Special Rational Solutions of the Fifth Painlevé Equation and their Asymptotic Behaviour

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ABSTRACT

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In this thesis, the first goal is to formulate a generating function and compute its moments alongside with the corresponding Hankel determinant. When the latter is non zero, we will prove that for Painleve 5, we can construct a Lax pair whose solution is a combination of the solution of the Riemann Hilbert Problem (RHP) and the generating function. An ingredient of that solution, called the Hamiltonian will be used to construct the Tau function which solves the ODE Painleve V. As such, it will be easy to show that when the Hankel determinant vanishes, the RHP is not solvable, and its zeroes correspond to the poles of the rational solution of the ODE Painleve V i.e. the Tau function.

On the other hand, an asymptotic analysis will be conducted to prove that the domain of the poles of the rational solution of the ODE Painleve V (its domain of non analyticity) defines a well shaped region with boundaries on the complex plane as the size of the square Hankel matrix goes to infinity.

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Introduction

The six Painlevé equations were classified by Painlevé and his student Gambier [5] more than a century ago. This was a result of the search for second order ODEs in the complex plane whose solutions, roughly speaking, have the property that all movable singularities are isolated poles. This property has now become known and referred to as the *Painlevé property*.

While this might have remained a purely mathematical investigation, it was much later recognized that these equations have significant applications in mathematical physics, with the resurgence in the '80s with the works connecting with Ising model and conformal field theory [36, 37]. Another momentous resurgence happened in the '90s when Tracy and Widom [38]used a special second Painlevé transcendent (the Hastings-McLeod solution [39]) to describe the fluctuations of the larges eigenvalue of a large random Hermitean matrix.

Amongst special solutions of the Painlevé equation, a natural interest is devoted in the literature to the simplest solutions, namely, rational functions; the literature is extensive and seems to start with [46] who discussed rational solution of the second Painlevé equation and defined a special sequence of polynomials that are now called *Vorob'ev-Yablonskii* after their discoverers (it appears that Yablonskii defined them slightly earlier but the reference is difficult to find [47]). Rational solutions also appear in semiclassical limits of integrable PDEs; in the one–dimensional sine–Gordon equation near a separatrix, for example, one finds that a suitable scaling of the solution is expressible in terms of rational solution of the second Painlevé equation [15]

For all but the first Painlevé equation there exist rational solutions: although there does not seem to be a full and complete classification in all cases, either the full classification of rational solutions exist or there are constructions of special families of rational solutions (for the Painlevé II [46, 47, 28], for the Painlevé III,V, VI [35, 34, 44] Painlevé IV [41]

The literature that investigates the asymptotic behaviour of the rational solutions and the pole distribution thereof is more recent, probably due to the interest spurred by numerical investigations and the appearance of well defined patterns; for the zeros of Okamoto polynomials (which are poles of rational solutions of PIV) see [40], for the zeros of Vorobev–Yablonskii polynomials and Painlevé II see [16, 17, 8], for the second Painlevé hierarchy see [7].

The approach to asymptotic analysis relies on the formulation of an associated Riemann–Hilbert problem, which is a boundary value problem for a piecewise analytic matrix valued function matrix. There are two logical distinct approaches that can be used in the asymptotic analysis. We can categorize them under the following banners:

1. the isomonodromic approach;
2. the orthogonal polynomial (OP) approach.

The isomonodromic approach relies on the general fact that any Painlevé equation appears as the compatibility between a 2×2 system of ODEs with rational coefficient in the complex plane and an additional PDE in an auxiliary parameter [25]. The different solutions are parametrized by (generalized) monodromy data of the ODE, which is the starting point for the Riemann–Hilbert analysis. Typically the degree of the rational solution appears explicitly as one of the parameters in the monodromy data and can be used as large parameter in the asymptotics.

The second approach was used, possibly for the first time, in [8] and then also applied to the generalized Vorob'ev–Yablonski polynomials in [7], and it is also the approach we follow in this paper. The main connection between OPs and equations of Painlevé type was established in [9], where it was shown that Hankel determinants built out of the moments of "semiclassical" moment functionals are always isomonodromic tau functions in the sense of [25]. It was a remark (Rem 5.3 ibidem) that special choices of semiclassical moment functionals lead automatically to tau functions of Painlevé equations (all, except possibly for Painlevé I). In genereal, however, these solutions correspond to transcendental solutions (like for example the solutions of PII constructed out of deteremінants of derivatives of Airy functions, see [27]).

It is possible to further restrict the setup of orthogonal polynomials in such a way that the moments of the moment functional become *polynomials* in a parameter, which then guarantees that the Hankel determinant (automatically an isomonodromic tau function) is a polynomial tau function of an equation of Painlevé type. This is what works "behind the scenes" of [8].

The advantage of this reformulation in terms of associated Orthogonal Polynomials is that there is a solid and well developed framework for studying their large degree asymptotics, with an extensive literature that starts with [21].

Before going into any further detail let us still discuss the known literature and results about the rational solutions of the fifth Painlevé equation.

1 Generalities and motivating results from Painleve II (PII) equation

1.1 Characteristics of the poles of a particular rational solution of PII

Proposition 1.1 *Let us consider the equation Painleve II:*

$$u_{xx} = xu + 2u^3 + \alpha, \quad \textit{with } \alpha \in \mathbb{C}$$

When the parameter $\alpha \in \mathbb{Z}$, its solutions can be completely described in terms of the Vorob'ev–Yablonski monic polynomial $Y_n(x)$ as follows (see [2]) :

$$u(x,n) = \frac{d}{dx}\left\{ \ln\left[\frac{Y_{n-1}(x)}{Y_n(x)}\right]\right\} \textit{ where } n = \alpha \in \mathbb{N}, \quad \textit{(it can be extented to } n \in \mathbb{Z}$$
$$\textit{by taking } u(x,-n) = -u(x,n))$$

$$u(x,n) = \frac{d}{dx}\left\{ \ln\left[\frac{Y_{n-1}(x)}{Y_n(x)}\right]\right\} = \frac{Y'_{n-1}}{Y_{n-1}} - \frac{Y'_n}{Y_n} = \sum_{i=1}^{n-1} \frac{1}{(x-a_i)} - \sum_{i=1}^{n} \frac{1}{(x-b_i)} \tag{1.1}$$

corresponding to the factorization of the Vorob'ev–Yablonski monic polynomials

$$Y_{n-1}(x) = (x-a_1)(x-a_2)(x-a_3)...(x-a_{n-1}), \qquad a_i \neq a_j \ \ \forall \ \ i \neq j \tag{1.2}$$
$$\text{and } \ Y_n(x) = (x-b_1)(x-b_2)(x-b_3)...(x-b_n), \qquad b_i \neq b_j \ \ \forall \ \ i \neq j. \tag{1.3}$$

Therefore, the zeros of the consecutive VY polynomials $Y_{n-1}(x)$ and $Y_n(x)$ are the poles of the solution $u(x,n)$ of the equation Painleve II with residue 1 and -1 respectively.

Proposition 1.2 ([46, 47]) *The Vorob'ev–Yablonski polynomials $Y_n(x)$ for all $n \in \mathbb{N}$ can be computed via the differential-difference relation*

$$Y_{n+1} = \frac{xY_n^2(x) - 4[Y_n^{"}(x)Y_n(x) - (Y_n'(x)^2]}{Y_{n-1}(x)}$$
$$\textit{with intial values } \ Y_0(x) = 1, \textit{and } \ Y_1(x) = 1$$

In figure 1 below, we can observe the plot of the poles for the solution $u(x,n)$ for $n = 10$.

As $n \to \infty$ these poles are contained within a region of almost triangular shape on the the complex plane (see [1]). For ODE Painleve V, we will try to produce similar results for our class of rational solutions of PV and analyse them asymptotically.

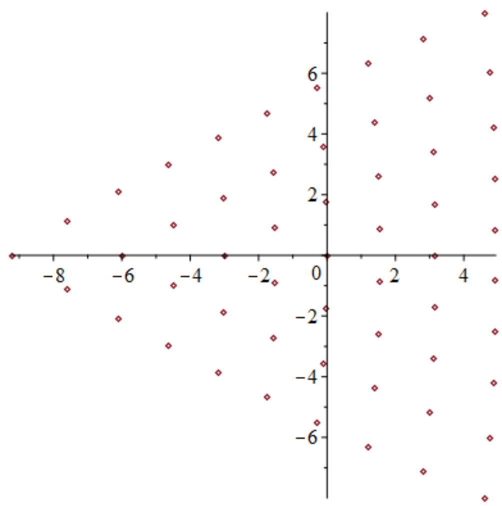

Figure 1: Plot of the poles the solution of ODE Painleve II

1.2 Relationship between VY and Orthogonal Polynomials

Proposition 1.3 *The VY polynomial $Y_n(x)$ in Prop. (1.2) admits the following determinant representation (see [1])*

$$Y_{n-1}^2(x) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \left[\frac{(2k)!}{k!}\right]^2 \det(\mu_{a+b-2}(x))_{a,b=1}^n$$

where the functions $\mu_k(x)$ are the taylor expansion coefficients of the function

$$f(t,x) = \exp\left(\frac{-t^3}{3} + tx\right) = \sum_{k=0}^{\infty} \mu_k(x) t^k. \text{ Namely, } \mu_k(x) = \left.\frac{f^{(k)}(t,x)}{k!}\right|_{t=0}$$

Now, by Cauchy differentiation formula, we obtain

$$\mu_k(x) = \frac{1}{2\pi i} \oint \frac{f(z,x)}{z^{k+1}} \mathrm{d}z \tag{1.4}$$

where the contour of the integral is the unit circle oriented counterclockwise around the origin.
With the change of variable $\zeta = \dfrac{1}{z}$, we define $-\dfrac{z^3}{3} + zx = -\dfrac{1}{3\zeta^3} + \dfrac{x}{\zeta} := -\theta(\zeta, x)$.
Since $z = \dfrac{1}{\zeta}$ and $\mathrm{d}z = -\dfrac{\mathrm{d}\zeta}{\zeta^2}$, we obtain an equivalent formula for (1.4):

$$\mu_k(x) = -\oint \zeta^k \underbrace{\frac{1}{2\pi i} \mathrm{e}^{-\theta(\zeta,x)} \frac{\mathrm{d}\zeta}{\zeta}}_{=\mathrm{d}\eta(\zeta,x)}, \qquad \mathrm{d}\eta(\zeta, x) \text{ being the new measure.} \tag{1.5}$$

Definition 1.4 *The $\mu_k(x)'s$ (1.4) or (1.5) are called "moments functions" and we used them to build the Hankel determinant D_n as follows:*

$$\det(\mu_{a+b-2}(x))_{a,b=1}^{n} = \begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-2}(x) \end{vmatrix} = D_n$$

Proposition 1.5 *Given any (possibly complex valued) measure $\mathrm{d}\mu(z,x)$ depending on a parameter x, and the corresponding moment functions $\mu_k(x) := \oint_\gamma z^k \mathrm{d}\mu(z;x)$, let us consider the polynomial in z:*

$$P_n(z) = \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ 1 & z & z^2 & \dots & z^n \end{vmatrix}}{D_n}$$

Notice that D_n is the cofactor of z^n. $P_n(z)$ is called a non-hermitian orthogonal polynomial, i.e. $\forall n \in \mathbb{N}$, it satisfies

$$\langle P_n(z), P_k(z) \rangle = \oint_\gamma P_n(z) P_k(z) \mathrm{d}\eta = \begin{cases} h_n(x), & k = n \\ 0, & k < n \end{cases} \tag{1.6}$$

where $\mathrm{d}\eta$ is a complex-valued measure, and γ is a closed contour on the complex plane surrounding the origin.

<u>Proof.</u> We will first show that proving 1.6 is equivalent to proving

$$\langle P_n(z), z^k \rangle = \oint_\gamma P_n(z) z^k \mathrm{d}\eta = \begin{cases} h_n(x), & k = n \\ 0, & k < n \end{cases} \tag{1.7}$$

($\Longrightarrow$) If 1.6 is true, then $\{P_n(z)\}_{n=0}^{\infty}$ is an independent set of orthogonal polynomials, hence we can write for example $z^k = \sum_{i=0}^{\infty} a_i P_i$ with $a_i \in \mathbb{C}$

$$\langle P_n(z), z^k \rangle = \sum_{i=0}^{\infty} a_i \langle P_n(z), P_i(z) \rangle = \begin{cases} a_n h(x) \equiv \tilde{h_n}, & k = n \\ 0, & k < n \end{cases}$$

($\Longleftarrow$) Conversely, (WLOG $k \leq n$), if 1.7 is true then

$$\begin{aligned}\langle P_n(z), P_k(z)\rangle &= \langle P_n(z), z^k + b_{k-1}z^{k-1} + b_{k-2}z^{k-2} + ... + b_1 z + b_o\rangle, \quad b_i \in \mathbb{C} \quad i \leq k-1 \\ &= \langle P_n(z), z^k\rangle + \sum_{i=0}^{k-1} b_i \langle P_n(z), z^i\rangle = \langle P_n(z), z^k\rangle, \quad \text{since} \quad k-1 < n, \forall k \leq n\end{aligned}$$

$$= \begin{cases} \tilde{h_n}, & k = n \\ 0, & k < n \end{cases}$$

So, to check if $P_n(z)$ is an orthogonal polynomial, it suffices to verify relation (1.7).

Let us write

$$\begin{aligned}P_n(z) &= \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ 1 & z & z^2 & \dots & z^n \end{vmatrix}}{D_n} \\ &= \frac{1}{D_n}(A_n z^n + A_{n-1}z^{n-1} + ... + A_1 z + A_0) = \sum_{i=0}^{n} A_i z^i\end{aligned}$$

where A_i is the cofactor of z^i. Then

$$\begin{aligned}\oint_\gamma P_n(z) z^k \mathrm{d}\eta &= \langle P_n, z^k\rangle \\ &= \frac{1}{D_n}\sum_{i=0}^{n} A_i \langle z^i, z^k\rangle = \frac{1}{D_n}\sum_{i=0}^{n} A_i \oint_\gamma z^i z^k \mathrm{d}\eta = \frac{1}{D_n}\sum_{i=0}^{n} A_i \mu_{k+i} \\ &= \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ \mu_k & \mu_{k+1} & \mu_{k+2} & \dots & \mu_{k+n} \end{vmatrix}}{D_n}\end{aligned}$$

$$= \begin{cases} \tilde{h_n}, & k = n \\ 0, & k < n, \qquad \text{(since two rows are repeated in the determinant)} \end{cases} \quad \blacksquare$$

1.3 The Riemann Hilbert Problem (RHP) and connection with orthogonal polynomials

1.3.1 Conditions of the Riemann Hilbert Problem

We want to find a 2×2 matrix valued function $\Phi(z)$ satisfying the following conditions :

1. the matrices $\Phi(z)$ and $\Phi(z)^{-1}$ are defined and holomorphic in $\mathbb{C}-\gamma$

2. it satisfies the boundary value condition $\Phi(z_+) = \Phi(z_-)\begin{bmatrix}1 & \mathrm{e}^{-V(z)}\\ 0 & 1\end{bmatrix}$ Here $\mathrm{e}^{-V(z)} \equiv W(z)$ is such that the complex-valued measure $\mathrm{d}\eta(x,z)$ can be written $\mathrm{e}^{-V(z)}\mathrm{d}z$; $W(z)$ is called the "weight".

3. As $z \to \infty$, for some $n \in \mathbb{N}$, the matrix $\Phi_n(z)$ has the Taylor series expansion in power of z^{-1} of the form:

$$\begin{aligned}\Phi(z) &= \left[\mathbb{I}_{2\times2} + \mathcal{O}(z^{-1})\right]\underbrace{\begin{bmatrix}z^n & 0\\ 0 & z^{-n}\end{bmatrix}}_{z^{n\sigma_3}} \equiv \Phi_n(z) \quad \text{with} \quad \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}1 & \mathcal{O}(z^{-1})\\ \mathcal{O}(z^{-1}) & 1\end{bmatrix}\begin{bmatrix}z^n & 0\\ 0 & z^{-n}\end{bmatrix}\\ &= \begin{bmatrix}z^n & \mathcal{O}(z^{-n-1})\\ \mathcal{O}(z^{n-1}) & z^{-n}\end{bmatrix} = \begin{bmatrix}A_n(z) & B_n(z)\end{bmatrix} \qquad (\dagger)\end{aligned}$$

   where $A_n(z)$ and $B_n(z)$ denote the columns of $\Phi_n(z)$.

This formulation and the the following explicit formula for its solution is due to Fokas-Its-Kitaev [23].

1.3.2 Uniqueness of the solution of the Riemann Hilbert Problem

First, let us show that the solution of the RHP is unique.
By condition 3), $\det(\Phi_n(z_+)) = \det\Phi_n(z_-)$ which implies that $\det(\Phi_n(z))$ is continuous across γ. By Morera's theorem, it is analytic in the neighbourhood of γ. In combination with condition 1), we conclude that $\det(\Phi_n(z))$ is entire. Moreover, by condition 3) $\det(\Phi_n(z)) \to 1$ as $z \to \infty$, hence, by Louiville's theorem, $\det(\Phi_n(z)) \equiv 1 \neq 0$, so $\Phi_n(z)$ is invertible.
Now let us assume that there exists another solution of the RHP $\tilde{\Phi}_n(z)$. Such solution must also satisfy $\det(\tilde{\Phi}_n(z)) \equiv 1$. Since $\tilde{\Phi}_n^{-1}(z)$ exists, so let us define $\kappa(z) = \Phi_n(z)\tilde{\Phi}_n^{-1}(z)$.
By Condition 2),

$$\kappa(z_+) = \Phi_n(z_+)\tilde{\Phi}_n^{-1}(z_+) = \Phi_n(z_-)z^{n\sigma_3}z^{-n\sigma_3}\tilde{\Phi}_n^{-1}(z_-) = \Phi_n(z_-)\tilde{\Phi}_n^{-1}(z_-) = \kappa(z_-)$$

So, κ is continuous on γ and analytic in its neighbourhood, by Morera's theorem. Together with condition 0), we conclude that each entry of κ is entire.
In addition, condition 3) says that as $z \to \infty$, $\kappa = (\mathbb{I}_{2\times2} + \mathcal{O}(z^{-1}))z^{n\sigma_3}z^{-n\sigma_3}(\mathbb{I}_{2\times2} + \tilde{\mathcal{O}}(z^{-1}))^{-1} \to \mathbb{I}_{2\times2}$. Therefore, by Liouville's theorem, $\kappa \equiv 1$, hence $\Phi_n(z) = \tilde{\Phi}_n(z)$ (uniqueness is proved).

1.3.3 Solution of the Riemann Hilbert Problem

Proposition 1.6 *The unique solution of the RHP on* Φ *is*

$$\Phi_n(z) = \begin{bmatrix} P_n(z) & B_n^{(1)}(z) \\ \tilde{P}_{n-1}(z) & B_{n-1}^{(2)}(z) \end{bmatrix} = \begin{bmatrix} P_n(z) & \frac{1}{2\pi i}\oint_\gamma \frac{P_n(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w \\ \tilde{P}_{n-1}(z) & \frac{1}{2\pi i}\oint_\gamma \frac{\tilde{P}_{n-1}(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w \end{bmatrix} \tag{1.8}$$

Proof. Following [23] we now write the solution of RHP explicitly in terms of the orthogonal polynomials. By condition 2),

$$\begin{aligned} \begin{bmatrix} A_n(z_+) & B_n(z_+) \end{bmatrix} &= \begin{bmatrix} A_n(z_-) & B_n(z_-) \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-V(z)} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A_n(z_-) & A_n(z_-)\mathrm{e}^{-V(z)} + B_n(z_-) \end{bmatrix}. \end{aligned}$$

Hence we conclude that

$$A_n(z_+) = A_n(z_-), \quad \text{and} \tag{1.9}$$

$$B_n(z_+) - B_n(z_-) = A_n(z_-)\mathrm{e}^{-V(z)}, \quad z \in \gamma. \tag{1.10}$$

The Sokhotskii Plemelj's formula suggests that the function

$$B_n(z) = \frac{1}{2\pi i}\oint_\gamma \frac{A_n(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w, \qquad z \notin \gamma \quad (\dagger\dagger)$$

satisfies relation 1.10. Let us show that no other function satisfies it. Taking $B_n(z)$ as another solution, we have:

$$\begin{aligned} &\begin{cases} B_n(z_+) - B_n(z_-) & = A_n(z_-)\mathrm{e}^{-V(z)} \\ \tilde{B}_n(z_+) - \tilde{B}_n(z_-) & = A_n(z_-)\mathrm{e}^{-V(z)} \end{cases} \\ \Rightarrow\ & B_n(z_+) - B_n(z_-) = \tilde{B}_n(z_+) - \tilde{B}_n(z_-) \\ \Rightarrow\ & \underbrace{B_n(z_+) - \tilde{B}_n(z_+)}_{=\beta_n(z_+)} = \underbrace{B_n(z_-) - \tilde{B}_n(z_-)}_{=\beta_n(z_-)} \end{aligned}$$

$\Rightarrow \beta_n(z) = B_n(z) - \tilde{B}_n(z)$ has no discontinuity on the contour γ therefore, by Morera's theorem, it is an analytic function in the neighbourhood of γ.
Since we were given that $\Phi(z) = \begin{bmatrix} A_n(z) & B_n(z) \end{bmatrix}$ is analytic in $\mathbb{C} - \gamma$, we deduce that $B_n(z)$ in particular, but also $\tilde{B}_n(z)$ (which is expressed in terms of $A_n(z)$ in ($\dagger\dagger$) with singularities on γ) are both analytic in $\mathbb{C} - \gamma$. Hence $\beta_n(z)$ is also analytic in $\mathbb{C} - \gamma$
In conclusion, $\beta_n(z)$ is an entire function.
Moreover, we know that from ($\dagger$) and from ($\dagger\dagger$) that $B_n(z)$ and $\tilde{B}_n(z)$ both go to zero as $z \to \infty$.
Hence By Liouville theorem $\beta_n(z) \equiv 0 \ \ \forall z \in \mathbb{C}$, which implies that $B_n(z) = \tilde{B}_n(z)$
On the other hand, relation (1.9) implies that $A_n(z)$ has no discontinuity on γ. Again, by Morera's Theorem, it is analytic in the neighbourhood of γ. As we were given that $\Phi_n(z)$(in particular $A_n(z)$)

is analytic in $\mathbb{C} - \gamma$, we conclude that $A_n(z)$ is entire.
($\dagger$) suggests that the first entry ($A_n^{(1)}(z)$) and the second entry ($A_n^{(2)}(z)$) of $A_n(z)$ are bounded respectively by z^n and $\mathcal{O}(z^{n-1})$ as $z \to \infty$; by a version of Liouville theorem, they correspond respectively to a monic polynomial of degree n and a polynomial (not necessarily monic) of degree $n-1$.
Let us denote $P_n(z) = A_n^{(1)}(z)$ and $\tilde{P}_{n-1}(z) = A_n^{(2)}(z)$. We will now show that $P_n(z)$ and $\tilde{P}_{n-1}(z)$ are in fact Orthogonal Polynomials.

Let us consider $B_n^{(1)}(z) = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{P_n(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w$, the first entry of the column $B_n(z)$ of $\Phi_n(z)$.
As $z \to \infty$, $w \in \gamma \to \infty$. Since γ is a contour around z i.e.($w \neq z$), $u(w) = \dfrac{1}{w-z}$ is analytic and its taylor expansion as $w \to \infty$ can be obtained by writing

$$u(w) = \frac{1}{-z\left(1 - \dfrac{w}{z}\right)} = -\frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{w}{z}\right)^j = -\sum_{j=0}^{\infty}\frac{w^i}{z^{j+1}}$$

Therefore, we have .

$$B_n^{(1)}(z) = -\frac{1}{2\pi i}\oint_\gamma \sum_{j=0}^{\infty}\frac{w^j P_n(w)\mathrm{e}^{-V(w)}}{z^{j+1}}\mathrm{d}w \tag{1.11}$$

$$= -\frac{1}{2\pi i}\sum_{j=0}^{\infty}\frac{1}{z^{j+1}}\oint_\gamma w^j P_n(w)\mathrm{e}^{-V(w)}\mathrm{d}w \quad \text{as} \quad z \to \infty \tag{1.12}$$

The swap of integral and sum in 1.12 above is justified by the fact that $w \in \gamma$ with γ being a compact set, the sum $\displaystyle\sum_{j=0}^{\infty}\frac{w^j P_n(w)\mathrm{e}^{-V(w)}}{z^{j+1}}$ is also finite as $z \to \infty$. Hence $\displaystyle\oint_\gamma\sum_{j=0}^{\infty}\frac{w^j P_n(w)\mathrm{e}^{-V(w)}}{z^{j+1}}\mathrm{d}w < \infty$.
By Tonelli Fubini, the swap of the contour integral and the sum is possible.

As we know from the condition 3) of the RHP, $B_n^{(1)}(z)$ behaves near ∞ like $\mathcal{O}(\frac{1}{z^{n+1}})$, so for $j+1 \geq n+1$ i.e. $j \geq n$, $B_n^{(1)}(z) \neq 0$ and $B_n^{(1)}(z) = 0$ for $j < n$. In other terms:

$$-\frac{1}{2\pi i}\oint_\gamma w^j P_n(w)\underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} = \begin{cases} \tilde{h}_n \Longrightarrow \displaystyle\oint_\gamma w^j P_n(w)\underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} = h_n \in \mathbb{C} \setminus \{0\}, & j = n \\ 0, \quad j < n \end{cases} \tag{1.13}$$

This shows that the relation x1.7 is satisfied. This is why $P_n(z)$ is an orthogonal polynomial.

■

Similarly, for $B_n^{(2)}(z) = \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{\tilde{P}_{n-1}(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w$, the second entry of the column $B_n(z)$ of $\Phi_n(z)$. By the same manipulation, we obtain

$$B_n^{(2)}(z) = -\frac{1}{2\pi i}\sum_{j=0}^{\infty}\frac{1}{z^{j+1}}\oint_\gamma w^j \tilde{P}_{n-1}(w)(w)\mathrm{e}^{-V(w)}\mathrm{d}w \quad \text{as} \quad z \to \infty \tag{1.14}$$

From condition 3) of the RHP, $B_n^{(2)}(z)$ evolves like $\frac{1}{z^n}$, so for $j+1 \geq n$ i.e. $j \geq n-1$, $B_n^{(2)}(z) \neq 0$ and $B_n^{(2)}(z) = 0$ for $j < n-1$. In other terms:

$$-\frac{1}{2\pi i}\oint_\gamma w^j \tilde{P}_{n-1}(w) \underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} = \begin{cases} 1 & j = n-1 \\ 0, & j < n-1 \end{cases} \tag{1.15}$$

Relation 1.7 is satisfied. This is why $\tilde{P}_{n-1}(z)$ is an orthogonal polynomial.

The solution (we proved it is unique) of RHP which features $P_n(z)$ exists if $D_n \neq 0$. Conversely, if $D_n \neq 0$ then we can construct an orthogonal polynomial $P_n(z)$ of degree n to build the first row of the solution $\Phi_n(z)$ which satisfies the condition 1), 1), and 2) of the RHP. Notice that the orthogonal polynomial, $P_n(z) = \dfrac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ 1 & z & z^2 & \dots & z^n \end{vmatrix}}{D_n}$ is monic of degree n since the Hankel determinant $D_n \neq 0$ and D_n is the cofactor of z^n in the ;
On the other hand, we can observe that if $D_n = 0$, a monic orthogonal polynomial **of degree** n, $P_n(z)$ cannot be defined (cannot be found), and hence the RHP does not have a solution by the fredholm alternative (i.e. the case where we have infinitely many solution when $D_n = 0$ is not possible).

2 Lax pair leading to the solution of Painleve V

2.1 Definition and characteristics of a Lax pair

Definition 2.1 *The Lax Pair is the pair of matrices* $\Big(A(z;t), B(z;t)\Big)$ *of two matrices depending rationally on* z *and analytically on* t *such that the system of two ordinary differential linear equations*

$$\begin{cases} \Psi_z = A(z,t)\Psi \\ \Psi_t = B(z,t)\Psi, \end{cases}$$

is compatible. Namely there is a joint solution $\Psi(z;t)$ *which satisfies the system. This is equivalent to the compatibility condition* $\partial_z\partial_t\Psi = \partial_t\partial_z\Psi$, *namely,* $\partial_t A - \partial_z B + [A, B] = 0$.

The goal is to explain how to find matrices A and B such that the compatibility condition yields the Painleve V equation. For this we will rely on the results of [26].
Let us reconsider the expression of the moment functions used to build the Hankel determinant in the solution of Painleve II:

$$\mu_k(x) = -\oint_\gamma \zeta^k \underbrace{\frac{1}{2\pi i}\mathrm{e}^{-\theta(\zeta,x)}\frac{\mathrm{d}\zeta}{\zeta}}_{=\mathrm{d}\eta(\zeta,x)}$$

where the contour γ is an enlarged circle (after the change of variable $z = 1/\zeta$) around the origin traversed clockwise. The $(-)$ sign before the integral makes it back to a counter-clockwise (positive) direction and we can write

$$\mu_k(x) = \oint_\gamma \zeta^k \underbrace{\frac{1}{2\pi i} \mathrm{e}^{-\theta(\zeta,x)} \frac{\mathrm{d}\zeta}{\zeta}}_{=\mathrm{d}\eta(\zeta,x)} .$$

To obtain solutions of Painlevé V we replace $\mathrm{d}\eta$ by a different complex valued measure

$$d\nu = \left(1 - \frac{1}{\zeta}\right)^\theta \zeta^{-m} \mathrm{e}^{-\dfrac{x}{\zeta}} \mathrm{d}\zeta. \tag{2.1}$$

The moment functional corresponds to the generating function $g(t, x, \theta) = (1 - t)^\theta t^m \mathrm{e}^{-xt}$ where x is a parameter, and θ is another parameter independent of x.
The moment functions are now given by

$$\mu_k(x) = \oint_\gamma \zeta^k \underbrace{\left(1 - \frac{1}{\zeta}\right)^\theta \zeta^{-m} \mathrm{e}^{-\dfrac{x}{\zeta}} \mathrm{d}\zeta}_{=d\nu(\zeta,x,\theta)} \tag{2.2}$$

where the contour γ is a circle around the origin, large enough to contain the points $\zeta = 0$ and $\zeta = 1$.
The measure is given by $d\nu = \mathrm{e}^{-V(\zeta)}\mathrm{d}\zeta$, so that the weight is

$$W(\zeta) \equiv \mathrm{e}^{-V(\zeta)} = \left(1 - \frac{1}{\zeta}\right)^\theta \zeta^{-m} \mathrm{e}^{-\dfrac{x}{\zeta}} \mathrm{d}\zeta \tag{2.3}$$

$$\text{i.e. } \ V(\zeta) = \frac{x}{\zeta} - \theta \ln \left(1 - \frac{1}{\zeta}\right) + m \ln(\zeta) \tag{2.4}$$

Now, we will prove that by setting some well constructed function as the solution of the Lax pair, we can derive matrices A and B of the pair such that the compatibility condition of the Lax pair yields equation Painleve V. More practically, we will simply compare the matrices A and B with that are found for the equation Painleve V in [26].

Let us define the matrix Ψ as follows:

$$\tilde{\Psi}(z, t) = \Psi(z) := \Phi(z) \begin{bmatrix} \mathrm{e}^{-\dfrac{V(z)}{2}} & 0 \\ 0 & \mathrm{e}^{\dfrac{V(z)}{2}} \end{bmatrix} = \Phi(z)\mathrm{e}^{\dfrac{-V(z)}{2}\sigma_3} \tag{2.5}$$

where $\Phi(z)$ is the solution of the RHP of Prop. 1.6. Our goal is to prove the following Proposition.

Proposition 2.2 *The matrix $\Psi(z; x)$ in (2.5) satisfies the pair of first order PDEs*

$$\left\{ \begin{array}{l} \partial_z \Psi(z; x) = A(z; x)\Psi(z; x) \\ \partial_x \Psi(z; x) = B(z; x)\Psi(z; x). \end{array} \right. \tag{2.6}$$

where the matrices A, B have the form

$$A(z;x) := \frac{x}{z^2} G_0\sigma_3 G_0^{-1} + G_0 \left(\frac{-m-\theta}{z}\sigma_3 + \frac{x}{z}\left[G_0^{-1}G_1, \sigma_3\right] \right) G_0^{-1} + \frac{-m}{z-1} H_0\sigma_3 H_0^{-1} + \frac{n-m}{z}\sigma_3 \tag{2.7}$$

$$B(z;x) := -\frac{1}{z} G_0\sigma_3 G_0^{-1} \tag{2.8}$$

where $G_0 := \Phi(0;x)$, $G_1 := \Phi'(0;x)$ and $H_0 := \Phi_n(1;x)$. Moreover it satisfies the following expansions near the points $z = 0, 1, \infty$:

$$\Psi(z;x) = \begin{cases} \mathcal{O}^\times(1) z^{\frac{-m-\theta}{2}\sigma_3} \mathrm{e}^{-\frac{x}{2z}\sigma_3}, & z \to 0 \\ \mathcal{O}^\times(1)(z-1)^{\frac{\theta}{2}\sigma_3}, & z \to 1 \\ \left(\mathbf{1} + \mathcal{O}(z^{-1})\right) z^{\left(n-\frac{m}{2}\right)\sigma_3}, & z \to \infty \end{cases} \tag{2.9}$$

where $\mathcal{O}^\times(1)$ denote a locally analytic (near the points $z = 0, 1$, respectively) and analytically invertible matrix–valued expression.

Proof. It is simple to verify directly from the jump condition (2) in section 1.3.1 that the matrix $\Psi(z)$ satisfies also a jump relation for $z \in \gamma$ where the matrix of the jump is *independent* of z and x. This means that $\Psi_n'(z)$ also satisfies the same relation and therefore $A(z;x) := \Psi_n'(z;x)\Psi_n(z;x)^{-1}$ extends analytically across γ. Therefore, we can write:

$$\begin{aligned} A &= \left[\Phi_n'(z)\mathrm{e}^{\frac{-V(z)}{2}\sigma_3} + \Phi_n(z)\left(\frac{-V'(z)}{2}\sigma_3\right)\mathrm{e}^{\frac{-V(z)}{2}\sigma_3}\right]\mathrm{e}^{\frac{V(z)}{2}\sigma_3}\Phi_n^{-1}(z) \\ &= \Phi_n'(z)\Phi_n^{-1}(z) - \frac{V'(z)}{2}\Phi_n(z)\sigma_3\Phi_n^{-1}(z) \\ &= \Phi_n'(z)\Phi_n^{-1}(z) + \frac{1}{2}\left[\frac{x}{z^2} + \theta\left(\frac{1}{z-1} - \frac{1}{z}\right) - \frac{m}{z}\right]\Phi_n(z)\sigma_3\Phi_n^{-1}(z) \end{aligned}$$

By studying the analyticity of A on $\mathbb{C}$, we now prove that A is in rational function of z.

First, let us verify that A has indeed no jump on γ, as anticipated above.
In fact, since $V(z) = \frac{x}{z} - \theta\ln\left(1 - \frac{1}{z}\right) + m\ln(z)$ has singularities only at 0, 1 enclosed by γ, then $V(z)$ has no jump on γ $(V(z_+) = V(z_-) = V(z))$ and we can write :

$$\begin{aligned} A(z_+) &= \Phi_n'(z_+)\Phi_n^{-1}(z_+) - \frac{V'(z)}{2}\Phi_n(z_+)\sigma_3\Phi_n^{-1}(z_+) \\ &= \Phi_n'(z_-)\begin{bmatrix} 1 & \mathrm{e}^{-V(z)} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\mathrm{e}^{-V(z)} \\ 0 & 1 \end{bmatrix}\Phi_n^{-1}(z_-) \\ &+ \Phi_n(z_-)\begin{bmatrix} 0 & -V'(z)\mathrm{e}^{-V(z)} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -\mathrm{e}^{-V(z)} \\ 0 & 1 \end{bmatrix}\Phi_n^{-1}(z_-) - \frac{V'(z)}{2}\Phi_n(z_+)\sigma_3\Phi_n^{-1}(z_+) \end{aligned}$$

$$
\begin{aligned}
&= \Phi_n'(z_-)\Phi_n^{-1}(z_-) + \Phi_n(z_-)\begin{bmatrix}0 & -V'(z)\mathrm{e}^{-V(z)}\\ 0 & 0\end{bmatrix}\Phi_n^{-1}(z_-)\\
&- \frac{V'(z)}{2}\Phi_n(z_-)\underbrace{\begin{bmatrix}1 & \mathrm{e}^{-V(z)}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & -\mathrm{e}^{-V(z)}\\ 0 & 1\end{bmatrix}}_{=\begin{bmatrix}1 & -2\mathrm{e}^{-V(z)}\\ 0 & -1\end{bmatrix}}\Phi_n^{-1}(z_-)\\
&= \Phi_n'(z_-)\Phi_n^{-1}(z_-)\\
&+ \Phi_n(z_-)\left(\begin{bmatrix}0 & -V'(z)\mathrm{e}^{-V(z)}\\ 0 & 0\end{bmatrix} + \begin{bmatrix}-\dfrac{V'(z)}{2} & V'(z)\mathrm{e}^{-V(z)}\\ 0 & \dfrac{V'(z)}{2}\end{bmatrix}\right)\Phi_n^{-1}(z_-)\\
&= \Phi_n'(z_-)\Phi_n^{-1}(z_-) + \Phi_n(z_-)\begin{bmatrix}-\dfrac{V'(z)}{2} & 0\\ 0 & \dfrac{V'(z)}{2}\end{bmatrix}\Phi_n^{-1}(z_-)\\
&= \Phi_n'(z_-)\Phi_n^{-1}(z_-) - \frac{V'(z)}{2}\Phi_n(z_-)\sigma_3\Phi_n^{-1}(z_-)\\
&= A(z_-)
\end{aligned}
$$

By Morera's theorem, A is analytic across γ and it may have only isolated singularities at $z = 0, 1$.

To ascertain the nature of these singularities, it remains to study

$$
A = \Phi_n'(z)\Phi_n^{-1}(z) + \frac{1}{2}\left[\frac{x}{z^2} + \theta\left(\frac{1}{z-1} - \frac{1}{z}\right) - \frac{m}{z}\right]\Phi_n(z)\sigma_3\Phi_n^{-1}(z) \tag{2.10}
$$

on $\mathbb{C} \setminus \gamma$.

It is easy to see that because of condition 1) in section 1.3.1 on $\Phi_n(z)$, $\Phi_n'(z)\Phi_n^{-1}(z)$ and $\Phi_n(z)\sigma_3\Phi_n^{-1}(z)$ are analytic on $\mathbb{C} \setminus \gamma$ in the expression (2.10) for A. The same expression (2.10) shows that $A(z)$ has at most a double pole at 0, and a simple pole at 1. Therefore A is rational and can be written as

$$
A = \frac{A_0}{z} + \frac{\tilde{A}_0}{z^2} + \frac{A_1}{z-1} \tag{2.11}
$$

where A_0, $\tilde{A}_0$ and A_1 are constant matrices to be determined and be compared to the ones in [**?**] related to Painleve V.

To find them, let us express (2.10) in the neighbourhood of ∞ and 0

As $z \to \infty \quad \Phi_n(z) = \left(1 + \mathcal{O}(z^{-1})\right) z^{n\sigma_3}$. Hence as $z \to \infty$

$$(2.10) \Longrightarrow A \longrightarrow \mathcal{O}(z^{-2})\, \underbrace{z^{n\sigma_3} z^{-n\sigma_3} \underbrace{\left(1 + \mathcal{O}(z^{-1})\right)^{-1}}_{\to \mathbb{I}_{2\times 2}}}_{\to \mathbb{I}_{2\times 2}}$$

$$+ \underbrace{\left(1 + \mathcal{O}(z^{-1})\right)}_{\to \mathbb{I}_{2\times 2}} \frac{n\sigma_3}{z} \underbrace{z^{n\sigma_3} z^{-n\sigma_3} \underbrace{\left(1 + \mathcal{O}(z^{-1})\right)^{-1}}_{\to \mathbb{I}_{2\times 2}}}_{\to \mathbb{I}_{2\times 2}}$$

$$+ \underbrace{\frac{1}{2}\left[\frac{x}{z^2} + \theta\left(\frac{1}{z-1} - \frac{1}{z}\right) - \frac{m}{z}\right]}_{\to \mathcal{O}(z^{-2}) \text{ since } (z-1) \approxeq z \text{ at } \infty} \underbrace{\underbrace{\left(1 + \mathcal{O}(z^{-1})\right)}_{\to \mathbb{I}_{2\times 2}} z^{n\sigma_3} \sigma_3 z^{-n\sigma_3} \underbrace{\left(1 + \mathcal{O}(z^{-1})\right)^{-1}}_{\to \mathbb{I}_{2\times 2}}}_{\to \sigma_3}$$

$$\longrightarrow \mathcal{O}(z^{-2}) + \frac{n\sigma_3}{z} \longrightarrow 0 \qquad (\star)$$

On the other hand, as $z \to \infty \quad (2.11) \Longrightarrow A = \dfrac{\tilde{A}_0}{z^2} + \dfrac{A_0 + A_1}{z} \underbrace{\Longrightarrow}_{\text{by } (\star)} A_0 + A_1 = n\sigma_3.$

We now investigate the behaviour as $z \to 0$: this is done by taking the Laurent expansion of A and recalling that $\Phi(s)$ is analytic at $z = 0$ we get

$$\begin{aligned}
(2.10) \Longrightarrow A \longrightarrow\ & \Phi_n'(0)\Phi_n^{-1}(0) - \frac{(1/2)x\Phi_n(z)\sigma_3\Phi_n^{-1}(z)}{z^2} + \frac{(1/2)(\theta)\Phi_n(z)\sigma_3\Phi_n^{-1}(z)}{-1} \\
& - \frac{(1/2)(\theta+m)\Phi_n(z)\sigma_3\Phi_n^{-1}(z)}{z} \\
& = \mathcal{O}(1) - \frac{(1/2)x\Phi_n(z)\sigma_3\Phi_n^{-1}(z)}{z^2} - \frac{(1/2)(\theta+m)\Phi_n(z)\sigma_3\Phi_n^{-1}(z)}{z} \\
& = \mathcal{O}(1) - \frac{(1/2)x\left[\Phi_n(0) + z\Phi_n'(0) + \mathcal{O}(z^2)\right]\sigma_3\left[\Phi_n^{-1}(0) - z\Phi_n^{-1}(0)\Phi_n'(0)\Phi_n^{-1}(z) + \mathcal{O}(z^2)\right]}{z^2} \\
& - \frac{(1/2)(\theta+m)\left[\Phi_n(0) + z\Phi_n'(0) + \mathcal{O}(z^2)\right]\sigma_3\left[\Phi_n^{-1}(0) - z\Phi_n^{-1}(0)\Phi_n'(0)\Phi_n^{-1}(0) + \mathcal{O}(z^2)\right]}{z} \\
& - \frac{(1/2)(\theta+m)\left[\Phi_n(0) + z\Phi_n'(0) + \mathcal{O}(z^2)\right]\sigma_3\left[\mathcal{O}(z^2)\right]}{z} =
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}(1) + \frac{1}{z^2}\Big(- (1/2)x\Phi_n(0)\sigma_3\Phi_n^{-1}(0) + \mathcal{O}(z^2)\Big) \\
&+ \frac{1}{z}\Big((1/2)x\Phi_n(0)\sigma_3\Phi_n^{-1}(0)\Phi_n'(0)\Phi_n^{-1}(0) - (1/2)x\Phi_n'(0)\sigma_3\Phi_n^{-1}(0) \\
&- (1/2)(\theta+m)\Phi_n(0)\sigma_3\Phi_n^{-1}(0) + \mathcal{O}(z^2)\Big) \\
&= \mathcal{O}(1) - \frac{(1/2)x}{z^2}\Phi_n(0)\Big(\sigma_3 + \mathcal{O}(z^2)\Big)\Phi_n^{-1}(0) \\
&+ \frac{(1/2)}{z}\Phi_n(0)\underbrace{\Big(x\sigma_3\Phi_n^{-1}(0)\Phi_n'(0) - x\Phi_n^{-1}(0)\Phi_n'(0)\sigma_3 - (\theta+m)\sigma_3 + \mathcal{O}(z^2)\Big)}_{\to x[\sigma_3,\Phi_n^{-1}(0)\Phi_n'(0) - (\theta+m)\sigma_3] \quad as \quad z\to 0}\Phi_n^{-1}(0)
\end{aligned}
$$

$(\star\star)$

Again, as $z \to 0$, we have

$$
\begin{aligned}
&(2.11) \Longrightarrow A = \frac{A_0}{z} + \frac{\tilde{A}_0}{z^2} + \frac{A_1}{-1} \\
&\underbrace{\Longrightarrow}_{\text{by }(\star\star)} \begin{cases} A_0 = (1/2)x\Phi_n(0)\Big[\sigma_3, \Phi_n^{-1}(0)\Phi_n'(0) - (\theta+m)\sigma_3\Big]\Phi_n^{-1}(0) \\ \tilde{A}_0 = (1/2)x\Phi_n(0)\sigma_3\Phi_n^{-1}(0) \end{cases} \\
&\Longrightarrow A_1 = n\sigma_3 - (1/2)x\Phi_n(0)\Big[\sigma_3, \Phi_n^{-1}(0)\Phi_n'(0) - (\theta+m)\sigma_3\Big]\Phi_n^{-1}(0)
\end{aligned}
$$

The expression for $B(z;x)$ is found along similar lines.

The expansions near $z = 0, 1, \infty$ follow from the behaviour of the term $\mathrm{e}^{\frac{V(z)\sigma_3}{2}}$ near those points and the condition (3) in Section 1.3.1. ■

In the work of the Japanese school [26] the z–component of the Lax pair for Painlevé V has also two Fuchsian and one second rank singularities, but with the positions reversed. More specifically, the Lax pair proposed in [26] is as follows[1] (see formulas (C.38–C.45) in loc. cit.)

$$
\begin{aligned}
&\partial_w Q(w;t) = A_{JMU}(w;t)Q(w;t)\ , \qquad \partial_t Q(w;t) = B_{JMU}(w;t)Q(w;t) \\
&A_{JMU}(w;t) = \frac{t}{2}\sigma_3 + \frac{1}{w}\begin{bmatrix} Z+\frac{\theta_0}{2} & -U(Z+\theta_0) \\ \frac{Z}{U} & -Z-\frac{\theta_0}{2}\end{bmatrix} + \frac{1}{w-1}\begin{bmatrix} -Z-\frac{\theta_0+\theta_\infty}{2} & UY\left(Z+\frac{\theta_0-\theta_1+\theta_\infty}{2}\right) \\ -\frac{Z+\frac{\theta_0+\theta_1+\theta_\infty}{2}}{UY} & Z+\frac{\theta_0+\theta_\infty}{2}\end{bmatrix} \\
&B_{JMU}(w;t) = \frac{w}{2}\sigma_3 + \begin{bmatrix} 0 & -\frac{U}{t}\left(Z+\theta_0 - Y\left(Z+\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)\right) \\ \frac{1}{tUY}\left((Y-1)Z+\frac{\theta_0+\theta_1+\theta_\infty}{2}\right) & 0\end{bmatrix} \qquad (2.12)
\end{aligned}
$$

where $Z = Z(t), Y = Y(t), U = U(t)$ satisfy a nonlinear first order system of ODEs in t (C.40),

[1] We transcribe the results of [26] but we adapt their notation to our conventions. Note that the paper contains a couple of small typos: in (C.39) there should be an x in front of the first term of B and the sign of the $(1,2)$ entry of the next term should be the opposite.

which implies the fifth Painlevé equation for Y:

$$\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = \left(\frac{1}{2Y} + \frac{1}{Y-1}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}Y}{\mathrm{d}t} + \frac{(Y-1)^2(\alpha Y + \frac{Y}{\beta})}{t^2} + \frac{\gamma Y}{t} + \frac{\delta Y(Y+1)}{Y-1}$$
$$\alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2; \ \beta = -\frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2; \ \gamma = 1 - \theta_0 - \theta_1; \ \delta = -\frac{1}{2}. \tag{2.13}$$

The solution $Q(w;t)$ has the following formal expansions near $w = 0, 1, \infty$:

$$\begin{aligned}
Q(w;t) &= \mathcal{O}(1) w^{\frac{\theta_0\sigma_3}{2}}, && w \to 0 \\
Q(w;t) &= \mathcal{O}(1)(w-1)^{\frac{\theta_1\sigma_3}{2}}, && w \to 1 \\
Q(w;t) &= \left(\mathbf{1} + \frac{Q_1}{w} + \mathcal{O}(w^{-2})\right) w^{-\frac{\theta_\infty}{2}\sigma_3} \mathrm{e}^{\frac{t}{2}w\sigma_3}, && w \to \infty.
\end{aligned} \tag{2.14}$$

The Hamiltonian function for the Painlevé equation is given by

$$H_V = -\frac{1}{2}\mathrm{tr}\,(Q_1\sigma_3) \tag{2.15}$$

and the equation admits the so–called *sigma-form*: indeed, introducing the new function

$$\sigma(t) = tH_V - \frac{t}{2}(\theta_0 + \theta_\infty) + \frac{(\theta_0 + \theta_\infty)^2 - \theta_1^2}{4} \tag{2.16}$$

it can be verified that it satisfies ([26] formula (C.45))

$$\left(t\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right)^2 = \left(\sigma - t\frac{\mathrm{d}\sigma}{\mathrm{d}t} + 2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 - (\theta_\infty + 2\theta_0)\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 +$$
$$-4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \theta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right) \tag{2.17}$$

If we want to identify our Lax pair (2.6) with the Japanese one (2.12) we see that it suffices to map $w = \frac{1}{z}$ and normalize suitably our $\Psi(z;t)$

Proposition 2.3 *The map $w = \frac{1}{z}$ and*

$$Q_n(w;t) = \Phi_n(0;t)^{-1}\Psi\left(\frac{1}{w};t\right)\mathrm{e}^{-i\frac{\pi}{2}\boldsymbol{\rho}\sigma_3} \qquad \widehat{Q_n}(z;t) := Q_n\left(\frac{1}{z};t\right). \tag{2.18}$$

transforms the Lax pair (2.6) into (2.12) with parameters

$$\theta_0 = -2n + m; \qquad \theta_1 = \theta; \qquad \theta_\infty = -m - \theta \tag{2.19}$$

As $w \to \infty$ the expansion of $Q_n(w)$ reads as follows:

$$Q_n(w) = \left(1 + \frac{Q_{n,1}(\infty)}{w} + \mathcal{O}(\frac{1}{w^2})\right)\mathrm{e}^{T(w)} \tag{2.20}$$

where,

$$T(w) = \frac{1}{2}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} w + \frac{1}{2}\begin{bmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{bmatrix} \ln\left(\frac{1}{w}\right). \tag{2.21}$$

This corresponds to the parameters α, β, γ in (2.13) as follows

$$\alpha = \frac{(N\theta)^2}{2}; \quad \beta = -\frac{(N-m)^2}{2}; \quad \gamma = 1 + 2N-m - \theta. \tag{2.22}$$

Proof. The map $w = \frac{1}{z}$ maps $z = 0$ to $w = \infty$, $z = \infty$ to $w = 0$ and $z = 1$ to $w = 1$. Thus the exponents of (formal) monodromy $\theta_{\{0,1,\infty\}}$ are read off by matching the exponents in (2.9) and (2.14) as we explain in details below.

Finding the expression for θ_∞. Since $z = \dfrac{1}{w}$ to find θ_∞, we need to expand $\hat{Q}_n(z;x) := Q_n\left(\frac{1}{z};x\right)$ near $z = 0$ as the product of two factors:

- An analytic factor of the form $\left(1+\hat{Q}_{n,1}(0)\cdot z+\mathcal{O}(z^2)\right)$ equivalent to $\left(1+\dfrac{Q_{n,1}(\infty)}{w}+\mathcal{O}(\dfrac{1}{w^2})\right)$ in 2.20 such that we can read off the expression of the Halmitonian as *minus* the $(1,1)$ entry of $\hat{Q}_{n,1}(0)$

- An exponential factor of the form $e^{\hat{T}(z)}$ where $\hat{T}(z)$ is equivalent to 2.21 such that we can read off the expression of β_0 .

In the neighbourhood of $w \to \infty$, i.e. $z \to 0$ the expansion of $Q_n(w)$ gives:

$$
\begin{aligned}
Q_n(w) = Q_n(\frac{1}{z}) &= \hat{Q}_n(z) = \Phi_n^{-1}(0)\Psi_n(z) = \Phi_n^{-1}(0)\Phi_n(z)\mathrm{e}^{-\frac{V(z)}{2}\sigma_3} \\
&= \Phi_n^{-1}(0)\Phi_n(z)\left(1-\frac{1}{z}\right)^{\frac{\theta\sigma_3}{2}} z^{-\frac{m\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \Phi_n^{-1}(0)\Phi_n(z)\left(1-\frac{1}{z}\right)^{\frac{\theta\sigma_3}{2}} \underbrace{z^{\frac{\theta\sigma_3}{2}} z^{-\frac{\theta\sigma_3}{2}}}_{=1} z^{-\frac{m\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \Phi_n^{-1}(0)\Phi_n(z)(z-1)^{\frac{\theta\sigma_3}{2}} z^{-\frac{\theta\sigma_3}{2}} z^{-\frac{m\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \Phi_n^{-1}(0) \underbrace{\left[\Phi_n(0) + \Phi_n'(0)z + \mathcal{O}(z^2)\right]}_{\text{Taylor expansion of } \Phi_n(z) \quad \text{at} \quad 0} (z-1)^{\frac{(\theta)\sigma_3}{2}} \\
&\cdot z^{-\frac{(\theta+m)\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \underbrace{\left[\mathbb{I}_{2\times 2} + \Phi_n^{-1}(0)\Phi_n'(0)z + \mathcal{O}(z^2)\right](-1)^{\frac{(\theta)\sigma_3}{2}}(1-z)^{\frac{(\theta)\sigma_3}{2}}}_{\text{Analytic as } z\to 0} \\
&\cdot \overbrace{\mathrm{e}^{\frac{1}{2}-(\theta+m)\sigma_3 \;\; \ln(z) - \frac{1}{2}x\sigma_3(\frac{1}{z})}}^{= z^{\frac{-(\theta+m)\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}}} = Analytic \cdot \mathrm{e}^{\hat{T}(z)},
\end{aligned}
$$

Where we have set,

$$
\hat{T}(z) = \frac{1}{2}\beta_0\sigma_3 \ln(z) - \frac{1}{2}x\frac{\sigma_3}{z} \tag{2.23}
$$

Now, let us refine the analytic part of $\hat{Q}_n(z)$ to find out the expression of $\hat{Q}_{n,1}(0) = Q_{n,1}(\infty)$ We had:

$$
\begin{aligned}
Q_n(w) &= Q_n(\frac{1}{z}) = \hat{Q}_n(z) \\
&= \underbrace{\left[\mathbb{I}_{2\times 2} + \Phi_n^{-1}(0)\Phi_n'(0)z + \mathcal{O}(z^2)\right](-1)^{\dfrac{\theta\sigma_3}{2}}(1-z)^{\dfrac{\theta\sigma_3}{2}}}_{\text{Analytic as } z\to 0} \mathrm{e}^{\hat{T}(z)} \\
&= \underbrace{(-1)^{\dfrac{\theta\sigma_3}{2}}}_{:=C(\text{ constant})} \left[\mathbb{I}_{2\times 2} + \Phi_n^{-1}(0)\Phi_n'(0)z + \mathcal{O}(z^2)\right] \\
&\cdot \underbrace{\left(\mathbb{I}_{2\times 2} - \frac{(\theta)\sigma_3}{2}z + \mathcal{O}(z^2)\right)}_{=\text{Taylor expansion of } (1-z)^{\dfrac{(\theta)\sigma_3}{2}}} \mathrm{e}^{\hat{T}(z)} \\
&= C\left[\mathbb{I}_{2\times 2} + \left(\Phi_n^{-1}(0)\Phi_n'(0) - \frac{\theta\sigma_3}{2}\right)z + \mathcal{O}(z^2)\right]\mathrm{e}^{\hat{T}(z)}
\end{aligned}
$$

where

$$
\hat{T}(z) = \frac{1}{2}\beta_0\sigma_3 \ \ln(z) - \frac{1}{2}x\sigma_3(\frac{1}{z}) = \frac{1}{2}\begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix}\frac{1}{z} + \frac{1}{2}\begin{bmatrix} \beta_0 & 0 \\ 0 & -\beta_0 \end{bmatrix}\ln(z)
$$

in the neighbourhood of $z = 0$.
This implies that as $w \to \infty$, , we obtain:

$$
\hat{T}(z) = \hat{T}(\frac{1}{w}) := T(w) = \frac{1}{2}\begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix}\frac{1}{w} + \frac{1}{2}\begin{bmatrix} -(\theta+m) & 0 \\ 0 & (\theta+m) \end{bmatrix}\ln(\frac{1}{w}) \tag{2.24}
$$

and

$$
\hat{Q}(z) = Q_n(w) = C\left[\mathbb{I}_{2\times 2} + \left(\Phi_n^{-1}(0)\Phi_n'(0) - \frac{(\theta)\sigma_3}{2}\right)\frac{1}{w} + \mathcal{O}(\frac{1}{w^2})\right]\mathrm{e}^{T(w)} \tag{2.25}
$$

So by identification with the expression of the solution at ∞ of the Lax pair in [3] written as $Q_n(w) = \left[\mathbb{I}_{2\times 2} + Q_{n,1}(\infty)\frac{1}{w} + \mathcal{O}(\frac{1}{w^2})\right]\mathrm{e}^{T(w)}$
where $T(w) = \frac{1}{2}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}\frac{1}{z} + \frac{1}{2}\begin{bmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{bmatrix}\ln(\frac{1}{w})$
we obtain that:

$$
\theta_\infty : = \beta_0 = -(\theta+m) \tag{2.26}
$$

$$
x = -t \tag{2.27}
$$

$$
\text{and,} \quad \hat{Q}_{n,1}(0) = Q_{n,1}(\infty) = \Phi_n^{-1}(0)\Phi_n'(0) - \frac{(\theta)\sigma_3}{2} \tag{2.28}
$$

Finding the expression for θ_0. As $w \to 0$ the expansion of $Q_n(w)$ reads as follows in [26]:

$$
Q_n(w) = \mathcal{O}^\times(1)\cdot \mathrm{e}^{M(w)} \tag{2.29}
$$

where,

$$M(w) = \frac{1}{2}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} w + \frac{1}{2}\begin{bmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{bmatrix} \ln(w) \tag{2.30}$$

In the neighbourhood of $w \to 0$, i.e. $z \to \infty$ the expansion of $Q_n(w)$ gives:

$$\begin{aligned}
Q_n(w) = Q_n\left(\frac{1}{z}\right) &= \hat{Q}_n(z) = \Phi_n^{-1}(0)\Psi_n(z) = \Phi_n^{-1}(0)\Phi_n(z)\mathrm{e}^{-\frac{V(z)}{2}\sigma_3} \\
&= \Phi_n^{-1}(0)\Phi_n(z)\left(1 - \frac{1}{z}\right)^{\frac{\theta\sigma_3}{2}} z^{-\frac{m\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \underbrace{\Phi_n^{-1}(0)}_{:=C(constant)} \quad \Phi_n(z) \underbrace{\left(1 - \frac{1}{z}\right)^{\frac{\theta\sigma_3}{2}}}_{Analytic \text{ as } z\to\infty} z^{-\frac{m\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= Analytic \cdot \underbrace{\left[\mathbb{I}_{2\times 2} + \mathcal{O}(\frac{1}{z})\right] z^{n\sigma_3}}_{\text{expansion of } \Phi_n(z) \text{ as } z\to\infty \text{ (see condition 2) of the RHP on } \Phi)} \\
&\cdot \ \mathrm{e}^{-\frac{(m)\sigma_3}{2}} \mathrm{e}^{-\frac{x\sigma_3}{2z}}
\end{aligned}$$

$$\begin{aligned}
&= \underbrace{Analytic \cdot \left[\mathbb{I}_{2\times 2} + \mathcal{O}(\frac{1}{z})\right]}_{Analytic} \mathrm{e}^{-\frac{x\sigma_3}{2z}} z^{(n-\frac{m}{2})\sigma_3} \\
&= Analytic \cdot \mathrm{e}^{-\frac{1}{2}x\sigma_3\frac{1}{z} + \frac{1}{2}(2n-m)\sigma_3 \ln(z)} \\
&= Analytic \cdot \mathrm{e}^{\hat{M}(z)}
\end{aligned}$$

where,

$$\hat{M}(z) = \frac{1}{2}\begin{bmatrix} -x & 0 \\ 0 & x \end{bmatrix} \frac{1}{z} + \frac{1}{2}\begin{bmatrix} (2n-m) & 0 \\ 0 & -(2n-m) \end{bmatrix} \ln(\frac{1}{z}) \tag{2.31}$$

So by identification $x = -t$ and $\theta_0 := -(2n-m)$.

Finding the expression for θ_1. As $w \to 1$ the expansion of $Q_n(w)$ reads as follows:

$$Q_n(w) = \mathcal{O}^{\times}(1) \cdot \mathrm{e}^{N(w)} \tag{2.32}$$

where,

$$N(w) = \frac{1}{2}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} (w-1) + \frac{1}{2}\begin{bmatrix} \theta_1 & 0 \\ 0 & -\theta_1 \end{bmatrix} \ln(w-1) \tag{2.33}$$

and θ_1 is a parameter. To find it, we need to expand $\hat{Q}_n(z)$ near 1 as the product of two factors:

- An analytic factor ;
- An exponential factor of the form $\mathrm{e}^{\hat{N}(z)}$ where $\hat{N}(z)$ is equivalent to $N(w)$ such that we can read off the expression of θ_1 .

In the neighbourhood of $w \to 1$, i.e. $z \to 1$ the expansion of $Q_n(w)$ gives:

$$
\begin{aligned}
Q_n(w) = Q_n(\frac{1}{z}) = \hat{Q}_n(z) &= \Phi_n^{-1}(0)\Psi_n(z) = \Phi_n^{-1}(0)\Phi_n(z)\mathrm{e}^{-\frac{V(z)}{2}\sigma_3} \\
&= \Phi_n^{-1}(0)\Phi_n(z)\left(1-\frac{1}{z}\right)^{\frac{\theta\sigma_3}{2}} z^{-\frac{m\sigma_3}{2}}\mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \Phi_n^{-1}(0)\Phi_n(z)(-1)^{\frac{\theta\sigma_3}{2}}\left(\frac{1}{z}-1\right)^{\frac{\theta\sigma_3}{2}} \underbrace{z^{\frac{\theta\sigma_3}{2}} z^{-\frac{\theta\sigma_3}{2}}}_{=1} z^{-\frac{m\sigma_3}{2}}\mathrm{e}^{-\frac{x\sigma_3}{2z}} \\
&= \overbrace{(-1)^{\frac{\theta\sigma_3}{2}}\Phi_n^{-1}(0)\underbrace{\left[\Phi_n(1)+\Phi_n'(1)(z-1)+\mathcal{O}(z-1)^2\right]}_{\text{Taylor expansion of }\Phi(z)\text{ at }z=1} z^{-\frac{\theta\sigma_3}{2}} z^{-\frac{m\sigma_3}{2}}\mathrm{e}^{-\frac{x\sigma_3}{2z}}}^{Analytic} \\
&\quad \cdot \underbrace{\mathrm{e}^{\frac{x\sigma_3}{2}(z-1)}\mathrm{e}^{-\frac{x\sigma_3}{2}(z-1)}}_{=1} \quad \underbrace{(1-z)^{\frac{\theta\sigma_3}{2}}}_{=\left(\frac{1}{z}-1\right)^{\frac{\theta\sigma_3}{2}} z^{\frac{\theta\sigma_3}{2}}} \\
&= \underbrace{(-1)^{\frac{\theta\sigma_3}{2}}\mathcal{O}^{\times}(1)\cdot \overbrace{\mathrm{e}^{\frac{x\sigma_3}{2}(z-1)}}^{Analytic}}_{Analytic}\mathrm{e}^{-\frac{x\sigma_3}{2}(z-1)}(z-1)^{\frac{\theta\sigma_3}{2}} \\
&= \mathcal{O}^{\times}(1)\cdot \mathrm{e}^{-\frac{1}{2}x\sigma_3(z-1)+\frac{1}{2}\theta\sigma_3\ln(z-1)} \\
&= \mathcal{O}^{\times}(1)\cdot \mathrm{e}^{\hat{N}(z)}
\end{aligned}
$$

where,

$$
\hat{N}(z) = \frac{1}{2}\begin{bmatrix}-x & 0\\ 0 & x\end{bmatrix}(z-1) + \frac{1}{2}\begin{bmatrix}(\theta) & 0\\ 0 & -(\theta)\end{bmatrix}\ln(z-1) \tag{2.34}
$$

So by identification $x = -t$ and $\theta_1 := \theta$. ■

The following definition was given in [26].

Definition 2.4 *The expansion of $Q_n(w)$ as $w \to \infty$ (as stated in 2.20) exhibits in its second term a coefficient matrix $Q_{n,1}(\infty)$ whose $(2,2)$ entry is called the Hamiltonian and denoted H_V:*

$$
H_V := \Big(Q_{n,1}(\infty)\Big)_{2,2}. \tag{2.35}
$$

In terms of our original solution of the matrix RHP we can formulate the following Proposition.

Proposition 2.5 *Similar to the solution $Q_n(w)$ of the Lax pair in* [3] *(p.443), the solution $\hat{Q}_n(z)$ of our Lax pair exhibits in its second term a coefficient matrix of the form:*

$$\hat{Q}_{n,1}(0) = \begin{bmatrix} -H_V & * \\ * & H_V \end{bmatrix}$$

Proof. Recall from the proposition 1.6 stating the RHP solution, we obtained:

$$\Phi_n(0) = \begin{bmatrix} P_n(0) & B_n^{(1)}(0) \\ \tilde{P}_{n-1}(0) & B_{n-1}^{(2)}(0) \end{bmatrix}$$

. Setting $B_n^{(1)}(0) = B_n(0)$ and $B_{n-1}^{(2)}(0) = \tilde{B}_{n-1}(0)$
we have

$$\Phi_n^{-1}(0)\Phi_n'(0) = \begin{bmatrix} P_n'(0)\tilde{B}_{n-1}(0) - \tilde{P}_{n-1}'(0)B_n(0) & * \\ * & P_n(0)\tilde{B}_{n-1}'(0) - \tilde{P}_{n-1}(0)B_n'(0) \end{bmatrix}$$

We will show that:
if $\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{22} = H_V = P_n(0)\tilde{B}_{n-1}'(0) - \tilde{P}_{n-1}(0)B_n'(0)$,
then $\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} = -H_V = P_n'(0)\tilde{B}_{n-1}(0) - \tilde{P}_{n-1}'(0)B_n(0)$
i.e. $\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} + \left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{22} = 0$, similarly to the solution of the Lax pair in [3]

In fact we know that $\det \Phi_n(z) = 1$ and $\det \Phi_n(z) = P_n(z)\tilde{B}_{n-1}(z) - \tilde{P}_{n-1}(z)B_n(z)$
we have:

$$\begin{aligned} 0 &= \left(\det \Phi_n(z)\right)' = \left(P_n(z)\tilde{B}_{n-1}(z) - \tilde{P}_{n-1}(z)B_n(z)\right)' \\ \Rightarrow 0 &= P_n'(z)\tilde{B}_{n-1}(z) + P_n(z)\tilde{B}'{}_{n-1}(z) - \tilde{P}'{}_{n-1}(z)B_n(z) - \tilde{P}_{n-1}(z)B_n'(z) \\ \Rightarrow 0 &= \underbrace{P_n'(0)\tilde{B}_{n-1}(0) - \tilde{P}_{n-1}'(0)B_n(0)}_{=-H_V} + \underbrace{P_n(0)\tilde{B}_{n-1}'(0) - \tilde{P}_{n-1}(0)B_n'(0)}_{=H_V} \end{aligned}$$

.

■

Proposition 2.6 *In [26] (p.443), the Hamiltonian along with the parameters θ_0, θ_1 and θ_∞ is used to construct the sigma function as follows:*

$$\sigma(x) = xH_V + \frac{1}{2}(\theta_0 + \theta_\infty) + \frac{1}{4}[(\theta_0 + \theta_\infty)^2 - \theta_1]^2$$

The constructed sigma function satisfies the sigma-form of the PV equation (2.17).

2.2 Expressions for the Hamiltonian

We will now find the corresponding expressions of the Hamiltonian by using the expansion of the solution $\hat{Q}_n(z)$ (2.18) of our Lax pair (also related to the PV equation) that we constructed.

2.2.1 Finding the expression of the Hamiltonian in terms of the moment functions

The following proposition is simply a restatement of Prop. 2.5 for convenience.

Proposition 2.7 *From (2.28) and from the Proposition 2.5, the expression of the Hamiltonian is the opposite of the* $(1,1)$ *entry of the matrix*

$$\hat{Q}_{n,1}(0) = Q_{n,1}(\infty) = \Phi_n^{-1}(0)\Phi_n'(0) - \frac{\theta\sigma_3}{2}$$

$$i.e. \;\; H_V = -\left[\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} - \frac{\theta}{2}\right] \tag{2.36}$$

With the aid of Prop. 2.7 we now express the Hamiltonian H_V in terms of the moment functions. Recall that the solution of the RHP on Φ is (see Prop. 1.6)

$$\Phi_n(z) = \begin{bmatrix} P_n(z) & B_n^{(1)}(z) \\ \tilde{P}_{n-1}(z) & B_{n-1}^{(2)}(z) \end{bmatrix} = \begin{bmatrix} P_n(z) & \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{P_n(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w \\ \tilde{P}_{n-1}(z) & \dfrac{1}{2\pi i}\displaystyle\oint_\gamma \frac{\tilde{P}_{n-1}(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w \end{bmatrix}$$

Proposition 2.8 *From relation (2.36) and (2.40) the expression of the Hamiltonian is:*

$$H_V = -\left[\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} - \frac{\theta}{2}\right] = -\left[\frac{D_n'}{D_n} - \frac{\theta}{2}\right] = -\left[\partial_x \ln(D_n) - \frac{\theta}{2}\right] \tag{2.37}$$

Proposition 2.9 *The expansion at* 0, 1 *and* ∞ *of the solution of the constructed Lax Pair provides as follow the Hamiltonian, and the parameters needed to find the Tau function which solves the ODE Painleve V*

$$\begin{aligned} H_V &= -\left[\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} - \frac{\theta}{2}\right] = -\left[\frac{D_n'}{D_n} - \frac{\theta}{2}\right] = -\left[\partial_x \ln(D_n) - \frac{\theta}{2}\right] \\ \theta_0 &= -(\theta + m) \\ \theta_1 &= \theta \\ \theta_\infty &= -(2n - m) \end{aligned}$$

Proof.

We start with the computation of $\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11}$.x Recall that:

$$P_n(z) = \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ 1 & z & z^2 & \dots & z^n \end{vmatrix}}{D_n} \tag{2.38}$$

Also $P_n'(0)$ corresponds to the coefficient (cofactor) of z and we have,

$$P_n'(0) = \underbrace{(-1)^{n+2}}_{=(-1)^n} \frac{\begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix}}{D_n}$$

Recall that $\oint_\gamma w^i w^j \underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} = \mu_{i+j}$ and,

$$\left\langle w^{n-1}, \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \\ 1 & w & w^2 & \dots & w^{n-1} \end{vmatrix}}{D_{n-1}} \right\rangle = \oint_\gamma w^{n-1} P_{n-1}(w) \underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} := h_{n-1}.$$

Since

$$-\frac{1}{2\pi i}\oint_\gamma w^{n-1}\tilde{P}_{n-1}(w)\underbrace{\mathrm{e}^{-V(w)}\mathrm{d}w}_{\mathrm{d}\eta} = 1 \Longrightarrow -\frac{1}{2\pi i}\oint_\gamma w^{n-1}\underbrace{\frac{\tilde{P}_{n-1}}{D_{n-1}}}_{=P_{n-1}}\mathrm{d}\eta = \frac{1}{D_{n-1}}$$

$$\Longrightarrow h_{n-1} \equiv \oint_\gamma w^{n-1}P_{n-1}\mathrm{d}\eta = \frac{-2\pi i}{D_{n-1}}$$

So, $P_{n-1} = \dfrac{\tilde{P}_{n-1}}{D_{n-1}} \Rightarrow \tilde{P}_{n-1} = P_{n-1}D_{n-1} = \dfrac{-2\pi i P_{n-1}}{h_{n-1}} = \dfrac{-2\pi i D_{n-1}}{D_n}P_{n-1}$

Hence,

$$\tilde{P}_{n-1}(z) = -2\pi i\frac{D_{n-1}}{D_n} = \underbrace{\frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \\ 1 & z & z^2 & \dots & z^{n-1} \end{vmatrix}}{D_{n-1}}}_{=P_{n-1}}$$

which gives

$$\tilde{P}_{n-1}(z) = -2\pi i\frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \\ 1 & z & z^2 & \dots & z^{n-1} \end{vmatrix}}{D_n}$$

and $\tilde{P}'_{n-1}(0)$ corresponds to the coefficient (cofactor) of z and we have,

$$\tilde{P}'_{n-1}(0) = \frac{(-1)^{n+1}(-2\pi i)}{D_n}\begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix}$$

Recall that

$$B_n(0) = \frac{1}{2\pi i}\oint_\gamma \frac{P_n(w)\mathrm{e}^{-V(w)}}{w-z}\mathrm{d}w\bigg|_{z=0} = \frac{1}{2\pi i}\oint_\gamma \frac{P_n(w)\overbrace{\mathrm{e}^{-V(z)}\mathrm{d}w}^{=\mathrm{d}\eta}}{w} = \frac{1}{2\pi i}\langle P_n(w), w^{-1}\rangle$$

$$= \frac{1}{2\pi i}\oint_\gamma w^{-1}\sum_{j=0}^{n} A_j w^j \mathrm{d}\eta = \frac{1}{2\pi i}\sum_{j=0}^{n} A_j \oint_\gamma w^{j-1}\mathrm{d}\eta = \frac{1}{2\pi i}\sum_{j=0}^{n}\mu_{j-1}A_j$$

where A_j is the cofactor of z^j in the determinant in relation 2.38

$$= \frac{1}{2\pi i} \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-1} \end{vmatrix}}{D_n}$$

and similarly,

$$\tilde{B}_{n-1}(0) = \frac{1}{2\pi i}\langle \tilde{P}_{n-1}(w), w^{-1}\rangle = \frac{-2\pi i}{2\pi i} \frac{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \\ \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-2} \end{vmatrix}}{D_n}$$

Finally

$$\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} = P_n'(0)\tilde{B}_{n-1}(0) - \tilde{P}_{n-1}'(0)B_n(0)$$

$$= \frac{(-1)^{n+1}}{D_n^2} \begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix} \begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \\ \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-2} \end{vmatrix}$$

$$- \frac{(-1)^{n+2}}{D_n^2} \begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix} \begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \\ \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-1} \end{vmatrix}$$

$$
= \underbrace{\frac{(-1)^{2n}}{D_n^2}}_{\frac{1}{D_n^2}} \begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix} \underbrace{\begin{vmatrix} \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-2} \\ \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix}}_{\text{the last row is moved to the first and produced } (-1)^{n-1}}
$$

$$
\underbrace{-\frac{(-1)^{2n+2}}{D_n^2}}_{-\frac{1}{D_n^2}} \begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix} \underbrace{\begin{vmatrix} \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix}}_{\text{the last row is moved to the first and produced } (-1)^{n}}
$$

$$
= \frac{1}{D_n^2} \Bigg(\underbrace{\begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix}}_{=M^{[1][2]}} \underbrace{\begin{vmatrix} \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-2} \\ \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_{n-1}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix}}_{M^{[n+1][n+1]}}
$$

$$
- \underbrace{\begin{vmatrix} \mu_0(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-3}(x) \end{vmatrix}}_{=M^{[1,n+1][2,n+1]}} \underbrace{\begin{vmatrix} \mu_{-1} & \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-1}(x) \end{vmatrix}}_{:=M} \Bigg)
$$

Where $M^{[a,b][c,d]}$ is the determinant of the matrix M without the rows a,b and the columns c and d.
The Desnanot identity [6] suggests that :

$$
M^{[a,b][c,d]} \cdot M = M^{[a][c]} \cdot M^{[b][d]} - M^{[a][d]} \cdot M^{[b][c]} \tag{2.39}
$$

In our case we get

$$\left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} = \frac{1}{D_n^2}\left(M^{[1][2]} \cdot M^{[n+1][n+1]} - M^{[1,n+1][2,n+1]} \cdot M\right)$$
$$= \frac{1}{D_n^2}\left(M^{[1][2]} \cdot M^{[n+1][n+1]} - M^{[1][2]} \cdot M^{[n+1][n+1]} + M^{[1][n+1]} \cdot M^{[n+1][2]}\right)$$
$$= \frac{1}{D_n^2} M^{[1][n+1]} \cdot M^{[n+1][2]}$$
$$= \frac{1}{D_n^2} \underbrace{\begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-2}(x) \end{vmatrix}}_{=D_n} \begin{vmatrix} \mu_{-1} & \mu_1 & \dots & \mu_{n-1} \\ \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-2}(x) \end{vmatrix}$$

$$\Longrightarrow \left(\Phi_n^{-1}(0)\Phi_n'(0)\right)_{11} = \frac{D_n'}{D_n} \tag{2.40}$$

To prove the relation (2.40), let us notice that

$$\mu_j'(x) = \oint_\gamma \frac{d}{dx} w^j \underbrace{\mathrm{e}^{-(\frac{x}{w} - \theta \ln(1-\frac{1}{w}))} \mathrm{d}w}_{\mathrm{d}\eta} = \oint_\gamma w^{j-1} \underbrace{\mathrm{e}^{-(\frac{x}{w} - \theta \ln(1-\frac{1}{w}))} \mathrm{d}w}_{\mathrm{d}\eta} = \mu_{j-1} \tag{2.41}$$

and also,

$$D_n' = \begin{vmatrix} \mu_0'(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1'(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}'(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-2}(x) \end{vmatrix} + \begin{vmatrix} \mu_0(x) & \mu_1'(x) & \mu_2(x) & \dots & \mu_{n-1}(x) \\ \mu_1(x) & \mu_2'(x) & \mu_3(x) & \dots & \mu_n(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n'(x) & \mu_{n+1}(x) & \dots & \mu_{2n-2}(x) \end{vmatrix}$$
$$+ \dots\dots + \begin{vmatrix} \mu_0(x) & \mu_1(x) & \mu_2(x) & \dots & \mu_{n-1}'(x) \\ \mu_1(x) & \mu_2(x) & \mu_3(x) & \dots & \mu_n'(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(x) & \mu_n(x) & \mu_{n+1}(x) & \dots & \mu_{2n-2}'(x) \end{vmatrix}$$
$$= \begin{vmatrix} \mu_{-1} & \mu_1 & \dots & \mu_{n-1} \\ \mu_0(x) & \mu_2(x) & \dots & \mu_n(x) \\ \mu_1(x) & \mu_3(x) & \dots & \mu_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-2}(x) & \mu_n(x) & \dots & \mu_{2n-2}(x) \end{vmatrix}$$

All the terms in the first equality above equal 0 because the corresponding determinants contain two identical columns, except the first term, which proves relation (2.40).

■

2.3 Solving the ODE Painleve V by exploiting the solution of the constructed Lax Pair

In [26], App. C, the Hamiltonian H_V is used to construct the Tau function, which in turn, solves the ODE Painleve V. In our case, in order to verify that the Lax Pair that is constructed is a good alternative to the Lax Pair in [3], we will construct its corresponding Tau function and confirm that it solve the ODE Painleve V.
Recalling the results in [3]:
The solution of their Lax Pair at ∞ has the following expansion:

$$Q_n(w) = \left[\mathbb{I}_{2\times 2} + Q_{n,1}(\infty)\frac{1}{w} + \mathcal{O}(\frac{1}{w^2})\right]\mathrm{e}^{\tilde{T}(w)} \tag{2.42}$$

Proposition 2.10 *The sigma function $\sigma(x)$ expressed as follows:*

$$\sigma(x) = xH_V + \frac{1}{2}(\theta_0 + \theta_\infty) + \frac{1}{4}[(\theta_0 + \theta_\infty)^2 - \theta_1]^2$$

solves the ODE Painleve V

Computations with the software Maple confirms that the Tau function satisfies the ODE Painleve V (see annexe)

$$\begin{aligned}\Big(x\sigma"(x)\Big)^2 = &\Big[\sigma(x) - x\sigma'(x) + \Big(2\sigma'(x)\Big)^2 + \upsilon_0 + (\upsilon_1 + \upsilon_2 + \upsilon_3)\sigma'(x)\Big]^2 \\ &- 4\Big(\upsilon_0 + \sigma'(x)\Big)\Big(\upsilon_1 + \sigma'(x)\Big)\Big(\upsilon_2 + \sigma'(x)\Big)\Big(\upsilon_3 + \sigma'(x)\Big)\end{aligned}$$

with $\upsilon_0 = 0$, $\upsilon_1 = -\dfrac{\theta_\infty - \theta_1 + \theta_0}{2}$, $\upsilon_2 = -\theta_\infty$ and $\upsilon_3 = -\dfrac{\theta_\infty + \theta_1 + \theta_0}{2}$

The zeroes of the Hankel determinant are the points of non-analyticity (the poles) of the Tau function, rational solution of the ODE Painleve V. On the plot below we can observe that these zeroes seem to form a well shaped region on the complex plane.

```
> Zeros:= [fsolve(DD(N),complex)]:
> complexplot(Zeros/N,style=point);
```

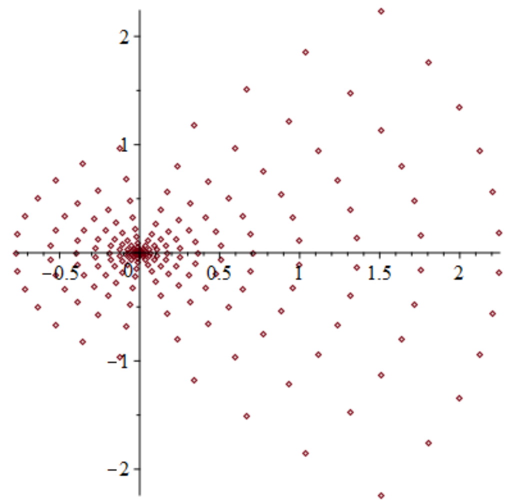

Figure 2: Plot of the Zeroes of Hankel Determinant corresponding to the poles the solution of ODE PV

The next Chapter we will discuss the asymptotic analysis of the shape of this region (its boundaries) as the size the Hankel matrix increases.

3 Asymptotic analysis of the poles of the σ function and Hamiltonian

In this part we adapt the Deift-Zhou steepest descent analysis [22, 21] to the analysis of the behaviour of the pole positions of σ, which by Prop. 2.9 correspond to the zeros of the polynomials $D_n(x)$. We are interested in their asymptotic location as $n \to \infty$ in terms of a scaled variable $s = n\,x$, in terms of which the zeros will be seen populating an asymptotically bounded region which we term "Eye of the Tiger" (EoT) and which is depicted in Fig. 3.

The goal is to explain the pattern of zeros computed numerically and shown as red dots in the Figures 21, 23, 24.

We will be able also to give an asymptotic estimate of the precise location of the zeros inside the EoT in terms of an implicit equation involving Jacobi's theta functions and elliptic integrals, see Theorem 4.26.

3.1 Construction of the g-function and transformations of the problem

Let us recall that the 2×2 matrix valued function $\Phi(z)$, solution of the $RHP\Phi$ (see Sect. 1.3.1) must satisfy the following conditions:

0) The matrices $\Phi(z)$ and $\Phi(z)^{-1}$ are defined and holomorphic in $\mathbb{C} \setminus \gamma$;

1) the matrix $\Phi(x)$ satisfies the boundary value condition

$$\Phi(z_+) = \Phi(z_-)\begin{bmatrix} 1 & \mathrm{e}^{-V(z)} \\ 0 & 1 \end{bmatrix}$$

where $\mathrm{e}^{-V(z)} \equiv W(z)$ is related to the measure of orthogonality by $\mathrm{d}\eta(x,z) = \mathrm{e}^{-V(z)}\mathrm{d}z$; $W(z)$ is called the weight. In our case

$$\mathrm{e}^{-V(z)} = \left(1 - \frac{1}{z}\right)^{\theta} z^{-m}\mathrm{e}^{-\dfrac{x}{z}} = \left(1 - \frac{1}{z}\right)^{\theta} z^{-m}\mathrm{e}^{-\dfrac{x}{z}} \tag{3.1}$$

Scaling regime. In our analysis we will set $x = -ns$ (the sign is of convenience only), and study the asymptotic behaviour of the RHP of Sect. 1.3.1 as $n \to \infty$ while s is kept fixed.

This means that we can write the weight function as follows:

$$\frac{\mathrm{d}\mu}{\mathrm{d}z} = z^{-m}\left(1 - \frac{1}{z}\right)^{\theta}\mathrm{e}^{\dfrac{ns}{z}} = z^{-m}\left(1 - \frac{1}{z}\right)^{\theta}\mathrm{e}^{nV(z)} \tag{3.2}$$

where we have defined $V(z) = \frac{s}{z}$.

So the boundary value condition becomes

$$\Phi(z_+) = \Phi(z_-)\begin{bmatrix} 1 & z^{-m}\left(1 - \dfrac{1}{z}\right)^{\theta}\mathrm{e}^{\dfrac{ns}{z}} \\ 0 & 1 \end{bmatrix} \tag{3.3}$$

We will make the following stipulation on the range of parameters m, θ.

Assumption 3.1 *We assume that the parameters m, θ satisfy:*

1. $m \in \mathbb{Z}$;

2. $\theta \in \mathbb{C} \setminus \mathbb{Z}$*.*

Furthermore, let us define for later convenience

$$Q(z) = z^{-m}\left(1 - \frac{1}{z}\right)^{\theta}. \tag{3.4}$$

Given our Assumptions 3.1 we see that $Q(z)$ has a branch cut that extends on an arc between 0 and 1. Moreover the fact that $\theta \notin \mathbb{Z}$ expresses that we need more than on sheet on the $z-$ plane to obtain "one copy" of the function $Q(z)$ on its plane, hence the branch cut. "Graphically", that branch cut does not intersect γ because the latter must enclose the branch points 0 and 1. We will call the branch cut ε.

2) As $z \to \infty$, for some $n \in \mathbb{N}$, the matrix $\Phi(z)$ has the Laurent series expansion of z^{-1} of the form: $\Phi(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right] z^{n\sigma_3} \equiv \Phi_n(z)$

As $n \to \infty$, this condition stipulates that the solution $\Phi_n(z)$ has a singularity at $z = \infty$. In order to have a version of condition 2) without a singularity as $n \to \infty$. Following the general strategy that was pioneered in [21] we now set out to transform the original RHP into a sequence of equivalent RHP's, the last of which will be amenable to an asymptotic analysis where the error terms can be estimated.

3.1.1 The g–function.

The first transformation requires the construction of a "normalizing function" which is commonly referred to as the g–function.

Definition 3.2 (The g–function and its properties) *The g–function is a locally bounded analytic function on $\mathbb{C} \setminus \Gamma$ where Γ is a union of oriented contours (to be determined) extending to infinity satisfying the properties listed hereafter.*

1. *The contour Γ can be written as $\Gamma = \Gamma_m \cup \Gamma_c \cup \Gamma_\infty$ (with Γ_m denoting the "main arc(s)", and Γ_c the "complementary arc(s)") where each of the components have pairwise disjoint relative interiors and both Γ_m, Γ_c consist of a finite union of compact arcs: $\Gamma_{\{m,c\}} = \bigsqcup \Gamma^{(j)}_{\{m,c\}}$. Finally Γ_∞ is a simple contour extending to infinity from a finite point, traversing eventually the negative real axis and oriented from infinity.*

2. *the contour $\gamma = \{|z| = R, R > 1\}$ can be homotopically retracted to $\Gamma_m \cup \Gamma_c$ in $\mathbb{C} \setminus [0,1]$, where $[0,1]$ here denotes a smooth simple arc connecting $z = 0, 1$ (not necessarily the straight segment).*

3. *for each $z \in \Gamma_m \cup \Gamma_c$ we have*

$$g(z_+) + g(z_-) = -\theta_0(z) - \ell + i\varpi_j, \qquad \varpi_j \in \mathbb{R}, \qquad z \in \Gamma^{(j)}_m \tag{3.5}$$

$$g(z_+) - g(z_-) = i\widehat{\varpi}_j, \qquad \widehat{\varpi}_j \in \mathbb{R}, \qquad z \in \Gamma^{(j)}_c \tag{3.6}$$

for some constants $\varpi_j, \widehat{\varpi}_j$ (different on each of the connected components[2] of $\Gamma_m,\ \Gamma_c$), while

$$g(z_+) - g(z_-) = 2i\pi, \quad z \in \Gamma_\infty. \tag{3.7}$$

[2]We will use different notation in the specific cases we discuss below.

4. as $z \to \infty$ in $\mathbb{C} \setminus \Gamma$ we have

$$g(z) = \ln(z) + \mathcal{O}(z^{-1}). \tag{3.8}$$

5. the g–function satisfies the following inequalities:

(i) for all $z \in \Gamma_c$ we have

$$\mathrm{Re}\left(g(z_+) + g(z_-) + \frac{s}{z} + \ell\right) \leq 0 \tag{3.9}$$

with the equality holding only at the endpoints of each component of Γ_c and possibly at isolated points within the relative interior of Γ_c;

(ii) for $z \in \Gamma_m$ we have

$$\mathrm{Re}\,(g(z_+) + g(z_-)) = -\mathrm{Re}\left(\frac{s}{z} + \ell\right). \tag{3.10}$$

(iii) the inequality below

$$\mathrm{Re}\left(g(z_+) + g(z_-) + \frac{s}{z} + \ell\right) \geq 0 \tag{3.11}$$

holds in an open neibourhood U of Γ_m with the equality holding only on Γ_m itself.

A useful auxiliary function is the "**effective potential**"

$$\varphi(z) = \frac{s}{z} + 2g(z) + \ell \tag{3.12}$$

We will see that Γ is a branch cut of $\varphi(z)$. The g–function will be constructed later in Prop. 3.6 and in Def. 4.2

3.1.2 First transformation: $\Phi \longrightarrow W$

Assuming that a suitable g–function has been constructed according to the Def. 3.2, we proceed with the first transformation.

Definition 3.3

$$W(z) := \mathrm{e}^{n\frac{\ell}{2}\sigma_3}\Phi(z)\mathrm{e}^{-n\left(g(z)+\frac{\ell}{2}\right)\sigma_3}$$

where $\ell \in \mathbb{C}$ and g have the properties in Def. 3.2.

As a consequence of the properties of g in Def. 3.2 and of the initial RHP satisfied by Φ in Sect. 1.3.1, the 2×2 matrix valued function $W(z)$ defined in 3.3 is a solution of the *RHPW* specified below.

Proposition 3.4 *The 2×2 matrix valued function $W(z)$, solution of the RHPW satisfies the following conditions:*

0) *$W(z)$ is defined and holomorphic in $\mathbb{C}-\gamma$;*

1) $\forall \quad z \quad \in \quad \gamma$;

$$W(z_+) = W(z_-)\begin{bmatrix} e^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & Q(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & e^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}$$

2) *As* $\quad z \to \infty, \quad W(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$

Proof.

0) $W(z)$ is defined and holomorphic in $\mathbb{C}-\gamma$, in particular because in condition I), we have that g is analytic on $\mathbb{C}-\Gamma$ and $\Gamma \subset \gamma$

1) $Q(z_+) = Q(z_-)$ since Q does not have a branch cut on γ as stated before, obviously $e^{\frac{ns}{z}}$ also does not, thus

$$\begin{aligned}
W(z_+) &= e^{n\frac{l}{2}\sigma_3}\underbrace{\Phi(z_-)\begin{bmatrix} 1 & Q(z)e^{\frac{ns}{z})} \\ 0 & 1 \end{bmatrix}}_{\Phi_+(z)} e^{-n\left(g(z_+)+\frac{l}{2}\right)\sigma_3} \\
&= e^{n\frac{l}{2}\sigma_3}\Phi(z_-)\,\overbrace{e^{-n\left(g(z_-)+\frac{l}{2}\right)\sigma_3}e^{n\left(g(z_-)+\frac{l}{2}\right)\sigma_3}}^{=\mathbb{I}_{2\times 2}}\begin{bmatrix} 1 & Q(z)e^{\frac{ns}{z})} \\ 0 & 1 \end{bmatrix} e^{-n\left(g(z_+)+\frac{l}{2}\right)\sigma_3} \\
&= \overbrace{W(z_-)}^{e^{n\frac{l}{2}\sigma_3}\Phi(z_-)e^{-n\left(g(z_-)+\frac{l}{2}\right)\sigma_3}} e^{n\left(g(z_-)+\frac{l}{2}\right)\sigma_3}\begin{bmatrix} 1 & Q(z)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} e^{-n\left(g(z_+)+\frac{l}{2}\right)\sigma_3} \\
&= W(z_-)\begin{bmatrix} e^{n\left(g(z_-)-g(z_+)\right)} & Q(z)e^{\frac{ns}{z}}e^{n\left(g(z_-)+g(z_+)+l\right)} \\ 0 & e^{-n\left(g(z_-)-g(z_+)\right)} \end{bmatrix} \\
&= W(z_-)\begin{bmatrix} e^{n\left(g(z_-)-g(z_+)\right)} & Q(z)e^{\frac{ns}{z}}e^{n\left(g(z_-)+g(z_+)+l\right)} \\ 0 & e^{-n\left(g(z_-)-g(z_+)\right)} \end{bmatrix}
\end{aligned}$$

On the other hand $\varphi(z) = \frac{s}{z} + 2g(z) + l$, so we have:

$$\begin{cases} g(z_-) - g(z_+) = \frac{1}{2}\Big(\varphi(z_-) - \varphi(z_+)\Big) \\ g(z_-) + g(z_+) = \frac{1}{2}\Big(\varphi(z_-) + \varphi(z_+)\Big) - l - \frac{s}{z} \end{cases}$$

Finally, for condition 1), we obtain

$$W(z_+) = W(z_-)\begin{bmatrix} \mathrm{e}^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & Q(z)\mathrm{e}^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & \mathrm{e}^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}$$

2) $W(z) = \mathrm{e}^{n\frac{l}{2}\sigma_3}\Phi(z)\mathrm{e}^{-n\left(g(z)+\frac{l}{2}\right)\sigma_3}$. As $z \to \infty$,

$$\begin{aligned} W(z) &= \mathrm{e}^{n\frac{l}{2}\sigma_3}\overbrace{\left(\left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]z^{n\sigma_3}\right)}^{=\Phi(z) \quad \text{as} \quad z\to\infty}\mathrm{e}^{-n\left(\overbrace{\ln(z) + \mathcal{O}(z^{-1})}^{=g(z) \quad \text{as} \quad z\to\infty}+\frac{l}{2}\right)\sigma_3} \\ &= \mathrm{e}^{n\frac{l}{2}\sigma_3}\left(\overbrace{\left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]}^{\to\mathbb{I}_{2\times 2} \quad \text{as} \quad z\to\infty}\overbrace{\mathrm{e}^{n\ln(z)\sigma_3}}^{=z^{n\sigma_3}}\mathrm{e}^{-n\ln(z)\sigma_3}\quad\overbrace{\mathrm{e}^{-\mathcal{O}(z^{-1})\sigma_3}}^{\to\mathbb{I}_{2\times 2} \quad \text{as} \quad z\to\infty}\quad\mathrm{e}^{-n\frac{l}{2}\sigma_3}\right. \\ &= \mathrm{e}^{n\frac{l}{2}\sigma_3}\left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]\mathrm{e}^{-n\frac{l}{2}\sigma_3} \end{aligned}$$

Thus, as $\quad z \to \infty, \quad W(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$

■

Notice that after the transformation $\Phi \to W$,the function W as $n \to \infty$ tends to the identity matrix in condition 2) and thus does not have a singularity .
Now that the function φ is being used as a proxy for the function g, Γ, the branch cut of φ will be used as a proxy for the branch cut of g .

3.1.3 Derivation of an appropriate g–function

Definition 3.5 *We define the following effective potential*

$$\varphi(z, s) := \int_{-\frac{is}{2}}^{z} \frac{2}{w^2}\sqrt{w^2 + \frac{s^2}{4}}\mathrm{d}w = 2\ln\left(\frac{2iz}{s} + i\sqrt{\frac{4z^2}{s^2} + 1}\right) - 2\sqrt{1 + \frac{s^2}{4z^2}} \tag{3.13}$$

$$\varphi'(z, s) = \varphi'(z) := \frac{2}{z^2}\sqrt{z^2 + \frac{s^2}{4}}. \tag{3.14}$$

Motivation for the definition above
Let us set $V(z) := -\frac{s}{z}$. From the Definition 3.2 we know that for $z \in \Gamma_m$:

$$g(z_+) + g(z_-) = V(z) - \ell$$

$$\Rightarrow g'(z_+) + g'(z_-) = V'(z)$$

$$\Rightarrow \left(g'(z_+)\right)^2 - \left(g'(z_-)\right)^2 = V'(z) \underbrace{\left(g'(z_+) - g'(z_-)\right)}_{\rho(z)}$$

$$\Rightarrow \left(g'(z)\right)^2 = \frac{1}{2\pi i} \int_\Gamma \frac{V'(\zeta)\rho(\zeta)}{\zeta - z} \mathrm{d}\zeta \qquad \text{(by the Sokhotski–Plemelj formula)}$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{\left(V'(\zeta) \overbrace{-V'(z) + V'(z)\Big)}^{=0} \rho(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

$$= \underbrace{\frac{1}{2\pi i} \int_{\Gamma_m} \frac{\left(V'(\zeta) - V'(z)\right)\rho(\zeta)}{\zeta - z} \mathrm{d}\zeta}_{:=R(z)} + \underbrace{\frac{1}{2\pi i} \int_{\Gamma_m} \frac{V'(z) \quad \overbrace{\rho(\zeta)}^{g'(z_+)-g'(z_-)}}{\zeta - z} \mathrm{d}\zeta}_{=V'(z)g'(z)}$$

(with $z \in \Gamma_m$)

So

$$R(z) + V'(z)g'(z) = \left(g'(z)\right)^2 \Rightarrow R(z) = \left(g'(z)\right)^2 - V'(z)g'(z) + \overbrace{\left(\frac{V'(z)}{2}\right)^2 - \frac{\Big[V'(z)\Big]^2}{4}}^{=0}$$

$$\Rightarrow \left[g'(z) - \frac{V'(z)}{2}\right]^2 = R(z) + \frac{\Big[V'(z)\Big]^2}{4} \tag{3.15}$$

Since $\varphi(z) = \overbrace{\frac{s}{z}}^{=-V(z)} + 2g(z) + \ell$, we have $\frac{1}{2}\varphi'(z) = g'(z) - \frac{V'(z)}{2}$,
hence,

$$(3.15) \Rightarrow \frac{1}{4}\Big(\varphi'(z)\Big)^2 = R(z) + \frac{\Big[V'(z)\Big]^2}{4} \tag{3.16}$$

So, in (3.16), the term $\frac{\Big[V'(z)\Big]^2}{4}$ does not have a jump and is a rational function.

Similarly, we will prove that $R(z)$ does not have a jump discontinuity and is itself at most a rational function

$$R(z+) - R(z-) = \frac{1}{2\pi i}\int_{\Gamma_m} \frac{\Big(V'(\zeta) - V'(z_+)\Big)\rho(\zeta)}{\zeta - z_+}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{\Gamma_m} \frac{\Big(V'(\zeta) - V'(z_-)\Big)\rho(\zeta)}{\zeta - z_-}\mathrm{d}\zeta$$

$$= 0 \quad ,$$

(Since $V'(z)$ does not have a jump in particular on Γ_m, $V'(z_+) = V'(z_-) = V'(z)$, and similarly $\frac{1}{\zeta - z_+} = \frac{1}{\zeta - z_-} = \frac{1}{\zeta - z}$ on Γ_m).

It follows that $R(z_+) = R(z_-)$. In conclusion, $R(z)$ and $\frac{\Big[V'(z)\Big]^2}{4}$ are at most rational functions, therefore (3.16) implies that $\frac{1}{4}\Big(\varphi'(z)\Big)^2$ is also a rational function of z. Therefore (3.16) $\Longrightarrow$

$$\frac{1}{4}\Big(\varphi'(z)\Big)^2 = \frac{1}{2\pi i}\int_{\Gamma_m} \frac{\Big(\frac{s}{\zeta^2} - \frac{s}{z^2}\Big)\rho(\zeta)}{\zeta - z}\mathrm{d}\zeta + \frac{s^2}{4z^4} = \frac{s}{2\pi i}\int_{\Gamma_m} \underbrace{\frac{-(\zeta^2 - z^2)}{\zeta^2 z^2(\zeta - z)}}_{=\frac{-\zeta - z}{\zeta^2 z^2}}\rho(\zeta)\mathrm{d}\zeta + \frac{s^2}{4z^4}$$

$$= \frac{s^2}{4z^4} - \frac{s}{z^2}\underbrace{\int_{\Gamma_m} \frac{\rho(\zeta)}{2\pi i\zeta}\mathrm{d}\zeta}_{:=A} - \frac{s}{z}\underbrace{\int_{\Gamma_m} \frac{\rho(\zeta)}{2\pi i\zeta^2}\mathrm{d}\zeta}_{:=B}$$

$$\Longrightarrow \Big(\varphi'(z)\Big)^2 = \frac{s^2}{z^4} - \frac{4As}{z^2} - \frac{4Bs}{z} \tag{3.17}$$

The expression above can also be seen as the expansion of $\left(\varphi'(z)\right)^2$ as $z \to \infty$ On the other hand, recall from condition II) that as $z \to \infty$, $\varphi(z) = 2\ln(z) + \mathcal{O}(\frac{1}{z})$

$\Rightarrow$ as $z \to \infty$, $\left(\varphi'(z)\right)^2 = \frac{4}{z^2} + \mathcal{O}(\frac{1}{z^4})$

By identification, we obtain $A = -\frac{1}{s}$ and $B = 0$ and we can write

$$\left(\varphi'(z)\right)^2 = \frac{s^2}{z^4} + \frac{4}{z^2} \Longrightarrow \varphi'(z) = \frac{2}{z^2}\sqrt{z^2 + \frac{s^2}{4}} \tag{3.18}$$

Proposition 3.6 *The effective potential given in Def. 4.2 defines a g function by $g(z) := \frac{1}{2}\left(-\frac{s}{z} - \ell - \varphi(z,s)\right)$ that satisfies all the conditions of Def. 3.2.*

Proof. A great simplification is achieved by observing that φ is really a function only of $\frac{z}{s}$, namely

$$\varphi(z; s) = \varphi\left(\frac{z}{s}; 1\right), \tag{3.19}$$

and hence it suffices to describe the domain and properties of $\varphi_0(z) := \varphi\left(\frac{z}{s}; 1\right)$ which is given by

$$\varphi_0(z) = 2\ln\left(2iz + i\sqrt{4z^2+1}\right) - 2\sqrt{1 + \frac{1}{4z^2}}, \quad \varphi_0'(z) = \frac{\sqrt{4z^2+1}}{z^2}. \tag{3.20}$$

The determination of the root is chosen such that $\varphi_0'(z) \simeq \frac{1}{z}$ at $z = \infty$, with a branch-cut connecting the branchpoints $\pm\frac{i}{2}$ to be determined below. The language of *vertical trajectories of quadratic differentials* of [43] is useful in this discussion: by definition these are the arcs of curves where $\mathrm{Re}\,\varphi_0$ is constant, which, in the plane of the variable $\xi(z) := \varphi_0(z) = \int^z \varphi_0'(w)\mathrm{d}w$ are (by definition) vertical segments, whence the terminology. We start by observing that $\underset{z=\infty}{\mathrm{res}}\, \varphi_0'(z)\mathrm{d}z = -2i\pi$ and $\underset{z=0}{\mathrm{res}}\, \varphi_0'(z)\mathrm{d}z = \pm 2i\pi$ (with the sign depending on whether the branch-cut leaves $z = 0$ to the left or to the right) and hence, no matter how we choose the branch-cut Γ_m (connecting the branchpoints) we have that

$$\text{the function } \mathrm{Re}\,\varphi_0(z) \text{ is single–valued, harmonic in } \mathbb{C}\setminus\Gamma\cup\{0\} \text{ and continuous in } \mathbb{C}\setminus\{0\}; \tag{3.21}$$

$$\text{for } |z| \text{ sufficiently large } \mathrm{Re}\,\varphi_0(z) = \ln|z| + \text{ harmonic and bounded.} \tag{3.22}$$

The observation (3.22) implies that the level-curves of $\mathrm{Re}\,\varphi_0$ are deformed circles for $|z|$ sufficiently large. One can verify that changing the determination of both radicals in (3.20) has the effect of flipping the sign of $\mathrm{Re}\,\varphi_0$ and hence that $\mathrm{Re}\,\varphi_0$ is a well–defined harmonic function on the Riemann surface of the radical

$$w^2 = 1 + 4z^2. \tag{3.23}$$

Furthermore $\mathrm{Re}\,\varphi_0$ is an odd function under the holomorphic involution that maps (w, z) to $(-w, z)$. This means that the level sets $\mathrm{Re}\,\varphi_0 = 0$ are well defined on the z–plane; they consist in vertical trajectories issuing from the points $\pm\frac{i}{2}$ and forming the pattern illustrated in Fig. 4.

We choose the branch-cut of the radical as the arc of Fig. 4 joining $\pm\frac{i}{2}$ in the right half–plane. With this choice we have that

$$\varphi_0'(z) \simeq -\frac{1}{z^2} + \mathcal{O}(1), \quad z \to 0. \tag{3.24}$$

and in general $\varphi'(z;s) = \frac{1}{s}\varphi_0'\left(\frac{z}{s}\right)$ satisfies

$$\varphi'(z;s) = -\frac{s}{z^2} + \mathcal{O}(1). \tag{3.25}$$

Verification of the properties of φ and range of validity. It suffices to verify the properties for $s = 1$ since changing $s \in \mathbb{C}$ amounts simply to a complex homotethy $z \mapsto sz$. We choose Γ_m as the arc joining $\pm\frac{i}{2}$ in the right plane, and Γ_c as an arc joining $\pm\frac{i}{2}$ in the left plane, and inside the region bounded by the imaginary axis and the contour $-\Gamma_m$ (see Fig. 4). Finally we choose Γ_∞ as the ray $(-i\infty, -\frac{i}{2}]$. We then proceed with the verification of the properties in Corollary **??**:

1. on the sole connected component Γ_m, we have $\varphi_0(z_+) + \varphi_0(z_-) = 0$ since the two boundary values differ by a vanishing period of φ_0';

2. on Γ_c we similarly have $\varphi(z_+) = \varphi(z_-)$;

3. on Γ_∞ we have $\varphi_0(z_+) = \varphi_0(z_-) - \operatorname*{res}_{w=\infty} \varphi_0'(w)\mathrm{d}w = \varphi_0(z_-) + 4i\pi$

4. Since Γ_m is defined as the zero level set of $\operatorname{Re}\varphi_0$, we have $\operatorname{Re}\varphi_0 \equiv 0$ on Γ_m by definition;

5. in the unbounded doubly–connected region outside of the "apricot" in Fig. 4 we have $\operatorname{Re}\varphi_0 = \ln|z| + \mathcal{O}(1)$ near $z \to \infty$; thus inevitably $\operatorname{Re}\varphi_0 > 0$ in the whole region (which, we remind, is bounded by the zero levelsets of $\operatorname{Re}\varphi_0$);

6. In the right hemi-apricot, the sign must be also positive because $\operatorname{Re}\varphi_0(z) = \operatorname{Re}\left(\frac{1}{z} + \mathcal{O}(1)\right)$;

7. by the same token, the sign is negative in the left hemi-apricot.

Thus all conditions except possibly the condition no. 2 in Def. 3.2 are verified, namely, we still need to verify that the union $\Gamma_m \cup \Gamma_c$ is homotopic to a circle $|z| = R$, $R > 1$ in the cut plane $\mathbb{C} \setminus [0, 1]$.

Since the levelsets in Fig. 4 are scaled by s, this latter condition is fulfilled as long as the point $z = 1$ lies inside the re-scaled apricot. This holds clearly for $|s|$ sufficiently large, and it fails precisely when the point $z = 1$ lies on either Γ_m or Γ_c, namely when

$$\operatorname{Re}\varphi(1;s) = 0 = \operatorname{Re}\varphi_0\left(\frac{1}{s}\right). \tag{3.26}$$

■

The set of points on the $s-$plane such that $\operatorname{Re}(\varphi(1,s)) = 0$ i.e. $\{s \in \mathbb{C} : \operatorname{Re}(\varphi(z_0 = 1, s)) = 0\}$ form a closed contour of what we will call the "Eye Of Tiger" (EoT, in short) because of its shape on the $s-$ plane as shown in Fig. 3 Now, let $\Xi_0 = \{z_0 \in \mathbb{C} : \operatorname{Re}(\varphi(z_0, s \in EoT)) = 0\}$. Notice that $z_0 = 1 \in \Xi_0$. Ξ_0 describes an apricot-looking closed contour shape figure as shown in Fig. 4.

In other words, as s moves on the contour of the Eye Of the Tiger on the $s-$plane, z moves on Ξ_0 to satisfy $\operatorname{Re}(\varphi(z_0, s)) = 0$

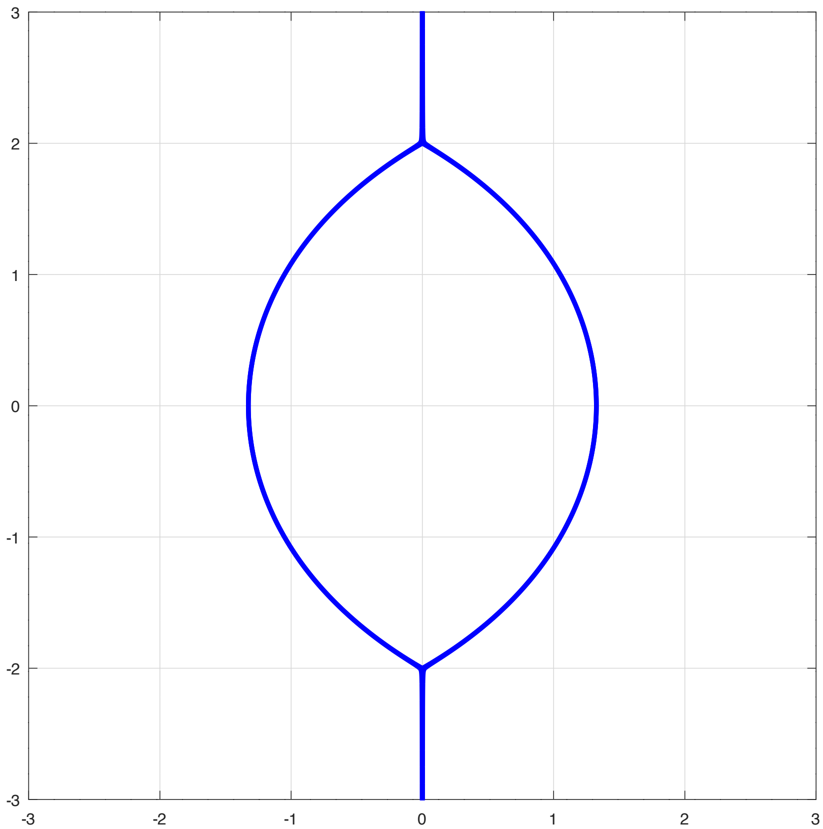

Figure 3: $EoT = \{s \in \mathbb{C} : \mathrm{Re}\,(\varphi(z_0 = 1, s)) = 0\}$

More generally, let us define $\Xi = \{z, s \in \mathbb{C} : \mathrm{Re}\,(\varphi(z, s)) = 0\}$. After we plot on the z plane the curve such that $\mathrm{Re}\,(\varphi(z, s)) = 0$, the change of the shape of the curve as $s \in \mathbb{C}$ varies as follows:

On the picture above:

- As s moves outside of the contour of the Eye Of the Tiger (in the neighbourhood of the EoT) on the $s-$plane, for $\mathrm{Re}\,(\varphi(z, s)) = 0$ to be satisfied , we simply observe the rotation of Ξ_0 which inflates to a generic Ξ.
- As s moves inside of the contour of the Eye Of the Tiger on the $s-$plane, for $\mathrm{Re}\,(\varphi(z, s)) = 0$ to be satisfied , we simply observe the rotation of Ξ_0 which deflates (shrinks) to a generic Ξ.

Depending on $s \in \mathbb{C}$ rotating inside the closed contour of EoT or outside of it, $z = 1$ will be outside or inside of Ξ on the $z-$ plane.
FACT: The contour Ξ contains two points a_+ and a_- which are symmetric with respect to the origin.

FACT : Since $\varphi(z)$ has branch points at $a_- = -i\dfrac{s}{2}$ and $a_+ = i\dfrac{s}{2}$, the contour of Ξ which goes from a_- to a_+ can be taken as a branch cut of $\varphi(z, s)$. Let us call it $\Gamma_{a_-}^{a_+}$.
Thus, $\Gamma_{a_-}^{a_+} \subset \Gamma$ such that $\Gamma = \Gamma_{a_-}^{a_+} \cup \Gamma(\infty)$. To solve the RHP, the jump on γ is our focus in condition 1). So it is convenient to retract γ on Γ_m such that, $\Gamma_{a_-}^{a_+} \subset \Gamma \subset \gamma$. With this configuration, we call $\Gamma_m = \Gamma$ the Main Arc. The contour of γ which is not superposed is called the Complementary arc Γ_c, and it is located on the region of the $z-$ plane where $\mathrm{Re}\,(\varphi(z, s)) < 0$ as pictured below on a

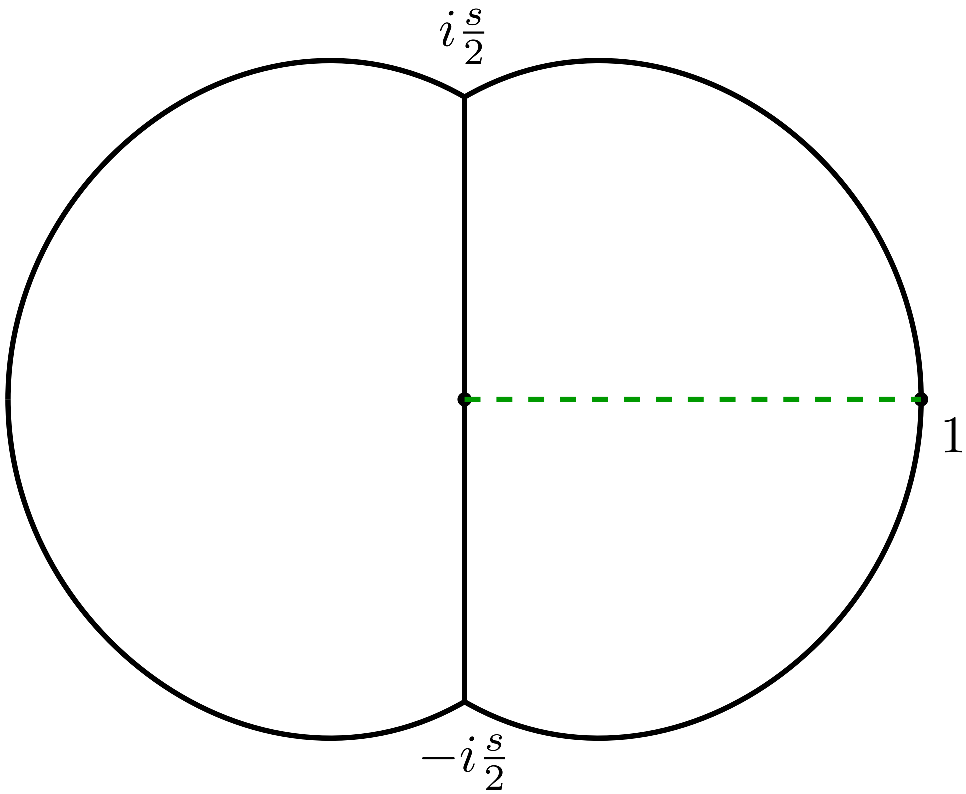

Figure 4: Apricot-shape, $\Xi_0 = \{z_0 \in \mathbb{C} : \mathrm{Re}\,(\varphi(z_0, s \in EoT)) = 0\}$

couple of the Apricot-shape figures as $s \in \mathbb{C}$ moves around the EoT :

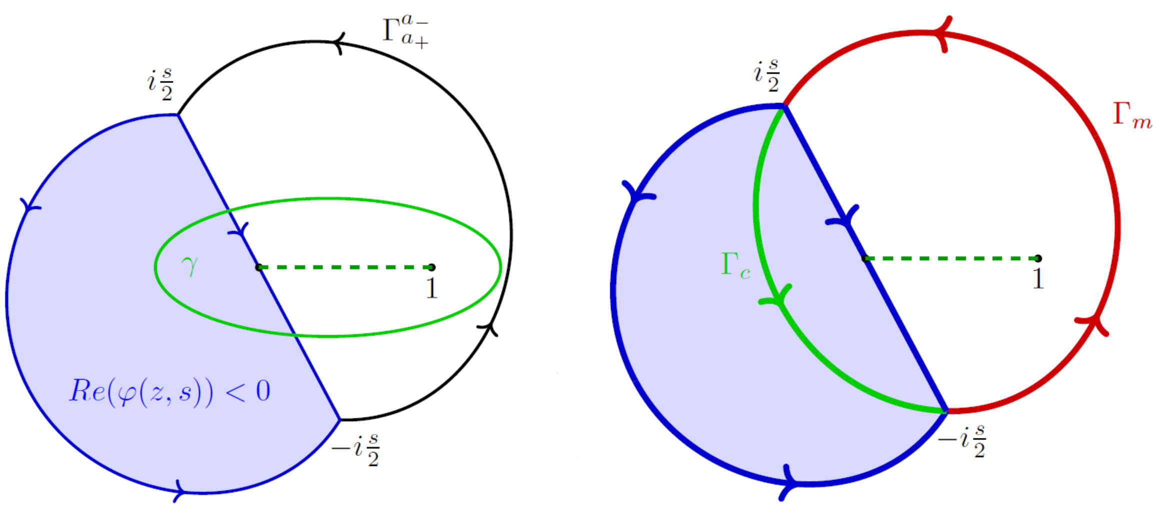

Figure 6: The Apricot-shape for some $s > 0$ on the EoT

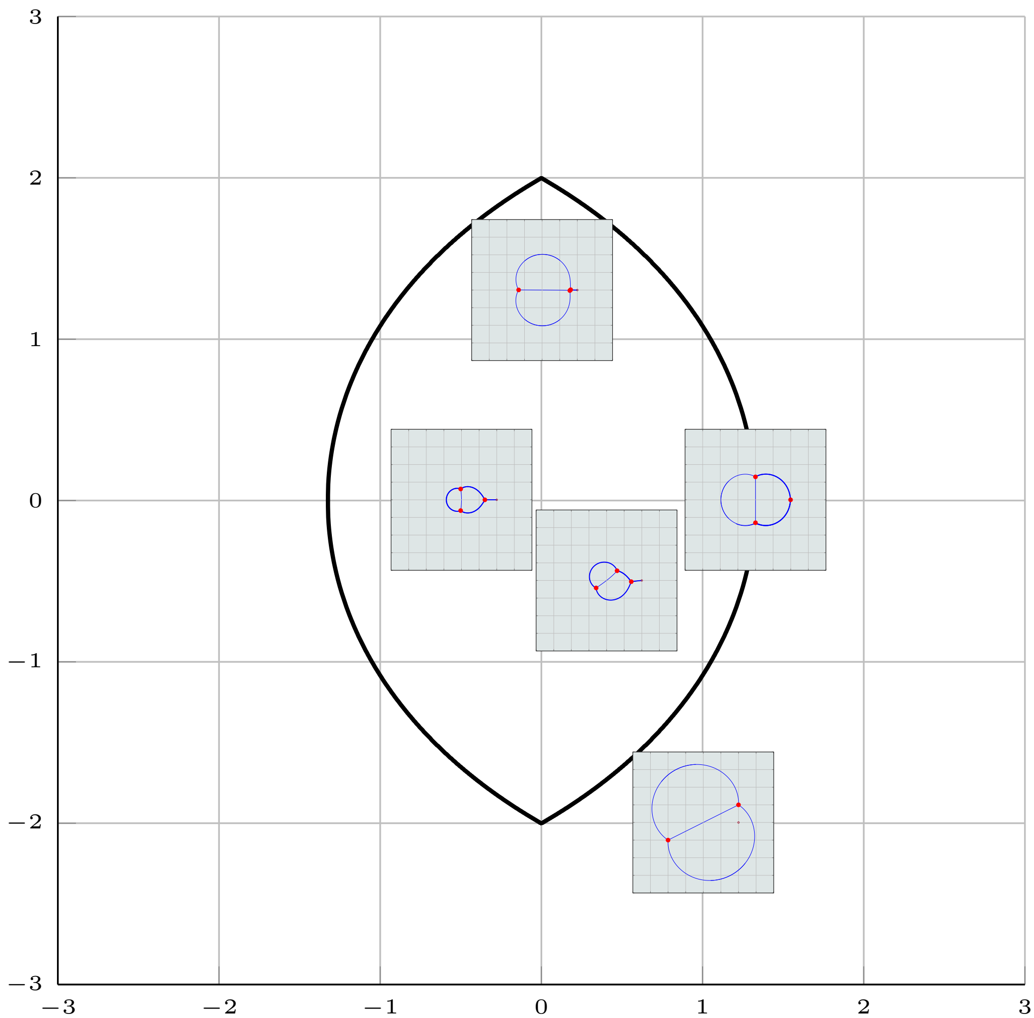

Figure 5: Variation of the Apricot-shape as $s \in \mathbb{C}$ moves around the EoT

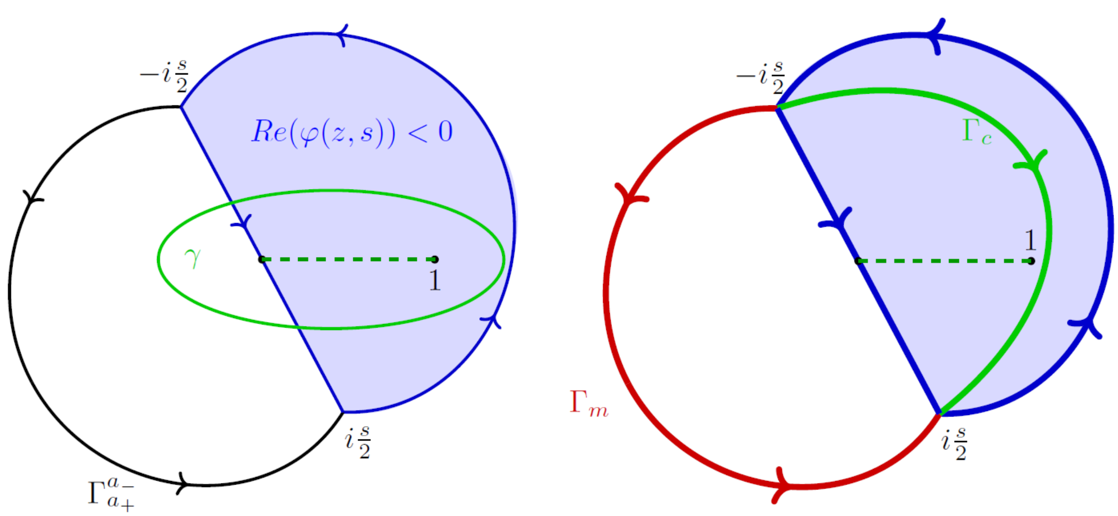

Figure 7: The Apricot-shape for some $s < 0$ on the EoT

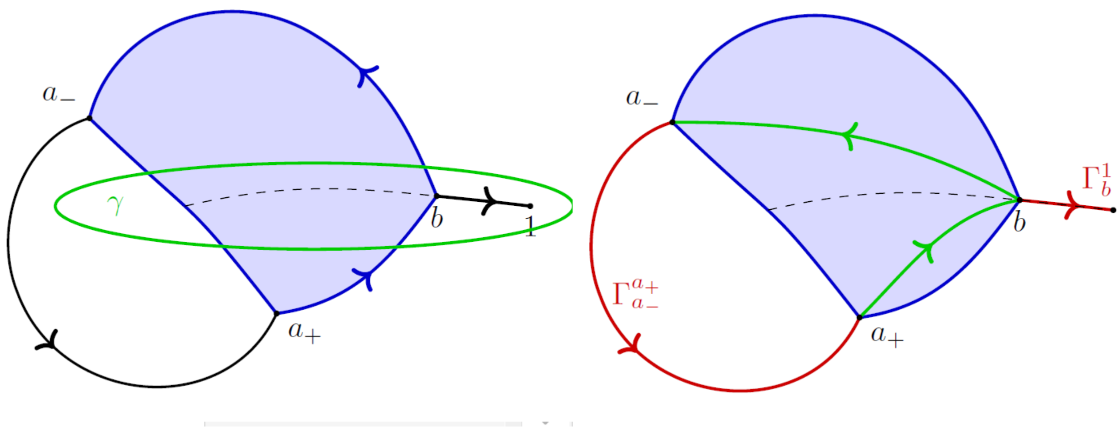

Figure 8: The Apricot-shape for some $s < 0$ inside the EoT

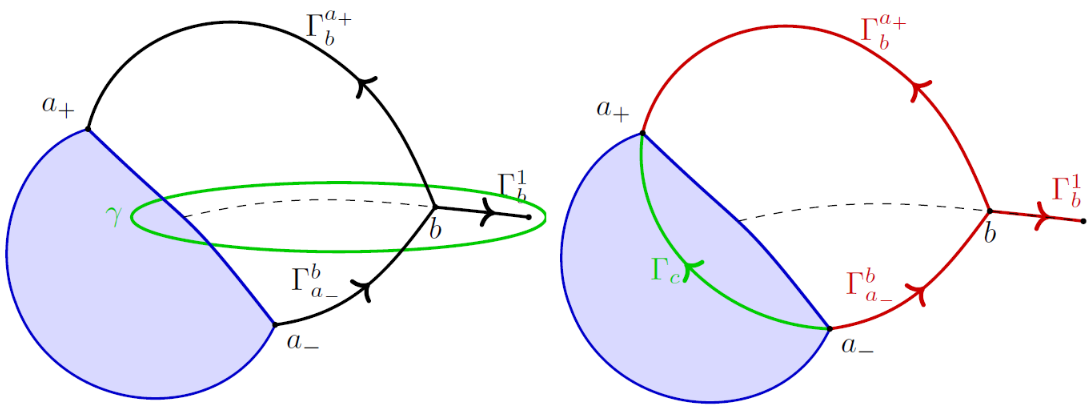

Figure 9: The Apricot-shape for some $s > 0$ inside the EoT

Notice that on figures 8 and 9, the branch cut of Q passes through γ and will generate a jump on it, which forces to redefine the jump condition 1) in the RHPW. This case will be discussed in the subsection 3.3

3.2 Discussion of the solution of RHPW when s is outside the Eye of the Tiger

From the previous discussion we know that φ has a branch cut, hence a jump discontinuity on $\Gamma_m \subset \gamma$. When s is outside the EoT, we typically obtain the figures 6 and 7 on the $z-$ plane.

3.2.1 Refining the first transformation: $\Phi \longrightarrow W$

Proposition 3.7 *The* 2×2 *matrix valued function* $W(z)$ *defined in Def. 3.3, solution of the RHPW satisfies the following conditions:*

0) *$W(z)$ is defined and holomorphic in $\mathbb{C}\setminus\gamma$*

1) *On $\gamma = \Gamma_c \cup \Gamma_m$ we have:*

$$W(z_+) = \begin{cases} W(z_-), & \text{on } \Gamma_c \\ W(z_-) \begin{bmatrix} e^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & Q(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & e^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}, & \text{on } \Gamma_m \end{cases}$$

2) *As $z \to \infty$ we have the asymptotic behaviour*

$$W(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

Proof. Recall the conditions of the first transformation *RHPW* ,$\Phi \longrightarrow W$ written in Prop. 3.4. We retract γ onto $\Gamma = \Gamma_m \cup \Gamma_c$ in $\mathbb{C}\setminus[0,1]$ and re-examine the jump conditions on the two components separately. See Fig. 6 and Fig. 7.

On $\gamma\setminus\Gamma_m = \Gamma_c$. On $\Gamma_c = \gamma\setminus\Gamma_m$, the effective potential φ does has a jump, so $\varphi(z_-)-\varphi(z_+) = 0$ and hence $e^{\pm\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} = 1$.

Moreover, Γ_c is contained in a region of the $z-$ plane where $\mathrm{Re}\left(\varphi(z)\right) < 0$, so

$$\begin{aligned} Q(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} &= Q(z)e^{n\varphi(z)} = Q(z)e^{n\left[\mathrm{Re}\left(\varphi(z)\right)+i\mathrm{Im}\left(\varphi(z)\right)\right]} \\ &= Q(z)\underbrace{e^{n\left[\mathrm{Re}\left(\varphi(z)\right)\right]}}_{\to 0 \quad \mathrm{as} n\to\infty} e^{i\left[n\mathrm{Im}\left(\varphi(z)\right)\right]} = 0 \end{aligned}$$

because $\tilde{Q}(z) \neq \infty$ as $z \neq 0, \infty$ when $z \in \Gamma_c$. Hence for $z \in \Gamma_c$,

$$W(z_+) = W(z_-)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = W(z_-)$$

Therefore we obtain:

$$W(z_+) = \begin{cases} W(z_-), & \text{on } \Gamma_c \\ W(z_-) \begin{bmatrix} e^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & Q(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & e^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}, & \text{on } \Gamma_m \end{cases}$$

■

Proposition 3.8 *From Proposition 3.7, relation 3.29, the jump condition for the matrix W on Γ can be factorized as follows:*

$$W(z_+) = W(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_-)}}{Q(z)} & 1 \end{bmatrix} m_Q \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_+)}}{Q(z)} & 1 \end{bmatrix} \tag{3.27}$$

where

$$m_Q := \begin{bmatrix} 0 & Q(z)\mathrm{e}^{\dfrac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ -\dfrac{\mathrm{e}^{-\dfrac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)}}{Q(z)} & 0 \end{bmatrix} \tag{3.28}$$

Proof.
Let us analyse the jump on Γ_m
The following equality is used to rewrite condition 1) in Proposition 3.7

$$\begin{bmatrix} \mathrm{e}^{a} & \mathrm{e}^{b} \\ 0 & \mathrm{e}^{-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-a-b} & 1 \end{bmatrix}\begin{bmatrix} 0 & \mathrm{e}^{-b} \\ \mathrm{e}^{-b} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{a-b} & 1 \end{bmatrix} \tag{3.29}$$

■

Proposition 3.9 *Using that $\varphi(z_+)+\varphi(z_-)=0$ on Γ_{a-}^{a+}, the jump condition for the transformation W on Γ_{a-}^{a+} can be rewritten as follows:*

$$W(z_+) = W(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_-)}}{Q(z)} & 1 \end{bmatrix}\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_+)}}{Q(z)} & 1 \end{bmatrix}$$

$\Longrightarrow$

$$\underbrace{W(z_+)\begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{e}^{-n\varphi(z_+)}}{Q(z)} & 1 \end{bmatrix}}_{T(z_+)} = \underbrace{W(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_-)}}{Q(z)} & 1 \end{bmatrix}}_{T(z_-)}\underbrace{\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}}_{m_Q} \tag{3.30}$$

Proof. On $\Gamma_{a_-}^{a_+}$ with $-\dfrac{is}{2} = a_-$, and $\dfrac{is}{2} = a_+$

$$\varphi(z_+) = 2\int_{a_-}^{z_+} \frac{\sqrt{w_+^2 + \frac{s^2}{4}}}{w^2}\mathrm{d}w = 2\int_{a_-}^{z_+} \frac{\sqrt{(w_+ - a_-)(w_+ + a_+)}}{w^2}\mathrm{d}w$$

$$\varphi(z_-) = 2\int_{a_-}^{z_-} \frac{\sqrt{w_-^2 + \frac{s^2}{4}}}{w^2}\mathrm{d}w = 2\int_{a_-}^{z_-} \frac{\sqrt{(w_- - a_-)(w_- + a_+)}}{w^2}\mathrm{d}w$$

Remark: It is important to notice that any main arc branch cut can be equivalently considered as a segment branch cut. In fact, since the main arc and the segment must have the same branch points (see figure 11 below), if we denote $\mathcal{D}$ the region between the two, it suffices to "move the cut" by redefining

$$\tilde{\varphi}(z) = \begin{cases} \varphi(z) & \text{if} \quad z \notin \mathcal{D} \\ -\varphi(z) & \text{if} \quad z \in \mathcal{D} \end{cases} \tag{3.31}$$

This redefinition can be used any time the jump discontinuity of a function (in this case $\tilde{\varphi}$) around a segment is of the form $\tilde{\varphi}(z_+) = \tilde{\varphi}(z_-)$

After the arc branch cut has "been moved" to a segment branch cut (a_-, a_+), let $\mid w_+ - a_- \mid := r_- =\mid w_- - a_- \mid$, $\mid w_+ - a_+ \mid := r_+ =\mid w_- - a_+ \mid$.

If $arg(w_+ - a_-) = 0$, then $arg(w_+ - a_+) = \pi$, $arg(w_- - a_-) = 0$, $arg(w_- - a_+) = -\pi$.

Let $r :=\mid w_- \mid=\mid w_+ \mid \Rightarrow w_- = w_+ = r\mathrm{e}^{i0} = r$. See figure 10 below.

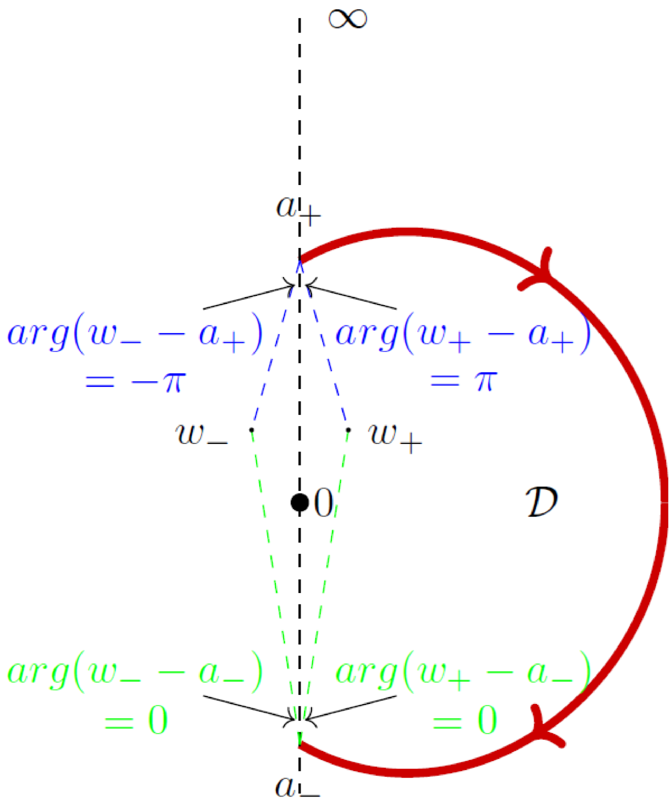

Figure 10: Analytic continuation of φ'.

So, $$\tilde{\varphi}(z_+) = 2\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}\mathrm{e}^{i\frac{0+\pi}{2}}}{w_+^2}dw = 2i\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}}{r^2}dw$$

Similarly, $$\tilde{\varphi}(z_-) = \int_{a_-}^{z_-} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}\mathrm{e}^{i\frac{0-\pi}{2}}}{w^2}dw = -2i\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}}{r^2}dw$$

Therefore $\tilde{\varphi}(z_+) = -\tilde{\varphi}(z_-)$ on the segment branch cut (a_-, a_+).

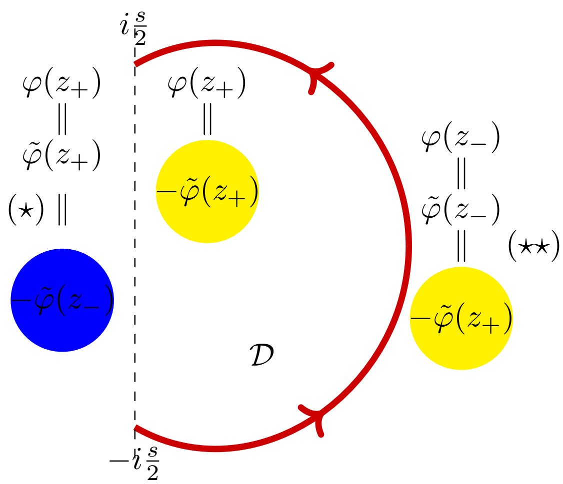

Figure 11: The equality $(\star)$ stems from our computations above. The equality $(\star\star)$ is a consequence of $(\star)$. The other equalities are obtained from the redefinition $\varphi \leftrightarrow \tilde{\varphi}$ above in 3.31. Notice the jump of $-\tilde{\varphi}(z_+)$ across the dashed line (new branch cut) labelled by the change of color. $-\tilde{\varphi}(z_+)$ is unchanged across the red cut (old branch cut) labelled by the unchanged yellow color

From the redefinition we made, we can see from fig. 11 above that we can revert back to the main arc and deduce that:

$$\varphi(z_+) = -\varphi(z_-) \Longrightarrow \varphi(z_+) + \varphi(z_-) = 0 \quad \text{on} \quad \Gamma_{a_-}^{a_+} \tag{3.32}$$

At $z = \infty$

$$\varphi(z_+) = \tilde{\varphi}(z_+) = 2\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}} e^{i\frac{0+0}{2}}}{w_+^2} dw = \tilde{\varphi}(z_+) = 2\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}}{r^2} dw \tag{3.33}$$

Similarly,

$$\varphi(z_-) = \tilde{\varphi}(z_-) = 2\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}} e^{i\frac{0+0}{2}}}{w_-^2} dw = \tilde{\varphi}(z_+) = 2\int_{a_-}^{z_+} \frac{(r_+)^{\frac{1}{2}}(r_-)^{\frac{1}{2}}}{r^2} dw \tag{3.34}$$

Thus $\varphi(z_-) = \varphi(z_+)$ on Γ_∞. As a consequence, we restrict the branch cut Γ to $\Gamma_{a_-}^{a_+}$

■

From the relation (3.30) above, we can notice that $W(z_+) = W(z_-)M_Q$ as $n \to \infty$ and with $z \neq a_+, a_-$. The factorization and rearrangement of matrices in (3.30) is suggestive of the next step in the analysis which we can term the "lens opening" technique.

3.2.2 Second transformation: lens opening

We can define a region (or lens) around $\Gamma_{a_-}^{a_+}$ bordered on the left and on the right respectively by two curves $\mathcal{L}_+$ and $\mathcal{L}_-$ that can be arbitrarily chosen as long as they stay in the region where

$\mathrm{Re}\left(\varphi(z) > 0\right)$ as shown in Fig. 12.

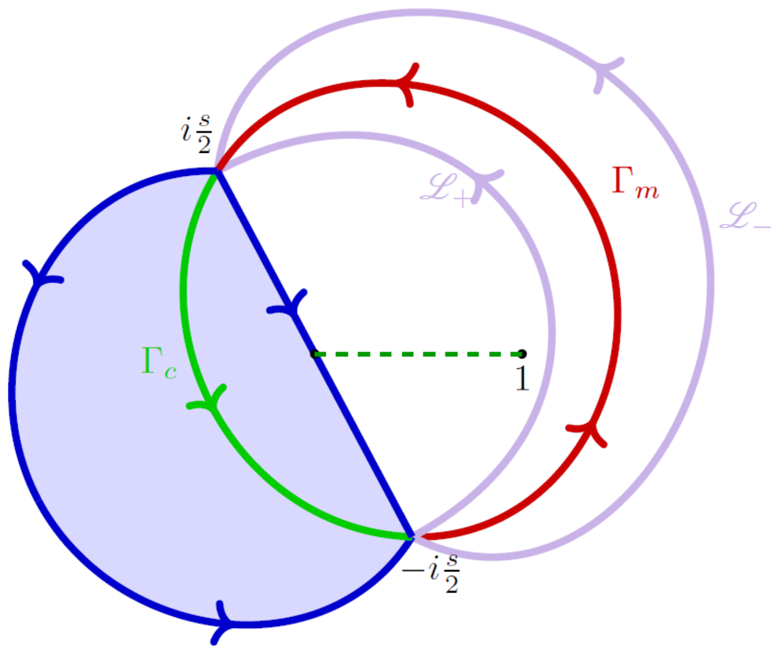

Figure 12: One branch cut lens opening

Definition 3.10 *Let T be a transformation on that region such that, as suggested in relation (3.30):*

$$T(z) := \begin{cases} W(z), & \textit{outside of the regions (lenses)} \\ W(z)\begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}, & \textit{on } \mathcal{R}_+ \\ W(z)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}, & \textit{on } \mathcal{R}_- \end{cases}$$

where $\mathcal{R}_+$ and $\mathcal{R}_-$ are respectively the region on the left and the right of Γ_{a-}^{a+}, such that relation (3.30) gives $T(z_+) = T(z_-)\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}$ on Γ_{a-}^{a+}, see Fig. 13.

Proposition 3.11 *The transformation T satisfies the jump conditions on the lenses $\mathcal{L}_\pm$ as follows*

$$T(z_+) = T(z_-) \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix} \tag{3.35}$$

Proof.

⊙ On $\mathcal{L}_+$

$T(z_+) = W(z)$, since that $W(z)$ defined in 3.3 does not have a jump on the lens $\mathcal{L}_+$

$T(z_-) = W(z) \begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}$ since that $W(z)$ and $\varphi(z)$ do not have a jump on the $\mathcal{L}_+$

Hence $T(z_+) = T(z_-) \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}$.

Similarly,

⊙ On $\mathcal{L}_-$

$T(z_+) = W(z) \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}$ since that $W(z)$ and $\varphi(z)$ do not have a jump on the lens $\mathcal{L}_-$

$T(z_-) = W(z)$ since that $W(z)$ does not have a jump on the lens $\mathcal{L}_-$.

Hence $T(z_+) = T(z_-) \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}$.

■

3.2.3 Model problem and approximation

Let us leave the scheme of the lens opening. The rationale behind the next construction is that the jump conditions on the lenses will be shown to be exponentially close to the identity jump, and hence, in first approximation, can be *heuristically* neglected (the full justification of this heuristics to be delayed to further analysis later on). The result of this is the RHP described hereafter.

Definition 3.12 *We define $M_Q(z)$ as the solution of the following RHP, that does not depend on n. The matrix has a jump discontinuity only on the branch cut $\Gamma_{a_-}^{a_+}$, such that:*

0) *$M_Q(z)$ defined and holomorphic in $\mathbb{CP}^1 \setminus \gamma$.*

1) *On* γ

$$M_Q(z_+) := \begin{cases} M_Q(z_-), & \text{on } \gamma \setminus \Gamma_{a-}^{a+} \\ M_Q(z_-)\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix} & \\ & \text{on } \Gamma_{a-}^{a+} \end{cases}$$

2) *As* $z \to \infty$

$$M_Q(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

3) *All entries are bounded by* $|z - a_\pm|^{-\frac{1}{4}}$ *as* $z \to a_\pm$ *(respectively).*

3.2.4 Error analysis

The goal of this section is to conclude the approximation analysis. We will prove that for s varying in closed subsets not intersecting the EoT region, the RHP for Φ is solvable for n sufficiently large, which implies that $D_n(ns)$ is non-zero. Thus, by exclusion, all zeros of $D_n(ns)$ must eventually fall within a neighbourhood of the EoT region.

The logic of the proof is to show that there exists a well defined transformation

$$E(z) = T(z)M_Q(z)^{-1} \tag{3.36}$$

that connects $T(z)$ and $M_Q(z)$ and that $E(z)$ tends to the identity pointwise. The logic is quite common in the literature, see e.g. [7, 8, 11, 16, 17], and hence we can be a bit cursory in the details.

Proposition 3.13 *There exists a well defined transformation* $E(z) = T(z)M_Q(z)^{-1}$ *that connects* $T(z)$ *and* $M_Q(z)$*. The Riemann Hilbert Problem for* E*, i.e.* $RHPE$ *is as follows:*

0) $E(z)$ *defined and holomorphic in* $\mathbb{C} \setminus (\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-)$

1)

$$E(z_+) = \begin{cases} E(z_-), & \text{on } \gamma \\ E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big), & \text{on } \mathcal{L}_- \cup \mathcal{L}_+ \end{cases}$$

where $G(z) = M_Q(z_-)\begin{bmatrix} 0 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 0 \end{bmatrix}(M_Q(z_-))^{-1}$

2)

$$As \quad z \to \infty, \quad E(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

Proof. The Riemann Hilbert Problem for E, i.e. $RHPE$ is as follows:

0) $E(z)$ defined and holomorphic in $\mathbb{C} - (\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-)$, this is justified from the definitions of $T(z)$ and $M_Q(z)$ (note that $\infty \in \gamma$).

1) On $\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-$

- On $\gamma - \Gamma_{a_-}^{a_+}$
  $E(z_+) = T(z_+)(M_Q(z_+))^{-1} = T(z_-)(M_Q(z_-))^{-1}$ since T and M_Q don't have a jump on $\gamma - \Gamma_{a_-}^{a_+}$. Finally $E(z_+) = E(z_-)$

- On $\Gamma_{a_-}^{a_+}$

$$\begin{aligned} E(z_+) = T(z_+)(M_Q(z_+))^{-1} &= T(z_-)\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}^{-1}(M_Q(z_-))^{-1} \\ &= T(z_-)(M_Q(z_-))^{-1} = E(z_-) \end{aligned}$$

- On $\mathcal{L}_- \cup \mathcal{L}_+$

$$\begin{aligned} E(z_+) = T(z_+)(M_Q(z_+))^{-1} &= T(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{e^{-n\varphi(z)}}{Q(z)} & 1 \end{bmatrix}\overbrace{(M_Q(z_-))^{-1}}^{=(M_Q(z_+))^{-1}} \\ &= T(z_-)\underbrace{(M_Q(z_-))^{-1}M_Q(z_-)}_{=\mathbb{I}_{2\times 2}}\left(\mathbb{I}_{2\times 2} + \begin{bmatrix} 0 & 0 \\ \dfrac{e^{-n\varphi(z)}}{Q(z)} & 0 \end{bmatrix}\right)(M_Q(z_-))^{-1} \\ &= T(z_-)\Bigg(\underbrace{(M_Q(z_-))^{-1}M_Q(z_-)}_{=\mathbb{I}_{2\times 2}}\mathbb{I}_{2\times 2}(M_Q(z_-))^{-1} \\ &\qquad + (M_Q(z_-))^{-1}\underbrace{M_Q(z_-)\begin{bmatrix} 0 & 0 \\ \dfrac{e^{-n\varphi(z)}}{Q(z)} & 0 \end{bmatrix}(M_Q(z_-))^{-1}}_{:=G(z)}\Bigg) \\ &= \underbrace{T(z_-)(M_Q(z_-))^{-1}}_{=E(z_-)} + \underbrace{T(z_-)(M_Q(z_-))^{-1}}_{=E(z_-)}G(z) \\ &= E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big) \end{aligned}$$

2)

$$\text{As} \quad z \to \infty, \quad E(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

■

The following theorem will allow us to suppress the function $G(z)$

Theorem 3.14 ***Small Norm Theorem for RHP's:***
Let E be a transformation (function) satisfying the following RHP:

$$\begin{cases} E(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right] \quad As \quad z \to \infty \\ E(z_+) = E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big), \qquad\qquad on\ a\ contour \sum \end{cases}$$

Let $N(z) = \sqrt{\sum (G_{i,j}(z))^2}$. Suppose that $N(z) \in L^p\Big(\sum, |\mathrm{d}z|\Big)$ with $1 \leqslant p \leqslant \infty$. If $\|N(z)\|_\infty$ is small enough, then the RHP has a solution.

In our case,
$\sum = \mathcal{L}_- \cup \mathcal{L}_+$. We have

$$G(z) = M_Q(z_-)\begin{bmatrix} 0 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} & 0 \end{bmatrix}(M_Q(z_-))^{-1} \tag{3.37}$$

Since $\sum$ is compact and $\dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)}$ is bounded on it, $N(z) \in L^p\Big(\sum, |\mathrm{d}z|\Big)$ with $1 \leqslant p \leqslant \infty$.
$\forall z \in \sum, \quad Q(z) \neq 0.$
In addition
$\forall z \in \sum - \{a_\pm\}$, $\mathrm{Re}\,(\varphi(z)) > 0 \Rightarrow$ as $n \to \infty$, $\dfrac{\mathrm{e}^{-n\varphi(z)}}{Q(z)} \to 0$.
Hence $\|N(z)\|_\infty$ is small enough.
However, for $z = a_\pm$, $\mathrm{e}^{-n\varphi(z)} = 1$ and $\|N(z)\|_\infty$ can not be made small enough.

Therefore we need to apply the small norm theorem to a contour that excludes the points $z = a_+$ and $z = a_-$. To achieve that, let us redefine $T(z)$ in the neighbourhood of those points: let us $\mathcal{D}_+$ and $\mathcal{D}_-$ be two disks with radius δ, $\forall \delta > 0$ in the neighbourhoods. If $\tilde{T}(z)$ is the redefinition of $T(z)$, we will have $E(z) := \tilde{T}(z)(M_Q(z))^{-1}$

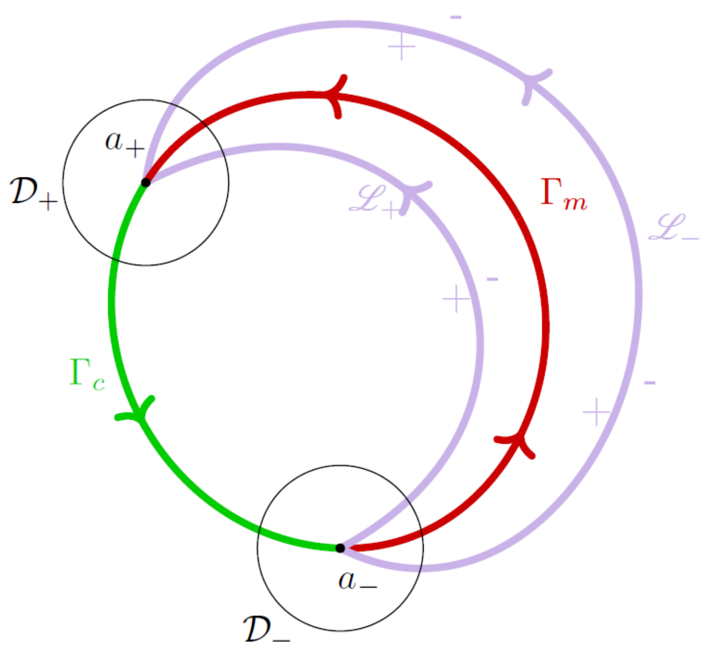

Figure 13: Disks opening around the branch points, and configuration of the jumps of the transformation T on the arcs and lenses

Let us start the procedure by first considering only the disks $\mathcal{D}_+$ and $\mathcal{D}_-$.

Definition 3.15 *Let's define a $P(z,n)$ as follows:*

$$P(z,n) = \begin{cases} P_{\mathcal{D}_\pm}(z,n), & \textit{inside} \quad \mathcal{D}_\pm \\ T(z), & \textit{on} \quad \partial\mathcal{D}_\pm \end{cases}$$

where $P_{\mathcal{D}_\pm}(z,n)$ is some transformation which has the same jump as $T(z)$ on the main arc $\Gamma_{a_-}^{a_+}$ and the lenses $\mathcal{L}_\pm$ as $T(z)$ inside $\mathcal{D}_\pm$
Now, let us define :

$$\tilde{T}(z) := \begin{cases} T(z), & \textit{on} \quad \mathbb{C} - \mathcal{D}_\pm \\ P(z,n)(T(z))^{-1}M_Q(z), & \textit{on} \quad \mathcal{D}_\pm \end{cases}$$

The existence of appropriate parametrices is a delicate but standard construction and can be found in [7, 8, 11, 16, 17]. We will not report their construction here.

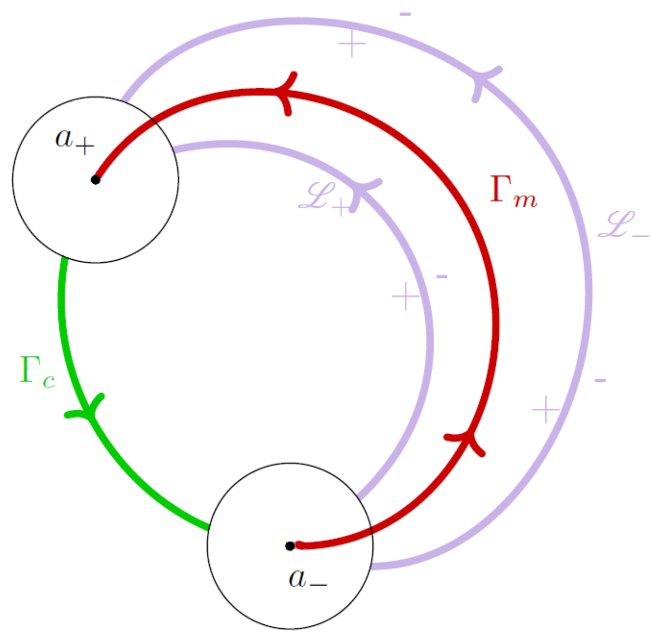

Figure 14: Configuration of the jumps of the transformation $\tilde{T}$ on the arcs, lenses and the disks

It follows from the definition that:

$$E(z) = \tilde{T}(z)(M_Q(z))^{-1} = \begin{cases} T(z)(M_Q(z))^{-1}, \quad \text{on} \quad \mathbb{C} - \mathcal{D}_\pm \\ P(z,n)(T(z))^{-1} = \begin{cases} P_{\mathcal{D}_\pm}(z,n)(T(z))^{-1} := \tilde{P}_{\mathcal{D}_\pm}(z,n), \\ \qquad\qquad\qquad\qquad \text{inside} \quad \mathcal{D}_\pm \\ \mathbb{I}_{2\times 2} + \mathcal{O}(\frac{1}{n}), \quad \text{on} \quad \partial\mathcal{D}_\pm \end{cases} \end{cases} \tag{3.38}$$

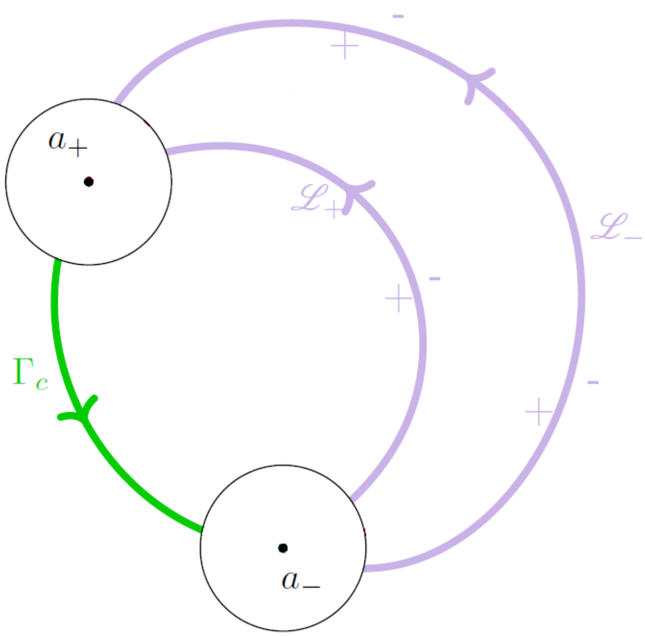

Figure 15: Configuration of the jumps of the transformation E on the arcs, lenses and the disks

Notice that:

⊙ $\tilde{P}_{\mathcal{D}_\pm}(z,n)$ as defined above, does not have a jump on both the main arc and the lenses inside $\mathcal{D}_\pm$

⊙ since $a_\pm \notin (\mathbb{C} - \mathcal{D}_\pm)$, and $E(z) = T(z)(M_Q(z))^{-1}$ on $\mathbb{C} - \mathcal{D}_\pm$ the RHP for E has a solution (as we derived before)

In particular, $E(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$ As $\quad z \to \infty$.

⊙ inside $\mathcal{D}_\pm$ and on $\partial\mathcal{D}_\pm$, the RHP for E is:

$$E(z_+) = E(z_-)(\left(\mathbb{I}_{2\times 2} + \underbrace{0_{2\times 2}}_{=G(z)}\right)$$

$G(z) = 0_{2\times 2}$ thus the RHP for $E = \tilde{T}(z)(M_Q(z))^{-1}$ has a solution (i.e. exists) by the small norm theorem.

Comments:

◦ Since the Riemann Hilbert Problem for the transformation E has a solution (i.e. E exists), we can know consider the RHP of transformation $M_Q(z) = (E(z))^{-1}\tilde{T}(z)$ which satisfies the conditions as defined in Def. 3.12 and solve it. Notice that the expressions of $M_Q(z)$ inside and outside the disks are different, although they both have the same jump matrix on $\Gamma_{a_-}^{a_+}$.

◦ Finding a solution to the RHPM_Q is equivalent to finding a solution to the RHPΦ. In order to find an explicit solution to the RHPM_Q, we will need a further transformation $M_Q \longrightarrow M$ to the Model problem, which is easier to solve explicitly.

3.2.5 Solution of the RHP for M_Q of Def. 3.12

In order to obtain a Riemann Hilbert Problem of Def. 3.12 we define a new matrix $M(z)$, with a *constant* jump matrix on Γ_{a-}^{a+} and no jump at ∞. Let us consider:

Definition 3.16 *Let us define*

$$M(z) := \mathrm{e}^{-S(\infty)\sigma_3} M_Q(z) \mathrm{e}^{S(z)\sigma_3}$$

The function $S(z)$ is the Szegö function and its properties are derived below, in particular (3.41) and given in Prop. 3.18 below.

We will solve for $S(z)$, called the Szego function, such that $M(z)$, if it exists, satisfies all the conditions above, in particular, has a constant jump on Γ_{a-}^{a+}.

Proposition 3.17 *The function $S(z)$ defined in Def. 3.16 has the properties:*
⊙ *$S(z)$ is analytic and bounded on $\mathbb{CP}^1 - \Gamma_{a-}^{a+}$*
⊙ *$S(z)$ has finite boundary values along Γ_{a-}^{a+} which satisfy*
$S(z_+) + S(z_-) = \ln(Q(z))$

Proposition 3.18 *The Szegö function is given by the expression*

$$S(z) := \frac{J(z)}{2\pi i} \int_{\Gamma_{a-}^{a+}} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w, \qquad z \notin \Gamma_{a-}^{a+}, \tag{3.39}$$

where $J(z)$ is as in (3.42). It is analytic and bounded on $\mathbb{CP}^1 \setminus \Gamma_{a-}^{a+}$ and satisfies the boundary relation

$$S(z_+) + S(z_-) = \ln(Q(z)), \qquad z \in \Gamma_{a-}^{a+}. \tag{3.40}$$

Proposition 3.19 *Let's consider the transformation $M(z) = \mathrm{e}^{S(\infty)\sigma_3} M_Q(z) \mathrm{e}^{-S(z)\sigma_3}$ as defined in Def. 3.16. Then*

0) *M is defined and holomorphic in $\mathbb{CP}^1 - \gamma$, (note that $\infty \in \gamma$)*

1) *On* γ

$$M(z_+) = \begin{cases} M(z_-), & \text{on } \gamma - \Gamma_{a_-}^{a_+} \\ M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \\ & \text{on } \Gamma_{a_-}^{a_+} \end{cases}$$

2) *As* $z \longrightarrow \infty$, $M(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big]$

3) *All entries are bounded by* $|z-a_\pm|^{-\frac{1}{4}}$ *as* $z \to a_\pm$ *(respectively).*

Proof.

Analyticity condition on $\mathbb{C}\setminus\gamma$. We need to impose that $S(z)$ is bounded and analytic on $\mathbb{C}\setminus\gamma$. This condition leads us to conclude that $M(z) = \mathrm{e}^{-S(\infty)\sigma_3} M_Q(z)\mathrm{e}^{S(z)\sigma_3}$ is analytic on $\mathbb{C}\setminus\gamma$ because M_Q is.

Jump Condition on γ.

- on $\gamma\Gamma_{a_-}^{a_+}$,

$$\begin{aligned} M(z_+) &= \mathrm{e}^{-S(\infty)\sigma_3} M_Q(z_+)\mathrm{e}^{S(z_+)\sigma_3} \\ &= \mathrm{e}^{-S(\infty)\sigma_3} M_Q(z_-)\mathrm{e}^{S(z_+)\sigma_3} = M(z_-) \end{aligned}$$

- on $\Gamma_{a_-}^{a_+}$,

$$\begin{aligned} M(z_+) &= \mathrm{e}^{-S(\infty)\sigma_3} M_Q(z_+)\mathrm{e}^{S(z_+)\sigma_3} \\ &= \mathrm{e}^{-S(\infty)\sigma_3}\tilde{T_-}(z)\,\overbrace{\mathrm{e}^{S(z_-)\sigma_3}\mathrm{e}^{-S(z_-)\sigma_3}}^{=\mathbb{I}_{2\times 2}}\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}\mathrm{e}^{S(z_+)\sigma_3} \\ &= M(z_-)\mathrm{e}^{-S(z_-)\sigma_3}\begin{bmatrix} 0 & Q(z) \\ -\dfrac{1}{Q(z)} & 0 \end{bmatrix}\mathrm{e}^{S(z_+)\sigma_3} \\ \Longrightarrow M(z_+) &= M(z_-)\begin{bmatrix} 0 & Q(z)\mathrm{e}^{-\left(S(z_+)+S(z_-)\right)} \\ -\dfrac{\mathrm{e}^{\left(S(z_+)+S(z_-)\right)}}{Q(z)} & 0 \end{bmatrix} \end{aligned}$$

Therefore, for the jump condition with a constant matrix to be satisfied, we must impose to have $S(z)$ bounded on $\Gamma_{a_-}^{a_+}$, and

$$S(z_+) + S(z_-) = \ln(Q(z)) \quad \text{i.e.} \quad M(z_+) = M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.41}$$

Construction of the Szegö function $S(z)$**.** Let us define for convenience

$$J(z) := \varphi'(z) = \sqrt{z^2 + \frac{s^2}{4}}. \tag{3.42}$$

We previously established in relation (3.32), namely that $\varphi(z_+) = -\varphi(z_-)$ on $\Gamma_{a_-}^{a_+}$.
This implies that $\varphi'(z_+) = -\varphi'(z_-)$ i.e. $J(z_+) = -J(z_-)$.
So diviting both sides of (3.41) by $J(z_+)$ we obtain the following

$$\frac{S(z_+)}{J(z_+)} + \frac{S(z_-)}{J(z_+)} = \frac{\ln(Q(z))}{J(z_+)} \tag{3.43}$$

$$\Rightarrow \frac{S(z_+)}{J(z_+)} - \frac{S(z_-)}{J(z_-)} = \frac{\ln(Q(z))}{J(z_+)} \tag{3.44}$$

$$\Longrightarrow \text{(by Sokhotski–Plemelj theorem on } \frac{S(z)}{J(z)}\text{)} \tag{3.45}$$

$$S(z) = \frac{J(z)}{2\pi i} \int_{\Gamma_{a_-}^{a_+}} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w, \quad z \notin \Gamma_{a_-}^{a_+} \tag{3.46}$$

Notice that the expression of $S(z)$ provided by Sokhotski–Plemelj is indeed bounded on $\Gamma_{a_-}^{a_+} - \{a_+, a_-\}$ as we want: in fact, as $z \notin \Gamma_{a_-}^{a_+}$, $S(z)$ is bounded because the contour of integration $\Gamma_{a_-}^{a_+}$ is compact, and the integrand of $S(z)$ is bounded on it. It is also bounded at the endpoints as a consequence of the expansions of Cauchy integrals, see [24], Ch. V.

Expansion at ∞**.** The function $S(z)$ is bounded at infinity because $J(z) = \mathcal{O}(z)$ and the Cauchy integral is $\mathcal{O}(z^{-1})$, as $|z| \to \infty$. In particular it has a limiting value

$$S(\infty) = -\int_{\Gamma_{a_-}^{a_+}} \frac{\ln(Q(w))}{J(w_+)} \mathrm{d}w. \tag{3.47}$$

Consequently we see that $M(z)$ is also bounded at infinity and correctly normalized:

$$M(z) = \mathrm{e}^{S(\infty)\sigma_3} M_Q(z) \mathrm{e}^{-S(z)\sigma_3} \Longrightarrow M(\infty) = \mathrm{e}^{-S(\infty)\sigma_3} \underbrace{\left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]}_{M_Q(\infty)} \mathrm{e}^{S(\infty)\sigma_3}$$

$$\Longrightarrow \quad \text{as} \quad z \to \infty, M(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

■

We give the explicit construction of M in the next section.

3.2.6 Solution of the Model Problem (RHPM) in Prop. 3.19

A further diagonalization in condition 1) gives:
$M(z_+) = M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = M(z_-)P\begin{bmatrix} i & 0 \\ -0 & -i \end{bmatrix}P^{-1}$ after diagonalization, where P is the matrix of eigenvectors of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Proposition 3.20 *A solution of the RHPM is :*

$$M(z) = \left(\frac{z-a_+}{z-a_-}\right)^{\frac{\sigma_2}{4}} = \frac{1}{2}\begin{bmatrix} \varrho + \dfrac{1}{\varrho} & -i\left(\varrho - \dfrac{1}{\varrho}\right) \\ i\left(\varrho - \dfrac{1}{\varrho}\right) & \varrho + \dfrac{1}{\varrho} \end{bmatrix} \tag{3.48}$$

where $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ *and* $\varrho = \left(\dfrac{z-a_+}{z-a_-}\right)^{\frac{1}{4}}$.

let us verify that $M(z)$ as given above solves the RHPM:

¶$_1$) It easy to see that ϱ is analytic and locally bounded on $\mathbb{CP}^1 - \Gamma_{a_-}^{a_+}$, so is $M(z)$. Near the two points $a_\pm$ we see that all the entries are indeed bounded by $|z-a_\pm|^{-\frac{1}{4}}$ as required by the Prop. 3.19.

¶$_2$) • From previous discussion on the jump of $\sqrt{(z-a_-)(z-a_+)}$ on $\Gamma_{a_-}^{a_+}$ similar computations show that

$$\begin{cases} \varrho_+ = \left(\dfrac{z_+-a_+}{z_+-a_-}\right)^{\frac{1}{4}} = (\dfrac{r_+}{r_-})^{\frac{1}{4}} e^{i\frac{\pi}{4}} \\ \varrho_- = \left(\dfrac{z_--a_+}{z_--a_-}\right)^{\frac{1}{4}} = (\dfrac{r_+}{r_-})^{\frac{1}{4}} e^{-i\frac{\pi}{4}} \end{cases} \tag{3.49}$$

Thus ϱ has a jump discontinuity only on $\Gamma_{a_-}^{a_+}$ and $\varrho_+ = i\varrho_- \Rightarrow \varrho_- = -i\varrho_+$.

- Now, let us verify the jump condition,

$$
\begin{aligned}
M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} \varrho_- + \dfrac{1}{\varrho_-} & -i\left(\varrho_- - \dfrac{1}{\varrho_-}\right) \\ i\left(\varrho_- - \dfrac{1}{\varrho_-}\right) & \varrho_- + \dfrac{1}{\varrho_-} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -i\left(\varrho_+ - \dfrac{1}{\varrho_+}\right) & -\left(\varrho_+ + \dfrac{1}{\varrho_+}\right) \\ \left(\varrho_+ + \dfrac{1}{\varrho_+}\right) & -i\left(\varrho_+ - \dfrac{1}{\varrho_+}\right) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \varrho_+ + \dfrac{1}{\varrho_+} & -i\left(\varrho_+ - \dfrac{1}{\varrho_+}\right) \\ i\left(\varrho_+ - \dfrac{1}{\varrho_+}\right) & \varrho_+ + \dfrac{1}{\varrho_+} \end{bmatrix} \\
&= M(z_+)
\end{aligned}
$$

¶$_3$) It is easy to see that as $z \to \infty$ $\varrho \to 1$ and $M(z) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, hence $M(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$

All the 3 conditions of RHPM are satisfied for $M(z)$ and this proves Prop. 3.19.

Short discussion of uniqueness. Another solution of the RHPM could be $\tilde{M}(z) = \underbrace{\left(\mathbb{I}_{2\times 2} + \dfrac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{(z-a_\pm)^k}\right)}_{:=L(z)} M(z)$, where k is an integer and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a constant matrix. In fact $\tilde{M}(z)$ satisfies all the first three $(0,1,2)$ conditions of the RHPM in Prop. 3.19 because:

¶¶$_1$ $L(z)$ (as defined in the expression of $\tilde{M}(z)$ above) is analytic and bounded on $\mathbb{CP}^1 - \Gamma_{a-}^{a+}$, so is $\tilde{M}(z)$, (note that $L(z)$ has only a singularity at $a_\pm$, because k is an integer, and $a_\pm \notin \mathbb{CP}^1 - \Gamma_{a-}^{a+}$.

¶¶$_2$ The singularities of $L(z)$, $a_\pm \in \Gamma_{a-}^{a+}$, so $\tilde{M}(z)$ has only a jump on Γ_{a-}^{a+} coming from $M(z)$

¶¶$_3$ As $z \to \infty$, $L(z) \to \mathbb{I}_{2\times 2}$, hence $\tilde{M}(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$

The uniqueness of the solution thus depends crucially on the condition 3) of Prop. 3.19. We now prove that with condition 3), the solution $M(z)$ to the RHPM is unique.
Let $\hat{M}(z)$ be another solution satisfying conditions 0), 1), 2) and 3) of the RHPM.
Notice that $\det(M(z)) = \det(\hat{M}(z)) = 1$:
In fact, from condition 1) $\det(M(z_+)) = \det(M(z_-))\underbrace{\det(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})}_{=1}$ on γ.

So $\det(M(z_+)) = \det(M(z_-)$ on γ. In addition, condition 0) $\Rightarrow$ $\det(M(z))$ and $\det(\hat{M}(z))$ are analytic on $\mathbb{C} \setminus \gamma$.

It follows that they both are analytic on $\mathbb{C}$. Moreover as $z \to \infty$, condition 2) $\Longrightarrow \det(M(z)) = 1 = \det(\hat{M}(z))$. Thus, $\det(M(z))$ and $det(\hat{M}(z))$ are analytic and bounded, by Liouville's theorem, $\det(M(z))$ and $\det(\hat{M}(z))$ are constant $\forall z \in \mathbb{CP}^1$.
Hence $\det(M(z)) = \det(\hat{M}(z)) = 1 \neq 0 \Rightarrow \hat{M}(z)$ is invertible $\forall z \in \mathbb{CP}^1$. Let us consider $X(z) = M(z)\hat{M}^{-1}(z)$.

$$\begin{aligned} X(z_+) &= M(z_+)\hat{M}^{-1}(z_+) = M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(\hat{M}(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1} \\ &= X(z_-) \quad \text{on } \gamma \end{aligned}$$

- It follows that $X(z)$ has no branch cut. It follows that $X(z)$ has at most poles ad $a_\pm$.
- Moreover, by condition 1), $M(z)$ and $\hat{M}^{-1}(z)$ have a jump discontinuity on $\Gamma_{a_-}^{a_+}$.

The two statements above imply that $X(z) = M(z)\hat{M}^{-1}(z)$, has at most poles at $a_\pm$.
However, with condition 3), each entry of $X(z)$ is bounded by
$\mathcal{O}\left(\frac{1}{(z-a_\pm)^{1/4}}\right).\mathcal{O}\left(\frac{1}{(z-a_\pm)^{1/4}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{z-a_\pm}}\right)$ at a_+ and a_- which implies a branch cut between a_+ and a_-. In other words, if $X(z)$ ever has singularities at a_+ and a_-, they must be branch points. This contradicts the fact that $X(z)$ has at most poles, therefore $X(z)$ has either no poles or removable poles at a_+ and a_-.
Hence $X(z)$ is analytic on $\mathbb{CP}^1$.
Furthermore, as $z \to \infty$ condition 2) $\Rightarrow \hat{M}(z) \to \mathbb{I}_{2\times 2}$, so is $\hat{M}^{-1}(z)$, therefore $X(z) = M(z)\hat{M}^{-1}(z) \to \mathbb{I}_{2\times 2}$. Therefore $X(z)$ is analytic and bounded on $\mathbb{CP}^1$. By Liouville's theorem $X(z)$ is constant equal to $\mathbb{I}_{2\times 2}$. It follows that $M(z) = \hat{M}^{-1}(z)$, thus $M(z)$ is unique under condition 3).

4 Discussion of the solution of RHPW inside the Eye of the Tiger

The boundary of EoT is described precisely by the condition that $z = 1$ belongs to the two sub-arcs of the zero-level set of $\mathrm{Re}\,\varphi(z;s)$ forming the "rind" of the apricot. As we move s inside the EoT we cannot use the same effective potential described in the previous section because the fifth condition in Def. 3.2 ceases to be verified.

The idea is to treat $z = 1$ as a "hard–edge" in the terminology that has come to pass in the literature about random matrix theory [11, 14]. We thus postulate the following form for $\varphi'(z;s)$

$$\varphi'(z;s) = \frac{2}{z^2}\sqrt{z^2 + \frac{s^2}{4} + \frac{Az^2}{z-1}} = \frac{2\sqrt{z^2(z-1+A) + \frac{s^2}{4}(z-1)}}{z^2\sqrt{z-1}} \tag{4.1}$$

The parameter $A = A(s)$ is chosen by the condition that all periods of $\varphi'(z;s)\mathrm{d}z$ on the Riemann surface of the radical are purely imaginary, which is necessary so that $\mathrm{Re}\,\varphi$ is continuous across the cuts; the Riemann surface of $\varphi'(z;s)$ is an elliptic curve branched at $z = 1$ and the other three roots of the radical in the numerator:

$$\mu^2 = (z-1)\left(z^2(z-1+A) + \frac{s^2}{4}(z-1)\right). \tag{4.2}$$

We denote these roots as b, a_+, a_- with b the closest root to $z = 1$; an expression in terms of Cardano's formulæ is possible but not necessary.

Now the complex parameter $A(s)$ is determined implicitly by the two real equations

$$\mathrm{Re}\int_b^1 \varphi'(z;s)\mathrm{d}z = 0 \qquad \mathrm{Re}\int_b^{a_+} \varphi'(z;s)\mathrm{d}z = 0. \tag{4.3}$$

Under these conditions it then follows that the real part of

$$\varphi(z;s) = \int_{a_-}^{z} \varphi'(w;s)\mathrm{d}w \tag{4.4}$$

is a well defined (single valued) harmonic function on the Riemann surface minus the preimages of the points $z = 0$ on the two sheets.

Determination of Γ_m and Γ_c. By the same argument already used in the genus zero case, the level sets $\mathrm{Re}\,\varphi(z;s)$ are well defined; they consists of the vertical trajectories of the quadratic differential $Q = \varphi'(z;s)^2\mathrm{d}z^2$ [43]

$$Q = 4\frac{z^2(z-1+A) + \frac{s^2}{4}(z-1)}{z^4(z-1)}. \tag{4.5}$$

The main arcs Γ_m are sub-arcs of the zero levelset of $\mathrm{Re}\,\varphi$ and we need to discuss their qualitative topology before proceeding.

The critical points are the three simple zeros (generically) and the simple pole $z = 1$; from each simple zero issue three vertical trajectories, while from the simple pole only one. The union of the trajectories is a connected planar graph and the unbounded region is conformally equivalent to the unit punctured disk (with the puncture at infinity) by the map

$$\zeta = \mathrm{e}^{-\frac{\varphi(z;s)}{4i\pi}} \tag{4.6}$$

which maps the exterior region into the disk $|\zeta| < 1$, with $z = \infty$ mapped to $\zeta = 0$. Some observations are in order

1. the level sets of $\operatorname{Re}\varphi$ depend only on s^2 and they are conjugated if we conjugate s;
2. one of the zeros of φ' is connected by a vertical trajectory to $z = 1$; we denote this zero by $z = b$; the other two zeros are one in the upper and one in the lower half plane. We denote them by $a_\pm$, respectively.

While the level sets of $\operatorname{Re}\varphi$ depend on s^2 alone, we must choose the branch-cuts Γ_m differently according to $\operatorname{Re} s > 0$ or $\operatorname{Re} s < 0$; the reason is that the sign distribution of $\operatorname{Re}\varphi$ differs according to the two cases.

This is seen by the following reasoning

- In the outside region $\operatorname{Re}\varphi \simeq 2\ln|z| + \mathcal{O}(1)$ and hence $\operatorname{Re}\varphi > 0$;
- near the origin we must have $\varphi(z; s) = -\frac{s}{z} + \mathcal{O}(1)$ and hence for $\operatorname{Re} s > 0$ the right "lobe" is where $\operatorname{Re}\varphi < 0$; viceversa it is the left one if $\operatorname{Re} s < 0$.

The branch-cut Γ_m is then singled out by the fact that $\operatorname{Re}\varphi$ is not differentiable, namely, $\operatorname{Re}\varphi$ has the same sign (positive) on both sides.

Collecting these observations, we thus have determined that

1. For $\operatorname{Re} s > 0$ the branch-cut Γ_m consists of the three arcs of the vertical trajectories connecting $z = b, 1$ and $z = b, a_+$ and $z = b, a_-$.
2. For $\operatorname{Re} s < 0$ the branch-cut Γ_m consists of the two arcs of the vertical trajectories connecting $z = b, 1$ and $z = a_+, a_-$ (passing to the left of the origin).

See Fig. 5.

Proposition 4.1 *The 2×2 matrix valued function $\Phi(z)$, solution of the RHPΦ must satisfy the following conditions:*

0) *$\Phi(z)$ defined and holomorphic in $\mathbb{CP}^1 \setminus \gamma$*

1) *The jump condition gives:*

$$\Phi(z_+) = \begin{cases} \Phi(z_-) \begin{bmatrix} 1 & Q(z)\mathrm{e}^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} & \text{if} \quad z \in \gamma \setminus \gamma_b^1 \\ = \Phi(z_-) \begin{bmatrix} 1 & \overbrace{(1 - \mathrm{e}^{2i\pi\theta})}^{:=\varsigma} Q(z_-)\mathrm{e}^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} & \text{if} \quad z \in b_1 \end{cases}$$

$$:= \Phi(z_-) \begin{bmatrix} 1 & \tilde{Q}(z)\mathrm{e}^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix}$$

where $\tilde{Q}(z)\mathrm{e}^{\frac{ns}{z}}$ is defined as the generic final expression of the $(1,2)$-entry of the matrix above $\forall z \in \gamma$.

2) *As $z \to \infty$, for some $n \in \mathbb{N}$, the matrix $\Phi(z)$ has the Laurent series expansion of z^{-1} of the form: $\Phi(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right] z^{n\sigma_3} \equiv \Phi_n(z)$*

Proof.

We define $b \in \mathbb{C}$ such that the path from $z = b$ to $z = 1$ (let us call it b_1) is $\varepsilon \cap \Gamma = b_1$. The steepest descent method is also applied to this case to find the solution of the $RHP\Phi$. The only difference with the previous case is that the function $Q(z) = z^{-m}\left(1 - \frac{1}{z}\right)^{\theta}$ as defined in relation (3.4) "technically" has now a jump on b_1 as shown on Figure 16 below. Since m is an integer z^{-m} does not have a jump on $b_1 \Rightarrow (z_+)^{-m} = (z_-)^{-m}$.
However θ is not an integer and hence,

$$\begin{cases} Q(z_+) = z^{-m}\left(1 - \dfrac{1}{z_+}\right)^{\theta} = z^{-m}\dfrac{(z_+ - 1)^{\theta}}{(z_+)^{\theta}} = z^{-m}\left(\dfrac{r_1}{r_2}\right)^{\theta} e^{i\pi\theta} \\ Q(z_-) = z^{-m}\left(1 - \dfrac{1}{z_-}\right)^{\theta} = z^{-m}\dfrac{(z_- - 1)^{\theta}}{(z_-)^{\theta}} = z^{-m}\left(\dfrac{r_1}{r_2}\right)^{\theta} e^{-i\pi\theta} \end{cases}$$

$$\Rightarrow Q(z_+) = e^{2i\pi\theta} Q(z_-) \tag{4.7}$$

where $r_1 = |z_+ - 1| = |z_- - 1|$ and $r_2 = |z_+| = |z_-|$

Let us recall that the 2×2 matrix valued function $\Phi(z)$, solution of the $RHP\Phi$ must satisfy the following conditions specified in Sec. 1.3.1. In the present situation they read as follows

0) $\Phi(z)$ is defined and holomorphic in $\mathbb{CP}^1 \setminus \gamma$

1) on γ, it satisfies the boundary value condition (see figure 16)

$$\Phi(z_+) = \begin{cases} \Phi(z_-)\begin{bmatrix} 1 & Q(z)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} & \text{if} \quad z \in \gamma \setminus b_1 \\ \Phi(z_-)\begin{bmatrix} 1 & Q(z_-)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & Q(z_+)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix}^{-1} & \text{if} \quad z \in b_1 \end{cases}$$

The arc b_1 is the subset of γ that will be superposed (flattened) on $b1$. (See figure 16)

To "make a jump" on b_1, Φ has to jump on $\left(\gamma_b^1\right)^-$ and $\left(\gamma_b^1\right)^+$ successively, hence the product of the two jump matrices for the jump condition on b_1. The two matrices are inverse of each other because they have opposite domains of definition (with respect to the sides of γ_b^1) as shown on the figure 16 above.
Recall that with respect to b_1, $Q(z_+) = e^{2i\pi\theta}Q(z_-)$, hence, the jump gives:

$$\Phi(z_+) = \begin{cases} \Phi(z_-)\begin{bmatrix} 1 & Q(z)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} & \text{if} \quad z \in \gamma \setminus \gamma_b^1 \\ = \Phi(z_-)\begin{bmatrix} 1 & \overbrace{(1 - e^{2i\pi\theta})}^{:=\varsigma} Q(z_-)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} & \text{if} \quad z \in b_1 \end{cases} \tag{4.8}$$

$$:= \Phi(z_-)\begin{bmatrix} 1 & \tilde{Q}(z)e^{\frac{ns}{z}} \\ 0 & 1 \end{bmatrix} \tag{4.9}$$

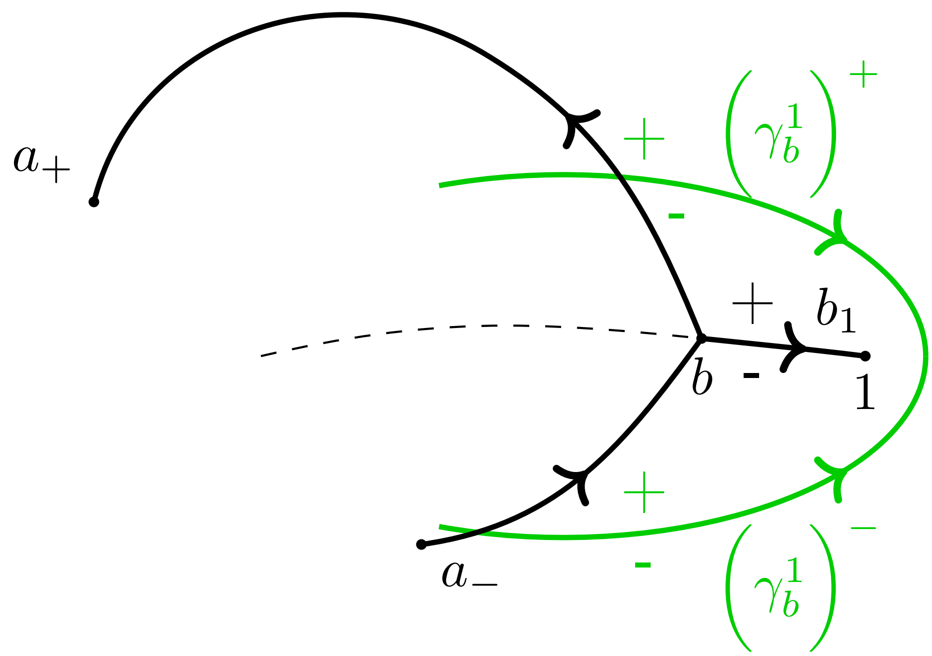

Figure 16: Decomposition of the jump of the transformation Φ on the branch cut Γ_b^1

where $\tilde{Q}(z)\mathrm{e}^{\frac{ns}{z}}$ is defined as the generic final expression of the $(1,2)$-entry of the matrix above $\forall z \in \gamma$.

2) As $z \to \infty$, for some $n \in \mathbb{N}$, the matrix $\Phi(z)$ has the Laurent series expansion of z^{-1} of the form: $\Phi(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big] z^{n\sigma_3} \equiv \Phi_n(z)$

■

4.1 First transformation $\Phi \longrightarrow W$

Similarly to the previous case (outside the EoT), and under the same conditions as defined in Def. 3.3 for the functions g and φ, we proceed to the first transformation as $\Phi \longrightarrow W$.

$$W(z) = \mathrm{e}^{n\frac{\ell}{2}\sigma_3}\Phi(z)\mathrm{e}^{-\left(g(z)+\frac{\ell}{2}\right)\sigma_3} \tag{4.10}$$

On the $s-$ plane, when $s \in \mathbb{C}$ is inside the EoT (as described at the end of the Section 3.1), we typically obtain on the $z-$ plane the figures 17 and 18 below (see pictures on the left), where $z = 1$ is outside Ξ on the $z-$ plane. Recall that $\Xi = \{z \in \mathbb{C} : \mathrm{Re}\,(\varphi(z,s)) = 0\}$.

Note that in this case ε, the branch cut of $Q(z)$ (between 0 and 1) is partly outside of Ξ. However ε must be inside the contour γ. This forces a subset of the closed contour γ (near $z = 1$) to be outside of Ξ as well.
Moreover, similarly to the previous case (i.e. when s is outside the EoT), for the jump of g (or φ) on γ to make sense, γ is deformed and passes through the branch cut Γ of φ.

Definition 4.2 *We define:*

$$\varphi'(z) = \frac{2}{z}\sqrt{\frac{(z-a_+)(z-a_-)(z-b)}{z-1}} \tag{4.11}$$

as a function with the branch cuts along Γ_m *as discussed in Section 4.1.*

Since Γ_m has branch points at a_-, a_+, b, and 1, this implies that γ along with Γ_m passes through those branch points as well (in particular the two superposed contours cannot cross ε as γ encloses ε).

Let us call "temporarily " $\hat{\Xi}$, the contour of Ξ which goes from a_- to a_+ such that $\mathrm{Re}\,(\varphi(z)) > 0$ on its left and on its right(recall that $\mathrm{Re}\,(\varphi(z)) = 0$ as $z \in \Xi$). We can choose to deform (superpose) once again γ along with Γ towards $\hat{\Xi}$ as pictured on figures 17 and 18, such that $\Gamma = \hat{\Xi} \cup \Gamma(\infty)$. Let us rename $\hat{\Xi} = \Gamma_{a_-}^{a_+}$ such that $\Gamma = \Gamma_{a_-}^{a_+} \cup \Gamma(\infty)$ now.

As a result, γ passes through the arcs $\Gamma_{a_-}^{b}$, $\Gamma_{b}^{a_+}$ and $\Gamma_b^1 \equiv b_1$ on figure 17 (see picture on the right). On figure 18 (see picture on the right), γ passes through the arcs $\Gamma_{a_-}^{a_+}$ and Γ_b^1. These arcs in each figure are called the Main arcs noted Γ_m.
The contour of γ which is not superposed is called the complementary arc Γ_c, and it is located on the region of the $z-$ plane where $\mathrm{Re}\,(\varphi(z,s)) < 0$.

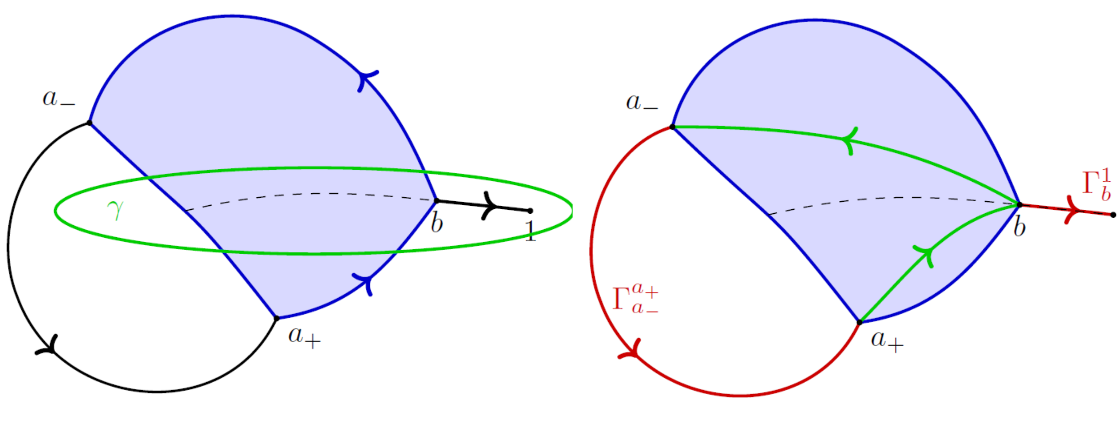

Figure 17: The retraction of the contour of integration (left pane) onto Γ when s is inside the EoT. Case for $\mathrm{Re}\,(s) > 0$.

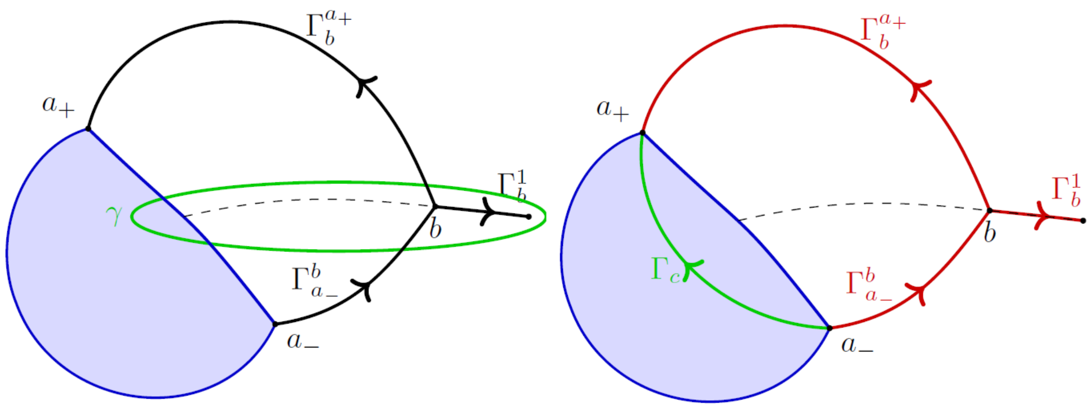

Figure 18: The retraction of the contour of integration (left pane) onto Γ when s is inside the EoT. Case for $\mathrm{Re}\,(s) < 0$.

Proposition 4.3 *Let us consider the transformation from $\Phi(z)$ to $W(z)$ as defined in Def. 3.3. The 2×2 matrix valued function $W(z)$ defined in 3.3 and solution of the RHPW must satisfy the following conditions:*

0) *$W(z)$ defined and holomorphic in $\mathbb{CP}^1 - \gamma$, (note that $\infty \in \gamma$)*

1) *Jump condition*

$$W(z_+) = \begin{cases} = W(z_-), \quad \text{on } \gamma - \Gamma \\ W(z_-) \begin{bmatrix} \mathrm{e}^{\frac{n}{2}\left(\varphi(z_-) - \varphi(z_+)\right)} & \tilde{Q}(z)\mathrm{e}^{\frac{n}{2}\left(\varphi(z_-) + \varphi(z_+)\right)} \\ 0 & \mathrm{e}^{-\frac{n}{2}\left(\varphi(z_-) - \varphi(z_+)\right)} \end{bmatrix}, \quad \text{on } \Gamma \end{cases}$$

2) *As $z \to \infty$ we have the limiting behaviour*

$$W(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big].$$

Proof.
The *RHPW* from proposition 3.7 is now adapted with $\tilde{Q}(z)$ as follows:

0) $W(z)$ defined and holomorphic in $\mathbb{CP}^1 - \gamma$, (note that $\infty \in \gamma$)

1) Jump condition on γ

$$W(z_+) = W(z_-)\begin{bmatrix} e^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & \tilde{Q}(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & e^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}$$

On $\gamma - \Gamma$

On $\gamma - \Gamma$, φ does not have a jump, so $\varphi(z_-) - \varphi(z_+) = 0$

$\Rightarrow e^{\pm\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} = 1.$

FACT: Moreover, $\gamma - \Gamma$ is contained in a region of the $z-$ plane where $\mathrm{Re}\left(\varphi(z)\right) < 0$, so

$$\begin{aligned}\tilde{Q}(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} &= \tilde{Q}(z)e^{n\varphi(z)} = \tilde{Q}(z)e^{n\left[\mathrm{Re}\left(\varphi(z)\right)+i\mathrm{Im}\left(\varphi(z)\right)\right]} \\ &= \tilde{Q}(z)\underbrace{e^{n\left[\mathrm{Re}\left(\varphi(z)\right)\right]}}_{\to 0 \quad \mathrm{as} n\to\infty} e^{i\left[n\mathrm{Im}\left(\varphi(z)\right)\right]} = 0\end{aligned}$$

(because $\tilde{Q}(z) \neq \infty$ as $z \neq 0, \infty$ when $z \in \gamma - \Gamma$)

Hence on $\gamma - \Gamma$,

$$W(z_+) = W(z_-)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = W(z_-)$$

Therefore we obtain:

$$W(z_+) = \begin{cases} = W(z_-), & \text{on } \gamma - \Gamma \\ W(z_-)\begin{bmatrix} e^{\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} & \tilde{Q}(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ 0 & e^{-\frac{n}{2}\left(\varphi(z_-)-\varphi(z_+)\right)} \end{bmatrix}, & \text{on } \Gamma \end{cases}$$

2)

$$\text{As} \quad z \to \infty, \quad W(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

■

Let us analyse the jump on Γ. We will use again the following equality to rewrite condition 1):

$$\begin{bmatrix} e^a & e^b \\ 0 & e^{-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-a-b} & 1 \end{bmatrix}\begin{bmatrix} 0 & e^{-b} \\ e^{-b} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ e^{a-b} & 1 \end{bmatrix}$$

Thus we have th jump condition on $\Gamma \implies$ taking the form

$$W(z_+) = \tag{4.12}$$

$$= W(z_-) \begin{bmatrix} 1 & 0 \\ \dfrac{e^{-n\varphi(z_-)}}{\tilde{Q}(z)} & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 0 & \tilde{Q}(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ -\dfrac{e^{-\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)}}{\tilde{Q}} & 0 \end{bmatrix}}^{:=m_{\tilde{Q}}} \begin{bmatrix} 1 & 0 \\ \dfrac{e^{-n\varphi(z_+)}}{\tilde{Q}(z)} & 1 \end{bmatrix} \tag{4.13}$$

The main arc $\Gamma = \Gamma_{a-}^{a+} \cup b_1$ as a union of branch cuts is where φ has a jump discontinuity
The result from relation 3.32 obtained on Γ_{a-}^{a+} can be generalized as follows on any branch cut:

$$\varphi(z_+) + \varphi(z_-) = \int_l^{z_+} \varphi'(w_+)\mathrm{d}z + \int_l^{z_-} \varphi'(w_-)\mathrm{d}z = 0 \tag{4.14}$$

where l and z belong to the same branch cut In particular, for $z \in \Gamma_b^{a+}$:

$$\varphi(z_+) + \varphi(z_-) = 0$$

Proposition 4.4 *In order compute $\varphi(z_+) + \varphi(z_-)$ on Γ_b^1 and Γ_b^{a+} (i.e with the additional branch cut b_1) we will need the following the Boutroux conditions defined as follows:*

$$\begin{cases} \Omega_1 := \displaystyle\int_b^{a_+} \varphi'(z_+)\mathrm{d}z \in i\mathbb{R}_- \\ \Omega_2 := \displaystyle\int_b^{1} \varphi'(z_-)\mathrm{d}z \in i\mathbb{R}_+ \end{cases} \tag{4.15}$$

Proposition 4.5

$$m_{\tilde{Q}} = \begin{bmatrix} 0 & \tilde{Q}(z)e^{\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)} \\ -\dfrac{e^{-\frac{n}{2}\left(\varphi(z_-)+\varphi(z_+)\right)}}{\tilde{Q}} & 0 \end{bmatrix} \tag{4.16}$$

$$\Rightarrow m_{\tilde{Q}} = \begin{cases} \begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}} & 0 \end{bmatrix} & On\ \ \Gamma^{b}_{a_-} \\ \begin{bmatrix} 0 & \tilde{Q}(z)\mathrm{e}^{n\Omega_1} \\ -\dfrac{\mathrm{e}^{-n\Omega_1}}{\tilde{Q}} & 0 \end{bmatrix} & On\ \ \Gamma^{1}_{b} \\ \begin{bmatrix} 0 & \tilde{Q}(z)\mathrm{e}^{n\Omega_2} \\ -\dfrac{\mathrm{e}^{-n\Omega_2}}{\tilde{Q}} & 0 \end{bmatrix} & On\ \ \Gamma^{a+}_{b} \end{cases} \tag{4.17}$$

Irrespectively of the branch cut, relation (4.13) is written as:

$$W(z_+) = W(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_-)}}{\tilde{Q}(z)} & 1 \end{bmatrix} m_{\tilde{Q}} \begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_+)}}{\tilde{Q}(z)} & 1 \end{bmatrix}$$

Proof.(of the above proposition (4.5))

Recalling that $\varphi(z) = \displaystyle\int_{a_-}^{z} \varphi'(w)\mathrm{d}w$, we obtain:

For $z \in \Gamma^1_b$

$$\varphi(z_+) + \varphi(z_-) = \left[\int_{a_-}^{b} \varphi'(w_+)\mathrm{d}w + \int_{b}^{a_+} \varphi'(w_+)\mathrm{d}w \right.$$
$$\left. + \int_{a_+}^{b} \varphi'(w_-)\mathrm{d}w + \int_{b}^{z_+} \varphi'(w_-)\mathrm{d}w \right] + \left[\int_{a_-}^{b} \varphi'(w_-)\mathrm{d}w + \int_{b}^{z_-} \varphi'(w_+)\mathrm{d}w \right]$$
$$= \int_{b}^{a_+} \varphi'(w_+)\mathrm{d}w + \int_{a_+}^{b} \varphi'(w_-)\mathrm{d}w \quad \left(\text{By using relation } 3.32 \text{ , see Remark below} \right)$$

$$= 2\int_b^{a_+} \varphi'(w_+)\mathrm{d}w = 2\Omega_1, \quad \Bigg(\text{since on} \quad \Gamma_{a_+}^b \quad \varphi(w_+) + \varphi(w_-) = 0$$

$$\Rightarrow \varphi'(w_-) = -\varphi'(w_+) \Rightarrow \int_{a_+}^b \varphi'(w_-)\mathrm{d}w = \int_b^{a_+} \varphi'(w_+)\mathrm{d}w\Bigg)$$

<u>Remark</u>

- By using relation 3.32, notice that the sum of the integrals of the same color cancel out.
- Around an arc which is not a branch cut, the contour integral is zero by Cauchy theorem.

So we obtained $\underline{\varphi(z_+) + \varphi(z_-) = 2\Omega_1 \text{ on } \Gamma_b^1}$

<u>For $z \in \Gamma_b^{a_+}$:</u>

$$\varphi(z_+) + \varphi(z_-) = \left[\int_{a_-}^b \varphi'(w_+)\mathrm{d}w + \int_b^{z_+} \varphi'(w_+)\mathrm{d}w\right] + \left[\int_{a_-}^b \varphi'(w_-)\mathrm{d}w\right.$$
$$\left.+ \int_b^1 \varphi'(w_-)\mathrm{d}w + \int_1^b \varphi'(w_+)\mathrm{d}w + \int_b^{z_-} \varphi'(w_-)\mathrm{d}w\right]$$
$$= \int_b^1 \varphi'(w_-)\mathrm{d}w + \int_1^b \varphi'(w_+)\mathrm{d}w \quad \Bigg(\text{The sum of the integrals}$$
$$\text{of the same color cancel out for the same reason as the previous computation}\Bigg)$$
$$= 2\int_1^b \varphi'(w_+)\mathrm{d}w = 2\Omega_2 \quad \Bigg(\text{For the same reason as the previous computation}\Bigg)$$

So $\underline{\varphi(z_+) + \varphi(z_-) = 2\Omega_2 \text{ on } \Gamma_b^{a_+}}$

■

The above proposition (4.5) implies that:

$$\underbrace{W(z_+)\begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{e}^{-n\varphi(z_+)}}{\tilde{Q}(z)} & 1 \end{bmatrix}}_{T(z_+)} = \underbrace{W(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z_-)}}{\tilde{Q}(z)} & 1 \end{bmatrix}}_{T(z_-)} m_{\tilde{Q}} \tag{4.18}$$

From the relation (4.18) above, we can notice that $W(z_+) = W(z_-)m_{\tilde{Q}}$ as $n \to \infty$ and with $z \neq a_+, a_-, 1, b$. More details will be given in the following lines with the Lens opening technique.

4.2 Second transformation: lens opening

We can define a region (or lens) around $\Gamma_{a_-}^{a_+}$ and b_1 bordered on the left and on the right respectively by two curves $\mathcal{L}_+$ and $\mathcal{L}_-$(to be arbitrarily chosen so that so that they stay in the region where $\mathrm{Re}\left(\varphi(z) > 0\right))$ as follows:

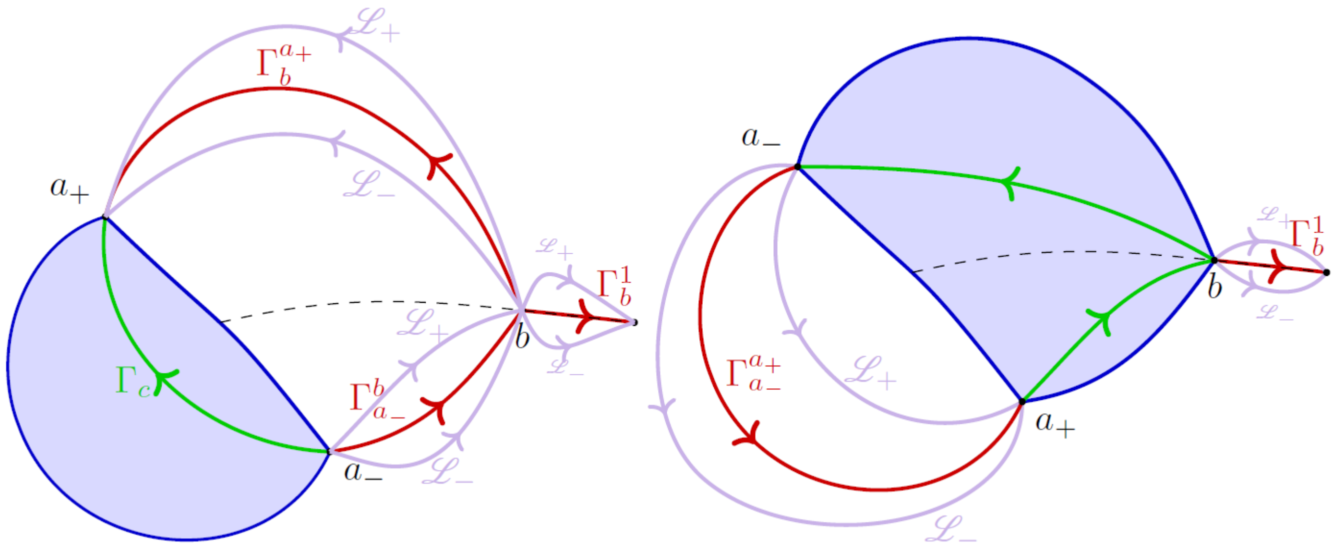

Figure 19: The lenses around Γ when s is inside the EoT according to the sign of $\mathrm{Re}\,(s)$.

Definition 4.6 *Let T be a transformation on that region such that,as suggested in relation (3.30):*

$$T(z) := \begin{cases} W(z), & \textit{outside of the regions (lenses)} \\ W(z)\begin{bmatrix} 1 & 0 \\ \dfrac{-\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 1 \end{bmatrix}, & \textit{on } \mathcal{R}_+ \\ W(z)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 1 \end{bmatrix}, & \textit{on } \mathcal{R}_- \end{cases}$$

where $\mathcal{R}_+$ and $\mathcal{R}_-$ are respectively the region on the left and the right of each branch cut, such that relation (3.30) gives

$$T(z_+) = T(z_-)\begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix} \textit{ on } \Gamma_{a_-}^b \cup \Gamma_b^1 \cup \Gamma_b^{a+}$$

Proposition 4.7 *The transformation T has the jump conditions on the lenses $\mathcal{L}_\pm$ as* $T(z_+) = T(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{e^{-n\varphi(z)}}{\tilde{Q}(z)} & 1 \end{bmatrix}$

Proof. See proof of Prop. 3.11 ■

4.2.1 Model problem and approximation

Let us leave the scheme of the lens opening. The rationale behind the next construction is analogous to the one used in Sec. 3.2.3, namely, that the jump conditions on the lenses will be shown to be exponentially close to the identity jump, and hence, in first approximation, can be *heuristically* neglected. The result of this is the RHP described hereafter.

Definition 4.8 *We define $M_{\tilde{Q}}(z)$ as a transformation not depending on n and that has a jump discontinuity only on the branch cut $\Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b$, such that:*

0) *$M_{\tilde{Q}}(z)$ defined and holomorphic in $\mathbb{CP}^1 - \gamma$, (note that $\infty \in \gamma$).*

1) *On γ*

$$M_{\tilde{Q}}(z_+) := \begin{cases} M_{\tilde{Q}}(z_-), & \text{on } \gamma - \left(\Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b\right) \\ M_{\tilde{Q}}(z_-)\begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix} & \text{on } \Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b \end{cases}$$

2)

$$As \quad z \to \infty, \quad M_{\tilde{Q}}(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$$

3) *All entries of $M_{\tilde{Q}}(z)$ are bounded by $|z-c|^{-\frac{1}{4}}$ as $z \to c$ and c is any of $1, b, a_+, a_-$.*

The conclusion of the approximation is given by defining a final transformation

$$E(z) = T(z)(M_{\tilde{Q}}(z))^{-1} \tag{4.19}$$

that connects $T(z)$ and $M_{\tilde{Q}}(z)$.

Proposition 4.9 *There exists a well defined transformation* $E(z) = T(z)M_{\tilde{Q}}(z)^{-1}$ *that connects* $T(z)$ *and* $M_{\tilde{Q}}(z)$. *The Riemann Hilbert Problem for* E, *i.e.* $RHPE$ *is as follows:*

0) $E(z)$ *defined and holomorphic in* $\mathbb{C} - (\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-)$

1)

$$E(z_+) = \begin{cases} E(z_-), & on\ \gamma \\ E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big), & on\ \mathcal{L}_- \cup \mathcal{L}_+ \end{cases}$$

where $G(z) = M_{\tilde{Q}}(z_-) \begin{bmatrix} 0 & 0 \\ \dfrac{e^{-n\varphi(z)}}{\tilde{Q}(z)} & 0 \end{bmatrix} (M_{\tilde{Q}}(z_-))^{-1}$

2)

$$As \quad z \to \infty, \quad E(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big]$$

Proof.
The Riemann Hilbert Problem for E, i.e. $RHPE$ is as follows:

0) $E(z)$ defined and analytic in $\mathbb{C} - (\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-)$, this is justified from the definitions of $T(z)$ and $M_{\tilde{Q}}(z)$ (note that $\infty \in \gamma$).

1) On $\gamma \cup \mathcal{L}_+ \cup \mathcal{L}_-$

- On $\gamma - \Big(\Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b\Big)$

$E(z_+) = T(z_+)(M_{\tilde{Q}}(z_+))^{-1} = T(z_-)(M_{\tilde{Q}}(z_-))^{-1}$ since T and $M_{\tilde{Q}}$ don't have a jump on $\gamma - \Big(\Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b\Big)$. Finally $E(z_+) = E(z_-)$

- On $\Gamma^b_{a_-} \cup \Gamma^1_b \cup \Gamma^{a_+}_b$

$$\begin{aligned} E(z_+) = T(z_+)(M_{\tilde{Q}}(z_+))^{-1} &= T(z_-)\begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix}\begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix}^{-1}(M_{\tilde{Q}}(z_-))^{-1} \\ &= T(z_-)(M_{\tilde{Q}}(z_-))^{-1} = E(z_-) \end{aligned}$$

- On $\mathcal{L}_- \cup \mathcal{L}_+$

$$
\begin{aligned}
E(z_+) = T(z_+)(M_{\tilde{Q}}(z_+))^{-1} &= T(z_-)\begin{bmatrix} 1 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 1 \end{bmatrix} \overbrace{(M_{\tilde{Q}}(z_-))^{-1}}^{=(M_{\tilde{Q}}(z_+))^{-1}} \\
&= T(z_-)\underbrace{(M_{\tilde{Q}}(z_-))^{-1} M_{\tilde{Q}}(z_-)}_{=\mathbb{I}_{2\times 2}} \left(\mathbb{I}_{2\times 2} + \begin{bmatrix} 0 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 0 \end{bmatrix}\right)(M_{\tilde{Q}}(z_-))^{-1} \\
&= T(z_-)\Bigg(\underbrace{(M_{\tilde{Q}}(z_-))^{-1} M_{\tilde{Q}}(z_-)}_{=\mathbb{I}_{2\times 2}} \mathbb{I}_{2\times 2}(M_{\tilde{Q}}(z_-))^{-1} \\
&\qquad + (M_{\tilde{Q}}(z_-))^{-1} \underbrace{M_{\tilde{Q}}(z_-)\begin{bmatrix} 0 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 0 \end{bmatrix}(M_{\tilde{Q}}(z_-))^{-1}}_{:=G(z)}\Bigg) \\
&= \underbrace{T(z_-)(M_{\tilde{Q}}(z_-))^{-1}}_{=E(z_-)} + \underbrace{T(z_-)(M_{\tilde{Q}}(z_-))^{-1}}_{=E(z_-)} G(z) \\
&= E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big)
\end{aligned}
$$

2)

$$
\text{As} \quad z \to \infty, \quad E(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big]
$$

■

The following theorem will allow us to estimate the function $G(z)$ and the discrepancy between $E(z)$ and the identity matrix.

Theorem 4.10 (Small Norm Theorem for RHP's:) *Let E be a matrix satisfying the following RHP:*

$$
\begin{cases}
E(z) = \Big[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\Big] \quad As \quad z \to \infty \\
E(z_+) = E(z_-)\Big(\mathbb{I}_{2\times 2} + G(z)\Big), \qquad\qquad \text{on a contour} \sum
\end{cases}
$$

Let $N(z) = \sqrt{\sum (G_{i,j}(z))^2}$. Suppose that $N(z) \in L^p\Big(\sum, |\mathrm{d}z|\Big)$ with $1 \leqslant p \leqslant \infty$. If $\|N(z)\|_\infty$ is small enough, then the RHP has a solution.

In our case,
$\sum = \mathcal{L}_- \cup \mathcal{L}_+$. We have

$$G(z) = M_{\tilde{Q}}(z_-) \begin{bmatrix} 0 & 0 \\ \dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} & 0 \end{bmatrix} (M_{\tilde{Q}}(z_-))^{-1} \tag{4.20}$$

Since $\sum$ is compact and $\dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)}$ is bounded on it, $N(z) \in L^p\left(\sum, |\mathrm{d}z|\right)$ with $1 \leqslant p \leqslant \infty$.
$\forall z \in \sum, \quad \tilde{Q}(z) \neq 0$.
In addition,
$\forall z \in \sum - \{a_\pm, b, 1\}$, $\mathrm{Re}\,(\varphi(z)) > 0 \Rightarrow$ as $n \to \infty$, $\dfrac{\mathrm{e}^{-n\varphi(z)}}{\tilde{Q}(z)} \to 0$.
Hence $\|N(z)\|_\infty$ is small enough. However, for $z = a_\pm, b, 1$, $\mathrm{e}^{-n\varphi(z)} \nrightarrow 0$ as $n \to \infty$ and $\|N(z)\|_\infty$ can not be made small enough.

Therefore we need to apply the small norm theorem to a contour that excludes the points $z = a_+$, $z = a_-$, $z = b$ and $z = 1$. To achieve that, let us redefine $T(z)$ in the neighbourhood of those points: let us $\mathcal{D}_+$, $\mathcal{D}_-$, $\mathcal{D}_b$ and $\mathcal{D}_1$ be four disks with radius δ, $\forall \delta > 0$ in the neighbourhoods. If $\tilde{T}(z)$ is the redefinition of $T(z)$, we will have $E(z) := \tilde{T}(z)(M_{\tilde{Q}}(z))^{-1}$
Let us start the procedure by first considering only the disks $\mathcal{D}_+$, $\mathcal{D}_-$, $\mathcal{D}_b$ and $\mathcal{D}_1$.

Definition 4.11 (Local Parametrices) *Let's define a $P(z,n)$ as follows:*

$$P(z,n) = \begin{cases} P_{\mathcal{D}_{\pm,b,1}}(z,n), & \textit{inside} \quad \mathcal{D}_{\pm,b,1} \\ T(z), & \textit{on} \quad \partial\mathcal{D}_{\pm,b,1} \end{cases}$$

where $P_{\mathcal{D}_{\pm,b,1}}(z,n)$ is some transformation which has the same jump as $T(z)$on the main arcs $\Gamma^b_{a_-}$, Γ^1_b, Γ^{a+}_b and the lenses $\mathcal{L}_\pm$ as $T(z)$ inside $\mathcal{D}_\pm$
Now, let us define :

$$\tilde{T}(z) := \begin{cases} T(z), & \textit{on} \quad \mathbb{C} - \mathcal{D}_{\pm,b,1} \\ P(z,n)(T(z))^{-1}M_{\tilde{Q}}(z), & \textit{on} \quad \mathcal{D}_{\pm,b,1} \end{cases}$$

The existence of appropriate parametrices is a delicate but standard construction and can be found in [7, 8, 11, 16, 17]. We will not report their construction here.

It follows from the definition that:

$$E(z) = \tilde{T}(z)(M_{\tilde{Q}}(z))^{-1} = \begin{cases} T(z)(M_{\tilde{Q}}(z))^{-1}, \quad \text{on} \quad \mathbb{C} - \mathcal{D}_{\pm,b,1} \\ P(z,n)(T(z))^{-1} = \begin{cases} P_{\mathcal{D}_{\pm,b,1}}(z,n)(T(z))^{-1} := \tilde{P}_{\mathcal{D}_{\pm,b,1}}(z,n), \\ \qquad\qquad\qquad\qquad \text{inside} \quad \mathcal{D}_{\pm,b,1} \\ \mathbb{I}_{2\times 2} + \mathcal{O}(\frac{1}{n}), \quad \text{on} \quad \partial\mathcal{D}_{\pm,b,1} \end{cases} \end{cases} \tag{4.21}$$

Notice that:

⊙ $\tilde{P}_{\mathcal{D}_{\pm,b,1}}(z,n)$ as defined above, does not have a jump on both the main arc and the lenses inside $\mathcal{D}_{\pm,b,1}$

⊙ since $a_\pm, b, 1 \notin (\mathbb{C} - \mathcal{D}_{\pm,b,1})$, and $E(z) = T(z)(M_{\tilde{Q}}(z))^{-1}$ on $\mathbb{C} - \mathcal{D}_{\pm,b,1}$ the RHP for E has a solution (as we derived before).

In particular, $E(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$ As $\quad z \to \infty$.

⊙ inside $\mathcal{D}_{\pm,b,1}$ and on $\partial\mathcal{D}_{\pm,b,1}$, the RHP for E is:

$$E(z_+) = E(z_-)(\left(\mathbb{I}_{2\times 2} + \underbrace{0_{2\times 2}}_{=G(z)}\right)$$

$G(z) = 0_{2\times 2}$ thus the RHP for $E = \tilde{T}(z)(M_{\tilde{Q}}(z))^{-1}$ has a solution (i.e. exists) by the small norm theorem.

Comments:

◦ Since the Riemann Hilbert Problem for the transformation E has a solution (i.e. E exists), we can know consider the RHP of transformation $M_{\tilde{Q}}(z) = (E(z))^{-1}\tilde{T}(z)$ which satisfies the conditions as defined in Def. 4.8 and solve it. Notice that the expressions of $M_{\tilde{Q}}(z)$ inside and outside the disks are different, although they both have the same jump matrix on $\Gamma^b_{a_-}$, Γ^1_b, and $\Gamma^{a_+}_b$.

◦ Finding a solution to the RHP$M_{\tilde{Q}}$ is equivalent to finding a solution to the RHPΦ. In order to find an explicit solution to the RHP$M_{\tilde{Q}}$, we will need a further transformation $M_{\tilde{Q}} \longrightarrow M$ to the Model problem, which is easier to solve explicitly.

4.3 Third transformation: $M_{\tilde{Q}} \longrightarrow M$

In order to obtain a Riemann Hilbert Problem of Def. 4.8 we define a new matrix $M(z)$, with a *constant* jump matrix on Γ_m and no jump at ∞. Let us consider the following definition.

Definition 4.12 *Let us define*

$$M(z) := \mathrm{e}^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z)\mathrm{e}^{S(z)\sigma_3}$$

The function S(z) is the Szegö function defined in Def. 4.14.

Analyticity of $M(z)$ on $\mathbb{C}\setminus\gamma$. We need to impose that $S(z)$ is bounded and analytic on $\mathbb{C}\setminus\gamma$. This condition leads us obtain the desired conclusion that $M(z) = \mathrm{e}^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z)\mathrm{e}^{S(z)\sigma_3}$ is analytic on $\mathbb{C} - \gamma$ because $M_{\tilde{Q}}$ is.

Jump Condition of $M(z)$ on γ.

- $\underline{\text{On } \gamma - \overbrace{(\Gamma_{a_-}^{b} \cup b_1 \cup \Gamma_{b}^{a+})}^{=\tilde{\Gamma}}}$ (We define $\tilde{\Gamma} := \Gamma_{a_-}^{b} \cup b_1 \cup \Gamma_{b}^{a+}$)

$$\begin{aligned} M(z_+) &= \mathrm{e}^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z_+) \mathrm{e}^{S(z_+)\sigma_3} \\ &= \mathrm{e}^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z_-) \mathrm{e}^{S(z_+)\sigma_3} = M(z_-) \end{aligned}$$

- $\underline{\text{On } \tilde{\Gamma}}$

$$\begin{aligned} M(z_+) &= \mathrm{e}^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z_+) \mathrm{e}^{S(z_+)\sigma_3} \\ &= \mathrm{e}^{-S(\infty)\sigma_3} \tilde{T}_-(z) \overbrace{\mathrm{e}^{S(z_-)\sigma_3} \mathrm{e}^{-S(z_-)\sigma_3}}^{=\mathbb{I}_{2\times 2}} \begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix} \mathrm{e}^{S(z_+)\sigma_3} \\ &= M(z_-) \mathrm{e}^{-S(z_-)\sigma_3} \begin{bmatrix} 0 & \tilde{Q}(z) \\ -\dfrac{1}{\tilde{Q}(z)} & 0 \end{bmatrix} \mathrm{e}^{S(z_+)\sigma_3} \end{aligned}$$

$$\Rightarrow M(z_+) = \begin{cases} M(z_-) \begin{bmatrix} 0 & \tilde{Q}(z) \mathrm{e}^{-\left(S(z_+)+S(z_-)\right)} \\ -\dfrac{\mathrm{e}^{\left(S(z_+)+S(z_-)\right)}}{\tilde{Q}} & 0 \end{bmatrix}, & \text{0n } \Gamma_{a_-}^{b} \\ M(z_-) \begin{bmatrix} 0 & \tilde{Q}(z) \mathrm{e}^{-\left(S(z_+)+S(z_-)\right)} \mathrm{e}^{n\Omega_1} \\ -\dfrac{\mathrm{e}^{\left(S(z_+)+S(z_-)\right)}}{\tilde{Q}} \mathrm{e}^{-n\Omega_1} & 0 \end{bmatrix}, & \text{0n } \Gamma_{b}^{1} \\ M(z_-) \begin{bmatrix} 0 & \tilde{Q}(z\mathrm{e}^{-\left(S(z_+)+S(z_-)\right)}) \mathrm{e}^{n\Omega_2} \\ -\dfrac{\mathrm{e}^{\left(S(z_+)+S(z_-)\right)}}{\tilde{Q}} \mathrm{e}^{-n\Omega_2} & 0 \end{bmatrix}, & \text{0n } \Gamma_{b}^{a+} \end{cases}$$

Therefore, for the jump condition with a constant matrix to be satisfied, we must impose the condition to have $S(z)$ bounded on $\tilde{\Gamma}$,
and

$$S(z_+) + S(z_-) = \begin{cases} \ln(\tilde{Q}(z)) & \text{, on } \Gamma_{a_-}^{b} \cup \Gamma_{b}^{a+} \\ \ln(\tilde{Q}(z)) - \nu & \text{, on } \Gamma_{b}^{1} \end{cases} \tag{4.22}$$

for some constant ν that will be found below. The jump conditions for $M(z)$ above become thus

$$M(z_+) = \begin{cases} M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \text{0n } \Gamma^b_{a_-} \\ M(z_-)\begin{bmatrix} 0 & e^{n\Omega_1+\nu} \\ -e^{-n\Omega_1-\nu} & 0 \end{bmatrix}, & \text{0n } \Gamma^1_b \\ M(z_-)\begin{bmatrix} 0 & e^{n\Omega_2} \\ -e^{-n\Omega_2} & 0 \end{bmatrix}, & \text{0n } \Gamma^{a+}_b \end{cases} \tag{4.23}$$

Proposition 4.13 *The function $S(z)$ defined in Definition has the properties:*
⊙ *$S(z)$ is analytic and bounded on $\mathbb{CP}^1 - \tilde{\Gamma}$*
⊙ *$S(z)$ has finite boundary values along $\tilde{\Gamma}$ which satisfy*

$$S(z_+) + S(z_-) = \begin{cases} \ln(\tilde{Q}(z)) & , on \quad \Gamma^b_{a_-} \cup \Gamma^{a+}_b \\ \ln(\tilde{Q}(z)) - \nu & , on \quad \Gamma^1_b \end{cases} \tag{4.24}$$

with ν given in (4.30).

Definition 4.14 *We define*

$$S(z) := \frac{J(z)}{2\pi i}\left[\int_{\tilde{\Gamma}\backslash b_1} \frac{\ln(Q(w))}{J(w_+)(w-z)}\mathrm{d}w + \int_b^1 \frac{\ln(Q(w)) - \nu}{J(w_+)(w-z)}\mathrm{d}w\right], \tag{4.25}$$

for $z \notin \tilde{\Gamma}$, where $J(z) := \sqrt{(z-a_+)(z-a_-)(z-b)(z-1)}$. All integrations are along the arcs in $\tilde{\Gamma}$ (depending on the case $\mathrm{Re}\,(s) > 0$ or $\mathrm{Re}\,(s) < 0$). The constant ν is defined in (4.30).

Proposition 4.15 *Let's consider the transformation*
$M(z) = e^{-S(\infty)\sigma_3} M_{\tilde{Q}}(z) e^{S(z)\sigma_3}$ as defined in Def. 4.12. The 2×2 matrix valued function W(z) solution of the RHPM must satisfy the following conditions:
0) *M is defined and holomorphic in $\mathbb{CP}^1 - \gamma$*
1) *On γ*
- *On $\gamma \setminus \tilde{\Gamma}$: M is defined and holomorphic.*
- *On $\tilde{\Gamma}$:*

$$M(z_+) = \begin{cases} M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \mathit{0n}\ \ \Gamma^b_{a-} \\ M(z_-)\begin{bmatrix} 0 & \mathrm{e}^{n\Omega_1+\nu} \\ -\mathrm{e}^{-n\Omega_1-\nu} & 0 \end{bmatrix}, & \mathit{0n}\ \ \Gamma^1_b \\ M(z_-)\begin{bmatrix} 0 & \mathrm{e}^{n\Omega_2} \\ -\mathrm{e}^{-n\Omega_2} & 0 \end{bmatrix}, & \mathit{0n}\ \ \Gamma^{a+}_b \end{cases}$$

$$\textit{with}\ \nu = 2\pi i\left(\theta + \frac{\displaystyle\int_b^0 \frac{1}{J(w_+)}\mathrm{d}w}{\displaystyle\int_{b_+}^{1_+} \frac{1}{J(w_+)}\mathrm{d}w}\right)$$

2) *As* $z \longrightarrow \infty$ $M(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]$

Proof. of the propositions 4.13, 4.15 and motivation for the definition 4.14 above

Construction of the Szegö function $S(z)$. Let us define for convenience

$$J(z) := \sqrt{(z-a_+)(z-a_-)(z-b)(z-1)} \tag{4.26}$$

with the branch-cuts on Γ_m; thus $J(z_+) = -J(z_-)$ on each sub-branch of $\tilde{\Gamma}$ (i.e. $\Gamma^b_{a_-}$, Γ^1_b and Γ^{a+}_b). We will use (impose) the condition of analyticity and boundedness of $S(z)$ on $\mathbb{CP}^1 - \tilde{\Gamma}$ (in particular at ∞) to find ν.

Let us compute $S(z)$. We know that:

$$S(z_+) + S(z_-) = \ln(\tilde{Q}(z)) + \nu\chi_{b_1}, \quad z \in \tilde{\Gamma}, \tag{4.27}$$

where χ_{b_1} denotes the characteristic function of the arc b_1. Therefore for each sub-branch of $\tilde{\Gamma}$

$$
\begin{aligned}
&\frac{S(z_+)}{J(z_+)} + \frac{S(z_-)}{J(z_+)} = \frac{\ln(\tilde{Q}(z)) + \nu\chi_{b_1}}{J(z_+)} \\
\Rightarrow &\frac{S(z_+)}{J(z_+)} - \frac{S(z_-)}{J(z_-)} = \frac{\ln(\tilde{Q}(z)) + \nu\chi_{b_1}}{J(z_+)}
\end{aligned}
$$

$$
\implies \text{(by Sokhotski–Plemelj theorem on } \frac{S(z)}{J(z)}) \text{ we obtain}
$$

$$
\begin{aligned}
S(z) = \frac{J(z)}{2\pi i}\Bigg[&\underbrace{\int_{\Gamma_{a_-}^{b}} \frac{\ln(\tilde{Q}(w))}{J(w_+)(w-z)}\mathrm{d}w}_{:=I_1} + \underbrace{\int_{\Gamma_b^1} \frac{\ln(\tilde{Q}(w)) + \nu}{J(w_+)(w-z)}\mathrm{d}w}_{:=I_2} + \\
&\underbrace{\int_{\Gamma_b^{a_+}} \frac{\ln(\tilde{Q}(w))}{J(w_+)(w-z)}\mathrm{d}w}_{:=I_3}\Bigg], \qquad z \notin \tilde{\Gamma}
\end{aligned}
$$

Notice that the expression of $S(z)$ provided by Sokhotski–Plemelj is indeed bounded on $\tilde{\Gamma} - \{a_\pm, b, 1\}$ as we want: in fact, as $z \notin \tilde{\Gamma}$, $S(z)$ is bounded because the contour of integration $\tilde{\Gamma}$ is compact, and the integrand of $S(z)$ is bounded on it.

Back to the computation of $S(z)$, first let us focus on the expression of $\tilde{Q}(z)$ on Γ_b^1. Recall the jump relation 1)that we obtained in Prop. 4.1. We have:

$$
\tilde{Q}(z_-) := \varsigma Q(z_-) \Longrightarrow \ln(\tilde{Q}(z_-)) = \ln(\overbrace{Q(z_-)}^{=\mathrm{e}^{-2\pi i\theta}Q(z_+)}) + \ln(\varsigma) = -2i\pi\theta + \ln(Q(z_+) + \ln(\varsigma) \tag{4.28}
$$

By arbitrarily taking (the direction) $\Gamma_b^1 = \Gamma_{b_-}^{1-}$ we can write:

$$
\begin{aligned}
I_2 &= \frac{1}{2}\Big[\int_{b_-}^{1_-} \frac{\ln(\tilde{Q}(w_-))+\nu}{J(w_+)(w-z)}\mathrm{d}w + \int_{b_-}^{1_-} \frac{\ln(\tilde{Q}(w_-))+\nu}{J(w_+)(w-z)}\mathrm{d}w\Big]\\
&= \frac{1}{2}\Big[\int_{b_-}^{1_-} \frac{\ln(Q(w_-))+\ln(\varsigma)+\nu}{J(w_+)(w-z)}\mathrm{d}w + \int_{b_+}^{1_+} \frac{-2i\pi\theta+\ln(Q(w_+))+\ln(\varsigma)+\nu}{J(w_+)(w-z)}\mathrm{d}w\Big]\\
&= \frac{1}{2}\Big[\int_{b_-}^{1_-} \frac{\ln(Q(w_-))+\ln(\varsigma)+\nu}{J(w_+)(w-z)}\mathrm{d}w - \int_{1_+}^{b_+} \frac{\ln(Q(w_+))+\ln(\varsigma)+\nu}{J(w_+)(w-z)}\mathrm{d}w\\
&+ \int_{b_+}^{1_+} \frac{-2i\pi\theta+\nu}{J(w_+)(w-z)}\mathrm{d}w\Big]\\
&= \frac{1}{2}\Big[\oint_{\Gamma_b^1} \frac{\overbrace{\ln(Q(w_-))-\ln(Q(w_+))}^{=-2i\pi\theta}}{J(w_+)(w-z)}\mathrm{d}w + \int_{b_+}^{1_+} \frac{-2i\pi\theta+\nu}{J(w_+)(w-z)}\mathrm{d}w\Big]\\
&= \frac{1}{2}\Big[\oint_{\Gamma_b^1} \frac{-2i\pi\theta}{J(w_+)(w-z)}\mathrm{d}w + \int_{b_+}^{1_+} \frac{-2i\pi\theta+\nu}{J(w_+)(w-z)}\mathrm{d}w\Big],
\end{aligned}
$$

$\Gamma_{a_-}^b \cup \Gamma_b^{a_+} \subset \gamma - \gamma_b^1$. So, on $\Gamma_{a_-}^b \cup \Gamma_b^{a_+}$ we have $\tilde{Q}(w)) = Q(w)$, hence:

$$
I_1 = \frac{1}{2}\oint_{\Gamma_{a_-}^b} \frac{\ln(Q(w))}{J(w_+)(w-z)}\mathrm{d}w \quad \text{and} \quad I_3 = \frac{1}{2}\oint_{\Gamma_b^{a_+}} \frac{\ln(Q(w))}{J(w_+)(w-z)}\mathrm{d}w
$$

On the other hand, we know that :

$$= \left(I_2 - \frac{1}{2} \int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)(w-z)} \mathrm{d}w \right)$$

$$\overbrace{\frac{1}{2} \oint_{\Gamma_{a_-}^b} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w}^{=I_1} + \overbrace{\frac{1}{2} \oint_{\Gamma_b^1} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w}$$

$$+ \underbrace{\frac{1}{2} \oint_{\Gamma_b^{a_+}} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w}_{=I_3} + \frac{1}{2} \oint_{\varepsilon_b^0} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w$$

$$= \frac{1}{2} \oint_{w \gg 1} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w \tag{4.29}$$

Let us discuss the behaviour of the integrand at ∞.

For the denominator, as $w \to \infty$
$J(w_+) = \sqrt{(w-a_-)(w-a_+)(w-b)(w-1)} \sim w^2$ and $w - z \sim w$
so $J(w_+)(w-z) \sim w^3$ and,

$$\lim_{w\to\infty} \frac{\ln(Q(w))}{J(w_+)(w-z)} = \lim_{w\to\infty} \frac{\ln\left((z^{-m})(1-\frac{1}{z})^{\theta}\right)}{w^3} = \lim_{w\to\infty} \frac{\ln(z^{-m})}{w^3} \overbrace{=}^{l'Hop} \lim_{w\to\infty} \frac{\alpha}{w^4} = 0$$

Therefore, as $w \to \infty$ the integrand $\dfrac{\ln(Q(w))}{J(w_+)(w-z)} \sim \dfrac{1}{w^{3-b}} \to 0$ with $b < 1$.

So we obtain that $\displaystyle \frac{1}{2} \oint_{w \gg 1} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w = 0$ and with relation (4.29) this implies that:

$$I_1 + I_2 + I_3 = \frac{1}{2} \int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)(w-z)} \mathrm{d}w - \frac{1}{2} \oint_{\varepsilon_b^0} \frac{\ln(Q(w))}{J(w_+)(w-z)} \mathrm{d}w$$

Hence,

$$S(z) = \frac{J(z)}{2\pi i}(I_1 + I_2 + I_3)$$
$$= \frac{J(z)}{\pi i}\Big[\int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)(w-z)}\mathrm{d}w - \oint_{\varepsilon_b^0} \frac{\ln(Q(w))}{J(w_+)(w-z)}\mathrm{d}w\Big]$$

As $z \to \infty$, $J(z) \sim z^2$ and $\frac{1}{w-z} \sim -\frac{1}{z}$, hence as $z \to \infty$:

$$S(z) = -\frac{c.z}{\pi i}\Big[\int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)}\mathrm{d}w - \oint_{\varepsilon_b^0} \frac{\ln(Q(w))}{J(w_+)}\mathrm{d}w\Big] + \mathcal{O}(1), \text{ where } c \in \mathbb{C} \text{ is a constant}$$

Since $M(z)$ is $S(z)$ bounded everywhere in particular on $\Gamma(\infty)$ (i.e. at ∞), we must set the coefficient of z to 0 i.e. $\displaystyle\int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)}\mathrm{d}w - \oint_{\varepsilon_b^0} \frac{\ln(Q(w))}{J(w_+)}\mathrm{d}w = 0$ and find ν accordingly.

It implies that:

$$\int_{b_+}^{1_+} \frac{-2\pi i\theta + \nu}{J(w_+)}\mathrm{d}w - \Bigg(\int_{b_+}^{0_+} \frac{\ln(Q(w_+))}{J(w_+)}\mathrm{d}w + \underbrace{\int_{0_-}^{b_-} \frac{\ln(Q(w_-))}{J(w_+)}\mathrm{d}w}_{-\int_{b_-}^{0_-} \frac{\overbrace{\ln(Q(w_-))}^{=-2i\pi\theta+\ln(Q(w_+))}}{J(w_+)}\mathrm{d}w}\Bigg) = 0$$

$$\Rightarrow -2\pi i\Big(\int_{b_+}^{1_+} \frac{\theta}{J(w_+)}\mathrm{d}w + \int_{b}^{0} \frac{1}{J(w_+)}\mathrm{d}w\Big) = -\int_{b_+}^{1} \frac{\nu}{J(w_+)}\mathrm{d}w$$

$$\Rightarrow \nu = 2\pi i\left(\theta + \frac{\displaystyle\int_b^0 \frac{1}{J(w_+)}\mathrm{d}w}{\displaystyle\int_{b_+}^{1_+} \frac{1}{J(w_+)}\mathrm{d}w}\right) \tag{4.30}$$

<u>**Expansion of $M(z)$ at ∞:**</u>

$$M(z) = \mathrm{e}^{S(\infty)\sigma_3} M_Q(z)\mathrm{e}^{-S(z)\sigma_3} \implies M(\infty) = \mathrm{e}^{-S(\infty)\sigma_3} \underbrace{\left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right]}_{M_Q(\infty)} \mathrm{e}^{S(\infty)\sigma_3}$$

$$\implies \text{ as } \; z \to \infty, M(z) = \left[\mathbb{I}_{2\times 2} + \mathcal{O}(z^{-1})\right], \;\; \text{Since} \;\; S(z) \;\; \text{is bounded at} \quad \infty$$

■

4.4 Solution of the Model Problem (RHPM)

The Riemann surface defined by $\xi^2 = (z - a_-)(z - a_+)(z - b)(z - 1)$ is an elliptic curve of genus 1. Let ω_1 and ω_2 be the two periods associated to it.
For all (z, ξ) on the elliptic curve, let us define the Abel map:

$$u(z, \xi) = u(z) := \int_{a_-}^{z} \frac{1}{2J(w)\omega_1}\mathrm{d}w \tag{4.31}$$

where $J(w)$ is defined in relation 4.26

$$\text{and} \;\; \omega_1 := \int_b^1 \frac{1}{J(w_+)}\mathrm{d}w \;\; , \;\; \omega_2 := \int_b^{a_+} \frac{1}{J(w_-)}\mathrm{d}w. \quad \text{Let us define} \quad \tau := \frac{\omega_2}{\omega_1} \tag{4.32}$$

Let us consider $\Theta(u(z), \tau)$ the Riemann $\Theta-$ function. It satisfies the properties:

$$\begin{cases} \Theta(u(z), \tau) = 0 \quad \text{for} \quad u(z) = \dfrac{\tau + 1}{2} + k + l\tau \quad \text{with } \forall k, l \in \mathbb{Z} \\ \Theta(u(z) + k + l\tau, \tau) = \exp\left(-2\pi i l u(z) - i\pi l^2\tau\right)\Theta(u(z), \tau) \end{cases} \tag{4.33}$$

Proposition 4.16 *Let* $M(z) = \begin{bmatrix} A(z) \\ B(z) \end{bmatrix}$ *where* $A(z)$ *and* $B(z)$ *are row vectors forming the matrix. Up to scalar normalization, the solution* $M(z)$ *of the Model Problem when* $s \in EoT$ *is such that :*

$$A(z) := \begin{bmatrix} A_1(z) & A_2(z) \end{bmatrix}$$

$$= \left[\frac{i\Theta\Big(u(z) - u(\infty) - \frac{\tau+1}{2} + G\Big)h(z)}{\Theta\Big(u(z) - u(\infty) - \frac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z)}} \quad \frac{\Theta\Big(-u(z) - u(\infty) - \frac{\tau+1}{2} + G\Big)h(z)}{\Theta\Big(-u(z) - u(\infty) - \frac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z)}} \right] \tag{4.34}$$

and,

$$B(z) := \begin{bmatrix} B_1(z) & B_2(z) \end{bmatrix}$$

$$= \left[\frac{i\Theta\Big(u(z) + u(\infty) - \frac{\tau+1}{2} + G\Big)h(z)}{\Theta\Big(u(z) + u(\infty) - \frac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z)}} \quad \frac{\Theta\Big(-u(z) + u(\infty) - \frac{\tau+1}{2} + G\Big)h(z)}{\Theta\Big(-u(z) + u(\infty) - \frac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z)}} \right] \tag{4.35}$$

where $h(z) = \dfrac{1}{\Big((z-a_-)(z-a_+)(z-b)(z-1)\Big)^{1/4}}$ *and* G, K *are constants to be determined so that* $M(z)$ *as stated above satisfies the conditions 0), 1) and 2) of the RHPM in Proposition 4.15.*

Finding G and K so that condition 2) is verified. We now set out to find the actual expressions for G, K; the results are contained in Prop. 4.23.

Proposition 4.17 *As* $z \to \infty$ *we have:*

$$M(z) \longrightarrow \begin{bmatrix} \dfrac{-i\Theta\Big(-\frac{\tau+1}{2} + G\Big)}{2\omega_1\Theta'\Big(-\frac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(\infty)}} & 0 \\ 0 & \dfrac{\Theta\Big(-\frac{\tau+1}{2} + G\Big)}{2\omega_1\Theta'\Big(-\frac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(\infty)}} \end{bmatrix} \tag{4.36}$$

Note that $\Theta'\Big(-\dfrac{\tau+1}{2}\Big)\neq 0$ *because the zeroes of* Θ *are all simple.*

Proof.

We know that as $z\longrightarrow\infty$ $M(z)=\Big[\mathbb{I}_{2\times 2}+\mathcal{O}(z^{-1})\Big]$. So we should have $\lim_{z\to\infty}A_1(z)=1$. Clearly $\lim_{z\to\infty}h(z)=0$.

Therefore, we have,

- $\lim_{z\to\infty}A_1(z)=\dfrac{i\Theta\Big(-\dfrac{\tau+1}{2}+G\Big).0}{\Theta\Big(-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi Ku(\infty)}}$. By relation (4.33) of the Riemann $\Theta-$ function, $\Theta\Big(-\dfrac{\tau+1}{2}\Big)=\Theta\Big(\dfrac{\tau+1}{2}-1-\tau\Big)=0$ by taking $k=l=1$. So $\lim_{z\to\infty}A_1(z)=\dfrac{0}{0}$ and by l'hopital rule, we have:

$$\lim_{z\to\infty}A_1(z)=$$

$$=\lim_{z\to\infty}i\frac{u'(z)\Theta'\Big(u(z)-u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z)+\Theta\Big(u(z)-u(\infty)-\dfrac{\tau+1}{2}+G\Big)h'(z)}{\mathrm{e}^{-i\pi Ku(z)}\Big[u'(z)\Theta'\Big(u(z)-u(\infty)-\dfrac{\tau+1}{2}\Big)-i\pi Ku'(z)\Theta\Big(u(z)-u(\infty)-\dfrac{\tau+1}{2}\Big)\Big]}$$

$$=i\frac{\Theta'\Big(-\dfrac{\tau+1}{2}+G\Big)\overbrace{h(\infty)}^{=0}+\Theta\Big(-\dfrac{\tau+1}{2}+G\Big)\lim_{z\to\infty}\dfrac{h'(z)}{u'(z)}}{\mathrm{e}^{-i\pi Ku(\infty)}\Big[\Theta'\Big(-\dfrac{\tau+1}{2}\Big)-i\pi K\underbrace{\Theta\Big(-\dfrac{\tau+1}{2}\Big)}_{=0}\Big]}$$

As $z\to\infty$, $h(z)\sim\dfrac{1}{z}\Rightarrow h'(z)\sim-\dfrac{1}{z^2}$. Also, $z\to\infty$, $u'(z)\sim\dfrac{1}{2\omega_1 z^2}$ hence $\dfrac{h'(z)}{u'(z)}\sim-\dfrac{1}{2\omega_1}$ as $z\to\infty$

Therefore, $\lim_{z\to\infty}A_1(z)=-i\dfrac{\Theta\Big(-\dfrac{\tau+1}{2}+G\Big)}{2\omega_1\Theta'\Big(-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi Ku(\infty)}}$.

- $\lim_{z\to\infty}A_2(z)=\dfrac{\overbrace{\Theta\Big(-2u(\infty)-\dfrac{\tau+1}{2}+G\Big)}^{<\infty\quad\text{TO BE PROVED}}.0}{\underbrace{\Theta\Big(-2u(\infty)-\dfrac{\tau+1}{2}\Big)}_{\neq 0\quad\text{TO BE PROVED}}\mathrm{e}^{i\pi Ku(\infty)}}=0$

Similarly, we have:

- $\lim_{z\to\infty} B_1(z) = i\dfrac{\overbrace{\Theta\Big(2u(\infty) - \frac{\tau+1}{2} + G\Big)}^{<\infty \quad \text{TO BE PROVED}}.0}{\underbrace{\Theta\Big(2u(\infty) - \frac{\tau+1}{2}\Big)}_{\neq 0 \quad \text{TO BE PROVED}}\mathrm{e}^{-i\pi K u(\infty)}} = 0$

- $\lim_{z\to\infty} B_2(z) = \dfrac{\Theta\Big(-\frac{\tau+1}{2} + G\Big).0}{\Theta\Big(-\frac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(\infty)}}$. By relation (4.33) of the Riemann $\Theta-$ function,

$\Theta\Big(-\frac{\tau+1}{2}\Big) = \Theta\Big(\frac{\tau+1}{2} - 1 - \tau\Big) = 0$ by taking $k = l = 1$. So $\lim_{z\to\infty} B_2(z) = \frac{0}{0}$ and by l'hopital rule, we have:

$$\lim_{z\to\infty} B_2(z) =$$
$$= \lim_{z\to\infty} \frac{-\Theta'\Big(u(\infty) - u(z)) - \frac{\tau+1}{2} + G\Big)h(z) + \Theta\Big(u(\infty) - u(z) - \frac{\tau+1}{2} + G\Big)\frac{h'(z)}{u'(z)}}{\mathrm{e}^{i\pi K u(z)}\Big[-\Theta'\Big(u(\infty) - u(z) - \frac{\tau+1}{2}\Big) + i\pi K\Theta\Big(u(\infty) - u(z) - \frac{\tau+1}{2}\Big)\Big]}$$

$$= \frac{-\Theta'\Big(-\frac{\tau+1}{2} + G\Big)\overbrace{h(\infty)}^{=0} + \Theta\Big(-\frac{\tau+1}{2} + G\Big)\lim_{z\to\infty}\frac{h'(z)}{u'(z)}}{\mathrm{e}^{i\pi K u(\infty)}\Big[-\Theta'\Big(-\frac{\tau+1}{2}\Big) + i\pi K\underbrace{\Theta\Big(-\frac{\tau+1}{2}\Big)}_{=0}\Big]}$$

As $z \to \infty$, $h(z) \sim \frac{1}{z} \Rightarrow h'(z) \sim -\frac{1}{z^2}$. Also, $z \to \infty$, $u'(z) \sim \frac{1}{2\omega_1 z^2}$ hence $\frac{h'(z)}{u'(z)} \sim -\frac{1}{2\omega_1}$ as $z \to \infty$

Therefore, $\lim_{z\to\infty} B_2(z) = \dfrac{\Theta\Big(-\frac{\tau+1}{2} + G\Big)}{2\omega_1\Theta'\Big(-\frac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(\infty)}}$.

■

The above matrix (in proposition 4.17) is invertible if and only if the Θ in the numerator is nonzero (all other terms are automatically different from zero).

$$\Theta\Big(-\frac{\tau+1}{2} + G\Big) \neq 0 \tag{4.37}$$

Summarizing we have proved the following proposition.

Proposition 4.18 *The model problem solution $M(z)$ is solvable if and only if*

$$\Theta\Big(-\frac{\tau+1}{2}+G\Big)\neq 0 \Leftrightarrow G\neq 0 \mod (\mathbb{Z}+\mathbb{Z}\tau). \tag{4.38}$$

Finding G and K so that the jump condition 1) is verified. Recall that:
1) On γ

- <u>On $\gamma\setminus\tilde{\Gamma}$</u> : M is defined and holomorphic.
- <u>On $\tilde{\Gamma}=\Gamma^b_{a_-}\cup\Gamma^1_b\cup\Gamma^{a+}_b$</u> :

We will need to use the following properties of the Abel map u which are and the function $h(z)$ on $\tilde{\Gamma}$:

Proposition 4.19 *On $\tilde{\Gamma}=\Gamma^b_{a_-}\cup\Gamma^1_b\cup\Gamma^{a+}_b$, the Abel map function satisfies:*

$$u(z_+)=-u(z_-)+\begin{cases}0 & on\quad \Gamma^b_{a-}\\ -\tau & on\quad \Gamma^1_b\\ -1 & on\quad \Gamma^{a+}_b\end{cases} \tag{4.39}$$

and the function $h(z)$ satisfies:

$$\begin{cases}h(z_+)=ih(z_-) & on\quad \Gamma^b_{a-}\\ h(z_+)=-ih(z_-) & on\quad \Gamma^1_b\cup\Gamma^{a+}_b\end{cases} \tag{4.40}$$

Proposition 4.20 *On $\Gamma^b_{a_-}$ it is verified that $M(z_+)=M(z_-)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$*

Proof. of proposition 4.20
▷ On $\Gamma^b_{a_-}$

$$A_1(z_+) = \frac{i\Theta\Big(u(z_+) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_+)}{\Theta\Big(u(z_+) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_+)}}$$

$$= \frac{-\Theta\Big(-u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(-u(z_-) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}} = -A_2(z_-) = 0.A_1(z_-) - A_2(z_-)$$

$$A_2(z_+) = \frac{\Theta\Big(-u(z_+) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_+)}{\Theta\Big(-u(z_+) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_+)}}$$

$$= \frac{i\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}} = A_1(z_-) = A_1(z_-) + 0.A_2(z_-)$$

$$B_1(z_+) = \frac{i\Theta\Big(u(z_+) + u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_+)}{\Theta\Big(u(z_+) + u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_+)}}$$

$$= \frac{-\Theta\Big(-u(z_-) + u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(-u(z_-) + u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}} = -B_2(z_-) = 0.B_1(z_-) - B_2(z_-)$$

$$B_2(z_+) = \frac{\Theta\Big(-u(z_+) + u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_+)}{\Theta\Big(-u(z_+) + u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_+)}}$$

$$= \frac{i\Theta\Big(u(z_-) + u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(u(z_-) + u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}} = B_1(z_-) = B_1(z_-) + 0.B_2(z_-)$$

In summary, on $\Gamma^b_{a_-}$,

$$M(z_+) = \begin{bmatrix} A_1(z_+) & A_2(z_+) \\ B_1(z_+) & B_2(z_+) \end{bmatrix} = \begin{bmatrix} A_1(z_-) & A_2(z_-) \\ B_1(z_-) & B_2(z_-) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.41}$$

■

Proposition 4.21 *On Γ_b^1 it is verified that*

$$M(z_+) = M(z_-)\begin{bmatrix} 0 & -\mathrm{e}^{-2i\pi\left(G-\frac{K\tau}{2}\right)} \\ \mathrm{e}^{2i\pi\left(G-\frac{K\tau}{2}\right)} & 0 \end{bmatrix} \tag{4.42}$$

Proof.

▷ On Γ_b^1

$$\begin{aligned} A_1(z_+) &= \frac{i\Theta\left(u(z_+)-u(\infty)-\frac{\tau+1}{2}+G\right)h(z_+)}{\Theta\left(u(z_+)-u(\infty)-\frac{\tau+1}{2}\right)\mathrm{e}^{-i\pi K u(z_+)}} \\ &= \frac{\Theta\left(-u(z_-)-\tau-u(\infty)-\frac{\tau+1}{2}+G\right)h(z_-)}{\Theta\left(-u(z_-)-\tau-u(\infty)-\frac{\tau+1}{2}\right)\mathrm{e}^{i\pi K u(z_-)}\mathrm{e}^{i\pi K\tau}} \end{aligned}$$

By letting

$$\begin{cases} U_n(z_-) := -u(z_-)-u(\infty)-\frac{\tau+1}{2}+G & \text{in the numerator} \\ U_d(z_-) := -u(z_-)-u(\infty)-\frac{\tau+1}{2} & \text{in the denominator} \end{cases} \tag{4.43}$$

and by relation (4.33) of the Riemann $\Theta-$ function, we have:

$$\Theta\left(U_{n,d}(z_-)-\tau\right) = \mathrm{e}^{-2i\pi l U_{n,d}(z_-)+i\pi l^2\tau}\Theta\left(U_{n,d}(z_-)\right) \tag{4.44}$$

with $l=-1$

Thus

$$A_1(z_+) = \frac{\Theta\left(U_n(z_-)\right)h(z_-)}{\Theta\left(U_d(z_-)\right)\mathrm{e}^{-2i\pi l\overbrace{\left(U_d(z_-)-U_n(z_-)\right)}^{=-G}}\mathrm{e}^{i\pi K u(z_-)+i\pi K\tau}}$$

$\Longrightarrow$

$$\hookrightarrow A_1(z_+) = \frac{\Theta\left(-u(z_-) - u(\infty) - \frac{\tau+1}{2} + G\right)h(z_-)}{\Theta\left(-u(z_-) - u(\infty) - \frac{\tau+1}{2}\right)\mathrm{e}^{i\pi K u(z_-)}}\mathrm{e}^{-2i\pi l\left(G+\frac{K\tau}{2l}\right)}$$

$$= A_2(z_-)\mathrm{e}^{-2i\pi l\left(G+\frac{K\tau}{2l}\right)} = 0.A_1(z_-) + A_2(z_-)\mathrm{e}^{-2i\pi l\left(G+\frac{K\tau}{2l}\right)}$$

$$= 0 \cdot A_1(z_-) + A_2(z_-)\mathrm{e}^{2i\pi\left(G-\frac{K\tau}{2}\right)} \quad \text{since} \quad l = -1$$

$$A_2(z_+) = \frac{\Theta\left(-u(z_+) - u(\infty) - \frac{\tau+1}{2} + G\right)h(z_+)}{\Theta\left(-u(z_+) - u(\infty) - \frac{\tau+1}{2}\right)\mathrm{e}^{i\pi K u(z_+)}}$$

$$= \frac{-i\Theta\left(u(z_-) + \tau - u(\infty) - \frac{\tau+1}{2} + G\right)h(z_-)}{\Theta\left(u(z_-) + \tau - u(\infty) - \frac{\tau+1}{2}\right)\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K\tau}}, \quad \text{notice that} \quad l = 1$$

$$= \frac{-i\Theta\left(u(z_-) - u(\infty) - \frac{\tau+1}{2} + G\right)h(z_-)}{\Theta\left(u(z_-) - u(\infty) - \frac{\tau+1}{2}\right)\mathrm{e}^{2i\pi l G}\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K\tau}} \quad \text{by appropriately using a}$$

$\Theta -$ function property (as before)

$\Longrightarrow$

$$\hookrightarrow A_2(z_+) = \frac{-i\Theta\left(u(z_-) - u(\infty) - \frac{\tau+1}{2} + G\right)h(z_-)}{\Theta\left(u(z_-) - u(\infty) - \frac{\tau+1}{2}\right)\mathrm{e}^{-i\pi K u(z_-)}}\mathrm{e}^{-2i\pi l\left(G-\frac{K\tau}{2l}\right)}$$

$$= -A_1(z_)\mathrm{e}^{-2i\pi l\left(G-\frac{K\tau}{2l}\right)} = -A_1(z_-)\mathrm{e}^{-2i\pi l\left(G-\frac{K\tau}{2l}\right)} + 0 \cdot A_2(z_-)$$

$$= -A_1(z_-)\mathrm{e}^{-2i\pi\left(G-\frac{K\tau}{2}\right)} + 0 \cdot A_2(z_-), \quad \text{since} \quad l = 1$$

$$
\begin{aligned}
B_1(z_+) &= \frac{i\Theta\Big(u(z_+)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_+)}{\Theta\Big(u(z_+)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_+)}} \\
&= \frac{\Theta\Big(-u(z_-)-\tau+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)-\tau+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}\mathrm{e}^{i\pi K\tau}}, \quad \text{notice that} \quad l=-1 \\
&\Longrightarrow \\
\hookrightarrow B_1(z_+) &= \frac{\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}}\mathrm{e}^{-2\pi i\left(lG+\frac{K\tau}{2}\right)} \\
&= B_2(z_-)\mathrm{e}^{-2\pi i\left(lG+\frac{K\tau}{2}\right)} = 0\cdot B_1(z_-)+B_2(z_-)\mathrm{e}^{-2\pi i\left(lG+\frac{K\tau}{2}\right)} \\
&= 0\,B_1(z_-)+B_2(z_-)\mathrm{e}^{2\pi i\left(G-\frac{K\tau}{2}\right)}, \quad \text{since} \quad l=-1
\end{aligned}
$$

$$
\begin{aligned}
B_2(z_+) &= \frac{\Theta\Big(-u(z_+)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_+)}{\Theta\Big(-u(z_+)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_+)}} \\
&= \frac{-i\Theta\Big(-u(z_-)+\tau+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)+\tau+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K\tau}}, \quad \text{notice that} \quad l=1 \\
&\Longrightarrow \\
\hookrightarrow B_2(z_+) &= \frac{-i\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}}\mathrm{e}^{-2\pi i\left(lG-\frac{K\tau}{2}\right)} \\
&= -B_1(z_-)\mathrm{e}^{-2\pi i\left(lG-\frac{K\tau}{2}\right)} = -B_1(z_-)\mathrm{e}^{-2\pi i\left(lG-\frac{K\tau}{2}\right)}+0\cdot B_2(z_-) \\
&= -B_1(z_-)\mathrm{e}^{-2\pi i\left(G-\frac{K\tau}{2}\right)}+0\cdot B_2(z_-), \quad \text{since} \quad l=1
\end{aligned}
$$

In summary, on Γ_b^1,

$$
\begin{aligned}
M(z_+) &= \begin{bmatrix} A_1(z_+) & A_2(z_+) \\ B_1(z_+) & B_2(z_+) \end{bmatrix} \\
&= \begin{bmatrix} A_1(z_-) & A_2(z_-) \\ B_1(z_-) & B_2(z_-) \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{e}^{-2i\pi\left(G-\frac{K\tau}{2}\right)} \\ \mathrm{e}^{2i\pi\left(G-\frac{K\tau}{2}\right)} & 0 \end{bmatrix}
\end{aligned}
$$

■

On the other hand the jump condition 1) on Γ_b^1 in the RPHM suggests that

$$
M(z_+) = M(z_-) \begin{bmatrix} 0 & \mathrm{e}^{n\Omega_1+\nu} \\ -\mathrm{e}^{-n\Omega_1-\nu} & 0 \end{bmatrix}, \tag{4.45}
$$

setting $W_1 = n\Omega_1 + \nu$ and knowing that $-1 = \mathrm{e}^{i\pi}$, $l = -1$
we can deduce that

$$
\begin{cases} W_1 = i\pi - 2\pi i\left(G - \dfrac{K\tau}{2}\right) \\ i\pi - W_1 = 2\pi i\left(G - \dfrac{K\tau}{2}\right) \end{cases}
$$

$$
\Longrightarrow G = \frac{1}{2} - \frac{1}{2\pi i}(n\Omega_1 + \nu) + \frac{K\tau}{2} \tag{4.46}
$$

Proposition 4.22 *On Γ_b^{a+} it is verified that*

$$
M(z_+) = M(z_-) \begin{bmatrix} 0 & -\mathrm{e}^{i\pi K} \\ \mathrm{e}^{-i\pi K} & 0 \end{bmatrix} \tag{4.47}
$$

Proof.

▷ On Γ_b^{a+}

$$
\begin{aligned}
A_1(z_+) &= \frac{i\Theta\left(u(z_+) - u(\infty) - \dfrac{\tau+1}{2} + G\right)h(z_+)}{\Theta\left(u(z_+) - u(\infty) - \dfrac{\tau+1}{2}\right)\mathrm{e}^{-i\pi K u(z_+)}} \\
&= \frac{\Theta\left(-u(z_-) - 1 - u(\infty) - \dfrac{\tau+1}{2} + G\right)h(z_-)}{\Theta\left(-u(z_-) - 1 - u(\infty) - \dfrac{\tau+1}{2}\right)\mathrm{e}^{i\pi K u(z_-)}\mathrm{e}^{i\pi K\tau}}
\end{aligned}
$$

By letting

$$\begin{cases} U_n(z_-) := -u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G & \text{in the numerator} \\ U_d(z_-) := -u(z_-) - u(\infty) - \dfrac{\tau+1}{2} & \text{in the denominator} \end{cases} \tag{4.48}$$

and by relation (4.33) of the Riemann $\Theta-$ function, we have:

$$\Theta\Big(U_{n,d}(z_-) - 1\Big) = \Theta\Big(U_{n,d}(z_-)\Big) \tag{4.49}$$

Thus

$$A_1(z_+) = \frac{\Theta\Big(U_n(z_-)\Big)h(z_-)}{\Theta\Big(U_d(z_-)\Big)\mathrm{e}^{i\pi K u(z_-)+i\pi K}}$$

$\Longrightarrow$

$$\hookrightarrow A_1(z_+) = \frac{\Theta\Big(-u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(-u(z_-) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}}\mathrm{e}^{-i\pi K}$$
$$= A_2(z_-)\mathrm{e}^{-i\pi K} = 0 \cdot A_1(z_-) + A_2(z_-)\mathrm{e}^{-i\pi K}$$

$$A_2(z_+) = \frac{\Theta\Big(-u(z_+) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_+)}{\Theta\Big(-u(z_+) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_+)}}$$
$$= \frac{-i\Theta\Big(u(z_-) - 1 - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(u(z_-) - 1 - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K}}$$
$$= \frac{-i\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K}} \quad \text{by appropriately using a}$$

$\Theta-$ function property (as before)

$\Longrightarrow$

$$\hookrightarrow A_2(z_+) = \frac{-i\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2} + G\Big)h(z_-)}{\Theta\Big(u(z_-) - u(\infty) - \dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}}\mathrm{e}^{i\pi K}$$
$$= -A_1(z_)\mathrm{e}^{i\pi K} = -A_1(z_-)\mathrm{e}^{i\pi K} + 0 \cdot A_2(z_-)$$

$$
\begin{aligned}
B_1(z_+) &= \frac{i\Theta\Big(u(z_+)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_+)}{\Theta\Big(u(z_+)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_+)}} \\
&= \frac{\Theta\Big(-u(z_-)-1+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)-1+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}\mathrm{e}^{i\pi K}} \\
&\Longrightarrow \\
\hookrightarrow B_1(z_+) &= \frac{\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(-u(z_-)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_-)}}\mathrm{e}^{-i\pi K} \\
&= B_2(z_-)\mathrm{e}^{-i\pi K} = 0\cdot B_1(z_-)+B_2(z_-)\mathrm{e}^{-i\pi K}
\end{aligned}
$$

$$
\begin{aligned}
B_2(z_+) &= \frac{\Theta\Big(-u(z_+)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_+)}{\Theta\Big(-u(z_+)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z_+)}} \\
&= \frac{-i\Theta\Big(u(z_-)-1+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(u(z_-)-1+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}\mathrm{e}^{-i\pi K}} \\
&\Longrightarrow \\
\hookrightarrow B_2(z_+) &= \frac{-i\Theta\Big(u(z_-)+u(\infty)-\dfrac{\tau+1}{2}+G\Big)h(z_-)}{\Theta\Big(u(z_-)+u(\infty)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z_-)}}\mathrm{e}^{i\pi K} \\
&= -B_1(z_-)\mathrm{e}^{i\pi K} = -B_1(z_-)\mathrm{e}^{i\pi K}+0\cdot B_2(z_-)
\end{aligned}
$$

In summary, on Γ_b^{a+},

$$
\begin{aligned}
M(z_+) &= \begin{bmatrix} A_1(z_+) & A_2(z_+) \\ B_1(z_+) & B_2(z_+) \end{bmatrix} \\
&= \begin{bmatrix} A_1(z_-) & A_2(z_-) \\ B_1(z_-) & B_2(z_-) \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{e}^{i\pi K} \\ \mathrm{e}^{-i\pi K} & 0 \end{bmatrix}
\end{aligned}
$$

$\blacksquare$

Proposition 4.23 *The expressions for K and G in the solution of the Model Problem stated in proposition 4.16 are as follow*

$$\begin{cases} K = \dfrac{W_2}{i\pi} - 1 = \dfrac{n\Omega_2}{i\pi} - 1 \\ G = \dfrac{1}{2} - \dfrac{1}{2\pi i}(n\Omega_1 + \nu) + \dfrac{K\tau}{2} \end{cases} \implies G = -\frac{1}{2\pi i}\Big(n\Omega_1 + \nu - n\Omega_2\tau\Big) - \tau + \frac{1}{2} \tag{4.50}$$

where

$$\Omega_1 = \int_b^{a_+} \varphi'(z_+)\mathrm{d}z \in i\mathbb{R}_-, \quad \textit{and} \quad \Omega_2 = \int_b^{1} \varphi'(z_-)\mathrm{d}z \in i\mathbb{R}_+ \tag{4.51}$$

$$\begin{cases} \left(\dfrac{n\Omega_2}{2i\pi} - 1\right) + \dfrac{\mathrm{Im}\left(-\dfrac{\nu}{2\pi i}\right)}{\mathrm{Im}\,(\tau)} = l \in \mathbb{Z} \\ \mathrm{Re}\,(\tau)\left(\dfrac{n\Omega_2}{2i\pi} - 1 - l\right) + \mathrm{Re}\left(-\dfrac{\nu}{2\pi i}\right) - \dfrac{n\Omega_1}{2i\pi} + \dfrac{1}{2} = k \in \mathbb{Z} \end{cases} \tag{4.52}$$

and ν is given by (4.30).

Proof.

On the other hand the jump condition 1) on Γ_b^1 in the RPHM suggests that $M(z_+) = M(z_-)\begin{bmatrix} 0 & \mathrm{e}^{n\Omega_2} \\ -\mathrm{e}^{-n\Omega_2} & 0 \end{bmatrix}$, setting $W_2 = n\Omega_2$ and knowing that $-1 = \mathrm{e}^{i\pi}$
we can deduce that

$$W_2 = i\pi + i\pi K$$

$$\implies K = \frac{W_2}{i\pi} - 1 \tag{4.53}$$

From relations 4.46 and 4.53 we have:

$$\begin{cases} K = \dfrac{W_2}{i\pi} - 1 = \dfrac{n\Omega_2}{i\pi} - 1 \\ G = \dfrac{1}{2} - \dfrac{1}{2\pi i}(n\Omega_1 + \nu) + \dfrac{K\tau}{2} \end{cases} \implies G = -\frac{1}{2\pi i}\Big(n\Omega_1 + \nu - n\Omega_2\tau\Big) - \tau + \frac{1}{2} \tag{4.54}$$

Recall that the Boutroux conditions implies that $\Omega_1, \Omega_2 \in i\mathbb{R}$,

$$\text{Let} \quad \Omega_1 = ia, \quad \text{and} \quad \Omega_2 = ib \quad a, b \in \mathbb{R} \tag{4.55}$$

Let us find a and b.
We establish before that the RHPM is not solvable if $G = 0 + k + l\tau$,with $k, l \in \mathbb{Z}$. so to find not

possible values for a and b, let us assume that $G = 0 + k + l\tau$. 4.54 implies that

$$\begin{aligned}
0 + k + l\tau &= -\frac{1}{2\pi i}\left(n\Omega_1 + \nu - n\Omega_2\tau\right) - \tau + \frac{1}{2}\\
&\Rightarrow -\frac{an}{2\pi} + \frac{bn\tau}{2\pi} - \frac{\nu}{2\pi i} = \tau - \frac{1}{2} + k + l\tau\\
&\Rightarrow \tau\left(\frac{bn}{2\pi} - 1 - l\right) - \frac{\nu}{2\pi i} = \frac{an}{2\pi} - \frac{1}{2} + k\\
&\Rightarrow \mathrm{Re}\,(\tau)\left(\frac{bn}{2\pi} - 1 - l\right) + \mathrm{Re}\left(-\frac{\nu}{2\pi i}\right)\\
&\quad + i\left[\mathrm{Im}\,(\tau)\left(\frac{bn}{2\pi} - 1 - l\right) + \mathrm{Im}\left(-\frac{\nu}{2\pi i}\right)\right] = \frac{an}{2\pi} - \frac{1}{2} + k
\end{aligned}$$

$$\Longrightarrow \begin{cases} \mathrm{Im}\,(\tau)\left(\frac{bn}{2\pi} - 1 - l\right) + \mathrm{Im}\left(-\frac{\nu}{2\pi i}\right) = 0\\ \mathrm{Re}\,(\tau)\left(\frac{bn}{2\pi} - 1 - l\right) + \mathrm{Re}\left(-\frac{\nu}{2\pi i}\right) = \frac{an}{2\pi} - \frac{1}{2} + k \end{cases}$$

$$\Longrightarrow \begin{cases} \left(\frac{bn}{2\pi} - 1\right) + \dfrac{\mathrm{Im}\left(-\frac{\nu}{2\pi i}\right)}{\mathrm{Im}\,(\tau)} = l \in \mathbb{Z}\\ \mathrm{Re}\,(\tau)\left(\frac{bn}{2\pi} - 1 - l\right) + \mathrm{Re}\left(-\frac{\nu}{2\pi i}\right) - \frac{an}{2\pi} + \frac{1}{2} = k \in \mathbb{Z} \end{cases}$$

$\forall k, l \in \mathbb{Z}$, the values of a and b which satisfy the above equations correspond to values of $\Omega_1 = ia$ and $\Omega_2 = ib$ such that the RHPM with the following jump condition is not solvable.

■

$$M(z_+) = \begin{cases} M(z_-)\begin{bmatrix} 0 & 1\\ -1 & 0 \end{bmatrix}, & 0\text{n } \Gamma_{a_-}^{b}\\ M(z_-)\begin{bmatrix} 0 & \mathrm{e}^{n\Omega_1+\nu}\\ -\mathrm{e}^{-n\Omega_1-\nu} & 0 \end{bmatrix}, & 0\text{n } \Gamma_b^1\\ M(z_-)\begin{bmatrix} 0 & \mathrm{e}^{n\Omega_2}\\ -\mathrm{e}^{-n\Omega_2} & 0 \end{bmatrix}, & 0\text{n } \Gamma_b^{a_+} \end{cases}$$

Notice that in the case where the branch cut of φ is superposed to b_1 such that $\Gamma_{a_-}^{a_+} \cap \Gamma_b^1 = \emptyset$ as shown on figure 20, the jump condition of M in the RHPM is reduced to that on $\Gamma_{a_-}^{b}$ in the previous case. In this current case we have:

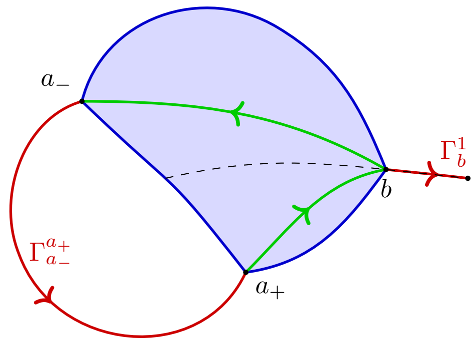

Figure 20: The contour Γ.

$$M(z_+) = M(z_-)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{on} \quad \Gamma_{a_-}^{a_+} \cup \Gamma_b^1 \tag{4.56}$$

The solution can be found by a simple diagonalization of the jump matrix (similarly to the case where s is outside of the Eye Of the Tiger)

4.5 Conditions on non solvability of the RHP inside the EoT

While it's logical to conclude that the initial Riemann Hilbert Problem $RHP\Phi$ is solvable when the $RHPM$ is solvable, it's not correct to assume that the non solvability of the $RHPM$ for some values of $s \in EoT$ implies that of the initial $RHP\Phi$.
Therefore, we need to consider directly the conditions on $s \in EoT$ for non solvability of the initial $RHP\Phi$.

Proposition 4.24 *The Hamiltonian in the asymptotic case is written as*

$$H_{V,\infty} = -\left[\left(\Phi_\infty^{-1}(0)\Phi'_\infty(0)\right)_{11} - \frac{\theta}{2} + \frac{k}{2}\right] = -\left[\frac{D'_\infty}{D_\infty} - \frac{\theta}{2} + \frac{k}{2}\right] \tag{4.57}$$

$$= -\left[\partial_x \ln(D_\infty) - \frac{\theta}{2} + \frac{k}{2}\right] \tag{4.58}$$

where, $n \to \infty$ and $H_{V,\infty}$, D_∞ and $\Phi(\infty)$ are computed when n is very large ($n \neq \infty$).
The initial $RHP\Phi$ is non solvable at the points of the $\mathbb{C}$ where the Hamiltonian $H_{V,\infty}$ has a pole.

Proof.
Recall that by the Fredholm alternative, the $RHP\Phi$ is not solvable if and only if the associated Hankel determinant $D_n(s) = 0$. In the non-asymptotic scheme, recall from 2.9 that

$$H_{V,n} = -\left[\left(\Phi_n^{-1}(0)\Phi'_n(0)\right)_{11} - \frac{\theta}{2} + \frac{k}{2}\right] = -\left[\frac{D'_n}{D_n} - \frac{\theta}{2} + \frac{k}{2}\right] = -\left[\partial_x \ln(D_n) - \frac{\theta}{2} + \frac{k}{2}\right] \tag{4.59}$$

So, in the non-asymptotic scheme, the initial $RHP\Phi$ is non solvable at the points of the $\mathbb{C}$ where the Hamiltonian $H_{V,n}$ has a pole.

■

Proposition 4.25 *Let*

$$\hat{M}(z) = \begin{bmatrix} \dfrac{iz\Theta\Big(u(z)-u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(u(z)-u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z)}} & \dfrac{z\Theta\Big(-u(z)-u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(-u(z)-u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z)}} \\ \dfrac{iz\Theta\Big(u(z)+u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(u(z)+u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(z)}} & \dfrac{z\Theta\Big(-u(z)+u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(-u(z)+u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(z)}} \end{bmatrix}$$

Given that in the expression of $\hat{M}(z)$ above, τ and G are function of s, $\Big(M^{-1}(0)M'(0)\Big)_{11}$ must also be a function of s.

$$\text{Poles of}\quad H_{V,\infty} = \text{Poles of}\quad \Big(\Phi_\infty^{-1}(0)\Phi_\infty'(0)\Big)_{11} = \text{"Poles" of}\quad \Big(M^{-1}(0)M'(0)\Big)_{11} \tag{4.60}$$

$$= \text{"Poles" of}\quad \Big(\hat{M}^{-1}(0)\hat{M}'(0)\Big)_{11} \tag{4.61}$$

Proof.

So, let's compute $H_{V,\infty}$. To achieve that, we need to compute $\Phi_\infty^{-1}(0)\Phi_\infty'(0)$ from (in terms of) the solution $M(z)$ of the Model problem for the asymptotic case where $s \in EoT$

Recall from 4.12 that

$$M(z) = \mathrm{e}^{-\mathrm{S}(\infty)\sigma_3}\mathrm{M}_{\tilde{\mathrm{Q}}}(\mathrm{z})\mathrm{e}^{\mathrm{S}(\mathrm{z})\sigma_3} \tag{4.62}$$

Where $M_{\tilde{Q}}(z)$ is defined in 4.8. We are interested in computing $\Phi_\infty^{-1}(0)\Phi_\infty'(0)$ and notice that $z = 0$ is outside of the lenses and the disks.
Since $M_{\tilde{Q}}(z) = T(z)$ outside of the disks (by the condition 1) in the proposition 4.9 and the small norm theorem) and $T(z) = W(z)$ outside of region $\mathcal{R}_\pm$ of the lenses (see relation 4.6), where the transformation $W(z)$ is defined in 4.3, at $z = 0$, the relation 4.62 becomes:

$$M(z) = \mathrm{e}^{-S(\infty)\sigma_3}W(z)\mathrm{e}^{\mathrm{S}(\mathrm{z})\sigma_3}$$

$$\Longrightarrow W(z) = \mathrm{e}^{\mathrm{S}(\infty)\sigma_3}\mathrm{M}(\mathrm{z})\mathrm{e}^{-\mathrm{S}(\mathrm{z})\sigma_3} \tag{4.63}$$

Furthermore, from definition 3.3

$$W(z) := \mathrm{e}^{n\frac{\ell}{2}\sigma_3}\Phi(z)\mathrm{e}^{-n\left(g(z)+\frac{\ell}{2}\right)\sigma_3}$$

$$\implies \Phi(z) = \mathrm{e}^{-n\frac{l}{2}\sigma_3}W(z)\mathrm{e}^{n\left(g(z)+\frac{l}{2}\right)\sigma_3} \tag{4.64}$$

Relations 4.63 and 4.64 imply that:

$$\begin{aligned}\Phi(z) &\simeq \mathrm{e}^{-n\frac{l}{2}\sigma_3}\underbrace{\mathrm{e}^{\mathrm{S}(\infty)\sigma_3}\mathrm{M(z)}\mathrm{e}^{-\mathrm{S(z)}\sigma_3}}_{\simeq W(z)}\mathrm{e}^{n\left(g(z)+\frac{l}{2}\right)\sigma_3}\\ &= \mathrm{e}^{\left(S(\infty)-n\frac{\ell}{2}\right)\sigma_3}M(z)\mathrm{e}^{\left(n\left[g(z)+\frac{\ell}{2}\right]-S(z)\right)\sigma_3}\end{aligned}$$

$$\begin{aligned}\Rightarrow \Phi^{-1}(z)\Phi'(z) = {}& \mathrm{e}^{-\left(n\left[g(z)+\frac{l}{2}\right]-S(z)\right)\sigma_3}M^{-1}(z)\\ &\cdot\left(M'(z)\mathrm{e}^{\left(n\left[g(z)+\frac{l}{2}\right]-S(z)\right)\sigma_3}+M(z)\Big(ng'(z)-S'(z)\Big)\sigma_3\ \mathrm{e}^{\left(n\left[g(z)+\frac{l}{2}\right]-S(z)\right)\sigma_3}\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow \Phi^{-1}(z)\Phi'(z) = {}& \mathrm{e}^{-\left(n\left[g(z)+\frac{l}{2}\right]-S(z)\right)\sigma_3}\Big(M^{-1}(z)M'(z)+(ng'(z)-S'(z))\sigma_3\Big)\\ &\cdot\mathrm{e}^{\left(n\left[g(z)+\frac{l}{2}\right]-S(z)\right)\sigma_3}\end{aligned}$$

From proposition 3.17, $S(z)$ is analytic at $z=0$. Moreover, from definition 3.3, $g(z)$ is also analytic at $z=0$.
Therefore, only $M(z)$ contributes in the expression $\Phi(z)$ written in **??** to find the poles of

$$H_{V,\infty} = -\left[\Big(\Phi_\infty^{-1}(0)\Phi_\infty'(0)\Big)_{11} - \frac{\theta}{2}+\frac{k}{2}\right]$$

In other words,

$$\text{Poles of}\quad H_{V,\infty} = \text{Poles of}\quad \Big(\Phi_\infty^{-1}(0)\Phi_\infty'(0)\Big)_{11} = \text{"Poles" of}\quad \Big(M^{-1}(0)M'(0)\Big)_{11} \tag{4.65}$$

Given that in the expression of $M(z)$ below (see relation 4.66), τ and G are function of s, $\Big(M^{-1}(0)M'(0)\Big)_{11}$ must also be a function of s.

Hence, in relation **??** above, "Poles" of $\Big(M^{-1}(0)M'(0)\Big)_{11}$ **means** the values of s in the s-plane (i.e. EoT plane) such that $\Big(M^{-1}(0)M'(0)\Big)_{11}$ is not defined.

At $z=0$, the $\Big(M(0)\Big)_{11}=\infty$. To avoid this, let us consider the following expression $\hat{M}(z)$ at $z=0$

$$\hat{M}(z)=\begin{bmatrix} \dfrac{iz\Theta\Big(u(z)-u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(u(z)-u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi Ku(z)}} & \dfrac{z\Theta\Big(-u(z)-u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(-u(z)-u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi Ku(z)}} \\ \dfrac{iz\Theta\Big(u(z)+u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(u(z)+u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi Ku(z)}} & \dfrac{z\Theta\Big(-u(z)+u(0)-\dfrac{\tau+1}{2}+G\Big)h(z)}{\Theta\Big(-u(z)+u(0)-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi Ku(z)}} \end{bmatrix} \tag{4.66}$$

It can be proved that

$$M^{-1}(0)M'(0)=\hat{M}^{-1}(0)\hat{M}'(0) \tag{4.67}$$

■

In conclusion we can state

Theorem 4.26 *The points of non solvability of the ODE Painleve5 inside the EoT are found in the neighbourhood where* $\Big(\hat{M}^{-1}(0)\hat{M}'(0)\Big)_{11}$ *has poles i.e. when*

$$\Theta\Big(-\frac{\tau(s)+1}{2}+G(s)\Big)=0, \tag{4.68}$$

where $G(s)$ *is the expression in Prop. 4.23.*

Proof.
Let us compute $\hat{M}^{-1}(0)\hat{M}'(0)$ instead.

While the entries $\Big(\hat{M}(0)\Big)_{12}=\Big(\hat{M}(0)\Big)_{12}=0$, we use l'Hopital rule to compute $\Big(\hat{M}(0)\Big)_{11}$ and $\Big(\hat{M}(0)\Big)_{22}$. Given that $\Theta\Big(-\dfrac{\tau+1}{2}\Big)=0$, we obtain:

$$\hat{M}(0) = \begin{bmatrix} \dfrac{i\Theta\Big(-\dfrac{\tau+1}{2}+G\Big)h(0)}{u'(0)\Theta'\Big(-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{-i\pi K u(0)}} & 0 \\ 0 & \dfrac{\Theta\Big(-\dfrac{\tau+1}{2}+G\Big)h(0)}{-u'(0)\Theta'\Big(-\dfrac{\tau+1}{2}\Big)\mathrm{e}^{i\pi K u(0)}} \end{bmatrix} \tag{4.69}$$

From relations 4.31, it can be verified that $u'(0) \neq 0$ and $h(0) \neq 0$. Hence $\Big(\hat{M}(0)\Big)_{11}$ and $\Big(\hat{M}(0)\Big)_{22}$ in the above relation 4.69) are finite and not zero. Thus, we can compute $\hat{M}^{-1}(0)$ as follows:

$$\hat{M}^{-1}(0) = \frac{-u'(0)\Theta'\Big(-\dfrac{\tau+1}{2}\Big)}{\Theta\Big(-\dfrac{\tau+1}{2}+G\Big)h(0)} \begin{bmatrix} \dfrac{i}{\mathrm{e}^{i\pi K u(0)}} & 0 \\ 0 & \dfrac{1}{\mathrm{e}^{-i\pi K u(0)}} \end{bmatrix}$$

Moreover,

$$\begin{aligned} \hat{M}'(0) &= \frac{d}{ds}\hat{M}(0) \\ &= \begin{bmatrix} \dfrac{*}{\Big[u'(0)\Theta'\Big(-\dfrac{\tau(s)+1}{2}\Big)\mathrm{e}^{-i\pi K u(0)}\Big]^2} & 0 \\ 0 & \dfrac{*}{\Big[-u'(0)\Theta'\Big(-\dfrac{\tau(s)+1}{2}\Big)\mathrm{e}^{i\pi K u(0)}\Big]^2} \end{bmatrix} \end{aligned}$$

Finally,

$$\begin{aligned} \hat{M}^{-1}(0)\hat{M}'(0) = & \frac{-1}{\Theta\Big(-\dfrac{\tau(s)+1}{2}+G(s)\Big)h(0)u'(0)\Theta'\Big(-\dfrac{\tau(s)+1}{2}\Big)} \\ & \cdot \begin{bmatrix} \dfrac{i}{\mathrm{e}^{i\pi K u(0)}} & 0 \\ 0 & \dfrac{1}{\mathrm{e}^{-i\pi K u(0)}} \end{bmatrix} \begin{bmatrix} \dfrac{*}{\mathrm{e}^{-2i\pi K u(0)}} & 0 \\ 0 & \dfrac{*}{\mathrm{e}^{2i\pi K u(0)}} \end{bmatrix} \end{aligned}$$

$\Longrightarrow$

$$\hat{M}^{-1}(0)\hat{M}'(0) = \frac{-1}{\Theta\Big(-\frac{\tau(s)+1}{2}+G(s)\Big)h(0)u'(0)\Theta'\Big(-\frac{\tau(s)+1}{2}\Big)} \cdot \begin{bmatrix} \frac{*}{e^{-i\pi K u(0)}} & * \\ * & * \end{bmatrix} \tag{4.70}$$

$$\text{where } \Theta'\Big(-\frac{\tau(s)+1}{2}\Big) \neq 0 \tag{4.71}$$

■

As $n \to \infty$, the shape of the EoT gets more precise and bigger on the $x-$ plane (recall that $x = ns$). Thus, on the $s-$plane , the poles of the Hamiltonian are rescaled by $\frac{1}{n}$ so that they get closer to the origin $s = 0$ as $n \to \infty$, and enclosed by the limiting shape of the EoT.

4.6 Pictures and numerical validation

The following pictures show the plot of the poles of the Hamiltonian in the $s-$plane as $n \to \infty$

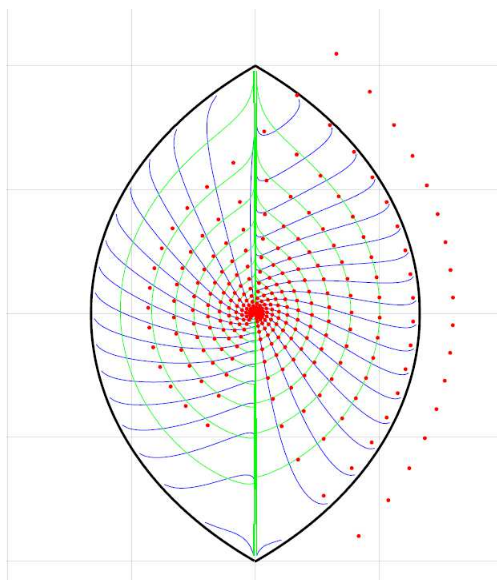

Figure 21: The zeroes for $(n, \beta) = (17, 3 + 1i)$

Notice that the figure above shows some roots outside of the limiting shape. This is explained by the fact that the value of θ is still too large relative to n; if we plotted the zeroes for the same value of theta but much larger n the zeroes would eventually fall within the EoT. The grid of green and blue lines in the figure and the figures below represents the levelsets determined by the quantization conditions expressed by the equations (4.52), plotted numerically. The zeroes are located approximately at the intersection of the grid. We did not provide a full justification or estimate of rate of approximation. However the approximation appears quite strict even for relatively small values of n. This appears to be a common phenomenon in this type of computations, see [7, 8, 11, 16, 17].

As n gets bigger, we get the following plots for the zeros of $D_n(ns)$

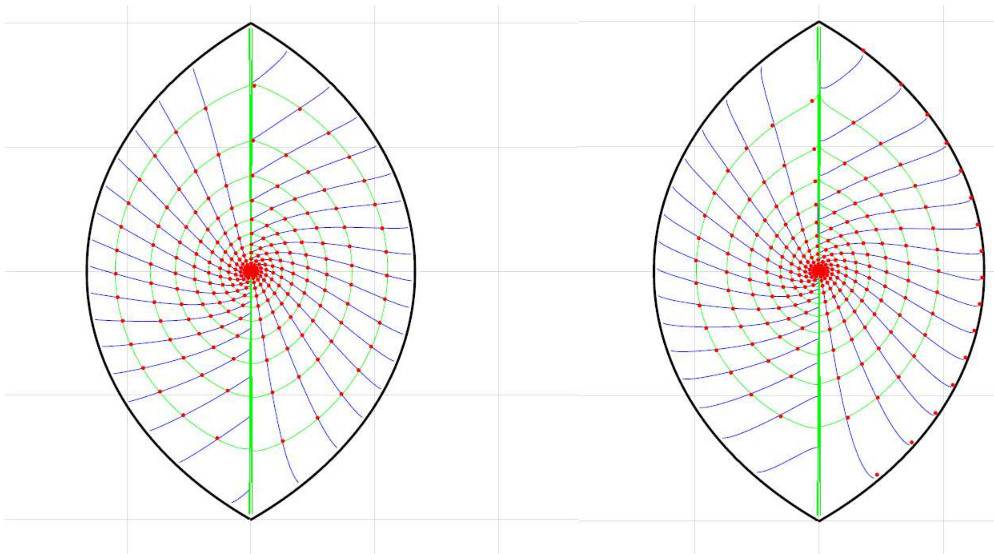

Figure 22: The zeroes for $(n, \beta) = \left(16, \frac{3}{100} + \frac{13i}{100}\right), \ \left(16, \frac{101}{100} + \frac{13i}{100}\right)$

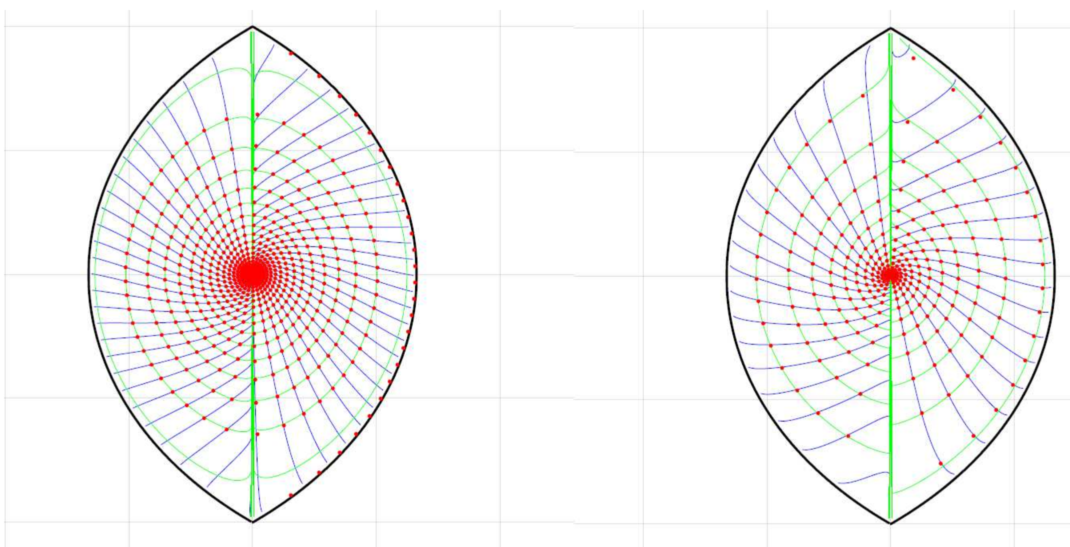

Figure 23: The zeroes for $(n, \beta) = \left(26, \frac{101}{100}\right), \quad \left(17, \frac{1}{2} + \frac{i}{2}\right)$.

For n very large, we get a plot for the zeros of $D_n(ns)$ as follows

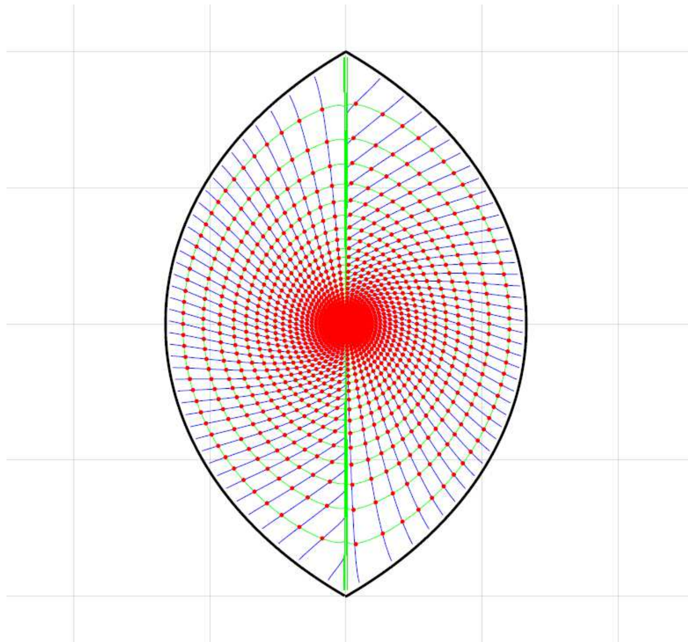

Figure 24: The zeroes for $(n, \beta) = \left(40, \frac{3}{100}\right)$

References

[1] Bertola M. and Bothner T.(2014). *Zeros of Large Degree Vorob'ev–Yablonski Polynomials via a Hankel Determinant Identity.* International Mathematics Research Notices, rnu239, 70 pages.

[2] Vorob'ev, A. "On rational solutions of the second Painlev´e equation." Differential Equations 1 (1965): 58–9 (in English).

[3] Bertola M. (2018). *Approximation theory: Pade approximants, orthogonal polynomials and Riemann Hilbert approach to asymptotic questions* [Lecture notes].

[4] Jimbo M., and Miwa T. (1980). *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II.* Kyoto, JAPAN: Research Institute for Mathematical Sctences, Kyoto Umverslty, Kyoto 606, Japan

[5] Bothner T., Miller Peter D., and Sheng Y., *Rational solutions of the Painleve-III equation* (2018) arXiv:1801.04360v1 [math.CA].

[6] Wikipedia contributors.(2024, March 20). Desnanot–Jacobi identity. In Wikipedia, The Free Encyclopedia. Retrieved 20:15, March 20, 2024, from http://en.wikipedia.org/wiki/Dodgson condensation

[7] F. Balogh, M. Bertola, T. Bothner, "Hankel determinant approach to generalized Vorob'ev-Yablonski polynomials and their roots", *Constr. Approx.* **44** (2016), no. 3, 417–453.

[8] M. Bertola, T. Bothner. "Zeros of large degree Vorob'ev-Yablonski polynomials via a Hankel determinant identity." *Int. Math. Res. Not. IMRN*, (19):9330–9399, 2015.

[9] M. Bertola, B. Eynard, and J. Harnad. "Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions." *Comm. Math. Phys.*, **263**(2):401–437, 2006.

[10] M. Bertola, A. Katsevich, A. Tovbis. "Singular value decomposition of a finite Hilbert transform defined on several intervals and the interior problem of tomography: the Riemann-Hilbert problem approach." *Comm. Pure Appl. Math.*, 69(3):407–477, March 2016.

[11] M. Bertola and S. Y. Lee. "First colonization of a hard-edge in random matrix theory." *Constr. Approx.*, 31(2):231–257, 2010.

[12] M. Bertola and M. Y. Mo. "Commuting difference operators, spinor bundles and the asymptotics of orthogonal polynomials with respect to varying complex weights." *Adv. Math.*, 220(1):154–218, 2009.

[13] Pavel Bleher and Alexander Its. Double scaling limit in the random matrix model: the Riemann-Hilbert approach. *Comm. Pure Appl. Math.*, 56(4):433–516, 2003.

[14] A. Borodin, P. J. Forrester, "Increasing subsequences and the hard-to-soft edge transition in matrix ensembles", *J. Phys. A*, **36** (2003), no. 12, 2963–2981.

[15] R. J. Buckingham, P. D. Miller, "The sine-Gordon equation in the semiclassical limit: critical behavior near a separatrix", *J. Anal. Math.* **118** (2012), no. 2, 397–492.

[16] R. Buckingham, P. Miller, "Large-degree asymptotics of rational Painlevé-II functions: noncritical behaviour", *Nonlinearity* **27** (2014), 2489-2577.

[17] R. Buckingham, P. Miller, "Large-degree asymptotics of rational Painlevé-II functions: critical behaviour", *Nonlinearity* **28** (2015), 1539–1596.

[18] R. Buckingham, P. Miller, "Large-degree asymptotics of rational Painlevé-IV solutions by the isomonodromy method", *Constr. Approx.* **56** (2022), no. 2, 233–443

[19] P. A. Clarkson, C. Dunning, "Rational Solutions of the Fifth Painlevé Equation. Generalised Laguerre Polynomials", arXiv:2304:01579

[20] P. Deift, *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*, Courant lecture notes, 1999.

[21] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.* **52** (1999), 1335-1425.

[22] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.*, **137** (1993), 295-368.

[23] A. Fokas, A. Its and A. Kitaev, Discrete Painlevé equations and their appearance in quantum gravity, *Comm. Math. Phys.* **142** (2) (1991), 313-344.

[24] F. D. Gakhov. "Boundary value problems", Dover Publications, Inc., New York, 1990. Translated from the Russian, Reprint of the 1966 translation.

[25] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. "Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function." *Phys. D*, 2(2):306–352, 1981.

[26] Michio Jimbo and Tetsuji Miwa. "Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II." *Phys. D*, 2(3):407–448, 1981.

[27] N. Joshi, K. Kajiwara, M. Mazzocco, "Generating function associated with the determinant formula for the solutions of the Painlevé II equation", *Astérisque* (2004), no. 297, 67–78.

[28] K. Kajiwara, Y. Ohta, Determinant structure of the rational solutions for the Painlevé II equation, *Journal of Mathematical Physics* **37** (1996), 4693-4704.

[29] A. V. Kitaev, C. K. Law, and J. B. McLeod. Rational solutions of the fifth Painlevé equation. *Differential Integral Equations*, 7(3-4):967–1000, 1994.

[30] H. L. Krall and Orrin Frink. A new class of orthogonal polynomials: The Bessel polynomials. *Trans. Amer. Math. Soc.*, 65:100–115, 1949.

[31] F. Marcellán, I.A. Rocha, "On semiclassical linear functionals: integral representations", *Proceedings of the Fourth International Symposium on Orthogonal Polynomials and their Applications (Evian-Les-Bains, 1992)*, **57**, no. 1-2, 239–249 (1995).

[32] F. Marcellán, I.A. Rocha, "Complex path integral representation for semiclassical linear functionals", *J. Appr. Theory* **94** (1998) 107–127.

[33] P. Maroni, "Prolégomènes á l'étude des polynômes semiclassiques", *Ann. Mat. Pura Appl.* 149 (1987) 165–184.

[34] T. Masuda, Y. Ohta, K. Kajiwara, "A determinant formula for a class of rational solutions of Painlevé V equation", *Nagoya Math. J.*, **168** (2002), 1–25.

[35] Masatoshi Noumi and Yasuhiko Yamada. Umemura polynomials for the Painlevé V equation. *Phys. Lett. A*, 247(1-2):65–69, 1998.

[36] B. M. McCoy, C. Tracy, T. T. Wu, "Connection between the KdV equation and the two-dimensional Ising model", *Phys. Lett. A* **61** (1977), no. 5, 283?284.

[37] B. M. McCoy, C. Tracy, T. T. Wu, "Painlevé functions of the third kind", *J. Mathematical Phys.* **18** (1977), no. 5, 1058?1092.

[38] C. Tracy, H. Widom, "Level–spacing distributions and the Airy kernel", *Comm. Math. Phys.*, **159** (1994), no. 1, 151–174.

[39] S. P. Hastings, J. B. McLeod, "A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation", *Arch. Rational Mech. Anal.* **73** (1980), no. 1, 31?51.

[40] V. Yu. Novokshenov, A.A. Shchelkonogov, "Double scaling limit in the Painlevé IV equation and asymptotics of the Okamoto polynomials", *Amer. Math Soc. Transl. Ser. 2, 233*, Adv. Math. Sci, **66**, (2014), 199–210.

[41] K. Okamoto, "Studies on the Painlevé equations III, second and fourth Painlevé equations, PII and PIV", Math. Ann. **275** (1986) 222-254.

[42] J. Shohat, "A differential equation for orthogonal polynomials", Duke Math. J., **5** (1939), 401–417

[43] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[44] H. Umemura, "Special polynomials associated with the Painlevé equations I", *Ann. Fac. Sci. Toulouse Math.* (6) **29** (2020), no. 5, 1063–1089

[45] M. Vanlessen. Strong asymptotics of Laguerre-type orthogonal polynomials and applications in random matrix theory. *Constr. Approx.*, 25(2):125–175, 2007.

[46] A. Vorob'ev, "On rational solutions of the second Painlevé equation", *Diff. Eqns* **1** (1965), 58-9 (in Russian).

[47] A. Yablonskii, On rational solutions of the second Painlevé equation, *Vesti Akad. Navuk. BSSR Ser. Fiz. Tkh. Nauk.* **3** (1959), 30-5 (in Russian).