

i. $\det(1 - Tu) = \sum_{n=0}^\infty c_n T^n$, *where c_n is defined as above;*

ii. $\det(1 - Tu)$ *is an entire function of T with values in $\mathcal{L}(E, E)$;*

iii. If $u_i \to u$ with $u \in \mathcal{C}(E, E)$. Then $\det(1 - Tu_i) \to \det(1 - Tu)$ *coefficient-wise;*

iv. If u is of finite rank, then $\det(1 - Tu)$ *coincides with the polynomial defined in the classical sense.*

Corollary 7.12. *If $u, v \in \mathcal{C}(E, E)$, then*

$$\det(1 + u + v + uv) = \det(1 + u) \det(1 + v).$$

Generally, we have $\det((1 - Tu)(1 - Tv)) = \det(1 - Tu) \det(1 - Tv)$.

Corollary 7.13. *Suppose that $u \in \mathcal{C}(E, F)$, $v \in \mathcal{C}(F, E)$. Then we have*

$$\det(1 - Tu \circ v) = \det(1 - Tv \circ u)$$

We will need the following lemma later.

Lemma 7.14. *Let $I = I' \cup I''$ be a disjoint union. Suppose u is completely continuous on $E = c(I)$, which sends $E' = c(I')$ to E'. Let u' be the restriction of u to E', and u'' be the induced map on the quotient E/E'. Then we have*

$$\det(1 - Tu) = \det(1 - Tu') \det(1 - Tu'')$$

Definition 7.15. The Fredholm resolvant $R(T, u)$ is defined to be

$$R(T, u) := \frac{\det(1 - Tu)}{(1 - Tu)} = \sum_{m=0}^\infty v_m T^m$$

where $v_m \in \mathcal{L}(E, E)$ are polynomials in u.

Proposition 7.16 ([Ser62, Proposition 10])**.** *$R(T, u)$ is an entire function of T with values in $\mathcal{L}(E, E)$.*

From now on, we will fix a compact operator $u \in \mathcal{C}(E, E)$ and simply write $P(T) = P_u(T)$ for $\det(1-Tu)$.

Proposition 7.17. *Suppose $\lambda \in K$. Then $1 - \lambda u$ is invertible in $\mathcal{L}(E, E)$ if and only if $P(\lambda) \neq 0$.*

Definition 7.18. For a function $F(T) = \sum b_m T^m$, define $\Delta^s f = \sum \binom{m+s}{m} b_{m+s} T^m$ for any $s \in \mathbb{N}$. A zero $\lambda \in K$ of f is said to have order $h \in \mathbb{N}$ if $\Delta^s(\lambda) = 0$ for all $s < h$ and $\Delta^h f(\lambda) \neq 0$. Note that if the characteristic of K is 0, we have $\Delta^s = \frac{1}{s!} \frac{d^s}{dT^s}$.

Proposition 7.19 (Riesz theory)**.** *If $\lambda \in K$ is a zero of $P(T) = \det(1 - Tu)$ of order h. Then the space E can be decomposed uniquely as a direct sum of u-invariant closed subspaces*

$$E = N(\lambda) \oplus F(\lambda)$$

such that

i. $(1 - \lambda u)$ *is invertible on* $F(\lambda)$*;*

ii. $(1 - \lambda u)$ *is nilpotent on* $N(\lambda)$*. More precisely,* $(1 - \lambda u)^h N(\lambda) = 0$ *and* $N(\lambda)$ *is of dimension* h*.*

In [Col97], Coleman generalized Serre's results to Banach modules. Most of the results stay the same after some minor modifications.

Definition 7.20. For a polynomial $Q(T)$ of degree d, we introduce the notation $Q^*(T) := T^d Q(T^{-1})$.

As a corollary of Proposition 7.19, Coleman proved

Theorem 7.21. *Suppose that $P_u(T) = Q(T)S(T)$, where $Q(T) = 1 + \cdots$ is a polynomial of degree h whose leading coefficient is a unit (i.e., $Q^*(0)$ is a unit) in A, and Q is relatively prime to S. Then there is a unique decomposition of M into u-invariant closed sub-modules*

$$M = N(Q) \oplus F(Q)$$

such that

i. $Q^*(u)$ *is invertible on* $F(Q)$*;*

ii. $Q^*(u)$ *is zero on* $N(Q)$*. Moreover,* $N(Q)$ *is of rank* h*, and the characteristic polynomial of* u *on* $N(Q)$ *is* $Q^*(T)$*.*

Remark 7.22. When the vector space E is finite dimensional, I will use the term "characteristic polynomial" to denote $\det(T - u)$ as usual. I will still call $\det(1 - Tu)$ "characteristic power series" even if it is a polynomial.

In order to understand more about the projection $E \twoheadrightarrow N(\lambda)$, we need to recall part of the proof of Proposition 7.19 (c.f. [Ser62]).

Proof. By assumption, we have $\Delta^s P(\lambda) = 0$ for all $s < h$, and $\Delta^h P(\lambda) = c \neq 0$. Consider the identity

$$(1 - Tu)R(T, u) = P(T)$$

and apply Δ^s, we get

$$(1 - Tu)\Delta^s R(T, u) - u\Delta^{s-1} R(T, u) = \Delta^s P(T)$$

Now put $w_s = \Delta^s R(\lambda, u)$, we have the equations

$$\begin{aligned}
(1 - \lambda u)w_0 &= 0 \\
(1 - \lambda u)w_1 - uw_0 &= 0 \\
&\vdots \\
(1 - \lambda u)w_{h-1} - uw_{h-2} &= 0 \\
(1 - \lambda u)w_h - uw_{h-1} &= c
\end{aligned}$$

We deduce from these equations that $(1 - \lambda u)^{s+1} w_s = 0$ for $s < h$.

Set $e = c^{-1}(1 - \lambda u)w_h$ and $f = -c^{-1}uw_{h-1}$. Then $e + f = 1$ and $fe^h = 0$ since $(1 - \lambda u)^h w_{h-1} = 0$ (w_s is power series in u, so it commutes with u).

Consider the equation $(e + f)^h = 1$, and let

$$\begin{aligned}
p &= e^h \\
q &= he^{h-1}f + \cdots + hef^{h-1} + f^h
\end{aligned}$$

Then $p+q=1$ and $pq=qp=0$. The maps p and q are the desired projections to $F(a)$ and $N(a)$ respectively. More precisely, $\ker p = \operatorname{Im} q = N(\lambda)$ and $\ker q = \operatorname{Im} p = F(\lambda)$. Hence we have the decomposition

$$E = N(\lambda) \oplus F(\lambda)$$

As $(1-\lambda u)^h q = 0$, we see that $(1-\lambda u)^h = 0$ on $N(\lambda)$. On the other hand, $(1-\lambda u)^h (w_h)^h = c^h p$, so $1-\lambda u$ is invertible on $F(\lambda) = \operatorname{Im} p$.

The rest of the proof is omitted. Details on proving that $\dim N(\lambda) = h$ can be found in [Buz07]. □

Remark 7.23. Looking at the formulae $f = c^{-1} u w_{h-1}$ and

$$q = he^{h-1} f + \cdots + hef^{h-1} + f^h$$

in the proof of Proposition 7.19, we see that the projector q has no constant term in u. This fact will be useful later when we apply finite polynomial cohomology to compute the Abel-Jacobi maps.

Now we are able to prove lemma 7.3

Proof. By assumption, for any power series $G(T)$ such that $G(u)$ converges, we have

$$\langle m_1, G(u) m_2 \rangle = \langle G(\phi) m_1, m_2 \rangle$$

as long as the expression $G(\phi) m_1$ also converges. So it suffices to show that $G(\phi)\eta = \eta$ when $G(T)$ is a power series such that $G(u)$ is the projector $e^{u=\alpha}$ on M.

Suppose that u has α as one of its eigenvalues, then α^{-1} is a root of $P(T) = \det(1 - Tu)$. On the finite dimensional subspace $N(\alpha^{-1})$, u has characteristic polynomial $(T-\alpha)^h$. For simplicity, we will assume that it is also the minimal polynomial.

By Proposition 7.16, the resolvant $R(T, u)$ is an entire function of T. In particular, $R(\alpha^{-1}, u)$ converges in $\mathcal{L}(E, E)$. We write $R(T)$ for the power series $R(\alpha^{-1}, T)$. For any eigenvector ω of u with eigenvalue α, the element $R(\alpha^{-1}, u)\omega = R(\alpha)\omega$ converges. Hence, the power series $R(T)$ a least converges for $|T| \leq |\alpha| = p^{-a}$.

In other words, we may view $R(T)$ as an element in the Tate algebra $K\langle T/\alpha \rangle$, and so are $w_s(T) = \Delta^s R(\alpha^{-1}, T)$ and the power series $G(T)$ that satisfies $G(u) = q = he^{h-1} f + \cdots + hef^{h-1} + f^h$ (recall that this is the projector to $N(\alpha^{-1})$).

As the polynomial $(T-\alpha)^h$ is regular, we can apply Weierstrass division on the Tate algebra $K\langle T/\alpha \rangle$ and write

$$G(T) = (T-\alpha)^h S(T) + r(T)$$

where $S(T) \in K\langle T/\alpha \rangle$ and $r(T)$ is a polynomial of degree $< h$. Since $G(u)$ is a projector to $N(\alpha^{-1})$, we must have $G(u)|_{N(\alpha^{-1})} = 1$. This implies $r(u) = 1$ on $N(\alpha^{-1})$. Since we assume $(T-\alpha)^h$ is the minimal polynomial on $N(\alpha^{-1})$, we must have $r(T) = 1$.

Now, for every $m \in M$, we have

$$\begin{aligned}
\langle \eta, e^{u=\alpha} m \rangle &= \langle \eta, G(u) m \rangle \\
&= \langle G(\phi)\eta, m \rangle \\
&= \langle [(\phi-\alpha)^h S(\phi) + 1]\eta, m \rangle \\
&= \langle \eta, m \rangle.
\end{aligned}$$

Note that the expression $[(\phi-\alpha)^h S(\phi) + 1]\eta$ converges.

The lemma then follows as the projector $e^{\leq a}$ is just the finite sum of all $e^{u=\gamma}$ with $\operatorname{ord}_p(\gamma) \leq a$. □

8 Appendix. Coleman integration and F-isocrystals

This appendix aims to provide some complimentary materials for §5, such as rigid cohomology, Coleman integration and F-isocrystals. We will try to keep in a manner of minimalism and refer most of the results to the listed references.

8.1 Coleman integration

In this part, we will recall the construction of Coleman integration. Our expectation is that one should be able to verify its connection with finite polynomial cohomology appearing in §5.3 without too much efforts after reading this section. For the main references, Coleman's original papers [Col82] and [Col85] are already good and clear. The note of Besser [Bes12] also gives a friendly introduction to this topic. Unfortunately, we need to assume basic knowledge of rigid geometry, which can be found in [Ber96] and [Ber97].

Throughout this appendix, we let K be a finite unramified extension of $\mathbb{Q}_p$, $V = \mathcal{O}_K$ be its ring of integers, $\mathfrak{m}$ be the maximal ideal and κ be the residue field. The valuation on K is fixed such that $|p| = p^{-1}$.

We first start with a simple example of rigid cohomology. The closed disk $B[0,1] := \{z \in K \mid |z| \leq 1\}$ is an affinoid (rigid analytic space) over K with structure sheaf given by the Tate algebra

$$K\langle T\rangle := \left\{ \sum_{n\geq 0} a_n T^n \mid a_n \in K, \lim_{n\to\infty} a_n = 0 \right\}.$$

If one naively computes the cohomology of the complex $K\langle T\rangle \to K\langle T\rangle \cdot dT$, one immediately sees that H^1 of this complex is non-zero, and in fact it is of infinite dimension. To remedy this problem, we consider the dagger algebra

$$K\langle T\rangle^\dagger := \left\{ \sum_{n\geq 0} a_n T^n \mid a_n \in K, \lim_{n\to\infty} a_n r^n = 0 \text{ for some } r > 1 \right\},$$

which can be viewed as the collection of functions that converge on some disks slightly larger than $B[0,1]$ (hence the name overconvergent). This idea was first proposed by Monsky and Washnitzer in [MW68]. The cohomology (sometimes referred as Monsky–Washnitzer cohomology) of the resulting complex

$$K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger \cdot dT$$

then has the desired cohomology group at degree 1.

The above idea inspired Berthelot's construction of the rigid cohomology, which will be briefly illustrated below (c.f. [Ber96, § 1]).

Let X_κ be a smooth scheme, separated of finite type over κ. By a rigid datum (X_κ, Y_κ, P) for X_κ, we mean the following objects: a formal V-scheme P and a proper κ-scheme Y_κ such that

1. Y_κ is a closed subscheme of P;
2. X_κ is an open κ-subscheme of Y_κ;
3. P is smooth on a neighborhood of X_κ.

In this situation, one has tubes $]X_\kappa[$ and $]Y_\kappa[$ inside the (rigid analytic) generic fiber P_K. An (admissible) open subset U in $]Y_\kappa[$ is called a strict neighborhood of $]X_\kappa[$ (in $]Y_\kappa[$) if $]X_\kappa[\subset U$ and $\{U,]Y_\kappa[\backslash]X_\kappa[\}$ forms an admissible cover of $]Y_\kappa[$.

For any strict neighborhood U of $]X_\kappa[$ in $]Y_\kappa[$, we write $j_U : U \to]Y_\kappa[$ for the inclusion map. We then define a functor

$$j^\dagger := \varinjlim_U j_{U,*} j_U^*$$

from the category of abelian sheaves on $]Y_\kappa[$ to itself, where U runs through all strict neighborhoods of $]X_\kappa[$ in $]Y_\kappa[$.

Example. Let $X_\kappa = \mathbb{A}^1_\kappa$, $Y_\kappa = \mathbb{P}^1_\kappa$ and P be the formal completion of $\mathbb{P}^1_V$ along its special fiber. Then (X_κ, Y_κ, P) is a rigid datum. In this situation, we have $]X_\kappa[= B[0,1]$ and one can verify that $H^0(]Y_\kappa[, j^\dagger\mathcal{O}_{]Y_\kappa[}) \cong K\langle T\rangle^\dagger$, and the complex $K\langle T\rangle^\dagger \to K\langle T\rangle^\dagger \cdot dT$ computes the hypercohomology of $j^\dagger\Omega^\bullet_{]Y_\kappa[}$ on $]Y_\kappa[$.

Definition 8.1. Let X_κ be a smooth, separated scheme of finite type over κ and (X_κ, Y_κ, P) be a rigid datum. The rigid cohomology $H^i_{\mathrm{rig}}(X_\kappa)$ is defined to be the i-th hypercohomology of $\Omega^{\dagger,\bullet}_{]X_\kappa[} := j^\dagger\Omega^\bullet_{]Y_\kappa[}$ on $]Y_\kappa[$.

Remark 8.2. Of course, one needs to verify that this definition is well-defined and check the functoriality. We would like to leave the details to [Ber97, § 1]. We only remark that when X_κ is affine, $H^i_{\mathrm{rig}}(X_\kappa)$ agrees with the "formal cohomology" of Monsky–Washnitzer mentioned earlier (c.f. [Ber96, § 4]).

We now shift our attention to the integration of differential forms on a smooth variety X_K defined over K. We will assume that X_K admits a smooth model X over V (*i.e.*, is of good reduction).

Theorem 8.3 ([Ber96, § 4] and [Ogu90, § 6])**.** *Under the above assumption, one has comparison isomorphisms*

$$H^i_{\mathrm{rig}}(X_\kappa) \cong H^i_{\mathrm{dR}}(X_K) \quad \textit{for all } i.$$

The functoriality of H^i_{rig} provides an action of Frobenius ϕ on H^i_{rig} (c.f. [Col85, Appendix] or [Chi98, § 2]). Naturally, we are interested in the behavior of this Frobenius.

Theorem 8.4 ([CLS98])**.** *If X_κ is smooth and proper, the eigenvalues of ϕ on $H^i_{\mathrm{rig}}(X_\kappa)$ are pure of Weil weight i.*

Corollary 8.5. *If X_κ is an affine open subscheme of a smooth proper curve, the eigenvalues of ϕ on $H^1_{\mathrm{rig}}(X_\kappa)$ are of Weil weights* 1 *and* 2*.*

We are now able to introduce Coleman integration. We start with the case where X_K is an affine open subscheme of a smooth proper scheme Y_K with good reduction. In this case, we fix the rigid datum $(X_\kappa, Y_\kappa, \hat{Y})$ where $\hat{Y}$ is the formal completion of the smooth model Y of Y_K along its special fiber.

Definition 8.6. A function f on a rigid analytic space (e.g. X_K^{an} or $Y_K^{\mathrm{an}} =]Y_\kappa[$) is called locally analytic if it is rigid analytic on each of its residue classes.

Theorem 8.7 ([Col85, Theorem 2.1])**.** *Given a closed differential form $\omega \in H^0(]Y_\kappa[, j^\dagger\Omega^1_{]Y_\kappa[})$, there is a polynomial $P(T) \in \mathbb{C}_p[T]$ whose roots are not roots of unity, such that $P(\phi)\omega = 0$ in the cohomology group $H^1_{\mathrm{rig}}(X_\kappa)$. Moreover, there is a locally analytic function f_ω on $]X_\kappa[$, unique up to an additive constant, such that*

i. $df_\omega = \omega$,

ii. $P(\phi)f_\omega$ is rigid analytic, i.e., it belongs to $H^0(]X_\kappa[, \mathcal{O}_{]X_\kappa[})$ (even $H^0(]Y_\kappa[, j^\dagger\mathcal{O}_{]Y_\kappa[})$).

Such a function f_ω is called a Coleman integral of ω.

Remark 8.8. One can extend the above result to a proper scheme by covering it with affine open subschemes. The details are explained in [Col85, § II].

8.2 F-isocrystals and cohomology

In this section we will focus on (overconvergent) F-isocrystals, which are the "coefficients" of our theory of finite polynomial cohomology with coefficients in §5.4.

There are several ways to introduce this topic. We will follow the one which is closer to Berthelot's original idea. This approach is more explicit, albeit the functoriality needs more elaborations. The main references are [Ber96, § 2], [CL99] and [Ked22]. A more functorial approach can be found in [Ogu84].

We will keep the notations of a rigid datum (X_κ, Y_κ, P) as in last section.

Definition 8.9. A convergent isocrystal on X_κ is a coherent sheaf E on the tube $]X_\kappa[$ together with an integrable connection ∇. An overconvergent isocrystal on X_κ is a coherent sheaf E on a strict neighborhood U of $]X_\kappa[$ (or alternatively, a coherent $j^\dagger\mathcal{O}_{]Y_\kappa[}$-module) together with an integrable connection ∇.

For an overconvergent isocrystal E on X_κ, one similarly defines the rigid cohomology $H^i_{\mathrm{rig}}(X_\kappa, E)$ to be the i-th hypercohomology of the de Rham complex

$$0 \to E \xrightarrow{\nabla} E \otimes_{j^\dagger\mathcal{O}_{]Y_\kappa[}} j^\dagger\Omega^1_{]Y_\kappa[} \xrightarrow{\nabla} E \otimes_{j^\dagger\mathcal{O}_{]Y_\kappa[}} j^\dagger\Omega^2_{]Y_\kappa[} \to \cdots.$$

We let $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa, E)$ be the (hyper-)derived complex of the above complex, which computes $H^i_{\mathrm{rig}}(X_\kappa, E)$.

Theorem 8.10 ([Ked06, Theorem 1.2.1])**.** *The K-vector spaces $H^i_{\mathrm{rig}}(X_\kappa, E)$ are finite dimensional.*

Remark 8.11. If X_κ itself is proper, we may take $Y_\kappa = X_\kappa$, and the definitions of convergent and overconvergent isocrystals are the same. Moreover, when X_κ is proper and admits a smooth model X over $\mathcal{O}_K$, then $P_K = X_K^{\mathrm{an}}$ is the rigid analytic space associated with the generic fiber X_K. By applying GAGA (between rigid analytic geometry and algebraic geometry), the data (E, ∇) on X_K^{an} correspond to a coherent sheaf on X_K with an integrable connection. By abuse of notations, we will denote this data by the same symbols (E, ∇) and even call it an isocrystal on X (or X_K). Moreover, one has a comparison isomorphism $H^i_{\mathrm{rig}}(X_\kappa, E) \cong H^i_{\mathrm{dR}}(X_K, E)$. Lastly, we would like to recall that there are derived complexes $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K, E)$ and $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K, E)$ which compute $H^i_{\mathrm{dR}}(X_K, E)$ and $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K, E)$ respectively.

We now consider the Frobenius action.

Definition 8.12. Let $\sigma : X_\kappa \to X_\kappa$ be the absolute Frobenius. An overconvergent F-isocrystal (E, ϕ) is an overconvergent isocrystal E on X_κ together with an isomorphism of isocrystals $\phi : \sigma^*E \cong E$. In particular, the map ϕ induces an automorphism on the cohomology group $H^i_{\mathrm{rig}}(X_\kappa, E)$ for all i, which will be again denoted by ϕ.

As we will need to control the Frobenius actions on the cohomology groups, we make the following definition.

Definition 8.13. Let X be a proper smooth V-scheme. An overconvergent F-isocrystal E on X is said to be of pure weight $w \in \mathbb{Z}$ if the action of the Frobenius ϕ on $H^i_{\mathrm{rig}}(Z_\kappa, \iota_Z^* E)$ is pure of Weil weight $w + i$ for any proper smooth morphism $\iota_Z : Z \to X$.

As illustrated in §5.4, this pure weight assumption is important if one wants to consider functoriality of finite polynomial cohomology between different schemes. In particular, it allows us to construct the push-forward map for a proper closed subscheme Z of X.

Remark 8.14. When X is proper, the sheaf $\mathcal{O}_{]X_\kappa[}$ is of pure weight 0. In this case our definition is somewhat analogous to Deligne's purity theorem on étale cohomology.

Example. A unipotent F-isocrystal with only one slope $m \in \mathbb{Z}$ on a proper smooth scheme X_κ is of pure weight $2m$ (c.f. [CL99, § 3]).

Remark 8.15. Similarly to the case of structure sheaf, Coleman also developed an integration theory for overconvergent F-isocrystals with certain conditions on the Frobenius action (c.f. [Col94]).

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