Deformation of Convex Hypersurfaces in Euclidean Space by Powers of Principal Curvatures

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ABSTRACT

Deformation of Convex Hypersurfaces in Euclidean Space by Powers of Principal Curvatures

Meraj Hosseini

The results presented in this thesis contribute to the understanding of the evolution of smooth, strictly convex, closed hypersurfaces in $\mathbb{R}^{n+1}$ driven by non-symmetric speeds on the principal curvatures. The preservation of convexity, the occurrence of singularities, and the asymptotic behavior of the flows are studied. After an introduction to geometric flows, Chapter 3 focuses on the analysis of the short-term and long-term behavior of a contraction flow governed by a non-symmetric speed for rotationally symmetric hypersurfaces. Our investigation reveals two key findings. Firstly, we establish that the flow maintains convexity throughout the deformation process. Secondly, we observe the development of a singularity within a finite time, leading to the convergence of every such strictly convex hypersurface to a single point. To investigate the asymptotic behavior of the flow, we employ a proper rescaling technique of the solutions. Through this rescaling, we demonstrate that the rescaled solutions converge subsequentially to the boundary of a convex body. In the fourth chapter, we extend our study to the short-term and long-term behavior of a non-symmetric expansion flow in $\mathbb{R}^{n+1}$. We show that, starting with a smooth, strictly convex, rotationally symmetric, closed hypersurface, the flow preserves convexity while expanding infinitely in all directions. Depending on certain parameters within the speed function, we establish that the existence time of the flow can be either finite or infinite.

We also investigate the asymptotic behavior of the flow through a suitable rescaling process and demonstrate the subsequential convergence of the solutions to the boundary of a convex body in the Hausdorff distance. In the fifth chapter, we introduce the most general version of the flow studied in the Chapter 3. We address the barriers and challenges encountered when transitioning from a symmetric speed to a non-symmetric speed, and present our strategies to tackle some of these difficulties.

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Contents

Chapter 1

Motivations and Prerequisites

Curvature flows are dynamic processes that drive the evolution of geometric objects based on a speed function determined by their curvature. Examining the progression of shapes under curvature flows provides valuable insights into intrinsic geometric structures and their transformative behaviours. Consequently, these insights find practical applications in image processing, shape analysis, and surface modeling, enhancing our ability to analyze and manipulate geometric data. For well-known curvature flows such as the mean curvature flow or the Gauss curvature flow, the speed function of a hypersurface is typically a symmetric function of the principal curvatures. This symmetry assumption carries over to the natural generalizations of these flows. However, when attempting to model the deformation of a geometric object afresh, it becomes natural to consider a non-symmetric speed function. This choice arises from the recognition that the forces influencing the object's deformation may lack global uniformity, resulting in an asymmetric speed function. By incorporating asymmetry into the speed function, a more adaptable and realistic depiction of the object's deformation process can be achieved. This thesis investigates a geometric flow governed by a non-symmetric speed function, focusing on its short-time existence and asymptotic behaviour. Our analysis aims to understand how convexity is maintained or altered throughout the evolution. Furthermore, we investigate the occurrence of singularities in the flow and explore their nature. By employing a proper rescaling technique, we examine the asymptotic behaviour of the flow, studying how the surface behaves in the long run. Consider the evolution equation governed by the equation:

$$\frac{\partial X(\cdot,t)}{\partial t} = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\alpha_2}(x,t)\cdots\kappa_n^{\alpha_n}(x,t)\nu(x,t), \tag{1.1}$$

where X denotes the embedding of a smooth, strictly convex hypersurface within the ambient space of $\mathbb{R}^{n+1}$, the quantities $\kappa_1(x,t) \leq \kappa_2(x,t) \leq \ldots \leq \kappa_n(x,t)$ represent the principal curvatures of the hypersurface, $\nu(x,t)$ denotes the outer unit normal vector to the evolving convex set $K(t)$, whose boundary is the evolving hypersurface, at the point $X(x,t)$, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers. When $n = 1$ and $\alpha_1 = 1$, we have the simplest form of the flow which is known as the curve shortening flow

$$\frac{\partial X(\cdot,t)}{\partial t} = -\kappa(x,t)\nu(x,t). \tag{1.2}$$

Based on the works of Gage, Hamilton, and Grayson, [23, 20, 19] the curve shortening flow evolves a smooth, closed, simple curve to a round point. It is a natural question to inquire whether a generalized version of the curve shortening flow exhibits the same graceful behaviour as the original curve shortening flow. More precisely, it is natural to ask

- Do solutions of the generalized flow exist?
- If solutions exist, does the flow preserve their convexity?
- Do the solutions asymptotically converge to a sphere too?

The Gauss curvature flow is widely recognized as a notable generalization of the curve shortening flow to higher dimensions:

$$\frac{\partial X(\cdot,t)}{\partial t} = -G(\cdot,t)\nu(\cdot,t),$$

where $G(\cdot,t)$ is the Gauss curvature and $\nu(\cdot,t)$ is the outer normal at $K(t)$. Tso's work [55] demonstrated that the Gauss curvature flow exhibits convergence to a point within a finite time. Additionally, the rescaled solutions of the flow converge to a convex hypersurface. Chow initially studied a generalization of the Gauss curvature flow, described by the equation:

$$\frac{\partial X(x,t)}{\partial t} = -G^{\beta}(x,t)\nu(x,t), \tag{1.3}$$

where G represents the Gaussian curvature, and β is a positive number, [13]. Chow's research revealed that when $\beta = \frac{1}{n}$, the rescaled solutions of this flow converge to a sphere. Andrews extended this understanding by demonstrating that for $\frac{1}{2} \le \beta \le \frac{2}{3}$, the flow in $\mathbb{R}^3$ asymptotically converges to a sphere, [13]. Building upon these insights, Brendle, Choi, and Daskalopoulos [9] further established that for $\beta \ge \frac{1}{n+2}$, the flow in $\mathbb{R}^{n+1}$ exhibits roundness. The flow we described by equation (1.1) can be regarded as a generalization of the flow by powers of Gaussian curvature. It can also be considered as a more challenging generalization of the Gaussian flow due to the asymmetric and non-homogeneous nature of the speed term in equation (1.1). The outcomes presented are limited by the technical challenges of the problem that led to some conjectures. Nonetheless, the value lies in the pioneering approaches employed to substantiate these assertions. In the first chapter, we focus on the following generalization of the curve shortening flow in the plane:

$$\frac{\partial X(\theta,t)}{\partial t} = -\gamma(\theta)\kappa^{\alpha}(\theta,t)N(\theta), \tag{1.4}$$

where X is the position vector of the curve with respect to θ, the angle of the outward normal $N(\theta) = (\cos\theta, \sin\theta)$ to the curve with respect to the horizontal direction, $\alpha \in (0,1)$ is an arbitrary real number and $\gamma : [0, 2\pi] \longrightarrow (0,\infty)$ is a smooth π-periodic, time independent function. For $\gamma = 1$, and $\alpha = 1$, the equation describes the curve shortening flow. In case $\alpha = 1$, and non-constant γ, Gage, and Gage-Li have studied the flow, ([19], [21]) and showed, among other things, that the flow evolves each strictly convex smooth curve to a point with its shape approaching a Minkowski isoperimetrix determined in a certain sense by γ. Andrews showed [4] that, asymptotically, solutions converge to a limiting shape when $\alpha \in (\frac{1}{3}, 1)$, however rescaled solutions are not necessarily convergent when $\alpha \in (0, \frac{1}{3})$ as compactness of the rescaled solutions may not hold. He states that if the isoperimetric ratio of the domains bounded by the solutions remains bounded for all time, then the compactness follows. However, this is not possible to detect in general. When γ is $\frac{\pi}{2}$-periodic, we show, in the first chapter, that rescaled solutions converge to a limiting shape which have been obtained independently by others, ([4],[11]), in a slightly different context. However, we choose to present them here as this was the starting point of another generalization to higher dimensions. More precisely, we show the following:

Theorem 1.0.1. *Let $\alpha \in (0,1)$ be a real number, and let $\gamma : [0, 2\pi] \longrightarrow (0,\infty)$ be a smooth $\frac{\pi}{2}$-periodic function. Then, there exists a planar strictly convex body $\widetilde{K}$ such that*

$$\gamma(\theta) = \frac{\widetilde{h}(\theta)}{\widetilde{\kappa}^{\alpha}(\theta)},$$

and every smooth, strictly convex, $\frac{\pi}{2}$-periodic curve that evolves by equation (2.1), $X_t = \gamma\kappa^{\alpha}N$, shrinks to a point in a finite time. Moreover, if the family of evolving curves is renormalized to enclose domains of constant area, they will converge sequentially in the Hausdorff metric to the boundary of a convex body $\widetilde{K}$ as above.

This behaviour is related to the to the L_p-Minkowski problem, [32]. While major contributions have been made to the L_p-Minkowski problem, see ([32],[34],[19], [21],[50], [49], [54]), it is not solved completely. To be more precise, in case $p \geq 0$, while certain conditions are imposed, the problem is not solved for every $p \geq 0$. Let $p \geq 0$ be a fixed real number and $\psi : [0, 2\pi] \longrightarrow (0, \infty)$ be a π-periodic, smooth function of class C^2. The L_p-Minkowski problem, corresponding to this data asks for $h : [0, 2\pi] \longrightarrow (0, \infty)$ such that for every $\theta \in [0, 2\pi]$

$$h^{1-p}(\theta)\big(h''(\theta) + h(\theta)\big) = \psi(\theta).$$

Using the asymptotic behaviour of the flow obtained in the previous theorem, for $\frac{\pi}{2}$-periodic data, we prove

Theorem 1.0.2. *Let $p \in (-\infty, 0)$, and let γ be a $\frac{\pi}{2}$-periodic, positive, C^2 function. Then, there exists a $\frac{\pi}{2}$-periodic support function $h : [0, 2\pi] \longrightarrow (0, \infty)$ which is a solution to the L_p-Minkowski problem on $\mathbb{S}^1$*

$$h^{1-p}(h + h'') = \psi,$$

where $\psi = \gamma^{1-p}$.

The results for curves have been also obtained by other authors, [16, 54, 11]. For us, the motivation was to understand the curvature flows techniques in the lowest dimensional case and to extend this to a nonsymmetric flow on surfaces. In the pioneering work of Li and Lv ([31]), contraction of convex hypersurfaces by non-homogeneous speeds was studied. Among their findings, it was demonstrated that such a flow leads convex hypersurfaces to converge to a single point in finite time. Notably, in hyperbolic space, the concept of roundness was established for sufficiently pinched initial hypersurfaces using high powers of the speed. Subsequently, McCoy

[36] extended these results to Euclidean spaces. Since flows with non-homogeneous speeds can be intricate, previous works, including the aforementioned ones, often consider initial hypersurfaces to be surfaces of revolution. To explore the evolution of a surface under a non-symmetric and non-homogeneous speed, in the second chapter, we specifically investigate the behaviour of a smooth surface of revolution in $\mathbb{R}^3$, and then we extend the obtained results to higher dimensions. More precisely, we evolve a smooth surface of revolution $\partial K_0 \subset \mathbb{R}^3$ using the following equation:

$$\frac{\partial X(x,t)}{\partial t} = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\alpha_2}(x,t)\nu(x,t), \tag{1.5}$$

where κ_1 represents the axial curvature, κ_2 represents the radial curvature, and α_1 and α_2 are positive real numbers. Subsequently, we delve into the inquiries concerning the generalization of curve shortening flows. One of our objectives lies in substantiating the existence of admissible solutions which, concurrently, means preservation of convexity throughout the course of the flow. Additionally, we substantiate the emergence of singularities after finite time undertaking a comprehensive analysis of these singularities. Moreover, throug an appropriate rescaling technique, we assert the convergence of the flow solutions to the boundary of a convex body, thereby adressing the flow's long term behaviour. In sum, in the second chapter, we prove:

Theorem 1.0.3. *Suppose that a smooth, strictly convex, embedded surface $\partial K_0 \subset \mathbb{R}^{n+1}$ is axially symmetric with an even profile curve. Then solutions to the equation (5.1) exist on a maximal time interval $[0, \omega)$, and they shrink to a point as $t \to \omega$. Furthermore, if solutions are rescaled to enclose domains of constant volume, the rescaled solutions converge sequentially, in the Hausdorff metric, to the boundary of a convex body.*

In the third chapter of this thesis, we introduce and explore the short-term and long-term behaviour of an expanding flow governed by a non-symmetric speed function that depends on the radii of curvatures. Specifically, we consider the expansion of a strictly convex surface of revolution, denoted as $\partial K_0 \subset \mathbb{R}^{n+1}$, using the following equation:

$$\frac{\partial X}{\partial t}(x,t) = \left(\frac{1}{\kappa_1}\right)^{\alpha_1}(x,t)\left(\frac{1}{\kappa_2}\right)^{\alpha_2}(x,t)\dots\left(\frac{1}{\kappa_n}\right)^{\alpha_n}(x,t)\nu(x,t),$$

where $\kappa_1 = \kappa_{axi}, \kappa_2 = \cdots = \kappa_n = \kappa_{rad}$ are principal curvatures, $\alpha_1, \dots, \alpha_n$ are arbitrary positive real numbers, and ν is the outer unit normal. Throughout this chapter, we analyze the behaviour of the flow, both the short term and the long term. We investigate how the surface expands and evolves under the influence of the non-symmetric speed function, examining the changes in its geometric properties. By studying the flow's dynamics and employing appropriate techniques from the theory of parabolic partial differential equations, we aim to gain insights into the expansion process and its asymptotic behaviour over time. In this chapter, we prove:

Theorem 1.0.4. *Consider a smooth, strictly convex embedded surface ∂K_0 that exhibits axial symmetry in $\mathbb{R}^{n+1}$. For equation (4.3), there exist solutions within a maximal time interval $[0, T)$, where T is finite when $\alpha + \beta > 1$ and infinite when $\alpha + \beta \leq 1$. As the flow progresses, these solutions will preserve convexity and will expand to infinity. Furthermore, if the profile curve is even and the solutions are rescaled to enclose domains of constant volume, they will converge sequentially, in the Hausdorff, metric towards the boundary of a convex body.*

In the fourth chapter of this thesis, the focus is on the most general form of the flow. While a suitable framework for the comprehensive study has not been fully developed, the chapter presents partial progress. It is crucial to acknowledge that a complete study of this problem would require new development of several theories to reach its full completion. We plan to continue the study of this problem in further work. Specifically, in this chapter, we consider a strictly convex hypersurface

∂K_0 undergoing curvature flow governed by equation (1.1). We address several key questions such as existence of solutuons, beginning with the scenario where ∂K_0 is a compact, smoothly embedded, strictly convex hypersurface of dimension $n \geq 2$ in $\mathbb{R}^{n+1}$ represented by the embedding $X_0 : \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$. Our investigation revolves around the evolution of the family of maps $X(\cdot, t)$ according to equation (1.1), where $\kappa_1(x,t) \leq \cdots \leq \kappa_n(x,t)$ denote the principal curvatures, $\nu(x,t)$ represents the outer normal to $K(t)$ at $X(x,t)$, and $\alpha_1 \leq \cdots \leq \alpha_n$ are positive real numbers. Our long term objective is to demonstrate the existence of solutions, establish the preservation of convexity, and investigate the singularities. Here, we prove short time existence of solutions:

Theorem 1.0.5. *The solutions to the flow (1.1) exist on some time interval* $[0, \omega)$ *with* $\omega < \infty$ *before developing a singularity.*

This is not trivial as all known results assume the symmetry of the speed function.

Chapter 2

Planar curvature flows and their applications in convex geometry

2.1 Introduction

We consider the following flow described by the partial differential equation:

$$\frac{\partial X(\theta, t)}{\partial t} = \gamma(\theta)\kappa^{\alpha}(\theta, t)N, \tag{2.1}$$

where X is the position vector of the curve with respect to θ, the angle of the inward normal $N = -(\cos\theta, \sin\theta)$ to the curve with respect to the horizontal direction, $\alpha \in (0, 1)$ is an arbitrary real number and $\gamma : [0, 2\pi] \longrightarrow (0, \infty)$ is a smooth π-periodic function.

Equation (2.1) can be interpreted as a generalization to the flow known as the curve shortening flow. Indeed, for $\gamma = 1$, and $\alpha = 1$, the equation describes the curve shortening flow on which there is an extensive literature, see [12]. In case $\alpha = 1$, and non-constant γ, Gage, and Gage-Li have studied the flow in a couple of papers ([19], [21]) and showed, among other things, that the flow evolves each strictly convex smooth curve to a point with its shape approaching a Minkowski isoperimetrix determined in a certain sense by γ.

In [4], Andrews showed that, asymptotically, solutions converge to a limiting shape when $\alpha \in (\frac{1}{3}, 1)$, however rescaled solutions are not necessarily convergent when $\alpha \in (0, \frac{1}{3})$ as compactness of the rescaled solutions may not hold. He states that if the isoperimetric ratio of the domains bounded by the solutions remains bounded for all time, then the compactness follows. However, in general, this is not possible to detect from the initial data. We prove that rescaled solutions converge to a limiting shape when γ is $\frac{\pi}{2}$-periodic and we will comment shortly why this is a worth considering in connection to a class of Minkowski problems. Other authors [4, 16], have used similar conditions to study the flow for some specific intervals α, our result applies for all $\alpha \in (0, 1)$. More precisely, we prove:

Theorem 2.1.1. *Let $\alpha \in (0,1)$ be a real number, and let $\gamma : [0, 2\pi] \longrightarrow (0, \infty)$ be a smooth $\frac{\pi}{2}$-periodic function. Then, there exists a planar strictly convex body $\widetilde{K}$ such that*

$$\gamma(\theta) = \frac{\widetilde{h}(\theta)}{\widetilde{\kappa}^{\alpha}(\theta)},$$

and every smooth, strictly convex, $\frac{\pi}{2}$-periodic curve that evolves by equation (2.1), $X_t = \gamma\kappa^{\alpha}N$, shrinks to a point in a finite time. Moreover, if the family of evolving curves is renormalized to enclose domains of constant area, they will converge sequentially in the Hausdorff metric to the boundary of a convex body $\widetilde{K}$ as above.

The classical Minkowski problem asks for a closed convex hypersurface whose Gauss curvature has been given. In the discrete setting, Minkowski himself solved the problem ([39], [40]), and, later, Bonnesen and Fenchel found that the given proof by Minkowski can be extended to the case of convex hypersurfaces in Euclidean spaces of any dimension [17], followed later by Alexandrov who proved the problem in an innovative way ([1], [2], [3]). Finally, by Nirenberg's [41] and A. V. Pogorelov's ([42], [43]) contributions addressing the regularity of the problem, the Minkowski problem was solved completely.

A generalization of Minkowski problem, known as the L_p-Minkowski problem, asks under what conditions on a given measure μ on $\mathbb{S}^n$, there exists a convex body K whose L_p surface area measure is μ. For $p = 1$, the problem is equivalent to the classical Minkowski problems which is already solved. Major contributions have been made to the L_p-Minkowski problem. Lutwak solved the even L_p-Minkowski problem in $\mathbb{R}^n$ for all $p \geq 1$ except for $p = n$ [32]. In [34], the generalized problem was studied and solved for $p = n$ by Lutwak, Yang and Zhang. Even in the plane, there is an extensive literature on this problem. The case $p = 0$ has been studied in the plane by Gage in [19] and by Gage and Li in [21], and for an atomic measure by Stancu in ([50], [49], [48]). For $p \neq 0$ the problem has been studied in the plane by Umanskiy [54], for $-2 < p < 0$ when the measure is not necessarily positive by Chen [11], and for $p < -2$ and $\frac{\pi}{k}$-periodic data ($k > 1$) which is positive at one point by Dou and Zhu [16]. So, in case $p < 0$, while certain conditions are imposed on the measure, the problem is solved partially, not for every $p < 0$. However, we impose some conditions on data, we solve the problem for every $p < 0$ when solvable. To be more precise, when data is $\frac{\pi}{2}$-periodic, we solve the smooth planar problem for every $p < 0$. In this regard, using the asymptotic behaviour of the flow obtained in the previous theorems, we prove:

Theorem 2.1.2. *Let $p \in (-\infty, 0)$, and let γ be a $\frac{\pi}{2}$-periodic, smooth, positive, C^2 function. Then, there exists a $\frac{\pi}{2}$-periodic support function $h : [0, 2\pi] \longrightarrow (0, \infty)$ which is a solution to the L_p-Minkowski problem on $\mathbb{S}^1$*

$$h^{1-p}(h + h'') = \psi,$$

where $\psi = \gamma^{1-p}$.

To see the connection between the solutions to the flow and the solutions to the

L_p-Minkowski problem in the plane for $p < 0$, note the following with $\alpha = 1/(1-p)$. We remark that solutions to the L_p-Minkowski problem are obtained as asymptotic shapes to corresponding curvature flows.

Let $\psi : [0, 2\pi] \longrightarrow (0, \infty)$ be a π-periodic smooth function, and $p < 0$. The L_p-Minkowski problem associated to this data is equivalent to the study of positive solutions to the following ordinary differential equation

$$h^{1-p}(h''(\theta) + h(\theta)) = \psi, \qquad \theta \in [0, 2\pi]. \tag{2.2}$$

So, if h is a $\mathcal{C}^2$ positive, periodic solution to this equation, h is the support function for a convex body whose p-surface area measure is $\psi d\theta$.

Let K be a compact, strictly convex body and symmetric with respect to the origin, and let $x : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ be the Gauss parametrization of its boundary, ∂K. The support function of ∂K is defined by

$$h(N) = -\big\langle x(N), N \big\rangle \qquad \text{for every } N = -(\cos\theta, \sin\theta) \in \mathbb{S}^1. \tag{2.3}$$

The curvature and support function are related by

$$\frac{1}{\kappa(\theta)} = \tau[h] = h_{\theta\theta}(\theta) + h(\theta),$$

where $h_{\theta\theta} = \frac{\partial^2 h}{\partial \theta^2}$, and S is the radius of curvature. We described the vector valued form of the flow by $X_t = \gamma(\theta)\kappa^{\alpha}(\theta, t)N$, therefore the scalar form of the flow is

$$h_t(\theta, t) = -\gamma(\theta)\kappa^{\alpha}(\theta, t), \tag{2.4}$$

where $X_t(\theta, t) = \frac{\partial X(\theta,t)}{\partial t}$, and $h_t(\theta, t) = \frac{\partial h(\theta,t)}{\partial t}$. We call a family of closed, simple, smooth, and strictly convex curves, $\{\partial K(., t)\}$, parametrized by time, a solution to the flow (2.4) (to the flow (2.1)) if their support function (vector position) satisfies the scalar form of the flow (the vector valued form of the flow). Every solution to the flow (2.4) corresponds to a solution to the vector valued form of the flow (2.1), and vice versa. Also, if a centrally symmetric curve is evolved by the flow, solutions remain symmetric with respect to the origin as long as they exist because the points of the opposite normal have the same curvature, and therefore they are contracted with the same speed.

In Section 2, we consider the basic properties of the flow and show that the total time of existence of the flow, T, is finite, during the interval of existence, the curvature is finite, and the area enclosed by curves goes to zero as $t \to T$. In Sections 3 and 4, we show that, when data is $\frac{\pi}{2}$-periodic, if we renormalize solutions to enclose domains of constant area, th normalized solutions converges to a limiting shape which solves the L_p-Minkowski problem for $p \in (-\infty, -2) \cup (-2, 0)$.

We end the introduction by commenting that many of the techniques detailed here are standard in the literature, but we choose to include them here for completion.

2.2 Short time behaviour of the flow

In this section, we show a solution to (2.1) exists for a short time and conclude that the solution exists as long as the area enclosed by the curve is not zero.

Lemma 2.2.1. *Suppose that ∂K_0 is a closed, simple, smooth, and strictly convex curve, which is the boundary of a convex body K_0. Then, there exists a family of smooth, simple, closed curves $\partial K(., t)$, $t \in [0, T)$, with $T > 0$ and $\partial K_0 = \partial K(., 0)$ whose support functions satisfy*

$$h_t(\theta, t) = -\gamma(\theta)\kappa^{\alpha}(\theta, t).$$

Proof. We consider the linearization of the equation above. Suppose that $\epsilon > 0$ is a small real number, and that $\delta \in (-\epsilon, \epsilon)$. Let $\phi : [0, 2\pi) \times [0, T) \longrightarrow \mathbb{R}$ be an arbitrary smooth positive function, and let

$$\bar{h}(\theta, t, \delta) = h(\theta, t) + \delta\phi(\theta, t).$$

The linearization of the scalar equation of the flow is thus

$$\phi_t(\theta, t) = \alpha\gamma(\theta)\kappa^{\alpha+1}(\theta, t)\phi_{\theta\theta}(\theta, t) + \alpha\gamma(\theta)\kappa^{\alpha+1}\phi(\theta, t), \tag{2.5}$$

which for a strictly convex smooth initial conditions is strictly parabolic and, in turn, by the theory of parabolic equations, the short time existence of solutions follows.

□

Lemma 2.2.2. *Assuming that the initial condition is the same as in previous lemma, then the solution $\partial K(., t)$ remains strictly convex on $[0, T)$.*

Proof. Let $u = \gamma\kappa^{\alpha}$. For every $t \in [0, T)$, let

$$u_{\min}(t) = \min_{\theta} u(\theta, t).$$

By taking derivative with respect to t of both sides of $\frac{1}{\kappa} = h_{\theta\theta} + h$, we get

$$\kappa_t = \kappa^2((\gamma\kappa^{\alpha})_{\theta\theta} + \gamma\kappa^{\alpha}).$$

The evolution of u is then

$$\begin{aligned} u_t &= \alpha\gamma\kappa^{\alpha-1}\kappa_t = \alpha\gamma\kappa^{\alpha+1}(\gamma\kappa^{\alpha} + (\gamma\kappa^{\alpha})_{\theta\theta}) \\ &= \alpha\kappa u(u + u_{\theta\theta}). \end{aligned}$$

Now, let $t \in [0, T)$ be arbitrary, and $\theta_o = \theta_o(t)$ be such that $\min_{\theta} u(\theta, t) = u(\theta_0, t)$.

Since $u_{\theta\theta}(\theta_0, t) > 0$, we infer the time evolution of the minimum, in the sense of forward differences when needed:

$$(u_{\min})_t(\theta_0, t) = \alpha\kappa(\theta_0, t)u(\theta_0, t)(u(\theta_0, t) + u_{\theta\theta}(\theta_0, t)) \geq 0$$

and

$$\min_\theta u(\theta, t) \geq \min_\theta u(\theta, 0) > 0.$$

As a result, there exists $c > 0$ such that $\min_\theta \kappa(\theta, t) \geq c$ for every $t \in [0, T)$. So, the solution remains convex as long as it exists and the curvature of the evolving curves has a uniform lower bound for all time of existence of the flow. □

Lemma 2.2.3. *If the area enclosed by the curve, $\partial K(., t)$, is not zero, then*

$$\max_\theta \kappa(\theta, t) < \infty.$$

Proof. Suppose that the area enclosed by the curve is not zero at some time t. Let the origin be in the interior of the domain bounded by $\partial K(\ ., t)$. Then, by our considerations, there exists a $\rho > 0$ such that the circle centered at the origin and radius 2ρ is included in the domain bounded by $\partial K(\ ., t)$. For simplicity, in what follows, we will drop the arguments θ and t when there is not risk of confusion. Following a standard technique in curvature flows, let

$$\Phi(\theta, t) = \frac{-h_t}{h - \rho} = \frac{\gamma\kappa^\alpha}{h - \rho}$$

and note that, since the support function is now greater than 2ρ, there exists some time interval $[t, t' := t + \delta)$ during which the area enclosed by the curve remains strictly positive. Assume that this δ is the time it takes the support function to decrease by at most half, thus remains larger than $\rho/2$.

Consider

$$\Phi_{\max}(t) = \max_{\theta}\{\Phi(\theta, t)\}.$$

For fixed $t_0 \in [0, t')$, at the point (θ_0, t_0) where $\Phi_{\max}$ is reached, we have

$$\Phi_\theta(\theta_0, t_0) = \frac{-h_{t\theta}}{h-\rho} + \frac{h_t h_\theta}{(h-\rho)^2} = 0, \tag{2.6}$$

$$\Phi_{\theta\theta}(\theta_0, t_0) = \frac{-h_{\theta\theta t}}{h-\rho} + \frac{h_{t\theta} h_\theta}{(h-\rho)^2} + \frac{h_{\theta t} h_\theta + h_t h_{\theta\theta}}{(h-\rho)^2} + \frac{-2(h_\theta)^2 h_t}{(h-\rho)^3} \leq 0, \tag{2.7}$$

and

$$\Phi_t(\theta_0, t_0) = \frac{-h_{tt}}{h-\rho} + \frac{h_t^2}{(h-\rho)^2} \geq 0. \tag{2.8}$$

By (5.1), we have $\frac{h_{t\theta} h_\theta}{(h-\rho)^2} = \frac{h_t (h_\theta)^2}{(h-\rho)^3}$, thus

$$\Phi_{\theta\theta} = \frac{-h_{\theta\theta t}}{h-\rho} + \frac{h_t h_{\theta\theta}}{(h-\rho)^2} \leq 0. \tag{2.9}$$

Also, recall that

$$h_{tt} = (-\gamma\kappa^\alpha)_t = -\gamma\alpha\kappa_t\kappa^{\alpha-1}$$

and

$$\kappa_t = -\kappa^2(h_t + h_{\theta\theta t}).$$

Applying the previous equations, in particular (2.9), we have

$$
\begin{aligned}
0 \le \Phi_t &= \frac{-h_{tt}}{h-\rho} + \frac{h_t^2}{(h-\rho)^2} = \frac{\gamma\alpha\kappa_t\kappa^{\alpha-1}}{h-\rho} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{-(\gamma\alpha\kappa^{\alpha+1})(h_t + h_{\theta\theta t})}{h-\rho} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{-(\gamma\alpha\kappa^{\alpha+1})(h_t)}{h-\rho} - \frac{\gamma\alpha\kappa^{\alpha+1}(h_{\theta\theta t})}{h-\rho} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\alpha\kappa^{2\alpha+1}}{h-\rho} - \frac{\gamma\alpha\kappa^{\alpha+1}(h_{\theta\theta t})}{h-\rho} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&\le \frac{\gamma^2\alpha\kappa^{2\alpha+1}}{h-\rho} - \frac{\gamma\alpha\kappa^{\alpha+1}(h_{\theta\theta}h_t)}{(h-\rho)^2} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\alpha\kappa^{2\alpha+1}}{h-\rho} + \frac{\gamma^2\alpha\kappa^{2\alpha+1}h_{\theta\theta}}{(h-\rho)^2} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\alpha\kappa^{2\alpha+1}}{(h-\rho)^2}(h-\rho+h_{\theta\theta}) + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\alpha\kappa^{2\alpha+1}}{(h-\rho)^2}(h-\rho+h_{\theta\theta}) + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\alpha\kappa^{2\alpha}}{(h-\rho)^2} - \rho\frac{\gamma^2\alpha\kappa^{2\alpha+1}}{(h-\rho)^2} + \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2} \\
&= \frac{\gamma^2\kappa^{2\alpha}}{(h-\rho)^2}(\alpha+1-\rho\alpha\kappa).
\end{aligned}
$$

Therefore,

$$\kappa \le \frac{1+\alpha}{\rho\alpha}, \tag{2.10}$$

concluding the proof. □

Proposition 2.2.1. *Let $[0,T)$ be the maximal time interval on which the solution of equation (2.4) exists. Then, the area enclosed by the convex curve at time t, $A(t)$ satisfies*

$$A(t) \to 0 \qquad as \quad t \to T.$$

Proof. Suppose by contradiction that $\lim_{t\to T} A(t) > 0$. Then, by Lemma 2.2.3 and

Lemma 2.2.2,

$$0 < \min_{\theta} \kappa(\theta, 0) \le \min_{\theta} \kappa(\theta, T) \le \max_{\theta} \kappa(\theta, T) < \infty.$$

Therefore, the flow equation, the equation (2.4), is strictly parabolic at time T, and in turn can be extended for a short time beyond T. This contradicts the assumption that $[0, T)$ is the maximal time interval on which the solution exists. So,

$$A(t) \to 0 \qquad \text{as} \qquad t \to T.$$

□

Lemma 2.2.4. *The total time of existence of the flow is finite.*

Proof. Let Γ_0 be a circle centred at the origin with radius ρ_0 containing the initial curve ∂K_0 in its interior. Evolve Γ_0 by the flow

$$h_t = -[\min_{\theta} \gamma(\theta)]\kappa^{\alpha}$$

to get the evolving circle $\Gamma(., t)$, more precisely to get the solution

$$\Gamma(t) = \left\{ (x(t), y(t)) \in \mathbb{R}^2 \mid x^2(t) + y^2(t) = (\rho(t))^2 := \left(\rho_0^{\alpha+1} - [\min_{\theta} \gamma(\theta)](\alpha + 1)t \right)^{\frac{2}{\alpha+1}} \right\}.$$

As $t \to \omega := \frac{\rho_0^{\alpha+1}}{\min_{\theta} \gamma(\alpha+1)}$, we have that $\rho(t) \to 0$ and therefore the circle solution to the flow defined above exists only for a finite time ω when the circle becomes a single point. We now claim that for every t, $\partial K(t) \subset Int(\Gamma(t))$ and, thus, the total existence time T of the flow for $\partial K(t)$ satisfies an upper bound ω, $T \le \omega$. By contradiction, suppose that this is not true, so $\partial K(t)$ lags behind. Then, there exists a time t_0 when at some θ_0, $\Gamma(\theta_0, t_0)$ touches from outside $\partial K(\theta_0, t_0)$ for the first time. Because the curves will have the same normal at that point, they will be tangent at θ_0 and thus, the containment implies

$$\kappa(\theta_0, t_0) \geq \frac{1}{\rho(t_0)},$$

and so the curves' speeds by their corresponding flows at (θ_0, t_0) are in the relation

$$-\gamma\kappa(\theta_0, t_0)^\alpha \leq -\min_\theta \gamma(\theta) \left[\frac{1}{\rho(t_0)}\right]^\alpha.$$

Therefore, the flow of $\partial K(t)$ at θ_0 moves faster toward the origin than the circle. So, the circle cannot pass the convex body at any time, the so-called avoidance principle, and we conclude that the solution for our original flow would become singular at most at time ω. □

2.3 Normalized evolution equations and compactness of normalized solutions

Using a standard technique in homogeneous curvature flows, we rescale the support function of the flow by

$$\widetilde{h}(\theta, t) = \frac{h(\theta, t)}{\sqrt{A(t)}},$$

creating a normalized support function, and define the new time variable by

$$\tau = -\frac{1}{2} \ln \frac{A(t)}{A(0)}.$$

By direct calculations, at every time $\tau \in (0, \infty)$, we get $\widetilde{\kappa}(\theta, \tau) = \sqrt{A(\tau)}\kappa(\theta, \tau)$, and

$$\widetilde{A}(\tau) = \frac{1}{2} \int_0^{2\pi} \frac{\widetilde{h}(\theta, \tau)}{\widetilde{\kappa}(\theta, \tau)}\, d\theta = 1.$$

By differentiating both sides of $A(t) = \frac{1}{2}\int\limits_0^{2\pi} \frac{h}{\kappa} d\theta$, the area enclosed by the solution to the flow at time t, with respect to t, we get

$$\begin{aligned}\frac{dA(t)}{dt} &= \frac{1}{2}\int\limits_0^{2\pi}\left(\frac{h_t}{\kappa} - \frac{h}{\kappa^2}\kappa^2(\gamma\kappa^\alpha + (\gamma\kappa^\alpha)_{\theta\theta})\right)d\theta \\ &= \frac{1}{2}\int\limits_0^{2\pi}\left(\frac{h_t}{\kappa} - h(\gamma\kappa^\alpha + (\gamma\kappa^\alpha)_{\theta\theta})\right)d\theta\,. \end{aligned} \tag{2.11}$$

Using integration by parts, we infer from equation (2.11) that

$$\begin{aligned}\frac{dA(t)}{dt} &= \frac{1}{2}\int\limits_0^{2\pi}\left(\frac{h_t}{\kappa} - h\gamma\kappa^\alpha - h_{\theta\theta}(\gamma\kappa^\alpha)\right)d\theta \\ &= \frac{1}{2}\int\limits_0^{2\pi}\left(\frac{h_t}{\kappa} - (h + h_{\theta\theta})(\gamma\kappa^\alpha)\right)d\theta \\ &= -\int\limits_0^{2\pi}(\gamma\kappa^{\alpha-1})\,d\theta\,. \end{aligned}$$

Now by chain rule, we get the evolution equations of the normalized flow:

$$\widetilde{h}_\tau = \widetilde{h} - \frac{\gamma\widetilde{\kappa}^\alpha}{\frac{1}{2}\int\limits_0^{2\pi}\gamma\widetilde{\kappa}^{\alpha-1}\,d\theta},$$

and

$$\widetilde{\kappa}_\tau = \widetilde{\kappa}^2\frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int\limits_0^{2\pi}\gamma\widetilde{\kappa}^{\alpha-1}d\theta} - \widetilde{\kappa}.$$

For the rest of this section, we assume $\gamma : [0, 2\pi] \longrightarrow (0, \infty)$ and $\partial\widetilde{K}_0$ are $\frac{\pi}{2}$-periodic.

Proposition 2.3.1. *The normalized evolving curves, $\partial\widetilde{K}(\ .\ ,\tau)$, remain in a compact annulus for all time τ, $0 \le \tau < \infty$.*

Proof. First, we reproduce a technique used in Lemma 4.2 of [27] to show $\{\widetilde{h}(\theta,\tau)\}_\tau$

is uniformly bounded from above. Let $\tau \in (0, \infty)$ be arbitrary, $\widetilde{h} := \widetilde{h}(\tau)$, and $\widetilde{h} = \widetilde{h}_0 + \sum_{n=1}^{\infty} \widetilde{h}_n \cos(4n\theta)$ be the cosine series of $\widetilde{h}$. Then,

$$\frac{1}{\widetilde{\kappa}} = \widetilde{h}_{\theta\theta} + \widetilde{h} = \widetilde{h}_0 + \sum_{n=1}^{\infty}(1 - 16n^2)\widetilde{h}_n \cos(4n\theta).$$

We infer from

$$0 \leq \int_0^{2\pi} \frac{1 \pm \cos(4n\theta)}{\widetilde{\kappa}} d\theta = 2\pi\widetilde{h}_0 \pm \pi\widetilde{h}_n(1 - 16n^2)$$

that for every $n \in \mathbb{N}$,

$$|\widetilde{h}_n| \leq \frac{2\widetilde{h}_0}{16n^2 - 1}.$$

This implies

$$\begin{aligned} 2 = \int_0^{2\pi} \frac{\widetilde{h}}{\widetilde{\kappa}} d\theta &= 2\pi\widetilde{h}_0^2 - \pi\sum_{n=1}^{\infty}(16n^2 - 1)\widetilde{h}_n^2 \\ &\geq 2\pi(1 - 2\sum_{n=1}^{\infty}\frac{1}{16n^2 - 1})\widetilde{h}_0^2 := 2C\widetilde{h}_0^2. \end{aligned}$$

Thus, $\widetilde{h}_0 \leq \sqrt{C}$. We have that

$$\widetilde{h}(\tau, \theta) = \frac{1}{2}(\widetilde{h}(\tau, \theta) + \widetilde{h}(\tau, \theta + \pi)) \leq \frac{1}{4}L(\partial\widetilde{K}(\tau)) = \frac{\pi}{2}\widetilde{h}_0 \leq \frac{\pi}{2}\sqrt{C}.$$

Since $\tau \in (0, \infty)$ is arbitrary and C is independent of τ and θ, $\{\widetilde{h}(\theta, \tau)\}$ is uniformly bounded from above. We claim that the support function is uniformly bounded from below. To see this, let $\tau \in (0, +\infty)$, and let $\widetilde{E}(\tau)$ be the John's ellipsoid of $\widetilde{K}(\tau)$. If $\rho = \rho(\tau)$, and $\eta = \eta(\tau)$ are respectively the shortest and the longest semi-axis of $\widetilde{E}(\tau)$, then, from $\widetilde{E}(\tau) \subset \widetilde{K}(\tau) \subset \sqrt{2}\widetilde{E}(\tau)$, John's Theorem (see [22], Theorem

4.2.12), we have

$$C \geq \max_{\theta} \widetilde{h}(\theta, \tau) \geq \eta \quad \text{and} \quad \min_{\theta} \widetilde{h}(\theta, \tau) \geq \rho. \tag{2.12}$$

Since area of $\widetilde{K}(\tau)$ is 1 and $\widetilde{E}(\tau) \subset \widetilde{K}(\tau) \subset \sqrt{2}\widetilde{E}(\tau)$, we infer that

$$\pi\rho\eta \leq 1 \leq 2\pi\rho\eta \leq 2\pi\rho C. \tag{2.13}$$

From here, $\rho \geq 1/(2\pi C)$, and so $\tilde{h}_K(\theta, \tau) \geq \min_\theta \widetilde{h}(\theta, \tau) \geq \rho \geq 1/(2\pi C)$, thus we obtain a uniform lower bound on $\tilde{h}_K$ concluding the proof. □

Corollary 2.3.1. *The normalized curvature function is uniformly bounded from above and below for all time τ.*

Proof. The normalized solution remains in a fixed annulus, so the normalized support function is bounded from both sides. We apply the technique from Stancu-Vikram [52], Proposition 3.4, to show the curvature is bounded from above. Let $t \in (0, T)$, and let $E(t)$ be the John's ellipsoid of $K(t)$. If ρ, and η are respectively the shortest and the longest semi-axis of $E(t)$, then

$$\max_{\theta} \widetilde{h}(\theta, t) \leq \sqrt{2}\eta \quad \text{and} \quad \min_{\theta} \widetilde{h}(\theta, t) \geq \rho.$$

Since $\widetilde{h}(\theta, t) = \frac{h(\theta,t)}{\sqrt{A(t)}}$, we have that $h(\theta, t) \geq \rho\sqrt{A(t)}$. We infer from Lemma 2.2.3, equation (2.10), that

$$\kappa(\theta, t) \leq \frac{2(1+\alpha)}{\rho\alpha\sqrt{A(t)}}.$$

Since $\widetilde{\kappa}(\theta, t) = \kappa(\theta, t)\sqrt{A(t)}$, we infer that

$$\widetilde{\kappa}(\theta, t) \leq \frac{2(1+\alpha)}{\rho\alpha}.$$

Since area of $\widetilde{K}(t)$ is 1, we infer from $\widetilde{E}(t) \subset \widetilde{K}(t) \subset \sqrt{2}\widetilde{E}(t)$ John's Theorem (see

[22], Theorem 4.2.12) that

$$\pi\rho\eta \leq 1 \leq \sqrt{2}\pi\rho\eta.$$

Thus

$$\widetilde{\kappa}(\theta,t) \leq \frac{2(1+\alpha)}{\rho\alpha} \leq \frac{2\sqrt{2}(1+\alpha)}{\alpha}\pi \max_{\theta} \widetilde{h}(\theta,t).$$

Since support function is bounded from above, the curvature is bounded from above on $(0,T)$. There is a one-to-one correspondence between $0 < t < T$, and $0 < \tau < \infty$, so the curvature is bounded from above on $0 < \tau < \infty$. □

2.4 Asymptotic behaviour of the flow

In this section, we study the asymptotic behaviour of the flow. We suppose the data is $\frac{\pi}{2}$-periodic and show the curves converge to a limiting shape which solves the L_p Minkowski problem. Define the entropy of the (un-normalized) flow by

$$\mathcal{E}(t) := A^{(\alpha-1)/2}(t)\int_0^{2\pi} \gamma(\theta)k^{\alpha-1}(\theta,t)\,d\theta.$$

Proposition 2.4.1. *The entropy satisfies the following inequality along the flow*

$$\frac{d}{dt}(\mathcal{E}(t)) \geq 0. \tag{2.14}$$

Furthermore, the entropy is constant if and only if the evolving curve is the boundary of a convex body solution to a L_p-Minkowski problem with $p < 0$.

Proof. To see this, note that

$$\frac{d\mathcal{E}(t)}{dt} = \frac{(\alpha-1)}{2} A^{\frac{\alpha-3}{2}}(t) A_t(t) \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha-1}(\theta,t)\, d\theta +$$

$$(\alpha-1) A^{\frac{\alpha-1}{2}}(t) \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t)[\gamma(\theta)\kappa^{\alpha}(\theta,t) + (\gamma(\theta)\kappa^{\alpha}(\theta,t))_{\theta\theta}]\, d\theta$$

$$= \frac{(1-\alpha)}{2} A^{\frac{\alpha-3}{2}}(t)(A_t(t))^2 -$$

$$(1-\alpha) A^{\frac{\alpha-1}{2}}(t) \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t)[\gamma(\theta)\kappa^{\alpha}(\theta,t) + (\gamma(\theta)\kappa^{\alpha}(\theta,t))_{\theta\theta}]\, d\theta \geq 0.$$

The last inequality holds because it is equivalent to a Minkowski's inequality. Indeed,

$$\frac{(1-\alpha)}{2} A^{\frac{\alpha-3}{2}}(t)(A_t(t))^2 - (1-\alpha) A^{\frac{\alpha-1}{2}}(t) \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t)[\gamma(\theta)\kappa^{\alpha}(\theta,t) + (\gamma(\theta)\kappa^{\alpha}(\theta,t))_{\theta\theta}]\, d\theta \geq 0$$

holds if

$$\frac{(1-\alpha)}{2}(A_t(t))^2 - \frac{(1-\alpha)}{2} \int_0^{2\pi} \frac{h(\theta,t)}{\kappa(\theta,t)}\, d\theta \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t)[\gamma(\theta)\kappa^{\alpha}(\theta,t) + (\gamma(\theta)\kappa^{\alpha}(\theta,t))_{\theta\theta}]\, d\theta \geq 0.$$

This is the same as

$$\left(\int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t) s[h(\theta,t)] d\theta \right)^2 \geq \int_0^{2\pi} h(\theta,t) s[h(\theta,t)] d\theta \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha}(\theta,t) s[\gamma(\theta)\kappa^{\alpha}(\theta,t)] d\theta.$$

This inequality is derived from the Minkowski's inequality ([46], Theorem 6.2.1) that states for two convex bodies with support functions h_1 and h_2

$$\left(\int_0^{2\pi} h_1(\theta,t)s[h_2(\theta,t)]d\theta\right)^2 \geq \int_0^{2\pi} h_1(\theta,t)s[h_1(\theta,t)]d\theta \int_0^{2\pi} h_2(\theta,t)s[h_2(\theta,t)]d\theta.$$

If we replace one of the support functions by an arbitrary smooth function, the inequality still holds. For an elaboration on this, see [4], proof of the lemma I1.5. The equality holds if there is a constant $\lambda > 0$ such that $\lambda\gamma\kappa^\alpha = h$. We have

$$\sqrt[\alpha]{\lambda\gamma} = \frac{h^{\frac{1}{\alpha}}}{\kappa} = h^{\frac{1}{\alpha}}(h_{\theta\theta} + h) = h^{1-p}(h_{\theta\theta} + h),$$

where $p = \frac{\alpha-1}{\alpha}$. So, the equality holds if K, up to rescaling, is a solution to a L_p-Minkowski problem. We note that since $\alpha \in (0,1)$, $p = \frac{\alpha-1}{\alpha} \in (-\infty, 0)$. □

Corollary 2.4.1. *The entropy of the normalized flow is non-decreasing and is constant if and only if the evolving curve is, up to normalization, a solution to the corresponding L_p-Minkowski problem.*

Proof. We have, strictly from the normalization and the fact $\tau = \tau(t)$, that

$$\begin{aligned}\widetilde{\mathcal{E}}(t) &= \widetilde{A}^{\frac{\alpha-1}{2}} \int_0^{2\pi} \gamma(\theta)\widetilde{\kappa}^{\alpha-1}(\theta,t)d\theta \\ &= \int_0^{2\pi} \gamma(\theta)(\sqrt{A}\kappa)^{\alpha-1}(\theta,t)d\theta \\ &= A^{\frac{\alpha-1}{2}} \int_0^{2\pi} \gamma(\theta)\kappa^{\alpha-1}(\theta,t)d\theta \\ &= \mathcal{E}(t).\end{aligned}$$

Therefore

$$\frac{d\widetilde{\mathcal{E}}(t)}{dt} = \frac{d\mathcal{E}(t)}{dt}.$$

Since $\frac{dt}{d\tau} = -\frac{2A(t)}{A_t} > 0$ and $\frac{d\mathcal{E}(t)}{dt} \geq 0$, we infer

$$\frac{d\widetilde{\mathcal{E}}(\tau)}{d\tau} = \frac{d\widetilde{\mathcal{E}}(t)}{dt} \times \frac{dt}{d\tau} \geq 0.$$

□

Proposition 2.4.2. *The entropy of the normalized flow is bounded from above and below.*

Proof. Since the entropy is increasing, it is bounded from below by its value at time zero. Now, we want to show the entropy is bounded from above. Suppose by contradiction that there is $\{\tau_n\} \nearrow \infty$ such that $\widetilde{\mathcal{E}}(\tau_n) \nearrow \infty$. Since

$$\widetilde{\mathcal{E}}(\tau) \leq \max_{[0,2\pi]} \gamma(\theta) \int_0^{2\pi} \frac{1}{\widetilde{\kappa}^{1-\alpha}(\theta,\tau)} d\theta,$$

we infer that

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{1}{\widetilde{\kappa}^{1-\alpha}(\theta,\tau_n)} d\theta = \infty.$$

Jensen's inequality implies that

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{1}{\widetilde{\kappa}(\theta,\tau_n)} d\theta = \infty.$$

By Proposition 2, the support function of the normalized solutions is bounded from below. Let $C > 0$ be such that for every $\tau > 0$ and every $\theta \in [0, 2\pi]$,

$$\widetilde{h}(\theta,\tau) \geq C.$$

Therefore

$$C \int_0^{2\pi} \frac{1}{\widetilde{\kappa}(\theta,\tau_n)} d\theta \leq \int_0^{2\pi} \frac{\widetilde{h}(\theta,\tau)}{\widetilde{\kappa}(\theta,\tau_n)} d\theta = 2.$$

As $n \to \infty$, the left hand side goes to infinity while the right hand side is fixed. This is a contradiction. So, the entropy is bounded from above. $\square$

Lemma 2.4.1. *There exists a constant* $\widetilde{\mathcal{E}} > 0$ *such that as* $\tau \to \infty$*, we have*

$$\widetilde{\mathcal{E}}(\tau) \to \widetilde{\mathcal{E}}.$$

This is a direct consequence of the monotonicity and the upper bound of the normalized entropy.

Lemma 2.4.2. *There exists a sequence of times,* $\{\tau_n\} \nearrow \infty$*, such that* $\lim_{n\to\infty} \widetilde{\mathcal{E}}_\tau(\tau_n) = 0$*. In other words,*

$$\liminf_{\tau\to\infty} \widetilde{\mathcal{E}}_\tau = 0.$$

Proof. By Corollary 2.4.1, $\widetilde{\mathcal{E}}_\tau(\tau) \geq 0$. Therefore, $\liminf_{\tau\to\infty} \widetilde{\mathcal{E}}_\tau \geq 0$. Suppose that $\liminf_{\tau\to\infty} \widetilde{\mathcal{E}}_\tau > 0$. So, there exists an $\epsilon > 0$, such that $\liminf_{\tau\to\infty} \widetilde{\mathcal{E}}_\tau(\tau) \geq \epsilon$. Thus, there is $\tau_0 \in (0, \infty)$ such that for every $\tau \geq \tau_0$

$$\widetilde{\mathcal{E}}_\tau(\tau) \geq \epsilon,$$

By taking integral, from τ_0 to τ, $\tau_0 \leq \tau$, from both sides of this equation, we get

$$\frac{1}{2}(\widetilde{\mathcal{E}}(\tau) - \widetilde{\mathcal{E}}(\tau_0)) \geq \epsilon(\tau - \tau_0).$$

This is a contradiction because the left hand side is bounded from above, due to the boundedness of the normalized entropy, Proposition 2.4.2, while the right hand side goes to ∞ as τ tends to ∞. $\square$

Lemma 2.4.3. *For any solution to the flow, the gradient of the normalized curvature function is uniformly bounded from above and below.*

Proof. Let $\Phi = (\gamma\widetilde{\kappa}^\alpha)^2 + (\gamma\widetilde{\kappa}^\alpha)_\theta^2$, and let $\tau \in [0, \infty)$ be arbitrary. The idea of the proof is inspired by Gage-Hamilton [20]. In order to show that the gradient of the normalized curvature is bounded from above and below, it is enough to show that Φ is bounded from above. Let $(\theta_0, \tau_0) \in [0, 2\pi] \times [0, \tau]$ be such that $\Phi(\theta_0, \tau_0) = \sup_{[0,2\pi]\times[0,\tau]} \Phi(\theta, \tau)$. Without loss of generality suppose that $\tau_0 > 0$. We claim that $(\gamma\widetilde{\kappa}^\alpha)_\theta(\theta_0, \tau_0) = 0$. To see this, suppose on the contrary that $(\gamma\widetilde{\kappa}^\alpha)_\theta(\theta_0, \tau_0) \neq 0$. At (θ_0, τ_0), we have

$$\begin{aligned}\Phi_\theta &= 2(\gamma\widetilde{\kappa}^\alpha)(\gamma\widetilde{\kappa}^\alpha)_\theta + 2(\gamma\widetilde{\kappa}^\alpha)_\theta(\gamma\widetilde{\kappa}^\alpha)_{\theta\theta} \\ &= 2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\gamma\widetilde{\kappa}^\alpha) + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}] = 0.\end{aligned}$$

Since $(\gamma\widetilde{\kappa}^\alpha)_\theta(\theta_0, \tau_0) \neq 0$, at (θ_0, τ_0) we have

$$(\gamma\widetilde{\kappa}^\alpha) + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta} = 0. \tag{2.15}$$

Using equation (2.15), at (θ_0, τ_0), we get

$$\begin{aligned}\Phi_{\theta\theta} &= 2(\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}[(\gamma\widetilde{\kappa}^\alpha) + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}] + 2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}] \\ &= 2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}] \leq 0,\end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
0 \le \Phi_\tau &= 2(\gamma\widetilde{\kappa}^\alpha)(\gamma\widetilde{\kappa}^\alpha)_\tau + 2(\gamma\widetilde{\kappa}^\alpha)_\theta(\gamma\widetilde{\kappa}^\alpha)_{\theta\tau} \\
&= 2(\gamma\widetilde{\kappa}^\alpha)(\alpha\gamma\widetilde{\kappa}^{\alpha-1}\widetilde{\kappa}_\tau) + 2(\gamma\widetilde{\kappa}^\alpha)_\theta(\alpha\gamma\widetilde{\kappa}^{\alpha-1}\widetilde{\kappa}_\tau)_\theta \\
&= 2(\gamma\widetilde{\kappa}^\alpha)(\alpha\gamma\widetilde{\kappa}^{\alpha-1}[\widetilde{\kappa}^2\frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} - \widetilde{\kappa}])+ \\
&2(\gamma\widetilde{\kappa}^\alpha)_\theta(\alpha\gamma\widetilde{\kappa}^{\alpha-1}[\widetilde{\kappa}^2\frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} - \widetilde{\kappa}])_\theta = \\
&- 2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 + 2(\gamma\widetilde{\kappa}^\alpha)_\theta(\alpha\gamma\widetilde{\kappa}^{\alpha+1}\frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} - \alpha\gamma\widetilde{\kappa}^\alpha)_\theta = -2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 + \\
&2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\alpha\gamma\widetilde{\kappa}^{\alpha+1})_\theta\frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} + (\alpha\gamma\widetilde{\kappa}^{\alpha+1})\frac{(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} - \alpha(\gamma\widetilde{\kappa}^\alpha)_\theta] \\
&= -2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 + 2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\alpha\gamma\widetilde{\kappa}^{\alpha+1})\frac{(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}} - \alpha(\gamma\widetilde{\kappa}^\alpha)_\theta] \\
&= -2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 - 2\alpha(\gamma\widetilde{\kappa}^\alpha)_\theta^2 + (\alpha\gamma\widetilde{\kappa}^{\alpha+1})[2\alpha(\gamma\widetilde{\kappa}^\alpha)_\theta\frac{(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}}].
\end{aligned}
$$

At (θ_0, τ_0), we have that

$$-2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 - 2\alpha(\gamma\widetilde{\kappa}^\alpha)_\theta^2 < 0,$$

and by equation (2.16),

$$2(\gamma\widetilde{\kappa}^\alpha)_\theta[(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}] \le 0,$$

therefore

$$0 \le \Phi_\tau = -2\alpha(\gamma\widetilde{\kappa}^\alpha)^2 - 2\alpha(\gamma\widetilde{\kappa}^\alpha)_\theta^2 + (\alpha\gamma\widetilde{\kappa}^{\alpha+1})[2\alpha(\gamma\widetilde{\kappa}^\alpha)_\theta\frac{(\gamma\widetilde{\kappa}^\alpha)_\theta + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta\theta}}{\frac{1}{2}\int\gamma\widetilde{\kappa}^{\alpha-1}}] < 0.$$

This is a contradiction. So, $(\gamma\widetilde{\kappa}^\alpha)_\theta(\theta_0, \tau_0) = 0$. Consequently,

$$\sup_{[0,2\pi]\times[0,\tau]}\Phi(\theta,\tau) \le \max\{\sup_{[0,2\pi]}\Phi(\theta,0), \sup_{[0,2\pi]\times[0,\tau]}(\gamma\widetilde{\kappa}^\alpha)^2\}.$$

The normalized curvature function is uniformly bounded from above and below on $[0, 2\pi] \times [0, \infty)$, therefore Φ is uniformly bounded from above, and in turn, $\widetilde{\kappa}_\theta$ is uniformly bounded from above and below.

□

Now, we are ready to give a proof of Theorems 2.1.1 and 2.1.2.

Proof. Let $\alpha = \frac{1}{1-p}$. By Lemma 2.4.2, there exists a sequence of time $\{\tau_n\} \nearrow \infty$ such that

$$\lim_{n\to\infty} \widetilde{\mathcal{E}}_{\tau_n} = 0.$$

We infer from

$$\begin{aligned}
\widetilde{\mathcal{E}}_\tau &= (\alpha - 1) \int_0^{2\pi} \gamma \widetilde{\kappa}^{\alpha-2} \widetilde{\kappa}_\tau d\theta \\
&= (\alpha - 1) \int_0^{2\pi} \gamma \widetilde{\kappa}^{\alpha-2} [\widetilde{\kappa}^2 \frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int \gamma\widetilde{\kappa}^{\alpha-1}} - \widetilde{\kappa}] d\theta \\
&= (\alpha - 1) \int_0^{2\pi} \left[\gamma\widetilde{\kappa}^\alpha \frac{\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}}{\frac{1}{2}\int \gamma\widetilde{\kappa}^{\alpha-1}} - \gamma\widetilde{\kappa}^{\alpha-1}\right] d\theta \\
&= 2(\alpha - 1) \frac{1}{\widetilde{\mathcal{E}}} \int_0^{2\pi} [\gamma\widetilde{\kappa}^\alpha(\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}) - \gamma\widetilde{\kappa}^{\alpha-1} \frac{1}{2} \int_0^{2\pi} \gamma\widetilde{\kappa}^{\alpha-1}] d\theta \\
&= 2(\alpha - 1) \frac{1}{\widetilde{\mathcal{E}}} \left[\int_0^{2\pi} (\gamma\widetilde{\kappa}^\alpha(\gamma\widetilde{\kappa}^\alpha + (\gamma\widetilde{\kappa}^\alpha)_{\theta\theta}) d\theta - \frac{1}{2} (\int_0^{2\pi} \gamma\widetilde{\kappa}^{\alpha-1} d\theta)^2 \right] \geq 0
\end{aligned}$$

that

$$\lim_{n\to\infty} \left(\int_0^{2\pi} \gamma(\theta) \widetilde{\kappa}^\alpha(\theta, \tau_n) s[\widetilde{h}(\theta, \tau_n)] d\theta \right)^2 \tag{2.17}$$

$$= \lim_{n\to\infty} \int_0^{2\pi} \widetilde{h}(\theta, \tau_n) s[\widetilde{h}(\theta, \tau_n)] d\theta \int_0^{2\pi} \gamma(\theta) \widetilde{\kappa}^\alpha(\theta, \tau_n) s[\gamma(\theta) \widetilde{\kappa}^\alpha(\theta, \tau_n)] d\theta.$$

Since $\widetilde{h}(\theta, \tau_n)$, $\widetilde{\kappa}(\theta, \tau_n)$, $\widetilde{h}_\theta(\theta, \tau_n)$, and $\widetilde{\kappa}_\theta(\theta, \tau_n)$ are uniformly bounded, we infer from $\frac{1}{\widetilde{\kappa}} = \widetilde{h} + \widetilde{h}_{\theta\theta}$ that $\widetilde{h}_{\theta\theta}(\theta, \tau_n)$ and $\widetilde{h}_{\theta\theta\theta}(\theta, \tau_n)$ are uniformly bounded. Let $\theta_1 \in [0, 2\pi]$ and $\theta_2 \in [0, 2\pi]$ be arbitrary. For every $n \in \mathbb{N}$, there exist $\theta_0^n \in [0, 2\pi]$ such that

$$|\widetilde{h}_{\theta\theta}(\theta_1, \tau_n) - \widetilde{h}_{\theta\theta}(\theta_2, \tau_n)| = |\widetilde{h}_{\theta\theta\theta}(\theta_0^n, \tau_n)||\theta_2 - \theta_1|.$$

Let $M > 0$ be such that $|\widetilde{h}_{\theta\theta\theta}(\theta, \tau_n)| \leq M$ for every $n \in \mathbb{N}$ and every $\theta \in [0, 2\pi]$. Then

$$|\widetilde{h}_{\theta\theta}(\theta_1, \tau_n) - \widetilde{h}_{\theta\theta}(\theta_2, \tau_n)| \leq M|\theta_2 - \theta_1| \qquad n \in \mathbb{N}.$$

So, the sequence $\{\widetilde{h}_{\theta\theta}(\theta, \tau_n)\}$ is uniformly bounded and equicontinous. So, it possesses a uniformaly convergent subsequence, say $\{\widetilde{h}_{\theta\theta}(\theta, \tau_{n_j})\}$. The same way, we can show $\{\widetilde{h}_\theta(\theta, \tau_{n_j})\}$ has a uniformly convergent subsequence, say $\{\widetilde{h}_\theta(\theta, \tau_{n_{j_k}})\}$. Finally $\{\widetilde{h}(\theta, \tau_{n_{j_k}})\}$ possesses a uniformly convergent subsequences, say $\{\widetilde{h}(\theta, \tau_{n_{j_{k_l}}})\}$. So, there exist a subsequence of $\{\tau_n\}$, for simplicity we denote it by $\{\tau_n\} \nearrow \infty$ again, such that $\{\widetilde{h}_{\theta\theta}(\theta, \tau_n)\}$, $\{\widetilde{h}_\theta(\theta, \tau_n)\}$ and $\{\widetilde{h}(\theta, \tau_n)\}$ are uniformly convergent. Suppose that

$$\widetilde{h}(\theta, \tau_n) \to \widetilde{h}(\theta),$$

then

$$\widetilde{h}_{\theta\theta}(\theta, \tau_n) \to \widetilde{h}_{\theta\theta}(\theta),$$

and therefore

$$\widetilde{\kappa}(\theta, \tau_n) \to \widetilde{\kappa}(\theta) \quad \text{uniformly},$$

where $\widetilde{\kappa} = \frac{1}{\widetilde{h}_{\theta\theta} + \widetilde{h}} > 0$. Therefore $\widetilde{h}$ is the support function of a strictly convex body, say $\widetilde{K}$. Uniform convergence of the support functions and curvature functions, along with the fact that

$$\lim_{n\longrightarrow\infty}\left(\int_0^{2\pi}\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta,\tau_n)s[\widetilde{h}(\theta,\tau_n)]d\theta\right)^2=$$

$$\lim_{n\longrightarrow\infty}\int_0^{2\pi}\widetilde{h}(\theta,\tau_n)s[\widetilde{h}(\theta,\tau_n)]d\theta\int_0^{2\pi}\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta,\tau_n)s[\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta,\tau_n)]d\theta,$$

implies

$$\left(\int_0^{2\pi}\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta)s[\widetilde{h}(\theta)]d\theta\right)^2=\int_0^{2\pi}\widetilde{h}(\theta)s[\widetilde{h}(\theta)]d\theta\int_0^{2\pi}\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta)s[\gamma(\theta)\widetilde{\kappa}^{\alpha}(\theta)]d\theta.$$

So, for the strictly convex body $\widetilde{K}$, the Minkowski type inequality becomes equality. Therefore, there exists a $\lambda>0$ such that $\gamma\widetilde{\kappa}^{\alpha}=\lambda\widetilde{h}$. Rearranging the terms, we get

$$\psi=\sqrt[\alpha]{\gamma}=\frac{(\lambda\widetilde{h})^{\frac{1}{\alpha}}}{\widetilde{\kappa}}.$$

If we rescale the limit body, $\widetilde{K}$, by λ, then

$$\psi=\frac{\widetilde{h}^{\frac{1}{\alpha}}}{\widetilde{\kappa}}=\widetilde{h}^{\frac{1}{\alpha}}(\widetilde{h}_{\theta\theta}+\widetilde{h})=\widetilde{h}^{1-\frac{\alpha-1}{\alpha}}(\widetilde{h}_{\theta\theta}+\widetilde{h})=\widetilde{h}^{1-p}(\widetilde{h}_{\theta\theta}+\widetilde{h}).$$

Therefore, we get a non-degenerate convex body satisfying

$$\psi=\widetilde{h}^{1-p}(\widetilde{h}''+\widetilde{h}).$$

This concludes the proof. □

Chapter 3

Contraction of strictly convex hypersurfaces of revolution in $\mathbb{R}^{n+1}$, $n \geq 2$

3.1 Introduction

The first study of the "curve shortening" flow dates back to Mullins [38]. Within decades different generalizations of the curve shortening in the plane were studied by many authors, most notably Gage and Hamilton [20], [19], [21], Andrews [4], [6], [7], as well as many others [10], [19], [21], [45], [48], [53]. We refer the reader to [12] for a more comprehensive account and list of references. The mean curvature flow is a well-known generalization of the curve shortening flow to higher dimensions. Huisken showed that the mean curvature flow converges to a round point in finite time [24]. The Gaussian curvature flow, another generalization of the curve shortening flow to higher dimensions, described by the following partial differential equation was considered by Tso [55]

$$\frac{\partial X(x,t)}{\partial t} = -K(x,t)\nu(x,t), \tag{3.1}$$

where X is an embedding of a smooth, strictly convex hypersurface in $\mathbb{R}^n$ and K is its Gaussian curvature. Starting from a convex hypersurface, Tso [55] showed that Gaussian curvature flow converges to a point in finite time, and rescaled solutions converge to a convex hypersurface. The following generalization of the Gaussian curvature flow was first studied by Chow, [13],

$$\frac{\partial X(x,t)}{\partial t} = -K^{\beta}(x,t)\nu(x,t), \tag{3.2}$$

where K is the Gaussian curvature, and $\beta > 0$. Chow showed when $\beta = \frac{1}{n}$ rescaled solutions, as in the case of the mean curvature flow, converge to a sphere. When $\beta \neq \frac{1}{n}$, there is extensive literature studying the asymptotic behaviour of the flow (see [9] and references therein).

In this chapter, after undertaking an exhaustive examination of a variant of the flow occurring within the three-dimensional space $\mathbb{R}^3$, we generalize the obtained results to include the higher-dimensional settings. Consider in $\mathbb{R}^3$ the following generalization of the flow studied by Chow, equation (3.2):

$$\frac{\partial X(x,t)}{\partial t} = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\alpha_2}(x,t)\cdots\kappa_n^{\alpha_n}(x,t)\nu(x,t) \tag{3.3}$$

where $\kappa_1(x,t) \leq \kappa_2(x,t) \leq \ldots \leq \kappa_n(x,t)$ are the principal curvatures and $\nu(x,t)$ is the outer normal to $K(t)$ at $X(x,t)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers. Contrary to the speed in the flow considered by Chow, Gaussian curvature, the speed in the evolution equation (3.3) is non-homogeneous and non-symmetric. Therefore, equation (3.3) can be interpreted as a more challenging generalization of the Gaussian flow. In the paper by Li and Lv [31] contraction of convex hypersurfaces by non-homogeneous speeds was studied for the first time. In their paper, the speed functions, in addition to other properties, are essentially required to be symmetric and homogeneous of degree one. Among other things, it was shown that such a flow evolves convex hypersurfaces to a point in finite time, and in fact, in hyperbolic space roundness was proved for sufficiently pinched initial hypersurfaces by high powers of the speed. McCoy extended these results to hypersurfaces in Euclidean space. He showed that, with sufficient initial curvature pinching, the flow converges in finite time to points that are asymptotically spherical [36]. For a review of the studied non-homogeneous flows in Euclidean space see [35] and the references therein. In all these works, in addition to other properties, the speed function is required to be symmetric and homogeneous of degree 1, while in this paper the speed, $f(\kappa_1, \ldots, \kappa_n) = \kappa_1^{\alpha_1} \ldots \kappa_n^{\alpha_n}$, is neither symmetric nor necessarily homogeneous of degree 1.

We restrict our attention to the case in which the initial surface, ∂K_0, viewed as the boundary of a smooth convex body K_0, is an axially symmetric surface smoothly embedded in $\mathbb{R}^3$. So, there exists a function $u_0 : [0, 1] \to \mathbb{R}$, meeting the x-axis orthogonally, strictly positive and smooth on $(0, 1)$, with $u(0, 0) = u(1, 0) = 0$, such that $X_0 : (0, 1) \times \mathbb{S}^1 \to \mathbb{R}^3$, defined by $X_0(x, v) = (x, u_0(x)v)$, parametrize ∂K_0. We show that there exist strictly positive, smooth functions $u(\cdot, t) : (a_t, b_t) \to \mathbb{R}^{\geq 0}$, meeting the x-axis orthogonally, with $u(a_t, t) = u(b_t, t) = 0$ satisfying the following equation, the scalar form of the equation (5.1):

$$\frac{\partial u(x,t)}{\partial t} = -\frac{(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\frac{\alpha_1+3\alpha_2-1}{2}}} \quad \text{on } (a_t, b_t) \times [0, \omega). \tag{3.4}$$

We show that, in finite time, solutions degenerate into a point, time at which hypersurfaces develop a singularity, and, under suitable initial conditions, the blow-up of the solutions converge to a convex hypersurface. More precisely, suppose that the profile curve of the initial surface is even, that is, $u_0(x+p) = u_0(p-x)$ for every $x \in (c, b_0)$, where p is the midpoint of the interval (a_0, b_0). Then, we have the following result:

Theorem 3.1.1. *Suppose that a smooth strictly convex embedded closed surface $\partial K_0 \subset \mathbb{R}^3$ is axially symmetric with an even profile curve. Then, solutions of the equation (5.1) exist on a maximal time interval $[0, \omega)$, and they shrink to a point as $t \to \omega$. Furthermore, if solutions are rescaled to have fixed axial length and enclose domains of constant volume, they will converge sequentially in the Hausdorff metric to the boundary of a convex body.*

3.2 Preliminaries

Consider a smooth, strictly convex, axially symmetric surface $\partial K_0 \subset \mathbb{R}^3$, boundary of a convex body K_0, generated by revolving a strictly positive, smooth, concave function $u : [0, a] \longrightarrow \mathbb{R}$ with $u(a) = u(0) = 0$ about x-axis. Consider the parametrization $X : (0, a) \times [0, 2\pi] \longrightarrow \mathbb{R}^3$ defined by

$$X(x, \theta) = \big(x, u(x) \cos\theta, u(x) \sin\theta\big). \tag{3.5}$$

Then, for every $x \in (0, a)$ at $\big(x, u(x)\big) \in \partial K$, the principal curvatures are equal to

$$\kappa_1(x) = \kappa_{rad}(x) = \frac{1}{u(x)\sqrt{1 + u_x^2(x)}}, \quad \kappa_2(x) = \kappa_{axi}(x) = -\frac{u_{xx}(x)}{(1 + u_x^2(x))^{\frac{3}{2}}}. \tag{3.6}$$

We note that since K is smooth, $\kappa_{\min}, \kappa_{\max}$ are continuous and poles are umbilical points, then at poles we still have κ_{rad} and κ_{axi} defined by

$$\kappa_1 = \kappa_{rad}(p) = \lim_{x \to p} \kappa_{rad}(x), \quad \text{and} \quad \kappa_2 = \kappa_{axi}(p) = \lim_{x \to p} \kappa_{axi}(x), \tag{3.7}$$

for $p \in \{0, a\}$. In this paper, for $\alpha_1, \alpha_2 > 0$ fixed, we consider the following flow

$$\frac{\partial X}{\partial t}(p,t) = -\kappa_1^{\alpha_1}(p,t)\kappa_2^{\alpha_2}(p,t)\nu(p,t), \qquad p \in \partial K. \tag{3.8}$$

We infer from equation (3.5) that for every $x \in (0, a)$,

$$\frac{\partial X}{\partial t}(x,\theta) = (0, u_t \cos\theta, u_t \sin\theta).$$

If follows from

$$\nu(x,\theta,t) = \frac{(-u_x, \cos\theta, \sin\theta)}{\sqrt{1+u_x^2}}$$

that

$$\begin{aligned} -\kappa_{rad}^{\alpha_1}(x,t)\kappa_{axi}^{\alpha_2}(x,t) &= \Big\langle \frac{\partial X}{\partial t}(x,\theta), \nu(x,\theta,t)\Big\rangle \\ &= \Big\langle (0, u_t\cos\theta, u_t\sin\theta), \frac{(-u_x, \cos\theta, \sin\theta)}{\sqrt{1+u_x^2}}\Big\rangle = \frac{u_t}{\sqrt{1+u_x^2}}. \end{aligned} \tag{3.9}$$

Thus, the scalar evolution equation for the graph function u is

$$\begin{aligned} u_t &= -\sqrt{1+u_x^2}\,\kappa_{rad}^{\alpha_1}(p,t)\kappa_{axi}^{\alpha_2}(x,t) \\ &= -\frac{(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\frac{3\alpha_2+\alpha_1-1}{2}}}. \end{aligned} \tag{3.10}$$

Since the behaviour of the flow around the poles is governed by similar equations, it is sufficient to study the behaviour of the flow around one of the poles. More precisely, let $u_{\max} = u(x_0)$, and define $w : \big(-u(x_0), u(x_0)\big) \to [0, x_0)$ as follows:

$$y \mapsto \begin{cases} u^{-1}(y) & 0 \le y \le u_{\max} \\ u^{-1}(-y) & -u_{\max} \le y < 0. \end{cases}$$

About $0 = (0,0,0) \in \partial K$, consider the parametrization $\bar{X} : B(0, u_{\max}) \longrightarrow \mathbb{R}^3$ defined by

$$(y,z) \mapsto \big(v(y,z), y, z\big) \tag{3.11}$$

where $v(y,z) = w(\sqrt{y^2+z^2}) = (u^{-1})(\sqrt{y^2+z^2})$. It follows from

$$g^{ij} = \delta^{ij} - \frac{v_i v_j}{1+|\nabla v|^2}, \quad \text{and} \quad h_{ij} = \frac{v_{ij}^2}{\sqrt{1+|\nabla v|^2}} \tag{3.12}$$

that

$$(g^{ij}) = \begin{bmatrix} \frac{1+v_z^2}{1+v_y^2+v_z^2} & -\frac{v_y v_z}{1+v_y^2+v_z^2} \\ -\frac{v_y v_z}{1+v_y^2+v_z^2} & \frac{1+v_y^2}{1+v_y^2+v_z^2} \end{bmatrix}, \quad \text{and} \quad (h_{ij}) = \begin{bmatrix} \frac{v_{yy}}{\sqrt{1+v_y^2+v_z^2}} & \frac{v_{yz}}{\sqrt{1+v_y^2+v_z^2}} \\ \frac{v_{yz}}{\sqrt{1+v_y^2+v_z^2}} & \frac{v_{zz}}{\sqrt{1+v_y^2+v_z^2}} \end{bmatrix}. \tag{3.13}$$

When $z = 0$, we have that

$$(g^{ij}) = \begin{bmatrix} \frac{1}{1+v_y^2} & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad (h_{ij}) = \begin{bmatrix} \frac{v_{yy}}{\sqrt{1+v_y^2}} & 0 \\ 0 & \frac{v_{zz}}{\sqrt{1+v_y^2}} \end{bmatrix}. \tag{3.14}$$

As a result, along the curve

$$\gamma = \Big\{ \big(y, v(y,0)\big) : y \in \big(-u_{\max}, u_{\max}\big) \Big\}, \tag{3.15}$$

the principal curvatures are

$$\kappa_{rad}(y,0) = \frac{v_{zz}}{(1+v_y^2)^{\frac{1}{2}}}, \quad \text{and} \quad \kappa_{axi}(y,0) = \frac{v_{yy}}{(1+v_y^2)^{\frac{3}{2}}}, \tag{3.16}$$

and the normal is

$$\nu(y,0) = \frac{(v_y, 0, -1)}{\sqrt{v_y^2+1}}.$$

By direct calculations, we get

$$(0, 0, v_t(y, t)) = \frac{\partial \gamma}{\partial t}(y, 0, t) = -\kappa_{rad}^{\alpha_1}(y, 0, t)\kappa_{axi}^{\alpha_2}(y, 0, t)\nu(y, 0, t).$$

Therefore,

$$\begin{aligned} &-\kappa_{rad}^{\alpha_1}(y, t)\kappa_{axi}^{\alpha_2}(y, t) = \Big\langle \frac{\partial \gamma}{\partial t}(y, 0, t), \nu(y, \theta, t) \Big\rangle \\ &= \Big\langle (0, 0, v_t(y, t)), \frac{(v_y, 0, -1)}{\sqrt{v_y^2 + 1}} \Big\rangle = \frac{-v_t}{\sqrt{1 + v_y^2}}. \end{aligned} \tag{3.17}$$

So, the scalar evolution equation of the graph function v is

$$v_t = \sqrt{1 + v_y^2}\,\kappa_{rad}^{\alpha_1}(y, 0, t)\kappa_{axi}^{\alpha_2}(y, 0, t) = \frac{1}{(1 + v_y^2)^{\frac{\alpha_1 + 3\alpha_2 - 1}{2}}} v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}. \tag{3.18}$$

We say smooth solutions for the equation (3.8) exists if there exist smooth, strictly positive, concave functions $u(\cdot, t) : (a_t, b_t) \longrightarrow \mathbb{R}$ with $u(a_t, t) = u(b_t, t) = 0$ satisfying equation (3.10) such that for every t the curve $\gamma(t)$, defined by equation (3.49), is smooth.

3.3 The free boundary problem

If we consider the inside and the outside of the closed surface, we can regard the surface as the interface separating two phases that are transforming, and the deformation of a surface according to equation (3.8) can be viewed as a free boundary problem. Solutions of equation (3.4) correspond to strictly convex hypersurfaces, the free boundaries, moving according to the equation (3.8). In this section, we show that solutions of equation (3.4) exist, their convexity is preserved, and after a finite time, the surfaces of revolution described by the solutions converge to a point. We show that curvatures develop a singularity when the volume enclosed by the solution becomes zero. Starting from a strictly convex, smooth surface of revolution the solutions always exist. We have

Lemma 3.3.1. *Suppose that the initial conditions are the same as in Theorem 3.1.1. Then, there exists* $T > 0$ *such that smooth solutions to the flow exist on* $[0, T)$.

Proof. We consider a linearization of the equation (3.10). Suppose that $\epsilon > 0$ is a small real number and $\tilde{u} = u + \epsilon\phi$ is a solution of the equation (3.10) where $\phi : [0, a] \times [0, \omega) \longrightarrow \mathbb{R}$ is a smooth function. Since

$$\frac{\partial \tilde{u}}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}}\phi_{xx} + \frac{2\beta u_x}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}\phi_x + \frac{\alpha_1}{u(1+u_x^2)^{\beta}}\phi$$

we infer from $\tilde{u}_t = u_t + \epsilon\phi_t$ that

$$\phi_t = \frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}}\phi_{xx} + \frac{2\beta u_x}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}\phi_x + \frac{\alpha_1}{u(1+u_x^2)^{\beta}}\phi.$$

Since $\frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}} > 0$ at time $t = 0$ the equation is parabolic and it follows from classical theory of parabolic equations that solutions of equation (3.10) exist. Now, we consider the evolution equation of $v_0 : \big(-\max u(0), \max u(0)\big) \to \mathbb{R}$:

$$v_t = \frac{1}{(1+v_y^2)^{\frac{\alpha_1+3\alpha_2-1}{2}}} v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}. \tag{3.19}$$

Suppose that $\epsilon > 0$ is a small real number and $\tilde{v} = v + \epsilon\phi$ is a solution of the equation (3.19) where ϕ is a smooth function. Since

$$\frac{\partial \tilde{v}_t}{\partial \epsilon} = -\Big(\frac{3\alpha_2+\alpha_1-1}{2}\Big)\frac{4(v_y+\epsilon\phi_y)}{(1+2(v_y+\epsilon\phi_y)^2)^{\frac{3\alpha_2+\alpha_1+1}{2}}}(v_{yy}+\epsilon\phi_{yy})^{\alpha_2}(y,t)v_{zz}^{\alpha_1}(y,t)\phi_y$$
$$+ \alpha_2\frac{1}{(1+2(v_y+\epsilon\phi_y)^2)^{\frac{3\alpha_2+\alpha_1-1}{2}}}(v_{yy}+\epsilon\phi_{yy})^{\alpha_2-1}(y,t)v_{zz}^{\alpha_1}(y,t)\phi_{yy},$$

it follows from $\tilde{v}_t = v_t + \epsilon\phi_t$ that

$$\phi_t = \frac{\alpha_2(v_{yy})^{\alpha_2-1}v_{zz}^{\alpha_1}}{(1+2(v_y)^2)^{\frac{3\alpha_2+\alpha_1-1}{2}}}\phi_{yy} - \Big(\frac{3\alpha_2+\alpha_1-1}{2}\Big)\frac{4v_y(v_{yy})^{\alpha_2}v_{zz}^{\alpha_1}}{(1+2(v_y)^2)^{\frac{3\alpha_2+\alpha_1+1}{2}}}\phi_y. \tag{3.20}$$

Since equation (3.20) is parabolic, smooth solutions to the equation (3.19) exist for some time interval. Since the solution of equation (3.20) satisfies equation (3.10) on its domain, we infer from the uniqueness of solutions that the solutions of equation (3.10) and equation (3.19) are the same on their common domain. We infer that smooth solutions of equation (3.8) exist for some time interval. □

Lemma 3.3.2. *Under the initial conditions specified in the Theorem 3.1.1, the flow preserves the convexity of the evolving surfaces as long as the solutions exist.*

Proof. We set $H = u_t$ and differentiate equation (3.10) with respect to time to get the evolution equation of the speed:

$$\begin{aligned} H_t = &\frac{\alpha_2(-u_{xx})^{\alpha_2-1}u^{\alpha_1}(1+u_x^2)^{\beta}}{u^{2\alpha_1}(1+u_x^2)^{2\beta}}H_{xx} + \frac{2\beta u_x u^{\alpha_1}(-u_{xx})^{\alpha_2}(1+u_x^2)^{\beta-1}}{u^{2\alpha_1}(1+u_x^2)^{2\beta}}H_x \\ &+ \frac{\alpha_1 u^{\alpha_1-1}(-u_{xx})^{\alpha_2}(1+u_x^2)^{\beta}}{u^{2\alpha_1}(1+u_x^2)^{2\beta}}H. \end{aligned} \tag{3.21}$$

We choose $r > 0$ to be big enough such that

$$\kappa_1^{\alpha_1}\kappa_2^{\alpha_2}(x,0) > \Big(\frac{1}{r}\Big)^{\alpha_1+\alpha_2}. \tag{3.22}$$

Suppose that the maximum of $u(\cdot,0)$ happens at q and consider the sphere generated by revolving $(x-q)^2 + y^2 = r^2$ around x-axis. By direct calculation, we get

$$y_x = \frac{-(x-q)}{\sqrt{r^2-(x-q)^2}}.$$

Since maximum of u happens at q, $u_x(q) = 0$. Since u_x and y_x are continuous and $u_x(q) = y_x(q) = 0$ we infer from equation (3.22) that there exist $\epsilon > 0$ such that for every $x \in (q-\epsilon, q+\epsilon)$ we have

$$\sqrt{1+u_x^2}\kappa_1^{\alpha_1}\kappa_2^{\alpha_2} > \sqrt{1+y_x^2}\left(\frac{1}{r}\right)^{\alpha_1+\alpha_2}. \tag{3.23}$$

Now, let small $\delta > 0$ be arbitrary, and let

$$0 < e = \min_{x\in[q+\epsilon, a-\delta]} \sqrt{1+u_x^2}.$$

For every $x \in [q+\epsilon, a-\delta]$, since $x \le a-\delta$ we infer that

$$\frac{(a-\delta-q)^2}{r^2-(a-\delta-q)^2} \ge \frac{(x-q)^2}{r^2-(a-\delta-q)^2} \ge y_x^2 = \frac{(x-q)^2}{r^2-(x-q)^2}. \tag{3.24}$$

If we choose $r > 0$ to be big enough, then for every $x \in [q+\epsilon, a-\delta]$

$$u_x^2 \ge \frac{(a-\delta-q)^2}{r^2-(a-\delta-q)^2} \ge \frac{(x-q)^2}{r^2-(a-\delta-q)^2} \ge y_x^2 = \frac{(x-q)^2}{r^2-(x-q)^2}. \tag{3.25}$$

Since u is an *even* function, we infer from equations (3.22) and (3.25) that for every $\delta > 0$ small enough there exist $r > 0$ big enough such that for every $x \in [\delta, a-\delta]$ we have

$$-H(x,0) = \sqrt{1+u_x^2}\kappa_{rad}^{\alpha_1}\kappa_{axi}^{\alpha_2}(x,0) \ge -y_t(x,0) = \sqrt{1+y_x^2}\left(\frac{1}{r}\right)^{\alpha_2+\alpha_1}. \tag{3.26}$$

Let $u(\cdot,t) : (a_t, b_t) \longrightarrow \mathbb{R}$ and $y(\cdot,t) : (c_t, d_t) \longrightarrow \mathbb{R}$ be the solutions corresponding, respectively, to evolving surfaces and shrinking balls. Since supersolutions dominate subsolutions, the equation (3.26) will be preserved on $[0,t]$ for every $t \in [0,\omega)$. Therefore, for every $t \in [0,\omega)$, and every $x \in (a_t, b_t)$, we have

$$\kappa_{axi}(x,t), \kappa_{rad}(x,t) > 0.$$

We apply the same argument to the poles. Namely, consider the curve v_0. As we have seen the principal curvatures along $v(y,0) = v_0$ are

$$\kappa_1 = \kappa_{rad}(y,0) = \frac{v_{zz}}{(1+v_y^2)^{\frac{1}{2}}}, \quad \text{and} \quad \kappa_2 = \kappa_{axi}(y,0) = \frac{v_{yy}}{(1+v_y^2)^{\frac{3}{2}}}. \tag{3.27}$$

Since minimum of v_0 happens at 0, $v_y(0,0) = 0$. Let $r > 0$ be big enough that equation (3.22) holds, and consider the sphere generated by revolving $(x-c)^2 + y^2 = r^2$ around x-axis. We note that

$$x_y = \frac{-y}{\sqrt{r^2 - y^2}}.$$

Since v_y, x_y are continuous and $v_y(0,0) = x_y(0,0) = 0$ we infer, from equation (3.22), that there exist $\epsilon > 0$ such that for every $y \in (-\epsilon, \epsilon)$ we have

$$\sqrt{1+v_y^2}\,\kappa_1^{\alpha_1}\kappa_2^{\alpha_2} > \sqrt{1+x_y^2}\left(\frac{1}{r}\right)^{\alpha_1+\alpha_2}. \tag{3.28}$$

Let $t \in [0, \omega)$ be arbitrary, and let $\epsilon > 0$ be small enough. As we have seen, equation (3.18) gives the evolution equation of v_0. We find the evolution equation of the speed v_t. Consider $\eta : [-\epsilon, \epsilon] \times [0,t] \longrightarrow \mathbb{R}$ defined by

$$\eta := v_t = \frac{1}{(1+v_y^2)^{\frac{\alpha_1+3\alpha_2-1}{2}}} v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}. \tag{3.29}$$

We set $\frac{\alpha_1+3\alpha_2-1}{2} = \mu$, and differentiate η with respect to time to get the evolution equation for the speed of the evolving curves $v(y,t)$:

$$\begin{aligned}\eta_t(y,t) &= \left[\frac{-2\mu v_y v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}}{(1+v_y^2)^{\mu+1}}\right]\eta_y + \left[\frac{\alpha_1 v_{yy}^{\alpha_2} v_{zz}^{\alpha_1-1}}{(1+v_y^2)^{\mu}}\right]\eta_{zz} + \left[\frac{\alpha_2 v_{yy}^{\alpha_2-1} v_{zz}^{\alpha_1}}{(1+v_y^2)^{\mu}}\right]\eta_{yy} \\ &= \left[\frac{\alpha_2 v_{yy}^{\alpha_2-1} v_{zz}^{\alpha_1}}{(1+v_y^2)^{\mu}}\right]\eta_{yy} + \left[\frac{-2\mu v_y v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}}{(1+v_y^2)^{\mu+1}}\right]\eta_y + \frac{\alpha_1 \eta_{zz}}{v_{zz}}\eta.\end{aligned} \tag{3.30}$$

So, the evolution equation of η is parabolic with uniformly bounded continuous coefficients on $[-\epsilon, \epsilon] \times [0,t]$. Since supersolutions dominate subsolutions equation (3.28) holds on $[-\epsilon, \epsilon] \times [0, t]$. In turn, the principal curvatures remain positive. $\square$

3.4 Singularity

In this section, we first show that ω is the first time when the volume enclosed by $K(t)$ is zero. Then, we show that as t goes to ω, the solutions shrink to a single point. We will then infer from Blaschke's selection theorem that the solutions approach a certain shape as $t \to \omega$ which we will study under a constant volume normalization.

Proposition 3.4.1. *Suppose that the initial conditions are the same as in Theorem 3.1.1, and suppose that $[0, \omega)$ is the maximal time interval on which the solutions exist. Then*

$$V(t) \to 0 \text{ as } t \to \omega.$$

Proof. Suppose for contradiction that $V(\omega) \neq 0$. Therefore, there exist $[a, b]$ with $a < b$ such that $[a, b] \subset (a_t, b_t)$ for every $t \in [0, \omega)$. Suppose that $[a, b]$ is the longest interval with this property, and suppose $K(\omega)$ is the convex body with the profile curve $u(x, \omega) = \inf_{t \in [0, \omega)} u(x, t)$. Since $u(x, \omega)$ is concave, is differentiable almost everywhere and, from the convexity of the surface, is twice differentiable almost everywhere. Therefore, there exist $x_0 \in (a, b)$ such that

$$\kappa_{axi}(x_0, \omega) := \sup_{t \in [0, \omega)} \kappa_{axi}(x_0, t) = \sup_{t \in [0, \omega)} \frac{(-u_{xx})}{(1 + u_x^2)^{\frac{3}{2}}} < \infty. \tag{3.31}$$

Since $u(x_0, t)$ is bounded from below

$$\sup_{t \in [0, \omega)} \kappa_{rad}(x_0, t) = \frac{1}{u(1 + u_x^2)^{\frac{1}{2}}} < \frac{1}{u} \leq \frac{1}{\epsilon} < \infty, \tag{3.32}$$

for some $\epsilon > 0$. Since $u_x^2(x_0, \omega)$ exists, we infer from the last two inequalities that, for some $C > 0$,

$$\sup_{t\in[0,\omega)} -u_t(x_0,t) = \sup_{t\in[0,\omega)} \sqrt{1+u_x^2}\kappa_{axi}^{\alpha}(x_0,t)\kappa_{rad}^{\beta}(x_0,t) < C. \tag{3.33}$$

We distinguish two cases:

Case 1: Since $u(\cdot,\omega)$ is even, there exist two points, say x_0 and x_1, with the same speed, around the midpoint of (a,b) such that the equation (3.33) holds at these points. For every $t \in [0,\omega)$, consider $H : [x_0,x_1]\times[0,t] \to \mathbb{R}$ defined in previous lemma, $H = u_t$. Since the evolution equation of H, the equation (3.21), is parabolic with uniformly bounded coefficients, it follows from maximum principle that

$$\max_{[x_0,x_1]\times[0,t]} -u_t \leq \max\Big\{\max_{t\in[0,\omega)} -u_t(x_0,t), \max_{[x_0,x_1]} -u_t(x,0)\Big\} < \infty. \tag{3.34}$$

So, the speed $u_t : [x_0,x_1]\times[0,\omega) \longrightarrow \mathbb{R}$ is bounded on $[x_0,x_1]$. We claim that the principal curvatures remain bounded from above on $[x_0,x_1]$. To see this, we note that the lower bound on u implies an upper bound on the radial curvature. If the axial curvature becomes infinite at a certain point, the boundedness of the product of curvatures implies the radial curvature tends toward zero at that point. The only way for the radial curvature to approach zero is if the derivative at that point becomes infinite. However, this is impossible as $u(\cdot,\omega)$ is concave, and even if the derivative does not exist, it cannot be infinite.

Case 2: For every $t \in [0,\omega)$, consider $\eta : [-c,c]\times[0,t] \to \mathbb{R}$ defined by equation (3.29) where

$$0 < c = u(x_0,\omega).$$

Since for every $t \in [0,\omega)$, the evolution equation of η, the equation (3.30), is parabolic with uniformly bounded coefficients on $[-c,c]\times[0,t]$, we conclude that

$$\max_{[-c,c]\times[0,t]} \eta \le \Big\{ \max_{t\in[0,\omega)} \eta(c,t),\ \max_{y\in[-c,c]} \eta(y,0) \Big\} < \infty. \tag{3.35}$$

So, the speed $\eta : [-c, c] \times [0, \omega) \to R$ is bounded on $[-c, c]$. Consequently, the product of curvatures at $y = 0$, which is an umbilic point and corresponds to $x = a$, remains bounded.

Since the choice of x_0 and x_1 is flexible, we infer from cases 1 and 2 that the principal curvatures of $K(\omega)$ are bounded from above, and in turn, the flow can be continued beyond ω. This is impossible as $[0, \omega)$ is the maximal time interval on which solutions exist. So, as the final time is approached, the volume of the domain enclosed by the solutions tends to zero. □

Proposition 3.4.2. *The flow converges to a point.*

Proof. To see this, suppose that it is not true, so $u(x, t)$ will degenerate into a segment. Since $K(t)$ is symmetric with respect to the x-axis, this segment will either lie on the x-axis or be parallel to the y-axis. Consider the first case where the flow degenerates into a segment on the x-axis, say to $[a, b]$ with $a < b$. By direct calculation, we find the evolution equation of u_{xx}:

$$\begin{aligned} u_{xt} &= \frac{\alpha_2(-u_{xx})^{\alpha_2-1}u_{xxx}}{u^{\alpha_1}(1+u_x^2)^{\beta}} + \frac{\left[\alpha_1 u_x u^{\alpha_1-1}(1+u_x^2)^{\beta} + 2\beta u_x u_{xx} u^{\alpha_1}(1+u_x^2)^{\beta-1}\right](-u_{xx})^{\alpha_2}}{u^{2\alpha_1}(1+u_x^2)^{2\beta}}, \\ &= \frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}}u_{xxx} + \frac{2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}u_{xx} + \frac{\alpha_1(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_x \end{aligned} \tag{3.36}$$

and, in turn,

$$
\begin{aligned}
u_{xxt} = & \frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}}u_{xxxx} + \frac{2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}u_{xxx} + \frac{\alpha_1(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_{xx} \\
& - \frac{(\alpha_2-1)\alpha_2(-u_{xx})^{\alpha_2-2}}{u^{\alpha_1}(1+u_x^2)^{\beta}}u_{xxx}^2 - \frac{\alpha_1\alpha_2 u_x(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_{xxx} + \frac{2\alpha_2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}u_{xxx} \\
& - \frac{2\beta(-u_{xx})^{\alpha_2+1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_{xx} - \frac{2\beta\alpha_2 u_x u_{xxx}(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_{xx} + \frac{2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}u_{xx} \\
& + \frac{2\beta(\alpha_1+1)u_x^2(-u_{xx})^{\alpha_2}}{u^{\alpha_1+2}(1+u_x^2)^{\beta}}u_{xx} - \frac{4\beta^2 u_x^2(-u_{xx})^{\alpha_2+1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta+1}}u_{xx} + \frac{\alpha_1\alpha_2(-u_{xx})^{\alpha_2-1}u_{xxx}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}u_x \\
& - \frac{\alpha_1^2(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta+1}}u_x^2 + \frac{2\alpha_1(\beta+1)(-u_{xx})^{\alpha_2+1}}{u^{\alpha_1}(1+u_x^2)^{\beta+2}}u_x^2.
\end{aligned}
\tag{3.37}
$$

Moreover, let $v = u_{xx}$. Then

$$
\begin{aligned}
v_t = & \frac{\alpha_2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1}(1+u_x^2)^{\beta}}v_{xx} + \frac{2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}v_x + \frac{\alpha_1(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}}v \\
& + \Big[- \frac{(\alpha_2-1)\alpha_2(-u_{xx})^{\alpha_2-2}u_{xxx}}{u^{\alpha_1}(1+u_x^2)^{\beta}} - \frac{\alpha_1\alpha_2 u_x(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}} + \frac{2\alpha_2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+1}} \Big] v_x \\
& + \Big[- \frac{2\beta(-u_{xx})^{\alpha_2+1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}} - \frac{2\beta\alpha_2 u_x u_{xxx}(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}} + \frac{2\beta u_x(-u_{xx})^{\alpha_2}}{u^{\alpha_1+1}(1+u_x^2)^{\beta}} \Big] v \\
& + \frac{2\beta(\alpha_1+1)u_x^2(-u_{xx})^{\alpha_2}}{u^{\alpha_1+2}(1+u_x^2)^{\beta}}v - \frac{4\beta^2 u_x^2(-u_{xx})^{\alpha_2+1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta+1}}v + \frac{\alpha_1\alpha_2(-u_{xx})^{\alpha_2-1}u_x}{u^{\alpha_1}(1+u_x^2)^{\beta+1}}v_x \\
& + \frac{\alpha_1^2 u_x^2(-u_{xx})^{\alpha_2-1}}{u^{\alpha_1+1}(1+u_x^2)^{\beta+1}}v - \frac{2\alpha_1(\beta+1)u_x^2(-u_{xx})^{\alpha_2}}{u^{\alpha_1}(1+u_x^2)^{\beta+2}}v.
\end{aligned}
\tag{3.38}
$$

Since $u(x,t)$ degenerates into a segment, at every point of (a,b), say at $x_0 \in (a,b)$, u_{xx} tends to zero, and at endpoints u_{xx} tends to negative infinity. For every $t \in [0,\omega)$, consider $u_{xx} : \Omega_t := [a,b] \times [0,t] \to \mathbb{R}$, and let Γ_t be the parabolic boundary of Ω_t. We have

$$
u_{xx}(x_0,t) \le \max_{\Gamma_t} u_{xx}(x,t). \tag{3.39}
$$

As $t \to \omega$, the right-hand side of the equation (3.39) tends to either a negative number or negative infinity while the left-hand side tends to zero. So, this case cannot occur.

Consider the case where the flow degenerates into a segment parallel to the y-axis, say to $[-c, c] \times \{\bar{x}\}$. We define the function $\eta : \Sigma_t := [-c, c] \times [t_0, t] \longrightarrow \mathbb{R}$ by

$$\eta = v_t = \frac{1}{(1 + v_y^2)^{\frac{\alpha_1 + 3\alpha_2 - 1}{2}}} v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}. \tag{3.40}$$

The evolution equation of η, equation (3.30), is parabolic with uniformly bounded continuous coefficients on Σ_t:

$$\begin{aligned} \eta_t(y, t) &= \left[\frac{-2\mu v_y v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}}{(1 + v_y^2)^{\mu+1}}\right]\eta_y + \left[\frac{\alpha_1 v_{yy}^{\alpha_2} v_{zz}^{\alpha_1 - 1}}{(1 + v_y^2)^{\mu}}\right]\eta_{zz} + \left[\frac{\alpha_2 v_{yy}^{\alpha_2 - 1} v_{zz}^{\alpha_1}}{(1 + v_y^2)^{\mu}}\right]\eta_{yy} \\ &= \left[\frac{\alpha_2 v_{yy}^{\alpha_2 - 1} v_{zz}^{\alpha_1}}{(1 + v_y^2)^{\mu}}\right]\eta_{yy} + \left[\frac{-2\mu v_y v_{zz}^{\alpha_1} v_{yy}^{\alpha_2}}{(1 + v_y^2)^{\mu+1}}\right]\eta_y + \frac{\alpha_1 \eta_{zz}}{v_{zz}}\eta. \end{aligned}$$

At $y = 0$, solutions are umbilic, so both principal curvatures and, in turn, the speed tends to zero at this point. Since at the endpoints, the speed tends to infinity an argument similar to the one in the previous part shows this is impossible. Therefore, the solutions cannot degenerate into segments. Consequently, they shrink to a point as $t \to \omega$. □

3.5 Asymptotic behaviour of the flow

While it seems plausible to prove the previous results even when relaxing the condition that the initial profile curve is even, when studying the asymptotic behaviour of the flow, this condition plays a crucial role. We demonstrate that by properly rescaling the solutions, starting from an even initial data, rescaled solutions converge to a convex hypersurface.

Proposition 3.5.1. *Suppose the initial conditions are the same as specified in Theorem 3.1.1. Then, for every sequence of times $\{t_n\} \nearrow \omega$, the properly rescaled solutions possess a subsequence that converges to a convex shape.*

Proof. Since the profile curve of the initial surface is contracting with the same speed at points x equally distanced from the midpoint of (a_0, b_0), for every $t \in [0, \omega)$, the profile curve corresponding to the solution at time t is even. We note that for every $t \in [0, \omega)$, there exist $x_t \in (a_t, b_t)$ such that

$$V(t) = \pi \int_{a_t}^{b_t} u^2(x,t)dx = \pi(b_t - a_t)u^2(x_t, t).$$

Let p be the point on the x-axis to which solutions converge. Consider

$$I_t = (c_t, d_t) := \left(\frac{\pi(a_t - p)u^2(x_t,t)}{V(t)}, \frac{\pi(b_t - p)u^2(x_t,t)}{V(t)} \right) = \left(\frac{a_t - p}{b_t - a_t}, \frac{b_t - p}{b_t - a_t}\right),$$

and define $T : (a_t, b_t) \longrightarrow I_t$ by

$$x \mapsto \frac{\pi(x-p)u^2(x_t,t)}{V(t)} = \frac{x-p}{b_t - a_t}.$$

Since solutions are even, the speed at the endpoints are the same, but of the opposite signs, $0 \neq \frac{\partial a_t}{\partial t} = -\frac{\partial b_t}{\partial t}$. In turn,

$$\lim_{t \longrightarrow \omega} \frac{a_t - p}{b_t - a_t} = -\frac{1}{2}, \quad \text{and} \lim_{t \longrightarrow T} \frac{b_t - p}{b_t - a_t} = \frac{1}{2}. \tag{3.41}$$

As a result $I_t \longrightarrow (-\frac{1}{2}, \frac{1}{2})$ as $t \longrightarrow \omega$. Consider the rescaling $\widetilde{u} : I_t \longrightarrow \mathbb{R}^{\geq 0}$ defined by

$$y \mapsto \sqrt{\frac{\pi(b_t - a_t)}{V(t)}} u \left(\frac{yV(t)}{\pi u^2(x_t,t)} + p, t \right). \tag{3.42}$$

We note that rescaled solutions enclose a domain of constant volume. More precisely,

$$\widetilde{V}(t) = \pi \int_{c_t}^{d_t} \frac{\pi(b_t - a_t)}{V(t)} u^2\left(\frac{yV(t)}{\pi u^2(x_t,t)} + p, t\right)dy = \frac{\pi(b_t - a_t)}{V(t)} \pi \int_{a_t}^{b_t} u^2(x,t) \frac{\pi u^2(x_t,t)}{V(t)} dx$$

$$= \frac{\pi u^2(x_t,t)(b_t - a_t)}{V(t)}\pi = \pi.$$

If the rescaled solutions degenerate, then since $I_t \to (-\frac{1}{2}, \frac{1}{2})$, they will degenerate into the segment $(-\frac{1}{2}, \frac{1}{2})$. However, this contradicts the fact that the rescaled solutions enclose constant volume π. In addition, since $I_t \to (-\frac{1}{2}, \frac{1}{2})$ and volume is constant, the rescaled solutions are bounded from above. Therefore, rescaled solutions are included in a compact annulus, and it follows from Blaschke selection theorem that $\{\widetilde{K}(t_n)\}_n$ is subsequentially convergent to a convex body $\widetilde{K}$. □

3.6 Extension to $\mathbb{R}^{n+1}$

To delve into the analysis of flow in higher dimensions, we establish the appropriate framework. Let $\partial K_0 \subset \mathbb{R}^{n+1}$ denote a smooth, strictly convex, axially symmetric hypersurface that serves as the boundary of a convex body K_0. This surface is generated by rotating a strictly positive, smooth, and concave function $u : [0, a] \longrightarrow \mathbb{R}$, satisfying $u(a) = u(0) = 0$, around the x-axis. Consider the parametrization $X : (0, a) \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^{n+1}$ defined as follows:

$$X(x, \omega) = (x, u(x)\omega), \tag{3.43}$$

where ω belongs to the $(n-1)$-dimensional unit sphere, $\mathbb{S}^{n-1}$. Through explicit computations, the matrices representing the metric, second fundamental form, and Weingarten map of ∂K_0 can be derived as follows:

$$(g^{ij}) = \begin{bmatrix} 1 + u_x^2 & 0 \\ 0 & u^2\bar{g}_{ij} \end{bmatrix}, (h_{ij}) = \begin{bmatrix} \frac{-u_{xx}}{\sqrt{1+u_x^2}} & 0 \\ 0 & \frac{u}{\sqrt{1+u_x^2}}\bar{g}_{ij} \end{bmatrix}, (h_i^j) = \begin{bmatrix} \frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} & 0 \\ 0 & \frac{1}{u\sqrt{1+u_x^2}}\bar{\delta}_{ij} \end{bmatrix}.$$

The principal curvatures are given by

$$\kappa_1(x) = -\frac{u_{xx}(x)}{(1+u_x^2(x))^{\frac{3}{2}}}, \quad \kappa_j(x) = \frac{1}{u(x)\sqrt{1+u_x^2(x)}}, j = 2, 3, \ldots, n. \tag{3.44}$$

We note that since ∂K_0 is smooth, $\kappa_{\min}, \kappa_{\max}$ are continuous and , at poles, are the principal curvatures are all equal, so we still have κ_{rad} and κ_{axi} defined by

$$\kappa_1 = \kappa_{axi}(p) = \lim_{x\to p} \kappa_{axi}(x), \quad \text{and} \quad \kappa_j = \kappa_{rad}(p) = \lim_{x\to p} \kappa_{rad}(x), j = 2, 3, \ldots, n,$$

for $p \in \{0, a\}$. For $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$, consider the following flow

$$\frac{\partial X}{\partial t}(x, \omega, t) = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\alpha_2}(x,t)\ldots\kappa_n^{\alpha_n}(x,t)\nu = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\beta}(x,t)\nu \tag{3.45}$$

where $\beta = \sum_{i=2}^{n} \alpha_i$. By taking the derivative with respect to t of both sides of equation (3.43), we obtain:

$$u_t(x) = \left\langle (0, u_t(x)\omega), \frac{(-u_x, \omega)}{1+u_x^2} \right\rangle = \left\langle \frac{\partial X}{\partial t}(x, \omega, t), \nu \right\rangle = -\kappa_1^{\alpha_1}(x,t)\kappa_2^{\beta}(x,t). \tag{3.46}$$

By employing a comparable analysis to that conducted in $\mathbb{R}^3$, we can investigate the behaviour of the flow around poles. Specifically, due to the governing equations' similarity, it suffices to examine the flow behaviour around a single pole. Let $u_{\max} = u(x_0)$, and introduce the function $w : \big(-u(x_0), u(x_0)\big) \to [0, x_0)$ with the following definition:

$$y \mapsto \begin{cases} u^{-1}(y) & 0 \le y \le u_{\max} \\ u^{-1}(-y) & -u_{\max} \le y < 0. \end{cases}$$

About $0 = (0,0,0) \in \partial K$, consider the parametrization $\bar{X} : B(0, u_{\max}) \to \mathbb{R}^{n+1}$ defined by

$$(y = x_1, \ldots, x_n) \mapsto \big(v(x_1, \ldots, x_n), x_1, \ldots, x_n\big) \tag{3.47}$$

where $v(x_1, \dots, x_n) = w(\sqrt{x_1^2 + \dots + x_n^2})$. We have that

$$g^{ij} = \delta^{ij} - \frac{v_i v_j}{1 + |\nabla v|^2}, \quad \text{and} \quad h_{ij} = \frac{\frac{\partial^2 v}{\partial x_i \partial x_j}}{\sqrt{1 + |\nabla v|^2}} \tag{3.48}$$

and, when $z = 0$, we have

$$(g^{ij}) = \begin{bmatrix} \frac{1}{1+v_y^2} & 0 & \dots & 0 \\ 0 & 1 & 0 \dots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad (h_{ij}) = \begin{bmatrix} \frac{\frac{\partial^2 v}{\partial y^2}}{\sqrt{1+v_y^2}} & 0 & \dots & 0 \\ 0 & \frac{\frac{\partial^2 v}{\partial x_2^2}}{\sqrt{1+v_y^2}} & 0 \dots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & \frac{\frac{\partial^2 v}{\partial x_n^2}}{\sqrt{1+v_y^2}} \end{bmatrix}.$$

As a result, along the curve

$$\gamma = \Big\{ \big(y, 0, \dots, 0, v(y,0)\big) : y \in \big(-u_{\max}, u_{\max} \big) \Big\}, \tag{3.49}$$

the principal curvatures are

$$\kappa_1 = \kappa_{axi}(y,0) = \frac{v_{yy}}{(1+v_y^2)^{\frac{3}{2}}}, \ \kappa_j = \kappa_{rad}(y,0) = \frac{v_{zz}}{(1+v_y^2)^{\frac{1}{2}}}, \ j = 2, \dots, n, \tag{3.50}$$

where $v_{zz} = \frac{\partial^2 v}{\partial x_j^2}$ for $j = 2, \dots, n$ and $v_{yy} = \frac{\partial^2 v}{\partial x_1^2}$. The outer normal along the graph of v is given by

$$\nu(y,0) = \frac{(v_y, 0, \dots, 0, -1)}{\sqrt{v_y^2 + 1}}.$$

By direct calculations, we get

$$(0, 0, v_t(y,t)) = \frac{\partial \gamma}{\partial t}(y, 0, t) = -\kappa_{rad}^{\alpha_1}(y,0,t)\kappa_{axi}^{\alpha_2}(y,0,t)\nu(y,0,t).$$

Therefore, the scalar evolution equation of the graph function v is

$$v_t = \sqrt{1+v_y^2}\kappa_1^{\alpha_1}(y,0,t)\kappa_2^{\beta}(y,0,t) = \frac{1}{(1+v_y^2)^{\frac{\beta+3\alpha_1-1}{2}}} v_{yy}^{\alpha_1} v_{zz}^{\beta}. \tag{3.51}$$

where $\beta = \sum\limits_{2}^{n} \alpha_j$. The existence of smooth solutions for equation (5.1) is established when a set of smooth, strictly positive, and concave functions $u(\cdot,t) : (a_t, b_t) \to \mathbb{R}$ is found, satisfying the conditions $u(a_t,t) = u(b_t,t) = 0$ and equation (3.10), while ensuring the smoothness of the curve $\gamma(t)$ defined by equation (3.49) for every value of t. The observation can be made that the dynamics of flow in both $\mathbb{R}^{n+1}$ and $\mathbb{R}^3$ can be effectively examined using identical equations, as they are governed by the same scalar form of evolution equations. We also note that the volume of a axially symmetric surface of revolution in $\mathbb{R}^{n+1}$ with profile curve u is given by $V = w_n \int\limits_a^b u^n(x)dx$ where w_n is the volume of the unit n-sphere. Therefore, by applying the same techniques and arguments presented in the previous sections, the following theorem follows:

Theorem 3.6.1. *Assume a smooth, strictly convex embedded surface $\partial K_0 \subset \mathbb{R}^{n+1}$ exhibits axial symmetry with an even profile curve. Solutions for equation (3.45) exist within a maximal time interval $[0,\omega)$. The solutions will shrink to a point as t approaches ω. Moreover, if solutions are rescaled to enclose domains of constant volume, they will converge in a sequential manner, in the Hausdorff metric, towards the boundary of a convex body.*

Chapter 4

Expansion of strictly convex hypersurfaces of revolution in $\mathbb{R}^{n+1}$ by positive powers of the radii of curvature

4.1 Introduction

In this chapter of the thesis, building upon the established framework introduced in Chapter 3 for studying a flow in higher dimensions, we undertake an exploration of the dynamic characteristics exhibited by an expanding flow. The evolution of this flow is governed by a non-symmetric speed function that relies on the axial and radial curvatures, denoted as κ_{axi} and κ_{rad} respectively. Our focus lies specifically on the expansion process of a strictly convex surface of revolution, denoted as $\partial K_0 \subset \mathbb{R}^{n+1}$ evolving by the following equation:

$$\frac{\partial X}{\partial t}(x,t) = \left(\frac{1}{\kappa_1}\right)^{\alpha_1}(x,t)\left(\frac{1}{\kappa_2}\right)^{\alpha_2}(x,t)\ldots\left(\frac{1}{\kappa_n}\right)^{\alpha_n}(x,t)\,\nu(x,t), \tag{4.1}$$

where $\kappa_1 = \kappa_{axi}$, $\kappa_2 = \cdots = \kappa_n = \kappa_{rad}$ are principal curvatures, $\alpha_1, \ldots, \alpha_n$ are arbitrary positive real numbers, and ν is the outer unit normal. To scrutinize the short-term and long-term behaviour of the flow, we rely on the evolution equation governing the scalar forms associated with the flow. More precisely, let $\partial K_0 \subset \mathbb{R}^{n+1}$ be a smooth surface of revolution generated by a strictly positive smooth curve $u : [0,1] \to \mathbb{R}$ with $u(0) = u(1) = 0$. Consider a parameterization of ∂K_0, $X : [0,1] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ given by

$$X(x, \omega) = \big(x, u(x)\omega\big). \tag{4.2}$$

At every $(x, \omega) \in (0,1) \times \mathbb{S}^{n-1}$, the outer normal, axial and radial curvatures are given by

$$v(x,t) = \frac{(-u_x, \omega)}{\sqrt{1+u_x^2}}, \quad \kappa_{rad}(x) = \frac{1}{u(x)\sqrt{1+u_x^2(x)}}, \quad \kappa_{axi}(x) = -\frac{u_{xx}(x)}{(1+u_x^2(x))^{\frac{3}{2}}}.$$

Since $\kappa_{axi} = \kappa_1$, $\kappa_{rad} = \kappa_2 = \cdots = \kappa_n$, if we assume $\alpha = \alpha_1$ and $\beta = \sum_{i=2}^{n} \alpha_i$, then, equation(4.1) can be simplified to:

$$\frac{\partial X}{\partial t}(x,t) = \left(\frac{1}{\kappa_{axi}}\right)^{\alpha}(x,t)\left(\frac{1}{\kappa_{rad}}\right)^{\beta}(x,t)\nu(x,t). \tag{4.3}$$

We have that

$$\begin{aligned}(\frac{1}{\kappa_{axi}})^{\alpha}(\frac{1}{\kappa_{rad}})^{\beta} &= \langle \frac{\partial X}{\partial t}, \nu \rangle = \langle \big(0, u_t\omega\big), \frac{\big(-u_x, \omega\big)}{\sqrt{1+u_x^2}} \rangle \\ &= \frac{u_t}{\sqrt{1+u_x^2}}\end{aligned}$$

The evolution equation of u is:

$$\begin{aligned}u_t = \sqrt{1+u_x^2}(\frac{1}{\kappa_{axi}})^{\alpha}(\frac{1}{\kappa_{rad}})^{\beta} &= \left(1+u_x^2\right)^{\frac{3\alpha+\beta+1}{2}} \frac{u^{\beta}}{(-u_{xx})^{\alpha}} \\ = \left(1+u_x^2\right)^{\mu} \frac{u^{\beta}}{(-u_{xx})^{\alpha}},&\end{aligned} \tag{4.4}$$

where $\mu = \frac{3\alpha+\beta+1}{2}$. Due to the analogous nature of the evolution equations governing the flow around poles, it suffices to investigate the flow dynamics in the vicinity of a single pole. Let $u_{\max} = u(x_0)$, and introduce the function $w : \big(- u(x_0), u(x_0)\big) \to [0, x_0)$ with the following definition:

$$y \mapsto \begin{cases} u^{-1}(y) & 0 \leq y \leq u_{\max} \\ u^{-1}(-y) & -u_{\max} \leq y < 0. \end{cases}$$

About $0 \in \partial K$, consider the parametrization $\bar{X} : B(0, u_{\max}) \to \mathbb{R}^{n+1}$ defined by

$$(y = x_1, \dots, x_n) \mapsto \big(v(x_1, \dots, x_n), x_1, \dots, x_n\big), \tag{4.5}$$

where $v(x_1, \dots, x_n) = w(\sqrt{x_1^2 + \cdots + x_n^2})$. When $z = 0$, it follows from

$$g^{ij} = \delta^{ij} - \frac{v_i v_j}{1 + |\nabla v|^2}, \text{ and } h_{ij} = \frac{\frac{\partial^2 v}{\partial x_i \partial x_j}}{\sqrt{1 + |\nabla v|^2}}, \tag{4.6}$$

that

$$(g^{ij}) = \begin{bmatrix} \frac{1}{1+v_y^2} & 0 & \dots & 0 \\ 0 & 1 & 0 \dots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix}, \text{ and } (h_{ij}) = \begin{bmatrix} \frac{\frac{\partial^2 v}{\partial y^2}}{\sqrt{1+v_y^2}} & 0 & \dots & 0 \\ 0 & \frac{\frac{\partial^2 v}{\partial x_2^2}}{\sqrt{1+v_y^2}} & 0 \dots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & \frac{\frac{\partial^2 v}{\partial x_n^2}}{\sqrt{1+v_y^2}} \end{bmatrix}. \tag{4.7}$$

As a result, along the curve

$$\gamma = \Big\{ \big(y, 0, \dots, 0, v(y, 0)\big) : y \in \big(- u_{\max}, u_{\max}\big) \Big\}, \tag{4.8}$$

the principal curvatures are

$$\kappa_{axi}(y,0) = \frac{v_{yy}}{(1+v_y^2)^{\frac{3}{2}}}, \kappa_{rad}(y,0) = \frac{v_{zz}}{(1+v_y^2)^{\frac{1}{2}}} \tag{4.9}$$

where $v_{zz} = \frac{\partial^2 v}{\partial x_j^2}$ for $j = 2, \ldots, n$ and $v_{yy} = \frac{\partial^2 v}{\partial x_1^2}$. The outer normal along the graph of v is

$$\nu(y,0) = \frac{(v_y, 0, \ldots, 0, -1)}{\sqrt{v_y^2+1}}.$$

By direct calculations, we get

$$(0, \ldots, 0, v_t(y,t)) = \frac{\partial \gamma}{\partial t}(y, 0, t).$$

We infer from

$$-v_t = \left\langle \frac{\partial \gamma}{\partial t}, \nu(y) \right\rangle$$

that the scalar evolution equation of the graph function v is

$$v_t = -\sqrt{1+v_y^2}(\frac{1}{\kappa_{axi}})^{\alpha}(\frac{1}{\kappa_{rad}})^{\beta} = -\left(1+v_y^2\right)^{\frac{3\alpha+\beta+1}{2}} v_{zz}^{-\beta} v_{yy}^{-\alpha} \tag{4.10}$$

where $\beta = \sum_{2}^{n} \alpha_j$. The existence of smooth solutions for equation (4.3) is established when a set of smooth, strictly positive, and concave functions $u(\cdot, t) : (a_t, b_t) \longrightarrow \mathbb{R}$ is found, satisfying the conditions $u(a_t, t) = u(b_t, t) = 0$ and equation (4.4), while ensuring the smoothness of the curve $\gamma(t)$ defined by equation (4.8) for every value of t. In this chapter, we prove the following theorem :

Theorem 4.1.1. *Consider a smooth, strictly convex embedded surface ∂K_0 that exhibits axial symmetry in $\mathbb{R}^{n+1}$. For equation (4.3), there exist solutions within a maximal time interval $[0, T)$, where T is finite when $\alpha + \beta > 1$ and infinite when $\alpha + \beta \leq 1$. As the flow progresses, these solutions will preserve convexity and will expand to infinity. Furthermore, if the profile curve is even and the solutions are rescaled* *to enclose domains of constant volume, rescaled solutions will converge sequentially in the Hausdorff metric towards the boundary of a convex body.*

4.2 Existence of solutions and preservation of convexity

Lemma 4.2.1. *There exists $\omega > 0$ such that solutions to the flow exist on $[0, \omega)$.*

Proof. We consider a linearization of the equation (4.4). Suppose that $\epsilon > 0$ is a small real number, $\tilde{u} = u + \epsilon\phi$ is a solution of the equation (4.4), where $\phi : [0, a] \times [0, T) \longrightarrow \mathbb{R}$ is an arbitrary smooth function. We have

$$\tilde{u}_x = u_x + \epsilon\phi_x,$$

$$\tilde{u}_{xx} = u_{xx} + \epsilon\phi_{xx}$$

and

$$\tilde{u}_t = u_t + \epsilon\phi_t. \tag{4.11}$$

Therefore

$$\tilde{u}_t = \Big(1 + \big(u_x + \epsilon\phi_x\big)^2\Big)^{\mu} \frac{\big(u + \epsilon\phi\big)^{\beta}}{(-u_{xx} - \epsilon\phi_{xx})^{\alpha}} \tag{4.12}$$

and

$$\begin{aligned} \frac{\partial \tilde{u}_t}{\partial \epsilon} &= 2\mu\Big(1 + \big(u_x + \epsilon\phi_x\big)^2\Big)^{\mu-1}\big(u_x + \epsilon\phi_x\big) \frac{\big(u + \epsilon\phi\big)^{\beta}}{(-u_{xx} - \epsilon\phi_{xx})^{\alpha}}\phi_x \\ &+ \beta\Big(1 + \big(u_x + \epsilon\phi_x\big)^2\Big)^{\mu} \frac{\big(u + \epsilon\phi\big)^{\beta-1}}{(-u_{xx} - \epsilon\phi_{xx})^{\alpha}}\phi \\ &+ \alpha\Big(1 + \big(u_x + \epsilon\phi_x\big)^2\Big)^{\mu} \frac{\big(u + \epsilon\phi\big)^{\beta}}{(-u_{xx} - \epsilon\phi_{xx})^{\alpha+1}}\phi_{xx}. \end{aligned} \tag{4.13}$$

By evaluating derivatives, with respect to ϵ, of both sides of equation (4.11) at $\epsilon = 0$ we get

$$\phi_t = \frac{\alpha\Big(1+u_x^2\Big)^{\mu}u^{\beta}}{(-u_{xx})^{\alpha+1}}\phi_{xx} + \frac{u_x u^{\beta}}{(-u_{xx})^{\alpha+1}}\phi_x + \beta\Big(1+\big(u_x\big)^2\Big)^{\mu}\frac{u^{\beta-1}}{(-u_{xx})^{\alpha}}\phi. \tag{4.14}$$

Since $\frac{\alpha\left(1+u_x^2\right)^{\mu}u^{\beta}}{(-u_{xx})^{\alpha+1}} > 0$ at time zero, we infer that smooth solutions to this equation exist for some time interval. Likewise, through the linearization of equation (4.8), we obtain a parabolic equation that guarantees the existence of solutions. Specifically, let us assume that $\epsilon > 0$ is a small real number, $\tilde{v} = v + \epsilon\phi$ is a solution of the equation (3.30), where, for some $\delta > 0$, $\phi : (-\delta, \delta) \times [0, T) \longrightarrow \mathbb{R}$ is an arbitrary smooth function. We have

$$\tilde{v}_y = v_y + \epsilon\phi_y,$$

$$\tilde{v}_{yy} = v_{yy} + \epsilon\phi_{yy},$$

$$\tilde{v}_{zz} = v_{zz}$$

and

$$\tilde{v}_t = v_t + \epsilon\phi_t. \tag{4.15}$$

By differentiating both sides of equation (4.15) with respect to ϵ and evaluating it at $\epsilon = 0$, we obtain the following expression:

$$\phi_t = \alpha\big(1+v_y^2\big)^{\mu}v_{zz}^{-\beta}v_{yy}^{-\alpha-1}\phi_{yy} \; - 2\mu\big(1+(v_y)^2\big)^{\mu-1}v_{zz}^{-\beta}v_{yy}^{-\alpha}v_y\phi_y.$$

Since the coefficient of ϕ_{yy} is positive, the existence of solutions can be inferred from the theory of parabolic equations. Therefore, smooth solutions to the flow exist for some time interval. □

Lemma 4.2.2. *Convexity is preserved throughout the time interval of existence of* *the flow.*

Proof. We set

$$H = u_t = \left(1 + u_x^2\right)^{\mu} \frac{u^{\beta}}{(-u_{xx})^{\alpha}} \tag{4.16}$$

and

$$Z = v_t = -\frac{\left(1 + v_y^2\right)^{\mu}}{v_{zz}^{\beta}} \frac{1}{v_{yy}^{\alpha}}. \tag{4.17}$$

By taking derivatives of both sides of these two equations, we obtain the following expressions:

$$H_t = \frac{\alpha\left(1 + u_x^2\right)^{\mu} u^{\beta}}{(-u_{xx})^{\alpha+1}} H_{xx} + \frac{2\mu u_x \left(1 + u_x^2\right)^{\mu-1} u^{\beta}}{(-u_{xx})^{\alpha}} H_x + \frac{\beta\left(1 + u_x^2\right)^{\mu} u^{\beta-1}}{(-u_{xx})^{\alpha}} H, \tag{4.18}$$

and

$$Z_t = \alpha \frac{\left(1 + v_y^2\right)^{\mu}}{v_{zz}^{\beta} v_{yy}^{\alpha+1}} Z_{yy} - \frac{2\mu\left(1 + v_y^2\right)^{\mu-1}}{v_{zz}^{\beta}} \frac{v_y}{v_{yy}^{\alpha}} Z_y + \beta \frac{Z_{zz}\left(1 + v_y^2\right)^{\mu}}{v_{zz}^{\beta+1} v_{yy}^{\alpha-1}} Z. \tag{4.19}$$

Based on the initial conditions $H(0) < \infty$ and $Z(0) > -\infty$, and considering that equations (4.18) and (4.19) represent parabolic equations with bounded and continuous coefficients, we can deduce, through the application of the maximal principle, that throughout the existence of the flow, it holds that $H(t) < \infty$ and $Z(t) > -\infty$. This guarantees that the surfaces remain strictly convex throughout the entire duration of the flow. □

4.3 Asymptotic behaviour of an expansion curvature flow

Lemma 4.3.1. *The expanding flow is characterized by unbounded growth of the volume enclosed by the evolving hypersurface.*

Lemma 4.3.2. *The expanding flow exhibits unbounded growth of the magnitude of the position vector of the evolving hypersurface in all directions.*

Proof. Consider the interval $[0, T)$ as the maximal time interval for the existence of solutions. Without loss of generality, we assume that T is finite ($T < \infty$). This assumption is justified by the fact that if T were infinite, we can introduce a new time variable $\tau < \infty$ defined as

$$\tau(t) = \frac{V(0)}{e^{V(0)}} - \frac{V(0)}{e^{V(t)}}.$$

Suppose, for the purpose of contradiction, that there exist directions along which the solution to the flow remains bounded. As a consequence, there exists an interval (a, b) such that $u(x, t)$ is smooth and bounded on $[a, b] \times [0, T)$. If the curvatures remain bounded on the interval $[a, b] \times [0, T)$, this contradicts the maximality of the time interval $[0, T)$, as all conditions are met to extend the flow beyond time T. Therefore it must be the case that the curvatures become unbounded on $(a, b) \times [0, T)$. However, the evolution equation is strictly parabolic, with coefficients that remain uniformly bounded, and is subject to boundary conditions insuring boundedness on the parabolic boundary, so the curvatures must remain bounded as well, concluding the proof. □

Proposition 4.3.1. *The expanding curvature flow exhibits distinct expansion properties based on the sum of powers α and β. When $\alpha + \beta \leq 1$, the flow expands infinitely* *over an unbounded time interval. However, when $\alpha+\beta > 1$, the flow expands infinitely within a finite time.*

Proof. Let B be an arbitrary sphere with radius r_0. Starting from this initial surface, B_0, the solutions of the expanding curvature flow defined by the equation

$$y_t = \sqrt{1+y_x^2}\,r^{\alpha+\beta}$$

are equivalent to the solutions of the simplified flow equation

$$r_t = r^{\alpha+\beta}.$$

We infer from

$$\int_{r_0}^{r(t)} r^{-(\alpha+\beta)} dr = t$$

that

$$r(t) = \begin{cases} r(0)e^t, & \alpha+\beta = 1 \\ \sqrt[1-(\alpha+\beta)]{r^{1-(\alpha+\beta)}(0) + [1-(\alpha+\beta)]t}, & \alpha+\beta < 1 \\ \sqrt[1-(\alpha+\beta)]{r^{1-(\alpha+\beta)}(0) - [(\alpha+\beta)-1]t}, & \alpha+\beta > 1 \end{cases} .$$

Given the positivity of real values α and β, satisfying $\alpha + \beta \leq 1$, we assert the perpetual existence of the flow over an infinite time horizon. To substantiate this, consider a sufficiently expansive sphere B_0 that encompasses ∂K_0. If ∂K_0 were to expand infinitely within a finite temporal interval, it would necessitate same expansion for B_0, thereby contradicting the infinite temporal existence of the flow initiated from B_0. Consequently, we conclude that the flow exists indefinitely over an infinite span of time. Now, let us suppose that $\alpha + \beta > 1$ and B_0 is sufficiently small to reside entirely within ∂K_0. In this case, the expanding spheres corresponding to the flow solutions originating from B_0 would reach infinity within a finite time, denoted as T. We assert that the flow initiated from ∂K_0 expands to infinity in finite time. To demonstrate this, suppose, for contradiction, that the flow expands to infinity within an infinite time interval. Thus, the flow would exist on the interval $[0, T)$, which represents the duration of existence for the flow initiated from B_0. However, this leads to a contradiction because $\partial K(T)$ encompasses the expanding balls and is a convex body, while the balls themselves expand to infinity as time approaches T. Therefore, the flow initiated from ∂K_0 would expand to infinity within a finite time interval. This completes the proof. □

In the next proposition, we propose a renormalization technique for the flow, which leads to sequential convergence of solutions towards the boundary of a convex body. Specifically, we perform a renormalization of the profile curve such that it encompasses regions of constant area. Subsequently, we demonstrate sequential convergence of the renormalized profile curves, and consequently convergence of the renormalized solutions to the boundary of a convex body.

Proposition 4.3.2. *Suppose that the initial profile curve is even. Then, the properly rescaled solutions converge to a convex hypersurface as time progresses.*

Proof. Let consider the function $y : (a_t, b_t) \to (\frac{a_t}{b_t - a_t}, \frac{b_t}{b_t - a_t})$ defined as

$$y(x) = \frac{x}{b_t - a_t}.$$

On $(\frac{a_t}{b_t - a_t}, \frac{b_t}{b_t - a_t})$ consider the renormalization $\tilde{u}(\cdot, t)$ of $u(\cdot, t)$ defined as

$$\tilde{u}(y, t) = \frac{b_t - a_t}{A(t)} u((b_t - a_t)y),$$

where $A(t)$ is the enclosed area by the profile curve:

$$A(t) = 2 \int_{a_t}^{b_t} u(x,t)dx.$$

We have that

$$\tilde{A}(t) = 2 \int_{\frac{a_t}{b_t - a_t}}^{\frac{b_t}{b_t - a_t}} \tilde{u}(y,t)dy = 2 \int_{\frac{a_t}{b_t - a_t}}^{\frac{b_t}{b_t - a_t}} \frac{b_t - a_t}{A(t)} u((b_t - a_t)y)dy = \frac{1}{A(t)} 2 \int_{a_t}^{b_t} u(x,t)dx = 1.$$

Suppose that $[0, T)$ is the maximal time interval on which solutions exist. Since solutions are even, the speed at the endpoints is the same, but of opposite signs, $0 \neq \frac{\partial a_t}{\partial t} = -\frac{\partial b_t}{\partial t}$. Therefore

$$\lim_{t \to T} \frac{a_t}{b_t - a_t} = -\frac{1}{2} \quad \text{and} \quad \lim_{t \to T} \frac{b_t}{b_t - a_t} = \frac{1}{2}. \tag{4.20}$$

If the rescaled solutions degenerate, the limiting interval of the profile curves, I_t, converges to $(-\frac{1}{2}, \frac{1}{2})$, indicating their collapse into the segment $(-\frac{1}{2}, \frac{1}{2})$. However, this contradicts the invariance of the enclosed area, which remains constant at 1 for the profile curves of the rescaled solutions. Moreover, as I_t approaches $(-\frac{1}{2}, \frac{1}{2})$ while preserving the constant enclosed area, the rescaled solutions remain bounded from above. Consequently, they exist within a confined annular region. By applying the Blaschke selection theorem, it follows that subsequential convergence of the rescaled solutions to a convex body. □

Chapter 5

Conclusion and future directions of study

5.1 Prologue

The preceding chapters have focused on understanding the contraction and expanding flows characterized by non-symmetric speed on the principal curvatures represented, respectively, by equations (3.45) and (4.1) for rotationally symmetric initial data. The convergence to a single point in the contraction flow, the convergence of rescaled solutions to a convex hypersurface when the initial hypersurface possesses even profile curves, the preservation of convexity, the phenomenon of infinite expansion in the expanding flow, and the convergence to the boundary of a convex body through appropriate rescaling collectively demonstrate the dynamics inherent of these deformations. The natural progression from these findings is to explore whether the properties of the flows remain valid when considering more general data than examined in the previous sections, or more general non-symmetric speed functions. This chapter briefly addresses the challenges that arise when dealing with such variations of the contraction flow previously studied in Chapter 3. Additionally, we present the strategies and approaches we have adopted to tackle some of these difficulties, opening avenues for further investigation and analysis.

5.2 A Contraction of strictly convex hypersurafces in $\mathbb{R}^3$

Let ∂K_0 be a compact, strictly convex hypersurface wihtout boundary that is smoothly embedded in $\mathbb{R}^3$ by the diffeomorphism $X_0 : \mathbb{S}^2 \to \mathbb{R}^3$. We consider the family of maps $X(\cdot, t)$ evolving according to

$$\frac{\partial X(\cdot, t)}{\partial t} = -\kappa_1^{\alpha_1}(x, t)\kappa_2^{\alpha_2}(x, t)\nu(x, t), \tag{5.1}$$

where $\kappa_1(x, t) \leq \kappa_2(x, t)$ are the principal curvatures, $\nu(x, t)$ is the outer normal to $K(t)$ at $X(x, t)$, and $\alpha_1 \leq \alpha_2$ are positive real numbers. Suppose that $f(\kappa_1, \kappa_2) = \kappa_1^{\alpha_1}\kappa_2^{\alpha_2}$, and suppose $S_+ \subset T^*M \otimes T^*M$ is the set of symmetric positive transformations. We define $F : S_+ \subset T^*M \otimes T^*M \to \mathbb{R}$ by

$$F(\mathcal{W}(x)) = f(\lambda(x)),$$

where $\lambda(x)$ is the vector of eigenvalues of $\mathcal{W}(x)$ in increasing order. Let ∂K be a smooth, closed, and strictly convex surface in $\mathbb{R}^3$. We define the curvature function $\kappa : \partial K \longrightarrow \mathbb{R}^2$ as $\kappa(x) = (\kappa_1(x), \kappa_2(x))$, where $\kappa_1 \leq \kappa_2$ represent the principal curvatures. If one considers the closed cone $\Gamma_+^{\leq} := \{(\lambda_1, \lambda_2) : 0 < \lambda_1 \leq \lambda_2\}$, then the flow for $f \circ \kappa$ on $\Gamma_+^{\leq}$ is not always differentiable due to the presence of umbilical points, as suggested by the Caratheodory Conjecture (see [26]). One possible approach is to define a smooth function f on $\Gamma_+ := \{(\lambda_1, \lambda_2) : 0 < \lambda_1 < \lambda_2\}$, and to do analysis related to the smoothness of f on Γ_+. Then by employing a limit argument and leveraging the continuity of principal curvatures, we can extend the obtained results concerning the boundedness of curvature ratios and curvature convexity to encompass umbilical points. We note that while f may not exhibit smoothness at umbilical points, f is still defined at these points, ensuring the global well-definedness of the flow. The existence of solutions to the flow defined by equation (5.1) is established when there are smooth, strictly convex hypersurfaces $\partial K(t)$ that satisfy the flow equation (5.1).

5.2.1 Properties of non-symmetric speed functions

The first challenge encountered when working with a non-symmetric speed pertains to the transition from the smooth speed function denoted as $f : \Gamma_+ \longrightarrow \mathbb{R}$ to a corresponding smooth function denoted as $F : S_+ \longrightarrow \mathbb{R}$ where $\Gamma_+ = \{(\lambda_1, \lambda_2) : \lambda_i > 0, i = 1, 2\}$ and S_+ stands for the set of positive definite symmetric matrices. In the case where the speed function f is symmetric, it is recognized that the existence of F can be ensured due to the capability of expressing f in terms of elementary symmetric functions. However, this desirable property does not necessarily hold true for a non-symmetric speed function. Hence, within this section, we present our strategies for addressing this specific challenge.

Lemma 5.2.1. *Let f and F be defined as above. Then*

(1) F is smooth.

(2) At any $X \in S^2_+$, the tensor $\dot{F}$ defined by $\dot{F}(B) = D_B F$ is positive definite.

Proof. (1) First, we show that for every $v = (v_1, v_2) \in \mathbb{R}^2$ with $v_1 < v_2$, F is differentiable (smooth) at $Diag(v)$. To see this let $\lambda : S^2_+ \longrightarrow \mathbb{R}^2$ be the eigenvalue function defined by

$$A \mapsto \big(\lambda_1(A),\ \lambda_2(A)\big)$$

where $\lambda_1(A) < \lambda_2(A)$. If $i \neq j$, we have that $\lambda(Diag(v) + te_{ij}) = \lambda(Diag(v))$. Therefore

$$\lim_{t \longrightarrow 0} \frac{F(Diag(v) + te_{ij}) - F(Diag(v))}{t} = 0.$$

If $i = j$, we have

$$\lim_{t \longrightarrow 0} \frac{F(Diag(v) + te_{ij}) - F(Diag(v))}{t} = \alpha_i v_j^{\alpha_j} v_i^{\alpha_i - 1}.$$

The same way it can be shown that F is actually smooth in this case. Now, suppose that $X \in S_+^n$ and U unitary is such that $U^{-1}XU = Diag(\lambda(X))$. We have that

$$F(X) = (f \circ \lambda)(X) = (f \circ \lambda)(U^{-1}XU).$$

Since X is invertible there exist an neighborhood of $X \in GL_n(\mathbb{R})$ on which for function G defined by $Z \mapsto UZU^{-1}$ we have that $G(X + Z) = U(X + Z)U^{-1} - UXU^{-1} = UXU^{-1}$. Therefore G is differentiable at X and $dG(X) = UXU^{-1}$; indeed, G is smooth. Since $f \circ \lambda$ is differentiable at $G(X)$ and G is differentiable at X, $F = f \circ \lambda \circ G$ is differentiable at X. The same way it can be shown F is twice differentiable (smooth) since f is smooth.

(2) Suppose that λ is the eigenvalue function defined in the first part of the proof, and $X \in S_+^2$. We note that the claim is true at $Diag(\lambda(X))$ since

$$\dot{F} = Diag\Big(\frac{\partial f(\lambda_1(X), \lambda_2(X))}{\partial \lambda_1}, \frac{\partial f(\lambda_1(X), \lambda_2(X))}{\partial \lambda_2}\Big).$$

Let $X = U^{-1}Diag(\lambda(X))U$. The function G, defined in the first part of the proof, preserves the positive definiteness since it is just a change of basis transformation. Therefore at $X \in S_+^2$, the tensor

$$\dot{F} = Diag\Big(\frac{\partial f(\lambda_1(X), \lambda_2(X))}{\partial \lambda_1}, \frac{\partial f(\lambda_1(X), \lambda_2(X))}{\partial \lambda_2}\Big)\Big(G(X)\Big)$$

is positive definite.

□

5.2.2 Convexity preservation and bounds on the ratio of curvatures

Our approach to studying the preservation of convexity and the asymptotic behavior of the flow involves applying the maximum principle to the evolution equation of $\frac{\mathcal{W}}{F}$, where $\mathcal{W}$ represents the Weingarten map and F is as defined in the previous section.

Proposition 5.2.1. *Suppose that for every i, $\alpha_i \geq 1$. Then, as long as the flow exists, hypersurfaces evolving by the equation (5.1) remain convex with $\dfrac{\kappa_{\max}}{\kappa_{\min}}$ bounded from above.*

Proof. In order to prove the proposition, we employ the same approaches in the Andrews' work [4]. Let $d\nu := \mathcal{W} : TK \longrightarrow TK$ be the Weingarten map. By taking covariant derivative of

$$h_{ij} = \big\langle D_{X_i}\nu, X_j \big\rangle = \big\langle X_i, \mathcal{W}(X_j) \big\rangle,$$

we get

$$\Big\langle X_i, \frac{\partial \mathcal{W}(X_j)}{\partial t} \Big\rangle = \frac{\partial h_{ij}}{\partial t} - \frac{\partial}{\partial t} \big\langle X_i, \mathcal{W}(X_j) \big\rangle = \frac{\partial h_{ij}}{\partial t} + 2F \big\langle \mathcal{W}(X_i), \mathcal{W}(X_j) \big\rangle$$

$$= \nabla_i \nabla_j F + F \big\langle \mathcal{W}(X_i), \mathcal{W}(X_j) \big\rangle.$$

On the other hand,

$$\Big(\operatorname{Hess}_{\nabla} F(\mathcal{W}) \Big)(u, v) = \nabla^2_{u,v} \Big[F(\mathcal{W}) \Big]$$
$$= \ddot{F}\Big(\nabla_u(\mathcal{W}), \nabla_v(\mathcal{W}) \Big) + \dot{F}\Big(\operatorname{Hess}_{\nabla} \mathcal{W} \Big).$$

Therefore

$$\nabla^2_{i,j} F = \ddot{F}\Big(\nabla_i(\mathcal{W}), \nabla_j(\mathcal{W})\Big) + \dot{F}\Big(\nabla^2_{i,j}\mathcal{W}\Big).$$

In turn,

$$\begin{aligned}
\langle X_i, \frac{\partial \mathcal{W}(X_j)}{\partial t}\rangle &= \nabla^2_{i,j} F + F\langle \mathcal{W}(X_i), \mathcal{W}(X_j)\rangle \\
&= \dot{F}\Big(\nabla^2_{i,j}\mathcal{W}\Big) + \ddot{F}\Big(\nabla_i(\mathcal{W}), \nabla_j(\mathcal{W})\Big) + F\langle \mathcal{W}(X_i), \mathcal{W}(X_j)\rangle \\
&= \frac{\partial F}{\partial \mathcal{W}^l_k}\Big\langle X_i, g^* \operatorname{Hess}_\nabla \mathcal{W}^l_k(X_j)\Big\rangle + \\
&\Big\langle X_i, g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)(X_j)\Big\rangle + F\Big\langle X_i, \mathcal{W}^2(X_j)\Big\rangle \\
&= \Big\langle X_i, \Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big)(X_j, \dot{F}) + g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)(X_j) + F\mathcal{W}^2(X_j)\Big\rangle.
\end{aligned}$$

Therefore

$$\frac{\partial \mathcal{W}(X_j)}{\partial t} = \Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big)(X_j, \dot{F}) + g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)(X_j) + F\mathcal{W}^2(X_j).$$

It follows from a form of Simons' identity and applying the correspondence g^* (see [5] proof of Lemma 3.13) that

$$\Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big)(X_j, \dot{F}) = \Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big)(\dot{F}, X_j) + \mathcal{W}(u)\dot{F}(\mathcal{W}^2) - \dot{F}(\mathcal{W})\mathcal{W}^2(u).$$

Therefore,

$$\begin{aligned}
\frac{\partial \mathcal{W}(X_j)}{\partial t} &= \Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big)(\dot{F}, X_j) + \mathcal{W}(u)\dot{F}(\mathcal{W}^2) + g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)(X_j) \\
&\quad + \Big(F(\mathcal{W}) - \dot{F}(\mathcal{W})\Big)\mathcal{W}^2(X_j).
\end{aligned}$$

Since $\dot{F}(\mathcal{W}) = \sum\limits_{i=1}^{n} \kappa_i \frac{\partial f}{\partial \kappa_i} = \left(\sum\limits_{i=1}^{n} \alpha_i\right) f = \underbrace{\left(\sum_{i=1}^{n} \alpha_i\right)}_{\alpha} F(\mathcal{W}) = \alpha F(\mathcal{W})$, we infer that

$$\frac{\partial \mathcal{W}}{\partial t} = \dot{F}\Big(g^* \operatorname{Hess}_\nabla \mathcal{W}\Big) + \mathcal{W}\dot{F}(\mathcal{W}^2) + \Big(\frac{1-\alpha}{\alpha}\Big)\mathcal{W}^2 \dot{F}(\mathcal{W}) + g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big).$$

We note that $g^*\ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)$ is positive definite. In other words, for every u,

$$0 \leq \frac{\partial^2 F}{\partial x^{rs} \partial x^{kl}} \nabla_u \mathcal{W}_r^s \nabla_u \mathcal{W}_k^l = \ddot{F}\Big(\nabla_u(\mathcal{W}), \nabla_u(\mathcal{W})\Big) = \Big\langle u, g^* \ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)(u)\Big\rangle.$$

To see this suppose that $\mathcal{W} : TM \longrightarrow TM$ is diagonal. Then $\nabla_u \mathcal{W}_r^s \neq 0$ only if $r = s$.

$$F(\mathcal{W}) = F\left(\begin{bmatrix} \mathcal{W}_1^1 & 0 \\ 0 & \mathcal{W}_2^2 \end{bmatrix}\right) = \left(\mathcal{W}_1^1\right)^{\alpha_1} \left(\mathcal{W}_2^2\right)^{\alpha_2}.$$

Since for each i, $\alpha_i \geq 1$, then f and in turn F is convex. Since $\nabla_u \mathcal{W}_r^s \neq 0$ if and only if $r = s$, and since F is convex we infer that

$$0 \leq \frac{\partial^2 F}{\partial x^{rr} \partial x^{kk}} \nabla_u \mathcal{W}_r^r \nabla_u \mathcal{W}_k^k.$$

Therefore $g^*\ddot{F}\Big(\nabla(\mathcal{W}), \nabla(\mathcal{W})\Big)$ is positive definite. We infer from parabolic maximum principle that infimum of F is increasing. It follows from evolution equations of $\frac{\mathcal{W}}{F}$, obtained from evolution equations for $\mathcal{W}$ and F, and parabolic maximum principle that the infimum over the unit ball in TM of $\frac{\mathcal{W}}{F}$ is increasing. Therefore there exists $C > 0$ such that $\kappa_{\min}(x) \geq CF(x)$. We claim $\frac{\kappa_{\max}}{\kappa_{\min}}$ is bounded from above. Consider an arbitrary sequence of time $t_n \nearrow T$. If $\kappa_{\max}(t_n)$ is bounded from above, then $\frac{\kappa_{\max}}{\kappa_{\min}}(t_n)$ is bounded from above. Suppose that $\kappa_{\max}(t_n)$ is not bounded from above, and $\kappa_{\max}(t_n) \geq 1$ for every n. Since $\alpha_i \geq 1$, if $\kappa_{\max} = \kappa_i$ for some i, then $\kappa_{\max}^{\alpha_i - 1} \geq 1$.

Let $\kappa_{\max} = \kappa_1$. Then

$$\prod_{i=2}^{n} \kappa_i^{\alpha_i} \geq (CF)^{\beta}$$

where $\beta = \sum_{i=2}^{n} \alpha_i$. Therefore

$$\kappa_{\max}^{\alpha_1 - 1} \prod_{i=2}^{n} \kappa_i^{\alpha_i} \geq (CF)^{\beta}.$$

In turn,

$$\frac{1}{C^{\beta}} F^{1-\beta} \geq \kappa_{\max}.$$

Since $\kappa_{\min} \geq CF$ we infer that

$$\frac{\kappa_{\max}}{\kappa_{\min}} \leq \frac{1}{C^{\beta+1} F^{\beta}}.$$

Since F is increasing and $\beta > 0$ we infer that $\frac{\kappa_{\max}}{\kappa_{\min}}(t_n)$ is bounded from above. Therefore $\frac{\kappa_{\max}}{\kappa_{\min}}$ is bounded. □

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