A Comparison of Students' Models of Knowledge to be Learned in an Introductory Linear Algebra Course with Results from Prior Research on Such Models in College Calculus Courses

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Abstract

A Comparison of Students' Models of Knowledge to be Learned in an Introductory Linear Algebra Course with Results from Prior Research on Such Models in College Calculus Courses

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Concordia University, 2024.

Research done from an institutional perspective has found students to develop non-mathematical practices in college calculus courses that emphasize routinization of knowledge. The knowledge students are expected to learn, as indicated by tasks determining their grade in the course, enables students to routinize techniques and use non-mathematical considerations, such as didactic and social norms from their course, to justify their techniques. Such research has mostly been done in the calculus context. To calibrate the study of the effects of institutionalized routinization of knowledge, I investigated these in the context of a course in a different domain of mathematics and regulated by institutional mechanisms similar to those regulating college calculus courses. To this end, I adapted, to an introductory college linear algebra course at a large urban North American university, the framework and methodology from a body of research that qualifies students' activity by attending to institutional mechanisms that regulate it. The framework appends to the Anthropological Theory of the Didactic (ATD) (Chevallard, 1985, 1999) notions from the Institutional Analysis and Development framework (IAD) (Ostrom, 2005). The ATD provides tools through which to model activities that occur in institutions and the IAD elaborates institutional mechanisms that regulate activity that occurs in institutions. I analyzed curricular documents to develop task-based interviews (TBI) that could draw out the nature of the knowledge students mobilize. I conducted interviews with ten students shortly after they had completed the course. The qualitative approach I used included an analysis of curricular documents to model knowledge to be learned in the course that relates to each TBI task, as well as an analysis to model the knowledge students mobilized in response to each TBI task. I found students mobilized non-mathematical practices: what they activated was conditioned by and delimited to knowledge normally expected of students in the course, and their mobilization contrasted in various ways with mathematics intrinsic to the tasks they were offered. I also propose an operationalization of the institutional notion of positioning previously proposed and examined as a mechanism regulating students' activity in didactic institutions.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis is an anthropological study of the effects of institutionalized routinization of knowledge in mathematics courses. Research has shown that what students learn in college calculus courses is a mixture of didactic, social, and mathematical norms established in these courses[1]. This mixture of norms does not attend to the calculus targeted by these courses and is rather fixed on algebraic considerations. Given the disconnect between the knowledge driving students' activity in college calculus courses and the mathematics targeted by these courses, research has qualified students as building *non-mathematical practices*. These practices can present as an obstacle to students' learning when they progress in the calculus stream as mathematical situations that had been the norm in previous courses are no longer so.

Research that brought to light the way students' activity consisted of non-mathematical (e.g., social, didactic) norms was done from the perspective that mathematics education research is a study of mathematical activity occurring in *institutions* (Chevallard, 1985). Students' approaches to acquiring knowledge and this knowledge itself are artefacts of the didactic institutions in which they exist[2]. This perspective looks to mechanisms of didactic institutions to explain students' practices and difficulties. These mechanisms include administrative and academic conditions and constraints that delimit the mathematics targeted by the teaching and learning in a course. The prevailing state of the college calculus institution is such that its emphasis is in routinization of mathematical knowledge. Research has shown how an emphasis on routinization regulates students' practices toward a non-mathematical direction.

College students have long been observed to routinize mathematics in ways that constrict their learning of the mathematics; this finding predates research taking an institutional perspective. This perspective came with increasing awareness in the 1990s of the role played by social, cultural, and institutional contexts on the teaching and learning of mathematics. A vast body of research done from other perspectives has found epistemological, conceptual, and cognitive sources for students' difficulties in calculus. Meanwhile, the institutional perspective on college students' learning has only addressed institutional sources of difficulties in the context of calculus courses. Given the difficulties that are known to be inherent to the learning of *calculus*, the role of the routinization that characterizes calculus learning merits further investigation. Does institutional emphasis

[1] A literature review is presented in Chapter 2.
[2] This perspective is addressed in my theoretical framework in Chapter 3.

on routinization regulate students' learning in similar ways regardless of the targeted mathematics?

1.1 Research questions and framework

The works of Lithner (2004); Barbé et al. (2005); Bergé (2008); Hardy (2009a); Brandes (2017); Broley (2020) constitute the main body of research to which I seek to contribute, though they are part of a broader field of study focused on routinization in college and university calculus courses. This field of study has produced classifications of tasks in calculus courses across the globe and found them to be amenable to routinization of techniques: textbook and assessment tasks can be solved by mimicking the template offered in solved examples that have similar surface-level features (Bergqvist, 2007; Brandes, 2017; Brehmer et al., 2016; Jäder et al., 2020; Lithner, 2004; Mac an Bhaird et al., 2017; Maciejewski & Merchant, 2015; Palm et al., 2006; Raman, 2004; Tallman et al., 2016, 2021). These studies have proposed that the learning environment sends students a message that encourages imitative strategies and neglect of knowledge that is not at the surface-level of a technique; in turn, other studies found this message to have been received. Such studies (e.g., Lithner, 2000; Hardy, 2009; Broley, 2020) investigated what students enact in response to calculus tasks and found students learn a restricted set of procedures, recognize tasks not by calculus-specific properties of their components but by surface-level properties such as algebraic symbols that typify certain types of calculus tasks, and do not learn the mathematical justifications that produce the procedures they mimic as these are unecessary for the short-term goal of passing their exams. Others have elaborated on the potential of routinization in earlier calculus courses to become an obstacle in real analysis courses later on in the calculus stream (Broley, 2020; see also Bergé, 2008; Raman, 2004).

The study of the effects of routinization on students' learning of calculus is rooted in the premise that students' learning is strongly influenced by mechanisms of institutions in which this learning happens—institutions including the course, the department coordinating a course, the university, etc. This is the foundational assumption of Chevallard's (1985) Anthropological Theory of the Didactic (ATD): that the learning of mathematics (that is subject of mathematics education research) is a product of the context in which it occurs. The focus of the ATD is on contexts in which there is didactic intent: an intent to teach knowledge and an intent for knowledge to be learned.

At the core of a didactic institution—one driven by a mandate to facilitate the teaching and learning of some given knowledge—is the didactic system: a triad made up of teacher, student, and *knowledge*. To understand the learning that gets accomplished in a didactic institution, the ATD looks to the knowledge transmitted in this institution and proposes *didactic transposition* as a mechanism that regulates its transmission.

The notion of didactic transposition captures the conditions and constraints that operate a didactic institution. This includes, for example, didactic intentions of its stakeholders (which range from broader society to subject-matter experts). In Section 3.1.1, I explain why this transposition of knowledge occurs and how, inevitably and not necessarily by design, knowledge is different at different layers of a didactic institution: relative to any morsel of mathematical knowledge targeted by a didactic intent, there is the knowledge of the scholarly experts, the knowledge of those in charge of the teaching, and the knowledge of those expected to learn.

Hardy (2009a) showed that students' knowledge is not a subset of the knowledge at other levels of didactic transposition. Hardy used *task-based interviews* (TBI) (Goldin, 2000) to identify students' *praxeologies* (Chevallard, 1999) for limit-finding tasks in a college calculus course: that is, models that describe students' activity in terms of the task being accomplished, the technique used to accomplish it, and the discourse that produces and justifies the technique. In the TBI, students completed tasks designed to elicit the knowledge they had built relative to limit-finding tasks as well as their perception of what they were expected to learn. In analysing students' activity and comments in the TBI, Hardy aimed to identify the techniques they had developed in their course and their justifications for using these techniques. Hardy found that students' praxeologies—that is, what they perceived a task to be, the techniques they used, and their justifications for using them—were not a subset of what a mathematician's or teacher's praxeologies for limit-finding tasks might be. Students' praxeologies were rather a mixture of (didactic, social, cognitive, and mathematical) norms of the college calculus course.

The framework Hardy (2009a) used to come to this finding took into account the institutional mechanisms elaborated in the ATD as well as mechanisms defined in a framework for the analysis of institutions (Ostrom, 2005). This combination was first elaborated by Sierpinska et al. (2008).

Sierpinska et al. (2008) and Hardy (2009a) point to two mechanisms in particular that regulate students' post-secondary learning of mathematics. Mechanism number one is borrowed from Chevallard's didactic transposition process: the knowledge students are expected to learn, as indicated by course assessments. Mechanism number two is borrowed from Ostrom's (2005) Institutional Analysis and Development framework (IAD): a student's positioning relative to their course. A student is positioned toward the objective they (un)wittingly have as participants in their course: it may be to pass the course, to develop understanding of a topic, to obtain a degree, or perhaps to achieve some professional aspiration. A student's position is not necessarily fixed—they may move in and out of it at different moments—but it accounts for how a student engages with 'mechanism number one': the knowledge they need to acquire to pass their course.

Sierpinska et al. (2008) describe how institutional mechanisms might encourage students to adopt certain positions over others: in courses such as college calculus, which is a prerequisite requirement to many university programs, students are more likely to have, as their main (if not only) objective, the goal to pass the course.

Through this combination of the ATD and the IAD, Hardy (2009a) and Broley (2020) identified positions available to students in different mathematics courses and mediating their acquisition of knowledge to be learned. For example, Hardy found that students whose comments in the TBI suggested they had positioned themselves as Students (to get a certain grade in the course) exhibited knowledge that predominantly reflected course norms. In her analysis of students' practices in a real analysis course, Broley identified in one student the position of a Mathematician in Training: the students' comments and activity reflected one whose objective was to join a community of mathematicians.

The research described in this thesis adopts the frameworks and TBI methodologies used in Hardy and Broley's studies of the praxeologies students build in their post-secondary mathematics courses. They had found students to build non-mathematical praxeologies and traced these to the routinization that is institutionally emphasized in calculus courses. My goal is to determine whether students build such praxeologies in other courses that emphasize routinization; as such, I adapted the frameworks and methodology to an introductory college linear algebra course (LA1) offered at a large North American urban university. I expected LA1 to be a suitable basis for comparison based on my experience in teaching this course. I knew, from experience, that the knowledge to be learned in the course—which I took to mean the knowledge needed to pass course assessments—would similarly be marked by routinization of knowledge. Apart from this similarity, this course is regulated by institutional rules[3] similar to those that apply to the college calculus course (Calculus 1) examined by Hardy (2009a): it is a prerequisite mathematics course students need to pass to gain entry to the same university courses that have college calculus as a prerequisite.

A preliminary analysis of tasks from midterm and final exams administered in recent years confirmed the knowledge to be learned in this introductory linear algebra course is similarly characterized by routinization. The preliminary analysis consisted in identifying types of tasks on exams and expected techniques. I used the course textbook to identify expected techniques; this was possible as exam tasks were of the same type as those in the textbook examples and textbook problems recommended on the course outline.

Once I determined this course was suitable as a target for comparison with the prior research, I established my research questions:

- What is the nature of what students are expected to learn in LA1? (Does it align with the nature of what students are expected to learn in Calculus 1?)
- What is the nature of the practices students develop in LA1?
- Research on the learning of calculus has found that students develop non-mathematical practices; are such practices replicated in linear algebra?

I ultimately reformulate these questions in terms of the ATD notion of praxeology; I present these reformulations in Section 3.4 after further elaborating on the theoretical framework for this research.

I used task-based interviews (Goldin, 2000) to identify the nature of students' practices. The first step was to design tasks that could stand to reveal students' praxeologies (that is, the tasks they perform, the techniques they use, and the discourse that produces and justifies their techniques). Based on the preliminary analysis of past exams, and from my experience in teaching the course, I knew students are expected to use techniques from

[3]I elaborate in Section 3.2.1 on the IAD concepts of rules and norms. For now, I distinguish between them as follows: rules are enforced and prescribe which actions are allowed and which are prohibited, whereas norms are precepts that establish what constitutes moral or prudent behavior in an institution.

the topic of linear systems and their solutions to solve various task types in the course. I therefore identified all exam tasks where these would be the expected techniques. I identified features of the tasks (e.g., mathematical objects involved, syntactic norms) to determine properties that aided in their routinization and inform my task design.

I designed eight tasks for use in the TBI. Seven of the tasks could be tackled using routinized linear-system techniques, but other knowledge from the course could be used more effectively (e.g., in terms of number of steps involved and in terms of suitability to the nature of the task). I added one task that did not correspond to any linear-system technique. Among the eight tasks, some resembled routine tasks from the course and others did not. Students were asked to think aloud as they attempted the task, and I followed a semi-structured script to guide my interventions; the aim of these interventions was to help elicit the techniques students were using and the justifications they had for their techniques.

To analyse the praxeologies students mobilized and determine their (non-)mathematical nature, I modelled the knowledge to be learned in the course (in terms of praxeologies: blocks of tasks, techniques, and justifying discourse) and which could be used to complete each task. These models confirmed the linear algebra course emphasized routinization, even if the available mathematics was amenable to routinization in a different way than how mathematics is routinized in calculus courses. In this thesis, I use "routinized" and "normative" and variants on their root words interchangeably. I found that tasks are routine (normal, normative, etc.) in that there is a limited set of task types that occur in LA1 final exams and the variety of tasks that occur in a final exam is stable from one semester to the next. Another praxeological element that is routinized is the way in which technologies are to be used: the types of tasks that are routinized limit what students need to know about each technology. In turn, students' praxeologies revealed surprising ways in which this routinization, along with the nature of the routinized algebraic representations and grading norms from the course, enable students to strip routines of their underlying reasoning and representations of their mathematical meaning. This showed through students' discourse as well as their struggle or failure to adapt routines to non-routine tasks.

I found students' praxeologies were often similar in nature to those observed in the research on calculus learning. The tasks they perceived were sometimes those that were normal in their course rather than the ones that were actually posed; the techniques they mobilized where conditioned by and delimited to the ones routinized in the course; and their justifications for the suitability or validity of their techniques were not reasoned based on the mathematics but rather on what was *usual* in the course. A few exceptions to this came about in select students' responses to certain triggers that signaled a permission to mobilize knowledge in a non-normative way.

By attending to the knowledge students mobilized and through reflection on the operationalization I proposed for the positioning framework elaborated by Sierpinska et al. (2008), Hardy (2009a), and Broley (2020), along with a posterior analysis of this operationalization, I determined how institutional mechanisms regulate the contribution of course norms of routinization to students' mobilization of non-mathematical praxeologies. That knowledge to be learned lends itself to routinization may not, on its own, suffice to push students to build non-mathematical praxeologies. Institutional rules incentivize Student positioning, however, and course norms indicate to Students what behaviors best serve their objective (to get a certain grade in the course). Course norms that institutionalize routinization of knowledge call for behaviors that contribute to the development of non-mathematical praxeologies.

The aim of this thesis was to sharpen the focus on the effect of institutional routinization in students' learning of mathematics. I did this by adopting a framework and methodology from a body of research that investigated routinization through an anthropological and institutional lens; I adapted this approach to examine students' learning in a course also marked by routinization but set in a different domain of mathematics. My findings highlight the importance, in the context of research on the effects of routinization (or other course norms, more generally) on students' learning, of coordinating an investigation of the knowledge transposed in a didactic institution with other institutional mechanisms (such as positioning) that regulate students' mobilization of knowledge.

1.2 Structure of this thesis

This thesis is structured as follows. In Chapter 2, I review literature pertinent to this research; this includes the body of research on calculus learning that motivated this research as well as a review of research on linear algebra education. In Chapter 3, I present my theoretical framework. I present my methodology in Chapter 4 and follow with the analysis of the data produced by this methodology in Chapter 5. The analysis is split into 9 sections, the first 8 of which correspond one-to-one with a TBI problem. My analysis of each TBI problem is divided into three parts: I first present my reference model[4], as a researcher, of the knowledge at stake in the problem; second, my model of the knowledge to be learned in LA1 that is pertinent for the problem; and third, an analysis of the knowledge students mobilized in response to the problem along with a synthesis of what this analysis implies about students' praxeologies. In the last section of the Analysis chapter, I present the analysis of students' positioning. The last chapter of this thesis is a discussion in four parts. In Section 6.1, I synthesize my results relative to each of my research questions. In Section 6.2, I discuss contributions of this work to research on linear algebra education. I finish in Section 6.3 with final remarks: my main conclusions, limitations of this research, and potential avenues for future work.

[4] I define the notion of reference model in Chapter 3.

Chapter 2

Literature Review

The aim of this literature review is to situate my study in the context of the research that led to the questions I aim to answer.

2.1 The affordances of the institutional perspective to the study of the teaching and learning of post-secondary mathematics

I start with an example of how institutional practices can shape the teaching (and learning) of post-secondary mathematics courses. These courses are marked by institutional strategies aiming to deal with the massification of tertiary education. One result is a diverse student population. This implies a diversity in mathematical foundations, interests, and goals. Administrative and academic measures answer the needs of a sizeable and diverse student population: one such measure is universities' offering of prerequisite math courses (e.g., remedial high-school algebra, single-variable calculus courses, an introductory linear algebra course on matrices and vectors). These are courses some students had not yet passed (e.g., in high-school or colleges) but must pass to gain entry into many university programs. The massification of tertiary education also impacts the allocation of financial and human resources. To manage the sizeable student populations of prerequisite math courses, for example, courses are split into several sections of 70 students each, all mediated by common syllabus, curricular documents, and assessment modes selected by a course examiner (a full-time faculty member of the mathematics department). Departmental funding needs may be such that teachers of different sections range from full-time faculty to graduate students teaching the course for the first time. A hefty curriculum has to be covered in a short amount of time (e.g., 3-5 hours per week for 13 weeks) and in some cases is delivered through "lectures" and "tutorials," with the former consisting of sessions (perhaps taught by a teaching assistant and not by the teacher) focused on types of problems set in the material to be covered in the corresponding lectures. These and other didactic conditions shape the teaching of the mathematics identified by educational ministry and departmental teaching committees as the target of these courses.

The aim of this section is to walk through the affordances of an institutional perspective on education, as used over the last few decades, to trace out how students' difficulties in mathematics courses more broadly, and in calculus courses more specifically, can root in institutional practices such as (but not limited to) those in the above example.

Before the turn of the century, research on the teaching and learning of mathematics mainly attended to cognitive and epistemological perspectives: researchers turned to aspects of human biology to make sense of the processes by which mathematics is learned. The institutional perspective emerged in the 1990s as recognition turned to the role of social, cultural, and institutional aspects in the teaching and learning of mathematics (Artigue et al., 2007, summarize the state of research on post-secondary mathematics at the time, including the beginning of research from sociocultural and anthropological perspectives). The shift to an institutional perspective drew on the increasing awareness of the role played by teaching and assessment practices in students' learning experiences (Artigue, 2022). In the context of calculus courses, this new perspective shifted attention toward institutional aspects that constrain and enable the teaching and learning of concepts such as limits (Hardy, 2009a,b) and the completeness property of $\mathbb{R}$ (Bergé, 2008), among other material targeted by calculus courses. This brought new understandings of sources of students' difficulties apart from those in the established library of cognitive and epistemological constraints on the teaching and learning of calculus: epistemological obstacles (Cornu, 1991; Davis & Vinner, 1986; Sierpinska, 1985), concept image (Tall & Vinner, 1981), difficulties rooted in the logic involved in limit definitions (Dubinsky & Yiparaki, 2000), difficulties inherent to the notion of rate of change (Thompson, 1994), difficulties inherent to types of representations (Monaghan, 1991; Richard, 2004), etc.

In this section, I attend first to how the institutional perspective has so far played out on research on teaching and learning in tertiary mathematics education: the basic premises of this perspective, the ways in which teaching and learning of mathematics are organized by institutional elements, and the affordances this perspective brings for addressing issues in post-secondary teaching and learning of mathematics. Second, I address the contributions of the institutional perspective to the study of the learning students accomplish in calculus courses.

Institution is used here in a broad sense: it refers to any structure that organizes social activity. One premise of the institutional perspective is that mathematics emerge from human practices, which, in turn, are institutional and sociocultural; another premise is that learning is both an individual activity and an institutional and sociocultural activity (Artigue et al., 2007). In light of these perspectives, the meaning of a mathematical object is determined by the institution in which it is regarded. These premises are at the basis of various theoretical frameworks set in the institutional perspectives.

I focus my discussion of the institutional perspective on the form it takes through the Anthropological Theory of the Didactic (ATD) and its affordances to research on post-secondary teaching and learning of mathematics. Chevallard (1985, 1991, 1992, 1999, 2002, 2019) developed this framework specifically for mathematics education research, though it has since been used to investigate institutionalized teaching and learning in other domains of human activity. I focus on the affordances of the ATD for two reasons, the first of which is subordinate to the second: first, it is the theoretical framework for the research I set to build on (Barbé et al., 2005; Bergé, 2008; Brandes, 2017; Broley, 2020; Hardy, 2009a; Sierpinska et al., 2008); and second, the ATD is recognized for its important role in the "increasing awareness of the role played by university teaching practices and assessment modes in the difficulties experienced by students" and in the "increasing influence of socio-cultural perspectives in mathematics education research at large" (Artigue, 2022).

I expand on the ATD in Section 3.1 but briefly review its three main assumptions for the purposes of this literature review. A first assumption is that of *didactic transposition*: knowledge that is actually taught and learned is transformed from knowledge that is expected to be taught, which itself is a transformation of some scholarly knowledge (Chevallard, 1985; see also Bosch & Gascón, 2006; Winsløw et al., 2014). This is not to say that any one of these is a subset of the other—Hardy (2009a) shows this indeed is not the case, with knowledge actually learned by students about limits of rational expressions, in a first differential calculus course in a North American university, existing in a different plane from knowledge to be taught or scholarly knowledge about such limits: students classify such limits according to their algebraic appearance, and not according to properties intrinsic to the calculus at stake (e.g., type of indetermination, type of technique to be applied, convergence or divergence, etc.). Knowledge to be taught (i.e., knowledge indicated by curricular documents such as ministerial descriptions of a course or course textbooks) is not simply selected from a library of scholarly knowledge; it is produced and transformed to fulfill certain didactic purposes (e.g., as in how the notion of function is transformed to fit the purposes of a grade 10 algebra course). Knowledge *actually* taught is transformed by virtue of considerations to which teachers must attend and conditions under which they operate (e.g., the teacher's knowledge, amount of class time, class sizes, students' prior mathematical knowledge, national tests, etc.). Other processes (e.g., expected outcomes, as communicated by teachers and curricular documents, etc.) eventually contribute to the knowledge actually learned by students in a course. The notion of didactic transposition therefore implies that to understand learning processes, a researcher must attend to different stages of a didactic transposition of knowledge.

The ATD's second assumption gives a tool through which to capture mathematical activity. The assumption is that human activity (and, therefore, mathematical activity) can be modeled by a *praxeology* (Chevallard, 1999): an organization of activity according to theoretical and practical blocks. The practical block $[t;\tau]$ consists of a *task* t accomplished in an activity and a *technique* τ through which to accomplish t. (These, too, are institutional: for example, Broley (2020) notes how students are expected to use algebraic manipulations to complete limit-finding tasks in a North-American college differential calculus course, whereas the expectation for limit-finding tasks in a first real analysis course in a mathematics degree is for students to use an ε-δ argument.) Chevallard (1999) contends that every activity includes a theoretical component $[\theta, \Theta]$: the discourse, or *technology* θ, that produces and justifies τ, and the *theory* Θ that gives legitimacy to θ. Hardy (2009a) found college calculus students' theoretical blocks need not be mathematical: a validation (θ) for their choice of technique was that it's how the teacher had shown to do a given task; the theory Θ framing this technology is that, as the authority in the course, the teacher determines what is valid or not. The notion of praxeology therefore allows for a concrete, fine-grained analysis of institutional mathematical activity (Winsløw et al., 2014).

The third assumption of the ATD is the importance of attending to the *ecology* of mathematical and didactic praxeologies (Chevallard, 2002; Winsløw et al., 2014): the conditions that enable the development of a praxeology in an institution and the constraints that impede it. Chevallard (2002, 2019) proposes a hierarchy of didactic codetermination—a scale of the levels at which exist conditions and constraints that impact the transformation of a praxeology: humanity, civilization, society, school, pedagogy (the supra-didactic levels), and discipline, domain, sector, theme, topic/question (the didactic levels). This hierarchy can help analyse the ecology of a mathematical and didactic praxeology. Artigue (2022) gives this example:

> Considering the field of functions, for instance, the teaching of a topic such as the variation of exponential functions is shaped by a diversity of conditions and constraints which go beyond those associated with its inscription in a particular theme (exponential functions), sector (transcendent functions of one real variable) and domain (Calculus or Analysis) of the mathematics discipline. It is also shaped by more global conditions and constraints for instance regarding the role given to digital tools (level of pedagogy), the curricular choices which may more or less emphasize connections between scientific disciplines and shape assessment practices (level of school). These choices, in turn, are constrained by society expectations, habits and values (level of society), which, for many of them, transcend a particular society (level of civilization or more in our globalized world).

The lens and tools of the ATD have helped to pinpoint the ways in which mathematics teaching practices and assessment modes have contributed to students' learning difficulties from primary to tertiary education. We'll take a walk through a series of studies done since the turn of the century to exemplify the affordances of the framework for examining post-secondary mathematics; given my overarching aim to review work pertinent to the teaching and learning of calculus, I narrow this walk-through to studies devoted to this discipline.

Praslon (2000) used the ATD to examine technical and conceptual breaches in the secondary-to-tertiary transition in France relative to the notion of derivative in the Calculus/Analysis domain. Artigue (2022) identifies this doctoral thesis as the first research using ATD to study the secondary-university transition. Praslon combined the ATD with already-established knowledge about the teaching and learning of Calculus/Analysis: constructs from cognitive and epistemological perspectives such as the tool and object dimensions of mathematical concepts, the notions of semiotic register, of procept, etc. Praslon constructed and analysed praxeologies from curricular documents, textbooks, teaching material, and assessments. A partial result of Praslon's analysis is the identification of the following breaches in the secondary-to-university transition in the domain of Calculus (relative to the notion of derivative), which informed his design of tasks set in the gap between the secondary and university practices relative to the notion of derivative:

- rapid increase in the introduction of new objects;
- routinization is harder due to a greater diversity of tasks;
- students have greater autonomy in problem-solving and in the choice and use of semiotic registers;
- a new balance between the tool and object dimensions of mathematical objects; and

- greater reliance on definitions for manipulation of objects, more systematic demonstration of results, and proofs acting in the role of mathematical methods.

Barbé et al. (2005) used the ATD to study how institutional restrictions could affect teachers' spontaneous practices relative to the teaching of limits of functions in Spanish high schools. This involved, first, characterizing the mathematical organization around the limits of functions in the knowledge *to be* taught; second, showing the effect of mathematical and didactic constraints on the didactic process; and third, following the observation of 14 sessions in a Spanish high school class (with 15 to 16-year-old students), showing how didactic restrictions shape the knowledge actually taught in class.

Barbé et al. (2005) turned to scholarly knowledge, knowledge to be taught, and knowledge actually taught to highlight the transformation process teachers have to produce to create a mathematical organization for the teaching of the limits of functions. The teacher is informed by the educational institution about what to teach through certain data (curricular documentation, textbooks, assessment tasks, national tests, etc.); these indicate praxeologies that are proposed to be taught as well as pedagogical elements.

Barbé et al. (2005) note that already, in the knowledge *to be* taught about limits of functions, what is left are traces left by two mathematical organizations in scholarly knowledge about limits of functions: one (MO1) is the algebra of limits (starting with the assumption that a limit of a function exists, MO1 is concerned with how to find the value of a limit) and the second (MO2) is the topology of limits (which is concerned with the problem of the existence of the limit of various functions types). MO1 and MO2 are closely related (e.g., the theoretical blocks that produce the calculation techniques for tasks in MO1 correspond to practical blocks in MO2). In textbooks, however, what is left of MO1 is only its practical block, meaning students are not expected to use the theory or technology that frame the practical; and what is left of MO2 are traces of its theoretical block (some definitions and expository comments)—its practical block is absent.

Barbé et al. (2005) identified two didactic consequences of the mathematical organization around limits of functions in the knowledge to be taught. When the teacher chooses what to teach (i.e. what tasks to demonstrate, which techniques to use, and what justifications are needed), the most likely scenario is that the teacher choose MO1 as the only knowledge to teach, as this circumvents the problem of the existence of limits. This does lead to another issue: the broader mathematical organization, where these two "local" mathematical organizations are related, is missing. This leaves the teacher with no motivation for teaching, for instance, the definition of limits of functions.

Knowledge actually taught was identified through students' notes and the teaching practices carried out by the teacher in the classroom. The task types Barbé et al. (2005) identified in what the teacher taught essentially served the technological function of justifying the tasks and techniques for which students would eventually held responsible; but the justifications were of a different nature than those found in the scholarly knowledge at the origin of the knowledge to be taught. For example, the observed teacher chose, as a first encounter with the mathematical organization around limits of functions, the task to find the slope of a straight line; this task served as preparation for a task of type "find the slope of the tangent to a curve in a given point." Altogether, the tasks serving this technological function belonged to the practical block of MO1 as they were essentially variations of the tasks in the practical block students would eventually be expected to know.

The aim of the above few paragraphs was to illustrate some of the didactic restrictions that can arise from the didactic transposition of knowledge about limits of functions, as well as the ways in which a teacher might act as a result of these restrictions. Considering the hierarchy of levels of didactic codetermination, the teacher's actions are at the narrow end of the scale: the teacher has no control over the problems to be raised in a course. For example, the mathematical questions to be asked in an educational institution are determined at the societal and school levels; the sector in which a question is asked is also out of the teacher's control (Barbé et al., 2005, note that the case of limits of functions may be decided to belong to the study of differentiability). The decisions available to a teacher are strictly in how to organize a limited mathematical organization around the limits of functions. Barbé et al. (2005) and Chevallard (2002) note that one of the major outcomes of teachers' limited authority over knowledge to be taught is the disappearance of the raison-d'être for the material being studied.

Barbé et al.'s 2005 study of constraints arising from didactic transposition and the different levels of didactic codetermination, and of how these constraints shape the knowledge to be taught and actually taught in a high-school calculus course, suggests the inevitability of potentially problematic characteristics of what teachers can teach about limits in similarly-structured educational institutions, including, for instance, an absence of mathematical justifications for the activity teachers offer their students. Hardy (2009a) studied students' experience with limits in a calculus course at a different educational institution.

Hardy (2009a) combined the ATD and a conceptualization of institutions from the Institutional Analysis and Development (IAD) framework Ostrom (2005) developed in political science to study the influence of institutional practices on students' perceptions of the knowledge to be learned about limits in a differential Calculus course at a North-American college. Given the heavy weight assigned to final examinations in students' final course grades, given the institutional rule of a common final exam for students enrolled in different sections of the same course, and given the norm that final exams are stable throughout the years (i.e., task types stay the same; functions in limits to be calculated may change but function *types* stay the same), she presumed limit-finding tasks from past final exams to be representative of the knowledge to be learned about limits in the course. She therefore used these tasks and text from the textbook to create a theoretical model of instructors' perception of knowledge to be learned; the praxeologies constructed in this model showed this knowledge consisted of task-technique blocks and an absent mathematical theoretical block (substituted, instead, by an auto-technological technology: the techniques are valid since they are what is to be taught in the course).

To produce the praxeologies that make up students' models of knowledge to be learned about limits, Hardy conducted 28 "task-based interviews" (Goldin, 2000). To this end, she designed tasks that visually resembled final exam tasks, but which differed conceptually. Students' responses revealed they produce "non-mathematical" praxeologies.

One way in which students' praxeologies were non-mathematical was that they identified tasks via surface-level features, usually in the algebraic representation of a function (e.g., rational expressions), instead of calculus-related features intrinsic to the tasks (such as, for example, the type of indetermination at stake). For example, one of the tasks was to find the limit

$$\lim_{x\to 1} \frac{x-1}{x^2+x}$$

The task is to find the limit of a rational function at a number in its domain. Direct substitution suffices. But students acted as if the task was to find the limit of an indeterminate form: they factored the algebraic expressions so as to find some common factor. Students did not pay attention to the entirety of the limit expression (i.e., the number at which the limit was to be evaluated) and focused instead on the algebraic expression: the denominator could be factored. This reflects a strategy that suffices in the knowledge to be learned (as indicated by final exam and textbook tasks): in these tasks, such algebraic expressions (polynomials that can be factored using one of a handful of factoring techniques taught in high school) *usually* occur in tasks where the goal is to evaluate an indeterminate form. This strategy corresponds to one of the mathematically-superficial strategies Lithner (2004) identified to be sufficient to complete most of the exercises in an undergraduate calculus textbook similar in content and pedagogy to popular American calculus textbooks (I return to Lithner's research in Section 2.2 in a broader discussion of the imitative reasoning made available to students through the routinization of tasks in calculus courses).

Students' explanatory comments confirm this feature of knowledge to be learned (i.e., that certain task types have a one-to-one relationship with certain algebraic expressions) as the source of the practice they exhibited in the interview: one justification they gave for their choice of technique was that it was what was "usually" done in the course when they encountered a certain type of algebraic expression. This comment is an example of a second feature Hardy (2009a) found to characterize students' praxeologies as non-mathematical: their theoretical blocks were of social, cognitive, or didactic nature.

Hardy (2009a)'s results were facilitated by her attention to another core theme of the ATD: the part of the ecology of the college Calculus course which consists of the conditions and constraints that come from students' personal experiences, aspirations, and needs. To this end, Hardy (2009a) used the institutional framework elaborated in Sierpinska et al. (2008) and which introduced the notion of institutional positioning. Sierpinska et al. (2008) had blended Chevallard's ATD and Ostrom's IAD frameworks (among others) to produce a framework for the institutional analysis of sources of frustration for students in prerequisite mathematics courses (PMC) at a large urban North-American university. The ATD addresses the formal positions of teacher and student in the educational institution. One construct from the IAD is the notion of the positions available to members of an institution depending on the objectives they aim to achieve. In the case of PMC, Sierpinska et al. (2008) identified the following positions available to students:

- a student behaves from the position of a Student when their actions aim to obtain the objective of passing a course;
- a student behaves from the position of a Learner when their actions aim to obtain the (cognitive) objective of gaining understanding;

- a student behaves from the position of a Client when their actions aim to obtain the objective of receiving a service for which they are paying;
- a student behaves from the position of a Person when their actions aim to obtain personal objectives that are external to the course.

Attending to students' comments in their task-based interviews helped Hardy conjecture about positions they had occupied as participants in their course; this helped to characterize the (non-mathematical nature of the) praxeologies students had developed around limit-finding tasks.

Another use of the ATD to study the teaching and learning of Calculus is in Bergé (2008)'s attempt to find clues for the increased failure rate as students transition from Calculus to Analysis courses. She studied the evolution in the treatment of the completeness property of $\mathbb{R}$ in four successive courses in the Calculus/Analysis stream at a university in Argentina. Using the lens of the ATD, she analysed the mathematical content of the four courses in terms of praxeologies and showed the four courses operated as disconnected institutions. Bergé (2008) noted the value of the completeness property comes through mainly in the work of validating mathematical work, but this is not made explicit in the courses. For example, in Course II, which aims to establish standards for validation, students are asked to prove statements that are obvious to them as they had been expected to treat the same statements by observation in Course I. The change in validation standards is mainly a reflection of changes in didactic contract, as the unreliable nature of geometric intuition and graphical representations is not baked into the tasks with which students are expected to engage. Altogether, Bergé (2008) found that each course had its own perspective on the completeness property, but this perspective was not made explicit, and students did not need to have this perspective to complete related tasks.

Bergé (2008)'s study of the evolution in the institutional practices around a mathematical property at the core of a sequence of Calculus/Analysis courses, like Praslon (2000)'s study of breaches in the secondary-to-tertiary transition relative to the notion of derivative in the Calculus/Analysis domain, fits into a body of research that investigates transition problems in mathematics education to find causes for increased failure and dropout rates in university mathematics. Research based in the ATD to study issues in the transition to and within university has found sources for students' difficulties in the impact of institutional practices on the mathematical organizations of knowledge at different stages of a didactic transposition (Artigue, 2022; Winsløw et al., 2014).

Winsløw et al. (2014) use the ATD to identify two types of transitions students must make as they move to and within university mathematics. Students arriving to a university mathematics program have praxeologies that consist only of a practical block; as they progress in their studies, students are expected to attend to theoretical blocks of the same praxeologies, in addition to the practical blocks for which they were previously responsible. This is a first transition. A second transition has to occur when students' practice has to inhabit what was once theory that justifed and produced their practice: for example, when students in Analysis courses are tasked with proving and engaging with properties that had been absent in their Calculus courses (despite their role in producing and justifying the tasks and techniques with which students engaged).

Brandes & Hardy (2018) elaborate on the placement of a multivariable Calculus course, a second-year course in a mathematics major's program at a large urban North American university, along the transitions outlined by Winsløw et al. (2014). This was a partial result of my master's thesis (Brandes, 2017), where I used the ATD notion of praxeology to model the knowledge students need to learn in this course. By contrasting praxeologies that model the knowledge to be taught (as proposed by the course outline and textbook) with those that model the knowledge to be learned (as indicated by tasks in final exams, which constitute the bulk of the weight of students' final grades in the course, and which had been given and stable over the span of a recent few years), I found the praxeologies expected of students in this second-year course of their mathematics degree align with the activities known to suffice in single-variable differential Calculus courses (Hardy, 2009a; Lithner, 2004): praxeologies sufficient for completing final exam tasks consisted of recognition of task types (from solved examples) and the recollection of appropriate techniques. This multivariable Calculus course therefore does not require students to have made the first of the transitions proposed by Winsløw et al. (2014).

The results reported by Barbé et al. (2005), Bergé (2008), Hardy (2009b), and Brandes & Hardy (2018), along with research on the routinization of tasks in Calculus courses and which enables imitative problem-solving strategies in students' work (I elaborate on this research in Section 2.2), shows that institutional practices in single-variable and multivariable Calculus courses at the start of university programs may enable the development of praxeologies that are non-mathematical. Activities offered to students lend themselves well to the use surface-level features of tasks as a guide for selecting an appropriate technique (as in the example I previously gave from Hardy's 2009a work), and students can pass their courses by learning routinized techniques and without acquiring any of the mathematical theories that produce and validate these techniques.

Broley (2020) set to examine the nature of praxeologies that make up students' activity as they progress in the Calculus/Analysis stream. Framing her study in the ATD, she analysed curricular documents in a first Real Analysis course at a large urban North-American university to determine the knowledge students are expected to learn (so as to pass the course); she used the praxeological models created in this analysis to inform her design of task-based interviews (Goldin, 2000) in a methodology inspired by Hardy (2009a). The purpose of these interviews was to draw out the practices students had developed in the course so as to identify the knowledge actually learned. The results from the 15 task-based interviews she conducted showed students' practices were (non-)mathematical (that is, at times mathematical and other times not, in the sense of reflecting practices that may be viewed as desirable in a general community of professional mathematicians). Broley (2020) identified two elements of the educational institutional at the origin of these (non-mathematical) practices. The first of these is the activities offered to students in the Real Analysis course as well as those offered in previous Calculus courses. Elaborating on a partial result of this doctoral thesis, Broley & Hardy (2022a) discuss a potential didactic obstacle in the compartmentalization of knowledge in the Calculus/Analysis stream: when types of tasks in the Analysis course (such as to "prove a specified limit") resemble types of tasks in Calculus courses (to "find a specified limit"), most students do not activate the knowledge they had been expected to learn in Analysis. A second institutional element Broley (2020) found to contribute to the variably (non-)mathematical nature of students' practices is in the positions (in the sense defined by Sierpinska et al., 2008) they may adopt in a university mathematics course institution and which may variable encourage students to engage in different ways (ranging from non-mathematical to mathematical) with the activities offered to them in their course.

Studies such as those of Barbé et al. (2005), Bergé (2008), Hardy (2009a), Brandes (2017), and Broley (2020) help to show the institutional (as opposed to, say, cognitive) origin of students' difficulties as they transition to and within post-secondary mathematics courses (Winsløw et al., 2014). Mathematical organizations are compartmentalized: topics are organized in terms of "point praxeologies" (praxeologies modeling activity centred on a single task type) and point praxeologies are not integrated into local praxeologies (types of tasks centred on a given technology) or regional praxeologies (point and local praxeologies unified by a given theory) (Winsløw et al., 2014). The compartmentalization of mathematical organizations (e.g., around the notions of limits or the completeness property of $\mathbb{R}$) within and across courses in the same domain (Calculus/Analysis) is such that it is up to students to connect seemingly-disconnected but conceptually related objects in their personal activity.

Bergé (2008), Broley & Hardy (2022a), and Hardy (2009b) show, however, how institutional practices within a course and even across courses set in the same mathematical domain might discourage such activity in students. Didactic constraints result in a mismatch between point praxeologies belonging to the same local or regional praxeologies; this mismatch, a result of institutional conditions, may well be the source of what may have once been construed as a "student misconception" originating in individuals (Winsløw et al., 2014). Broley's 2020 finding that students revert to techniques learned in Calculus for task types in Analysis that resemble Calculus ones (such as limit-proving and limit-finding tasks) gives an example of the potential fall-out from the compartmentalization of knowledge into point praxeologies across courses targeting the same material.

The institutional perspective, the form it takes through the ATD, and other theoretical additions (such as the positioning framework developed by Sierpinska et al., 2008, Hardy 2009a, and Broley, 2020) have increased awareness of the ways in which students' difficulties in post-secondary mathematics can originate in institutional practices, independently of cognitive or epistemological phenomena (Artigue, 2022; Winsløw et al., 2014). The constructs proposed by the ATD—didactic transposition, praxeologies, didactic codetermination—allow for a fine-grained analysis of the teaching and learning of mathematics. Institutional (mathematical and didactic) conditions and constraints can constrict the mathematical organizations available to be knowledge actually taught in courses (Barbé et al., 2005; Bergé, 2008), potentially stripping them down to point praxeologies that are no longer integrated in local and regional praxeologies that are needed to motivate the material students are expected to learn. Similar and other institutional (administrative, academic, didactic, mathematical) conditions and constraints (such as those described in Hardy, 2009a, and Sierpinska et al., 2008) further enable and encourage students to develop praxeologies that, at least in the Calculus courses examined by Hardy (2009a), Brandes (2017), and Broley (2020) and as suggested by other research on the routinization of tasks in Calculus courses (which I discuss in the next section of this literature review), need not be mathematical in nature.

2.2 The effect of routinization on Calculus students' practices

The studies I address in this section have used a partially sociocultural perspective to address the consensus that students' activity in Calculus courses—variously offered in the last years of high school, in the first year of university, or in the in-between, college, depending on the educational system—is characterized by rote learning and an absence of conceptual understanding (Artigue et al., 1990; Cox, 1994; Hiebert, 2003; Lithner, 2003; Orton, 1983; A. Schoenfeld, 1985; J. Selden et al., 1994; A. Selden et al., 1999; Verschaffel et al., 2000; White & Mitchelmore, 1996). While research from the late twentieth century attended to cognitive and epistemological difficulties inherent to Calculus to explain the dearth of conceptual understanding and overreliance on routinization of procedures observed in students (e.g., Sfard, 1991; Tall & Vinner, 1981), the emerging sociocultural perspective took to task the environment that shapes students' engagement with Calculus. This meant a focus on the nature of the school tasks to which students are exposed: tasks demonstrated or assigned to students in class, tasks assigned as homework or recommended as practice, tasks that show as examples and/or suggested problems in a course textbook, and tasks students are given on graded assessments. Altogether, this research has qualified Calculus school tasks as heavily routinized, lending themselves well to mimicry and allowing students to pass their courses with minimal engagement of mathematics intrinsic to the tasks they are given.

To illustrate how research has come to this conclusion, I start in Section 2.2.1 by addressing studies that have classified tasks in curricular materials (e.g., textbooks) by the type of reasoning they enable relative to mathematical properties intrinsic to the tasks. I then address in Section 2.2.2 studies that show learning environments centred on an emphasis on routine tasks in Calculus courses may indicate to students a message to prioritize imitative reasoning—that is, reasoning based in the copying of procedures for routinized tasks and which could allow students to avoid engaging with mathematical properties intrinsic to tasks. I follow in Section 2.2.3 with empirical studies that have documented students' practices to examine their responses to (non-)routine tasks and contrast these with the routinized nature of the majority of tasks in Calculus. These studies overwhelmingly qualify students' activity as rote, as failing to engage mathematical properties intrinsic to tasks they are given, and as failing to include mathematical (problem-solving) activity other than that of activating established procedures. I end in Section 2.2.4 with comments some of the researchers have made on potential implications of this routinized nature of students' activity in Calculus.

2.2.1 Classifications of Calculus school tasks

In this section, I focus on the classifications mobilized by researchers who have sought to characterize the activity enabled and encouraged by learning environments in Calculus courses. I address the results of these classifications in the following section.

Lithner (2004) classified (598) tasks in a Calculus textbook (similar in content and pedagogy to American calculus textbooks) by the level to which reasoning sufficient to complete them depended on the mathematics intrinsic to the components of the tasks. Motivation: "[s]ince it seems that superficial reasoning dominates among Swedish students and they spend most of their learning time with textbook exercises, it is important to try to understand what kind of reasoning that may be enhanced by these exercises." Starting from the premise that in school, it is acceptable to complete tasks using reasoning that is *plausible* but does not guarantee certainty or truth (unlike, for instance, scholarly mathematical activity), Lithner classified tasks according to one of the following:

- *identification of similarities* (IS), wherein the strategy choice is made by identifying in a task surface-level features that resemble those in an example, rule, definition, theorem, or other situation given in the textbook, and the strategy implementation is done by mimicking the procedure in the identified situation;
- *local plausible reasoning* (LPR), wherein at least one of the following holds:
  - the strategy choice is made as in IS, but where components between the given task and identified situation differ in one or a few local parts, and the problem-solver reasons as to the plausibility (i.e., uses "plausible reasoning") that the procedure can copied for the given task; or
  - the strategy implementation is done, like in IS, by mimicking the procedure in the identified situation, but also modifying the few locally-different steps;
- *global plausible reasoning* (GPR), wherein at least one of the following holds:
  - the strategy choice is made by attending to mathematical properties that are intrinsic to the components of the given task, and a solution is produced by plausible reasoning; or
  - the strategy implementation is done mainly by engaging in plausible reasoning based in intrinsic mathematical properties.

If a task could be completed by IS, Lithner classified it as such; if it could not but LPR sufficed, Lithner classified it as an LPR task; otherwise, the task was classified as requiring GPR. Using this approach, Lithner classified 85% of the textbook tasks as IS, 8% as LPR, and 7% as GPR. I return to the results of Lithner's classification in Section 2.2.2, where my aim is to attend to the activity enabled, encouraged, or sufficient for completing school Calculus tasks; the aim for this section is to review the classification systems proposed for analysing tasks to which Calculus students are exposed.

Lithner's classification of tasks has since developed into a conceptual framework about the reasoning that mathematics school tasks enable and/or encourage students to use. Several studies have contributed to the development of the framework and used it to classify tasks in other situations in mathematics education (Bergqvist, 2007; Brehmer et al., 2016; Jäder et al., 2020; Mac an Bhaird et al., 2017; Palm et al., 2006).

To examine the reasoning required of Swedish university students in mathematics on their exams, Bergqvist (2007) used Lithner's 2008 framework to classify over 200 tasks from 16 introductory Calculus course exams. Bergqvist (a student of Lithner's) showed these exams largely consisted of tasks that could be solved by imitative reasoning, one similar in nature to IS—Lithner's original classification had, at this point, grown to distinguish between *imitative reasoning* (IR) and *creative reasoning*, wherein the former includes IS and the latter GPR.

Palm et al. (2006) used Lithner's conceptual framework to classify 1186 assessment tasks in terms of their reasoning requirements in (8) Swedish national tests and in (52) Swedish teacher-made tests at the upper secondary level.

Mac an Bhaird et al. (2017) used Lithner's framework to compare three first-year Calculus courses in two Irish universities. They classified 632 tasks from a textbook using the procedure proposed by Lithner (2008) and Bergqvist (2007): construct, first, a solution to each task, compare it to course notes and textbook examples, and assess which of the reasoning types identified in the framework sufficed to solve the task.

Jäder et al. (2020) used Lithner's framework and same procedure to analyse 5700 tasks in secondary school mathematics textbooks from twelve countries on five continents. In all these studies, the motivation was to determine the reasoning students are encouraged or enabled to use in their course.

Brehmer et al. (2016) adapted Lithner's (2008) conceptual framework on mathematical reasoning to define a notion of "mathematical problem" (MP) as one which requires creative reasoning (plausible reasoning based in mathematical properties intrinsic to components of a task); Brehmer et al. (2016) used this adaptation to classify as MP or not 5722 calculus tasks in mathematics textbooks for Swedish upper secondary school.

Brandes (2017) classified tasks from 12 final exams given in a multivariable Calculus course at a large urban North-American university. The motivation was to examine how the activity encouraged in students of this course stands against the type of activity enabled by and encouraged in single-variable Calculus courses, as characterized by Barbé et al. (2005), Bergé (2008), Hardy (2009a), and Lithner (2000, 2004). To this end, Brandes (2017) aimed to identify the knowledge students needed to have to pass the course; in the case of this course, this amounts to passing the final exam given its heavy weight in the grading scheme. Using the conceptualization from the Anthropological Theory of the Didactic that any activity can be modeled via a *praxeology* (Chevallard, 1999), that is, in terms of the task(s) involved, the technique(s) used to complete the task(s), and the justification and dicourse that produces the technique(s), Brandes (2017) modeled the activity expected of students in terms of tasks, techniques students are expected to use, and the justifications they are expected to provide. Since tasks in final exams do not always indicate the expected techniques, Brandes (2017) organized the model of knowledge to be learned (as indicated by final exams) in parallel to a model of the knowledge to be taught in the course, as indicated by expository text in the textbook and textbook exercises recommended on the course outline. The course textbook was representative of textbooks used for multivariable Calculus courses in North America. Brandes created models of the activity needed to complete the exercises, turning to the expository text in the textbook section associated with a given exercise to identify the expected techniques and justifications.

The results of Brandes' (2017) classification helped to qualify the mathematical activity expected of students in the multivariable Calculus course in language similar to that of Lithner (2003, 2004) and J. Selden et al. (1994), that is, in terms of the level to which imitation strategies are enabled by course tasks and the extent to which students are required to engage mathematical properties intrinsic to components of these tasks.

Similarly to Lithner's 2008; 2017 framework, which classifies tasks by the cognitive load demanded by their conceptual focus (i.e., in terms of the extent to which a technique to a task is routinized in curricular documents), Maciejewski & Merchant (2015) classified tasks in mathematics courses to document their cognitive demand. Maciejewski & Merchant (2015) aimed to explore correlations between students' grades and their study approaches in undergraduate mathematics courses throughout all four North-American undergraduate years at a major Canadian university. Maciejewski & Merchant (2015) used Bloom's taxonomy, a tool for analyzing the cognitive demands of (educational) tasks (e.g., remembering, understanding, applying, analysing, etc.), to classify tasks students experience in final exams of mathematics courses from all four undergraduate years.

Tallman et al. (2016) set to classify tasks in a large number of final exams in Calculus courses at various post-secondary U.S. institutions after noting the gap in knowledge about these tasks in comparison to the classifications amassed about other courses or at other educational institutions (such as the body of research using Lithner's (2008; 2017) conceptual framework to characterize tasks in mathematics courses in Sweden). Tallman et al. (2016) developed the Exam Characterization Framework (ECF) to characterize tasks in a first post-secondary Calculus course ("Calculus 1"). THis framework classifies exam tasks according to three criteria: the cognitive demand required (a modification of Bloom's taxonomy), the mathematical representations used in the task statement and those expected in the solution (applied/modeling, symbolic, tabular, graphical, definition/theorem, proof, example/counterexample, explanation), and the format of the task (on a scale from open-ended to the availability of multiple-choice answers). Tallman et al. (2016) used the ECF to code 150 final exams in Calculus 1; Tallman et al. (2021) set to build on the results of the earlier study by adopting a conceptual focus as they analysed 254 Calculus 1 final exam (altogether comprising 4167 tasks) from various U.S. colleges and universities. Tallman et al. (2021) used the previous ECF in a first phase of analysis to categorize tasks by the item being assessed (e.g., modeling, derivatives, functions, Riemann sums, etc.) and conducted a second phase of analysis to produce precise descriptions of how meanings of these items are expected to be used to reason a solution for the task.

Raman's 2004 classification aims differed from those I addressed so far; Raman (2004) aimed to document how a specific mathematical concept (continuity) was treated in three courses set in the same domain (pre-Calculus, Calculus, and Analysis) so as to explore possible sources for students' difficulties relative to the concept in the learning environment. Raman classified tasks related to continuity in the three textbooks: one for pre-Calculus, one for Calculus, and one for Analysis. Raman chose the textbooks to be representative of popular texts in courses at each level. Motivation: "Many American students have difficulty making a transition from high school to college-level mathematics. This difficulty can be traced, at least in part, to students' beliefs about what mathematics is. Several studies have indicated how students' beliefs can conflict with the beliefs needed to succeed at a particular level (Schoenfeld, 1989; Schommer et al., 1992; Tall, 1992). The focus of this study is on one possible source for conflicting beliefs: the messages sent by high school and college-level mathematics textbooks which, for better or for worse, tend to have a strong influence on the way mathematics is taught and learned."

Raman (2004) aimed to decrypt epistemological messages about continuity from each of the texts. This included an analysis of how each text defined continuity of a function and of how students are expected to use this definition (as indicated by textbook tasks). Raman (2004) categorized tasks by their objective (e.g., in pre-Calculus, one set of tasks had as objective to determine whether functions given by their graphs were continuous or discontinuous over given intervals), by the mathematical objects students are expected to use to complete the task (e.g., in the pre-Calculus task in the previous example, this was graphs and the informal definition of a continuous function as one whose graph can be traced without lifting the pencil), by whether students are actually expected to use the definition to complete the task (or, for instance, a syntactic argument), and whether the tasks help to motivate the (need for the) concept of continuity. Using this categories, Raman (2004) found that each of the three textbooks used "a different definition of continuity with a different purpose to be used by students in three different ways," thereby sending conflicting messages about the concept.

The studies cited in this section conducted classifications of textbook and assessment tasks to document what these tasks indicate about baseline knowledge and ways of reasoning that suffice for students to pass Calculus courses in various educational systems in North-America and Europe. The classifications are largely conceptually-focused in that they set to identify the ways in which mathematical properties intrinsic to components of a task are needed to complete the task. In conjunction with the extent to which such properties are (un)necessary for the completion of these tasks, many of these studies developed frameworks describing their cognitive demand: Lithner's framework classifies tasks by whether they allow for imitative reasoning or require creative reasoning, and Maciejewski & Merchant (2015) and Tallman et al. (2016, 2021) used variations on Bloom's taxonomy to classify tasks by a similar range of cognitive activity (from remembering to creating). These frameworks share in their characterization of available or encouraged activity as ranging from imitation of templates available in curricular documents to production of procedures based on knowledge of mathematical properties intrinsic to components of a task. In the next section, I address the results of these studies along with those of others that examined what is expected of students in Calculus learning environments.

2.2.2 Messages to students from their learning environment that encourage imitative strategies

The results of the Calculus task classifications conducted in (Bergqvist, 2007; Brandes, 2017; Brehmer et al., 2016; Jäder et al., 2020; Lithner, 2004; Mac an Bhaird et al., 2017; Maciejewski & Merchant, 2015; Palm et al., 2006; Raman, 2004; Tallman et al., 2016, 2021) are consistent: the overwhelming majority of Calculus tasks could be solved by mimicking the template offered in the solution of curricular document tasks that have similar surface-level features. The studies variously refer to reasoning based in this strategy as imitative and algorithmic reasoning (Lithner, 2008, 2017), surficial approach (Maciejewski & Merchant, 2015) and problems enabling such strategies as non-mathematical (Brehmer et al., 2016); what the task-classification tasks have found is an emphasis on such problems in curricular and assessment documents. Reasoning based in mathematical properties intrinsic to components of a task is rarely needed; they can be circumvented as soon as a procedure is routinized.

So Calculus tasks can be qualified as *routine*: the tasks students are required to solve are made in the image of tasks to which they are exposed in class and in their textbooks. Tasks offered as practice problems in textbooks or as problems in graded assessments are repetitive in that they mimic one another (within textbook sections), are stable throughout the years (in the case of final exams, which usually consist of the same types of questions from one year to the next, e.g., as per Hardy's 2009a analysis of limit-finding tasks in Calculus exams at a large North-American college), and can be solved by copying templates given in class and in textbooks via examples, definitions, rules, and other situations—in most cases, though, by copying templates given in *solved examples*.

Task-classification studies had turned to textbook (and assessment) tasks to document how they may encourage the superficial student activity observed in various small-scale studies such as those of Artigue et al. (1990), Cox (1994), Lithner (2000, 2003), and Orton (1983). One reason for looking to textbook and assessment tasks is the acknowledgment of their role in shaping students' learning (Lithner, 2004). Bergqvist (2012), for example, cite national Swedish surveys that confirm, first, that students spend large amounts of their mathematics-studying time in solving textbook exercises, and second, that teachers rely heavily on textbooks as a basis from which to develop their lessons. Teachers use textbooks as guides for what content to teach and for how to organize this content (Haggarty & Pepin, 2002; Pepin & Haggerty, 2004). Institutional frameworks such as the ATD (Chevallard, 1985) examine the many layers involved in educational systems and which bequeath onto textbooks this important role in shaping what teachers actually teach.

The classification studies discussed in the previous section show that assessment tasks are made in the image of textbook tasks. This makes it *possible* for students to routinize techniques. Bergqvist (2012), Cox (1994), and Törner et al. (2014) point to elements of Calculus learning environments, other than the deterministic role of textbooks on teachers and students' practice, that may *encourage* students to routinize techniques.

In Cox's 1994 study of first-year university students' retention of core A-level mathematics skills, results show students focus on routine tasks and use superficial reasoning; interviews with A-level mathematics teachers brought up the matter that focus on routine tasks and superficial reasoning is a way of coping with the content overload in A-level syllabi. Teachers report a difficulty of achieving in-depth coverage of core topics because of this content overload, and note that strategic learning allows students to obtain even the highest grades on examinations. Even with the diversity of routine tasks, the excess breadth of the content allows students to drill only select content and still obtain good grades.

Törner et al. (2014) point to another element in the Calculus learning environment that may encourage students to invest in the IR made possible by the bulk of their textbook and exam tasks: a broader norm in national curricula that emphasis in the teaching of Calculus be procedural and unconcerned with the conceptual. From interviews with teachers from France, Germany, UK (England), Belgium, Italy, Greece and Cyprus, Törner et al. (2014) found the emphasis on procedure to be especially true in UK, Greece, and Cyprus. The teaching occurring in classes is usually aligned to national testing philosophies, indicated, for instance, by trends in final examinations. Törner et al. (2014) cite a trend of "teaching students to the test" which leads to a reduction and compartmentalization of content.

Some of these teaching practices may be explained in part by a finding from Bergqvist's 2012 interviews with university mathematics teachers who had produced the final exams analysed by Bergqvist (2007). The teachers were content with the main result of the previous study—that it's possible to pass their exams using only imitative reasoning. This main result of Bergqvist's 2007 study was replicated by Palm et al. (2006). Teachers interviewed by Bergqvist 2012 believed tasks requiring creative reasoning (CR) would be too difficult for students and lead to high failure rates. It is reasonable that such a belief, together with a concern with passing rates, would encourage teachers to construct exams heavy on IR.

In discussing whether CR tasks need automatically be more difficult to solve than imitative reasoning (IR) tasks (which may, for example, require heavy memorization), Bergqvist (2012) points to one characteristic of textbooks that may make it so in the current learning environment. Textbook design contributes to students' lack of engagement with this reasoning: tasks at the beginning of textbooks' exercise sections can be solved via IR, whereas only those at the end of the sections require CR and involve additional (e.g., conceptual) difficulties. This pattern is confirmed by Brehmer et al.'s 2016 classification of tasks in Swedish calculus textbooks. Brehmer et al. (2016) attended to tasks' location in chapters and to their conceptual load: 84.62% of CR tasks appeared at the end of the chapter, were given only in pure mathematical contexts, and were treated as problems that summarize all content to be learned in that chapter. Additionally, only 5.75% of the 5722 Brehmer et al. (2016) had analysed in several Calculus textbooks actually required CR—and the proportion was similar on a textbook-by-textbook basis. Brehmer et al. (2016) note this emphasis on procedural skill and operations is reflected in textbook studies in other countries. Given the placement of already-scarce CR tasks, students are less likely to reach them and therefore unlikely to engage with the reasoning and mathematical knowledge needed for these tasks. The message delivered by the placement of tasks may also contribute to students' avoidance of these tasks: Sidenvall et al.'s 2015 analysis of students' engagement with textbook tasks labelled by difficulty level found students did not attempt the more difficult tasks. Indeed, 84% of students' attempted tasks were among those labelled in the easiest level of difficulty, 16% belonged to the intermediate difficulty, and no attempt was made at tasks labelled as most difficult.

Calculus tasks make it possible for students to pass their courses by acquiring routinized procedure, and messages from authorities in their learning environment—textbooks, teachers, and, if available, past final exams—encourage students to do so. In Section 2.2.3, I address research that has shown Calculus students do predominantly develop the imitative-reasoning-type practices enabled and encouraged by the emphasis on routine tasks in their courses.

2.2.3 The practices students have been observed to enact in response to (non-)routine Calculus tasks

It is long known that students struggle with non-routine (or novel) tasks not only in introductory Calculus courses but in secondary mathematics as well (Orton, 1983; A. Schoenfeld, 1985; J. Selden et al., 1994; Verschaffel et al., 2000). Task-classification studies (Section 2.2.1) and those addressing elements of the learning environment related to the emphasis on routine tasks (Section 2.2.2) propose the struggle students experience with non-routine tasks can be attributed to the disproportionate emphasis by Calculus learning environments on routine tasks. In this section, I address research that has sought to empirically study students' activity in response to non-routine and routine tasks (that is, (non-)routine relative to their Calculus courses) to contrast their actual practices with the practices enabled and encouraged by the emphasis on routine tasks in their learning environments.

Cox (1994) administered tests to first-year university students in the UK to examine their retention of core A-level skills (including differentiation and integration) and followed up in discussions with students, with A-level teachers from several schools, and with academics with knowledge or experience of mathematics teaching. The results of the study suggest it is common for students to pass (and get good grades) on the A-levels by strategically learning the knowledge strictly needed to complete routine tasks and engaging with this knowledge at a superficial level.

A. Selden et al. (1999) found that students enrolled in an ordinary differential equations course after having completed 1-1.5 years of first Calculus course struggled with non-routine tasks: more than half were unable to solve any of the problems administered to them and most were unable to progress toward any solution. This was in spite of engagement with routine tasks in the study which confirmed the students' familiarity with calculus concepts needed to solve the non-routine problems.

In a study aiming to examine students' conceptual knowledge relative to rates, White & Mitchelmore (1996) found that students search for symbols to which they could apply known procedures, without regard for what the symbols represented in the given context, and identified procedures via the symbols used when they were taught.

Lithner (2000) administered two Calculus school tasks—neither purely routine nor non-routine—to students at the end of their first semester of university studies in mathematics in Sweden. Lithner found students' strategy to be controlled by reasoning based in established experiences (EE): experiences established in their learning environment rather than in mathematical properties intrinsic to a task. Lithner's (2003) investigation of undergraduate students' textbook-based homework activity revealed their strategy choices and strategy implementations were controlled by IS reasoning and not based in mathematical properties intrinsic to components in the given tasks. Lithner conjectured the focus on IS reasoning while completing homework with the aid of textbook is a preface to students' reliance on EE reasoning when no textbook is available (as observed by Lithner, 2000).

Boesen et al. (2010) used Lithner's (2008) conceptual framework on mathematical reasoning to analyse the relation between task types and the mathematical reasoning triggered in students by tasks in a national test situation in Sweden. This analysis revealed that test tasks that had important properties in common with textbook tasks triggered EE reasoning in students; test tasks that did not have important properties in common with textbook task triggered creative reasoning founded in mathematical properties intrinsic to components of the task. This finding suggests that students are not doomed to failure when confronted with non-routine Calculus tasks (as suggested by Selden et al., 1994), but that exclusive exposure to routine tasks—or the lack of requirement to ever engage in non-routine tasks—may rob students of opportunities to engage in creative mathematically-founded reasoning.

Sidenvall et al. (2015) used Lithner's (2008) conceptual framework on mathematical reasoning to analyse students' textbook task-solving in Swedish upper secondary school. The analysis revealed students correctly solved 80% of tasks they attempted using imitative strategies. The textbook tasks were labelled according to three levels of difficulties; students made no attempt at those labelled as most difficult and which may have required them to use creative mathematically-based reasoning.

Hardy (2009a) investigated the influence of routine tasks on students' practices. Hardy did this through a study of instructors' and students' perceptions of the knowledge to be learned about limits of functions in a single-variable differential Calculus course at a North American college. The study revealed that the institutional emphasis on routine tasks conditioned students to exclusively expect these types of tasks. For example, given four limit-finding tasks that a mathematician or college Calculus teacher would classify as the same type of task, students only classified three of them as the same type, going by the algebraic symbols used in their appearance to associate the tasks with the procedure usually used in routine tasks involving these symbols, rather than by the calculus implied by the task. Students' reasoning was algorithmic, in conformity with the predictions suggested by the task-classification studies discussed in Section 2.2.1. They identified tasks by surface-level features rather than mathematical properties intrinsic to their components. The absence of mathematical theoretical discourse enabled by the routine tasks, which enable students to rely only on algorithmic thinking and bypass using mathematical properties intrinsic to the components of the task, caused difficulty as students struggled with non-routine problems (as these could not be solved by the routinized techniques) and relied heavily on memorized steps—steps whose requisite order was difficult to remember in the absence of the mathematical discourse that produces them.

Broley & Hardy (2022a) point to another fallout from the routinization of techniques afforded by institutional practices in single-variable Calculus courses at the college and university levels. They report on a partial finding of Broley's (2020) study of the task-solving practices developed by students in a first course in Real Analysis—the first theory-heavy course in the university Calculus stream. An ideal model of students' progression along this stream proposes the focus on theory in later courses builds on familiarity gained by procedure in earlier courses; as elaborated by Broley & Hardy (2022a), however, a didactic obstacle to this idealized model is made evident when students are confronted with familiar tasks. In task-based interviews conducted with 15 students who had recently completed the Analysis course, Broley found that most students reverted to routinized Calculus techniques when given a type of Analysis task (e.g., "to prove a specified limit") that resembled a type of Calculus task (e.g., "to find a specified limit").

Hardy (2009a) and Broley (2020) incorporated into their study a conceptualization of institutions from a political science framework (Ostrom, 2005) and which allowed for an analysis of institutional mechanisms that can help to trace the way in which routine tasks shape students' practices. Two of the mechanisms, for example, are rules and norms: institutional rules describe the behavior required of members of the institution and sanctions exist to enforce members' compliance; institutional norms describe the behavior usually accepted and expected by members. Hardy (2009b) elaborates on the mixture of mathematical, social, cognitive, and didactic norms that constituted part of her (28) task-based interview participants' models of the knowledge to be learned. In comparing these students' models with teachers' models, Hardy (2009a) found that mathematical justifications and descriptions of tasks in teachers' models were substituted by non-mathematical (i.e., social, cognitive, didactic) elements. The latter reflected norms enabled by the exclusively-routine tasks students had been administered in their Calculus course. Broley (2020) cites a comment, made by one of her participants, which exemplifies how institutional practices other than routinization of tasks might nevertheless encourage students to strategize by routinizing tasks: in anticipation of the little amount of time available for problem-solving during final exams, students "grind problems at home" so they may recognize tasks and activate a routinized technique (i.e., as in Lithner's 2017 notion of algorithmic reasoning based in established experiences).

In response to the large body of research raising alarm on the emphasis on procedure in the first years of undergraduate mathematics courses, Maciejewski & Star (2016) warns of the false dichotomy insinuated by pitting emphasis on procedure against emphasis on conceptual knowledge. Research shows heavy *routinization* of procedure in tertiary mathematics education (Maciejewski & Merchant, 2015) restricts students' ability to engage with novel contexts. The issue is less so with emphasis on procedure than it is with its routinization, which enables rigid use of procedures and fails to require that procedures be learned with depth and flexibility (Maciejewski & Star, 2016). In a study of university students' justifications for the steps they take in row-reducing matrices (a procedures these students will have learned in their introductory linear algebra course), Maciejewski & Star (2019) found that even within this routinized activity, the freedom to make certain decisions along the way leaves room for students to mobilize various criteria for what they perceive as appropriate for a solution. Maciejewski & Star (2016) posit that flexible procedural knowledge can be taught: the argument is based on findings from an intervention designed to help undergraduate Calculus students use procedures flexibly. Participants in the intervention were students from two sections of the same course; one section was randomly assigned a treatment and another served as a control. Treatment consisted of a set of derivative-finding tasks which had students use alternative methods and compare resulting solutions. Control students were given a set of functions to differentiate. Treatment students were more likely to use flexible procedural knowledge (i.e., various procedures) without prompt than students from the control group.

My aim in this section was to outline the practices students have been observed to enact in response to (non-)routine Calculus tasks. Boesen et al. (2010), Broley & Hardy (2022a), Cox (1994), Hardy (2009a), Lithner (2000), Orton (1983), A. Schoenfeld (1985), J. Selden et al. (1994), A. Selden et al. (1999), Sidenvall et al. (2015), Verschaffel et al. (2000), and White & Mitchelmore (1996) show the institutional emphasis on routine Calculus tasks is indeed accompanied by students developing strategies that capitalize on this emphasis: students learn a restricted set of procedures, recognize tasks not by mathematical properties intrinsic to their components (such as type of indetermination, in the case of limit-finding tasks) but by surface-level properties such as algebraic symbols that resemble those used in textbook examples for a certain task type, and do not learn the mathematical justifications that produce the procedures they mimic as these are unecessary for the short-term goal of passing their exams. These strategies limit students' ability to adapt to different types of tasks and even to recognize that a given task is different from a routine one when these look alike (Broley & Hardy, 2022a; Hardy, 2009b). Routine tasks in Calculus enable students to use procedures in rigid ways and side-step acquiring knowledge that is ostensibly part of the intended curriculum.

2.2.4 The heavy routinization of Calculus tasks enables students to succeed by operating only along surface-level features of a restricted set of procedures; so what?

A little learning is a dangerous thing;
drink deep, or taste not the Pierian spring:
there shallow draughts intoxicate the brain,
and drinking largely sobers us again.

Alexander Pope, 1902

Some of the researchers cited in this section identified criticisms that could be levelled against the alarmist tone of research that has revealed the diagnosis of students' knowledge as superficial and their practices as imitative—conditions seemingly contracted by the routine tasks they are administered in their mathematics courses. In this section, I gather from the literature explanations that address these criticisms one by one. The last remark in this list is one I have made and which has determined the research objectives for this dissertation.

Criticism 1: Routinization is not non-mathematical. While some have classified as "mathematical" only tasks that require a modicum of creative reasoning (e.g., Brehmer et al., 2016), the implication in the body of research examining the routinization of Calculus school tasks is by no means that routinization itself is amathematical. It is, after all, an affordance of results in mathematics, and recall of similar tasks and related structures has an important role in problem-solving (Lithner, 2004; Maciejewski & Star, 2016; Pólya, 1945). The use of the term "mathematical" to describe only problems that require creative reasoning beyond imitation of algorithms for similar tasks, as in the work of Brehmer et al. (2016), or to describe a practice that is not exclusively restricted to imitative reasoning and reliance on surface-level features of tasks, as in the work of Broley & Hardy (2022b), is rather a nod to two characteristics a practice incited by a problem may (fail to) have: one, that it be guided by consideration of mathematical properties intrinsic to components of the problem, and two, that it involve various behaviors known to be productive for problem-solving in mathematics (Schoenfeld, 1985, and Mason, 2016, are but two examples among a large collection of research about problem-solving behaviors and ways of thinking useful in varied domains of mathematics).

It's not that routinization and the imitative reasoning it affords are de facto amathematical. The issue is that it's the only activity students are encouraged to develop by the tasks they are offered and the institutional conditions under which these are administered, such as limited time across the board - from class time, to time in which to acquire an excess of diverse content, to the time available during in-class exams (Broley, 2020; Cox, 1994; Sierpinska et al., 2008). Students' activity is restricted to reasoning guided by expected experiences (Lithner, 2000). This comes at the expense of plausible reasoning based in mathematical properties of the components of a task. When confronted with a task that looks like a familiar one but differs in substance, students resort to procedures that had always worked (in their course) for the superficially-similar task even as these procedures may be more complicated or altogether inappropriate for the given task (Broley, 2020; Hardy, 2009a). When confronted with tasks that are obviously non-routine, students struggle, having no established experience from which to draw an algorithmic approach (Hardy, 2009a; Lithner, 2000, 2003; J. Selden et al., 1994; A. Selden et al., 1999).

Some of the studies that exposed students to non-routine problems showed that students are not incapable of engaging in creative mathematically-based reasoning (Lithner, 2017). Boesen et al. (2010), for instance, showed that while Swedish national test tasks that resembled those from textbooks did trigger EE reasoning in students, tasks that did not have important properties in common with textbook tasks did trigger creative reasoning based in mathematical properties intrinsic to components of the task. Hardy (2009a) similarly noted from her participants' activity in task-based interviews that they were not "doomed to failure" when confronted with non-routine tasks, as the obviously non-routine nature of tasks prompted students to attempt to engage with them in non-imitative ways. These two studies found students to engage in non-imitative reasoning in a circumstance with a shared characteristic: students had no choice but to engage with these non-routine tasks. This does not typically occur in Calculus courses. Given the institutional emphasis on routine tasks, there is no motivation for students to opt, for example, for those end-of-section exercises known to require activity beyond that expected of students. Sidenvall et al.'s (2015) study gives evidence of what likely happens as students spend time studying for their course by doing textbook exercises (of which the overwhelming majority are amenable to algorithmic/imitative reasoning): given the choice between tasks known to be "easier" and those known to be "hardest," students simply do not attempt the latter.

No, routinization is not non-mathematical. It has heuristic value. But courses that exclusively encourage routinization enable students to avoid engaging with mathematical properties that are intrinsic to the tasks they are given. Students' difficulty with non-routine tasks that could be solved using concepts from their course show students pass their courses without having to acquire substantial portions of knowledge that are ostensibly part of the intended curriculum.

Criticism 2: Sure, students are limited in their activity at this level, but clearly some students—namely, those who go on to graduate studies in mathematics and beyond—manage to escape the shackles of routinized tasks. Sure. Those who go on to graduate studies in mathematics, however, form a significantly small portion of students required by university rules to complete Calculus courses. This leads to question whether the superficial knowledge and procedural rigidity students develop in these courses meet the objective behind the rule that students must pass first courses in Calculus to gain entry into various programs apart from mathematics. Second, several of the studies cited herein were conducted with an empirical problem in mind: transition issues marked by high failure and dropout rates as students progress from secondary to tertiary mathematics as well as within tertiary mathematics (Artigue et al., 2007; Artigue, 2022; Barbé et al., 2005; Bergé, 2008; Brandes, 2017; Broley, 2020; Raman, 2004; Winsløw et al., 2014). I address two transition issues suggested by the literature to be sourced in the routinization of Calculus tasks.

One consequence of the institutional and exclusive emphasis on routine tasks is that, in enabling students to mimic established procedures, students are not confronted with genuine problems in the sense proposed by Schoenfeld (1985) and Mason (2016). This is a missed opportunity to confront students with problems that could help to motivate the knowledge they are expected to learn. This is an issue as students transition into higher-level mathematics courses such as Analysis. For example, Broley (2020) found students can fail to grasp the need for using newer constructs such as formal definitions when routinized procedures from Calculus suffice. Bergé (2008) similarly noted the potential issue in knowledge being compartmentalized into routine tasks across four courses in the Calculus stream (from a single-variable differential Calculus course to a course on metric spaces). Bergé's (2008) analysis revealed that each course compartmentalizes knowledge about the completeness of $\mathbb{R}$ into certain types of tasks where the completeness property does not always need to be treated as intrinsic to the task. The link between the tasks across the courses (the link being their root in the completeness property) is not made explicit to the students, who are then left to their own devices to even become aware of a thread being attempted to be woven from one course to the next to develop in them a fuller conception of the completeness of $\mathbb{R}$. Raman (2004) has similarly found the treatment of the notion of continuity across courses (pre-Calculus, Calculus, and Analysis) to suffer from the curse of tasks that do not require students to engage with mathematical properties intrinsic to *continuity*, having students engage rather with other mathematical components instead (e.g., as in pre-Calculus where tasks that appear to be about classifying functions as continuous or not are in fact about identifying intervals on a graph—an exercise serving to strengthen algebraic concepts rather than that of continuity). The tendency to turn Calculus tasks into algebraic ones is similarly noted by (Hardy, 2009b), here pointing to how routine tasks can lead to the erasure of knowledge intrinsic to Calculus (the purported aim of the course).

A second consequence of the institutional and exclusive emphasis on routine tasks is in the message it sends to students about mathematics, about what it takes to learn mathematics, and about what it means to do mathematics. Studies have shown how students' beliefs can be at odds with what it takes to succeed at certain levels (Schoenfeld, 1989; Tall, 1992). Raman (2004) suggests that the epistemological messages from high school and college-level mathematics textbooks and the tasks they offer to students can contribute to conflicting beliefs, giving as one example the suggestion, in pre-Calculus textbooks, that definitions are appendages fit for amputation come task-solving time (with textbooks disguising an informal description of continuity as a definition and failing to call on this definition in the solutions illustrated for tasks of determining the continuity of functions), which eventually conflicts with the norm in Analysis textbooks whereby definitions directly produce the techniques through which similar types of tasks are completed.

Mesa et al. (2012) note another way in which (college algebra) textbooks can shape students' beliefs about mathematics: the disproportionate emphasis on less cognitively-demanding tasks limit students' opportunities to learn about skills needed to develop problem-solving strategies. Lithner (2000) similarly notes that the emphasis in textbooks in exercises that enable imitative reasoning can contribute to a belief that to do mathematics is to follow ready-made procedures. Schoenfeld (1992) notes the common belief that students cannot expect to actually grasp mathematics or engage in any behavior other than imitating what a teacher demonstrates. This belief is likely reinforced by textbook design norms such as that found by Brehmer et al. (2016) in the analysis of Swedish upper secondary school textbooks: tasks that cannot be solved by imitative reasoning (IR) and rather require creative mathematically-founded reasoning (CR) are scarce and pushed to the ends of exercise sections. Palm et al.'s (2006) reflection shows how the message sent by textbooks about IR and CR tasks is reinforced by teachers' tests: "memorisation of facts and procedures based on superficial properties of the tasks is a competence that is sufficient for handling mathematical tasks. In fact, this kind of superficial reasoning may to a large extent be interpreted to define the school subject mathematics." A belief that the construction of solutions is outside the purview of students, belonging solely to that of teachers and textbooks, may impinge on students' sense of agency (Sierpinska et al., 2008) and willingness to attempt constructive reasoning (Sidenvall et al., 2015).

In sum: the routinization that operates along surface-level features of tasks and techniques can contribute to students' difficulties as they transition within tertiary mathematics in at least two ways. One is the resulting compartmentalization of knowledge (into routine tasks), as it conflicts with the presumption teachers may have that students' familiarity with concepts from earlier courses helps them progress in their engagement with these concepts, when in reality the established experiences may even impinge on students' activity with the same concepts in later courses. Another is the beliefs that may be created and reinforced by the institutional emphasis on routine tasks: they enable and encourage students to mimic ready-made procedures, contributing to the belief any remotely constructive activity is beyond the scope of a student's capacity and conflicting with the activity that is eventually expected of students as they progress in university mathematics courses.

Caveat: studies focused on the emphasis of routine tasks at the post-secondary level have mainly considered pre-Calculus and Calculus courses, and studies that empirically contrasted these routine tasks against students' practices have only done so in the Calculus context. The literature on the institutional emphasis on routine tasks and its potential impact on students' practices at the early stages of university mathematics has been done almost exclusively within the domain of Calculus. In the last section of this literature review, I attest to the absence of research that has similarly examined the practices students develop in introductory Linear Algebra, the only other mathematics course as widely-required as a prerequisite to university programs as differential and integral single-variable Calculus. Given the vast documentation of the cognitive and epistemological difficulties inherent to concepts students are exposed to in these early Calculus courses, it is reasonable to wonder whether, in the mixture of cognitive, epistemological, and institutional features of Calculus courses that can explain students' disengagement from mathematical properties intrinsic to calculus tasks, the institutional emphasis on routine tasks has a dominating role in this cocktail.

Is there a similar institutional emphasis on routinization in a course set in a different domain of mathematics but at a similar stage of students' mathematics studies? Based on my experience teaching an introductory Linear Algebra course on matrices and vectors, offered in both colleges and universities in North America, it seems there is. This is also suggested by Maciejewski & Merchant (2015) in their report on a classification study of tasks in mathematics courses set across all four North-American undergraduate years. If there is an institutional emphasis on routinization in a non-Calculus course given at a similar stage in students' mathematical studies—does this emphasis present in students' practices in the same way it does in their Calculus courses?

2.3 Research on the Teaching and Learning of Linear Algebra

Research on the teaching and learning of linear algebra started in the late 1980s when researchers in mathematics education noted difficulties students were having with linear algebra concepts (e.g., Harel, 1989; Hillel and Sierpinska, 1993). The publication of the influential volume *On the Teaching of Linear Algebra*, edited by Dorier (2000a), came soon after amid a wealth of research about cognitive and conceptual sources for students' difficulties in the discipline. Stewart et al. (2019) give a more recent report on the global state of research in the field of linear algebra education. The report appears in a special issue of ZDM Mathematics education devoted to research on the teaching and learning of linear algebra. The issue was published as an extension of discussions at the 13th International Congress in Mathematics Education (ICME 13) and the ICME 13 series volume. Stewart et al. (2019) synthesize themes, questions, results, and perspectives in the papers from this issue as well papers published between 2008 and 2017 identified by surveying the 20 most cited English language journals covering mathematics education research, along with the International Journal for Research in Undergraduate Mathematics Education and Linear Algebra and its Applications.

I draw from Dorier (2000a) and Stewart et al. (2019) to summarize the frameworks most commonly used in the field (Section 2.3.1) and the themes most commonly addressed in this research (Section 2.3.2). A survey of papers published in 5 majors journals in mathematics education research[1] confirmed the trends in research on linear algebra education have held since Stewart's 2019 report. I follow with observations on gaps in the state of the field that my study seeks to fill (Section 2.3.3).

[1]ZDM Mathematics Education, Educational Studies in Mathematics, International Journal of Research in Undergraduate Mathematics Education, Journal of Mathematical Behavior, and Recherches en Didactique des Mathématiques [Research in the Didactics of Mathematics]

2.3.1 Frameworks commonly used for research on the teaching and learning of linear algebra

Some of the earliest attempts at identifying sources for students' difficulties in linear algebra did so through a historical and epistemological lens (Dorier, 2000b). One of the main results of this research looked at the last stage in the development of the field, after 1930: the axiomatization of linear algebra. The theoretical reconstruction and new axiomatic central theory did not, on its own, allow for the resolution of new problems (with some exceptions; Dorier & Sierpinska (2001) note that of problems in non-denumerable infinite dimension). The power of the development was rather in its unifying and generalizing effect and the consequent simplification of methods for solving problems in mathematics.

There are two stages in unification and generalization:

- recognition of similarities among components of a situation (e.g., objects and methods); and
- reorganization of old knowledge through the construction of an object that makes explicit a unifying and generalizing concept.

It's through these necessary stages in the unification and generalization process that Dorier (2000b) identifies a source for students' difficulties. If a problem is accessible to a student of an introductory linear algebra course, then that problem can be solved without axiomatic theory. These students' knowledge needs to somehow be assimilated for their similarities to become visible, and the gains brought by unification, generalization, and simplification are not within reach of novices. The structural aspect of linear algebra can therefore seem gratuitous to students.

The work done in the direction of epistemological and historical analyses frames studies that seek to generate solutions to problems of the type described above. Some push to axe theory of vector spaces from undergraduate linear algebra courses; for example, Hillel (2000) questions the purpose of including it in courses where the focus is in finite dimensensional spaces and where the isomorphism between $\mathbb{R}^n$ and n-dimensional spaces is on the lineup of knowledge to be taught. Others have explored strategies for equipping students with tools through which to grasp the need for axiomatic theory. For example, (Dorier et al., 2000) propose "meta-level activities," including explicit discourse from the teacher about axiomatic concepts, about the unifying and generalizing significance of the theory, and about its methodological affordances. Discourse from the teacher may help students acknowledge that axiomatic structure is a thing of importance to mathematicians, and some students may even be convinced of its significance, but discourse on its own is unlikely to turn the axiomatic structure of linear algebra "part of the 'cognitive furnishing' of the students' minds" (Dorier & Sierpinska, 2001). For such discourse to not diffuse into didactic ether, it needs to be accompanied by mathematical problems that expose students to the affordances of axiomatic theory.

Another lens through which sources for students' difficulties are examined deals with the language of linear algebra (e.g., Hillel, 2000; Duval, 2006). Apart from the difficulties students encounter when confronted by mathematical activity driven by formal language (definitions, theorems, etc.), there are those rooted in the geometric, the algebraic, and the abstract components of linear algebra (as Hillel, 2000, puts it) or the graphical, tabular, and symbolic registers of this language (Duval, 2006).

"Geometric" language consists of objects in two- and three-dimensional spaces (line segments, lines, planes, etc.); "algebraic" refers to objects in $\mathbb{R}^n$ more broadly (e.g., n-tuples, matrices, rank, linear systems and their solutions, etc.); and "abstract" is that of the broader theory of vector spaces (e.g., vector spaces, dimension, eigenheory, etc.). Hillel (2000) showed students struggled with the 'opaqueness' of each language (e.g., an n-tuple in $\mathbb{R}^n$ relative to a standard basis versus its representation relative to a non-standard basis) and with teachers' unexplained shifts between the languages.

Duval's (2006) notion of semiotic representations does with the productions needed to capture mathematical objects—objects that can't be directly perceived. Semiotic representation have double use: communication and cognition. The productions made using signs from a system of representation are needed to communicate and for mental processes. Duval distinguished two components acting together in any cognitive process involving a mathematical object: one, the comprehension or production of a representation (via signs), and two, the conceptual comprehension of the represented object. Duval and researchers who built on his framework claimed that, among the activities involved in the cognitive processing of an object, teaching does not address the conversions made to transform semiotic representations from one register to another (e.g., graphical, such as arrows, tabular, such as n-tuples, and symbolic, such as an abstract vector in a vector space).

Language and registers are among the characteristics of linear algebra that determine ways of thinking needed to understand it. Reflecting on research done so far on the subject, Dorier & Sierpinska (2001) identify cognitive flexibility (e.g., as explored in the work of Alves Dias and Artigue, 1995, which I give as a next example), trans-object level of thinking (Hillel & Sierpinska, 1993), theoretical and practical thinking (Sierpinska, 2000), and the synthetic-geometric, analytic-arithmetic, and analytic-structural modes of thinking in linear algebra (Sierpinska, 2000). To help convey some of the ideas addressed here, I elaborate on the first and last in this list.

Alves Dias & Artigue (1995) found students did not have the cognitive flexibility needed to move freely between semiotic registers (building on the work of Duval, 2006) and in conjunction with various concepts (e.g., such as the integration of the concepts of rank and duality in problems involving transformations between Cartesian and parametric representations of vector spaces). Alves Dias & Artigue (1995) observe that linear algebra textbooks generally do not expose students to tasks that require cognitive flexibility.

Sierpinska's (2000) modes of thinking run parallel to the three languages of linear algebra. Synthetic-geometric thinking is that needed to engage with objects that seem directly accessible (e.g., lines in 3-space); analytic (-arithmetic or -structural) is that needed to engage with objects constructed by a language and conceptual system. Sierpinska (2000) refers to studies that showed a point of difficulty for students is wielding control over the three different modes, knowing where each is appropriate and moving flexibly (Alves Dias & Artigue, 1995) between them.

Tall's (2013) three worlds of mathematical thinking—embodied, symbolic, and formal—similarly deal with ways of thinking needed to coordinate activity in linear algebra; Tall's three worlds are commonly used to frame research on the learning of linear algebra (Stewart et al., 2019). Tall (2013) distinguishes between the 'conceptual-embodied' world of perception (developed cognitively), the 'proceptual-symbolic' world of calculation and algebraic manipulation, and the 'axiomatic-formal' world of concepts (captured by definitions) and mathematical proof.

APOS theory (Dubinsky & McDonald, 2001) is a framework frequently used in research on the learning of linear algebra (Stewart et al., 2019) and in other mathematical fields as well. The theory classifies mental constructions an individual can have relative to a mathematical object: action, process, object, or schema. An *action* perception of an object is restricted to step-by-step instruction on how to perform an operation involving that object. A *process* is a mental construction wherein an individual can reflect on an action without being restricted to a specific object on which that action can be performed (e.g., a process construction of *function* views it an input-output machine, whereas an action construction is limited to a particular function, as represented by some specific formula). An *object* construction occurs when a process is viewed as a whole which itself can be acted upon, and a *schema* is developed as an individual forms and links actions, processes, objects, and other schemas relative to a given concept. Aydin's (2014) application of APOS theory to linear algebra education had students generate examples: this study found that students' focus for determining linear (in)dependence was mainly on row reduction processes, and not on the structure of linear combination relations between the vectors.

Finally, another type for framework used in much research on linear algebra education deals with how mathematics should be taught. One commonly used framework is Harel's (2008) DNR perspective (duality, necessity, and repeated reasoning), which posits three principles for the teaching of linear algebra (and mathematics more broadly) to respond to learners' intellectual need (e.g., as produced by a state of being puzzled) and respect the mathematical integrity of the content taught. Another frequently-used framework is that of Realistic Mathematics Education (RME), a framework refined by various researchers in the 1990s and which views mathematics as a human activity that can be meaningful for students. RME proposes that 'realistic' situations be centred in the didactic process—realistic in that students can imagine them.

2.3.2 Themes most commonly studied in research on the teaching and learning of linear algebra

In this section, I list the themes most commonly addressed in the research on linear algebra education. I emphasize themes more pertinent to this dissertation.

Earlier works on the teaching and learning of linear algebra contributed to the development of some of the conceptual and cognitive frameworks listed in the previous section. This was mainly research that looked to identify sources for students' difficulties in epistemological and historical analyses, in analyses of the language of linear algebra, and characterizations of thinking needed to understand linear algebra; in accordance with these analyses, some studies into the teaching of linear algebra proposed principles for its teaching and examined how they are violated in practice, others investigated the affordances and limitations of geometry-based linear algebra courses, others conducted (long-term) teaching experiments, and some examined student-tutor interactions in relation to a linear algebra course (Dorier, 2000a; Dorier & Sierpinska, 2001).

In the decades since, geometry has retained its prominence in research on the study of linear algebra. This includes research on the role of geometry in students' understanding of eigenvalues and eigenvectors, linear independence, and linear transformations; studies were framed in various frameworks (including, e.g., Sierpinska's (2000) three modes of thinking, Tall's (2013) three worlds of mathematical thinking, and Duval's (2006) semiotic representations). Some teaching experiments (e.g., with software) showed geometry could help students develop understanding and other results show students perform better in more routine algebraic problems (Stewart et al., 2019); this extends on the earlier observation made by Dorier & Sierpinska (2001) that, while geometry can help students grasp more abstract concepts, its introduction prior to algebraic concepts in a linear algebra course can be counterproductive. The relation between linear algebra and geometry, from an epistemological perspective, is not as natural as often presumed by teachers and researchers (the basics of vector space theory were produced in the context of linear equations or field theory—not in a geometric context—though the role of geometry in the genesis of linear algebra grew with the development of functional analysis) (Dorier, 2000a); Chartier's study (see Dorier, 2000a, pp. 262-264) of textbooks from various countries and across different periods, together with questioners for teachers at students at different levels of university, showed geometry can even act as an obstacle in the learning of linear algebra, with the use of geometry often being superficial.

Many of the papers surveyed by Stewart et al. (2019) were framed by APOS theory and only a few studies focused on proofs in linear algebra. Many papers focused on students' content-specific understandings. Much attention is given to topics related to linear combinations of vectors, especially those of span and linear independence. Some focus is on students' understanding of eigen theory, and very few studies focused on systems of equations; Stewart et al. (2019) note the scarcity of work done on students' geometric and algebraic understandings of systems and their solutions. There are similarly few papers investigating students' understandings of properties of linear transformations and orthogonality.

A last set of studies converge under the heading of research on instructional innovations and analyses, including the role of technology (Stewart et al., 2019). This includes general instructional approaches (such as flipped classrooms) and their relations with grades and student affect. Two studies on instructors' practices (reasoning, moves, role, and collaboration) suggest phenomena similar to that of routinization of tasks observed in calculus courses. Grenier-Boley (cited in Stewart et al., 2019) found that a priori institutionalization of ideas sometimes transformed exercises into "simple and isolated tasks" that stopped "students from being exposed to the main difficulties of these concepts." Rensa's study (see Stewart et al., 2019) of an engineering student's notes in a linear algebra class showed the student used these notes mainly to develop instrumental understanding (e.g., to find solutions). Other research on instructional approaches include application-based approaches (often drawing on the RME framework) and studies on the role of technology. Of note, Stewart et al. (2019) observe there is room for research on the use of textbooks and online videos as resources for learning, as well as of homework materials: their content, presentation of content, and how they are used by students and instructors.

2.3.3 Gaps in linear algebra education research that are pertinent for this dissertation

> It is commonly claimed in the discussions about the teaching and learning of linear algebra that linear algebra courses are badly designed and badly taught, and that no matter how it is taught, linear algebra remains a cognitively and conceptually difficult subject. This leads to (a) curricular reform actions, (b) analyzing the sources of students' difficulties and their nature, as well as (c) research based and controlled teaching experiments. (Dorier & Sierpinska, 2001)

Theory of vector spaces was introduced to secondary level education in the 1960s in many countries and constituted one of the main ussies of the reform of modern mathematics education of that period (Dorier, 2000a). The study of linear equations was overshadowed by formalism and axiomatic theory; these became a unifying and generalized model for all linear problems (finite and infinite-dimensional). It was thought that the mathematical simplicity offered by this formalism would translate to mathematics that is simpler and more accessible to students. It shortly became clear this formalism was a great source of difficulties for students and the reform movement was progressively abandoned in the early 1980s. Vector space theory was now consecrated for first-year mathematics studies in university (Dorier, 2000a).

A curricular reform movement for the teaching of linear algebra (at the tertiary level) began in the United States in 1989; in 1990, a group of mathematicians formed a study group (the Linear Algebra Curriculum Study Group, or LACSG) to propose recommendations based on teaching experiences and in research on algebra in the United States (Dorier & Sierpinska, 2001). Curricular changes similarly went underway on a global scale, and countries where the teaching of linear algebra had previously emphasized theory (e.g., as in France, Poland, and Morocco) turned to a focus on numerical computations. These changes were informed by the studies discussed in the seminal work edited by Dorier (2000a) and which has led to many of the frameworks that are now often used to frame studies in linear algebra education: Dorier's epistemological and historical analysis and resulting observations about difficulties inherent to the unifying and generalizing aspect of linear algebra, Hillel's modes of description and the problem of representation in linear algebra, Sierpinska's three modes of thinking and practical versus theoretical thinking, Harel's principles for the teaching of mathematics (his 2008 DNR framework developed in consequence to earlier work; in fact, Harel was the main mathematics education researcher in the LACSG), the need for cognitive flexibility, the role and place of geometry in the teaching of linear algebra, etc.

The frameworks and themes that typify research in linear education research are centred on conceptual and cognitive sources for students' difficulties in the domain and otherwise focus on teaching experiments. The institutional perspective that had started to gain traction in research on calculus education, in light of the increasing recognition in the 1990s of social and cultural contexts' role in students' learning trajectories, is practically absent from this research. Only one study among those surveyed by (Dorier, 2000a) used this perspective: Behaj's (1999) doctoral dissertation (discussed in the work of Dorier, 2000a) used the Anthropological Theory of the Didactic to investigate how university instructors' structuring of knowledge and institutional constraints may shape how their students structure their knowledge and use it for problem-solving. The experimental portion of the study included one-on-one interviews with teachers, asking them to describe how they usually present to students the notions of vector (sub)space, linear (in)dependence, generators, basis, rank, and dimension, as well as an experiment with students from second to fifth years of university mathematics and science programs in France and Morocco; these students were asked to create a lesson plan about the same concepts for first-year students, and to elaborate on how they would use examples, exercises, and proofs. They were also asked about their understanding of linear algebra and how it has evolved throughout their studies. One of the main results of the study was the importance of students' maturity and the time (over a long term) needed to learn; another main result is in the non-continuous nature of students' learning in university, in that it occurs in selective, intense periods of individual work.

None of the papers identified in Stewart et al. (2019)'s survey of research in linear education from 2008 to 2019 were framed by an institutional framework, and my search for such papers among 5 major mathematics education research journals only yielded a paper reporting on Behaj's (1999) doctoral dissertation. This is not to say that no studies have attended to curricular elements in linear algebra courses, but those that have are few and far in between[2]. Stewart et al. (2019) refer to a 2018 paper by Harel about varieties in the use of geometry in the expository parts of six popular introductory linear algebra textbooks, some of which are widely used in the United States. Stewart et al. (2019) also refer to a 2014 paper by Rensaa reporting on a study whose aim was to investigate how note-taking and other class notes may impact a student's rationale while studying for a course. The study attended to class notes taken by an engineering student as well as notes provided by the teacher to this student (e.g., solutions to mini-tests). Results showed that the main use of these notes was likely instrumentalist, that is, to be used to identify rules and ways to find correct answers.

I found two papers (published past Stewart et al.'s 2019 survey) that have investigated textbook use in linear algebra courses, both reporting on studies conducted by students of Vilma Mesa as part of a broader research project on student and instructor use of open software and textbooks in undergraduate mathematics. Castro-Rodríguez et al. (2022) examined how students engage with digital textbooks in a linear algebra course taught at four universities in the United States. Analysis targeted students' real-time viewing and use, students' responses to bi-weekly open-ended surveys, and other curricular documents (e.g., syllabi). Castro-Rodríguez et al. (2022) found engagement with the textbook dropped when the algebraic language and textbook content weren't institutionally legitimzed and when the textbook didn't directly correspond to students' goals for earning

[2] I know of one study, apart from Behaj's, that used an institutional perspective to investigate a higher-lever linear algebra course: De Vleeschouwer (2010) also used the Anthropological Theory of the Didactic.

a degree. Gerami et al. (2024) analysed an interactive textbook's presentation of the notion of span along with student responses to two "reading questions" associated with these expository texts; "reading questions" are problems intended to motivate students to read textbook content before attending a lesson in which that content is to be covered. Analysis targeted students' strategies for determining whether a vector was in the span of a given set of vectors, as indicated by students' input in the interactive textbook software. This analysis revealed students' strategies were not confined to those presented in the textbook; the authors note this result echoes earlier studies (such as that of Castro, 2022) which found students attempt the reading questions without reading the corresponding text and also use resources other than the textbooks themselves (e.g., perhaps peers or popular resources on the Internet).

As Stewart et al. (2019) point out, there is ample room for analysis of textbooks and assignments—to which I add, of curricular documents more generally—with a focus on their content and on how they are used by instructors and students. There is similarly ample room, more broadly, for analysis conducted from an institutional perspective to look for sources for the practices students develop (e.g., as in Aydin's 2014 finding that students prioritize row-reduction as a technique to verify linear dependence over verifying the structure of the linear combination relations among vectors) in the activities offered to students and the ways in which such activities may enable and encourage certain practices over others.

A lack of institutional perspective is one gap in the slurry of studies performed to investigate students' difficulties since the reform which overhauled introductory linear algebra courses in university in the 1990s, reducing their theoretical load and incorporating numerical competencies. Another gap is in the topics targeted by the studies, which perhaps favor content directly set in the language of vector spaces. Stewart et al. (2019) point out there is a strong research base in students' reasoning relative to the topics of span, linear independence, eigenvectors, and eigenvalues; some additional papers address the areas of geometric reasoning related to linear transformations and of vector spaces. Few papers investigate students' reasoning in the topics of linear systems and their solutions, properties of linear transformations, orthogonality, and least squares (Stewart et al., 2019).

Chapter 3

Theoretical Framework

I frame my work in Chevallard's (1985, 1992, 1999) Anthropological Theory of the Didactic (ATD) along with a perspective on institutional practices from Ostrom's (2005) Institutional Analysis and Development (IAD) framework and a theory of non-mathematical practices developed by Sierpinska et al. (2008), Hardy (2009a), and Broley (2020), studies framed by this same mixture of the ATD and IAD. In Section 1.1, I explained the pertinence of the ATD framework and its affordances to my questions about students' practices in post-secondary mathematics courses; I also elaborated the notion of non-mathematical practices developed through applications of the ATD and IAD to study similar questions, and stated my objective to investigate whether such practices develop in introductory linear algebra courses.

I elaborate further in Section 3.1 on the two major tenets of the ATD: the notions of didactic transposition (Section 3.1.1) and praxeology (Section 3.1.2). The perspectives afforded by the notions of didactic transposition and praxeology clarify the mechanisms regulating the situation I intend to study—the learning accomplished in the college linear algebra institution. These two notions give the setup needed for two more elements to my theoretical framework: the theory of non-mathematical practice (Section 3.3) and a perspective of institutional positioning borrowed from the IAD, a framework developed in political science (Section 3.2). I use notions from this framework to present an actionable iteration of the research questions in Section 3.4.

3.1 The Anthropological Theory of the Didactic

The ATD (Chevallard, 1985, 1992, 1999) proposes that the learning accomplished in a didactic institution is regulated by transformation of knowledge. The research programme defines mechanisms that operate this transformation of knowledge and offers a way to model the knowledge being exchanged. In Section 3.1.1, I explain these mechanisms, which Chevallard ties together in the notion of *didactic transposition*, and relate the idea to my overarching questions. The notion of didactic transposition is a first tenet of the ATD; the notion of *praxeology* is a second. Praxeology is a tool with which to model students' activity. I explain this in Section 3.1.2.

3.1.1 First tenet of the ATD: didactic transposition

The ATD describes learning that takes place in didactic systems: triads involving a teacher, a student, and knowledge at stake. The ATD maintains that knowledge targeted by the didactic system must be examined to study the learning taking place in it. In the case of mathematics education, a knowledge-centered approach means that the mathematics that is subject in school must be subject to investigation to examine the teaching and learning accomplished in the institution. In this section, I recount Chevallard's (1985, 1992) reasoning for this approach: the premise of didactic transposition. This premise leads to a definition of the researcher's role in examining the knowledge transposed in a given institution.

Why must the mathematics targeted by a didactic institution be investigated to understand the teaching and learning that occur in this institution?

A mathematician works on a problem for some time, attempts, fails, revisits, perhaps the problem changes, others work on the same or related problems, and somewhere along the way results are established and formalized with axioms, definitions, theorems, proofs, and the like. This and other, similarly-synthesized knowledge are selected as reference for the knowledge to be taught in schools. A popular axiom-definition-theorem presentation may seem to lend itself well for teaching and learning processes, but this structure of knowledge is free of the problems and difficulties that triggered the creation of this knowledge in the first place (Chevallard, 1985). The absence of the situations that necessitated the creation of this knowledge is an obstacle; it makes it difficult to teach this knowledge. Teachers, along with many others involved in the teaching establishment, therefore redesign this knowledge to make it palatable for the classroom environment (Brousseau, 2002).

The process described above oversimplifies but captures the essence of the phenomenon of *didactic transposition*: knowledge that is taught in schools does not originate in schools. It originates in other (perhaps academic) circles and is reorganized for the purpose of teaching. This *reorganization* is the phenomenon Chevallard targets with the theory of didactic transposition.

Chevallard introduced this theory at a time when didactics of mathematics was forming as a scientific field of study; its objectives were just getting established. In the 1970s, Brousseau introduced his *Theory of Didactic Situations* and shifted the focus in mathematics education research (in French, eventually in Spanish, and later, in English-speaking communities) to the didactic system made up of teacher, student, and milieu; up until then, the prevailing paradigm in mathematics education focused mostly on psychological elements of learning (Bosch & Gascón, 2006; Brousseau, 2002). Chevallard's work fit in the prevailing focus on the didactic system as a source of explanation for the learning students accomplish. Chevallard proposed a focus on the *ecology* of didactic systems (Chevallard, 1999).

Chevallard's focus on ecology consists, first, in looking at the conditions and constraints that shape the transposition of knowledge into teachable and learnable knowledge, and second, to look at the knowledge that results from this transposition to understand what teachers ultimately teach and what students ultimately learn.

The notion of didactic transposition draws its purpose from an axiomatic shift in the understanding of didactic systems: they are now seen as a triad between teacher, student, and knowledge, rather than the result of interaction between teacher and student. (Chevallard, 1988, contrasts the expressions "I teach *something*," "I teach," and "I teach *something* to *someone*.") To understand the teaching and learning accomplished in a didactic system, it's necessary to grasp the knowledge at stake. I borrow an example from Chevallard (1985): students trained to solve for x in equations of the type $2x = 12, 3x = 12, 4x = 12$ transfer the knowledge gained from this training to solve for x in equations of the type $2x = 0, 3x = 0, 4x = 0$; the expectation that the value of x vary in relation to that of its coefficient leads to the erroneous conclusion, from $2x = 0$, that $x = -2, \frac{1}{2}$, or $-\frac{1}{2}$. This erroneous conclusion respects the contract established by the previous task type ($ax = 12$). Difficulties observed at one stage of learning can be understood in light of the knowledge taught and learned previously.

The shift to view didactic systems as a teacher-student-knowledge triad calls for a study of this knowledge, but the knowledge at stake in a didactic institution is not a given nor constant. This premise is core to Chevallard's elaboration of a theory of didactic transposition. The mathematics in a didactic institution is in flux: there is scholarly mathematics, but this is distinct from the mathematics in the sphere of those in charge of disseminating it in a given institution, and this, in turn, is distinct from the mathematicsof those in charge of learning it in that institution. Mathematics targeted by a didactic system is transposed within that system; it transforms as it slides along a path between scholarly mathematics and the mathematics of those who learn it in a given didactic institution.

This transposition doesn't occur only by choice and design. It's a result of conditions and constraints on didactic systems. Such systems live on a tightrope strung between expert and socioeconomic needs and expectations. Consider elementary and high schools, for example: their mandate and functioning are mediated, revisited, and continuously prodded by stakeholders that include parents, education ministries, publication houses, subject-matter experts, and pontifications of various agents ranging from industry moguls to political pundits. Chevallard calls "noosphere" the collective of stakeholders of a given didactic institution, along with their mediations and concerns (Chevallard, 1985). This is where decisions are made—decisions that determine which parts of scholarly mathematics are the target of a didactic institution.

I'll sidestep, briefly, to identify the noosphere of one of the didactic institutions at stake in this research. The Quebec chapter of the college linear algebra (LA1) institution takes the form of a linear algebra course usually offered in CEGEPs[1]; the course is also offered at Quebec's universities for students who did not complete it in a CEGEP degree (e.g., mature, Canadian out-of-province, and international students) but need it to enter a given university program. The noosphere of this institution includes Quebec's Min-

[1] "Collège d'enseignement général et professionnel" - General and professional teaching college - is a post-secondary educational system unique to Quebec; it offers technical career-oriented programs as well as 2-year programs between high school and university education, which are mandatory for entry to university by Quebec students who are below 21 years of age.

istry of Education and of Higher Education, which establishes official course conditions (e.g., 75 hours of instruction over one semester at a CEGEP), objectives, and standards, and produces official documents to disseminate these - see, for instance, Figure 3.1 below. The noosphere also includes scholarly experts involved in the development of these standards and objectives; publication houses whose textbooks have become standards in the North American college linear algebra institution; mathematicians in departmental teaching committees; teachers in charge of the course; students, who are typically (young) adults when registered in these courses, and who have their own expectations as students and clients of these institutions; since the turn of the century, the noosphere includes a selection of well-established internet sensations (YouTube channels, websites) by whom students faithfully abide as go-to sources of information about what they are expected to study for their course, and whose teachings inform students' (and clients') expectations from their linear algebra course.

Program-Specific Component
Common Objectives and Standards

Code: 00UQ

Objective	*Standard*
Statement of the Competency	
Apply the methods of linear algebra and vector geometry to problem solving.	
Elements of the Competency	**Performance Criteria**
1. Express concrete problems as linear equations. 2. Solve systems of linear equations using matrices. 3. Establish connections between geometry and algebra. 4. Determine the equation of geometric loci (straight lines and planes) and find their intersections. 5. Calculate angles, lengths, areas and volumes. 6. Demonstrate propositions. 7. Make two- and three-dimensional drawings of loci.	• Proper use of concepts • Representation of situations in terms of vectors and matrices • Correct application of algorithms • Correct solution of systems of linear equations • Adequate representation of loci • Justification of the steps in the solution • Algebraic operations in conformity with rules • Accuracy of calculations • Correct interpretation of results • Use of appropriate terminology

Learning Activities

Discipline:	Mathematics
Weighting:	3-2-3
Credits:	2⅔
Periods of instruction:	75

Indications: Matrix and determinant: definitions, properties, operations, applications.
The Gauss-Jordan and inverse matrix methods of solving systems of linear equations.
Geometric and algebraic vectors: definition, representation, properties, operations, applications.
Products of vectors: dot, cross and scalar triple product.
Vector space: coordinate system, basis, dimension, linear combination, linear independence.
Geometric applications: straight lines and planes, intersections of loci, calculations of angles and distances.

Figure 3.1: Quebec ministry objectives and standards for LA1 (Ministère de l'Éducation et de l'Enseignement supérieur, 2017)

The decisions made in the noosphere determine what knowledge is targeted, what knowledge is to be taught, which knowledge actually gets taught, which knowledge is to be learned, and which knowledge eventually gets mobilized by students. The expectations of the stakeholders mediate, first, the knowledge that a didactic institution targets. Chevallard (1985) gives, for example of a mediating constraint on elementary mathematics education, the tension between *the mathematics parents already know* and *the mathematics of the teachers*. For a school to survive, for teachers to maintain their didactic purpose, the knowledge they teach cannot be knowledge that parents *could* teach their children themselves, *if only they had the time.* This example underscores an important property of the conditions and constraints that shape the transposition of knowledge: they are not necessarily the motivations held by, for instance, teachers, when they transpose mathematics into knowledge that they teach. (A teacher likely has reasons other than "I need to teach something my students' parents don't know" for the choices they make in class.) Chevallard includes conditions and constraints borne out of the need for didactic institutions to survive and function - these may include, but are not limited to, the motivations and intentions of those in the noosphere.

One moment at which the transposition of knowledge can be traced is when it is put down to text (Chevallard, 1985). For instance, one such moment is when objectives and standards are officialized, as in Figure 3.1. Another is the development or selection of a textbook for a given course. Knowledge to be taught is determined by the selection, presentation, and organization of knowledge in a given textbook; boundaries to the knowledge are established and a logical scaffolding is implied. This process explicitly defines the knowledge to be taught and suggests a norm for the progression of learning (Chevallard, 1985).

The process of rendering knowledge to text, in addition to delimiting what knowledge is *to be taught*, depersonalizes knowledge and thereby renders it legitimate: knowledge sanctioned by a publisher or social institution is depersonalized and legitimized when compared with knowledge a particular teacher chooses to teach (even if such knowledge is otherwise identical). The explicit definition of knowledge and its depersonalization render it into knowledge suitable for didactic purposes (Verret, 1975, as cited in Chevallard, 1985).

Knowledge rendered teachable - by virtue of being depersonalized and legitimized, delimited, made suitable for didactic planning - differs from knowledge that is ultimately taught and learned because of properties of what Chevallard (1985) calls *didactic time.* This refers to the time, and organization of knowledge within that time, needed for knowledge to be acquired. One of the main issues at play here is the tension between new and old knowledge: for instance, new knowledge draws its sense from previously established knowledge (e.g., one isn't ready to learn how to hold a violin by its neck without, at the least, knowing what a *violin* is, what its neck is, and what it means to hold something), but old knowledge may also contradict or become an obstacle in the acquisition of new knowledge. Another property of didactic time is that it is absent of the (scientific, historical) problems that called for the creation of the new knowledge.

The difficulties inherent to didactic time result in yet further transpositions of knowledge: teachable knowledge is transposed into knowledge actually taught, and the latter is transposed into knowledge students actually learn. An important factor at stake is that learning is institutionally mediated by ways of *verifying* the learning students accomplish: this process makes explicit what students are expected to learn, and, given students' goal to survive or succeed in the institution, influences what students actually learn.

The theory of didactic transposition thus defines the conditions and constraints that drive the mutation of knowledge targeted by a didactic institution, wherein there is intent to teach items that belong to some socially-sanctioned sphere of knowledge. Knowledge is transformed as it shifts back and forth between scholarly knowledge, knowledge to be taught, knowledge actually taught, knowledge to be learned, and knowledge actually learned.

A word to the notion of knowledge "learned" by students: to avoid overstepping with claims about knowledge (not) "acquired" (and privately held) by students, which can be intractable, I suggest instead to address the knowledge students *mobilize*. I suggest this in part due to the pragmatic matter that what students *mobilize* is currency available for study; while it is possible to draw out what knowledge students do or do not have, this is not always the case. The main reason for my suggestion to attend to what students mobilize, though, is that it is a blend of the knowledge they acquire and the knowledge they choose to use in response to certain prompts. I reflect further on the affordances of this distinction between what students mobilize and what they know in the Discussion and Conclusions chapter (see Section 6.1.3).

Finally, one last word about the theory of didactic transposition: the transposition doesn't occur in a one-directional linear flow. Conditions and constraints of the didactic system mutate scholarly mathematics into teachable knowledge - and this into knowledge that is taught and then that which is mobilized. But there is regurgitation. Knowledge mobilized influences knowledge taught (e.g., when a teacher shifts gears upon noticing some gap between what students are expected to have learned and what they are mobilizing) and so on and so forth.

Given the existence of didactic transposition, that is, the flow of knowledge between spheres of learners and teachers and experts, mediated by the overarching educational and societal systems in which a didactic system exists, the knowledge that is currency in this system must be examined to understand the teaching and learning accomplished in this system. To understand problems in a didactic system, the transposition of the knowledge at stake is a pivotal unit of analysis. The didactic researcher's task, then, is to examine the knowledge at its various stages of transposition.

How is the didactician to investigate the mathematics at stake in a didactic institution?

If the didactician's task is to examine the didactic transposition of mathematical knowledge, the next question is how the didactician is to go about this task. Given the theory that the knowledge transposed at each each layer of a didactic institution can elucidate students' learning, the didactician needs a backdrop against which to analyse the knowledge at each stage, a backdrop that helps compare knowledge at a particular stage with the didactic transposition of the knowledge across the institution. Such a backdrop can then help to answer a variety of questions:

> For instance, what are the 'limits of functions' taught at undergraduate level? Or what kind of 'proof' or 'problem solving' are I considering? Is it something existing in 'scholarly mathematics'? In what way? Does it exist as knowledge to be taught? Since when? In what terms? What kind of restrictions does it impose on the teachers' practice? On the students' practice? (Bosch & Gascón, 2006)

Chevallard (1985) proposes that the researcher make explicit a *reference model*: a model of knowledge at stake in an institution. Such a model incorporates scholarly knowledge as well as knowledge of those in charge of teaching and those expected to learn. The reference model should not be subject to the dominance of knowledge at any one of the stages of didactic transposition under investigation (e.g., knowledge to be taught, or scholarly knowledge), as such a model would fail in its function as a backdrop against which to analyse the knowledge at each stage. A reference model must account for all stages of the transposition of a given morsel of knowledge.

With the didactic researcher's task established - make a reference model to examine the knowledge at any stage of a didactic transposition - there is the question of *how* knowledge might be modeled. Now comes the second major tenet of the ATD: that doing mathematics, and, by extension, learning mathematics, is but one human activity among all others. This premise, along with assumptions about what human activity involves, brings forth the notion of *praxeology*: a way to model human activity, and therefore a way to model the knowledge that is currency in a didactic transposition.

3.1.2 Second tenet of the ATD: praxeologies

The second basic tenet of the ATD starts from the observation that doing mathematics is a human activity (Chevallard, 1999). This assumption places the act of doing mathematics within the social sphere—within the realm of institutions. I complement Chevallard's description of *institution* with Ostrom's (2005) Institutional Analysis and Development (IAD) framework. I borrow from the IAD framework an understanding of *institutionalized* human activity; the characterization of activity that occurs in institutions, as given by the IAD, has proved useful in the body of research to which this study belongs (Broley, 2020; Hardy, 2009a). A characterization of institutionalized activity is pertinent here given the aim to characterize students' activity (mobilization of knowledge) in didactic institutions (post-secondary mathematics courses). I elaborate on the mechanisms I borrow from the IAD framework in Section 3.2 and focus first on the implications of the assumption that doing mathematics is a human activity among all others.

Any human activity can be described in terms of the tasks it involves, the ways in which the tasks are performed, and the reasoning that produces, justifies, and explains why these ways-of-doing accomplish the desired tasks (Chevallard, 1999). Doing mathematics—or mobilizing mathematical knowledge—can therefore be described in terms of the task(s) involved, the technique(s) used to perform the task(s), and the theory that produces the technique(s) and validates the technique(s) (validates that they accomplish what they are expected to accomplish).

Chevallard (1999) proposes to model human activity according to the task(s) T involved (or, perhaps, a *type of task* T, such as, for instance, "to solve a linear system," and tasks $t \in T$, such as "to solve a linear system of 2 equations in 2 unknowns" and "to solve a linear system of m equations in n unknowns), the technique(s) τ needed to accomplish the task(s), and a theoretical block - made up of a technology θ and a theory Θ (which I elaborate on shortly) - that produces and justifies the techniques. $[T, \tau, \theta, \Theta]$: Chevallard calls this model of human activity a *praxeology*.

A praxeology therefore models human activity according to its practical block $[T, \tau]$ and its theoretical block $[\theta, \Theta]$. The practical is the "know-how" while the theoretical block is the "know-why." What's critical to the notion of praxeology is that it is institutional in nature (Chevallard, 1999); that is, the technique(s) and theory involved (and, for that matter, the task itself) in the performance of any task are relative to the institution in which an activity is performed. The same task may be the target of a performance in different institutions, each of which privileges a different technique for performing that task. This is because institutional goals, expectations, and preferences privilege certain techniques over others.

For instance, consider this task t: to solve a linear system of 2 equations in 2 unknowns. The task is performed in Quebec secondary mathematics classes, but also in the LA1 college linear algebra institution. In Quebec's high schools, the task is performed graphically by determining the slope and y-intercept of the line represented by each equation and graphing the lines represented by the equations (let's call this technique τ_1^{HS}) , and algebraically by using the so-called comparison, substitution, or elimination techniques ($\tau_2^{HS}, \tau_3^{HS}, \tau_4^{HS}$, respectively)[2]. For instance, to solve the system

$$\begin{array}{rcrcr} x & - & y & = & 5 \\ 2x & - & 2y & = & 10 \end{array}$$

graphically, τ_1^{HS} is to rewrite each equation in point-slope form,

$$\begin{array}{rcrcr} y & = & x & - & 5 \\ y & = & x & - & 5 \end{array}$$

wherein both lines, having the same slope 1 and y-intercept -5, can be seen to overlap, so the system is concluded to have infinitely many solutions. To solve the system by the method of comparison, the technique is to isolate one of the variables - say, y, and to determine the value of x for which the two expressions for y equal one another. In the current example, isolating x in $x - 5 = x - 5$ to the equation $0 = 0$. In the high-school algebra institution, the conclusion can now be drawn that the system has infinitely many solutions. To solve the system by the method of substitution, the technique is to isolate one of the variables in one of the equations - say, to isolate y in the first equation: $y = x - 5$, to substitute this expression for y into the second equation:

$$2x - 2(x - 5) = 10,$$

and to solve this last equation in terms of x:

$$2x - 2x + 10 = 10$$

$$10 = 10$$

$$10 - 10 = 10 - 10$$

$$0 = 0,$$

[2]These are the techniques listed and described on the page about *la résolution de systèmes d'équations linéaires* (solving systems of linear equations) in Alloprof (n.d.). Alloprof is a non-for-profit supported by the Montreal School Service Centre; it offers phone and internet services to help students in their studies. The services offered are developed by Quebec teachers and subject-matter experts, so I assume the techniques on the webpage are representative of the techniques used in Quebec high schools.

wherein it follows that the system has infinitely many solutions. The mathematical theoretical block supporting both algebraic techniques τ_2^{HS} and τ_3^{HS} includes the technology of substitution, the notion of equivalent equations, the algebraic operations that can produce equivalent equations, and the notion that the equation $0 = 0$ is true; these technologies may or may not be included in knowledge students are taught.

τ_2^{HS} and τ_3^{HS} are the institutionally-suggested techniques when the linear system includes an equation in which one of the unknowns is already isolated (Alloprof, n.d.). When this is not the case, that is, when neither equation includes an isolated variable and a few additional steps of algebraic manipulation are needed to achieve that form, the method of elimination (τ_4^{HS}) is the suggested technique: manipulate the equations algebraically so the coefficient of *one* of the unknowns is the same in both equations (alternatively, so the coefficients are negative inverses of one another); then "subtract (alternatively, add) the equations from one another" (as per the description normative in high schools), so as to eliminate that unknown (whose coefficients were the same or negative inverses of each other in the two equations):

Multiply both sides of the first equation by 2:

$$\begin{array}{rcrcr} 2x & - & 2y & = & 10 \\ 2x & - & 2y & = & 10 \end{array}$$

Subtract the second equation from the first equation:

$$\begin{array}{rrcrcr} & 2x & - & 2y & = & 10 \\ - & (2x & - & 2y & = & 10) \end{array}$$

The result of the subtraction is $0 = 0$, and the institutionally-accepted conclusion at this point is that the system has infinitely many solutions.

In LA1, only the graphical technique τ_1^{HS} and algebraic technique τ_4^{HS} for accomplishing the task t "to solve a linear system of 2 equations in 2 unknowns" are brought up as knowledge to be taught (or rather, reviewed); perhaps these techniques are privileged because they and the theoretical blocks by which they are framed are at the heart of the knowledge be taught in the course (as indicated by the course textbook). For instance, τ_4^{HS} is (in the knowledge to be taught in LA1) the technology that produces the technique of Gauss-Jordan elimination, τ_1^{LA1}, through which t may be performed. I'll build on the previous example:

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & -2 & 10 \end{array}\right]$$

Add -2 times row 1 to row 2:

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

This corresponds to the linear system

$$\begin{array}{rcrcr} x & - & y & = & 5 \\ & & 0 & = & 0 \end{array}$$

where x is a leading variable and y is a free variable, so, assigning a parameter to y, $y = t$, where $t \in \mathbb{R}$, leads to the general solution $(x, y) = (t + 5, t)$. Since the parameter can have any real value, the system has infinitely many solutions.

τ_1^{LA1} is justified by τ_4^{HS} and builds on the same theory as that on which τ_4^{HS} is based—the notion of equivalent equations and the algebraic operations that produce equivalent equations—but shifts to a new symbolic representation for these notions: an augmented matrix and row operations. The technique thus operates only on what's to be understood, in LA1, as the components that determine the solutions of a linear system (its coefficients and constant terms). This new focus on the coefficients and constant terms, and on the algebraic operations that produce equivalent equations, is expanded upon in LA1 in theory about linear systems of the form $Ax = b$, where $A \in M_{m\times n}(\mathbb{R}), x \in R^n$, and $b \in R^n$ $(m, n \in \mathbb{N})$; in turn, this theory produces other techniques for the task t (to solve a linear system in 2 equations and 2 unknowns) and the broader task type T, to solve a linear system of m equations in n unknowns.

The praxeologies that model how to solve linear systems of two equations in two unknowns are dependent on the institution in which the activity (solving linear systems) is performed. But, within a given didactic institution, the praxeology of a teacher differs from that of a student. Within each institution, the praxeologies held by any given individual are further dependent on the *institutional position* held by that individual (in the sense of Ostrom, 2005, which I address in Section 3.2): simply put, this refers to an individual's objectives as members of an institution. This take on institutional positions is more expansive than that in the ATD, which only takes into account officially-sanctioned positions in an institution (e.g., those of student and teacher). Examples of positions students might occupy, as identified by Hardy (2009a) and Broley (2020), include those of a Learner whose mission is to acquire knowledge; a Mathematician-in-Training whose mission is to develop the practices shared by a certain community of mathematicians; or a Student whose mission is to acquire grades and a degree. Hardy (2009a) and Broley (2020) found there is some relation between the position held by a student and the praxeologies they develop.

The example elaborated above highlights how techniques associated with similar tasks may differ across institutions. The theoretical blocks that frame these techniques also vary across institutions and institutional positions. This block is made up of technology θ that gives authority to, justifies, and produces the technique(s) to be used, and theory Θ, which is the discourse from which technology draws its authority and validity. Chevallard (1999) avows that the distinction between θ and Θ can be murky and suffers from a self-iterative property (theory may then have a theory justifying it, and this latter must have its own theory, and so on and so forth); nevertheless, Chevallard holds that the notion of praxeology need only be tailored to the case in which it is applied. To this end, some clarity can be brought by considering the institutional nature of a praxeology.

For instance, Hardy (2009a) found that students who had recently completed Calculus 1 had theoretical blocks that were non-mathematical in nature. One justification students gave for why they expected their techniques to work was the technology θ: this was how the instructor did things. The theory Θ here is that the instructor is an authority on what is mathematically accurate, as well as the authority on what it takes to pass that course. Chevallard (1999) notes the auto-technological nature of techniques: that a technique is used in a given institution, by virtue of being established as a technique, suggests it is accepted as a correct and appropriate tool for accomplishing a task. But the purpose of a technology is beyond giving authority to a technique; it's also to impart (to those who use said technique) why a technique works as it does, and this is where a non-mathematical theoretical block can prove problematic. Sierpinska (2007) addresses this in her discussion of prerequisite mathematics students' tendency to depend on their teachers' validation. I return to the notion of non-mathematical praxeology and its role in my theoretical framework in Section 3.3).

The theoretical block of the knowledge to be taught in LA1 about the task *to solve a linear system* and related technique *use Gauss-Jordan elimination*, as indicated by the course textbook, includes the technologies of augmented matrices, row operations, the notion of reduced row echelon form of an augmented matrix, and the claim that the algorithm described by Gauss-Jordan elimination produces the reduced row echelon form of the augmented matrix. The theory that gives validity to this technology is based in referential authority - the claim about the results of Gauss-Jordan elimination is true because it's been proven, presumably by a trusted source (as Chevallard, 1999, puts it, theory at times involves throwing responsibility elsewhere - "you'll see this in some other course," "mathematicians said so"). Referential authority aside, the technology in this theoretical block *does* impart why the technique of Gauss-Jordan elimination works as it does, and it also serves the third function of technology (Chevallard, 1999): to *produce* technique.

I give a last example to relay how technology and theory of a praxeology can be distinguished in light of the institutional nature of a praxeology. Recall the notion of the didactic researcher's model (discussed toward the end of Section 3.1.1); in my reference model of the LA1 activities related to the task *to solve a linear system* and related technique *use Gauss-Jordan elimination*, I view technology to be, essentially, the knowledge found within the definitions, theorems, and proofs in the course texts. These constitute the knowledge that produces the techniques to be used to complete tasks in LA1. These technologies take their legitimacy from the logico-formal structure of mathematics and the view that the axioms that underpin linear algebra and Euclidean geometry are founded in the physical reality humans inhabit; altogether, these form Θ. This theoretical block suits my need, as a researcher, to have a backdrop against which to analyse the knowledge transposed in the LA1 institution.

The notion of praxeology can serve to model activities of varied breadth. Chevallard (1999) classifies praxeologies by the breadth of the activity that they model. A single type of task, T, along with its related technique(s) τ, technology θ, and theory Θ, combine to form a *point praxeology* denoted by $[T, \tau, \theta, \Theta]$. A collection of point praxeologies may agregate under the umbrella of a given technology θ to form a *local praxeology* denoted by $[T_i, \tau_i, \theta, \Theta]$, and a set of these may cluster around a theory Θ and form a *regional praxeology* $[T_{ij}, \tau_{ij}, \theta_j, \Theta]$. A *global praxeology*, $[T_{ijk}, \tau_{ijk}, \theta_{jk}, \Theta_k]$, refers to the body of regional praxeologies coordinated by a set of theories within a given institution. These notions and notations can be useful to model the knowledge at stake in a didactic institution.

This completes what I need from the ATD to inform my theoretical framework. I now turn to two more components that I coordinate with the ATD to form this framework: the notion that students form non-mathematical praxeologies (see Section 3.3) and the notion of institutional positions from Ostrom's (2005) IAD framework. I conclude in Section 3.4 by rephrasing the research questions of this thesis in terms of the notions defined in this theoretical framework.

3.2 Positioning and institutions

To clarify how the mechanisms of mathematics course institutions regulate the practices of its participants, Hardy (2009a) took to Ostrom's (2005) perspective on institutions. This lens helped to reveal the normative quality to the practices that occur in mathematics courses (Chevallard, 1999; Hardy, 2009b): practices reflect what is *normal* in the given institution, in the sense of how things are usually done in this institution. In Section 3.2.1 I overview terms and notions from Ostrom's Institutional Analysis and Development framework needed to define the *positions* held by participants of an institution and how positions relate participants to the practices they form. I follow in Section 3.2.2 with a review of the repertoire of the use, conceptualization, and implementations of the notion of institutional positioning in mathematicseducation research.

3.2.1 The Institutional Analysis and Development (IAD) framework

In Ostrom's IAD framework, an institution is any rule-structured, repetitive, and ongoing situation in which humans interact. With this definition, universities, faculties within these universities, departments within these faculties, courses administered in these departments, and the mathematical fields of analysis and abstract algebra are institutions—as are family units, friendships, supermarkets, souks, and international governmental alliances. Individuals that engage in structured interactions "face choices regarding the actions and strategies they take, leading to consequences for themselves and for others" (Ostrom, 2005, p. 3). The IAD framework traces how rules shape interactions in rule-structured situations and gives tools with which to attend to how the structure of institutions regulates the behavior of its participants.

Institutions are structured via rules, strategies, and norms. In the IAD framework, a *rule* consists of the "enforced prescriptions concerning what actions (or outcomes) are required, prohibited, or permitted" (Ostrom, 2005, p.18). Rules allow certain actions and prohibit others. Individuals in the institution are assigned positions that require, prohibit, or permit them to enforce the rules; these individuals can be monitored or sanctioned should they fail to do so. Rules are not necessarily written nor created via formal procedures, but are usually "crafted by individuals to change the structure of repetitive situations that they themselves face in an attempt to improve the outcomes that they achieve" (Ostrom, 2005, p. 18). This aligns, for example, with Chevallard's (1985) notion that one of the drivers of didactic transposition is to improve, in some sense, the outcomes achieved by the didactic institution (e.g., students' learning of particular knowledge).

Strategies are the ways in which participants act within an ongoing and structured situation. For example, the prescription "to acquire the knowledge to be learned, consult solutions of tasks from past exams" is a strategy in the LA1 institution. Strategies are individual plans of actions that have been deemed effective for accomplishing tasks in a given institution. Strategies may be recommended plans of actions, but, unlike rules, are not "enforced, rescinded, or reinstated" (Ostrom, 2005, p. 17).

Norms are precepts that establish what constitutes moral or prudent behavior in a community. For instance, a norm in many human societies is for people to greet one another upon meeting and different societies have different strategies for acting upon this norm—shaking hands, hugging, or bowing. Norms are not subject to regulation (e.g., in the form of prosecution or revocation).

The structure of an institution by rules, strategies, and norms informs its participants on how to act within the institution. The rules, strategies, and norms define the behaviors and values accepted and expected in the institution; in short, they define institutional actions.

A participant in an institution has a set of actions available to him: depending on the action they choose to undertake at any moment, they may simultaneously occupy more than one institutional position at a time. For example, a student-participant in LA1 may simultaneously be in the position of *Learner*, as someone trying to understand why a point-normal equation represents a plane, and in the position of *Client*, as someone expecting their teacher to use a certain type of discourse to explain why point-normal equations represent planes. A *position* is a "slot" into and out of which a participant can move; it's the connection between a participant and the action they engage in.

3.2.2 A positioning framework developed in and for mathematics education research

The positioning framework I outline was first proposed by Sierpinska et al. (2008): they operationalized institutional mechanisms defined in the IAD to construct a positioning framework. This framework was proposed as a lens through which to investigate students' frustrations in prerequisite mathematics courses offered at the university level. Hardy (2009a) and Broley (2020) adopted and further developed this framework in their studies of students' models of knowledge to be learned in two courses in the university calculus stream.

Sierpinska et al. (2008), Hardy (2009), and Broley (2020) mobilized the notion of position defined in the ATD and in the IAD frameworks. The ATD (Chevallard, 1991, 1992) distinguishes between the formal positions of teacher and student and the impact of these positions on the knowledge held by members of the institution: a teacher's knowledge functions according to their charge of transmitting and evaluating knowledge and a student's according to their charge of acquiring knowledge. Ostrom's (2005) notion that members in an institution organize themselves according to *objectives* points to the availability of further, but informal, positions: a student may act according to an objective to get a certain grade; to gain certain knowledge; to gain entry into a professional or social community; etc. A position is a frame of reference that traces a student's actions to their objective(s). A student may occupy different positions at different moments.

3.2.2.1 Positions identified in the literature

Sierpinska et al. (2008) define the positions of Student, Learner, Client, and Person:

- The **Student** acts according to the goal to get a certain grade. In the case of a mathematics university course, since grades are determined by the teacher, it is the teacher and course materials (as opposed to the mathematics targeted by the course) that are authorities over what is accurate, pertinent, and appropriate. These authorities determine the criteria for grading. The Student's strategy is therefore to gain the knowledge deemed as accurate, pertinent, and appropriate by these authorities.

- The **Learner** acts with the goal to gain knowledge for the sake of gaining this knowledge. A Learner might therefore spend their time attempting to gain certain knowledge until they perceive themselves to have gained it (the criteria for whether knowledge has been gained, in this scenario, are not necessarily shared by anyone other than this student); Sierpinska et al. (2008) qualify this position as a cognitive one.

- The **Client** has expectations relative to a product they are purchasing. A student is said to act from the position of a client when their behavior reflects a concern with whether the product they receive aligns with their expectations for this product; for example, students may have expectations about what their teacher *ought* to expect of them on an exam, and these expectations guide the practices they choose to develop.

- The **Person** is mediated by concerns external to their experience in a didactic institution, which can nevertheless impact this experience (e.g. professional goals, personal difficulties, etc.).

Broley (2020) added three positions to this list following her investigation of students' practices in a Real Analysis course: the **Skeptic**, driven by expectations about the courses they (do and do not) need to achieve the purported aims of the program in which they are registered; the **Enthusiast**, driven by interest in the knowledge they believe they are to learn in a course; and the **Mathematician in Training**, driven by a professional goal to join a community of mathematicians.

Reflecting on Broley's (2020) definitions of Skeptic and Mathematician in Training, I view these as *types* of Client and Person, respectively, and which are relevant to note in the context of university mathematics courses (UMC) given their potential to regulate a student's practices. Reflecting on Broley's (2020) definition of an Enthusiast, I view it as potentially a type of Learner, but possibly a behavior both Learners and Students may exhibit.

Broley (2020) characterizes the Skeptic as a student who, as someone in a degree that forces them to take a course whose pertinence to the student is not clear, questions the value of the course to *themselves*. The student who inspired this position in Broley's work was in a program designed to prepare students to be actuaries. The student questioned the requirement that they must take a course in Real Analysis: they felt the mathematics (rules) targeted by the course (such as limit laws of real-valued functions) regurgitated what they'd already acquired in Calculus courses, and they did not view the underlying theory as pertinent to their objectives for being in the program. A Client is motivated by expectations they have relative to a product (in this case, a degree or a course) for which they are paying; I therefore view the Skeptic as a *type* of Client relevant in the context of UMC (and in post-secondary education more broadly).

I also view the Mathematician in Training as a *type* of a position defined by Sierpinska et al. (2008): specifically, the Person. Broley (2020) qualifies the Mathematician in Training as a student who aspires to join a community of mathematicians. A student with this objective tries to develop practices they perceive to be needed to gain entry into such a community. In turn, the position of *Person* is that of a member of society at large. A Person has aspirations ranging between and beyond social and professional. In the context of a didactic mathematics institution, a Person is a member (of society) for whom mathematicsis a part of the world (Sierpinska et al., 2008); I therefore view a Mathematician in Training as a Person for whom "this part of the world" (mathematics) is what they hope to be the setting for their profession.

Broley (2020) defines the Enthusiast as "devoting themselves to what they perceive to be the practices to be learned in [a] course." Broley (2020) identified this position through a participant who expressed "keen interest" in the knowledge they perceived they had to learn and defined the position of Enthusiast to capture the effect such interest can have on the practices a student develops. In the Methodology chapter, I elaborate an operationalization of the positioning framework that I propose to regulate its implementation. The choice to bring this operationalization came about after reflecting on how to qualify whether a participant is exhibiting signs of having occupied one position of another. In the operationalization I developed, I propose to consider students' behavior relative to norms from their course to determine the position reflected in their behavior (I explain this in more detail in Section 4.3.3 of the Methodology chapter). Reflecting on this, along with the emphasis on *devotion* as the characteristic trait that led Broley (2020) to propose an Enthusiast position, I consider the possibility that *enthusiasm toward or devotion to a practice* is a behavior that students can exhibit as Students or as Learners; the practice to which this devotion is held would indicate the position occupied (e.g., devotion to a cognitive aim would indicate a Learner position).

My view of the positioning framework developed thus far, first by Sierpinska et al. (2008), and then by Hardy (2009a) and Broley (2020), is that students in university mathematics courses may variously act from the positions of Student, Learner, Client(-Skeptic), and Person(-Mathematician in Training). Part of my contribution to this framework is the proposal that positions present through certain behaviors, that is, through students' activity, and that superimposing this activity with those sanctioned institutionally can help to determine the "slot" (position) occupied by an individual at any moment. I elaborate on how institutional mechanisms regulate participants' positions in the following section.

3.2.2.2 The regulatory effect by institutional mechanisms on positioning

When I claim a member of an institution "occupies a position," I do not mean they selected that position; I identify positions by a member's *objectives.* The terms *Student,* *Learner, Client(-Skeptic), Person(-Mathematician in Training)* refer to objectives a student claims to have or to objectives indicated by their behaviors. For example, one of my participants said that to study for LA1, he did all the problems in past final exams to which he had access. In LA1, it is a rule that the final exam makes up 90% of a student's final grade in the course and the norm is that final exams consist mostly of the same types of tasks from one semester to the next. This participant's activity geared toward obtaining a certain grade on the final exam and therefore in LA1. Since the target is the grade obtained in the course, I see this participant as having occupied the position of Student in engaging with the practice of doing past final exams.

A student's objectives are a result, in part, of who *they* are—their experiences, didactic and otherwise. In the context of a given didactic institution, however, the norms, strategies, and rules (Ostrom, 2005) of that institution also influence the objectives a student may have. In the following paragraph, I discuss how norms, strategies, and rules may shape a student's objectives and their activity for attaining these objectives. I use the case of prerequisite mathematics courses (PMC) as a basis for examples.

The requirement for students to pass PMC to gain entry into various university programs may encourage students to take on the positions of Student and/or Client. A student whose goal is to gain entry into a given program is incentivized to position themselves as a Students, as they need to pass the PMC; they may also be prompted to position themselves as Clients, given their goal to gain entry to a program they may perceive to have little to no footing in the content targeted by PMC. Sierpinska et al. (2008) document the frustration experienced by Clients in response to the requirement to pass PMC. This is one example of how an institutional mechanism may encourage students to occupy certain positions more than others.

Institutional mechanisms also encourage participants holding a given position to invest in certain activities over others. For example, an institutional rule of PMC is a heavy weight assigned to final exams in grading schemes (e.g., 90% of the final grade). This rule, together with the high volume of knowledge to be taught in PMC (Sierpinska et al., 2008), are institutional mechanisms that make it such that a Student's goal is best served by acquiring the knowledge strictly needed to pass their course. Indeed, in the absence of time needed and motivation to gain a high volume of knowledge (Sierpinska et al., 2008), Students are propelled to focus on surface-level features of the knowledge needed to complete final exams, as these features suffice to pass calculus PMC (Hardy, 2009a; Lithner, 2004).

3.2.2.3 The value of looking at students' positions in an anthropological framework

In what follows, I propose a synthesis of the positioning framework elaborated so far in mathematics education research (Broley, 2020; Hardy, 2009a; Sierpinska et al., 2008) with the context of the anthropological framework in which it has been developed. This synthesis sets the stage for the operationalization I propose in the Methodology chapter (Section 4.3.3) for examining students' positioning. In actuality, this synthesis came about as a result of the operationalization I developed for determining students' positions; the choice to develop this operationalization, in turn, was a result of difficulties encountered in identifying students' positions in this research and the resulting need to clarify how a student's position can present in their activity. The synthesis I propose affirms its consistency with the ATD lens and, in so doing, brings clarity to the positioning framework and highlights its affordances to the study of the effect of routinization on students' learning—and to the study of students' learning in mathematics courses more generally.

My view of activity, in line with that proposed in the ATD (Section 3.1.2), is that it consists of a theoretical and a practical block. This brings clarity to the positioning framework: a student's position, that is, the *objective* they hold in a given instance relative to their participation in a course, and properties of such a position, form the reasoning that produces what their behavior. By 'behavior,' I mean what a student tries to accomplish (their task) and how they go about accomplishing it (their technique).

Given the view of activity as institutional, in that its theoretical and practical blocks exist in a given institution, it follows that positions (as elements of theoretical blocks) are institutional. This does not necessarily mean that certain positions exist in some didactic institutions and not in others. It rather means that the way a position *presents* depends on the didactic institution being considered. For example, Studenting[3] could imply drastically different activity depending on the course a student is in; this is because a Student—defined by their objective to get a certain grade in their course—is incentivized to conform to course norms, and norms can vary from one course to another. Sierpinska et al. (2008) and Hardy (2009a) provide a characterization of Student activity in calculus courses that emphasize routinization: behaviors indicative of this position include identification of routine exam tasks and assimilation of surface-level features of expected techniques.

I addressed in Section 3.2.2.2 the regulatory effect by institutional mechanisms on positioning. One aspect of this regulation is in what it may mean to Student, to Learn, to Client, or to be otherwise positioned in a given course. This means, for example, that some course norms may bring behaviors of a Student closer to or further from those of a Learner. This highlights the importance of looking at positions in the context of a framework that looks at the effects of routinization on the learning accomplished by students.

3.3 Non-mathematical praxeologies

Praxeologies model how activities unfold in a given institution. In broad terms, a mathematical praxeology is one corresponding to what a community of mathematicians (in a certain domain) would understand the task, technique, and theoretical block as the way an activity goes. In the context of a particular mathematics course, a praxeology is mathematical if all its components (practical, theoretical) reflect the mathematics at stake in an activity. If any part of a praxeology deviates from this—for instance, if a theoretical block includes considerations such as what a teacher usually accepts as sufficient—a

[3]For literary convenience, I'll use position labels in grammatically flexible ways, maintaining the position labels as root words. By *Studenting* and *to Student*, for example, I refer to the state of occupying a Student position.

praxeology is non-mathematical. The theory that students develop non-mathematical praxeologies is based in the assumption that the teaching and learning of mathematics occur within *institutions*, and the assumption that norms of these institutions—as opposed to mathematical theory—regulate the mathematical activity of its students and teachers (Chevallard, 1985; Hardy, 2009b; Sierpinska et al., 2008).

Hardy's (2009a) integration of the ATD and IAD frameworks illustrates how norms of didactic institutions can and do regulate students' (non-)mathematical activity. When students are engaged with tasks in their mathematics courses, their activity is regulated by mechanisms of the didactic institution rather than (exclusively) by mechanisms of the mathematics at stake. Institutional norms affect every component of students' activity: what students perceive a task to be, the techniques they use, as well as the technology that produces and justifies their choice of technique and the theory that gives discourse to their technology. I explain how each of these components can be regulated by institutional norms in the following paragraphs.

Given a mathematical task, what students perceive the task to be can differ from what a teacher or a mathematician would perceive this task to be (Hardy, 2009b). For instance, in Hardy's (2009a) study, students faced with the task to find the limit

$$\lim_{x \to 1} \frac{x-1}{x^2+x}$$

acted as if the task was to find the limit of an indeterminate form: their technique included factoring the algebraic expressions, with the goal of finding some common factor. The task is to find the limit of a rational function at a number in its domain. Direct substitution suffices. But students did not pay attention to the entirety of the limit expression (i.e., the number at which the limit was to be evaluated) and focused instead on the algebraic expression: the denominator could be factored. In the college calculus course, such algebraic expressions (polynomials that can be factored using one of a handful of factoring techniques taught in high-school algebra) *usually* occur in tasks where the goal is to evaluate an indeterminate form. Further, limit-related tasks in the college calculus institution are routinized. The routine is to manipulate algebraic expressions; the focus is on the algebra, and not on limit expressions in their entirety. These institutional norms condition students to perceive tasks in certain ways and, in turn, regulate their choice of technique (Hardy, 2009b).

Apart from the way institutional norms shape students' perceptions of tasks and choice of technique, these norms also make up students' technology and theory. For example, to perform the previous task of finding the limit of a rational expression (at a given number), students' technique was to factor and to use direct substitution (both, and in no particular order). Students' technology (justification for their choice of technique) was that this was what was usually done in this type of circumstance. In other words, the technique is valid because it is the normative technique in the college calculus institution. Chevallard (1985) refers to the auto-technological nature of techniques that are used in an institution: a technique must be valid because it is the accepted technique. This normative quality of techniques in the college calculus institution extends to (and is reinforced by) teachers' models of the knowledge to be learned: Hardy (2009b) points out the norm to reuse the same types of tasks on final exams from one year to the next, and the norm that teachers expect students to use normative techniques correctly but do not expect students to use the mathematical theory that underlies these techniques.

If the normative practices of the college calculus institution (the expectation that students solve routinized exercises) are a source for students' strategies, then a theory that explains why this technology is reasonable is that teachers, textbooks, and solutions to past final exams are an authority on what is valid (Hardy, 2009b; Sierpinska et al., 2008). The absence of mathematical theory in students' mathematical activity leaves a void, which must therefore be filled (Sierpinska et al., 2008): human activity is regulated by *theory* (Chevallard, 1999), so *something* must take the place of the mathematical theory that would otherwise inform students' practice. Students' *institutional position* can fill this void. Indeed, the teacher has the power to determine a student's outcomes (as per Sierpinska et al., 2008, I use "power" in the sense of Ostrom, 2005, as a function of opportunity - potential outcomes in a situation - and the extent of control over this opportunity); hence, the teacher has authority on what is valid, and students are subordinate to this judgment. From the position of a Student, whose purpose is to pass their course (as opposed to the purpose of Learner, who aims to acquire knowledge), there's no *need* to assess mathematical validity; the appropriate action is to follow the teacher's example.

Apart from the impact of positioning on students' praxeologies, and that of normative tasks and techniques, there are also institutional strategies at play. Sierpinska et al. (2008) identify mechanisms of the prerequisite mathematics course institution that result in an absence of mathematical theory. The fast pace of PMC is an important factor (Sierpinska et al., 2008): the little amount of time allotted to cover significant quantities of mathematics, contrasted with the conceptual difficulties students are known to have with the mathematics at stake, mean that teachers might choose to not attend to theory and to prioritize examples instead. Others have also explored the potential of institutional strategies to regulate autonomy in students' mathematical practice; Boaler & Greeno (2000), for example, argue that "didactic-teaching" classrooms, wherein a teacher gives a lecture and students receive this lecture, reinforce this problem, and that discussion-based classrooms have the potential to give students greater autonomy in establishing mathematical truth.

One of the problematic consequences of a lack of mathematical theory is that it makes students dependent on predetermined routines (Boaler & Greeno, 2000; Sierpinska et al., 2008) and they lose autonomy in their mathematical practice. For example, if they stick to solved examples too rigidly, they may be unable to solve a problem where the underlying mathematical theory is the same but surface characteristics are different. This 'worst case scenario' aside, though, Hardy (2009b) points out that a lack of autonomy over mathematical theory gives an arbitrary quality to the normative techniques: they are a list of steps, and there are no guiding principles as to when to apply which step - no guiding principles to help students remember the order in which to apply these steps. For example, without knowing why direct substitution or why factorization are appropriate techniques for evaluating limits, students struggle to remember whether to substitute or factor first or may not even know when factoring is (un)necessary, as in Hardy's (2009) study. A lack of autonomy over mathematical theory contributes to students' learning difficulties.

My research questions, which I first presented in Chapter 1 in broad terms, and which I shortly rephrase using language from this theoretical framework, are framed by this theory that students develop non-mathematical practices, and that such practices can be identified by *students' tasks* (that is, what students perceive a task to be), their choice of technique (when it's reflective of techniques students have been conditioned to use, rather than informed by mathematical properties of the written task), and the justifications they give for their choice of technique (when these justifications are free of mathematical theory).

3.4 The research questions rephrased in terms of this theoretical framework

This is the first iteration of the research questions targeted by this thesis (from Chapter 1):

- What is the nature of what students are expected to learn in LA1? (Does it align with the nature of what students are expected to learn in Calculus 1 and Analysis 1?)
- What is the nature of the practices students develop in LA1?
- Research on the learning of calculus has found that students develop non-mathematical practices; are such practices replicated in linear algebra?

The second iteration of the research questions takes into account language and notions from the theoretical framework:

1. What are the praxeologies expected when considering problems posed in linear algebra final exams?
2. What is the nature of the knowledge linear algebra students mobilize when they solve linear algebra tasks? What kind of (mathematical or non-mathematical) praxeologies do they activate?
3. Research on the learning of calculus has modeled students' practices and found them to consist of routinizing techniques and building non-mathematical praxeologies; are these practices replicated in linear algebra?

For the first research objective, I need a praxeological model of the knowledge needed to complete linear algebra final exams. By "praxeological model," I mean a description of the knowledge in terms of the task(s) to be performed, the techniques needed to perform these tasks, and the theoretical block that produces and justifies these techniques. To answer the second question, I need a praxeological model of the knowledge linear algebra students mobilize to solve LA1 tasks: given a task, what do *they* perceive it to be? What techniques do they use? What produces these techniques, and what is students' reasoning for the validity of these techniques?

For the third research objective, I need to contrast the practices students build in their linear algebra course with those they have been documented to build in calculus. Given the considerations outlined in of this theoretical framework, I need to account also for the institutional mechanisms that characterize calculus courses and those characteristic of the linear algebra course under investigation. This includes a comparison of the nature of the knowledge to be learned in the courses (e.g., to determine if routinization is a shared norm), the knowledge students mobilize, and the positions students seem to have occupied during their course.

Chapter 4

Methodology

In this chapter, I describe the research procedures for this study. I start with considerations from the theoretical framework that determined the research instruments I would use. I then describe these research instruments and finish with the data analysis procedures.

4.1 Considerations in light of the theoretical framework

To attend to the research questions, I analysed curricular documents and interviews with students who had completed LA1 in the version of this course offered at a large North American urban university. In this section, I address considerations that guided the methodology in light of the overarching goals and framework of this research.

4.1.1 What knowledge to target?

In this research, following in the notions proposed by the ATD, knowledge to be learned is the knowledge students need to complete exam tasks. At the institution from which I collected my data, two grading schemes are offered, so a student's final grade is the best of two options:

1. 10% for the assignments, 25% for the midterm exam, and 65% for the final exam (LA1);

2. 10% for the assignments, 10% for the midterm exam, and 80% for the final exam (LA1).

Students pass LA1 on the basis of their performance in the midterm and primarily final exams; this is why I consider midterm and final exams tasks to indicate the knowledge to be learned in LA1. The textbook, along with the course outline and past exams, communicate to all returning and new instructors the knowledge to be taught and what students are expected to learn; I therefore used the course textbook to identify the techniques students are expected to use to complete exam tasks. The textbook for LA1 at the institution in which I conducted interviews is popular in colleges throughout North America for equivalent courses.

For pragmatic purposes, I could not model *all* knowledge to be learned and *all* knowledge a student might mobilize, and for my research purposes, this was not necessary. The overarching goal was to determine the (non-)mathematical nature of students' (expected or mobilized) praxeologies, and to this end, I did not need an anthology of *all* knowledge to be learned, nor of *all* knowledge students might mobilize; it sufficed to have models of the knowledge about a few *core* blocks of content in the course—blocks of knowledge without which a LA1 student cannot pass their course.

4.1.2 How to find the knowledge students mobilize?

To identify knowledge students activate, I conducted Task-Based Interviews (TBI) (Goldin, 2000) with students who had completed LA1 a month prior to the interviews. These were semi-structured interviews in which I instructed students to think aloud as they performed tasks similar to the problems they had to solve on their LA1 final exam. I describe the task and interview designs in Section 4.2. In this section, I justify the choice to conduct TBI to achieve the aims of this research.

Goldin (2000) describes the use of task-based interviews as research instruments in mathematics education research. Such interviews involve at least one subject and an interviewer interacting relative to some task(s) introduced in some structured way. The subject's interactions with the task(s) and with the interviewer are used to make inferences about their "mathematical thinking, learning, and/or problem solving" (Goldin, 2000). TBI allow to elicit knowledge students develop apart from mathematical procedures, cognitive representations they hold, beliefs—information pertinent for qualifying learning that takes place in didactic institutions.

Hardy (2009a) and Broley (2020) used TBI to investigate students' practices in relation to institutional norms. Their framework was the same combination of the ATD and IAD framework described in Chapter 3. From this perspective, they viewed the knowledge students are expected to learn to be the knowledge needed to complete final exam tasks (given their weight in determining students' performance in their course). To achieve their research aims, part of their work was to develop models of students' knowledge; they used TBI to create data whose analysis would yield these models.

Hardy (2009a) and Broley (2020) analysed course documents (e.g., exams, textbooks) to inform their construction of tasks that could elicit the knowledge students had learned in their course, as well as students' perception of what they were expected to learn in their course. They used textbook and exam tasks to determine what is normally expected of students, and used these norms to inform their TBI design. For example, Hardy designed tasks that resembled a routine limit-finding task from a college calculus course but which differed in the calculus needed to complete it: the resemblance was in the type of rational function at stake (one with a common factor in numerator and denominator) and the difference was in the type of limit at stake (the limit was to be taken at a point of continuity of the function, whereas the normative task with such functions always involved an indeterminacy that would require students to factor out the common factor). The intent of such task design is to determine whether students' response would reflect the course norm (always factor out the common factor) or mathematics intrinsic to the task. This is just one example among various design principles Hardy and Broley used to create tasks that would help to situate students' knowledge relative to course norms and to determine how amenable these are to the mathematics targeted in a course.

In addition to the design of the TBI tasks, Hardy and Broley used a semi-structured script (Goldin, 2000) to guide their questions and interventions throughout the interviews. The purposes of these questions and interventions were to clarify what students were doing and to elicit the justifications they had for what they were doing. Comments students made throughout their interview, along with their activity in response to the tasks, allowed for inferences about the non-mathematical nature of their practices: taken together, students' comments and activity brought out considerations to which they paid attention and which were not about (and at time inconsistent with) the mathematics (e.g., students spoke of what was "usual" in their course).

4.2 Research instruments

To identify knowledge students activate, I conducted Task-Based Interviews (TBI) (Goldin, 2000) with students who had recently completed LA1 at a large North American urban university. In these interviews, I asked students to think aloud as they worked on problems similar to the ones they had to solve to pass their course. The tasks in these problems were similar enough to those given in the final exams, so participants were able to engage with them, but I designed them so they would reveal which knowledge participants do or do not have in relation to these tasks (as discussed in Section 4.1.2).

4.2.1 The tasks

In this section, I first explain the high-level reasoning and process that guided my design of these problems and then present specific expectations relative to each of the problems.

4.2.1.1 Principles for task design

The TBI problems needed to be recognizable to participants so they could engage with them, but also designed so as to reveal the knowledge students choose to mobilize and allow me to make inferences about the nature of the praxeologies they mobilize.

To create problems that are recognizable, I created problems that involve praxeological components (e.g., tasks, technologies) of tasks in midterm and final exam problems. I looked to 4 midterm and 6 final exams given in LA1 from the years 2014 to 2019 to identify tasks that would be recognizable to students and to identify the knowledge likely to be routinized in students' praxeologies.

After a preliminary review of a few past final exams for LA1, in which I identified the tasks and types of tasks to be performed, I decided to focus most attention to tasks that relate to topics of linear systems and their solutions. Tasks, techniques, and theory pertaining to solving linear systems appear in various topics in the course, so the practices students developed in relation to this topic are a pertinent target. I then identified all exam tasks where the required technique would relate to this topic and to make note of properties typical to mathematical objects in these tasks (e.g., linear systems have coefficient matrices $A \in M_{m\times n}(\mathbb{Z})$ where $m, n \in \{2, 3, 4\}$). This amounted to 87 of the 116 tasks on the final exams. I identified the following task types:

- To solve a linear system.
- To determine the number of solutions of a linear system.
- To find a basis for the solution space of a homogeneous system.
- To find the algebraic representation of a geometric object that is determined by the intersection of geometric objects whose algebraic representations are given or can be found using techniques and/or theoretical constructs from real vector spaces.
- To solve a vector equation.
- To determine whether a set of vectors is linearly independent.
- To show a set of vectors is a basis for $\mathbb{R}^3$.

The purpose of this process - identifying tasks that involve techniques originating from the solution of linear systems - was not to create a *complete* model of the knowledge to be learned about linear systems on LA1 exams; the goal was only to identify tasks that would be recognizable and that might trigger routinized knowledge. For each of these exam problems, I identified the technique used in LA1 to solve it, as indicated by the course textbook; this helped to identify knowledge that stands to be routinized. These tasks, together with their related techniques, informed my design of tasks that could generate useful data in the TBI. By "useful data," I mean responses that would reveal participants' routinized knowledge, but also whether they have mathematical praxeologies that would give them autonomy in making choices of which techniques to apply.

The main objective that guided my task design was to determine whether students produce their techniques on the basis of mathematics at stake in a task or on the basis of what they were used to associating with that task. I expected that students' spontaneous reaction to the problems would reveal the type of practice they had developed in relation to routinized knowledge. I wanted to determine whether students would mobilize other knowledge in addition.

I designed one set of problems to visually resemble routine tasks but which either cannot be solved by the routine technique or can be solved significantly more efficiently by non-routine use of knowledge to be learned. The idea was to see if students are triggered (or 'deceived') by surface-level features to mobilize normative technique that is inappropriate for a task (in that it would not achieve the objective a student might seek). The guiding principle for the design of such task types mimics that proposed by Hardy (2009a).

I designed a second set of problems to include explicitly non-routine components yet be amenable to routine approaches that are computationally much heavier than non-routine approaches drawing on LA1 knowledge. The goal was to see whether students would favor computationally hefty routine approaches, and, if they did, two more things: first, whether they would be able to apply routines to non-routine tasks (where surface-level features of techniques may not apply); and second, whether they would be able to mobilize less routine knowledge as well.

4.2.1.2 Application of the task design principles to each TBI problem

I designed 8 problems for the TBI. In this section, I briefly describe the reasoning behind the design of each. In Chapter 5, where I elaborate my analysis, I address the knowledge to be learned in LA1 and which is relevant to each of the TBI problems; this includes more detail on the routines I had identified in past exam tasks and which informed specific choices in the design of each task (e.g., such as choice of scalars and matrix sizes).

Problem 1

Solve the following equation for C.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} BC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This resembles the task of type "to solve a matrix equation" in which the task is to manipulate matrices using matrix operations. In LA1, by applying properties of matrix addition, multiplication, and scalar multiplication, students can isolate the matrix for which they are meant to solve. These operations do not suffice to complete Problem 1. The knowledge needed is that the product of matrices is invertible if and only if all matrices in the product are invertible. All matrix symbols in this task are recognizable to students, and the notion of matrices being invertible or not is part of the knowledge to be learned in the course, but is never relevant in the LA1 task of type "to solve a matrix equation."

I wanted to see if students' spontaneous reaction would be to mobilize the normative technique for this task, and whether and if students would be able to mobilize any other knowledge productively.

Problem 2

The coefficient matrix below is invertible. Solve the system:

$$\begin{bmatrix} 9 & 16 & 3 & 4 \\ 5 & 6 & 0 & 8 \\ -2 & 3 & 0 & 4 \\ 3 & 6 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 2 \\ -3 \end{bmatrix}$$

This resembles the task of type "to solve a linear system," for which the usual technique in LA1 is to use row-reduction on the augmented matrix or, if explicit instruction is given, to use Cramer's rule. The affirmation in Problem 2 that the coefficient matrix is invertible is meant to trigger the knowledge that the system has one solution, and consideration of the scalars at stake a trigger to find the solution by observation. I considered the possibility that students might not even consider solving the system by observation, as this is never done in LA1, and if they do, that they may struggle to do so; but I did wonder whether the affirmation that the matrix is invertible might prompt students to use Cramer's rule, and in that case, whether they would have the technical dexterity to capitalize on the proportionality between the first column of the coefficient matrix and the column of constants to the right of the equation.

In light of routines associated with techniques for solving linear systems, I expected two possibilities. One, that the statement that the coefficient matrix is invertible may deceive students into using (lengthy) matrix inversion to solve the system, given the LA1 task type most visually similar to this one and which is solved by multiplying both sides of the equation by the inverse of the coefficient matrix. The second possibility was that students would row-reduce the augmented matrix, in an ironic twist on the purpose of row-reduction as a technique meant to produce an equivalent system whose solution can be readily observed.

Problem 3

> Show that $(w_1, w_2, w_3) = (29, -9, 3.2) \times (11, 2.1397, 41)$ is a solution of the following system.
>
> $$\begin{array}{rcrcrcl} 29x & - & 9y & + & 3.2z & = & 0 \\ 11x & + & 2.1397y & + & 41z & = & 0 \end{array}$$

The task resembles the LA1 task of finding a basis for the solutoin space of a homogeneous linear system of two equations in two unknowns, where the technique is to reduce the augmented matrix so as to find a general solution. Problem 3 differs from this task in that it does not instruct to find a basis of the solution space of the system, let alone its general solution. Problem 3 is also distinct in its inclusion of an object that never appears in the "find a basis" task—a cross product. I wondered if students would mobilize geometric knowledge about cross products and planes or even make a comment about the repetition of the scalars in the cross product vectors and coefficients of the equations. Given that tasks and techniques in which cross products appear always require students to compute the cross product, I expected students to compute the cross product and plug it into the system; I wondered if the non-integer scalars would prompt students to seek a different technique.

Problem 4

> Find a non-trivial solution of the following system:
>
> $$\begin{array}{rcrcrcl} -5.2x & + & 2y & + & \pi z & = & 0 \\ 4x & - & 1.3y & + & 4z & = & 0 \end{array}$$

This problem is intentionally placed right after Problem 3. I wondered if students would address any similarity between the two tasks, and if so, what similarity they would note. I wondered if students would use what they did in Problem 3 to produce a technique for Problem 4—and if so, what justification they would have for doing so. I wondered if students would just mobilize the technique for the basis-finding task whose algebraic symbols resemble those in this task—or if the non-integer scalars would prompt students to seek a different technique (and if so, for what reason).

Problem 5

> Given $k \in \mathbb{R}$, the vectors $(-k, 1, 1), (-1, 1, k)$, and $(1, 0, 1)$ form a parallelepiped of volume 0. Find the values of k for which the vectors are linearly independent.

This task is non-routine but does include a component that students could latch on to as a prompt to engage in a routine task: the instruction to find some condition under which some vectors are linearly independent. Students may see this as similar to the LA1 task to check whether a given set of vectors (often three in $\mathbb{R}^3$) is linearly independent. Given this similarity, I expected students to mobilize the technique normative for this task: either reduce the appropriate matrix (built from the given vectors) to its reduced row echelon form or compute the determinant (of the appropriate matrix) whose (non-)zero value would indicate whether the vectors are linearly (in)dependent. I expected that students may struggle to use elementary row operations accurately with one entry being an unknown, though one LA1 task does have students do this for a different task type.

I wondered if, rather than or at least in addition to engaging in the computationally-heavier routine approach, students would mobilize the geometry that is an intrinsic component of the task. From the affirmation that the vectors form a parallelepiped of volume 0, would they infer the vectors are coplanar? If yes, would they (attempt to) mobilize this to make an inference about the linear dependence of the vectors?

Problem 6

> Solve the following system of equations:

$$\begin{array}{rcrcrcl} x^2 & + & x & + & 1 & = & 0 \\ 2x^2 & + & 4x & - & 6 & = & 0 \end{array}$$

The task is ostensibly non-routine; the equations are not even linear. Solving quadratic equations *is*, however, a routine from high-school algebra, which is prerequisite to LA1. I wondered if students would solve the first equation, find it has no (real) solution (I presumed students would only know of real numbers at this stage in their mathematics education), and conclude the system has solution. But I also wondered if students would be 'tricked,' given the overarching context of the interview, to sidestep this approach and instead adapt the usual system-solving technique: row-reduction. If this were to be the case, I wanted to see students if students would be able to mobilize the technique to a non-routine system.

Problem 7

> Determine the number of solutions of this system of equations:

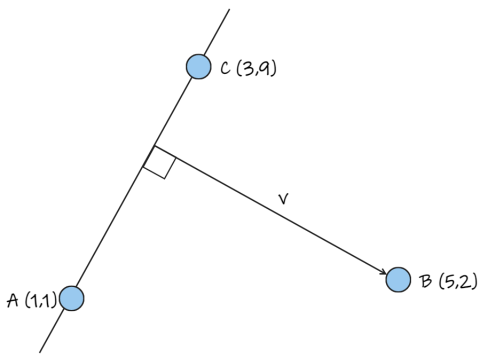
$$(x, y) = (1, 3) + t(1, 5)$$

$$(x, y) = (3, 7) + r(-2, 1)$$

This problem coordinates a non-routine algebraic representation into the routine LA1 task “to determine the number of solutions of a linear system of equations.” The routine task has its linear equations in point-normal form—never in vector equation form—and is usually solved by row reduction. It’s possible to find point-normal equations correspoding to these vector equations, though I wondered whether students would be able to do so. I rather expected students to equate corresponding components in the vector equations and use high-school techniques for solving linear systems (substitution, etc.). Such an approach would most closely resemble routines students are accustomed to in LA1. I wondered whether students would mobilize the geometric representation of vector equations to observe the given equations represent lines that are not parallel, and conclude they have exactly one point of intersection, and the system has exactly one solution. Students do not usually have to mobilize geometric representations of vector equations, though an explanation of why vector equations correspond to lines is part of the knowledge to be taught in the course.

Problem 8

> Find the length of the vector $\vec{v}$, which has B as terminal point and is orthogonal to the line that goes through the points A and C.



There was no intent to ‘trick’ students into any routine technique here—this is rather an open problem designed to examine how students go about a task that does not prescribe a technique (by virtue of resembling some routine task in LA1).

This problem can be solved using the formula for the distance between a point and a line in R^2. It can also be solved by mobilizing the view of $\vec{v}$ as the component of $\overrightarrow{AB}$ orthogonal to $\overrightarrow{AC}$. It’s possible to view the length of $\vec{v}$ as the height of triangle ABC relative to a base that can be calculated using a LA1 formula for norms, and it’s possible find the area of the triangle using a LA1 formula for areas of parallelepipeds in 2-space. It’s also possible to find the area of the triangle using a LA1 formula for areas of parallelepipeds in 3-space, but this would require mobilizing a projection of $\mathbb{R}^2$ into $\mathbb{R}^3$ (e.g., by working with the points $(1, 1, 0)$, $(5, 2, 0)$, and $(3, 9, 0)$. None of the techniques described here are routinized in LA1 in the sense that, when students are to use any of the formulas involved in these techniques, *they are instructed to do so* either explicitly or implicitly (by virtue of a course norm that a certain task type is always completed using a certain formula).

A rather more normative, but still not routinized technique for Problem 8 would be to produce a linear system whose solution would be the initial point of $\vec{v}$. This isn't routinized in that tasks in LA1 do not require students to use geometric representations to produce equations. But it's rather more normative than the techniques proposed above because, in LA1, it is a norm that linear systems are used in many different task types, including some that are nominally about lines or planes ("nominally" in the sense that a task statements might refer to a given equation as an equation of a "line" or "plane"). I expected that most students would mobilize a technique of this sort, but wondered what knowledge they would mobilize to produce relevant equations.

In all, I wanted to see how students might mobilize normative knowledge in non-normative ways when given the opportunity. The opportunity, here, is the unavailability of routine technique to be mimicked.

4.2.2 The interviews

I conducted 10 task-based interviews with students who had completed LA1 about a month prior to the interview. I planned 2 hours for each interview; most interviews lasted the full 2 hours. In these interviews, I presented students with the tasks in Section 4.2.1.2, one by one and in the same order as above, and instructed students to think aloud as they worked on these tasks. I provided blank paper, pencils, pens, and a calculator.

The interviews were semi-structured; I designed an interview protocol (see Appendix A) to guide my interventions. The general considerations for interventions were as follows:

1. If a student does not know a definition or formula, give it to them.
2. If a student is quiet for 1 minute, remind them to think out loud.
3. If a student asks a direct question, keep in mind the goal to see how they do a problem and the reason they give for doing it that way, and respond accordingly (e.g., turn the question back to them; if it's a request for validation of what they are doing, respond to "do what [they] think is most appropriate," and when they finish, ask why they chose that method and whether it is what they would do on an exam.
4. If a student is stuck
    a) on something that requires knowledge not taught in LA1 but which I expect them to know from previous courses (e.g., finding solutions of a quadratic equation), provide the information needed to proceed. Otherwise,
    b) ask what they would have done if they were stuck on this problem on an assignment or exam.
    c) If they are stuck on something that requires knowledge taught in LA1, give a series of increasingly directive hints (without saying what to do) without giving the answer away (Appendix A include problem-specific guidelines made in mind of the objectives behind the design of each TBI problem)

d) If they are still stuck, suggest moving on to another problem and returning to this one if there is time after they've attempted all the other problems.

5. If a student going in a wrong or overly time-consuming direction,

   a) do not let them go on for more than 10 minutes.

   b) If what they are trying to do is not clear, ask for clarification.

   c) Ask what it is they're hoping will happen.

   d) Acknowledge what they are trying to do and ask if they can think of another approach. If they cannot, suggest moving on to another problem and returning to this one if there is time after they've attempted all the other problems.

6. If you must improvise, keep in mind the goal is to see how the participants solve the problems and why they do what they do as they solve.

7. While a student is attempting a problem, ask about how they solve problems for themselves and for exams. E.g., given a set of vectors that are linearly independent, what do they need, as an individual, to be convinced that the vectors are or aren't linearly independent?

8. After a student attempts a problem, time permitting,

   a) Ask follow-up questions about anything the participant had produced (e.g., ask for clarification if reasoning was unclear, ask if what they did is something they would have done on an exam, ask if they would have received full marks, ask if they would be convinced by what they did if they were just doing it on their own, ask if they had thought of any other approaches to the problem and, if so, why they used one approach rather than another; if not, ask if they can think of another approach).

9. Once all problems had been attempted and problem-specific follow-up questions asked, time-permitting,

   a) Ask follow-up questions I was unable to ask about specific problems.

   b) If a student had exhibited strong emotions at any point, ask them to explain how they felt during the interview or about some of the problems.

   c) If a participant had mentioned, while solving the problems, that they would use the computer to accomplish some tasks, offer a computer and ask if they can show what they would have done.

I made this guide in anticipation of potential responses students might have to the problems. Broley (2020) elaborates six anticipated scenarios for a given problem: a student may be immediately stuck, off track/on track to a lengthy production, stuck during, and may have unexplained or unclear thinking, unexpected production, or meaningful production. I used Broley's (2020) protocol for her task-based interviews as a starting point to reflect on potential responses, adapting them to the linear algebra context as needed (e.g., I anticipated that given the nature of the mathematics at stake, students were more likely to be on track to a lengthy solution—or production—than one that is altogether off track).

The interviews were audio-recorded and all materials written by the participants during the interview collected.

4.3 Data analysis procedure

In this section, I describe my analysis procedures for identifying knowledge to be learned, for identifying and qualifying the knowledge students mobilized, and for identifying and qualifying students' positions. My data analysis is in Chapter 5 and organized as follows: the ith section corresponds to Problem i of the TBI ($1 \leq i \leq 8$) and, in the last section, I present my analysis of students' positioning.

4.3.1 Analysis procedure for identifying knowledge to be learned

I made models of knowledge to be learned (KtbL) in LA1 that is relevant for each TBI problem. To produce these models, I first identified the exam tasks targeting praxeologies relevant for each problem.

To identify exam tasks targeting praxeologies relevant for each problem, I first identified LA1 technologies pertinent to each problem. For instance, for Problem 1, I identified "matrix equation" and "matrix inverse." I then identified all tasks, from 4 midterm and 6 final exams given between 2014 and 2019 in LA1, that involved these technologies. I tagged each such task by a code to trace it to its location in the exam in which it was given. I recorded the task stated in the problem, the problem statement's (mathematical) components (e.g., scalars, symbols used, fields or vector spaces involved, any LA1-specific knowledge explicitly stated in the problem), and the TBI problem technology to which the task relates. The data produced in this way was organized in an Excel table; this table, along with others produced from it (described below) are not included in an appendix in this document because they are not amenable to PDF format, but are in a shareable digital file that can be viewed upon request.

I filtered the tables I produced to view all exam tasks related to each LA1 technology. This showed, for example, that there were 7 "matrix equation" tasks, all of the type "to solve a matrix equation for a matrix X" where one side of the equation was an expression involving X and 1-3 matrix operations involving other (given) matrices, and the other side of the equation was a given matrix; in all cases, X could be isolated by multiplying by the inverse of another matrix and possibly using 1 or 2 other matrix operations (e.g., addition or using a distributivity property). Wherever a LA1 technology was associated with a larger number or variety of exam tasks, I produced pivot tables (an Excel function) to facilitate finding the task types to which the tasks belonged.

For each task type, I identified the technique(s) students would have been expected to activate. In some cases, the exam task instructed students on which technique to use (e.g., "use Cramer's rule"). Otherwise, I turned to expository text (including examples) in the textbook to determine the expected techniques. In most cases, the exam task corresponded (almost) identically to a task in a textbook example. I used these solved examples as templates for the techniques expected of students, but also relied on my experience grading exams in LA1 in conjunction with other LA1 instructors to inform my description of what's expected of students on an exam (e.g., from this experience, I knew instructors do not usually expect students to include justifications that are given in solved examples). I similarly identified any theoretical block elements (technologies, theory) that students need to have to deliver on a technique (e.g., formulas) or may be expected to address in their exam submissions.

Through the procedure described above, I created models of knowledge students are expected to learn and which relates to each TBI problem: the models are praxeological, indicating tasks to be completed, techniques through which these tasks are to be completed, and any theoretical block components that may be needed. I present these models in Section 5.i.2 (for each $i = 1, \ldots, 8$) as part of the analysis of data pertaining to Problem i.

The analysis of KtbL that relates to each TBI task corresponds to 75% of the tasks in the exams to which I had access; I take this into account in the discussion of the results (presented in Sections 5.i.2, for each $i = 1, \ldots, 8$) to make observations about the nature of the praxeologies students are expected to develop in LA1.

4.3.2 Analysis procedure for identifying and qualifying the knowledge students mobilized

In this section, I first describe the procedure I used to identify the knowledge students mobilized in response to each TBI problem, and second my procedure for qualifying it as (non-)mathematical.

I identified students' mobilized knowledge from transcripts of their TBI and what they wrote on paper as they attempted each problem. I time-stamped sections of their written productions to match students' written activity with the transcripts. This was the data I analysed to identify and qualify students' mobilized knowledge.

The analysis procedure for identifying students' mobilized knowledge, in response to a given TBI problem, and qualifying it as (non-mathematical), consisted of three steps. I organize this section accordingly.

4.3.2.1 Step 1 of the analysis procedure for identifying and qualifying students' mobilized knowledge

I first split students' activity into steps in chronological order. I considered a unit of their activity a "step" when:

- it indicated the task the student was attempting to complete, as indicated by the technique they activated or comments they made (e.g., as in Problem 1, where students' multiplication by symbols for inverses of matrices indicated the task they were attempting to complete was "to isolate a matrix in an equation"); or
- the student had not identified a task to undertake and was stuck, as indicated by their comments and/or lack of activated technique.

I categorized a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step.

In one column, categorized each step in terms of the task-technique pairing (i.e., the practical block) the student had activated. In a second column, I described the student's engagement with the practical block indicated in the first column. For example, if the task a student had activated was to isolate a matrix in an equation, and their technique was to multiply by inverses, then column 2 indicated more detail about what the student activated in this technique (e.g., if they had computed any inverses, if they had isolated a matrix by expressing it in terms of *symbols* for the inverse of a matrix, but without actually computing the inverse; I also indicated if the student had completed the task or only started to enact the practical block in column 1). In a third column, I copy-pasted from the transcripts any comments they had made which indicated their theoretical block (i.e., the reasoning producing or justifying the practical block they had activated). In a fourth column, I included comments a student had made in that step and which didn't fit into a description of their practical or theoretical block, but which seemed to indicate a characteristic of their positioning. In a fifth (and sometimes additional) column(s), I included screenshots of a student's written production if it was needed to convey their engagement with a practical block.

As I progressed from parsing one student's activity into steps to parsing another student's activity in this way, some practical blocks recurred. When this happened, I consolidated students' "engagement with a practical block" in the row corresponding to that practical block.

This constituted the first step of analysis of students' mobilized knowledge. This produced (Excel) tables of substantial size, as I included large blocks of text from the transcripts—essentially, I had parsed students' transcript (portion related to Problem i) into chronological steps, each conveying one praxeology the student had activated. For practical purposes, I produced one such table for each step; that is, I produced a table that included what all participants had activated in Step 1 of their engagement with Problem i, another table that included what all participants had activated in Step 2 of their engagement with Problem i, etc. Some students' activity consisted of more steps than others'. The tables corresponding to each step sometimes included practical blocks that show in previous steps.

The first step of analysis of students' mobilized knowledge is organized in 8 Excel files, each conveying the paths of participants' activity as they worked on each TBI problem. The content of these files is not included in an appendix in this document because it is not amenable to PDF format; the files can be shared upon request.

4.3.2.2 Step 2 of the analysis procedure for identifying and qualifying students' mobilized knowledge

The second step of analysis of students' mobilized knowledge was to produce a table that would summarize the paths of participants' activity as they engaged with Problem i (for each $i = 1, \ldots, 8$). Each such table, captioned "Paths of LA1 Students' Activity in Problem i," is presented at the start of Section 5.i.3 (for each $i = 1, \ldots, 8$), where I lay out my analysis of the knowledge students activated in response to Problem i. This summary table consists of a two main columns, each split into further columns.

The first main column indicates practical blocks $[t, \tau]$ that appeared in students' mobilized knowledge. In some cases, students' task/technique corresponded to the same practical block but differed in some way; in such cases, the first main column was split into two columns. For example, Table 5.1 (summarizing paths of students' activity in Problem 1) shows many students attempted to isolate C (in an equation of the form $M_1AM_2M_3BC = I$) by multiplying both sides of the equation by inverses; among these students, there were students who could again be grouped by their choice to multiply by inverses of the matrices M_1, A, M_2, M_3, B; other students, in later steps, suggested to multiply by C^{-1}, as they believed this would isolate C^{-1} and allow them to find C by computing the inverse of C^{-1}.

The second main column includes a summary of participants' engagement with the practical blocks (from the first main column). This is organized by the steps previously identified. Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I grouped students according to Step 1 and color-code the groups to help trace students' paths thereafter.

The purpose of the summary table for each TBI problem was to have a single display that captures the paths of students' activity in each problem. This table does not include participants' theoretical blocks, but these can be found by consulting the tables described in Section 4.3.2.1.

4.3.2.3 Step 3 of the analysis procedure for identifying and qualifying students' mobilized knowledge

The final step of the analysis procedure produced the analysis of the knowledge students activated in response to Problem i, as presented in Section 5.i.3 (for each $i = 1, \ldots 8$).

In this final step, I used the summary table described in Section 4.3.2.1, together with comments from students' theoretical blocks (recorded in the tables described in Section 4.3.2.1), to infer the praxeologies students had mobilized in response to each problem. I then qualified the praxeologies students mobilized toward Problem i (for each i) as (non-)mathematical by comparing their praxeologies with the models of knowledge to be learned in LA1 that is relevant to the given problem (presented in Section 5.i.2), as well as with those described in my reference model for that problem (presented in Section 5.i.1). I describe the notion of a reference model as a research instrument in Section 3.1.1. The model presented in Section 5.i.1 is my model, as a researcher, of all knowledge at stake in LA1 relative to Problem i; it accounts for knowledge at various levels of didactic transposition. The goal of comparing students' praxeologies with those that model KtbL and those from the reference model was to determine whether students' practical and/or theoretical blocks reflect, exlusively, the mathematics at stake in a problem, or whether they incorporate other knowledge (that determined by course norms); additionally, I aimed to determine whether course norms would reflect or contrast with the mathematics intrinsic to a problem.

The analysis procedure here consisted of a guideline for the types of inferences I sought to make. For example, I took students' spontaneous response to each problem to indicate norms they had formed in LA1. I attended to the trajectories in their activity, after their initial response, to determine whether students would or could activate any other knowledge, and if so, what it was. I compared students' knowledge from different steps of their activity, and when relevant with their activity in other problems, to make reliable inferences about what students knew or did not know (whenever possible) and what they were mobilizing.

In addition to students' practical blocks, I also attended to comments they had made. I did this to gain insight about the techniques they were activating. I also attended to students' comments, contrasting them with the totality of their activity pathway—relative to a given problem but also relative to praxeologies students had activated in response to other problems—to infer, whenever possible, what students knew or did not know (whenever possible), to contrast these with what they mobilized, and also to make inferences about their theoretical blocks (i.e., and whether these consisted of mathematics intrinsic to a problem or were a mixture of didactic, social, and mathematical norms from their course).

These general principles meant I conducted comparisons of different moments from a student's activity until I reached a 'stable' model of the knowledge they had mobilized (and, when possible, of what they did or did not know). By 'stable,' I mean that it captured students' mobilization of knowledge not only in given moments of their activity, but overall, throughout this activity.

I organized the results by sections that group students by praxeologies that shared certain components (practical or theoretical). I coordinated my analysis in different sections to infer and qualify the (non-)mathematical nature of students' praxeologies.

4.3.3 Analysis procedure for identifying positions

In this section, I explain the analysis procedure I created to infer positions students had occupied during their tenure as LA1 students, as well as positions students *might* be prompted to occupy by certain task features.

The data targeted by this analysis are students' mobilized praxeologies, as elaborated in Sections $5.i.3$ ($i = 1, \ldots, 8$), rather than the transcripts and written productions from the TBI. This is because, to determine if a student's activity or comment indicates the position of, say, a Student, or that of a Client, or a Learner, or some other potential position, I need to consider the comment or activity in relation to norms of the didactic institution. This is due to the institutional relativity of the positions available to students in a didactic institution. I explain. The position of a Student is defined by a students' objective to pass their course (with a certain grade). To achieve this position, the student needs to determine what is expected of them. What's expected of students can change from one course to another and be more or less amenable to, for example, the objective of a Learner. While some comments students make may clearly indicate a position they occupy, the knowledge they mobilize can only indicate a position they had occupied in LA1 if it is held up against what was expected of students in LA1. Since this is the analysis completed in Sections 5.i.3 ($i = 1, \ldots, 8$), I can maintain consistency by using this analysis rather than reenacting a similar analysis for the purpose of identifying positions.

To analyse to infer students' positioning from the analysis presented in Sections 5.i.3 ($i = 1, \ldots, 8$), I first classified instances of a participant's activity (or comments) in terms of behaviors that put the activity (or comments) in relation with course norms. For example, I classified the instances described in

> Problem 5; sections 2.5.3.2.1, 2.5.3.3.3; P2 was able to perform the task by activating this representation but only once I had prompted him to do so; further, he did not believe he would get full marks if he submitted such an "analysis" for grades but did think his calculations would award him full marks, and, further yet again, he said this "analysis" would not have convinced him if he had been working on his own ("usually, I do calculations"); P2's model of what's expected of students in LA1 included τ_{42} but not the geometric representation: the calculations, he said, would grant him full marks on a submission in the course, but the explanation of the geometry would not.

in terms of this "behavior":

> believes students are expected to demonstrate calculations, not use concepts

Next, I reflected on whether a behavior contributes to the objectives of a Student, Learner, Client, Person, or possibly an as-of-yet unelaborated position. Once I determined that a behavior contributes to the objectives of a certain Position (I use the term as a placeholder for Student/Learner/Client/etc.), I identified how the behavior contributes to this Position by classifying it by a property of that position. For example, I identified the behavior listed above by this position property:

> belief about expectations of students produced by normative LA1 KtbL

Some instances were amenable to classification by more than one behavior. Some behaviors, in turn, were amenable to classification by more than one position property; this is because the position properties were not necessarily mutually exclusive (e.g., there is overlap between "lacking agency and sense of agency over mathematics at stake" [1] and "surface-level grasp of KtbL"). I classified a behavior by the position property to which they contributed most directly (in the given instance), and other times it was appropriate to classify a behavior by more than one position property.

As I progressed through each section, classifying instances by behaviors, position property, and position, I endeavored to use already-identified behaviors and position properties when possible. I also examined whether identified behaviors (or position properties) were replicates of other behaviors (or position properties) and, when this occurred, synthesized the behaviors into a single one. I similarly synthesized behaviors (or position properties) that were not replicates of one another but shared in an overarching trait.

Constraints unrelated to this work prevented further synthesis of the behaviors and position properties I used to infer, from a given instance of a student's activity, a position

[1] Sierpinska et al. (2008) describe the former as lacking the capacity to use the mathematics as needed and the latter as lacking confidence in one's capacity to use this mathematics.

they had occupied in LA1 or a position they may be prompted to occupy by certain tasks. Further synthesis could be achieved, for instance, by using triangulation to classify the instances chosen for analysis by the identified behaviors and position properties.

The operationalization of the positioning framework, as described above, was the first step for analyzing students' positioning. The result is organized in an Excel file. The content of this file is not included in an appendix in this document because it is not amenable to PDF format; the file can be shared upon request.

The second step consisted in presenting the data produced in the first step in ways that can help to answer questions I had about students' positioning. One question, for example, was whether some TBI problems triggered only certain positions (e.g. that of a Student) while other TBI problems had triggered others (e.g., Learner) as well. to this end, I used the pivot table function in Excel to organize the data from the first step in various ways. For example, to answer the question just posed, I produced a pivot table that indicated, for each TBI problem, which positions it had triggered in students, as well as the number of instances associated with each position (for that problem). The analysis I present in Section 5.9 is organized according to charts; each chart corresponds to a pivot table designed to answer a particular question I had about students' positioning.

Chapter 5

Analysis LA1

This chapter is organized as follows: Section i corresponds to Problem i of the TBI (for each $i = 1, \ldots, 8$) and in Section 9 I present my analysis of students' positioning. The answers to the research questions are in this chapter. I present the praxeologies expected of students, when considering LA1 exam tasks, in Sections 5.i.2 (for each $i = 1, \ldots, 8$): these are the models of knowledge to be learned that relates to each of the TBI problems. I present the praxeologies students mobilize and examine their (non-)mathematical nature in Sections 5.i.3 (for each i).

5.1 LA1 Problem 1

The following was the first problem presented to the 10 LA1 students in the TBI:

Solve the following equation for C.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} BC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5.1.1 Reference model for LA1 Problem 1

I define, in Section 3, the ATD construct of reference model (Chevallard, 1985) as a backdrop against which to examine the knowledge students are expected to learn in a didactic institution as well as the knowledge they eventually mobilize. Additionally, there is the ATD tenet that any human activity (including mathematical) can be described by a praxeological model (Chevallard, 1999): a model that identifies task(s) t to be accomplished in that activity, techniques τ through which to accomplish a task, and a theoretical block $[\theta, \Theta]$ consisting of technology θ that frames and produces tasks and techniques and theory Θ that frames and produces technology θ. A *reference model* is the didactic researcher's model of knowledge pertinent to a given task. In the case of mathematical activity, this model includes knowledge from various levels of didactic transposition, from knowledge of scholars, to that of teachers, to that of students; the reference model is meant to be used as a backdrop against which to analyse knowledge at any stage of this transposition.

The task t in Problem 1 is to solve a matrix equation in $M_{2\times2}(\mathbb{R})$. The technologies underlying this task combine to form what's commonly called matrix algebra. Without rehashing a comprehensive account of this field in linear algebra, I list some of the technologies pertinent to Problem 1, given the nature of the equation at stake (wherein the left-hand side is a product of 2×2 matrices and the right-hand side is I_2): the definitions of matrices, of linear maps between vector spaces and of matrix multiplication, of equality between matrices, the notions of identity matrix, of inverses of matrices, and any technology (such as the inversion algorithm, determinants, formulas for the inverse of a matrix A when it exists, such as $A^{-1} = \frac{1}{\det A}\mathrm{adj}(A)$, and various theorems) related to inverses of matrices and their existence. I will broadly refer to this list of technologies by θ[1]. The theory Θ from which springs this technology is the axiomatic and logical discourse that frames the discipline of linear algebra.

Below, I refer by M_1, M_2, and M_3 to the matrices that are given in Problem 1:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; M_3 = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

As the right-hand side of the equation in Problem 1 is the identity matrix, the task is to solve for a matrix C whose product with $M_1AM_2M_3B$ is I_2. M_3 is not invertible (e.g., $\det M_3 = 0$); hence, the product $M_1AM_2M_3BC$ is not invertible and so cannot equal I_2 (an invertible matrix), no matter the entries of C. So there is no matrix C for which $M_1AM_2M_3BC = I_2$ is a true equation.

Another variation of the argument might go like this: since M_3 is not invertible, $M_1AM_2M_3B$ is not invertible, and so there is no matrix C for which $(M_1AM_2M_3B)C$ is I_2. The argument presented in this and the previous paragraphs (and any other variation thereof) forms one technique τ_1 by which to perform task t.

Another technique through which to complete the task t is τ_2: assign variables to the entries of A, B, and C, and solve the system of equations obtained by equating the corresponding components of the matrices on either side of the equation. τ_2 can be completed without directly mobilizing the fact that the matrices at stake are not invertible, but the approach I propose does involve calculation of a determinant.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

The equation given in Problem 1 is thus equivalent to

$$\mathcal{A}\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{A} &= M_1AM_2M_3 &= \begin{pmatrix} 10a_{11} + 4a_{12} & 15a_{11} + 6a_{12} \\ -10a_{21} - 4a_{22} & -15a_{21} - 6a_{22} \end{pmatrix} \\ \mathcal{B} &= BC &= \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix} \end{aligned}.$$

[1]In the reference models for the TBI problems, I sometimes use the symbolic notation suggested in the ATD to refer to a single technology and sometimes to a collection of technologies. One reason for which I might denote a particular technology using symbolic notation is if it helps to communicate the results of my analysis—for example, if that technology is knowledge whose mobilization by students is, or is likely to be, a subject of investigation.

This corresponds to the following system of equations:

$$\begin{array}{rcrcll} \mathcal{A}_{11}\mathcal{B}_{11} & + & \mathcal{A}_{12}\mathcal{B}_{21} & = & 1 & (1) \\ \mathcal{A}_{21}\mathcal{B}_{12} & + & \mathcal{A}_{22}\mathcal{B}_{22} & = & 1 & (2) \\ \mathcal{A}_{11}\mathcal{B}_{12} & + & \mathcal{A}_{12}\mathcal{B}_{22} & = & 0 & (3) \\ \mathcal{A}_{21}\mathcal{B}_{11} & + & \mathcal{A}_{22}\mathcal{B}_{21} & = & 0 & (4) \end{array}$$

It can be shown this system has no solutions using the fact that the determinant of $\mathcal{A}$ is 0. (That $\det \mathcal{A} = 0$ can be shown by calculating it.) It is useful, first, to note the entries of $\mathcal{A}$ must be non-zero for the equation to hold. (This is stronger than necessary.)

If $\mathcal{A}_{21} = \mathcal{A}_{22} = 0$, then $\mathcal{AB}$ is a 2×2 matrix whose second row is made up of 0's, so the equation $\mathcal{AB} = I_2$ is false. If $\mathcal{A}_{22} = 0$ (that is, if $a_{21} \neq \frac{6}{15}a_{22}$) and $\mathcal{A}_{21} \neq 0$ (that is, $a_{21} = \frac{4}{10}a_{22}$), or vice-versa, there is a contradiction. Hence, $\mathcal{A}_{21} \neq 0$ and $\mathcal{A}_{22} \neq 0$. It similarly follows that $\mathcal{A}_{11} \neq 0$ and $\mathcal{A}_{12} \neq 0$.

Here is one way of showing the system above has no solutions: since $\det \mathcal{A} = 0$, it follows that

$$\mathcal{A}_{11}\mathcal{A}_{22} = \mathcal{A}_{12}\mathcal{A}_{12}.$$

Multiplying both sides of equation (3) by $\mathcal{A}_{22}$ yields

$$\begin{array}{rrcll} & \mathcal{A}_{11}\mathcal{A}_{22}\mathcal{B}_{12} + \mathcal{A}_{12}\mathcal{A}_{22}\mathcal{B}_{22} & = & 0 & \\ \Rightarrow & \mathcal{A}_{12}\mathcal{A}_{21}\mathcal{B}_{12} + \mathcal{A}_{12}\mathcal{A}_{22}\mathcal{B}_{22} & = & 0 & \text{since } \mathcal{A}_{11}\mathcal{A}_{22} = \mathcal{A}_{12}\mathcal{A}_{12} \\ \Rightarrow & \mathcal{A}_{12}(\mathcal{A}_{21}\mathcal{B}_{12} + \mathcal{A}_{22}\mathcal{B}_{22}) & = & 0 & \end{array}$$

Since $\mathcal{A}_{12} \neq 0$, this implies

$$\mathcal{A}_{21}\mathcal{B}_{12} + \mathcal{A}_{22}\mathcal{B}_{22} = 0,$$

which contradicts equation (2).

While τ_2 does not explicitly require knowledge that any matrix product involving a singular matrix is also singular, this knowledge does guide the calculations in τ_2 through the choice to consider $\det \mathcal{A}$. Ultimately, the failure of the equation to have a solution is driven by θ_1: any matrix product is singular if one of its factors is singular.

The reference model for Problem 1 is summarized by the praxeological models $[t; \tau_i; \theta, \theta_1; \Theta]$ $(i = 1, 2)$.

5.1.2 Knowledge to be learned in LA1 that relates to Problem 1

The type of task in which students are to solve a given matrix equation for a particular matrix (say, X) appears in 7 of the 10 past exams to which I had access from 2014 to 2019; I denote this task type by t_1. In LA1, t_1 has the following characteristics: the equation is usually in the form $AX = B$ or $AXB = C$ (or can be made to be in this form after one operation is applied), and any matrix by which X is multiplied is always invertible. The given matrices have 2-3 rows or columns and their entries are single-digit integers, usually with absolute value at most 4. I distinguish between t_1 and a different type of task that also involves matrix equations, but which is different in nature and which I will denote by t_2. This task type involves equations of the form $A\mathrm{x} = \mathrm{b}$, where

A is a matrix of size $m \times n$ where $m, n \leq 6$ (and if $m \neq n$, one is usually less than or equal to 4), x and b are column matrices, and the entries of x are unknowns. As with t_1, matrix entries (of A and b) are usually single-digit integers. Task t_2 is usually phrased in terms of linear systems: that is, the problem statement is to solve a linear system, and not "to solve an equation [between matrices]." I distinguish between t_1 and t_2 because of this difference in the problem statements but also because of the techniques assigned to them in LA1.

For t_1, the task to solve for a matrix X is synonymous with isolating X. To isolate X, the technique is to multiply both sides of the equation (on the appropriate side) by the inverses of the matrices by which X is multiplied. That this is the technique can be inferred, partially, from how the task is typically presented on LA1 exams: for 4 of the 7 tasks of type t_1 on past exams, the task was in part (b) of a problem where part (a) instructed students to find the inverse of some unrelated 3×3 matrix; and another one of these 7 tasks started off with the equation $(6A - 4I)^{-1} = B$ where B was a given and invertible 2×2 matrix. For example, to solve $AXB = C$, the technique is as follows:

$$A^{-1}(AXB)B^{-1} = A^{-1}CB^{-1} \Rightarrow X = A^{-1}CB^{-1}$$

Students needn't write, to get marks on an exam, that $A^{-1}A = I$, but this is the knowledge to be taught to students to justify the suitability of multiplying by a matrix inverse. And then what is left is to compute the product $A^{-1}CB^{-1}$. Hence, to complete tasks of type t_1, students are expected to know to isolate a matrix by multiplying by inverses of matrices by which it is multiplied, and so are also required to know how to find inverses of 2×2 or 3×3 matrices (students also need to know how to find the inverse of a matrix in other LA1 tasks in past exams: diagonalizing matrices or finding a large power of a given 2×2 matrix).

To find inverses of matrices, LA1 students are to be taught a formula for determining whether a matrix is invertible, a formula for determining its inverse if it exists, as well as an inversion algorithm (IA) by which they can simultaneously determine whether a matrix is invertible, and if it is, find its inverse.

Formula-wise, students are expected to know A is invertible if and only if $\det A \neq 0$. If A is invertible, then its inverse is given by the formula

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$$

or the version of this formula specific to 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

LA1 students are expected to know how to calculate determinants: 9 of the 10 exams I consulted included the task to find the determinant of a matrix, and 9 of the 10 exams required students to use Cramer's rule (and so to compute determinants).

Alternatively, to find the inverse of a matrix, students can use the IA, wherein they apply elementary row operations to reduce A to its reduced row echelon form (which I denote by $RREF(A)$) and the same operations are applied, in the same order, to I. If

$RREF(A)$ is found to have a row of 0's, then A is not invertible. Otherwise, $RREF(A)$ is an identity matrix; in this case, the matrix obtained (by applying the same elementary row operations to I), once A is reduced to I, is A^{-1}. Students are expected to know of certain technologies (e.g., elementary matrices) that produce the IA—for instance, there is a textbook section in which the syllabus-recommended problems are to find a row operation and the corresponding elementary matrix that would restore a given elementary matrix to the identity matrix—but students are not expected to demonstrate any knowledge about the IA on LA1 exams apart from a capacity to apply it (*if* they choose to use the IA instead of the determinant-adjoint formulas).

For t_2, the norm is that exam problems instruct students on which technique to use to solve linear systems $A\mathrm{x} = \mathrm{b}$; I discuss this further in my model of knowledge to be learned to do tasks such as that in Problem 2 (see Section 5.2.2). The most commonly-required technique is Gauss-Jordan elimination (this was required in tasks on all 10 of the past exams I consulted); students are also often instructed to use Cramer's rule (as in 9 of the 10 past exams I had) and on one exam were instructed to use A^{-1} to solve a given linear system. Knowledge to be taught about linear systems includes a proof that wields the technique for t_1 as technology that frames t_2: if the coefficient matrix A of a linear system $A\mathrm{x} = \mathrm{b}$ is invertible, then

$$A\mathrm{x} = \mathrm{b} \Leftrightarrow \mathrm{x} = A^{-1}\mathrm{b},$$

in which case $A^{-1}\mathrm{b}$ is the unique solution to the system.

In sum, the knowledge students are expected to know relative to matrix equations and matrix inverses is procedural and has a normative quality to it. What students must know about matrix inverses is procedural: to use formulas or an algorithm to determine whether a matrix is invertible, and if it is, what its inverse is. The procedure for matrix equations in LA1 is, normally, to multiply by the inverse of 1 or 2 invertible matrices. The knowledge to be taught explains the suitability of this technique (an inverse of a matrix A is defined as a matrix B for which $AB = BA = I$), and, perhaps, students might use this knowledge to verify whether a matrix they find is indeed the inverse of a given matrix.

While knowledge to be taught in LA1 does specify that an inverse of a matrix A is a matrix B for which $AB = BA = I$, LA1 tasks which students are required to complete *don't* engage students with the flip side of this definition: a matrix A is not invertible if there is no matrix B for which $AB = I$. Knowledge students are expected to learn about a matrix not being invertible is restricted to procedures. Is the determinant 0? Is the reduced row echelon form not an identity matrix? Theory and technology to be taught that produces the inversion algorithm, and in the lead up to the knowledge that a matrix is invertible if an only if its reduced row echelon form is I and if and only if its determinant is non-zero, includes the knowledge about how, if a matrix A is not invertible, then nor is AB or BA for any B. But, in LA1 knowledge-to-be-*learned*, the notion of whether a matrix is invertible is related to the notion of products of matrices through only one logical implication: *if* A is invertible, *then* $A^{-1}A = A^{-1}A = I$, and this is relevant for matrix-isolating tasks as $IB = B$ for any B.

In light of these considerations, the intent behind my design for Problem 1 is to see whether students fall into the trap: will they immediately start to multiply by inverses to isolate C? Will they realize the task involves a question about the invertibility of matrices, given that the left-hand side of the equation is a product in which C is a factor, and the right-hand side is I? And if so, will they be able to use any other LA1 knowledge about invertibility, once they know—apart from the fact that A and B are only invertible if one imposes that condition—M_3 is not invertible?

If students spontaneously mobilize the knowledge normally used to perform a LA1 task that, on one hand, resembles one of the TBI tasks, but on the other hand, differs from it in substance (e.g., as in Problem 1, which has the appearance of LA1 t_1 but is not the task to isolate a matrix), the behavior triggered in them by the task points at the norm(s) they had developed in LA1 in association with these tasks[2]. This, in turn, is evidence of the normative quality of the knowledge students mobilize after completing a course, and evidence of the gap between the normative behavior and behavior guided by knowledge of the mathematics at stake. Hence, students "falling into a trap" points to non-mathematical practice engendered by what students are expected to learn in LA1[3].

5.1.3 Knowledge LA1 students activated in response to Problem 1

Table 5.1 summarizes the paths of participants' activity as they worked on Problem 1. Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step. If a participant does not appear in the column for Step i ($i \geq 2$), it is because they did not engage in any new activity after Step $i-1$.

Participants were unable to complete Problem 1. Some (P1, P3, P4, P6) hypothesized there might be no solution to C, but were not certain and had a mathematically incorrect justification for this hypothesis: that C might not exist because $C = B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$ and M_3^{-1} does not exist. Participants' activity was mostly to isolate C by multiplying by inverses of matrices, at times even after they found these do not exist (as was the case for M_3: participants mobilized M_3^{-1} even after knowing it does not exist) or exist only conditionally (as is the case for A, B). I will refer to this technique by τ_1, to reflect that students mobilized the technique for the LA1 task t_1: to solve matrix equations of form $AXB = C$ (where A and B are invertible). I discuss how participants' engagement with Problem 1 was conditioned by τ_1 in Section 5.1.3.1. In preface to that section, I summarize students' engagement with Problem 1 below.

[2]A process studied in psychological research on decision-making is that of the priming effect (Bargh et al., 1996; Molden, 2014): the influence a stimulus (a prime) can exert on a person's behavior, even without their awareness of its effect; a stimulus can affect behavior by, for example, activating concepts a person had previously related with that stimulus, activating a given behavioral response, activating certain goals, or activating certain perceptions, e.g. in that a certain stimulus might direct what a person notices in a certain situation.

[3]By "what students are expected to learn" in a course, I do not refer to what a teacher might hope their students to acquire; I refer, rather, to the minimal core of knowledge students can mobilize to obtain a passing grade in that course.

Table 5.1: Paths of LA1 Students' Activity in Problem 1

Practical block $[t, \tau]$		Participant's engagement with $[t, \tau]$: Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	
τ_1: isolate C by multiplying both sides of EQ by inverses	multiply by (the symbols) M_1^{-1}, A^{-1}, M_2^{-1} and M_3^{-1} or $(M_2M_3)^{-1}$, and B^{-1}, or $(M_1AM_2M_3B)^{-1}$.	P1							enacts and gets stuck: gets $C = C^{-1}B^{-1}(M_2M_3)^{-1}A^{-1}M_1^{-1} \Rightarrow C^2 = B^{-1}(M_2M_3)^{-1}A^{-1}M_1^{-1}$, does not know how to manage the excess of C's
		P2							describes, finds M_i^{-1} $(i = 1, 2, 3)$, gets stuck upon finding M_3 is not invertible, and says "there's no way to solve this."
		P3							enacts; after I point out M_2M_3 is not invertible, P3 says there is not enough information to solve the problem.
		P4							enacts, then asks if to find a "numerical answer" (I say "whatever you think is appropriate"), says he doesn't think there's "enough information to have a final answer," leaves B^{-1} as is, and finds M_3^{-1} does not exist; concludes there is "no solution to C" because "if $[M_3]$ is not invertible, that means there's no inverse, but that's part of the expression for C." Hypothesizes "A and B are not given because there's no solution to the problem in the first place."
		P5							describes, finds M_1^{-1}, M_2^{-1}, uses IA on M_3 and finds M_3 has a RREF with a row of 0's ("so I kind of forget what to do once they get to a matrix that.. can't be put, like can't be reduced like that. Um, I guess let's put into RREF because I don't know what else to do"), multiplies both sides of equation by the matrix corresponding to $RREF(M_3)$ in the IA (it is incorrect as P5 did steps that are not elementary row operations). Gets $C = B^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix} M_2^{-1}A^{-1}M_1^{-1}$.
		P6							enacts, then asks if the goal is to find "a numerical solution" (I say "whatever you think is appropriate"); finds M_1^{-1}, M_2^{-1}, notes $\det M_3 = 0$, using IA finds M_3 reduces to a matrix with a row of 0's; finds M_2M_3, finds $\det M_2M_3 = 0$, determines it's not invertible; gets stuck ("I already know that C^{-1} is equal to this long string of numbers. But that doesn't solve anything [pause] [...] The determinant is zero so it's not invertible").
		P8							is initially confused because "usually [she] would do inverses" but does not know what A and B are (I say they are matrices); finds M_1^{-1}, multiplies both sides of EQ by M_1^{-1}, finds M_2M_3, asks if she "should get numbers in the end" (I say "whatever you think is appropriate"), multiplies both sides by A^{-1}, finds M_2M_3 is not invertible; stuck
		P9							enacts, finds M_1^{-1}, says "I don't know what A is so I leave A as A^{-1}, finds M_2^{-1}, uses IA to find M_3^{-1} but gets a row of 0's and says "I forgot how to get the inverse of this matrix," then finds $\det M_3 = 0$, saying this means M_3 does not have an inverse so C can't be found because M_3^{-1} is a factor of C
			P2						finds M_2M_3 and, applying IA incorrectly (choice of row operations incorrect and applied incorrectly), finds it is invertible; then isolates C
			P7						describes and concludes the method may or may not work since A and B are unknown
				P1					finds M_2M_3, M_1^{-1}, finds that M_2M_3 is not invertible, hypothesizes there is no solution to EQ because "normally" C would be $B^{-1}(M_2M_3)^{-1}A^{-1}M_1^{-1}$ but $(M_2M_3)^{-1}$ does not exist
	assign values to entries of A, B, then isolate C						P7		describes; says this would not work if B is not invertible, in which case an alternative method would be needed (assign unknowns to the entries of A, B, C, solve the linear system corresponding to EQ)
	multiply by (the symbol) C^{-1} to isolate C^{-1} and then find its inverse C		P1						enacts partially and abandons
			P5						enacts, reaches unclear conclusion: C^{-1} is some matrix but C "isn't a matrix" or "you can't inverse $[C^{-1}]$" because the expression for C^{-1} is a product involving a non-invertible matrix
			P6						claims $M_1AM_2M_3B = C^{-1}$ because $MN = I \Rightarrow N = M^{-1}$, and if he multiplies both sides of EQ on the right by C^{-1}, he'd get that $C^{-1} = C^{-1}$, "which doesn't help"; stuck
				P9					describes
	if A, B are invertible, isolate C		P4						enacts, stops once he finds M_2M_3 is not invertible, says is convinced there is no solution to C
assign entries to A, B, C and solve the system corresponding to the matrix EQ	assign unknowns to matrix entries	P10							refers to M_1, M_2, M_3, I by D, E, F, G, respectively, moves C and G to either side of EQ and attempts to solve EQ $GDAEFB = C$ by assigning the unknowns A, B, C, D to the entries of B and multiplying matrices, and then the unknowns E, F, G, H to the entries of A. Gets stuck. Did not mean that the entries $A, ..., H$ are the matrices $A, ..., H$.
			P3						assigns unknowns to the entries of A and B and finds the product $M_1AM_2M_3B$ (as a single matrix), abandons because "it makes it super difficult"; had hoped to determine if $M_1AM_2M_3B$ is invertible
			P9						describes as an alternative to attend to the problem that M_3 cannot be "remove[d]"
				P7					describes; later says to use this method if B is not invertible
use correspondence between EM (E an elementary matrix), and applying to M the elementary row operation that produces E		P7							describes and dismisses because the matrices by which C is multiplied are not all elementary
address θ: whether M being not invertible implies MN is not invertible				P3					P3 had mentioned in Step 2 wanting to figure out if $M_1AM_2M_3B$ is invertible, and I asked if she could think of a way to do that without doing all the calculations; P3 wonders about θ, but does not know
				P6					says he thinks that if M is not invertible, it's possible to multiply by some N such that MN is invertible; tries examples, this fails; begins an attempt at a proof by multiplying two general 2×2 matrices (with variables as entries); suspects θ is true because of my line of questioning
use knowledge about multiplication by matrices as linear transformations (symmetry about x or y axis)					P7				says that multiplying A by M_1 (as in $M_1AM_2M_3BC$, the right-hand side of the EQ) would be "like" doing a symmetry across the horizontal or vertical axis (in 2-space), and dismisses
use the notion of degree of freedom to gauge how many possibilities there are for C						P7			suggests, giving parabolas in $\mathbb{R}^2$ as an example in that 3 points specify a parabola, so there are 3 degrees of freedom in that situation; says there may be infinitely many combinations of A, B, C that satisfy EQ, asks "are you sure I have got a one unique solution to the C?"
use eigenvalues								P7	suggests (points out the right-hand side of EQ is I and brings up the formula $\lambda I - A = 0$), dismisses immediately.

$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $M_3 = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$, IA: inversion algorithm, $RREF(M)$: reduced row echelon form of M, RREF: reduced row echelon form, EQ: the given equation.

P10 was the only participant who did not activate τ_1. P10 struggled to come up with any technique through which to do Problem 1. She assigned unknowns to the entries of A, B, and C, multiplied some of the matrices, and got stuck. Three other participants also brought up the idea of assigning unknowns to the entries of A, B and C, though unlike P10, had a more explicit goal in mind for doing this. I will refer by τ_3 to the strategy of assigning entries to A, B, and/or C. P3 had hoped to use τ_3 to determine if $M_1AM_2M_3B$ is invertible (she knew, at this point, that M_3 was not) by representing it as a 2×2 matrix (as opposed to its representation as a product of 5 matrices). P3 abandoned this because she found the expressions involved "ma[de] it super difficult" (to complete the task). P9 and P7*'s goal for τ_3 was different. For them, τ_3 was an alternative to τ_1 once they abandoned τ_1 due to the presence of non-invertible matrices: P9 knew it is not possible to multiply by the inverse of M_3 as it does not exist, and P7* knew A and B are not necessarily invertible. (P9 did return to τ_1 in his third and last attempt at Problem 1; his turn-around to using the non-existent inverse of M_3 was to multiply both sides of the equation by C^{-1}, and then to find the inverse of C^{-1} once C^{-1} is isolated. P9 did not acknowledge any deficiency in this suggestion.)

Two students (P3 and P6) turned to the question of whether a product AB can be invertible if A is not invertible—and both were unable to answer this question. I denote the notion at stake by θ. Addressing θ was the third and last step of both participants' engagement with Problem 1.

P3 had just abandoned τ_3 as a technique for determining if $M_1AM_2M_3B$ is invertible, with M_3 not being invertible. P3 wondered about θ but was unable to wield knowledge through which to examine it, concluding that "if [...][4] [the product $M_1AM_2M_3B$] is invertible, then the solution is [$C = B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$]. But if it is not, it is impossible to get the solution." It is not clear if P3 meant there would not *be* any solution or if it would not be possible to "get" it: she had also said that "maybe [she has] to assume that [$M_1AM_2M_3B$] is invertible if [she] really want[s] to get C" and added, in response to the prompt "and the fact that [M_3 or M_2M_3] is not invertible," that she is "not sure."

P6 had fibbled with two takes on τ_1: first, he isolated C and expressed it in terms of inverses of M_1, A, M_2, M_3, and B, and got stuck once he found M_3 is not invertible; and then, he isolated C^{-1} by multiplying both sides of the equation by C^{-1}, but got stuck after having done so and asked for help. It was the prompt I gave P6 at this point that pointed him in the direction of θ. I pointed out two things P6 had said: that $M_1AM_2M_3B$ is the inverse of C since their product is I; that M_3 is not invertible; and I added there's a question about A and B: what if they are not invertible? This prompted P6 to ask if I'm "saying that there's no solution." He added that even if a matrix is not invertible, its product with another matrix could turn out to be invertible. He attempted to verify this. He tried two examples, multiplying M_3 by two matrices and found these products were not invertible. He then said that "two examples, where [he's] just trying to think of some random [numbers], it's not a proof" and attempted "a general case" as he believed it "better to abstract [the situation] into letters or variables." P6 had written two general 2×2 matrices with unknown entries, found their product, and seemed to try

[4] I use "[...]" to indicate that a part of the quote was omitted; I only remove parts whose omission from a quote does not change its potential analysis. I use "..." to indicate a momentary (e.g., 1-2 second) pause in a student's speech and "[pause]" to indicate any significantly longer pauses in their speech.

to find conditions under which something (not clear what) would equal 0. While P6 was unable to demonstrate anything, he concluded a product AB would not be invertible if B is not invertible, but this conclusion seemed rather to reflect inferences he made from social cues ("all these questions you're asking me, you're making me very uncertain") than knowledge about mathematics relevant to θ.

Finally, P7* was the only participant who brought up knowledge other than τ_1, assigning entries to A, B, C, or θ—but none of the knowledge he proposed proved helpful to the task. His spontaneous reaction to Problem 1 was to bring up multiplication by elementary matrices; the "identity matrix on the right-hand side of the equation" and the product of "many matrices" on the left-hand side brought to his mind a "method": from P7*'s description, I recognize a task type in the problems listed at the end of the course textbook section about elementary matrices and inverses[5]. The task type is to find A given an equation of the form $EA = B$ (or even $E_1 \cdots E_m A = B$, where m is some single-digit whole number), where the E's are elementary matrices. P7* characterized elementary matrices as a "recording [of elementary row] operations." P7* dismissed this approach as he noticed at least one of the matrices in Problem 1 is not an elementary matrix.

P7*'s second suggestion was τ_1 (to isolate C by multiplying by matrix inverses); his third τ_3 (to assign unknowns to the entries of A, B, C and solve the system corresponding to the matrix equation); his fourth was to use knowledge about multiplication by matrices as a linear transformation, and while he explained the sense in which multiplying by

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has the effect of instigating symmetry across one of the axes in $\mathbb{R}^2$, P7* did not address the fact that M_1 was not multiplying elements of $\mathbb{R}^2$ but elements of $M_{2\times 2}(\mathbb{R})$. He dismissed this suggestion too.

P7*'s fifth suggestion was to use the notion of degree of freedom to gauge how many possibilities there are for C. To explain what he meant, he brought up parabolas and the expression $ax^2 + bx + c$, saying there are 3 degrees of freedom when it comes to these objects. A parabola is fixed in 2-space once three points on the parabola are known. In application to Problem 1, P7* did not suggest anything more specific than this broad concept, but did speculate there may be infinitely many combinations of matrices A, B, and C that satisfy the equation, and asked if I was "sure [there is] one unique solution for C."

P7* next returned to τ_1 and τ_3, suggesting to assign values to A and B and then apply τ_1, but dismissed this again, saying it would not work if B is not invertible. P7* quickly dismissed his next and last suggestion: he pointed to the presence of I in the equation, brought up the formula $\lambda I - A = 0$, and then the notion of eigenvalues, but dismissed this suggestion almost as soon as he made it.

In sum, my participants' attempts at Problem 1 were limited to the knowledge normative in LA1 for completing task types t_1 (isolate a matrix by multiplying by inverses)

[5]It is possible this task appears in LA1 assignments, but I did not have access to these.

and t_2 (to solve the linear systems produced by finding the product $A\mathrm{x}$ in $A\mathrm{x} = \mathrm{b}$ and equating corresponding components). Given the design of Problem 1, the task at stake is not t_1, and t_2 would involve far more variables than LA1 students are accustomed to handling when solving a system of equations (and, to add to that, the resulting system is not linear). In what follows, I focus on the extent to which τ_1 conditioned participants' engagement with Problem 1, including how it shaped what participants made of M_3 not being invertible and of the possibility that A and B may not be invertible.

5.1.3.1 9 of 10 students' responses were conditioned by τ_1

All students but P10 isolated C spontaneously as a reaction to Problem 1, and discovered M_3 is not invertible only in the process of trying to find an expression for C more succinct than one of type "$B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$." Students reacted in different ways to the discovery that M_3 (or M_2M_3) is not invertible. (As for A and B, some students said they "assume[d]" them to be invertible, when asked questions similar to "what if A isn't invertible?") The differences between students' reactions notwithstanding, they did predominantly reflect τ_1 know-how. P8 got stuck upon the discovery that the product $M_1AM_2M_3B$ included a non-invertible factor; for P2, P7*, and P9, this meant C cannot be isolated via τ_1 and an alternative technique is needed to solve for C; P1, P4, and P5 determined C does not exist because C *is* $B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$ and this product involves matrices that do not exist; and P2, P3, and P6 toyed with the hypothesis that a product involving a non-invertible matrix could be invertible (a calculation error P2 made confirmed, to him, this false hypothesis as true). For all three participants, the hypothesis being true would make it possible to solve the equation via τ_1; the potential of the hypothesis to be false, however, left P3 feeling "confused" and P6 "uncertain."

5.1.3.1.1 P8 activated τ_1 and got stuck upon finding the product on the left-hand side of the equation involved a matrix that is not invertible P8's spontaneous reaction to Problem 1 was to "[try] to understand," though she claimed she "definitely should use inverses." Her initial confusion had to do with what A and B represented, but once I said they are matrices, she applied τ_1. When P8 got to M_2M_3, she found the product and then applied the inversion algorithm to find its inverse. A row of 0's appeared. It "doesn't make sense... so it's not invertible." A pause. "I don't know what to do next." Another pause. "I need a hint." I said the matrix on the right-hand side of the equation is an identity matrix and asked if that told her anything. For P8, this meant the product to the left of C "should be an inverse of C, because an inverse multiplied by the original matrix should give you an identity matrix." "Is that right," she asked. P8 was stuck.

P8 did not side-step the norm of applying τ_1 and was stuck in the logical fallacy that the given equation was a truth statement. When I asked how she knew the inversion algorithm works (that is, that it does what it's advertised to do), she knew (as some other students did too) it has "something to do with the fact that the matrix can be represented as a product of elementary matrices" (she struggled to give more of an explanation, though; she said "the inverse of elementary matrix should be... also an elementary matrix"). P8 did not return to the problem at hand after this explanation and I did not prompt her to do so.

5.1.3.1.2 For P2, P7*, and P9, the left-hand side of the equation being a product with a non-invertible factor implied τ_1 cannot be applied and an alternative technique must be used—but the alternatives they proposed invariably included variations on τ_1. P7* said τ_1 is not applicable because A and B are not known and may not be invertible. That said, this did not mean, for P7*, that there is no solution for C; it simply meant a different technique was needed to complete the task. Indeed, P7* made 7 (albeit failed) attempts at Problem 1. P2 and P9 came to a similar conclusion after finding M_3 is not invertible: an alternative to τ_1 must be used to solve for C.

When P2 found M_3 is not invertible, he said: "there's no way to solve this, because this one [M_3], the determinant equals 0, so I can't inverse it, so I can't take it to the other side to isolate C." The inference from M_3 not being invertible was that C could not be isolated—not that there was no solution: indeed, P2 went on to another approach through which to isolate C. He found $M_2 M_3$, and through an incorrect use of the inversion algorithm, determined the product is invertible and used its 'inverse' to isolate C. I asked P2 about A and B; he said he assumed they are invertible. What if they weren't? He would do what he did with $M_2 M_3$ to fix the M_3 issue: he'd multiply A or B by some matrix such that the product is invertible. What if he couldn't find such a matrix? "I would give up."

P9 knew to isolate C upon reading Problem 1: "so here I have to solve for C, so I have to have C on one side and all the other matrices on the other side." After he isolated C, he started to ask: "so I have to..." This seemed a question as to whether to actually do any calculations; I said "yeah, go ahead," and P9 proceeded to the inversion algorithm (IA). He first found M_1^{-1} and M_2^{-1}. Applying the IA to M_3, he found the matrix reduced to a matrix with a row of 0's and initially said he "forgot how to get the inverse of this [type of] matrix." He quickly realized the matrix is not, in fact, invertible, "because the determinant is 0," and came to a conclusion:

> I cannot find C, because I will have the matrix [M_3^{-1}] as a [factor of] C [...] C is equal to the inverse of the rest times this one. So I cannot find that if its determinant is 0.

By "we cannot find C," P9 did not intend that there is no solution for C, but that C cannot be isolated via τ_1. "So I would have to expand C... And then I would equate each [entry] from the matrix with the [corresponding] one on the other side. Because I cannot remove [M_3], so I would write C in terms of x_1, x_2, x_3, x_4. And then I would multiply this by that." This brings to mind the LA1 norm of solving linear systems corresponding to matrix equations of the form $A\mathrm{x} = \mathrm{b}$. I then asked P9 if he could think of "any other way of going about this [problem]." His (next and last) suggestion was to "find the inverse of C and then find the inverse again." This is a variant of τ_1: instead of isolating C, isolate C^{-1}. Then, find its inverse C.

Unlike P2 and P9, P7* did not find out M_3 is not invertible until I told him as much. This is because, as per his style throughout the interview, P7* rarely actually went through with calculations. I had told P7* that M_3 is not invertible toward the end of his engagement with Problem 1; he got stuck and, despite having offered 7 techniques through which to tackle the task, did not offer any knowledge appropriate for the given situation.

For P2, P7*, and P9, then, the information about M_3 was a signal about which procedure (not) to use to find the entries of C. They knew it is not possible to multiply by the inverse of a matrix with no inverse. However, their proposed alternatives invariably fell back on τ_1. P2 thought M_2M_3 would be invertible, and his incorrect application of the inversion algorithm made it seem it was; and so he applied τ_1, but multiplied by the apparent inverse of M_2M_3 as a salve to the non-existent inverse of M_3. P6, whose case I discuss further below, also thought it possible to overcome the M_3 obstacle in this way but ultimately, after having found M_2M_3 is not invertible, and following some prompts I had given and an attempt to theorize on the matter, hypothesized it may not be possible to multiply a non-invertible matrix by another matrix and obtain an invertible one. P9 also knew it is not possible to multiply by the inverse of a matrix with no inverse, but presumed C would be invertible and as an alternative to τ_1 inadvertently proposed τ_1 yet again: if you can't multiply by M_3, multiply by C^{-1}! P7* was less susceptible than others to the magnetism of τ_1, though he did bring it up twice: the first time (Step 2 of his engagement with Problem 1) he dismissed it due to the unknown quality of A and B, and the second time (Step 6), dismissed it for the same reason. Unlike the other participants, P7* did not return to τ_1 once he knew M_3 is not invertible, though he did get stuck. These participants sought a technical alternative to τ_1 but had none.

5.1.3.1.3 P1, P4, and P5 approached a thesis of C not existing, justifying it on a claim that C *is* a product that involves matrices that do not exist. These students did not have the knowledge needed to support their hypothesis accurately and one of them (P5) did not clearly state that C may not exist.

P5's spontaneous reaction to M_3 having a row echelon form with a row of 0's was to side-step the obstacle; "[she didn't] know what else to do," so she still used what she found through the inversion algorithm to isolate C. That is, after having applied the same operations to M_3 and I successively to find the reduced row echelon form of M_3, the matrix resulting from I was

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$$

and P5 used this matrix in the role of M_3^{-1} to isolate C. She explained this choice afterwards:

> I didn't know if it was 100%. I know I did something wrong, but I didn't know what I did wrong. So I just kind of went with it. Because that's what I would do on a test. Yeah. So, because I knew that this, this was weird, because it wasn't 1 0 0 1. [...] That was the only matrix I could end up with so I just went with it.

P5's activity therefore reflected that of a Student: the main goal is to produce a solution, regardless of whether it is correct or not—and even when knowing it might not be. P5 knew that "because the bottom rows have two zeros at the bottom, if you're going to multiply it with [M_3], you're not going to get an identity matrix because one of the bottoms... are going to end up being not right. There's no, like, 1 to multiply by, so it's automatically going to be zero." I asked, then, if she was saying that what she found was the inverse of M_3. P5 checked and confirmed the product is not I. I then confirmed

M_3 was not invertible and asked how that affects things. My intervention (and perhaps my "authority" over the mathematics, akin to that of a teacher or textbook for one who Students) prompted P5 to think beyond how to isolate C.

P5 brought up the argument that $C = B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$ but M_3 is not invertible: "you want to take C to this side and C isn't, technically, can't be invertible? Because if C is equal to all of these, this one's not invertible, then C's not invertible." P5 struggled to qualify what this meant about C and the equation: all she could say about C is that it "isn't, like, a matrix." ("I don't know if that makes sense.") Her mind had "go[ne] blank on the invertible thing."

P1 had a misstep in his initial and spontaneous mobilization of τ_1, and this led to the equation

$$C^2 = B^{-1}(M_2M_3)^{-1}A^{-1}M_1^{-1}.$$

(P1 had multiplied both sides of the original equation by $C^{-1}B^{-1}(M_2M_3)^{-1}A^{-1}M_1^{-1}$, and had not noticed that $C^{-1}C$ is I, not C.) P1 was stuck with this first approach. His second was to apply τ_1 again but this time by isolating C^{-1} (by multiplying both sides of the equation on the right by C^{-1}). He then said he had "another idea" and moved on to his third approach—which was, in fact, a reiteration of his first use of τ_1. This time, however, he multiplied both sides of the equation by one matrix at a time and calculated inverses before multiplying by them. And so he discovered M_2M_3 is not invertible. ("That makes things difficult.") "Maybe... the solution doesn't exist?" His reasoning: C, "normally, should be"

$$B^{-1}\begin{pmatrix}10 & 15\\ 4 & 6\end{pmatrix}^{-1}A^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

"but this one $\left[\begin{pmatrix}10 & 15\\ 4 & 6\end{pmatrix}^{-1}\right]$ doesn't exist. So maybe its solution doesn't exist also." It is not clear P1 was convinced by this argument ("we can't invert this one [pause] so... this equation is not right, but... Is that possible?") but he was unable to mobilize any other knowledge.

P4's spontaneous reaction to Problem 1 was also to activate τ_1. After he isolated C, he said he doesn't think there's "enough information to have a final answer," because A and B are not given, but then decided to "leave [B^{-1}] as that." He then got to M_3 and found its determinant is 0. "Okay, so it's not even invertible in the first place. [laughs] Okay, so there's no solution." Why? "If [M_3] is not invertible, that means there's no inverse here, but that's part of the expression for C." He then hypothesized that "A and B are not given because there's no solution to the problem in the first place." ("Probably, I would guess that. It's maybe a proof to us that the problem does not work.") P4 assessed A and B from the lens of a Student: their inclusion is designed as a hint to the problem-solver—they are not merely part of the equation, as their function is extramathematical.

When I prompted P4 to again explain the "link between" C and M_3, his explanation was again based in the procedure of τ_1: "we need to find C with our given information through the inverse of [M_3], and [M_3] is not invertible. So I cannot find C." Note the wording: "we *need* to find C [...] through the inverse of [M_3]" [emphasis added]. For P4, the task to solve for C was inherently to *isolate* C by multiplying by inverses, as is the norm in LA1.

P4, like P1, did not seem convinced by his argument: "I don't really like how it ended, the problem." What didn't he like? "Just... the expression contains... the expression for C contains a value that I cannot have. I don't like this. Doesn't sit well with me, and especially because I don't have information on A and B, but it gets me to think that maybe there is some trick about A and B that I should spot. [...] But still, [M_3] inverse is part of the expression and if [M_3] inverse does not exist then C doesn't exist." Much as this argument did not "sit well with" P4, he did not mobilize knowledge beyond the scope of τ_1. Indeed, after P4 mentioned that "maybe there is some trick about A and B that [he] should spot," I prompted him to consider the case in which A is not invertible. He said this would "further support [his] proposition." I gave another prompt: "let's say they're both invertible." P4 went again for τ_1, this time multiplying M_2M_3 first and discovering this matrix, too, is not invertible. This finally "convinced [him of his] answer," that is, that there is no solution for C, though still he did not propose any explanation apart from the one in which C is isolated using matrices that do not exist.

For P1, P4, and P5, that M_3 did not exist signaled there is no solution for C. But their justification was based in an application of τ_1 to a case in which it does not apply: they multiplied by M_3^{-1}, a matrix they knew does not exist, to get an expression for C—a matrix they suspected to not exist. In spite of P1 and P4's comments attesting to their dissatisfaction with this argument, they did not mobilize any other knowledge. I presume P1, P4, and P5 lacked the knowledge needed to justify why C does not exist. Further, the normative quality of τ_1 was such that they did not recognize, as P2, P7*, and P9 had (to an extent), that τ_1 cannot be applied as M_3 is not invertible. In LA1, matrix equations in the task t_1 are such that it is *always* possible to solve them via τ_1.

5.1.3.1.4 Two students (P3, P6) kept one foot in τ_1 as they gingerly toed a theoretical line of query: can a product involving non-invertible matrices be invertible? P3 and P6 were prompted onto this theoretical question by two things: first, that $M_1AM_2M_3B$ included M_3, a matrix that is not invertible, amid their attempt to use τ_1, and second, prompts I gave about what this meant for the equation. I give more detail about P3 and P6's disengagement from τ_1 below.

When I asked P3 about the possibility that A might not be invertible ("what if A is not invertible?"), she said it is invertible, "because [the product] of these matrices is [the identity matrix]." I then pointed out that the product she found for M_2M_3 is not invertible. P3 calculated its determinant. It was 0. I asked what this tells her about the equation. "It tells me there is not enough information to solve this problem."

P3 was stuck. I rephrased the task: "the goal is to find a matrix C that would make this equation true." Perhaps the prompt about what it takes to "make the equation true" implied the potential of the equation to *not* be true[6]. P3's first suggestion was to assign

[6]As participants' activity in Problem 1 reveals, the phrase "solve the equation," given a matrix equation, did not usually include the possibility of there being no solution; this contrasts with students' knowledge in Problem 6, where students knew that a real quadratic equation may not have solutions—which they justified with normative high-school knowledge about discriminants of quadratic equations. In LA1, matrix equations in the task t_1 ("to solve" a matrix equation) always have a solution.

variables to the entries of A and B so as to express the product $M_1AM_2M_3B$ as a matrix, and then check if this matrix is invertible. P3 found the product but abandoned the approach—there were many unknowns in the matrix entries ("wow, no, it makes it super difficult"). I asked if she could "think of any other way to reach that kind of conclusion, without doing all the calculations." She couldn't. While she did wonder whether a product involving a non-invertible matrix could be invertible, it's not clear she would have known what to make of the product not being invertible. She did know that if $M_1AM_2M_3B$ is invertible, then C could be isolated via τ_1. But it's not clear she'd have known what to make of it not being invertible: she said that "if it is not [invertible], it is impossible to get the solution," but did not elaborate on what she meant by "impossible to get the solution." She did not "remember exactly how to handle the non-invertible matrix, [it] ma[de] [her] really confused."

For P6, upon receiving Problem 1, the task was "basically [...] to go through all of these inverses into here, from the left, until I get to C, and then C is equal to whatever that is." That is, for P6, the task was "basically" t_1. He isolated C. He found M_1^{-1} and M_2^{-1}. He noted M_3 has determinant 0. Started the inversion algorithm. Upon finding it reduces to a matrix with a row of 0's, he started the algorithm from scratch. Same result. He swerved and calculated M_2M_3 instead. He wondered if it is invertible, and found it is not. "Did I make a mistake? Are you allowed to tell me?" He "[felt] like [he's] making a mistake" and said he was "getting a little bit sick" because "you can't invert something that [has] determinant zero." "So... What do I do here?" He knew he can't apply τ_1 "from the left" and said he "can take C out... From the right." P6 did not have a well-defined operation in mind: he said "you can factor it [C] but it doesn't do anything [...] because you can't do matrix division." For P6, C^{-1} was the product to the left of C in the equation "because if you multiply C by something, and it is equal to I, then it has to be [its] inverse." He briefly suggested to multiply both sides of the equation by C^{-1} on the right, but rejected this idea: "if I multiply by C^{-1}, I get... C^{-1} equals C^{-1}, if I do it from the right [pause] multiply it by the inverse of C^{-1}, just get C equals C, which doesn't help me now. Okay, I'm stuck."

He asked me to help. I responded: "the fact that here you have an identity matrix—you said you're thinking this would be the inverse of C. And you said that this, this one [either M_3 or M_2M_3], is not invertible." P6 asked if this was right and I confirmed it was. I added that "there's also the question about A and B: what if they're not invertible?" P6's inference was this: "so what you're saying is there's no solution?" This did not correspond to P6's understanding of the mathematics at stake: "but I thought, but it doesn't matter—like, if a matrix is not invertible, if you multiply by another matrix, it's not necessarily still not invertible? [I thought] that... it can become invertible. After you multiply it, surely..." He thought that multiplying a non-invertible matrix, say, A, by some appropriate matrix B, could 'fix' the numerical pattern that leads to a determinant being 0: "I think this matrix here [the product M_2M_3], I have it's not invertible because these add to 60 and these add to 60, so it's 60 minus 60 equals zero. So I just have to find something that, I don't know, *changes one value*" [emphasis added]. P6 then attempted an example wherein he multiplied M_2M_3 by a matrix to check if the determinant would be zero. It was (though I had pointed out a calculation error which had first led to a non-zero determinant). "Okay, so you're trying to tell me is that if something's not invertible, it's always not invertible, doesn't matter how much you multiply it." P6's wording—"*you're trying to tell me*"— brings to mind his earlier one ("so what you're saying is there's no solution?"); both suggest a lacking sense of agency[7]. I said I wanted to know what he thought about that and P6 said he "just chose a poor example." He tried another example. "Surely this doesn't add up to zero... It does!" But P6 knew examples are not a proof:

> I still don't think... because two examples like this, where I'm just trying to think of some random thing, it's not a proof of any kind. I still feel like there's definitely some kind of determinant, some kind of thing here where you would add zero, right? Or I wouldn't add to zero. What if this was like six? If it was like a higher even... or like, 32... this is... [multiplies M_2M_3 by another matrix on the left] this would be 14, 21... it wouldn't matter 'cause it's still the same issue.

P6 then attempted a proof ("if you want something to be more or less general, it's better just to abstract it into letters, or variables"). He found the product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and seemed to try to find conditions under which something would equal 0 ("let's say this is equal to 0... show that that's equal to 0"). And he "start[ed] to believe [pause] that if something's not invertible, then you can multiply by [another] matrix and make it invertible." Nevertheless, despite his belief that a matrix that is not invertible can be multiplied by some matrix such that the product is invertible, "all [the] questions [I was] asking [made him] very uncertain" and he concluded, in what seemed like deference to my authority, that multiplying a non-invertible matrix by another matrix would give a matrix that is not invertible.

P3 and P6's questioning was this: could $M_1AM_2M_3B$ be invertible even if M_2M_3 is not? If yes, then τ_1 could be applied. In the case that $M_1AM_2M_3B$ is not invertible, both participants threw a hypothesis about there being no solution to the equation, though P3 did not say so in such explicit terms—it's not clear if "if it is not [invertible], it is impossible to get the solution" was a statement about the existence of a solution so much as a statement about the *operation* of "get[ting]" a solution (this brings to mind Sfard's 1991 theory of reification, which maintains that operational approaches precede structural approaches in students' development of algebraic knowledge, in that the understanding of an algebraic expression, such as $1 + x$, as an object in and of itself can only come after students spend time perceiving this expression in terms of the operations implied within it—Sfard (1991) points out the precedence of operation over structure is reflected in the history of algebra as a domain, with thousands of years of focus on computational procedures preceding the advent of modern algebra).

The only explicit conclusions P3 and P6 made relative to the matrices that were not invertible were how they felt about these: P3, "confused" about how to handle a non-invertible matrix, and P6, "a little bit sick" as "you can't invert something that has

[7] Sierpinska et al. (2008) contrast "lacking agency"—lacking the capacity to use the mathematics as needed—with a "lacking sense of agency": lacking confidence in one's capacity to use this mathematics. One can lack agency yet have a sense of agency.

determinant 0." ("I'll be honest, I really thought that this interview is more gonna be... I'm feeling very uncertain all of a sudden," P6 said after attempting to investigate his theoretical query.)

5.1.3.1.5 Summary: participants struggled to extricate themselves from τ_1 even as they knew M_3 is not invertible. Participants were restricted by the LA1 norm of τ_1 for solving matrix equations. First, the various inferences they made, about the impact of M_3 not being invertible, were erroneously justified through τ_1. And second, their perception of the task of solving the equation for C was married to the notion of *isolating* C.

As P8 tried to mobilize τ_1, she got stuck when she found M_3 had no inverse.

P2, P7*, and P9 inferred that since M_3 has no inverse, τ_1 can't be applied and an alternative technique is needed. But the alternatives they proposed were not always alternatives. Indeed, P2's alternative was to find M_2M_3 so as to deal with *its* inverse instead; one of P9's alternatives was another variant of τ_1 (isolate C^{-1} instead of C); and P7* twice suggested τ_1 even though he'd already rejected the technique the first time around due to the unknown nature of A and B. Even though P2 and P9 knew τ_1 could not be applied using M_3, they toyed with the notion of applying a variant of τ_1 that, on surface, seemed to not involve M_3.

P1, P4, and P5 suspected the equation had no solution, but their justification was an abuse of τ_1: they claimed C *is* $B^{-1}M_3^{-1}M_2^{-1}A^{-1}M_1^{-1}$, and since M_3 is not invertible, this meant C may not exist.

P3 and P6 did extricate themselves from the LA1 norm of τ_1, but only after prompts I had given them, and still within the context of τ_1. They both broached a theoretical question: if M_3 is not invertible, can $M_1AM_2M_3B$ be invertible? They related this question to τ_1: if $M_1AM_2M_3B$ *is* invertible, then τ_1 could be applied and the mission accomplished. P3 and P6 did not mobilize mathematical knowledge that helped to answer the question, though P6 did attempt to do so (he briefly tried to check, computationally, if the product of two general 2×2 matrices could be invertible, if one of them was not invertible).

In addition to participants' inability to use knowledge *other* than τ_1 (and various misinterpretations of τ_1), participants' perception of the task (solve for C) was that the task was *to mobilize* τ_1—that is, to isolate C. A comment P4 made when he suspected the equation had no solution alludes to this: "I would not like the question to not have specified that... Like, 'if any'. Because 'solve the following equation,' it implies a solution." The perception that there *is* a solution was shared by most other students; this contrasts with participants' ready acceptance, in Problem 6, that "to solve a quadratic equation" may mean to conclude it has no solution.

It's not only P4's comments that suggest that for my participants, it was a given that the task was to isolate C. First, there's the matter that the *spontaneous* reaction to Problem 1 of 8 of my 10 participants was to isolate C via τ_1. But even as they sought alternatives, even as they thought they had "another idea" (P1), the idea turned out to be a variant of τ_1 that differed from their first suggestion only superficially (e.g., instead of multiplying both sides of the equation by M_1^{-1}, A, M_2^{-1}, etc., multiply both sides of the equation by C^{-1}). The perception that the task was to isolate C via τ_1 was such that many held onto τ_1, without reservation, even as they found M_3 has no inverse. Recall P9's reaction when he found M_3 reduced to a matrix with a row of 0's: "I forgot how to get the inverse of this [type of] matrix." His spontaneous reaction was not that M_3 had no inverse; it was that he, a student who had recently completed LA1, had forgotten "how to get the inverse." Recall P5's reaction to the same finding: she plodded on and applied τ_1 using what she'd found to correspond to the reduced row echelon form of M_3, despite knowing she "did something wrong"; she "didn't know what [she] did wrong, so [she] just kind of went with it. Because that's what [she] would do on a test." What she would do on a test is what would most likely get her marks. What would most likely get her marks is to plod on with τ_1 no matter what. Incorrect math (or not-math) was par for the course. Recall P6's reaction; he found $\det M_3 = 0$ and even though he knew "you can't invert something that [has] determinant zero," still applied the inversion algorithm to M_3; and when a row of 0's appeared, he started the algorithm all over again; and when this resulted in the same obstacle, he went for M_2M_3 instead, expecting it to be invertible and allow him to apply τ_1. M_2M_3 was similarly P3's hope for τ_1 and P2's resolution to the τ_1/M_3-has-no-inverse problem, as his incorrect application of the inversion algorithm made him believe M_2M_3 is indeed invertible. P2, P3, P5, P6, and P9 (half my participants) directed their efforts toward finding a way to accomplish τ_1; P1, P4, P5 were under the impression they accomplished the task via τ_1 (having isolated C using M_3^{-1} even as they knew it does not exist); and P8 offered no alternative to τ_1.

This activity is not surprising. In LA1, as I discuss in Section 5.1.2, matrix equations students have to solve typically have the form $AXB = C$ and are always satisfied by some matrix X. τ_1 is always the way to go[8]. This is different from types of equations with which students are confronted in previous math courses and even in LA1. It is a norm in LA1 that systems of linear equations, as in Problem 2, might have no solutions; and it is a norm in high-school algebra courses that quadratic equations, as in Problem 6, might have no solutions. Comments my participants made in the context of Problems 2 and 6 show they had this knowledge. But the norm in LA1 relative to matrix equations in the context of a task that looked like t_1 includes one option only: use τ_1. The practical block $[t_1, \tau_1]$ from knowledge to be learned in LA1 proved to be an epistemological obstacle for students: knowledge that was once effective began to be inadequate or inconsistent in a new scenario. In the absence of conditions that allow for τ_1, students still perceived the task to be "to apply τ_1" and abused it in the process. Knowledge from their experience with equations in other task types (such as "solve $A\mathrm{x} = \mathrm{b}$" or "solve $ax^2 + bx + c = 0$") emerged tentatively but was not supported by knowledge about matrices: P1, P4, and P5 hypothesized the equation may have no solution, but based this on the claim that C equaled a product of matrices that do not all exist; P6's inference that the equation may have no solution was not based on the mathematics at stake, but rather made from comments I made ("so what you're saying is there's no solution?"); and neither P3 nor P6 were able to support their hypothesis using mathematical knowledge.

[8] I exclude, here, matrix equations of type $A\mathrm{x} = \mathrm{b}$, where x and b are column matrices, and the former a matrix of unknowns, as these are normatively associated with systems of linear equations, which students most typically are expected to solve using Gauss-Jordan elimination, as I discuss in my model of knowledge to be learned in relation to Problem 2; see Section 5.2.2.

5.1.3.2 P3's theoretical inquiry and emotions expressed by P1, P3, P4, and P6 suggest the didactic potential of tasks like Problem 1

By "tasks like Problem 1," I refer to the trap it has set: it mimics a LA1 task, so a student who's recently completed LA1 is likely to perceive it as that LA1 task, but the normative LA1 technique cannot be applied, so the student is unable to complete the task using the normative technique. For Problem 1, the trap had students multiply by inverses. This trap triggered problematic emotions in some of my participants: P3 was "confused," P6 felt "a little bit sick" and "uncertain," and P1 and P4 seemed unsettled by their argument (that C does not exist because it equals a product that involves matrices that do not exist; P1 had acknowledged that since M_3 is not invertible, the equation $C = \ldots$ "is not right," and P4 said he did not "really like how it ended, the problem"), though they lacked the knowledge needed to refute this argument or produce another one. The trap, along with prompts I had given, had also triggered in P3 and P6 a question: if a matrix B is not invertible, could a product AB be invertible?

Most students did not problem-pose, but I take the problem-posing of P3 (and, to an extent, that of P6) as a sign of the *potential* of tasks like Problem 1. Its potential is two-fold. First, it has potential to trigger in students a shift from using techniques (here, τ_1) to engaging with the theory that produces these techniques, as in the transition from secondary to tertiary education; see Winsløw et al. (2014). Second, it has potential to prompt students to engage in mathematical behavior such as problem-posing (e.g., Hersant & Choquet, 2019; Schoenfeld, 1985).

To assess whether these potentials are rooted in the task itself, or if P3 and P6's problem-posing was simply rooted in qualities they had as participants, I take into account the positioning they displayed. The two participants' positioning throughout the interview differed in substantial ways from one another.

Throughout his interview, P6's comments and activity predominantly pointed to his having occupied the position of a Student in LA1 (e.g., seen through a lacking sense of agency relative to the mathematics at stake) but also the position of a Learner in mathematics learning contexts other than LA1 but within the university (he said he enjoyed reading mathematics books in his free time—he mentioned number theory—and that he sometimes approached professors in the postsecondary institution's Mathematics Department with questions about material he read; and at the end of his engagement with Problem 1, he said: "now you have to tell me the trick"). In Problem 1, the knowledge he mobilized was mainly that built as a Student in LA1. He commented about his expectations regarding the interview, attempted to glean information from my prompts as an interviewer, and asked for validation. "Are you allowed to tell me—I can just multiply [M_2 and M_3], right?"; "did I make a mistake? Are you allowed to tell me?" Was his approach right? Were his arguments correct? Can the interviewer tell him certain things? If I, as the authority on mathematics in the context of this TBI, asked certain questions, he inferred something he was doing or saying must be wrong: "all these questions you're asking me, you're making me very uncertain." "I'll be honest, I really thought that this interview is more gonna be... I'm feeling very uncertain all of a sudden." He sought my validation as interviewer but also my help: "so help me out here. How do you solve this?" These emotions reflect a lacking sense of agency brokered by the failure of knowledge that normally worked in LA1 ("I'm getting a little bit sick... you can't invert something that [has] determinant zero," and "I don't think I actually did any questions like this [...] you have to tell me for my own sanity... The determinant is zero so it's not invertible").

I bring up P6's positioning to make the point that his problem-posing (if B is not invertible, can AB be invertible?) did not build on relevant mathematics. It was shaped by two other factors. First, a conflict between LA1 norms and the properties of the task at stake: M_3 had no inverse, and he was used to multiplying by inverses in LA1. P6 had encountered a *problem* (Mason, 2016), in that something in the task bothered him (he felt "a little bit sick"). The second factor that led to P6's problem-posing was a lacking sense of agency, which I view as a result of his having occupied a position as Student in LA1, and a Learning behavior triggered by this lack of confidence. Indeed, the point at which he extracted himself from LA1 norms and engaged in problem-posing overlapped with expressions of concern about his performance in the interview and an attempt to decrypt what I (as the interviewer) implied when I spoke: "okay, so *you're trying to tell me* that if something's not invertible, it's always not invertible, doesn't matter how much you multiply it" [emphasis added].

The didactic potential of Problem 1 can't be presumed from P6's activity, given the positions he occupied. It can, however, be gleaned from P3's activity. P3's positioning throughout the interview was mostly that of a Student. The techniques she proposed reflected normative LA1 knowledge(-to-be-learned) and its limitations (e.g., that tasks lend themselves well to the strategy Lithner, 2004, dubs "identification of similarities" and which is what my participants had all done in reaction to Problem 1: find an example with surface-level features similar to those in a task to be solved, and mimic the procedure in the example; the didactic limitation of such tasks is that it limits the knowledge students *must* acquire to the procedures in such examples). The justifications she gave (when I asked how she knew certain techniques worked, for instance) were references to what her LA1 teacher had said or what the LA1 textbook had shown. This positioning was not unique to P3 (indeed, most of my participants presented with qualities belonging to a Student). This positioning, however, did not stop the potential of Problem 1. The problem, along with a prompt I gave and which shifted P3 from an operational perspective on "solving for C" (as in, isolating C) to a structural perspective (determining whether there is a C for which the equation can be true) (Sfard, 1991), led P3 to engage in problem-posing about theoretical knowledge: if a matrix is not invertible, can its product with another matrix be invertible?

The potential failure of τ_1 as a technique for a task they had initially identified as similar (Lithner, 2004) to a familiar LA1 task had triggered, in P1, P3, P4, and P6, emotions of confusion and uncertainty. This reaction shows the potential of a task like Problem 1 to pose a *problem* (Mason, 2016) to students, not in the sense of a task to be solved in a mathematics course, but in the sense of a source of frustration they may be compelled to resolve. This did not suffice to push students away from τ_1, however. P3, the only student who did so (and not as a result of her position), posed a theoretical query only after receiving an appropriate prompt—one that targeted the operational perspective that possibly accompanies students' resolve to use a technique like τ_1.

5.2 LA1 Problem 2

The following was the second problem presented to the 10 LA1 students in the TBI:

The coefficient matrix below is invertible. Solve the system:

$$\begin{bmatrix} 9 & 16 & 3 & 4 \\ 5 & 6 & 0 & 8 \\ -2 & 3 & 0 & 4 \\ 3 & 6 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 2 \\ -3 \end{bmatrix}$$

5.2.1 Reference model for LA1 Problem 2

The task T in Problem 2 is to solve a matrix equation of the form $A\mathrm{x} = \mathrm{b}$, where A is a square matrix (of order 4) with integer entries, x is a column matrix of unknowns, and b is a column matrix with integer entries. Given the definitions of matrix multiplication and of equality of matrices, this task is effectively the task to solve a linear system:

$$\begin{array}{rcrcrcrcr} 9w & + & 16x & + & 3y & + & 4z & = & -9 \\ 5w & + & 6x & & & + & 8z & = & -5 \\ -2w & + & 3x & & & + & 4z & = & 2 \\ 3w & + & 6x & + & y & + & z & = & -3 \end{array}$$

Problem 2 can be solved by inspection (τ_1): since the column of constants b, where

$$b = \begin{pmatrix} -9 \\ -5 \\ 2 \\ -3 \end{pmatrix},$$

is equal to minus one times the first column of the coefficient matrix A, $(-1, 0, 0, 0)$ (in its form as a column vector) is a solution of the system, and since the coefficient matrix is invertible, the system has a unique solution (see θ_1 below), so $(-1, 0, 0, 0)$ is the only solution of the system. The technology θ_1, framed by the theoretical discourse of matrix algebra and logic, is this:

$$\exists A^{-1} \Rightarrow A^{-1}(A\mathrm{x}) = A^{-1}\mathrm{b} \Rightarrow (A^{-1}A)\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow I\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$$

Not all systems can be solved by inspection; the technique τ_2 of row reduction addresses this issue. The process is produced by the principle that when the solutions of a linear system are not readily apparent, algebraic operations can produce an equivalent linear system whose solutions *are* readily apparent. This technique is framed by the technology $\theta_2 = [\theta_{21}/\theta_{22}$[9]$; \theta_{23}]$:

θ_{23} Every linear system $A\mathrm{x} = \mathrm{b}$ can be expressed in terms of an augmented matrix $[A|\mathrm{b}]$ (and vice-versa).

[9] I use a slash to indicate either technology could be used.

θ_{21} Elementary row operations correspond to algebraic operations that produce equivalent equations. Any matrix A can be reduced by elementary row operations to a row echelon form (REF) or to a unique reduced row echelon form (RREF): the algorithm of Gaussian elimination can be used to find a REF and that of Gauss-Jordan elimination to find RREF(A). (Other algorithms exist.) The REF and RREF correspond to linear systems, equivalent to the initial system, where the solutions are readily apparent. If $\exists A^{-1}$, then RREF$(A) = I$ so the augmented matrix can be reduced to one of the form

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & B_1 \\ 0 & 1 & \cdots & 0 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & B_n \end{array}\right)$$

Each unknown is therefore isolated in the linear system that corresponds to the RREF of the augmented matrix or can be made so via back-substitution (in the case of a REF). If $\nexists A^{-1}$, then any REF of A would have at least one row whose entries are all 0; if any such row, in the augmented matrix, corresponds to an equation of the form $0 = a$ where $a \neq 0$, then the system has no solutions; otherwise, any reduced form of the augmented matrix has variables that are free to have any value, and once their value is fixed, so is the value of the other variables. In this case, any free variables may be assigned parameters and the general solution of the system can be expressed in terms of these parameters. For example, if the RREF of the augmented matrix of a system in the unknowns x, y, z is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then y, z may be treated as free variables and therefore assigned parameters: $y = r, z = s,$ where $r, s \in \mathbb{R}$ and a general solution of the system is $(-2r - 3s + 4, r, s)$. In this case, the system has infinitely many solutions.

θ_{22} Patterns among entries of a given matrix can be taken advantage of to obtain a REF more conveniently than strict adherence to Gaussian elimination (when doing this process by hand). For example, in Problem 2, three of the entries in R_2 (row 2 of the augmented matrix) are equal to 2 times the corresponding entries in R_3, and three of the entries in R_1 are equal to 3 times the corresponding entries in R_4. Using the row operation $R_i + aR_j \to R_i$ to take advantage of the proportionalities can produce, for example, the solution

$$\left(\begin{array}{cccc|c} 9 & 16 & 3 & 4 & -9 \\ 5 & 6 & 0 & 8 & -5 \\ -2 & 3 & 0 & 4 & 2 \\ 3 & 6 & 1 & 1 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & -2 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & -9 \\ -2 & 3 & 0 & 4 & 2 \\ 3 & 6 & 1 & 1 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 0 & -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 3 & 0 & 4 & 0 \\ 0 & 6 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 0 & -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right)$$

From the last matrix and through back-substitution, it can be deduced that the system has $(-1, 0, 0, 0)$ as a unique solution.

I bring θ_{22} up because in the context of LA1, it is pertinent, as the matrices given in tasks of type T in the course are amenable to θ_{22}; students could benefit from it as it would allow them to more quickly finish tasks of these types on exams or when studying for their course, and the technology is mentioned in the course textbook. Students, however, are not required to use θ_{22} to get marks[10].

The technique τ_2 of row-reducing the augmented matrix is applicable to any task of type T and produces a general solution no matter what - whether the coefficient matrix is invertible or not. The two techniques I describe next are applicable only to the tasks of sub-type $T_{\exists A^{-1}}$: the same as task T, but where the coefficient matrix is known to be invertible or to have non-zero determinant.

One technique is τ_3: Cramer's rule. The theorem gives a formula for the value of each unknown: $x_i = \frac{\det A_i}{\det A}$. To calculate determinants, one technology is θ_{31}, the definition of a determinant as a cofactor expansion along any row or column (there is a theorem that ascertains the uniqueness of the number obtained by cofactor expansions) and θ_{32}, properties that relate row and column operations with determinants (which are proved via the cofactor-expansion definition of determinant). Given square matrices A and B of the same size, θ_{32} is made up of three parts:

θ_{321} If B can be obtained from A by doing a row (or column) operation of the type "swap rows R_i and R_j (or columns C_i and C_j) of A, where $i \neq j$", then $\det B = -\det A$.

θ_{322} If B can be obtained from A by doing a row (or column) operation of the type "multiply R_i (or C_i) by a scalar c." then $\det B = c \det A$.

θ_{323} If B can be obtained from A by doing a row (or column) operation of the type "add a scalar multiple of R_i to R_j (or of C_i to C_j), where $i \neq j$," then $\det B = \det A$.

In the case of Problem 2, the statement that the coefficient matrix A is invertible implies that Cramer's rule is applicable; to apply it, one calculates all the requisite determinants using any advantageous combination of θ_{31} (e.g., one of the columns has several entries that are 0) and θ_{32} (to produce a row/column with even more entries that are 0). For instance, since

$$\mathrm{b} = \begin{bmatrix} -9 \\ -5 \\ 2 \\ -3 \end{bmatrix}$$

is proportional to one of the columns in A, there is no need to calculate $\det A$. Indeed, from θ_{322} and θ_{323} and the theorem that $\exists A^{-1} \Rightarrow \det A \neq 0$, it follows that

[10] I infer this from my experience as a LA1 teacher, but also from the lack of explicit instruction to use θ_{22} in final exam tasks.

$$w = \frac{\det A_1}{\det A} = \frac{\begin{vmatrix} -9 & 16 & 3 & 4 \\ -5 & 6 & 0 & 8 \\ 2 & 3 & 0 & 4 \\ -3 & 6 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 9 & 16 & 3 & 4 \\ 5 & 6 & 0 & 8 \\ -2 & 3 & 0 & 4 \\ 3 & 6 & 1 & 1 \end{vmatrix}} = \frac{-\det A}{\det A} = -1$$

$$x = \frac{\det A_2}{\det A} = \frac{\begin{vmatrix} 9 & -9 & 3 & 4 \\ 5 & -5 & 0 & 8 \\ -2 & 2 & 0 & 4 \\ 3 & -3 & 1 & 1 \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} 9 & 0 & 3 & 4 \\ 5 & 0 & 0 & 8 \\ -2 & 0 & 0 & 4 \\ 3 & 0 & 1 & 1 \end{vmatrix}}{\det A} = \frac{0}{\det A} = 0,$$

where the determinant in the numerator can be found to be 0 by cofactor expansion along the second column. It similarly follows that $y = z = 0$.

The last technique (for performing a task of type T) which I will mention in my reference model is τ_4: find A^{-1}, multiply both sides of $A\mathrm{x} = \mathrm{b}$ to find that $\mathrm{x} = A^{-1}\mathrm{b}$ (as per θ_1), and then to find the product $A^{-1}\mathrm{b}$ using the definition of matrix multiplication. The inverse can be found via

θ_{41} the inversion algorithm: reduce A to its RREF and apply the same elementary row operations to I; the sequence of operations that produces RREF(A) from A ultimately produces A^{-1} from I, as can be proven by a theoretical discourse that uses the property that each elementary row operation corresponds to multiplication by an elementary matrix; or

θ_{42} the formula $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Given that both the inversion algorithm and τ_2 are made up of row-reducing the coefficient matrix, and given that τ_4 involves the additional step of multiplying A^{-1} by b, τ_2 is a more efficient approach. τ_4 is a valid approach, but its value is mostly in the theoretical discourse about matrix equations of the type $A\mathrm{x} = \mathrm{b}$ and what the invertibility of (or lack thereof) of A means for the solutions of the equation.

The theoretical discourse Θ about matrix equations of the type $A\mathrm{x} = \mathrm{b}$ that are targeted by T frames the techniques τ_i $(i = 1, 2, 3, 4)$ (and their related technologies) and is part of the knowledge to be taught in LA1. The theory includes the algebraic and logical discourse that frames and proves the following theorem: if $A \in M_{n\times n}(\mathbb{R})$ and $\mathrm{x}, \mathrm{b} \in M_{n\times 1}(\mathbb{R})$, then the following statements are equivalent:

(a) A is invertible.

(b) The equation $A\mathrm{x} = \mathrm{b}$ has a unique solution for any b.

(c) RREF$(A) = I_n$

(d) A is a product of elementary matrices.

(e) $\det A \neq 0$.

(f) The equation $A\mathrm{x} = \mathrm{b}$ is consistent for any b.

I summarize the reference model for Problem 2 in Table 5.2 on p.101.

Table 5.2: Reference model for LA1 Problem 2

Task type	Technique	Theoretical discourse		
T: to solve by hand $A\mathrm{x} = \mathrm{b}$ or the linear system to which it corresponds, where $A \in M_{m\times n}(\mathbb{R})$ and $\mathrm{x},\mathrm{b} \in M_{n\times 1}(\mathbb{R})$ $(m, n \in \mathbb{N} \cap [1, 6])$	τ_1 by inspection	θ_1	if $m = n$ and $\exists A^{-1}$ then any observed solution is the only one	Θ: The algebraic and logical discourse that frames the following statements and proves they are equivalent: (a) $\exists A^{-1}$ (b) $A\mathrm{x} = \mathrm{b}$ has a unique solution $\forall \mathrm{b}$ (c) $A = \prod_{i=1}^{k} E_i,\ k \in \mathbb{N}$, where E_i is an elementary matrix $\forall i \in 1, ..., k$ (d) $\det A \neq 0$. (e) The equation $A\mathrm{x} = \mathrm{b}$ is consistent $\forall \mathrm{b}$.
	τ_2 row reduction	θ_2 $[\theta_{21}, \theta_{22}; \theta_{23}]$	θ_{23} definition of augmented matrix and choice of row operations guided by θ_{21} (Gauss-Jordan and Gaussian elimination) or θ_{22} (numerical patterns of the entries of a given matrix)	
	τ_3 Cramer's rule (formula)	θ_3 $[\theta_{31}, \theta_{32}; \theta_{33}]$	θ_{33} if $m = n$ and $\exists A^{-1}$ then Cramer's rule (theorem) applies; calculation of determinants according to θ_{31} (cofactor-expansion definition of determinant) or θ_{32} (properties of determinants and row/column operations, $\theta_{32i}, i = 1, 2, 3$, on p.99)	
	τ_4 find A^{-1} and use $\mathrm{x} = A^{-1}\mathrm{b}$	θ_4 $[\theta_{41}, \theta_{42}; \theta_{43}; \theta_{44}]$	θ_{44} if $m = n$ and $\exists A^{-1}$ then $A\mathrm{x} = \mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$; calculation of A^{-1} using θ_{41} (the inversion algorithm) or θ_{42} ($A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$), calculation of $A^{-1}\mathrm{b}$ according to θ_{43} (definition of matrix multiplication)	
$T_{\exists A^{-1}}$: T when A is known to be invertible	$\tau_1, \tau_2, \tau_3, \tau_4$	$\theta_1, \theta_2, \theta_3, \theta_4$	$[\tau_2, \theta_{23}, \theta_{21}]$ is in general the more efficient technique, but numerical properties specific to a given matrix mean that $T_{\exists A^{-1}}$ can be accomplished more efficiently by other approaches: $[\tau_1, \theta_1], [\tau_2, \theta_{23}, \theta_{22}], [\tau_3, \theta_3]$ $[\tau_4, \theta_4]$ involves the same steps as $[\tau_2, \theta_2]$ and requires more calculations, so there is no benefit to τ_4 over τ_2 in the practical block for T; the benefit is in the related theoretical discourse or in other, related tasks (e.g. to solve $A\mathrm{x} = \mathrm{b}_i$ for several b_i, though this too can be done efficiently via τ_2).	Θ
$T_{\nexists A^{-1}}$: T when A is known to be non-invertible	τ_2	θ_2	If $\nexists A^{-1}$ and $\mathrm{b} \neq 0$ then the system has either infinitely many solutions or no solutions; if $\mathrm{b} = 0$ then the system has one solution ($\mathrm{x} = 0$) if $m > n$ and infinitely many if $m \leq n$	Θ and the discourse that proves that if $\nexists A^{-1}$ and $\mathrm{b} \neq 0$ then the system has either infinitely many solutions or no solutions.

5.2.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 2

Table 5.3 on p.103 summarizes the knowledge to be learned in LA1 to perform tasks of the type T (as defined in the reference model in Section 5.2.1: to solve $A\mathrm{x} = \mathrm{b}$ or the linear system to which it corresponds, by hand). On past exams, tasks that require students to solve a linear system either specify which technique to use (that is, task t_i states students are to use technique τ_i, $i = 2, 3, 4$), or the linear system involves a non-square coefficient matrix (as in task t_1), in which case, the only viable technique is τ_2. Tasks of type T in problems recommended on the course outline also correspond to one of t_i ($i = 1, 2, 3, 4$): they similarly indicate which technique to use, either because the problems appear at the end of the textbook section in which the related technique (and theoretical discourse) is covered, or because the problem explicitly states which technique to use.

Apart from being a task students are expected to be able to perform (for its own sake), solving systems by row reduction is the normative technique for many of the other types of tasks students have to perform in LA1: I refer by t_5 to the LA1 task type "to solve a linear system so as to accomplish a different LA1 type of task." In these other LA1 types of tasks, students either have to row-reduce a given linear system with an objective other than finding its solutions, or they have to use given information to produce a linear system that they then have to solve. These tasks types are the following[11]:

- to find the values of b for which the system $A\mathrm{x} = \mathrm{b}$ has no solutions, one solution, or infinitely many solutions;
- to find a basis for the solution space of a homogeneous linear system $A\mathrm{x} = 0$;
- to find intersections of lines and/or planes in $\mathbb{R}^3$;
- to solve vector equations (e.g., to find the values of the coefficients for which

  $$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = \mathrm{v}$$

  is true, where the vectors v_i, $\mathrm{v} \in \mathbb{R}^3$ are given);
- to check if a set of vectors is linearly independent (a problem which reduces to the task "check if

  $$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = 0$$

  implies that $c_i = 0 \forall i$," where the vectors $\mathrm{v}_i \in \mathbb{R}^3$ are given; in LA1, the task is accomplished by producing the matrix A whose column vectors are v_1, v_2, v_3 and then either row-reducing the augmented matrix $[A|0]$ or finding $\det A$);
- to show a set of vectors is a basis for $\mathbb{R}^3$ (here too, the problem can alternatively be tackled by checking if an appropriate determinant is non-zero);
- to find the eigenvectors corresponding to eigenvalues of a matrix.

Considering Problem 2 in light of the model of the knowledge to be learned to perform tasks of type T (Table 5.3 on p.103), I expected students to interpret it either as t_4 (because of the affirmation that the coefficient matrix is invertible) or as task $t = [T, \tau_2]$: to

[11] I gather these are the task types from my study of 10 past (final and midterm) LA1 exams to which I had access from the years 2014 - 2019; the solutions (normative to LA1) of these task types, as indicated by similar problems and examples in the course textbook, involve the production and/or resolution of a linear system.

Table 5.3: Model of knowledge to be learned to perform exam tasks of type T (as in LA1 Problem 2)

Task		Count	Technique		Theoretical discourse
$t_1 \in T_{\nexists A^{-1}}$	to find all solutions of a linear system $A \in M_{m\times n}(S), m \neq n$, $m, n \in \{3,4,5\}$ (with at least one of m, n equal to 3)	1	$[\tau_2, \theta_2]$		None expected
$t_2 \in [T, \tau_2]$	to use Gauss-Jordan elimination to find all solutions of a linear system where $A \in M_{3\times m}(S)$, $m \in \{3,4,5\}$, or $A \in M_{4\times 3}(S)$	11	$[\tau_2, \tau_{2-eq}, \theta_2]$	in LA1, row reduction (τ_2) includes τ_{2-eq}, writing one of the augmented matrices obtained by reduction in terms of the corresponding linear system and using equation notation to complete the task; reducing an augmented matrix to its RREF (the purpose of θ_{21}, Gauss-Jordan elimination) may not be required to successfully complete this task	the task requires that θ_{21} be used for marks to be obtained
$t_3 \in [T, \tau_3]$	to use Cramer's rule to solve a linear system, $A \in M_{3\times 3}(S)$	9	$[\tau_3, \theta_{31}/\theta_{32}]$		the task requires that τ_3 be used for marks to be obtained; some instructors may expect θ_{33} (acknowledgement that Cramer's rule is applicable after finding that $\det A \neq 0$), but the task explicitly instructs students to use Cramer's rule, so any such expectation can have only limited weight in the grading
$t_4 \in [T_{\exists A^{-1}}, \tau_4]$	to use A^{-1} to solve a linear system $A \in M_{3\times 3}(S)$	1	$[\tau_4, \theta_{41}/\theta_{42}, \theta_{43}]$		a previous task requires students to find A^{-1}; t_4 requires students to use it for marks to be obtained
t_5	to solve a linear system so as to accomplish a different LA1 type of task	30	τ_2	If the coefficient matrix is square and the task can be about the number of solutions of the system (instead of identifying the solution), τ_2 can be replaced by calculating the determinant of the coefficient matrix and using the technology $\det(A) \neq 0 \Leftrightarrow A\text{x} = \text{b}$ has a unique solution $\forall$ b. Students are free to choose their technique in these cases.	None expected

Count is the number of exam problems, among 4 midterm and 6 final examinations between 2014 - 2019, in which this was the task to accomplish. Total number of problems on these midterm and final examinations is 116.
This count does not account for the number of times these tasks appear in problems recommended on the course outline or in assignment problems. Apart from midterm and final exams, students perform tasks in assignments that correspond to the weeks in which the related praxeologies are to be taught in class.
A: the coefficient matrix of the linear system.
S: the set of scalars in LA1 exam tasks t_1, t_2, t_3: mostly integers in $[-5, 5]$, occasionally including an integer in $[-20, -7]$.
T, τ, and θ: the task types, techniques, and technology defined in the reference model (see Table 5.2 on p.101).

solve a linear system by row reduction[12]. Indeed, τ_2 is the normative technique for solving linear systems whenever no instruction is given as to which technique to use (as in tasks t_1 and t_5). Additionally, τ_2 is the technique required for t_2, a task that appeared on each of the 12 exams to which I had access. I therefore expected students to spontaneously use the normative techique τ_2 (row reduction) when given Problem 2 in the TBI, but also expected the affirmation that the coefficient matrix is invertible to prompt some students into the more time-consuming τ_4 (find A^{-1} and use $A^{-1}A\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$).

In the case of Problem 2, the affirmation that A is invertible indicates Cramer's rule ($[\tau_3, \theta_3]$) is applicable, and the proportionality between the first column of A and b is amenable to calculations (using θ_{32}) that can make finding the solutions quicker than row reduction. However, it is *not* a norm in LA1 to decide, among $\tau_1, \tau_2, \tau_3, \tau_4$, which technique to use to solve a linear system; the norm is to use Cramer's rule (τ_3) only upon instruction to use it. In contrast, row reduction is the normative technique for 42 of the 116 problems in the midterm and final exams to which I had access[13].

I wondered whether any of the participants would solve the system by inspection (as in τ_1 of the reference model), and if none of them would, whether this would be because they do not notice that b is equal to -1 times the first column of A, or in spite of noticing that.

5.2.3 Knowledge LA1 students activated in response to Problem 2

Table 5.4 summarizes the paths of participants' activity as they worked on Problem 2. Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. Students moved on to a new step (e.g., Step 2, Step 3) mostly after a prompt from myself, the interviewer (e.g., "can you think of any other approaches?" or pointing out A is invertible and asking if that makes them think of any other approach), though in some cases students shifted tasks/techniques without a prompt from the interviewer. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step. If a participant does not appear in the column for Step i ($i \geq 2$), it is because they did not engage in any new activity after Step $i-1$.

Among the 10 participants, 7 spontaneously began to row-reduce to solve the system, 1 spontaneously started to calculate $\det(A)$ to check if A is invertible (but immediately switched to row-reduction upon noticing the affirmation in the problem statement that A is invertible), 1 spontaneously began to use the inversion algorithm to solve the system by using $A\mathrm{x} = \mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$, and 1 spontaneously used Cramer's rule to find the solution of the system. Only this last participant came close to solving the system: this was P8. P8 found that $x = y = z = 0$ by activating $[T_{\exists A^{-1}}; \tau_3; \theta_{32}, \theta_{33}]$ and taking advantage of the proportionality between b and the first column of A; P8 determined w is equal to "some

[12] I distinguish t from t_2 as follows: while t_2, in LA1, explicitly states to use Gauss-Jordan elimination, t does not and rides instead on the norm that row-reduction is the way to go.

[13] This is the number of exam problems that correspond to tasks t_1, t_2, and t_5.

Table 5.4: Paths of LA1 Students' Activity in Problem 2

	Step 1		Step 2		Step 3		Step 4	
Practical block $[t, \tau]$	Participant	Type of engagement with $[t, \tau]$	Participant	Type of engagement with $[t, \tau]$	Participant	Type of engagement with $[t, \tau]$	Participant	Type of engagement with $[t, \tau]$
$[t; \tau_2, \tau_{2-eq}; \theta_{21}, \theta_{23}]$ row-reduce to find the solution set of the system	P1 P4 P5 P9 P10	enacts, does not complete t	P5	enacts theoretical block to predict the system has one solution	P3	describes τ_{2-eq}		
			P3	enacts τ_2, does not complete t				
	P6	enacts τ_2 but on A (instead of $[A\vert b]$), does not complete t	P2 P7*	describes how to use (incorrect choice of augmented matrix)				
	P7*	describes how to use	P8	mentions				
$[t_4; \tau_4; \theta_{41}, \theta_{44}]$ find A^{-1} so as to multiply each side of $Ax = b$ by A^{-1} and find the solution of the system	P2	partially enacts	P1 P6 P9	mentions				
calculate $\det(A)$ to check if A is invertible or to check the expected number of solutions	P3	enacts $\det(A)$ calculation, abandons once aware of the affirmation that A is invertible	P4	describes own theoretical block, concludes $\det(A)$ calculation is "useless" for type T tasks	P7*	describes how to use (determinant calculation for incorrect matrix: $[A\vert b]$ instead of A), dismisses		
$[T_{\exists A^{-1}}; \tau_3; \theta_3]$ use Cramer's rule to find the solution of the system	P8	enacts, nearly completes: finds $x = y = z = 0, w =$ some number			P4	mentions, dismisses		
					P9	describes how to use		
use the fact that A is invertible, but no task identified					P5	mentions		
					P8	mentions formula $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, dismisses (as not "helpful")		
talk about linear transformations (reflection, symmetry)							P7*	describes, dismisses (may not be "helpful")

number" but did not "really want to calculate it further because the only thing that's left is just algebra." None of the other participants identified a (correct or incorrect) value for any of the unknowns.

The knowledge participants activated in response to Problem 2, beyond their spontaneous reactions, still mostly related to one of the four categories of spontaneous activity:

τ_2 row-reduce to find the solution set of the system

τ_4 find A^{-1} so as to multiply each side of $Ax = b$ by A^{-1} and find the solution of the system

find $\det(A)$[14] calculate $\det(A)$ so as to check if A is invertible or to check the expected number of solutions, but dismiss before implementing

τ_3 use Cramer's rule to find the solution of the system

In what follows, I discuss the knowledge participants activated relative to each of these four categories. I do not discuss the knowledge activated by P7* in step 4 of his engagement with the problem (see Table 5.4 on p.105) except to note it had little to no basis in the mathematics at hand and mainly reflected P7*'s general behavior throughout the interview, in that he always suggested to activate a variety of technologies, not always relevant to the mathematics at stake, without describing a concrete way through which to activate a technology, and usually dismissing the suggestion soon after making it. I finish the analysis of participants' engagement with Problem 2 with an examination of their activity and explanations: the goal is to address the question of whether students' normative practices align with or impinge on the mathematics at stake in Problem 2.

5.2.3.1 Row reduction

All 10 participants brought up or engaged with row reduction as a technique for finding the solution set of the system; in terms of the reference model and model of knowledge to be learned, all participants brought up $[t, \tau_2]$ and engaged with parts of $[\tau_2, \tau_{2-eq}; \theta_{21}, \theta_{23}; \Theta]$. This was the spontaneous activity of 8 of the 10 participants: these 8 consist of 7 who immediately engaged in row-reduction upon reading the problem statement (P1, P4, P5, P6, P7*, P9, P10), as well as 1 participant (P3) who initially set to find the determinant of the coefficient matrix, but I categorize their spontaneous activity in the practical block $[t_1; \tau_2, \tau_{2-eq}]$ due to their subsequent explanation and activity.

P3's spontaneous reaction was to calculate $\det(A)$ to check if A is invertible; I intervened 2 minutes into this activity to ask the participant why she was looking for the determinant, and she then noticed the affirmation in the problem statement that A was invertible ("so I don't really need to check its determinant here [...] because it is in-invertible. That means the determinant [...] it's not zero."); she *then* spontaneously started to row-reduce. P3 later claimed her practice is to first check whether $\det(A) = 0$ to decide whether to row-reduce to get the RREF of the augmented matrix, or rather to reduce it to some REF, find the corresponding equations, and then use back-substitution ("I'm not really sure how to handle the [non-]invertible matrix and that is the only reason that I chose the different ways"). Given P3's claim she checks whether $\det(A) = 0$ to inform her approach to τ_2, I categorize P3's spontaneous activity as row reduction.

Apart from the 8 who spontaneously engaged in row reduction, the remaining 2 participants (P2 and P8) brought it up after a first prompt from the interviewer asking them if they could think of any other approach. P8 alleged that row reduction would have been her spontaneous activity if not for the affirmation that the coefficient matrix is invertible.

In the next section, I describe participants' engagement with $[t, \tau_2]$. I will follow by addressing threads I found in participants' techniques and explanations that can help to characterize the practices they had developed in relation to $[t, \tau_2]$.

5.2.3.1.1 Practical blocks of students' activity relative to $[t, \tau_2]$ All participants engaged with τ_2. They either implemented it or suggested it as a technique to accomplish task t. Here I describe the practical block of students' engagement with this technique.

7 of the participants implemented τ_2: P1, P3, P4, P5, P6, P9, and P10 did row operations on paper. 3 of them (P5, P6, P9) explicitly identified the linear equations in the system before engaging in row operations. Participants' row operations were guided by the goal of getting leading ones (mostly in R_1, but in some cases elsewhere) and 0's either above or below this leading 1; the choice of row operations and speed at which participants did their arithmetic (with and without calculators) were such that participants would not have finished row reducing the system within 12 minutes (for most of those who enacted τ_2, I intervened around 9 minutes into their activity as this became clear and I had allotted an average of 15 minutes per problem in the TBI).

P2, P7*, P8 engaged with τ_2 but did not implement it. P2 suggested an incorrect use of τ_2, where the augmented matrix would have involved the transpose of A instead of A; P7* stated the technique ("Gaussian elimination" of the 4×5 augmented matrix), described elements of Θ and types of outcomes that result from τ_2 (in terms of RREF of the matrix), and then proposed an incorrect use of τ_2, where the augmented matrix would have been a 3×5 submatrix of the original augmented matrix; P8, who had essentially solved the system in her first approach (using Cramer's rule), only mentioned the technique.

None of the participants managed to solve the system via τ_2 in the allotted time. I did ask all the participants, except for P8 (who had essentially solved the system previously, with Cramer's rule) what they expected to find as they kept going. I describe participants' responses in the following paragraphs.

P3 said that since the coefficient matrix is invertible, row-reducing the augmented matrix would lead to a matrix of the form $[I_4|B]$ where B is the column matrix made up of the values of the unknowns that solve the system. P1 and P2 also described this final result, but did not justify it (and I did not prompt for an explanation). P10 also described the same final result, and when I prompted to ask how she knew this would be the case, she did not refer to A being invertible; instead, she said she could verify the solution by plugging it in to the original equation.

P4, P5, P7* gave a general description of the potential results of row reduction: the case of 1 solution (using a description similar to P3's), the case of no solutions (the participants referred to the potential that one row would imply $0 = a$ where $a \neq 0$), or the case of infinitely many solutions (where one of the rows is made up entirely of zeros).

After I pointed out to P5 that A is invertible, P5 said this implies that $\det(A) \neq 0$ and that row-reducing the augmented matrix would lead to a matrix of the form $[I_4|B]$ where B is the column matrix made up of the values of the unknowns that solve the system (much like the description given by P1, P2, P3, and P10).

After I pointed out to P7* that A is invertible, P7* said this implies its reduced row echelon form would have a row of 0's and so the system would have infinitely many solutions. P7* wrote the following:

$$\begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It seems that P7* had confused "coefficient matrix" with the augmented matrix. In this instance, P7* spoke of augmented matrices as matrices that *can* be invertible; this error persists into P7*'s next suggestion, when he talks about the determinant of a 4×5 matrix (he writes $(|\ |)_{4\times 5}$ in reference to the determinant of the matrix). Prior to either suggestion, P7* had said that he "would try to figure out what's going to be the characteristics of the invertible matrices 'cause [he] kind of forg[o]t a lot about them." Nevertheless, that P7* associated the notions of invertibility and determinants with the notion of augmented matrices suggests a superficial grasp of the notion of augmented matrix.

Some time after I pointed out to P4 that A is invertible, he asked whether it was a rule that if the matrix is invertible, then the system has a solution; I turned the question back on him and he claimed that *if it is a rule* (my emphasis), then he would expect to find a solution.

When I asked P9 what he expected to happen if he kept going, he said there would be infinitely many solutions because the last matrix he had produced (before I interrupted his row reduction process) had

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

as its right-most column, and if $0t = 0$, then t can be any number, so there are infinitely many solutions. None of the rows in P9's last matrix were made up entirely of 0's or appeared to be leading to that result.

Finally, when I asked P6 what he expected to happen as he kept going, P6 started to answer but switched approaches almost immediately to τ_4 (multiplying both sides of $A\mathrm{x} = \mathrm{b}$ by A^{-1}), and I did not return to discuss τ_2.

Table 5.5 on p.109 summarizes participants' predictions.

5.2.3.1.2 The theoretical blocks that drive students' use of $[t, \tau_2]$ Below, I share the theoretical blocks that influence participants' use of τ_2. These theoretical blocks include the concerns and difficulties participants had when it comes to this technique: these concerns and difficulties seem to shape how participants engage with τ_2, so in the sense of the notion of praxeology (a way to model an activity in terms of its practical block and its theoretical block, the latter of which consists of the theory and technology that produce and justify the techniques in the practical block), these concerns and difficulties form what we'd call the theoretical block of participants' activity. I also include, in participants' theoretical blocks, the justifications they have for the validity of τ_2 as a technique for accomplishing the task t. I discuss these theoretical blocks, together with

Table 5.5: LA1 participants' predictions about what they would find if they continued with row reduction (τ_2) in Problem 2

Prediction P		Participants who made P	
		initially	after I said $\exists A^{-1}$
"Option 1 solution": a matrix of the form $[I_4\|B]$ where $w = B_{11}, x = B_{21}, \ldots$	because A is invertible	P3	P5
	no justification	P1, P2, P10	
3 options: "Option 1 solution"; a matrix where one row looks like $[0 \cdots 0\|a]$ where $a \neq 0$ so there are no solutions; or a matrix where one row looks like $[0 \cdots 0\|0]$ so there are infinitely many solutions	"the teacher emphasizing" these are the only 3 options, and from experience, had "never [gone] through a solution other than those three"	P4	
3 options: the left side of $RREF(A\|b)$ is I so there is 1 solution; a matrix where one row looks like $[0 \cdots a\|0]$ where $a \neq 0$ so there are no solutions; or a matrix where one row looks like $[0 \cdots 0\|0]$ so there are infinitely many solutions	1 solution since there's no row made up of 0's; no solutions since non-zero number can't equal 0; infinitely many solutions since, if 0=0, the variable can be any number and 0=0 still holds	P5	
2 options: "Option 1 solution"; or a matrix where one row looks like $[0 \cdots 0\|0]$ so there are infinitely many solutions		P7*	
there will be infinitely many solutions	because of the row of 0's in $RREF(A\|b)$ (incorrect properties: $\exists[A\|b]^{-1}$, "$\exists A^{-1} \Rightarrow RREF(A)$ has a row of 0's")		P7*
	because of 0's in the right-most column of the matrix he had last found, and the fact that "if $0t = 0$ then t can be any number"	P9	
if it is a rule that $\exists A^{-1} \Rightarrow A\mathrm{x} = \mathrm{b}$ has a solution, then there is a solution			P4
no prediction made	got distracted by thought of τ_4	P6	

the practical component of participants' activity, to examine whether their activity was driven by consideration of the specifics of the problem at hand, by mathematics inherent to t, or primarily by normative practices they've built in LA1, and whether the latter supports or impinges on the mathematics at stake.

5.2.3.1.2.1 Students use τ_2 and know it works because it's a norm in LA1. The prevalence of τ_2 in participants' activity is not surprising. As explained in 5.2.2, τ_2 is the normative technique for 42 of the 116 problems in the past midterm and final exams to which I had access. Given Problem 2, 8 of the participants spontaneously engaged with τ_2 and it was the second technique of choice for the participants who hadn't spontaneously engaged with it (P8 and P2). That said, both P8 and P2's explanations suggest their practice is also driven, to a significant extent, by the normative practices of LA1: P8 claimed she'd have used τ_2 from the get-go if not for the affirmation that A is invertible, and P2 proposed an incorrect use of τ_2 (he had suggested to row-reduce $[A^t|b]$) and was not confident in his suggestion because it *wasn't* the usual: "usually on a test they would put like uh, questions everyone can solve [. . .] I don't think other people would think this way."

That τ_2 was the spontaneous activity for nearly all participants suggests participants' approach to Problem 2 was driven in large part by their normative practice from LA1; the scalars I had chosen for the system make of τ_2 one of the less effective approaches for solving the system (indeed, it can be solved by inspection, and the proportionality between b and the first column of A make Cramer's rule amenable thanks to the potential to use column operations when calculating determinants). The explanations given by four participants (P4, P5, P8, and P10) further indicate that participants rely on τ_2 primarily because it the norm in the institution (LA1). (Meanwhile, P2 was unsure of his incorrect use of τ_2 and the only explanation he gave for this uncertainty was that his suggestion *wasn't* the norm in LA1.) I address P4, P5, P8, and P10's comments below.

According to P4, P8, and P10, using τ_2 is the "usual," it is the "obvious"; it is a knee-jerk reaction. P8 qualified τ_2 as "the usual": "if I wouldn't [have seen] it says it's invertible, I will [sic] probably think about the usual [. . .] basically, to solve the system of equations and to use either Gaussian elimination or Gauss-Jordan elimination." P10's only explanation for how she knew τ_2 would work to find the solutions was that "it's how [she'd] always done it"; she added that "there were some questions like this, in the past papers as well. [She] had seen some answers. Stuff like this. And also, normally if [she sees] a question like this, it's kind of obvious [she] has to do this." When P10 was asked if she could think of another approach to the problem, P10 suggested to "move" the matrix of unknowns "to the other side [of the equation]"; given this suggestion is not founded, in any way, in the mathematics involved in τ_2, I infer that P10's use of τ_2 is a norm acquired from LA1 and is not informed by any other knowledge. As P4 said: "the second I see Ax = b, oh ok, REF. I mean, I need to reach REF and start solving toward that."

Part of participants' reasoning for using τ_2 is auto-technological: they use it because it works. The reduced form of $[A|b]$ that is the target of τ_2 produces solutions. As P7* puts it: "eventually, the left four columns become [...] ideally it becomes something like identity matrix [...] so this right column is just the solution here. So that's why I know that [τ_2] works." P4 also addresses what row operations can produce: "in my ideal case, I would - or maybe not ideal [laughs], but in a normal case, I would, 0 0 0, and then expect to have a rough identity matrix, and I could just pick out my w, x, y, z." He also describes the possibility that parameters be involved in the solution set and the possibility that the system has no solutions. P6 characterizes the possibilities afforded by τ_2 as "nice": "[there are] some nice-looking, easy simplifications I can do here. Because I can just completely get a leading one in the second row with all zeros." Participants know τ_2 produces a "rough identity matrix" - a (reduced) row echelon form - and so, row operations are the "way to victory" (P4).

P4's explanations for how he knows τ_2 works - that it is his "way to victory" - came down to experience and what the teacher had taught in LA1. Asked how he knew using row operations on the augmented matrix would lead to a solution, P4's response was this: it's "just the way that we were taught; so yeah, so to find a unique solution, I need to reach REF or RREF and if REF I need to back substitute." In the context of LA1, back-substitution involves finding the linear system corresponding to a reduced form of an augmented matrix and then using substitution to solve the system of equations. To explain other knowledge he had about τ_2, P4 was again only able to rely on norms established in LA1, though he did attempt to refer to the mathematics at hand: he maintained that the possible results from τ_2 are that there would be one solution, "parametric solutions," or no solutions, and asked how he knows this, this was his response:

> I never studied or encountered a problem where it wasn't. But anyways also, I either have one solution, or I have countable solutions, which means parameter, or have no solutions. There's nothing else in the, my number line, I could have like one, I could have two, three, which means parameter. [...] the two solutions, three solutions fall under the parametric solution [despite prompts to try and have P4 clarify what he meant by this, his intent remains unclear] [...] the teacher emphasizing, was emphasizing a lot that those are my three cases. So it just stuck in my head like that. And also through, well, I never went through a solution that was other than those three.

In contrast to P4's knee-jerk reaction to the problem ("the second I see $A\mathrm{x} = \mathrm{b}$, oh ok, REF"), P5 seemed to not *recognize* the task at first; she only recognized it when she perceived it to be about a system of linear equations (as opposed to an equation between matrices). At first, she did not know the role played by the matrix on the right side of the equation. She wrote

$$9w + 16x + 3y + 4z =$$

and then multiplied A and x incorrectly:

$$\begin{bmatrix} 9w & 16x & 3y & 4z \\ 5w & 6x & 0 & 8z \\ -2w & 3x & 0 & 4z \\ 3w & 6x & & \end{bmatrix}$$

[...]make P5 realize that $A\mathrm{x} = \mathrm{b}$ corresponds to a linear system: "I'm trying to jog my memory [...] I don't know what to do with [the numbers on the right side of the equation] [...] Oh wait, I know what this is!" At this point, P5 recognized that $A\mathrm{x} = \mathrm{b}$

was a system of linear equations. Asked how she knows τ_2 would end up giving her the solutions, she gave a reason that aligned with P4's and P10's: "when you multiply like, this matrix by the second matrix, you're gonna get pretty much a system... it's like a system of equations to solve algebraically with – *this is the way I learned how to do it this year*" [emphasis added].

P4, P5, P8, and P10's comments reflect the normative quality of τ_2 as a technique for performing task t. P5's additional comments about the technique, and explanations given by other participants, point to another justification LA1 students have for the validity of τ_2 as a technique for solving linear system: a superficial grasp of its equivalence to τ_{2-eq}, the set of techniques students learn in high-school for solving linear systems. I address this justification below.

5.2.3.1.2.2 Students may know τ_2 is equivalent to and simpler than τ_{2-eq}, but even when they do, this knowledge is superficial. Participants P5, P6, and P9 justified using τ_2 with the claim that it's like solving linear systems using the equations (τ_{2-eq}) but is simpler, or nicer-looking. This justification aligns with the knowledge to be taught in LA1: τ_2 is introduced as an equivalent version of τ_{2-eq} that saves time because it does away with the need to write the unknowns and equal signs. As P6 put it, τ_{2-eq} is "the gross way" whereas τ_2 offers "nice-looking, easy simplifications." P9 addressed the utility of τ_2 in the case of larger linear systems: "over here, you're multiplying [A and x]. And you're doing it by the equation. So you're going to rewrite the equation and all the coefficients and stuff. So because it works for small matrices, the same process applied to bigger matrices where you cannot prove it, it's easier to do it by matrix, which is why I believe [it] is correct." To understand what P9 meant by "you cannot prove it," I turn to his previous comments about "doing it [solving the system] by equation" (that is, solving the linear system via τ_{2-eq}): I presume he was referring to the hassle (or, perhaps, the impossibility) of solving larger linear systems via τ_{2-eq}.

Before I address the knowledge some participants *did* have about the relation between τ_2 and τ_{2-eq}, I address the case of three participants (P2, P7*, and P10) whose engagement with τ_2 severed any connection to τ_{2-eq}. P2 suggested to apply τ_2 to $[A^t|b]$; P7* suggested to apply it to a 3×5 submatrix of the initial augmented matrix; P10 used τ_2 appropriately, but when asked if she can think of another approach, she suggested to "move" the matrix of unknowns "to the other side" ("Maybe. I don't know if it would work or not. But maybe, I dunno"). I address P2 and P7*'s approaches in the next two paragraphs.

P2 had spontaneously engaged with $[t_4, \tau_4; \theta_{41}, \theta_{44}]$ (use x = A^{-1}b) and expressed self-doubt: P2 found the row reduction in the inversion algorithm to be "tricky" and said he would "skip it if it were on a test." When I asked P2 if he could think of another approach, he suggested row reduction, but on the augmented matrix made up of the *transpose* of A (that is, P2 suggested to row-reduce $[A^t|\text{b}]$). To explain this, P2 said each row of A "is like" one of the variables, so once the row and columns are swapped, each column of A^T "[would be]" one of the variables. P2 was not confident in this approach, but did not pinpoint that the issue was an absence of θ_{23} (the definition of augmented matrix); instead, his lack of confidence in using τ_2 on $[A^t|\text{b}]$ had to do with the fact that it's not the norm in LA1. He was "not sure if this is right... the obvious, obvious way to do it is to invert [A] and multiply it by the matrix here" (as in his first approach, with τ_4); the second approach he suggested might not be right because "usually on a test they would put like uh, questions everyone can solve [...] [P2 didn't] think other people would think this way. Obvious way would be to invert this and multiply."

P7* did not enact $[t, \tau_2]$ but rather invoked elements of its mathematical practical and theoretical blocks: "the row operations [...] are reversible [...] so that means the initial augmented matrix I have and the final augmented matrix I simplify, or the reduced row echelon form, they are equivalent, in, in - I mean in terms of the rows. Which means the linear system has not had [...] structural change in between." He described the types of results that can be obtained via row operations: "eventually, the left four columns become [...] ideally, it becomes something like identity matrix [...] Which means, well, at this point, if I write in [the form of a matrix equation], it's going to be identity matrix times something equals to something, which is the last, the right column. So this right column is just the solution here." To clarify, P7* did not mean that *here*, the left side of the augmented matrix would reduce to I, rather that in general it reduces to a RREF: when I asked what P7* expected would be found here, if he'd get the identity matrix on the left side, he said it's possible the last row or two might be completely zero (in which case, "this matrix [would] have infinitely many solutions"). When I asked P7* if he could think of any other approach, his suggestion was still to use row operations, but this time he suggested to do so on a 3×5 submatrix: since A is invertible, "it means there's gonna be at least one row after the Gaussian elimination that's gonna be completely 0, which means I may just [...] cancel out [...] any row here and it will not [...] change the final solution [...] so it will make the calculation easier [...] by [...] doing this 3×5 elimination in the augmented matrix." P7* continued:

> A matrix is invertible, it means the vectors, the columns are [...] not linearly independent. They are linearly dependent. No, the row... wait a minute. Yes, let's assu... I'm not sure, but let's assume that the row columns, they are not. They are not linearly independent, which means I can just cancel any of them and still the others... the others may be linearly independent, but they may also be linearly dependent, but let's just use the other 3 to do the calculation and it will still work.

P7* refers to several concepts from linear algebra which have properties that relate them to one another: the notion of invertibility of a matrix, its reduced row echelon form, and linear dependence of column or row vectors of a matrix. P7*'s third suggestion for how to approach Problem 2 involved the determinant of the 4×5 augmented matrix. P7* has incorrect versions of these interrelated properties. Further, P7*'s attempt to apply properties of matrices to an *augmented matrix* shows P7* is glossing over the very definition of augmented matrix (θ_{23}); it's just a matrix like any other. This suggests that P7*'s grasp of the concepts he discusses - including τ_2 and any notion of equivalence of rows in this process - is superficial.

P6 similarly engaged with τ_2 in a way that was initially severed from τ_{2-eq} and θ_{23}, but, unlike P2 and P7*, acknowledged the error later on. Nevertheless, and despite P6's explicit claim that τ_2 and τ_{2-eq} are "functionally" equivalent, his activity in relation to Problem 2 suggests his knowledge of the relation between τ_2 and τ_{2-eq} is superficial. At first, P6 applied row reduction to the *coefficient* matrix, instead of the augmented matrix. While doing this, P6 expressed doubt about what he was doing and eventually stopped the approach: "now I'm just getting myself... Should I just do the systems of equations? I feel like I'm making a mistake... 1 2 3.. plus 2... is... honestly, I'm experiencing some real doubt here." He then switched to τ_{2-eq} and said "I know this is a little bit messy this way, but I like algebra." Later in the interview, P6 said he stopped because he had realized he should have been using the augmented matrix—but then asked if row reduction (τ_2) would have led him to a solution. I turned the question back to ask him what he thinks: "I'm not sure actually, I'm not, because... it doesn't mean - I mean, even if I did, it doesn't look like a very, like, pretty reduction."

The comment that it "doesn't look like a [...] pretty reduction" might have suggested that P6 is considering the efficiency of his techniques, but his next suggested approach - τ_4, finding A^{-1} by row reduction and multiplying each side of $A\mathrm{x} = \mathrm{b}$ by A^{-1} to isolate x, and his claims that this technique "should have been obvious, [it] seems like a much easier way to do it than [τ_2]," that "the math is probably faster to invert it," betray a superficial grasp of the procedure involved in τ_2 and τ_4. After all, both involve row reduction, and τ_4 involves more calculations since the row operations applied to A must be applied to I_4 in parallel (and then there's the multiplication of A^{-1} with b). Even if P6 wishes to act efficiently, a superficial grasp of these techniques may impede this inclination.

When I asked why row reduction might or might not work, P6 acknowledged that "eventually, you just get, like, the one coefficient and you get [inaudible] w x y etc. equals" and (after a prompt to explain why this happens) said this happens because "you're just setting these equations all equal to each other, and then like, subtracting them out," "you just like move this over to the side and say plus [inaudible] equals zero, minus two equals zero and set that equal to each other," "sort of like reduce until you have one equation left. I know, that's not exactly how it works. But it's kind of, that's sort of how I think about reduction, it's like that," "you're taking, like, some of this away from another and you're messing with the constants that they're, like, the equations are equal to."

P6's description of "setting equations all equal to each other," "subtracting them out," and reducing "until you have one equation left" are reminiscent of a high-school technique for solving linear systems called "method of elimination" or "method of reduction." When I asked P5 how she knew that the process of row reduction would end up solving the system, she explicitly referred to the method of elimination:

P5: You're solving like a system of equations. And that method is always going to work to solve a system of equations.

I: which method? Like the... this, with the augmented matrix?

P5: Yeah, it's pretty much like solving it like you would, like, elimination, for like a normal set of equations.

Mathematically, this method corresponds to the elementary row operation of adding a multiple of one row to another row, often used to get an entry of 0 in a desired location.

To illustrate the high-school technique of elimination, I borrow an example from Alloprof (n.d.)[15]. To solve the linear system

$$\begin{array}{rcrcr} 15y & - & 9x & = & 6 \\ -5y & - & 2x & = & -10 \end{array}$$

the technique is to "find the operation that makes the coefficients of y equal in both equations" (perhaps this is what P6 meant by "setting equations all equal to each other") and "[e]liminate by subtracting term by term the two equations to form a first-degree equation with one variable" (similarly to P6's "subtracting them out" and reducing "until you have one equation left") (Alloprof, n.d.):

$$\begin{array}{rcrcr} 15y & - & 9x & = & 6 \\ -(15y & + & 6x & = & 30) \\ \hline 0y & - & 15x & = & -24 \end{array}$$

This last equation is then solved for x and the value found is substituted into either one of the first equations to find the value for y.

On surface, the high-school technique of elimination looks different from its row-reduction equivalent, even if it is produced by the same algebraic and logical discourse:

$$\left(\begin{array}{cc|c} 15 & -9 & 6 \\ -5 & -2 & 10 \end{array}\right) \xrightarrow{3R_2+R_1 \to R_1} \left(\begin{array}{cc|c} 0 & -15 & -24 \\ -5 & -2 & 10 \end{array}\right)$$

This row-reduction of the augmented matrix mimics the following algebraic and logical manipulations of the system of equations:

$$\begin{array}{rcrcr} 15y & - & 9x & = & 6 \\ -5y & - & 2x & = & -10 \end{array} \Leftrightarrow \begin{array}{rcrcrcrcrcrl} (15y & - & 9x) & + & 3(-5y & - & 2x) & = & 6 & +3(-10) \\ -5y & - & 2x & & & & & = & -10 & \end{array}$$

$$\Leftrightarrow \begin{array}{rcrcr} & - & 15x & = & -24 \\ -5y & - & 2x & = & -10 \end{array}$$

If only the procedural components of the high-school technique of elimination are known, it can be difficult to bridge the relation between τ_2 and τ_{2-eq}. This can make it difficult for students to know their equivalence in more than a superficial manner. Indeed, the procedural component of the technique doesn't require knowledge of equivalent equations/systems (equations/systems whose solutions are the same) nor of any logic ("this equation is true and that equation is true if and only if ...").

None of the participants explained why τ_{2-eq} works as a method for solving linear systems. P7* did say that augmented matrices are equivalent to their RREF, and that this means that the corresponding linear system has not had "structural change" in between, but his subsequent explanations question whether P7* had the algebraic discourse that supports this claim; P7*'s mention of "structural change" could have been a nominal description.

[15] Alloprof is a not-for-profit digital service delivered by Quebec teachers and subject matter experts to help elementary and high-school students in their studies.

In addition to the potentially superficial grasp of how the row operations involved in τ_2 relate to the operations done in τ_{2-eq}, there is a normative quality to how students interpret what they find by reducing an augmented matrix with row operations: students have rules about types of rows that can appear in a reduced form of an augmented matrix and the number of solutions a system has. For example, I stopped P9's row-reduction process when he had finished producing the following matrix and asked what he expected to happen as he kept going:

$$\left(\begin{array}{cccc|c} 1 & 2 & \frac{1}{3} & \frac{1}{3} & -1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{13}{3} & \frac{59}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{17}{3} & 0 \end{array}\right)$$

He described the remainder of the process: get 0's, "get a diagonal," and "values on the side." P1, P2, P3, P4, P5, P7*, and P10 also initiated their predictions with a description of what a reduced augmented matrix might look like, before describing how they interpret the solutions from a reduced form. (P6 got distracted by his idea of using τ_4, and so did not make a prediction, and P8 had already nearly-solved the system using Cramer's rule, so I did not ask her to make a prediction of what would happen if she used τ_2.) In P9's case, he predicted the system would have infinitely many solutions because of the 0's in the rightmost column and because, if $0t = 0$, then t can be any number; but there was nothing in the work he had done so far to ensure any row would give rise to such an equation. Rather, the 0 on the right side of the augmented matrix seemed to have triggered a rule P9 had about reduced forms of augmented matrices. I prompted him to describe what the left side of the bar (of the augmented matrix) would look like, in case this might change his prediction. He drew the following to show what the left side of the reduced form would look like:

$$\begin{array}{cccc} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & \\ & & & z \end{array}$$

("And then over here, I have numbers, with zeros all around.") P9 was not the only participant to have a rule associating reduced forms of augmented matrices and the case of infinitely many solutions: P4, P5, and P7* said that if there is a row of the form $[0 \cdots 0|0]$, then the system would have infinitely many solutions. This is false. Consider, for instance, the systems corresponding to the following augmented matrices:

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

But the tasks students are usually given in LA1 do not involve such situations. Students' predictions, which inferred conclusions about linear systems from rules about types of rows that might appear in a row reduction, suggest there is a normative element to students' interpretation of reduced forms of augmented matrices. These rules normally work in the tasks students are expected to complete.

Some of the rules participants gave had other mathematical inaccuracies (more significant than in their failure to account for cases like those above); these, along with some participants' explanations, show these rules are not founded in a definition of augmented matrices. P5, for example, had the rule that if there is a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 5 & 0 \end{array}\right]$ then there would be no solutions "because a number can't equal 0." P5 brought this rule up again in Problem 4, when she said a row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 16 & 0 \end{array}\right]$$

would mean that $16 = 0$. She struggled to justify this and tried to appeal to the knowledge she was taught in LA1:

> I'm trying to think about how that would [pause] I learned like, why. Like, if one variable is a five and then it's equal to 0, 0 0 5 0. Like, why that makes sense. [...] Yeah, so there's no solutions if [pause] the variable... So is - this one, right here? With the [...] z section, [it's] five and that's equal to zero, then there's going to be no solutions in the matrix because you can't have a [non-zero] number equal to zero.

P5 interpreted the vertical bar in the augmented matrix as an equal sign and spoke of the entries (e.g., the "z section") as if they were values of the unknowns, rather than their coefficients; this brings to mind P9's representation of a reduced form of his augmented matrix (see above): he had denoted the entries along the main diagonal by w, x, y, z.

Eventually, P5 moved on to explain the reasoning behind the rule that a row of zeros implies there are infinitely many solutions. It was only in the context of this rule that she used the definition of augmented matrices, possibly aided by a rule that sounded like P9's "if $0t = 0$, then t can be any number." Indeed:

> If you have a row of zeros, then... that number can be, that variable can be any number, like all real numbers [...] because... it's a variable... or no, if the co - sorry these are for coefficients. If the coefficient is... zero, then, and it's equal to zero, then the variable can be any number and it's still going to be equal to zero.

P4, P5, and P7*'s predictions about τ_2 were descriptions of (seemingly mutually exclusive) cases of the types of rows one might find by reducing $[A|b]$; this, and the explanations they gave, show their practice is regulated by mathematically incorrect norms and are only superficially connected to τ_{2-eq}. P4's explanation for how he knows that "one solution," "parametric solutions," or "no solutions" are the only three options suggests from where students draw validity for their rules: "we never studied or encountered a problem where it wasn't. But anyways, also, I either have one solution, or I have countable solutions, which means parameter, or have no solutions [...] the teacher emphasizing, was emphasizing a lot that those are my three cases. So it just stuck in my head like that. And also through well, I never went through a solution that was other than those three."

In the knowledge to be taught in LA1, there is emphasis on the theorem that a linear system in R^n has either 1 solution, no solutions, or infinitely many solutions. This is reinforced by tasks students have to perform, wherein students either have to determine whether a system has 1 solution, no solutions, or infinitely many solutions, or to find values of augmented matrix entries for which the system has 1 solution, no solutions, or infinitely many solutions. P4, P5, and P7*'s predictions and explanations suggest students form rules about reduced forms of augmented matrices, rules that perhaps bypass the need for a definition of augmented matrices, and associate them with each of the possible cases. I hypothesize these rules came about from their experience in performing tasks of type T in LA1, as the options students described do capture the possibilities that arrise in the problems given in the course.

The rules proposed by some of my participants are not founded in the underlying mathematics, which helps to explain the brittle quality of these rules. P5 mis-remembered the type of row that would lead to a conclusion of no solutions, and deduced that if a matrix has a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 5 & 0 \end{array}\right]$, then there would be no solutions. P7* did not even mention the option of no solutions even though he was giving a general description of what might happen when an augmented matrix is row-reduced. And P9's prediction of infinitely many solutions was made purely on the basis of the 0's of the right side of the bar (in the augmented matrix). His prediction prioritized his rules over the specifics of the matrix he was dealing with. LA1 students may be getting by with their normative rules, without needing to make recourse to the mathematics at stake.

The only participant whose prediction spontaneously took into account a property that was specific to the problem at hand was P3. P3 invoked the invertibility of A to predict that by continuing τ_2, she would find a matrix of the form $[I_4|B]$ where $w = B_{11}, x = B_{21}, \ldots$. This spontaneous and accurate use of the property that A is invertible, however, may just reflect a practice P3 developed as a strategy for tackling a certain type of problem: P3 explained she has a practice of calculating $\det(A)$ prior to using row operations so as to decide, in advance, whether to reduce $[A|b]$ to a REF or to its RREF. If $\det(A) \neq 0$, then P3 knows that the RREF of A would be I, and P3 knows that if A is invertible, then $\det(A) \neq 0$. In this case, P3 reduces $[A|b]$ to its RREF. If $\det(A) = 0$, then P3 reduces it to a REF and uses back-substitution (with the corresponding equations) because she's "not really sure how to handle the [non-]invertible matrix."

P3 was not the only participant who seemed more at ease with reverting to the linear equations than using τ_2 on the augmented matrix in certain situations. P6 reverted to τ_{2-eq} when he noticed he should have used the augmented matrix, rather than the coefficient matrix, to solve the system. He said: "I've done way more algebra. So I just feel more comfortable doing it. Whereas [row reduction], I understand, functionally, is kind of the same." Indeed, elementary row operations correspond to the algebraic operations that can be used on an equation to produce an equivalent equation. But no elementary row operations correspond to the method of substitution (substituting the value/expression found for one unknown into other equations), which was part of what P6 suggested to do and what he claimed was "functionally [...] kind of the same" as row reduction. A combination of τ_2 and substitution (or, rather, back-substitution, as it's called in LA1) is part of the knowledge to be taught in LA1: when a REF is found, an accepted normative practice is to write the corresponding equations and then use substitution (with the equations) to find the solution(s). In the knowledge to be taught, this combination is marketed as sometimes quicker than using row operations until a RREF is found; but P6 made no reference to this. P6 suggested to use algebra out of habit. P3, meanwhile, said she usually reverts to the equations corresponding to a reduced form of an augmented matrix based on a rule she'd formed: this is the technique to use when the coefficient matrix is not invertible. In P3 and P6's cases, then, τ_{2-eq} is a crutch they can rely on, from the position of Students, to compensate for a lack of confidence or knowledge relative to τ_2. This practice is driven by personal preferences (e.g., for high-school normative techniques for solving linear equations) that are not founded in the mathematics at stake in a problem.

In LA1, the knowledge to be taught introduces and validates τ_2 (row reduction) as a technique equivalent to τ_{2-eq} (algebraic manipulations of systems of linear equations). But the theory that explains this equivalence is not part of the knowledge students are expected to learn, and the procedural elements of τ_{2-eq} which students used successfully in high-school do not lend themselves to τ_2 on the surface; this can make it difficult to know how τ_{2-eq} produces τ_2 as a technique if a student's related knowledge is only superficial (e.g., if it's restricted to the practical block of high-school τ_{2-eq}). The absence of the discourse that produces τ_2 from τ_{2-eq} is most glaring when the definition of augmented matrix (θ_{23}) is absent from students' proposed or enacted techniques for solving $A\mathrm{x} = \mathrm{b}$ (P2, row-reducing $[A^t|b]$, P7*, row-reducing 3×5 submatrix of $[A|b]$ or calculating $\det[A|b]$, and P6, row-reducing A) and from students' rules about reduced forms of augmented matrices and the number of solutions a system has (P5, for whom a row of the form $\begin{bmatrix} 0 & 0 & 0 & 5 \mid 0 \end{bmatrix}$ implies a system has no solutions because $5 \neq 0$). Students' engagement with augmented matrices seems superficial: they can produce them, use row operations to modify them, but they don't know why these row operations produce valid results, and they might even modify augmented matrices via strategies that are not based in θ_{23} (as in P7*'s deletion of a row of the augmented matrix or P2's suggestion to use the transpose of A).

Students' superficial grasp of augmented matrices and row operations on these matrices comes through even when students know τ_2 is equivalent, somehow, to techniques they've learned in high-school (P5, P6, P9); their vague explanations suggest they do not know what the equivalence is. And despite the equivalences that are shared by τ_2 and τ_{2-eq}, some students (P3, P6) prefer τ_{2-eq} for reasons that aren't based in the mathematics: the preference is based either in their habits (P6) or in confusion about the mathematics (e.g., when the coefficient matrix is non-invertible, as in P3's case).

That students' row reduction of augmented matrices operates on a superficial level shows particularly in the rules they develop to interpret the results they find by operating on augmented matrices. All students' predictions about what might happen as they complete τ_2 started with a description of what a reduced augmented matrix might look like, and ended with how they infer solutions from this reduced form (P1, P2, P3, P4, P5, P7*, P9, and P10). Four students' predictions (P4, P5, P7*, P9) brought out normative rules for reduced forms of augmented matrices (either it's I and the values of the unknowns are on the right, or there is a row of 0's and then there are infinitely many solutions; P4 and P5 recalled the possibility of no solutions, though P5 associated it with a row of the type $\begin{bmatrix} 0 & 0 & 0 & a \mid 0 \end{bmatrix}$ where $a \neq 0$). These students did not spontaneously consider the invertibility of A, and only P5 was able to come to a correct conclusion (that there would be one solution) once I pointed out A was invertible. P3 had spontaneously made this conclusion, but her spontaneous recognition of the link between the invertibility of A and $RREF(A|b)$ may have been supported by a strategy P3 used to cope with confusion about matrices whose RREF is not the identity matrices. In their predictions, some participants (P7*, P9) prioritized normative rules (that are not, in general, true) instead of considering the specifics of the matrix with which they were dealing (P7* conflated various incorrect rules - an invertible matrix has a RREF with a row of 0's, and an augmented matrix can be invertible - to predict $[A|b]$ would have a RREF with a row of 0's; P9 concluded from the 0's in the right-most column in the last matrix he'd produced that the RREF would have a row of 0's).

Altogether, these patterns suggest that the practice students form in relation to τ_2 is driven by normative rules that work in LA1 and some element of personal preference, all of which only superficially connect with τ_{2-eq}. The absence of a discourse that connects τ_2 with τ_{2-eq} may make it difficult for students to remember the rules correctly (P5, P7*). The absence of a discourse that connects these also impinges on the mathematical meaning students can attach to the process involved in τ_2. This brings to mind P4's comment about his initial perception of augmented matrices in LA1: "I did not know what I was looking at, you know what I mean? [...] When I first went to [LA1] and looked at matrices, I was looking at a box with numbers, not an actual problem."

5.2.3.2 Find A^{-1} and use $A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$ (τ_4)

Finding A^{-1} so as to isolate x in Ax = b (τ_4) was the next most frequently-made suggestion after row operations on the augmented matrix (τ_2), though τ_4 was nowhere nearly as popular as τ_2: 1 participant engaged in it spontaneously (P2) and 3 participants (P1, P6, and P9) mentioned τ_4 when asked if they could think of a second approach (they all had τ_2 as their spontaneous approach). While working on Problem 2, P1, P2, and P6 exclusively activated and mentioned τ_2 and τ_4; after mentioning τ_4 and giving his favorable opinion of this technique, P9 spontaneously brought up Cramer's rule and concluded it would be the ideal technique for the problem (we'll address his reasoning in Section 5.2.3.4).

None of these participants found a solution via τ_4. P2 was the only one to activate it and abandoned it because he found the inversion algorithm "tricky." The inversion algorithm (τ_{41}) involves using row operations to reduce A to its reduced row echelon form (RREF) and applying the same operations to I; once $RREF(A)$ is found (and if $RREF(A) = I$), the matrix produced by applying the operations to I is A^{-1}. P2's calculations had produced decimals, which to him signaled something was off: "[the] calculator is giving me a decimal answer so [pause] I think I'm doing something wrong. There is no way I should invert it. If the answer is alright because it's… gonna take like… a year." I asked P2 if he could think of another approach and he suggested to use row operations on the augmented matrix (albeit an incorrect one - he suggested $[A^t|b]$).

P9 specified he would use the inversion algorithm. P1 and P6 did not specify which technique they would use to find A^{-1}.

5.2.3.2.1 Theoretical blocks of students' activity relative to τ_4: P1, P2, P6, and P9 expressed opinions about the suitability and utility of τ_4 in the given context. I address these opinions in the next few paragraphs.

P2 and P6's discourse suggested they viewed the task in Problem 2 as t_4, the LA1 task in which students are required to use A^{-1} to solve a linear system given in the form of a matrix equation ($A\mathrm{x} = \mathrm{b}$). To P2 and P6, it was "obvious" that the task was to use the inverse of A to solve the system by multiplying both sides of $A\mathrm{x} = \mathrm{b}$ by A^{-1}. As P2 put it, "the obvious, obvious way to do it is to invert [A] and [to] multiply it by the matrix here"; after he rejected a different approach he had suggested (row-reducing $[A^t|b]$, which he wasn't sure about because he thought other people wouldn't "think this way"), he reinstated that the "obvious way would be to invert $[A]$ and multiply," that "the obvious [method] is $[\tau_4]$. Because when I [want to solve] an equation, I keep the variable here and I take everything to the left side." For P2, τ_4 is simply how a task of type "solve a matrix equation" is done—it is a norm. (This brings to mind participants' interpretation of the task in Problem 1: participants perceived the task to be to solve a matrix equation of a form $ABC = D$ by multiplying, successively, by A^{-1} and B^{-1}.) P6, for his part, had spontaneously engaged in τ_2 but was unsure of his approach: he kept wondering why it mattered that A is invertible, and abandoned his row-reduction approach when he realized he was reducing the coefficient matrix instead of the augmented matrix. When I asked P6 what he had hoped would happen via row-reduction, P6 brought up it was bothering him not to know why it matters that A was invertible, and suddenly proposed τ_4: "$[\tau_4]$ should have been obvious." P6 did not explain why it *should have* been obvious, but did list (his perception of) the benefits τ_4 has over τ_2.

P1 and P9 did not seem to view Problem 2 as a task of type t_4, as P2 and P6 seemed to have done. P1 rejected τ_4 because it would involve the additional step of multiplying A^{-1} and b; because of this, on an exam, he would stick to row-reducing the augmented matrix (τ_2), though at home, he said he would do both techniques for practice. P9 was enthusiastic about τ_4 as a technique for this problem— he claimed it is easier and faster than τ_2—but ended up proposing Cramer's rule as a more efficient approach.

In addition to their opinions on the suitability of τ_4 as an approach for Problem 2, P6 and P9 made claims about its utility. Whether P6 and P9 would stand by these claims if they thought them through or actually activated τ_4 is another matter, but their claims do, at least, point to what students might be concerned about when they solve tasks of type T (to solve linear systems).

First, P6 and P9 both claimed that the result obtained by τ_4 is easier to check than that obtained via τ_2: "once you get an inverse you can immediately know what's wrong if you just multiply it" (P6), "this way [finding A^{-1}], it was kind of very concrete. Either you really, you can check it and you got it right or you could check and you got it wrong" (P6), and "finding the inverse you could, it's a little bit easier, and you could verify if it's true or not" (P9), whereas "sometimes when you're solving like, other equations, you get an answer and even if you - like, there are usually ways to check but sometimes it's like, you can check and still feel unsure" (P6) and "to find if [the solutions found via τ_2 are] correct or not, it will take a lot of time to prove that it's right" (P9).

P6 and P9's claims that τ_4 yields a result that is easier to check than τ_2 is mathematically baseless; after all, if A^{-1} exists, then τ_2 would yield a unique solution, and validating it would involve less calculations than the multiplication needed to verify that the calculated inverse is indeed the inverse of A (as per P6 and P9's suggestions). Nevertheless, these comments do suggest that students are concerned about validating the solutions they find, and that students may not always know how to validate solutions they find by row reducing an augmented matrix. I hypothesize this uncertainty may rather be about the cases where a linear system has no solutions or infinitely many solutions.

Apart from their comparison of the results produced by the two techniques, P6 and P9 also viewed τ_4 as more suitable and useful an approach to Problem 2 than τ_2 on the basis that it is *faster* and *easier*. As P9 put it: "[τ_4] is way easier and [τ_2] [...] [is] very long and [. . .] [can have] many calculation mistakes along the way." P6 said that "[τ_4] seems like a much easier way to do it than [τ_2]" and that "the math is probably faster to invert it." This perception, too, is mathematically baseless. The inversion algorithm, for instance, involves the same row operations as those needed to row-reduce the augmented matrix, and while the augmented matrix reduction has row operations applied to A and b, the inversion algorithm has the operations applied to A and I_4 (so τ_4 involves 3 additional columns - and then there's the multiplication of A^{-1} and b). Only P2 and P9 explicitly involved the inversion algorithm (θ_{41}) in their activity, but I surmise P6 likely would have as well, as it's the normative technique in LA1 for finding the inverse of a matrix in A; the alternative is the formula $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, which students are rarely expected to use except in the case of 2×2 matrices, though in that context the normative formula appears different:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

I do note that as P9 described what the inversion algorithm would look like, he spontaneously brought up Cramer's rule and concluded that *this* would be "the fastest method." That said, his reasoning for Cramer's rule being faster was *not* founded in the specifics of the matrices involved in Problem 2 (I return to this in Section 5.2.3.4).

P6 and P9 did not into account the mathematics involved when they claimed τ_4 is faster and easier than τ_2, and P2 doesn't seem to have done so either when he suggested row reducing the augmented matrix as an alternative approach to τ_4 when he found the row operations involved in the inversion algorithm to be "tricky." P6 and P9's comparisons and P2's suggestion of a not-so-alternative approach point to the surface-level grasp students might have on elementary row operations and the LA1-normative techniques τ_2 and τ_4.

Finally, even if P6 and P9's claim about τ_4 being "faster" and "easier" than τ_2 is mathematically baseless, it does bring out students' concern with the arithmetic required in row operations and the time and mistakes it can involve. I recall here that of the 7 participants who activated τ_2 to do Problem 2, none had come close to finding the solutions to the system, even though most had spent at least 9 minutes doing row operations before I put a stop to their activity. Students struggle to use row operations efficiently; I hypothesize this relates not only to difficulties some may have with arithmetic but also to students' activity operating on a superficial level, where they blindly enact techniques from LA1, in an almost knee-jerk reaction and using surface-level characteristics to guide their calculations, without even *looking* at the specifics of tasks they are given (such as the values of the entries in a matrix). I return to this point in Section 5.2.3.6, as students commented on their concerns about time and calculations throughout their activity for Problem 2 (and not only in relation to τ_4, which was my focus for the current section).

Finally, the value of the technology invoked by τ_4 ($A\mathrm{x} = \mathrm{b} \Rightarrow A^{-1}A\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$) is, arguably, in its role in the theory that unifies matrix algebra with systems of linear equations; but if students' practice in LA1 exists mostly in the practical blocks produced by this theory, they may superficially produce other justifications (it's faster, it's easier) to find value in the different techniques (such as τ_4) they are expected to know.

5.2.3.3 Finding the determinant of A

Three participants (P3, P4, P7*) brought up the option of calculating $\det(A)$ and subsequently dismissed it. In this section, I will address each participant's engagement with the option of calculating $\det(A)$.

For P3, calculating $\det(A)$ was a spontaneous reaction to Problem 2. She stopped her calculation once she noticed (because of a prompt I gave) the affirmation that A is invertible and said it wasn't necessary to find $\det(A)$ because, if A is invertible, then she knows $\det(A)$ is non-zero; she then switched to τ_2 and her activity remained in row-reducing the augmented matrix thereafter. Her justification for calculating $\det(A)$ was that it helped her decide whether to row-reduce $[A|b]$ to its RREF (in the case that $\det(A) \neq 0$) or to a REF and then use back-substitution (in the case that $\det(A) = 0$). This strategy is redundant; row-reducing the augmented matrix would reveal whether the coefficient matrix is invertible. P3 said she does this because she's not sure how to handle non-invertible matrices. That she uses "$\det(A) \neq 0 \Leftrightarrow \exists A^{-1} \Leftrightarrow RREF(A) = I$" as a rule to guide her application of τ_2 is evidence of a superficial grasp of notions involved in the rule and in τ_2.

The equivalence $\det(A) \neq 0 \Leftrightarrow \exists A^{-1} \Leftrightarrow RREF(A) = I$ and underlying proof are core to the theory that produces and justifies the techniques students need to know. The equivalence itself is emphasized in the knowledge students are to be taught but they are not required to know its underlying proof, even though constructs involved in the proof are part of the knowledge students are to be taught and are expected to learn. For instance, elementary matrices (the building blocks of much of the theory) are knowledge to be taught and make an appearance in the knowledge students are expected to learn (at the very least on one assignment throughout the semester). It is up to instructors' discretion whether they cover the proofs of the equivalence (between determinants, invertibility, and reduced row echelon forms), and midterm and final exams do not include problems that require knowledge of these proofs. The equivalence between $\det(A)$ and the invertibility of A is likely reinforced by formulas students need to know in LA1, such as

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ and } A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A).$$

P3 was not the only student to bring up the full set of equivalences ($\det(A) \neq 0 \Leftrightarrow \exists A^{-1} \Leftrightarrow RREF(A) = I$) as a package deal in reaction to the affirmation that A is invertible: P5 brought it up when making her prediction that row-reducing $[A|b]$ will lead to $[I_4|B]$, where $w = B_{11}, x = B_{21}$, etc. The determinant of A is irrelevant to the bridge between invertibility of A and its RREF being I.

P4 also seemed triggered by the invertibility of A to think of $\det(A)$: his first reaction, upon reading the problem, was to say that since A is invertible, its determinant is non-zero. He again brought $\det(A)$ up after I pointed out the affirmation that A was invertible (later on in his engagement with Problem 2); his explanation ("*usually* [emphasis added] when I see invertible, it means oh ok, related to determinant. Because that was the first thought that I had, actually, before doing anything") suggested that, in P4's practice, there's a normative quality to the relation between $\det(A)$ and the existence of A^{-1}.

P4 explained that when he "saw 'invertible,' [he] tried to think if there's anything to do maybe to solve for w, x, y, z, if there's any more optimal way than solving it by hand" and said it "was actually a haste on [his] side, [he] should have maybe thought that there is more optimal way, because that's actually very important throughout the exams to see if there's any quicker way to solve the problem." But, from experience, he determined that $\det(A)$ would be of no use here: "[he has] solved tons of similar problems so this would end up being useless. And then if [he] actually found a no-solution solution or parameter solution [presumably, P4 refers to the possibility that $\det(A) = 0$) then [...] this [calculating $\det(A)$] [will have been] useless." He then wondered whether there is a rule that says that if A is invertible, then $A\mathrm{x} = \mathrm{b}$ has a solution, and concluded he didn't know. In Problem 1, P4 activated the technique of multiplying by inverses to solve an equation of the form $ABC = I$, where the task was to solve for C. Here, P4 was trying to *recall a rule* from the knowledge to be taught in LA1; from that position, he did not activate knowledge he *did* activate to complete a task that normally called for that knowledge (multiplying by inverses).

Like P4, P8 was also triggered by the invertibility of A to think of $\det(A)$. Upon reading the problem statement, she said: "invertible, solve this system... that's probably said for a reason. I think it's invertible if its determinant is not zero. But I could solve it without, without it as well... So I guess, if it's about matrix being invertible then I have to use the determinant, I think the Cramer's rules." (Unlike P4, P8 was able to identify a practical application of the relation 'determinant-invertible.')

P7*, finally, brought up the notion of determinant after I had pointed out to him that column b has some similarity with column 1 of A; I had asked if P7* can think of another approach, and he responded that "when the two columns, they are proportional, it means that the determinant it's going to be 0, which has a characteristic of, of it means that the matrix is in - invertible. Yes. When the determinant is zero, it is in - sorry, I always, I. It's not invertible. Yes, I think that may help. That may help." He also wrote "$(|\ |)_{4\times 5}$" on paper. When I pointed out his matrix has size 4×5, he agreed about the size, and when I asked "so determinants?" he immediately responded: "oh, no, no, no. Yes. Yes. So it's not a determinant. For sure. It has no determinant because it's not an n by n matrix. But still." He then moved on to another idea.

The image of proportional columns seems to have triggered the notion of determinants in P7*, and the same appears to have occurred with P4. When I prompted him to think of another approach (after he finished explaining why calculating the determinant is, in general, a useless strategy in the context of solving linear equations), P4 took another look at the matrix and finally noticed the proportionality between the first column of A

and b: "Ah, yeah! I did not notice that at all. This is the same as this, right? Oh no, it isn't. No, minus 9, minus. Oh! It's the minus of that! Okay. I need to pay more attention before starting problems." He continued: "Does this mean anything? Because what happened was, I realized that one of the columns I had was... the minus of my b column. [pause] Trying to think of what that could imply. [pause] I don't think I remember anything that I could benefit from that." 15 seconds later, P4 brought up Cramer's rule and dismissed it: "Oh yeah, and also I wouldn't - I just thought of this, I wouldn't solve it using Cramer because it's four by four, so the determinant would be hard to find. I'm trying to see if there's any relation between the other columns and the b matrix. But there isn't."

To summarize: P3 brought up $\det(A)$ because it's an object she uses to guide her application of τ_2, despite knowing the rule $\det(A) \neq 0 \Leftrightarrow \exists A^{-1} \Leftrightarrow RREF(A) = I$, and she stopped her calculations once she realized A is not invertible (and so $\det(A) \neq 0$); P4 brought it up because the invertibility of A automatically triggered in him the concept of determinant, but he saw no practical use in calculating the determinant of a coefficient matrix in the context of solving a linear system; and P7*'s notion of determinant was triggered by the image of proportional columns. P7* also brought up the relation between $\det(A)$ and the invertibility of A, though he fumbled with whether a determinant that is 0 implies that A is invertible or not invertible.

I discuss my inferences from participants' comments about the determinant of A in Section 5.2.3.6, where I draw from the different elements of participants' activity in Problem 2 to address their incapacity to behave efficiently despite having a will to do so.

5.2.3.4 Cramer's rule

Three participants brought up Cramer's rule as an approach to Problem 2: P8, who did so spontaneously, and P4 and P9, who brought it up only after I prompted them a second time if they could think of any other approach. This is not surprising. In LA1, students are always explicitly told when to use Cramer's rule to solve a linear system.

The participant who spontaneously used Cramer's rule upon reading the problem statement was the only one to come close to solving the system. P8 decided to use Cramer's rule because the affirmation that A is invertible, to her, suggested the problem was "about" A being invertible: "invertible, solve this system... that's probably said for a reason. I think it's invertible if its determinant is not zero. But I could solve it without, without it as well... So I guess, if it's about matrix being invertible then I have to use the determinant, I think the Cramer's rules."

P8 started by calculating $\det(A)$: this took 9.5 minutes and yielded an incorrect value. She then wrote out the entries for the matrix A_1 and asked if she could just describe what she'd do next. She said that w would be $\frac{\det(A_1)}{\det(A)}$ and, when asked if she had any idea what $\det(A_1)$ might be, without going through all the calculations, she said that maybe there'd be something to do with the first column of A_1 being "the same numbers [as in the] first column [of A], but with the opposite sign," using rules that relate row/column operations with the determinant that she knows from "the instructor," "the book," and from examples that demonstrate the rules. She described the appropriate rule, and eventually concluded that "$w =$ number" but did not "really want to calculate it further. But - because this, the only thing that's left is just algebra." Although she wasn't able to spontaneously activate the rule she had described to find $\det A_1$, she did find that $x = y = z = 0$ by activating the row/column-determinant properties she'd mentioned and the fact that column 1 of A is equal to -b. She explicitly wrote this for x:

$$x = \frac{\det(A_2)}{\det(A)}, A_2 = \begin{bmatrix} 9 & -9 & 3 & 4 \\ 5 & -5 & 0 & 9 \\ -2 & 2 & 0 & 4 \\ 3 & -3 & 1 & 1 \end{bmatrix},$$

concluded that $x = 0$ (specifically, she mentioned adding a column to another column, knew it would have no effect on the determinant, and mentioned a cofactor expansion along the column of 0's). She then inferred that $y = 0$, and $z = 0$. I note that P8 had not activated any of these properties to facilitate her calculations of $\det(A)$, which had taken nearly 10 minutes to find, even though she had considered them a minute into the start of her calculations ("maybe I should reduce it... No, I won't"); instead, she did a cofactor expansion along the column that had two entries that were 0, and stuck to cofactor expansions thereafter. She only activated the properties that relate determinants with row and column operations after writing out the entries in A_2, pausing, and making the following observation: "honestly, I don't remember. But this might look a bit weird. I think there was something that if a column or a row is proportional, like if two rows or columns are proportional, proportional then, they should be, they can be converted. Like I can make a column of zeros and then the determinant is zero." She only activated the rules she'd recalled about row/column operations and determinants after recalling another rule: "there was something that if a column or a row is proportional, like if two rows or columns are proportional, proportional then, they should be, they can be converted."

Apart from P8, only two other participants (P4 and P9) mentioned Cramer's rule. I discuss P4 and P9's activity (relative to Cramer's rule) in the next two paragraphs.

Both participants had row-reduction (τ_2) as their spontaneous reaction, and brought up Cramer's rule after (but not immediately after) I had pointed out the affirmation that the coefficient matrix is invertible. P9 had initially misread the problem and thought it stated the coefficient matrix was *not* invertible ("it's an ininvertible [non-invertible] matrix. So [...] you cannot multiply by the inverse"); once P9 was aware it was invertible, he spent a minute addressing τ_4 and then spontaneously brought up Cramer's rule. P9 described ways to calculate determinants: cofactor expansion, rules that relate row/column operations with determinants, and a LA1 rule for determinants of 3×3 matrices (a mnemonic device that helps to recall the result of cofactor expansion for matrices of this size, without actually doing a cofactor expansion), but did not actually do any calculations to find the value of any of the unknowns. P9 noticed here that column 1 of A is equal to $-$b but said this can't be used, and when asked if he could predict the value of x, P9 said he'd have to do the calculations to find it (and did not do them).

When P9 thought of Cramer's rule as an approach for Problem 2, he exclaimed that it "is the easiest" approach, that it's "the fastest method," that it is "faster and easier [than τ_4]. You just row reduce and then you have the matrix." But, despite his recollection of rules that relate row operations with determinants, his claim that the relation between column 1 of A and b can't be used puts to question why P9 thought Cramer's rule would be faster and easier than τ_4. Without the relation we'd designed into A and b, Cramer's rule would require P9 to calculate 5 different determinants ($\det(A)$ and each one of $\det(A_i)$, where A_i is the matrix obtained by substituting column i of A by b).

P4, the only other participant who mentioned Cramer's rule, dismissed it as a viable technique on the basis that the size of the matrix would make the determinant hard to find. P4 brought Cramer's rule up after I asked a second time if he could think of another approach, and after he had noticed the relation between column 1 of A and b. This was the interaction:

P4: Isn't a coefficient matrix a... Isn't it a special case?

I: What do you mean?

P4: Ah, yeah! I did not notice that at all. This [b] is the same as this [column 1 of A], right?

I: Yes.

P4: Oh no, it isn't. No, minus 9, minus. Oh, it's the minus of that. Okay. I need to pay more attention before starting problems. Does this mean anything? Because what happened was, I realized that one of the columns I had was... the minus of my b column. [pause] Trying to think of what that could imply. [pause] I don't think I remember anything that I could benefit from that.

I: Ok.

P4: Yeah.

I: Ok.

P4: Oh yeah, and also I wouldn't - I just thought of this, I wouldn't solve it using Cramer because it's four by four, so the determinant would be hard to find.

I: Ok.

P4: I'm trying to see if there's any relation between the other columns and the b matrix. But there isn't. So... yeah I think I would continue solving it... Even if it was a... some sort of coincidence. And I will just continue solving.

While both P4 and P9 had noticed that column 1 of A is equal to -b, neither participant used this to facilitate calculating the relevant determinants and neither ended up solving the system. They were hindered by the prospect of calculating determinants: P4 said it would be hard to find the determinant of a 4×4 matrix, and P9 did not activate the properties of determinants and row/column operations despite having described these "rules" he'd seen in his textbook.

P8 and P9 had both recollected the properties that relate column/row operations to determinants but were unable to activate them completely to calculate the values of w, x, y, and z. P8 only activated them to calculate x, y, and z after having retrieved another rule, a rule about determinants with columns that are proportional; P7* had also mentioned this rule after I had pointed out to him that b was similar to column 1 of A and asked him if he could think of another approach. P9 did not attempt to calculate the values of any of the unknowns despite having noticed that column 1 of A is equal to -b, and when I asked if he could predict the value of x, P9 did not attempt to do so. His answer (saying he'd have to calculate it) suggested he might have expected these calculations to be at least somewhat laborious. P8, who did succeed in activating the rules to quickly find that $x = y = z = 0$, did not want to do the calculations needed to find the value of w (despite my prompt for her to do so, after she'd found the values of the other unknowns).

P4, meanwhile, had a norm that determinants of 4×4 matrices are hard to find, and he did not activate any knowledge that could have helped him take advantage of the pattern he had noticed in A and b.

P8 and P9 struggled to activate the row/column operation-determinant relations they had recollected (P8 explained that "you have to be careful to not make any mistake here. And again, you should make sure that you add everything according to the row operations that you made. And again, sometimes you can forget that"), and P4 did not bring them up; instead, he had the norm that 4×4 matrices are hard to find. The relations between determinants and row/column operations may be difficult for LA1 students to remember. For example, consider P8's confusing description of one such rule:

> If I divide or multiply by something, then it's... So the new matrix that I created will be the determinant of that original matrix, A, multiplied but by that same number, but because I need to find the determinant of the... my original, both my original matrix and not the new that I created, I should do the opposite. So I should divide by the number.

These "rules" can be especially difficult to remember if students only know them superficially and do not know the reasoning, based in cofactor expansions, that explains a relation between the determinants of two matrices A and B, where A can be obtained from B via one row or column operation.

In LA1, students are taught to calculate determinants using cofactor expansions (which they're encouraged to wield on rows/columns that have the greatest number of 0's) and using properties that relate determinants with row and column operations. However, students are never explicitly told which technology to use; they are free to choose which one to activate, and the matrices whose determinants they are required to find are rarely of a size greater than 4×4. Indeed, in the past midterm and final exams to which I had access, each exam had one problem in which students were required to find the determinant of a matrix; one exam involved a 5×5 matrix (which had a row with three entries that were 0) and the rest all involved 4×4 matrices, the vast majority of which included at least one row or column with two entries that were 0. These tasks are amenable to cofactor expansion and, for someone acting from the position of a Student, do not justify acquiring difficult-to-remember rules that don't significantly reduce the labor needed to calculate determinants.

Cofactor expansion involves a predetermined algorithm (students only have to choose along which row or column to expand), while the relations between row/column operations and determinants do not—to be used fruitfully, it's necessary to consider the entries in a matrix; my participants' choice of row operations, when activating τ_2, suggests students do not consider the entries in a matrix before activating their techniques. Finally, P9 pointed out a third LA1 technique for calculating determinants (of 3×3 matrices specifically): this technique is a mnemonic device in the textbook for LA1, a visually-memorable rule determinants of 3×3 matrices (the rule is actually the result of a cofactor expansion of such matrices, but highlights instead a mnemonic pattern which students likely favor). Since students normally calculate determinants of 4×4 matrices, then, they can use this mnemonic device, along with the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and successfully compute determinants using one application of cofactor expansion along a row or column half of whose entries are 0's, and using zero application of any property about determinants of matrices that differ by one row/column operations.

5.2.3.5 What students made of the affirmation that the coefficient matrix is invertible

A feature of Problem 2 that distinguished it from the tasks normally given in LA1 of task type T (to solve a linear system) was the affirmation that the coefficient matrix was invertible. I wondered whether and how students would use this affirmation. For instance, if any student found a solution by inspection (an achievable option only if the entries in the matrices are considered *before* a method is selected), then the affirmation that the coefficient matrix is invertible would confirm the observed solution is the only one; otherwise, the affirmation that the coefficient matrix is invertible would confirm that Cramer's rule is applicable, and, if the entries in the matrices are considered, Cramer's rule can be seen to be more efficient than row-reducing the augmented matrix (since the first column of A is $-$b).

Neither situation arose in my interviews. P8 did choose to use Cramer's rule because of the affirmation that A is invertible, but this choice had nothing to do with the entries in the matrices; P8 inferred from the affirmation A is invertible that she "[has]" to use the property. P8's comments and activity gave no sign that P8 made this inference on the basis of anything other than the fact that A was affirmed to be invertible—perhaps as a 'hint' from an authority. When I asked P5, another participant, if she would do anything differently on an exam (her only suggestion was to use τ_2) her answer suggested she also interpreted the affirmation A as a 'hint' from an authority: she said she didn't know if τ_2 is the most "efficient" or "fastest" way, and when asked what would be faster, she said that "they tell you that the matrix is invertible. There must be some sort of faster way that [she] just [doesn't] know."

Participants were not able to use the affirmation A is invertible to act more effectively. But the affirmation did help to reveal the fragmented map of associations students link with the concept "$\exists A^{-1}$"—a fragmented map likely supported by the tasks students are expected to perform in LA1. I address these in the last paragraphs of this section.

For some students, the affirmation A is invertible triggered technical knowledge: P2 (spontaneously), P1, P6, and P9 suggested the technique τ_4 (find A^{-1} and multiply both sides of $A\mathrm{x} = \mathrm{b}$ by A^{-1}). τ_4 is the least efficient technique for the problem at hand. Notably, P4 did not suggest this technique (though he had used a similar one to solve Problem 1:

$$WXAYBC = I \Rightarrow (WXAYB)^{-1}(WXAYB)C = (WXAYB)^{-1}I,$$

and in that context there was no affirmation as to whether A or B are invertible). I highlight P4's case because he had overtly asked whether there is a rule about A being invertible and $A\mathrm{x} = \mathrm{b}$ having a solution. A potential distinction between P4 and P1, P2, P6, and P9 is that P4 was inquiring about a "rule," which, from the position of Student, he may perceive to be theory that is outside of his reach; meanwhile, P1, P2, P6, and P9 were activating a technique they needed to learn in LA1.

The affirmation A is invertible also revealed a ritualistic component of some participants' knowledge: for P3, P4, P5, P7*, and P8 the concept of invertibility triggered the image of a determinant being equal to 0 or not (P7* struggled to remember the accurate relation). P4's explanations suggest what may have inculcated this knowledge in students' memory: he said he knew this "rule" because his teacher had said it and "whenever [he] studied, [he] looked at [his] notes, and okay, determinant is not zero means that it's invertible." The equivalence $\det(A) \neq 0 \Leftrightarrow \exists A^{-1}$ is emphasized in the knowledge to be taught in LA1: it's at the core of a theorem that is revisited and expanded upon in 2 of the 4 chapters of knowledge to be taught in LA1. This equivalence may also be reinforced in the knowledge to be learned through formulas for A^{-1}: these formulas involve the expression $\frac{1}{\det(A)}$, and students are expected to memorize and use these formulas. For P3 and P5, the affirmation A is invertible also triggered the image of the reduced row echelon form of A being an identity matrix and they were able to deduce from this (when asked to make a prediction) that the system would have one solution. This knowledge ($\exists A^{-1} \Leftrightarrow RREF(A) = I$) may be a result of the inversion algorithm students need to know in LA1, wherein they discover whether A is invertible by finding out if its RREF is I. P5's explanation suggests this is indeed the case:

I: So how do you know that if it's invertible, you can get that form? 1 0 0 0 0 1 0 0? And so on.

P5: Um, because when you're trying to like inverse something, it's going to, like, remember the thing I said in the last problem, with the matrix is equal to 1 0 1?

She's referring here to the inversion algorithm she used in Problem 1.

The techniques and explanations participants proposed, in reaction to the affirmation A is invertible, were fragments of knowledge about invertibility, determinants, and the equation $A\mathrm{x} = \mathrm{b}$. This may seem like theoretical knowledge, but students' explanations (throughout Problem 2, and especially in relation to row-reductions of augmented matrices) suggests it rather reflects knowledge that is emphasized by techniques students need to learn in LA1 (row-reducing augmented matrices, the inversion algorithm, formulas for inverses). In any case, participants did not, in general, wield the affirmation A is invertible to behave more efficiently.

5.2.3.6 Summary: students struggle to optimize their approaches because their practice is restricted to superficial features of the problem of solving linear systems

In this section, I infer from my participants' activity and explanations that students' normative practice from LA1 may be restricted to superficial features of the problem of solving linear systems and that this might contribute to the difficulty they can have in solving linear systems accurately and efficiently. Students' dependence on superficial features of Problem 2 prevented them from making efficient choices—from the choice of technique all the way to choices of row and/or column operations when reducing augmented matrices, using the inversion algorithm, or calculating determinants. This helps to explain what seemed to be their main concern about tasks of type T (solving linear systems): the time and accuracy of calculations involved in τ_2, τ_3, and τ_4.

In Sections 5.2.3.1, 5.2.3.2, 5.2.3.3, and 5.2.3.4, I focused on participants' engagement with a task of type T. I examined their explanations for using τ_2, τ_3, and τ_4 and found students had fragments of knowledge from the theoretical blocks that frame them. Some students knew that τ_2 reflects the algebra done to solve systems of linear equations; several students knew the equivalence $\det(A) \neq 0 \Leftrightarrow \exists A^{-1}$; some students knew that a linear system has either 1 solution, no solutions, or infinitely many solutions; some students knew that if A is invertible then it can be reduced to an identity matrix; some students had the technology that produces τ_4: $\exists A^{-1} \Rightarrow A\mathrm{x} = \mathrm{b} \Rightarrow A^{-1}A\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$ (which, essentially, is the matrix multiplication technique they also activated in Problem 1); and some students knew that if A is invertible, then Cramer's rule is applicable.

But students did not know *how* τ_2 reflects the algebra done to solve linear equations, and some students' activity stripped τ_2 from any relation to linear equations when they brought up "augmented" matrices that were not, in fact, augmented matrices (e.g., $[A^t|b]$, or the rule that a row of type $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 5 & 0 \end{array}\right]$ means there are no solutions because $5 \neq 0$). And to explain the fragments of theory they knew (e.g. the relation between $\det(A), A^{-1}, RREF(A)$), participants were only able to rely on the authority of their teachers, textbook, and experience (e.g., from the inversion algorithm, there is the rule that associates A having an inverse with its potential to be reduced to I). One student had activated matrix multiplication to solve the equation $WXAYBC = I$ in Problem 1 but wondered whether there is a rule that says that if A is invertible, then the matrix equation $A\mathrm{x} = \mathrm{b}$ has a solution. Students had only superficial and disconnected fragments from Θ, the theoretical discourse that frames and produces the techniques for solving linear equations whose solutions aren't self-evident.

The fragments of theoretical knowledge students had are features of practical knowledge students typically use in LA1. For instance, when students are first taught the notion of determinant in LA1, it is through a theorem, early on in the semester, about 2×2 matrices and their invertibility. The theorem states that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if its determinant is non-zero and, in that case, the inverse of the matrix is $\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (students know 'you can't divide by 0'). The equivalence $\det(A) \Leftrightarrow \exists A^{-1}$ is established early on and emphasized throughout LA1: in a theorem that's recurrently revisited throughout the first half of LA1, through a similar formula for inverses of larger square matrices, and through the normative technique for several types of LA1 tasks. Another example of theoretical knowledge that, for LA1 students, is a feature of their practical knowledge: one participant referred to the inversion algorithm as evidence for the equivalence $\exists A^{-1} \Leftrightarrow RREF(A) = I$. And students knew that a linear system has either 1 solution, no solutions, or infinitely many, but explained this knowledge either by referring to the authority of their teachers and textbooks or notes, to the authority of the examples they've solved, or to the authority of incomplete or incorrect rules about what an augmented matrix might look like when it's reduced (rules that reflected students' experience with augmented matrices in LA1).

Considering the knowledge students are expected to learn in LA1 relative to tasks of type T, it's not surprising that students' knowledge of the theoretical discourse that frames these tasks is localized to features that come up in practice. Students are not expected to activate theoretical knowledge; they are expected to activate practical knowledge. But participants' activity, as they engaged with Problem 2, showed that the norms they develop in LA1 (such as automatically activating τ_2 to solve linear systems when no instruction is given as to what technique they should use) contribute to difficulties they have in activating practical knowledge.

Participants relied on superficial features of Problem 2 to guide their activity. When they were presented with Problem 2, they mostly responded in a way that reflected their normative technique as students from LA1 (use τ_2 to solve linear systems). Eight students spontaneously activated the normative LA1 technique for solving linear systems: row-reducing the augmented matrix (τ_2). One of these eight was unable to activate any other technique (P10). Another student spontaneously fell into the trap of solving the system by calculating the inverse of A, and when prompted for another approach, could only suggest to apply τ_2 to $[A^t|b]$ (and three participants who spontaneously engaged in τ_2 made τ_4 as their follow-up suggestion). τ_4 is a technique students are expected to learn in LA1, though the technology that produces it ($\exists A^{-1} \Rightarrow A\mathrm{x} = \mathrm{b} \Rightarrow A^{-1}A\mathrm{x} = A^{-1}\mathrm{b} \Rightarrow \mathrm{x} = A^{-1}\mathrm{b}$) has its value mainly in the theoretical discourse that supports the techniques for tasks of type T (e.g. if A is invertible then $A\mathrm{x} = \mathrm{b}$ has a unique solution for any b; one participant, who had not activated τ_4, wondered if there was such a rule). Another student, P8, used Cramer's rule, but only because the affirmation that A is invertible struck her as an indication that she *has* to use Cramer's rule. She did not consider the entries in the matrix before deciding on her approach, and even claimed later (after having found that $x = y = z = 0$) that τ_2 would probably have been a faster approach.

None of the participants considered the entries in the matrices before activating their technique. This is perhaps most evidenced by the fact that only three students actually addressed the similarity between the matrix b and the first column of A, and even they only did so toward the end of their activity with A, and only after questions from the interviewer intended to prompt their attention away from their spontaneous reactions.

That students did not take entries in the matrices into account—or did so only superficially—is also evidenced in their poor choice of row operations. When it came to τ_2 and τ_3, their choices rendered the task more laborious or increased their risk of error and dependence on a calculator. For instance, P5 got some 0's and leading 1's and then did operations that undid those 0's and leading 1's. Others (P2, P9, P10) divided the first row by the value of A_{11} to get a leading 1 in that row, and then used the leading 1 to get 0's in the column underneath it; but this produced many entries that were fractions (cue P9: "it's tricky," "it's... gonna take like... a year"). Some participants did take entries of the matrix into account, but their considerations seemed restricted to the case of the "entry whose value is 1": P1 used an entry that was a 1 from the get-go to get 0's in the same column, and P4 and P6 took advantage of entries (in the same column) whose difference was 1 to get an entry whose value was 1 (in that column). Altogether, participants consistently spent significant amounts of time on these reductions and got nowhere near solving the system after nearly 10 minutes of calculations.

If not for my interventions, most participants' activity seemed bound to remain in their row operations (with the exception of P7*, who, at the other extreme, bounded from one suggestion to the next without actually engaging in any operation). It's not surprising, then, that participants' primary concern when it came to solving linear equations was the accuracy of their calculations and the time it took to complete them. They commented about the calculations involved in Problem 2 ("that's harder than [the] exam," said P1) but also about how they'd prioritize these in an exam or as they practiced at home.

Participants commented on the accuracy of their arithmetic. P6 mentioned he doesn't always indicate which row operations he does (e.g., $R_1 + 2R_3 \to R_1$) and that this makes him "mistake-prone": "I just do in my head. [...] I just find... I mean, sometimes it makes me a little mistake-prone. It's harder to find my errors." P4 was concerned about arithmetic errors on exams: "to maximize my marks, I'm gonna need to probably go through it like five times to make sure I didn't make any arithmetic mistake. I would [...] try to... work slowly. Because I would not like to write 10 lines and then need to scratch everything out." P2 mentioned a Student technique for identifying his errors: "calculator is giving me a decimal answer so [pause] I think I'm doing something wrong," "it's, uh, usually when I get like uh. Lot of tricky numbers. I, I assume something is uh, wrong. Honestly, if it was on a test, I would skip it." P9, in reference to the mnemonic device for calculating determinants of 3×3 matrices, said that "when [he] was studying for the final, this was the method where [he] had least mistakes and was the fastest."

Speed seemed a top concern for participants. P5 said that "for the final - final exam, [she'd] just go a little faster, because this might not be the fastest way" but, when I asked what would be faster, did not have an answer (she guessed there'd be something to do with invertibility, but did not know what). P6 said he doesn't indicate, on exams, which row operations he does because this "eats into the time." P6's opinion of linear algebra was marred by how much time it eats up: "the thing I don't like about linear is that all these like basic operations seems to take a very long time." ("I just like algebra [...] I just like numbers like that. Like, the things that I read for fun are not linear. It's more like... abstract algebra. Number theory.") At least one participant relied on the time needed to perform a task to gauge the validity of his technique: P2 abandoned τ_4 because the inversion algorithm would take too long. The "calculator is giving me a decimal answer so [pause] I think I'm doing something wrong. There is no way I should invert it. If the answer is alright because it's... gonna take like... a year" (nevertheless, P2's only other suggested technique was τ_2, which still requires row operations to solve the system). On an exam, P2's strategy for dealing with time-consuming problems is to postpone them: "if it takes a long time on the test, I usually skip it and keep it for the [end] if I have time."

When prompted to think of other techniques, some students suggested techniques and exclaimed they'd likely be faster, but they clearly did not actually consider the mathematics involved when saying this. For example, when P6 thought of τ_4 (after engaging with τ_2), he said that "the math is probably faster to invert it." It isn't. After engaging with τ_2, P9 also decided τ_4 would be less time-consuming than τ_2: "it [τ_4] is way easier and [τ_2] [...] [is] very long and [...] [can have] many calculation mistakes along the way." After P8 found that $x = y = z = 0$ using Cramer's rule (τ_3), I asked if she could think of any other approach. She said that if it weren't for the affirmation that A is invertible, she'd have used her "usual" technique, τ_2, and said that "maybe that would actually be faster [than Cramer's rule, for Problem 2]. You know, probably." It isn't. P9, who had engaged with τ_2 and temporarily claimed τ_4 would be faster than τ_2, eventually claimed that "Cramer's rule is the easiest [approach]," it's "the fastest method," "Cramer's rule is faster and easier [than τ_2]. So you just row reduce and then you have the matrix." But P9 also said that the similarity between b and the first column of A can't be used, so given that Cramer's rule would normally involve the calculation of 5 determinants, P9's claim that "Cramer's rule is faster and easier [than τ_2]" seems superficial: it is not informed by the mathematics involved in Cramer's rule, in general, nor by the mathematics that could be involved in Problem 2 specifically.

I found only two instances in which participants accurately took into account the math involved in a technique to make an inference about the time needed to enact it. P1 said that on an exam, he would use τ_2 and not τ_4, because τ_4 involves another step (multiplication). Asked what he'd do differently or similarly on an exam, P4 said he wasn't sure he'd bother swapping rows to get leading 1's into their 'ideal' spots "because of the time," and that he'd "try to optimize how much [he] can combine operations," "[he tries] to combine as much row operations as possible in one step [...] try to show the least amount of matrices [he] could show but still in a.. a coherent way, so [instructors] can understand."

When P4 noticed, well into his activity in Problem 2, that the first column of A is the negative of b, he exclaimed that "[he needs] to pay more attention before starting problems." This brings to mind his earlier desire to behave optimally, which he expressed while discussing the relation $\det(A) \Leftrightarrow \exists A^{-1}$: "I should have maybe thought that there is more optimal way, because that's actually very important throughout the exams to see if there's any quicker way to solve the problem." P4's comments show he wants to behave optimally, but his activity in Problem 2 shows he is not equipped to do so. When he activated τ_2, he knew to avoid operations that would produce fractions, but did not otherwise take the entries in the given matrix into account to inform his choice of row operations. When asked what he expected to find at the end of the row-reduction process, he described the three potential options for a linear system (1 solution, no solutions, infinitely many solutions), and was not able to use the fact that A^{-1} is invertible to predict there would be one solution; he wondered if there was a rule that said that if A is invertible, then $A\text{x} = \text{b}$ has a solution. When he noticed that the first column of A was equal to -b, and thought of Cramer's rule, he said he wouldn't use Cramer's rule because finding the determinant of a 4×4 matrix is difficult. P4's normative knowledge got in the way of his desire for efficiency.

P4 seemed aware that he does not always behave optimally. In addition to his claim that "[he needs] to pay more attention before starting problems," his perception of instruction in LA1 suggests he feels he is missing strategies for using LA1 techniques efficiently:

> I would really like if the instruction gave us a strategy [he claimed he came up with the strategy of getting 0's and 1's on his own, from experience solving problems at home], because especially for the introduction courses, like [Calculus 1], [LA1], [Calculus 2], I think the problems are, I wouldn't, some are actually challenging but at least something like this. I think it's more primitive. So I think if I have maybe, manual, so oh okay, so, what I'm - because I did this on my own, so I want this to be zero, and then this to be zero, this to be zero, and then work.

He also seemed to feel he has a surface-level grasp of the tasks in LA1, and that this impinges on his capacity to tackle them efficiently: "I would like to, if the instruction has maybe more information about the, how I should think about the problem or the nature of the problem for us to see it as a problem, not as a bunch of numbers, I think that would greatly enhance my... problem approaching skills."

P4 was not the only participant who seemed aware he might be lacking knowledge and therefore behaving inefficiently. When P8 started to calculate $\det(A)$ (to use Cramer's rule), she considered using row-reduction and decided against it. She explained later why she stuck to cofactor expansion throughout her (9.5-minute) calculation (which had led to an incorrect answer):

> I know [cofactor expansion] is lengthier... I know there's a lot of numbers in algebra, but like reducing should be faster, but you have to be careful to not make any mistake here. And again, you should make sure that you add everything according to the row operations that you made. And again, sometimes you can forget that. Or just [you make a] mistake, and... I have a lot of these with algebra. So like, at least, like, if I will write everything here. I have fewer chances. But yeah, it's probably, it probably takes more time.

P8 had prioritized LA1 techniques that could be grasped superficially, even if she knew that other techniques could allow her to do calculations more efficiently. In the LA1 exam task in which students are asked to find the determinant of a matrix, the matrix usually has size 4×4, its entries are exclusively (smaller) single-digit integers, and it has a column or row with (at least one but usually) two entries that are 0. It's not *necessary*, in LA1, to develop efficient determinant-calculating techniques.

Students' descriptions of what they would do differently or similarly if they were doing Problem 2 at home, as practice, give a glimpse into practices that privilege surface-level features of LA1 problems. P1 said he would do both τ_2 and τ_4 for practice; but the bulk of the two techniques involves the exact same process of row-reducing A. P4 said he'd "probably go on a matrix calculator, make sure that he's doing it correctly." (No word about attempting to do anything efficiently or optimally, which he said he'd try to do on exams.) P5 said she'd do the same she had done in the interview (activate τ_2).

Participants' normative techniques for tasks of type T (to solve a linear system) involve using row operations to manipulate a "bunch of numbers" (P4) into a form that involves many 1's and 0's. Their practice is centered on a concern with getting to that final form quickly and without making calculation errors, but they do not take into account the specifics of the "bunch of numbers" they're given to guide their calculations; they automatically engage in the techniques that LA1 tasks had trained them to activate in response to certain triggers, and as they engage in their techniques, they either activate the given algorithms religiously (multiply R_1 trough by $\frac{1}{A_{11}}$ to get a leading 1 in entry A_{11}) or use surface-level features (such as existing 1's) to avoid producing fractions in their calculations.

Students were dependent on their normative techniques and these techniques did not include taking the features of a problem into account to guide their choice or implementation of technique. This led students to perform operations that may have been lengthier than what they were used to (as P1 put it: this was "harder than [the] exam"). The way the problem was designed, meanwhile, meant the given system could have been solved by inspection. But this is not a norm in LA1, and students struggled to get out of their norm. They are not used to *looking* at the "bunch of numbers" in the matrices they are given; their job is to algorithmically manipulate this bunch of numbers into submission. In the most extreme cases, some students' mistakes (P2 and P7* with their incorrect augmented matrices, P6's initial use of a coefficient matrix while using τ_2 and later question as to whether τ_2 would actually produce a solution, given that the reduction looked messy, and P5's interpretation of a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 5 & 0 \end{array}\right]$ as the equation $5 = 0$) suggest that, even as students know, on paper, that row operations and augmented matrices correspond to linear systems and the algebra that's used to solve them, students' normative engagement with linear systems and augmented matrices is superficial enough that it *suffices* to know augmented matrices as "a bunch of numbers"—a bunch of numbers divorced from their connection to the linear system they're meant to capture.

The purpose of row-reduction is to manipulate the matrix in a way that allows you to find a solution by inspection or a substitution that allows you to easily find the solutions of the system. It doesn't seem students knew this is what they're doing. They applied an algorithm; but a human who views the objective of row reduction as to simplify the situation may not automatically do row-reduction. Students are given tasks that have them use row-reduction to demonstrate their capacity to use it. Inspection or easy substitution are not techniques they are expected to use. If students were trained to view row reduction as a technique whose goal is to simplify, they might be more inclined to begin with an inspection of the system they are given. I see two possibilities: one, that students are unaware of the simplification objective of row-reduction; or they are aware, but they have a normative approach because they are Studenting and they know row reduction is what they are expected to do.

5.3 LA1 Problem 3

The following was the third problem presented to the 10 LA1 students in the TBI:

Show that $(w_1, w_2, w_3) = (29, -9, 3.2) \times (11, 2.1397, 41)$ is a solution of the following system.

$$\begin{array}{rcrcrcl} 29x & - & 9y & + & 3.2z & = & 0 \\ 11x & + & 2.1397y & + & 41z & = & 0 \end{array}$$

5.3.1 Reference model for LA1 Problem 3

Problem 3 is a task of the type "show that a given element is a solution of a given linear system," but I will focus this reference model on a more narrow task type T to which it belongs: "show that a given element is a solution of a homogeneous linear system of two equations in $\mathbb{R}^3$." I had actually designed Problem 3 with a yet more specific task t in mind: "show the cross product of normals of two planes (that contain the origin) is in the intersection of the two planes."

Planes in $\mathbb{R}^3$ are defined by equations of the form $ax + by + cz = d$. This definition is produced by the following reasoning in $\mathbb{R}^3$: a plane can be uniquely identified by a point it contains and a vector orthogonal to that plane. This reasoning can be supported by a visual aid (which is the reasoning given in LA1 in the knowledge to be taught). That this conception of planes is well-defined is confirmed algebraically using the dot-product definition of orthogonality (two vectors are said to be orthogonal if their dot product is zero; this, in turn, has its roots in the definition that two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are perpendicular if the smaller angle between them is $\frac{\pi}{2}$). If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two planes containing a point (x_0, y_0, z_0) and orthogonal to the vector (a, b, c), then $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide. Indeed, if $(x, y, z) \in \mathcal{P}_1$, then the vector with terminal point (x, y, z) and initial point (x_0, y_0, z_0) is parallel to $\mathcal{P}_1$ so by definition of vector addition and the negative of a vector, $(x - x_0, y - y_0, z - z_0)$ is parallel to $\mathcal{P}_1$ and therefore orthogonal to (a, b, c):

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0. \tag{5.1}$$

Similarly, if $(x, y, z) \in \mathcal{P}_2$ then equation (5.1) holds as well; any point in $\mathcal{P}_1$ is therefore in $\mathcal{P}_2$ and vice-versa, so the equation corresponds to a unique plane. Applying the definition of dot product to equation (5.1) produces what's called the "point-normal equation of a plane":

$$\begin{array}{rrcl} & a(x - x_0) + b(y - y_0) + c(z - z_0) & = & 0 \\ \Leftrightarrow & ax + by + cz - (ax_0 + by_0 + cz_0) & = & 0 \\ \Leftrightarrow & ax + by + cz & = & d, \quad \text{where } d = ax_0 + by_0 + cz_0. \end{array}$$

Any plane given by an equation of such a form has (a, b, c) as a normal (a vector orthogonal to the plane), and d is the dot product of this normal with a point on the plane. The conceptualization of a plane in terms of a normal, then, is key to the equation definition of a plane.

A dot-product representation helps to highlight the orthogonality between points on a plane through the origin and a normal of that plane:

$$ax + by + cz = 0 \Leftrightarrow (a, b, c) \cdot (x, y, z) = 0.$$

Any solution of a *system* of homogeneous linear systems is a point that belongs to each plane in the system; the dot-product representation highlights the property that (x, y, z) is a solution if and only if it is orthogonal to the normals of all the planes. In the context of a homogeneous linear system of two equations (as in Problem 3), the concept of cross product is particularly useful: given the system

$$\begin{array}{rcccl} \mathrm{n}_1 & \cdot & (x,y,z) & = & 0 \\ \mathrm{n}_2 & \cdot & (x,y,z) & = & 0 \end{array},$$

the cross product of the normals n_1 and n_2 is orthogonal to n_1, which is orthogonal to $\mathcal{P}_1$, so $\mathrm{n}_1 \times \mathrm{n}_2$ is parallel to $\mathcal{P}_1$; this plane goes through the origin and $\mathrm{n}_1 \times \mathrm{n}_2$ (the vector from the origin to the point $\mathrm{n}_1 \times \mathrm{n}_2$) is parallel to it, so $\mathrm{n}_1 \times \mathrm{n}_2$ is a point on $\mathcal{P}_1$. Similarly, $\mathrm{n}_1 \times \mathrm{n}_2$ is a point on $\mathcal{P}_2$. So the cross product is a solution of the system. (The algebraic reasoning for this is more direct, since two vectors are defined to be orthogonal if their dot product is 0.) Conversely, any solution P of the homogeneous linear system would be a point on $\mathcal{P}_1$ and a point on $\mathcal{P}_2$; since the vector P is parallel to both planes, it is orthogonal to both of their normals, so it is parallel to $\mathrm{n}_1 \times \mathrm{n}_2$. Hence, $\mathrm{n}_1 \times \mathrm{n}_2$ generates the solution space of the system.

This discourse is what produced my design of Problem 3. My reference model for the task of type t (show the cross product of normals of two planes (that contain the origin) is in the intersection of the two planes) consists of the technologies I used in the design of this problem (the technologies that produced the choice to have the task focus on the cross product of the normals of planes that go through the origin). I will refer to the collective of these technologies (the above discourse) by θ. The theory Θ at the backdrop of these technologies is the algebraic, geometric, and logical discourse (axioms, definitions, properties, theorems, proofs) that gives authority to those technologies and the view that the axioms that underpin linear algebra and Euclidean geometry are founded in the physical reality humans inhabit.

We'll now outline techniques for performing task t: to show the cross product of normals of two planes through the origin is in the intersection of the two planes.

The system in Problem 3, in terms of dot products, can be represented as follows:

$$\begin{array}{rcccl} (29,-9,3.2) & \cdot & (x,y,z) & = & 0 \\ (11,2.1397,41) & \cdot & (x,y,z) & = & 0 \end{array}$$

Since $(29,-9,3.2) \times (11,2.1397,41)$ is orthogonal to both $(29,-9,3.2)$ and $(11,2.1397,41)$, it is a solution of the given system. I will refer to this approach by τ_1: it is one technique through which to perform the task t.

Another technique, τ_2, leans on the geometric discourse from θ: since the first equation corresponds to the plane through the origin with normal (29,-9,3.2), and since the vector $\mathrm{w} = (29,-9,3.2) \times (11,2.1397,41)$ is orthogonal to that normal, this vector is parallel to the plane; the plane goes through the origin, so the endpoint of w is a point on the plane and therefore solves its equation. Similarly, w is also a solution to the second equation. So it solves the system.

Another technique is τ_3: to show that $(w_1, w_2, w_3) = (29, -9, 3.2) \times (11, 2.1397, 41)$ is a solution of the given system, find the values of w_i $(i = 1, 2, 3)$ by calculating the cross product and plug them into each equation to verify if the system is satisfied.

Techniques τ_1 and τ_2 are produced and justified by the theoretical block $[\theta, \Theta]$. The theoretical block $[\theta_3, \Theta_3]$ needed to activate τ_3 is a definition of cross product, the notion of what it means for an element to be a solution to an equation, and knowing how to multiply and add real numbers. My reference model, then, consists of the following praxeologies:

- $[t; \tau_1; \theta; \Theta]$;
- $[t; \tau_2; \theta; \Theta]$; and
- $[t; \tau_3; \theta_3; \Theta_3]$.

5.3.2 Knowledge to be learned in LA1 that relates to Problem 3

In LA1, tasks of type "show that a given element is a solution of a given linear system" do not appear on midterm and final exams nor on the problems recommended to students as practice on the course outline. It's possible students perform this task to verify their solution when they solve linear systems and find one solution. It follows that the subtype T (show a given element solves a homogeneous linear system of two equations in $\mathbb{R}^3$) and the task t (show the cross product of normals of two planes through the origin is in the intersection of two planes) do not show up on any midterm and final exam tasks either.

For my model of knowledge to be learned, I'll focus instead on tasks from LA1 that involve the mathematical constructs present (explicitly or implicitly) in Problem 3: homogeneous linear systems, cross products, dot products, and point-normal equations of planes. In LA1, as I will show below, tasks usually target one such construct at a time (contrarily to Problem 3, which combines them into a single task). To help trace the knowledge students activated in the TBI (examined in Section 5.3.3) to knowledge students are expected to know about these usually-disjointed constructs, I will use ATD praxeological notation to a higher degree of specificity than I had in the models of knowledge to be learned that is relevant for some of the other TBI problems (e.g., Problems 1 and 2).

I will start with t_4[16], the only LA1 task about homogeneous linear systems in past midterm and final exams and in the problems recommended as practice problems on the course outlines: to find the basis of the solution space of a homogeneous linear system. I found this task on 6 past exams; there, the task always included a system of 3 equations in 4 to 7 unknowns. In 4 of the 6 exam problems, the coefficient matrix was in RREF and in another one of the exam problems, it was in REF. The entries in the coefficient matrices on all exam problems were integers between -2 and 8 (and mostly closer to 1).

[16] t_1, t_2, and t_3 are not defined here nor in the reference model for Problem 3, but I index this task as t_4 to distinguish its related technique (τ_4) from the techniques denoted by τ_1, τ_2, and τ_3 in the reference model for this problem.

To perform t_4 in LA1, the technique τ_4 is to row-reduce the augmented matrix (this was only actually needed in one of the past exams, as the others gave coefficient matrices that were already in RREF or in a REF), find the general solution in terms of parameters, possibly express the parametric solutions in vector form (though instructors may not require this), and infer the basis from (the vector form of) the parametric solutions (e.g., if the general solution is found to be $x = t, y = 4t, z = 5t$, then the solutions are vectors of the form $(t, 4t, 5t) = (1, 4, 5)t$ so $\{(1, 4, 5)\}$ is a basis for the solution space). No theoretical discourse is needed.

There were two types of tasks related to cross products on past final exams: t_5, to find a vector orthogonal to the plane of a triangle in $\mathbb{R}^3$, given the vertices A, B, C of the triangle (this appeared in 2 exams); and t_6, to find the area of a triangle, given the vertices of the triangle (this appeared in 3 exams, one of which also had t_5). In both cases, the vertices were integers between -2 and 5.

To perform t_5, the technique τ_5 is first to find two vectors parallel to the plane (e.g., $\overrightarrow{AB}$ and $\overrightarrow{AC}$) and then compute their cross product. The technology $\theta_5 = [\theta_{51}, \theta_{52}, \theta_{53}]$[17] includes the formula for a vector given its initial and terminal points (θ_{51}), the definition of cross product or any mnemonic device that produces an accurate cross product (θ_{52}), and the property that the cross product of two vectors is orthogonal to both vectors (θ_{53}). The knowledge to be taught in LA1 includes a discourse that combines geometric and component definitions of vector, vector addition, and additive inverses to produce the formula $\overrightarrow{AB} = (x_b - x_a, y_b - y_a, z_b - z_a)$, but students need only know this formula and how to add and subtract vectors in component form. The knowledge to be taught in LA1 includes a component-based proof that shows (using dot products) $\mathrm{u} \times \mathrm{v}$ is orthogonal to u and v; this is not part of the knowledge students need to learn.

To perform t_6, the technique τ_6 is to use the formula for the area $\mathcal{A}$ of a parallelogram with edges AB and AC:

$$\mathcal{A} = \|\overrightarrow{AB} \times \overrightarrow{AC}\|.$$

To activate this technique, students need to know θ_{51} (the formula for the vector given its initial and terminal points), θ_{52} (the definition of cross product or any mnemonic device that produces an accurate cross product), and θ_6, the above formula for the area of a parallelogram and that the area of the triangle formed by vertices A, B, and C is half of $\mathcal{A}$. Students do not need to know the discourse that produces the formula for the area of a parallelogram, though it is included in the knowledge to be taught.

Only one LA1 exam included tasks that involved dot products, and this is the same exam that included both types of cross product tasks (t_5 and t_6). One task, t_7, was to "show that if u is orthogonal to v and w then u is orthogonal to $a\mathrm{v}+b\mathrm{w}$, where a and b are numbers." To perform this task, students need to know $\theta_7 = [\theta_{71}, \theta_{72}]$: properties of dot product, addition, and scalar multiplication of vectors, which mimic properties of multiplication and addition of real numbers (θ_{71}), and the definition that two vectors are

[17] For the model of knowledge to be learned that relates to geometry, I find it may be useful to use praxeological notation to a higher-level of specificity. This a priori reflection may help bring attention to the nature of the properties of geometric technologies that are retained in the praxeologies students mobilize.

orthogonal if their dot product is 0 (θ_{72}). The second task, t_8, was to find (u+v)·(2u-v) given only the numerical values of u·u, u·v, and v·v. To perform this task, students needed to know θ_{71}.

Only three LA1 exams included tasks that (potentially) involve point-normal equations.

One exam had the task t_9, to find the point-normal form of a plane that contains a given point $P = (x_0, y_0, z_0)$ and has a given normal $n = (a, b, c)$ (a second part of this task requires students to write the equation of the plane in the form "$ax + by + cz + d = 0$"). The technology needed to perform t_9 is θ_9, the formula for the point-normal equation of a plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

One exam had the task t_{10}, to find the equation of a plane $\mathcal{P}_1$ containing a given point P and parallel to a plane $\mathcal{P}_2$ $ax + by + cz = d$. The technique τ_{10} is to use the normal n of the plane $\mathcal{P}_1$ and the point P it contains to produce a point-normal equation for $\mathcal{P}_1$. To activate this techique, the technology is $\theta_{10} = [\theta_9, \theta_{10-1}]$, where θ_9 is the formula for the point-normal equation of a plane. θ_{10-1} is the knowledge that parallel planes have parallel normals.

Another exam had two tasks that involved planes in point-normal equations, but the techniques needed to accomplish these tasks did not require any knowledge about *point-normal equations*. One task included finding a line $\mathcal{L}$ of intersection of two planes (both given in the form $ax+by+cz = d$), using $\mathcal{L}$ to produce an equation for a plane containing $\mathcal{L}$ and a given point P; the other task instructed students to find the coordinates of the intersection of $\mathcal{L}$ and a third plane (also given in the form $ax+by+cz = d$). I won't detail all the techniques and technologies needed to perform this task in their entirety, as they are not all relevant for Problem 3. But one major technology (θ_{11}), that is needed for these tasks (which I will denote by t_{11}) is relevant to Problem 3: it is the knowledge that solutions of systems of linear equations in $\mathbb{R}^3$ correspond to intersections of planes; and that a line in $\mathbb{R}^3$ corresponds to the intersection of two planes, and therefore corresponds, algebraically, to the solution set of a system of two linear equations.

I summarize, in Table 5.6, the knowledge students need to learn in LA1 to engage with the mathematical constructs that appear (implicitly or explicitly) in Problem 3.

In light of the knowledge students need to learn about the mathematical constructs in Problem 3, I wondered whether the homogeneous linear system in the problem would prompt students to spontaneously engage in their usual linear-system-solving techniques (row-reducing the augmented matrix), that is, whether students would treat Problem 3 as a task of type t_4; or, perhaps, whether students would be triggered to calculate the cross product as tasks involving cross products in LA1 always require students to calculate them; or whether the cross product representation and the fact that this is a *homogeneous* linear system (where the constant to the right of the equations is 0) might be sufficient to trigger, in students, the relation between dot and cross products that is in some of the knowledge they need to learn (θ_{53}, θ_{72}).

Table 5.6: Model of the knowledge students need to learn in LA1 to engage with mathematical constructs in Problem 3

	Task	Count		Technique	Theoretical discourse
t_4	to find the basis of the solution space of a homogeneous linear system	6	τ_4	row-reduce the augmented matrix find the general solution in terms of parameters possibly express the parametric solutions in vector form infer the basis from (the vector form of) the parametric solutions	none
t_5	to find a vector orthogonal to the plane of a triangle in $\mathbb{R}^3$, given the vertices of the triangle	2	τ_5	use triangle vertices to find two vectors parallel to the plane and compute their cross product	$\theta_5 = [\theta_{51}, \theta_{52}, \theta_{53}]$ formulas (θ_{51} for vector between two points, θ_{52} for cross product) and θ_{53} property that u × v is orthogonal to both u and v
t_6	to find the area of a triangle $\mathbb{R}^3$, given the vertices of the triangle	3	τ_6	activate $[\theta_{51}, \theta_{52}, \theta_6]$ to find the area of the parallelogram formed by two edges of the triangle, divide by 2	$[\theta_{51}, \theta_{52}, \theta_6]$ where θ_6 is the formula $\mathcal{A} = \|\overrightarrow{AB} \times \overrightarrow{AC}\|$
t_7	show that if u is orthogonal to v and w then u is orthogonal to av+bw, where a and b are numbers	1		activate θ_7	$\theta_7 = [\theta_{71}, \theta_{72}]$ properties of dot product, addition, and scalar multiplication of vectors (θ_{71}); two vectors are orthogonal if their dot product is 0 (θ_{72})
t_8	to find (u+v)·(2u-v) given only the numerical values of u·u, u·v, and v·v	1		activate θ_{71}	θ_{71}
t_9	to find the point-normal form of a plane that contains a given point $P = (x_0, y_0, z_0)$ and has a given normal $n = (a, b, c)$	1		activate θ_9	θ_9 formula for the point-normal equation of a plane ($a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$)
t_{10}	to find the equation of a plane $\mathcal{P}_1$ containing a given point P and parallel to a plane $\mathcal{P}_2 : ax + by + cz = d$	1	τ_{10}	to use the normal n of the plane $\mathcal{P}_1$ and the point P it contains to produce a point-normal equation for $\mathcal{P}_1$	$\theta_{10} = [\theta_9, \theta_{10-1}]$, where θ_{10-1}: parallel planes have parallel normals
t_{11}	to find intersections of loci in $\mathbb{R}^3$ and use these to find equations of other loci	1			θ_{11} solutions of systems of linear equations in $\mathbb{R}^3$ correspond to intersections of planes; a line in $\mathbb{R}^3$ corresponds to the intersection of two planes, and therefore corresponds, algebraically, to the solution set of a system of two linear equations

5.3.3 Knowledge LA1 students activated in response to Problem 3

Table 5.7 summarizes the paths of participants' activity as they worked on Problem 3. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step. If a participant does not appear in the column for Step i ($i \geq 2$), it is because they did not engage in any new activity after Step $i-1$.

Table 5.7 reveals that most participants' activity landed in one of three categories: activating techniques and technologies normative to LA1 (calculating cross products using θ_{52} and/or row-reducing augmented matrices of homogeneous linear systems to find their parametric solutions, as in τ_4), being stuck trying to recall LA1 norms related to cross products (what teachers had said, past homework, practice exams), or being stuck trying to use or recall geometric properties of cross products or linear systems and their solutions (as planes and lines). Of the seven students who completed the task, four did so using θ_{52} and/or τ_4. Two (P2, P4) completed the task by combining these two: for them, the task was to check if the cross product was *one of* the (parametric) solutions of the system. I will denote this approach by $[\tau_4, \theta_{52}]$ in reference to these elements of knowledge to be learned in LA1; see Table 5.6. P2 and P4 found the components of the cross product, the parametric solutions of the system, and found a value of the parameter that generated (approximately) the cross product they had found (e.g., if $z = t$, then they looked at the z-component of the cross product to identify the value of t that would generate the cross product). Two other students (P5, P6) completed the task by calculating the cross product, plugging it into the left side of the equations, and finding this produces (approximately) 0; this is technique τ_3 from the reference model. Three students (P1, P7*, P9) completed the task by operating in a fourth category—to avoid operating θ_{52}: they activated τ_1 (from the reference model for Problem 3), which includes the definitions of dot product and orthogonality and the property that $\mathrm{u} \times \mathrm{v}$ is orthogonal to both u and v.

The three students who did not complete the task are P10, P8, and P3. P10 and P8 had also suggested $[\tau_4, \theta_{52}]$ but did not activate it and did not complete the problem, though they got stuck for different reasons. P10 had initially calculated the cross product and was confused because the cross product has three components while the system has two equations; I asked if she could think of another approach, and she suggested $[\tau_4, \theta_{52}]$, but was not sure and tried to think of a different approach. Eventually, I asked how she "normally check[s] if something is a solution of a system," and P10 responded: "not sure, maybe [pause] grades do it? Like I don't know. Not sure." P8 was also unsure of her suggestion to use $[\tau_4, \theta_{52}]$ (and had chosen not to actually calculate the cross product, after having written an expression for w_1, w_2, w_3); but P8's hesitation was because she was trying to recall, instead, theoretical knowledge about cross products. P3, finally, had calculated the cross product but did not see the point in doing this. Most of P3's activity in relation to Problem 3 was in trying to recall LA1 norms related to cross products and geometric representations of cross products and linear systems; she correctly described the system of two equations as corresponding to planes intersecting along a line, but also

Table 5.7: Paths of LA1 Students' Activity in Problem 3

Practical block $[t, \tau]$		Step 1 Participant	Step 1 Type of engagement with $[t, \tau]$	Step 2 Participant	Step 2 Type of engagement with $[t, \tau]$	Step 3 Participant	Step 3 Type of engagement with $[t, \tau]$	Step 4 Participant	Step 4 Type of engagement with $[t, \tau]$	Step 5 Participant	Step 5 Type of engagement with $[t, \tau]$
find cross product (CP)	no goal identified	P5	start to enact (incorrect vectors include unknowns)	P5	enact (finds CP) (I had prompted P5 to notice the vectors in her CP were incorrect)	P3	enact and dismiss (knowing the values doesn't show the CP solves the system)				
		P10	enact (finds CP)								
		P1	start to enact (writes expression for CP)								
		P8	start to enact (writes expression for CP)								
	$[\tau_4, \theta_{52}]$: to check if the CP is equal to one of the solutions in the system's parametric solutions (PS)	P2	enact (finds CP, finds PS, finds a parameter value that generates a solution approx. equal to the CP)	P8	suggest and start to enact, hesitates	P4	enact and get stuck (finds PS, generating vector does not equal CP)			P4	enact (observes CP is multiple of vector generating the general solutions); also gives planes/line interpretation of system (prompted for alternative approach, says this is geometric representation, approach would stay same)
				P10	suggest and get stuck						
		P4	start to enact (writes augmented matrix)	P4	start to enact (finds CP)						
	τ_3: to plug CP entries into equations to check it's a solution	P6	enact (finds CP; plugs into equation, does not get 0; finds error; concludes approach is fine)			P5	enact (CP doesn't give exactly 0 when plugged in, P5 assumes rounding error); completes task				
		P7*	describe								
		P9	start to enact (writes expression for CP, calculations for w_1)								
τ_1: definitions of dot product and orthogonality, and property that u × v is orthogonal to u and v				P1	enact						
				P7*	enact/describe						
				P9	enact/describe						
stuck	does not know what the task is	P3	tries to recall relation between CP and system; thinks system is missing an equation; rewrites first equation as a matrix equation $A\mathbf{x} = 0$								
	try to recall LA1 norms related to CPs			P3	I reworded the problem, P3: "I forgot how to handle this." Vague recollections from LA1 ("teacher" said...)			P5	I asked if P5 thought of an approach that wouldn't involve calculating the CP. Recalls from "past homework," "doing problems in the past": CP, 0's, vectors and coefficients trigger memory of relation between CP, dot product, orthogonality, and 0, but cannot activate anything	P5	I asked if P5 can think of geometric relation between CP, vector $(29, -9, 3.2)$, and vector $(11, 2.1397, 41)$; mentions right-hand rule, CP has something to do with height of vector (from right-hand rule), may relate to hypotenuse (dismisses this)
	try to use/recall geometric property of CP or linear equations (planes/lines)					P6	P6 had brought up τ_1 (from the reference model for Problem 3); I asked if he could use this to do the problem, but P6 cannot activate it. Does not know why "this being orthogonal [...] means these are equal to zero"	P4	the calculated CP is not equal to the vector found to generate the general solution; tries to recall what CP represents, recalls its norm is area of parallelogram made up of vectors, dismisses (not useful here)	P3	I asked if P3 can think of geometric relation between the CP, the vector $(29, -9, 3.2)$, and the vector $(11, 2.1397, 41)$; confusing interpretation of system and CP in terms of planes and intersections
						P8	I asked why P8 hesitates: "there probably was some geometric explanation or something like that, not geometric but... some proofs about the CP that I didn't read [...] I remember there were some parts that I skipped. In the book"	P3	represents the system as planes that intersect along a line, says the CP is another plane (confusing descriptions)		
plug components of $(11, 2.1397, 41)$ into first equation, components of other vector as well, doubts utility				P6	I asked if P6 can think of a way to do the problem without calculating the CP; P6 enacts and dismisses (does not see why it would "prove"/"mean anything")						

said the cross product was another plane intersecting with these two planes.

Problem 3 did not directly correspond to any normative task in LA1. At its surface, it most closely resembles the normative LA1 task t_4, wherein students are to find (a basis of) the solution space of a homogeneous linear system; the technique for that task is τ_4, which was the technique activated by P2, P4 and suggested by P8, P10, but with a different purpose in mind: their goal was to confirm that the cross product (once its components are found) is equal to one of the parametric solutions of the system. There is no need to produce all the solutions of the system to do Problem 3; τ_4 is superfluous to the task. But P2, P4, P8, and P10 spontaneously activated τ_4 and were unable to activate any other technique when asked if they could think of another approach—even when the question was whether they could think of an approach that wouldn't involve calculating the cross product or an approach that wouldn't involve finding *all* the solutions of the system. This shows their practice was driven by LA1 norms, and that these norms had participants behave counter to the mathematics at stake.

Apart from P2, P4, P8 and P10's roundabout application of LA1 norms, all ten participants' spontaneous activity also reflects an LA1 norm: every participant, as their intial step toward performing the task, wrote out the expressions needed to calculate w_1, w_2, w_3. For example, w_1 would be

$$\begin{vmatrix} -9 & 3.2 \\ 2.1397 & 41 \end{vmatrix}.$$

I discuss students' intentions in this first step in Section 5.3.3.1.

Nevertheless, features of the problem, which differ overtly from what students are used to in LA1, seem to have propelled students to operate or *try* to operate outside of their norms from LA1. In light of the activity and comments that were triggered in participants in response to Problem 3, I surmise that these features are responsible for students' attempts to operate outside of their norm and act more efficiently throughout their engagement with Problem 3. As we'll discuss in Sections 5.3.3.2 and 5.3.3.3, much of students' spontaneous activity and decisions was in reaction to hesitation regarding features of the problem. These features include the non-integer values of the scalars, the presence of a vector given in the form of a cross product (and not in terms of its components, in a task of type "check this is a solution of the system," where cross products do not usually appear in LA1), and the fact that the entries of vectors $\mathrm{u} = (29, -9, 3.2)$ and $\mathrm{v} = (11, 2.1397, 41)$ are the coefficients in the first and second equation, respectively.

Even though students' comments and activity were reactions to overtly abnormal features of the problem, only three students (P1, P7*, P9) were able to mobilize knowledge that allowed them to *not* depend on the technical LA1 norms of calculating cross products when they appear and of solving linear systems by row-reducing their augmented matrix; and these three students behaved in this abnormal way only in reaction to the abnormal features of the problem. Additionally, only four students (P5, P6, and again P7* and P9) proposed a technique (other than τ_1) appropriate to the given task: to check the given element is a solution by plugging its components into the equations (τ_3). This means that five students (P2, P3, P4, P8, and P10) were unable to mobilize LA1 knowledge to perform their task in a way that reflects only *the task* (and not their norms from LA1).

In Section 5.3.3.2, I examine students' spontaneous and non-spontaneous activity after their initial decision to write out the expressions for the components w_1, w_2, w_3 of the cross product; this activity, together with students' theoretical blocks (that is, the justifications, beliefs, reasonings that drove the techniques they activated to perform the task), suggests that the norms students developed in LA1 impinged on their ability to mobilize knowledge from the course (knowledge to be taught in LA1) to perform the task more effectively. I refer here to two norms in particular: the norm that row-reducing augmented matrices is a cure-all—it *always works*; and the norm that students need only operate at the surface-level when performing tasks, without knowing the mathematics that frames their techniques.

5.3.3.1 Students' initial spontaneous reaction was to compute the cross product

7 participants (P1, P2, P3, P5, P8, P9, P10) needed confirmation as to what the symbol $\times$ represented, and all participants spontaneously started to compute the cross product when they knew that's what it was. To compute the cross product, 6 participants (P1, P2, P6, P8, P9, P10) activated the following:

$$\mathrm{u} \times \mathrm{v} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

and made a cofactor expansion along the first row; 3 participants used the form of cross product given in its definition (I had given P3 a copy of the definition as she was stuck), and 1 participant did not actually get to computing it (P7*), but computing the cross product was part of his first suggested approach.

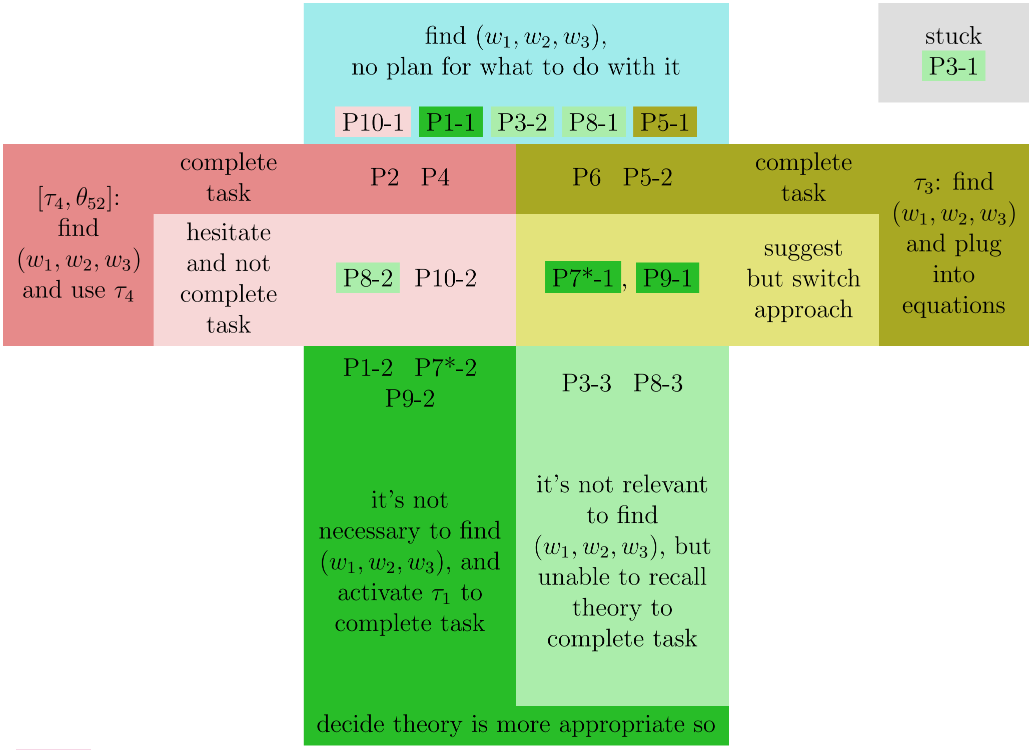
Table 5.8 summarizes participants' justifications in their choices to and not to find the components (w_1, w_2, w_3) of the cross product. The rest of this section outlines participants' activity and explanations in more detail; I invite the reader to consult the summary table first.

P3 only set to find the components of the cross product after I gave her the definition of cross product to help her get un-stuck, as she was unable to suggest any approach upon reading the problem and even after I reworded the question: she had initially tried to recall knowledge that would relate the cross product with the system, then thought the system was missing an equation (specifically, $0x + 0y + 0z = 0$), and then rewrote the first equation in the form of a matrix equation ($A\mathrm{x} = 0$). After I reworded the problem for P3, she said "she forgot how to handle this" and tried to recall what the "teacher" had said: "I remember like during the class. The teacher was like, keep talking about like, after this calculation it should not be 0 or [pause] oh, it's about independency." P3 was stuck so I gave her the definition of cross product; she said that "[she's] trying to remember... The way that [she] solved this kind of question before" and then calculated the components of the cross product. But P3 did not activate this to complete the task. She did not see the value in knowing the components:

P3: what I've done is just finding the values of w 1 2 3. I don't really think that that can be an exact

I: Why is it relevant to have the values?

Table 5.8: Summary of participants' justifications for and against finding (w_1, w_2, w_3)



P*n*-*m* : *m* indicates the pathway in P*n*'s reasoning relative to finding (w_1, w_2, w_3);
colorbox indicates P*n*'s final reasoning.

P3: Yeah, because like finding value [sic] is not really [...] they're not asking me to find the values, they're just asking, explain why this is the answer of this.

The rest of P3's activity relative to Problem 3 was to recall geometric properties of linear systems: each equation is a plane, the intersection of the planes is a line, which corresponds to the solution of the system, and the cross product is a plane that also intersects with the first two planes. She reiterated that knowing the components of the cross product wouldn't help because "calculation is just a calculation. It's not the theory."

Two participants (P2 and P4) found the cross product with the express goal of checking that it's equal to one of the solutions in the general solution of the system. To clarify: P2 and P4 calculated the cross product, found the parametric solutions of the system, and identified the parameter that would generate the cross product. This is approach $[\tau_4, \theta_{52}]$. P4 had briefly tried to activate theory about cross products when he thought his application of $[\tau_4, \theta_{52}]$ was faulty; but P4 was not able to activate relevant theory (he recalled that the norm of a cross product is the area of the parallelogram formed by the vectors) and, in response to prompts from the interviewer, concluded his application of $[\tau_4, \theta_{52}]$ was appropriate. When P2 and P4 concluded their approach, satisfied with their application of $[\tau_4, \theta_{52}]$, I asked if they could think of a way to approach the problem without finding all the solutions to the system. They could not. (Nor could P2 suggest a different approach when asked if he could think of a way to do the problem without calculating the cross product.)

Four participants (P1, P5, P8, P10) set to find the components of the cross product upon reading the problem statement and identified a reason for doing so only after they started to write out expressions for w_1, w_2, w_3. P1 and P8 did not end up calculating the cross product; P1 activated other knowledge instead and P8 attempted to activate other knowledge (more on this in the next paragraph). P5 and P10 did calculate the cross product; P5 used it to complete the task and P10 suggested how she could use it to complete the task (but did not use it; more on both cases in the paragraph after the next).

P1 and P8 *started* to do the necessary computations (they wrote the expressions needed to calculate w_1, w_2, w_3) but stopped short of doing any calculations; writing these expressions out seems to have prompted them to think of another approach. P1 instead activated τ_1 (cross product property of orthogonality) to complete the task and P8, after suggesting $[\tau_4, \theta_{52}]$, still hesitated. I asked P8 why she was hesitating: "there probably was some geometric explanation or something like that, not geometric but... some proofs about the cross product that I didn't read [...] I remember there were some parts that I skipped. In the book I mean." She continued: "I'm not sure how to... How the relation between the cross product and... the... the matrix. I'm not sure how... I can... what exactly it is that I could use." Asked if she could think of any other approach, P8 did not answer. Asked if she could use any of what she's written so far: "probably not."

P5 and P10 did the calculations needed to find the cross product. P10 struggled to know what to do with the cross product once she found it; she suggested $[\tau_4, \theta_{52}]$, but hesitated and did not actually do this. She then tried to think of what else she could do; after she was stuck for a minute, I asked how she "normally check[s] if something is a solution of a system" and she said: "not sure, maybe [pause] grades do it? Like, I don't know. Not sure." P5 plugged the components of the cross product into the equations to check that it satisfies them.

Finally, 3 participants (P6, P7*, P9) set out to find the cross product with the goal of plugging its components into the equation to check the cross product is a solution. P6 completed this approach, but P7* and P9 did not. P9 had started to write the expression for cross product and the calculations needed to find w_1 and spontaneously changed approaches: he activated τ_1 instead. P7* had only *described* this approach (as was his style throughout the interview), and when he thought of his second approach (to activate τ_1), concluded it was more elegant.

5.3.3.2 Most students' 'post-cross-product' activities were driven by LA1 norms that impinged on their ability to mobilize knowledge other than θ_{52} and τ_4

While participants' initial spontaneous activity was to calculate the cross product, participants' paths diverged as they started to enact these calculations. I identified three paths. One path had participants spontaneously mobilize normative LA1 knowledge and not spontaneously suggest any other knowledge (see Section 5.3.3.2.1): prompted to think of an approach that would involve less calculations (e.g., an approach that doesn't require finding the cross product), some of these participants were unable to suggest any (P2, P10), while others recalled there exists theoretical knowledge about cross product but were either unable to recall what it is or unable to mobilize it (P4, P5, P6). Another path was of students who spontaneously wanted to mobilize theoretical knowledge about cross products but did not have the knowledge (P3, P8) (see Section 5.3.3.2.2). A third path was of students who mobilized theoretical knowledge about cross product spontaneously and successfully (P1, P7*, P9) (see Section 5.3.3.2.3).

In my discussion of each type of path, I look to students' explanations to highlight how norms from LA1 may have inhibited students' capacity to engage with Problem 3 using knowledge other than how to calculate a cross product (θ_{52}) and how to solve a linear system by row-reducing its augmented matrix (τ_4).

5.3.3.2.1 Some students spontaneously mobilized normative LA1 knowledge and did not spontaneously suggest any other knowledge (P2, P10; P4, P5, P6) I identified two categories of students here.

5.3.3.2.1.1 Some of these students, when prompted to think of approaches that do not involve certain calculations, could not think of any (P2, P10). P10 and P2 were unable to suggest approaches other than the one they had initially made ($[\tau_4, \theta_{52}]$); the techniques they had suggested reflected a technique students usually activate in LA1 to perform tasks that look like Problem 3 (t_4, to find the solution space of homogeneous linear equations). P10's spontaneous and only activity was to compute the cross product and to then suggest τ_4 (row-reduce the augmented matrix to solve the system). P10 did not mobilize τ_4 and was unable to suggest any other technique. P2's spontaneous and only activity was the same, though P2 did mobilize $[\tau_4, \theta_{52}]$ and completed the task. When P2 was asked if he can think of a way of doing this without finding all the solutions, he said: "I'm not sure if there is another way. To do it. [...] Usually, when uh. I have to find like an answer. In vector form. I would solve it uh, in this way and I would find the. A vector in a parameter form and then I choose th- choose the parameter based on what the question wants. Yeah. So I'm not sure that there's another way." P2's answer is from the perspective of an LA1 student: "*usually*, when [...] I have to find" an answer, τ_4 is the way to go. Recall that row-reducing augmented matrices is the normative technique for 42 of the 116 exam problems to which I had access. P2 was unable to suggest another technique even after a prompt asking him if there is "a way of doing this without calculating the cross product," "without actually finding what the vector is." ("I don't think so.")

5.3.3.2.1.2 Some of these students, when prompted to think of approaches that do not involve certain calculations, recalled there is theoretical knowledge about cross products, but were either unable to recall what it is or unable to mobilize it (P4, P5, P6). P4, P5, and P6 had completed the task: P4 spontaneously activated $[\tau_4, \theta_{52}]$, and P5 and P6 spontaneously activated τ_3 (they found the cross product and plugged it into the equations). After P5 and P6 had finished their approach, I asked if they could think of one that would not involve calculating the cross product. After P4 had finished his approach, I asked if he could think of one that would not involve finding the all the solutions of the system. All three recalled that cross products have certain properties but either did not know what these are, or were unable to mobilize the properties they recalled.

The explanations given by P4, P5, and P6 when they were prompted to think of other approaches point at how the norms they'd developed from the position of Students in LA1 impinged on their ability to mobilize knowledge other than the formula for cross products (θ_{52}) and row-reduction of augmented matrices (τ_4).

P5 recalled there are things her teacher taught about cross product but does not remember what. She also remembered surface-level features of LA1 tasks involving cross products:

> From like, doing problems in the past that had me, like, jig [sic], which thing would make like the cross product zero, and like, the numbers were like, you just like flip... like a number. Or, like, put it like plug in numbers that are like similar. I don't know how to, like, explain it! Like it would be... Like I would have... Like, I'm just trying to remember like past homework. Like, we'd have like, two equations like this. And it'd be like, what would... I don't know if it was exactly like a similar question. It's like, find two vectors that would like... that are like perpendicular or parallel. or whatever. I don't know. I think I'm like confusing a bunch of different ideas together.

I followed by asking P5 if she could think of any geometric relation between the cross product, the vector (29, -9, 3.2), and the vector (11, 2.1397, 41). She could not. This is not surprising; P5's attempt at recollecting knowledge from LA1 drew on superficial features of techniques she had used in the course ("you just like flip... like a number," "plug in numbers that are similar"). This left her with a "confusing [...] bunch of [...] ideas." As for P5's vague recollection that cross products have something to do with vectors that are "perpendicular or parallel," this too was a byproduct of P5's experience solving problems in LA1 (and not of any underlying mathematical knowledge): asked what made her say the word perpendicular, P5 said "those are a lot of the problems that [she] did at the end of the term. And this is kind of around the time [she] remember[ed] like doing... things like that."

When I initially asked P6 if he could think of an approach that wouldn't involve calculating the cross product, he proposed an approach with no basis in the mathematics (plug components of $(11, 2.1397, 41)$ into first equation, do the same with the components of the other vector in the cross product). P6 did not see why doing this would be relevant and dismissed it: "I just don't see why that would indicate that the vectors are orthogonal. In like a 3d space." I then asked if P6 could think of a way to use the fact that the cross product would be orthogonal to the two vectors it's made up of, but he was unable to mobilize this. All he could mobilize was the formula $\cos 90 = 0$ (and he needed confirmation that this is the cosine, and not the sine). He explained his difficulty:

> I guess I just don't understand. Like, the point is that I don't understand why this being orthogonal, it means that these are equal to zero. This kind of like where I'm, I'm failing at the geometry part. It's like, I don't have that like, relationship in my brain. But I'd be happy for you to explain it to me afterwards." From P6's perspective, his difficulty is rooted in visualization: "the weakest part of my linear algebra was definitely... like the visualization aspect. So stuff like cross product and dot product and like, point to a plane, that kind of stuff... Was... I, if I could turn it into just like a numerical thing?

> I would do that. I just haven't done a lot of visualization. Geometry [has] always been like my worst part. I've been enjoying trig a lot recently. So I've been getting a bit better at it. But I'm still like, you know, if I look at an equation or something, I still have to sort of think for a while and figure out what shape is.

P6's comments suggest his position in LA1 was that of a Student: the tasks students are expected to do not require students to know the reasoning that establishes the relation between algebraic and geometric representations of planes, lines, cross products, or dot products and orthogonality. These relations are part of the knowledge *to be taught* (though I don't know what knowledge is actually taught), but tasks students have to perform do not require students to know of any of these relations beyond surface-level knowledge such as "planes have equations that look like $ax + by + cz = d$."

P4's spontaneous activity was $[\tau_4, \theta_{52}]$. P4 had initially interpreted his parametric solutions (PS) superficially (his solution set had the form tv, and v was not equal to (w_1, w_2, w_3)) so he got stuck. Being stuck prompted P4 to try to recall what a cross product represents geometrically; he recalled its norm is the area of the parallelogram formed by the vectors, and dismissed this as useless for the current problem. After I prompted P4 to explain why he thought his cross product is different from the parametric solutions he found, he eventually rectified his error and found his cross product was approximately the same as an element of the parametric solutions. After P4 finished with his approach, I asked if he could think of an approach that wouldn't involve finding the parametric solutions and he gave a geometric interpretation of the problem:

> These are planes, now that I think about it [...]. So this is the intersection of a plane, which is a line, the intersection of a plane, it's going to be a line, and this, the cross product of these two is parallel to the equation of the line. So this is just a geometric interpretation of my solution, but it's not a different way, I would solve it the exact same way [...] the math is the same [...] the math being I need to solve this, with b is equal to zero, I - I - b [inaudible] zero, I need to search, find for xyz and find the vector. So the math is same, but conceptually, this is how I would understand it.

P4's recollection is a memory P4 has from his experience in LA1:

> I recognize from memory that when I tried to set up the - a system equations like this one, I would recognize that I was trying to find the intersection of two planes, which is a line. And this - this is based on ex - exactly just one question that I solved. [inaudible] I remember that question. I don't think that yeah, other than the lecture information. A lot, a lot of my knowledge was gained from the past exam questions. So like one, one question I - I tackled was - would always be in the back of my head when approaching another problem. Especially because it's an intro course and the questions are similar to each other.

P4 "recognize[d] from memory" that the system in Problem 3 was of a certain type of task he'd previously accomplished. P4 had brought up that the norm of a cross product is the area of a parallelogram; P4's justification for his knowledge (of the geometry of a linear system) is based in his experience; P4 did not bring up (at any point in the 27 minutes spent on this problem) that a cross product is orthogonal to the vectors of which it is made; P4 had an unnecessarily bulky approach to the task of checking whether a given element of $\mathbb{R}^3$ is a solution of a linear system, and it was a combination of the only two normative LA1 techniques that relate to Problem 3. I agree with P4's self-assessment that most of his knowledge is "gained from past exam questions." The knowledge he shared reflected knowledge needed to do questions on past exams; the geometric interpretations he was able to give reflected the knowledge needed to perform a type of task he's found on practice exams. He knew that "the intersection of a plane, it's going to be a line, and this, the cross product of these two is parallel to the equation of the line" because this was the reasoning expected for that type of problem. P4 did not bring up, in the context of Problem 3, that the cross product of vectors is orthogonal to the vectors, and this is the technology needed to produce the knowledge that "the cross product of [the normals of a plane] is parallel to the [...] line."

P4 seems to have acquired his knowledge from the position of a LA1 Student: his task, from this position, is to perform tasks that appear in final exams, and he is entitled to draw authority for his techniques from teachers, textbooks, and solved problems. P4 consistently referred to these authorities when asked how he knew what he said was valid; P4 was unable to activate any technique other than a circuitous combination of a technique and a technology ($[\tau_4, \theta_{52}]$) that are normative in LA1; and the theoretical knowledge he was able to activate may only reflect the reasoning needed to do a type of task he expected to possibly encounter on a final exam. The issue is that P4's normative knowledge is restricted to the surface level of the techniques needed to perform normative LA1 tasks, and this inhibited his capacity to act in a way that was appropriate for the task at hand in Problem 3.

P5 and P6's comments, outlined earlier, suggest their knowledge from LA1, like P4's, is also limited to the surface level of the techniques needed to perform normative LA tasks: they knew cross products *have* certain geometric properties, but as LA1 students, they did not need to learn these. For P5, such properties were things the teacher had taught. And P6 was able to get by having only algebraic representations of cross products, dot products, points on planes, without knowing their geometric equivalents: "stuff like cross product and dot product and like, point to a plane, that kind of stuff... Was... I, if I could turn it into just like a numerical thing? I would do that."

5.3.3.2.2 Some students (P8, P3) spontaneously wanted to mobilize theoretical knowledge about cross product but did not have it. Instead, P8 and P3 were only able to recall such knowledge exists, referring to their textbook (P8) and teacher (P3), and P3 recalled surface-level features of problems she'd seen in LA1.

P8's initial spontaneous reaction was to compute the cross product; she wrote out the expressions needed to find the components but did not do calculations. When I asked what she expected to happen by calculating the cross product, she said she "didn't actually want to calculate it. [She] just wanted to see how it looks. In the end, it will give [her] a vector." She then suggested to mobilize τ_4, but again did not do any calculations and paused. When I asked why she hesitated, she said: "there probably was some geometric explanation or something like that, not geometric but... some proofs about the cross product that I didn't read [...] I remember there were some parts that I skipped. In the book I mean." She continued: "I'm not sure how to... How the relation between the cross product and... the... the matrix. I'm not sure how... I can... what exactly it is that I could use." Asked if she could think of any other approach, P8 did not answer. Asked if she can use any of what she'd written so far: "probably not."[18]

P3 was also eventually stuck because she thought there was theoretical knowledge that could be mobilized here. At first, P3 did not know what the task was and was unable to start, so I gave her the definition of cross product. Her spontaneous reaction to this was to calculate the cross product, but she dismissed this right after and did not see a point to knowing its components. For P3, the task was "not [...] to find the values," it was to "explain why this is the answer of this." For P3, "calculation is just a calculation. It's not the theory": "I feel like this is, this is requiring me to define... Why this. Um. Why *this* is... A solution of this system. But what I've—what I've done is just solving the problems, which means I just got the values of w 1 2 3, but it doesn't really mean that... I found... The... It doesn't mean that I can explain why those are the same." I then asked what P3 thinks it would take to explain that, and P3 proceeded to make her sketch. She sketched 3 planes that intersect along a line and explained: "this is a space and I would have... One single line. So I have to find. The equation of the line, right? Or. But from here I found... Wait, wait, wait. [pause] So I—so this is not exactly what—this is not the, the, wait. So this is not the, the equation of... The in-in... [I interjected: "intersection?"] Yeah intersection. But this is the, the space. Like another space. Third space. And they have. They all have... The intersection, at the same point." The planes refer to "the first two systems" and the line to the "solution." A third plane "is another plane which shared the same intersection with the system"; asked where this third plane comes from, P3 was unsure:

> The cross product to be [sic] defined by [pause] um, no... Where it comes from? Um [pause] no... This is not the value of xyz, and this is the only element of a vector here, like... So, that means that this is not the... This is not the equation of the intersection, and this is the plane which. Has the same intersection with the, with the system here.

This explanation reflects the surface-level knowledge that equations of the form $ax+by+cz=d$ correspond to planes and that the intersection of these planes are lines, which are then solutions of systems of two such equations. I maintain that, for P3, this is *surface-level* knowledge (as in, P3 does not have knowledge beyond what's written in the previous sentence) because of her suggestion that the cross product is somehow related to a third plane in this situation.

The features of Problem 3 seemed to have prompted P3 to *try* and use theoretical knowledge, but she did not have this knowledge. In my last attempt to help P3 get unstuck, I asked if she could think of any geometric relation between the cross product, the vector (29, -9, 3.2), and the vector (11, 2.1397, 41); she could not. P3's knowledge drew on surface-level features of tasks that are normative in LA1 and did not include any of the theory needed to produce the techniques for those normative tasks. When P3

[18]It's possible that P8 didn't mean that it's *not possible* to do the problem by calculating the cross product and using it somehow, but rather that she didn't like the calculations this would involve—I recall her hesitation, in Problem 2, to calculate the value for w (after explaining what the calculation would involve and finding that $x=y=z=0$ using Cramer's rule) because it would just be some more "algebra."

initially read Problem 3, she was quiet for a minute and then said: "I was trying to find some relationship between this [the cross product] and this [the system]." Her suggestions were then to add a third equation ($0x + 0y + 0z = 0$) or rewrite the first equation as a matrix equation ($A\mathrm{x} = 0$); she rejected both immediately. I then reworded the problem for her and her immediate reaction was that she did not remember how to do this type of problem ("I forgot how to handle this") and recalled LA1 tasks with surface-level features similar to Problem 3: "something should not be 0," "I remember like during the class. The teacher was like, keep talking about like, after this calculation it should not be 0 or [pause] oh, it's about independency." She was recalling what she knew about LA1 task types whose techniques focalized on "0"s. P3 could only (try to) mobilize surface-level knowledge from LA1 tasks, and this was not enough to support P3 in her attempt to accomplish Problem 3.

5.3.3.2.3 Some students mobilized theoretical knowledge about cross product spontaneously and successfully (P1, P7*, P9) in spite of their norms. The initial spontaneous reaction of P1, P7*, and P9 was to calculate the cross product. P7* and P9 had suggested to plug its components into the equations; P1 had not given a reason for finding the cross product. The three participants stopped themselves before actually doing any of the calculations and instead activated τ_1.

All three said they were convinced by this approach. They preferred this approach to their initial suggestion, as it had involved calculating the cross product. They abandoned their initial suggestions as soon as they thought of τ_1. P1 characterized that first approach as "useless" and said it would make him "cry"; P7* acknowledged his initial approach would have taken more time and that activating τ_1 is an "elegant way" to do the problem—he called it "outstanding."

P1, P7*, and P9's comments suggest they do not view activating τ_1 as part of their norm in the context of LA1. When I asked P1 what he'd have done if he had something like this on an exam, he said: "I would have done this [his first approach] first and then I'm gonna cry." He did say he'd have crossed it out and gone for his second approach, but still, he said his spontaneous reaction is to immediately do calculations. This is also the case for P7*: "I will definitely - if I have any idea of the second one, I will definitely use the second one because that's going to be, hm, outstanding, but I will not - you know, spend a lot of time thinking about the structures or figure out the patterns in this one. I will still use the first one because the first one is the first one that I [would] probably reach [in] my mind." As a LA1 student, it's not part of P7*'s norm to spend time "thinking about the structures" or "the patterns" in a problem.

P1 and P7* said their initial approach would suffice (and take more time) on a LA1 exam, but did say their second approach would be acceptable. P9's comments suggest he views mobilizing τ_1 as an extra—not what's expected in LA1, but something that can be accepted *in addition* to what's expected: on an exam, "the only thing [he would] do is calculate and then replace here and [he would] get zero eventually. [...] I can also write the analysis that this is because of this [τ_1]."

5.3.3.3 Students reacted strongly to the unusual features of the problem

Immediately or shortly after reading Problem 3, many participants explicitly commented on features of the problem that differ from what they usually see in LA1 tasks related to linear systems: participants commented on the non-integer numbers (P1, P2, P4, P6, and, toward the end of her engagement with Problem 3, P5), on the fact that the components of the vectors in the cross product were also the coefficients in the equations (P1, P4, P5), and on the cross product symbol itself (most students needed confirmation as to what the symbol was). There is no correlation between whether a participant explicitly commented on these patterns and their choice of approach to the problem (this had rather more to do with whether they were able to recall *and* activate τ_1); but other participants' hesitation in their approaches (P3, P8, P10) or choice to engage in an approach that bypasses the need to do any calculations (P1, P7*, P9) suggest that all participants were influenced, somehow, by this departure from the features normally seen in LA1.

The comments some of the participants made indicated that, to them, these features were a signal to operate outside of the norm—a signal to use knowledge other than how to calculate the components of a cross product or how to solve a linear system. To P1, the decimal was "confusing"; the 0's to the right of the equal signs and the fact that the cross product vector entries were the coefficients in the equations, to P1, indicated there was a "trick" to the problem. He identified this "trick" shortly after writing out an expression for the cross product sign (he later said calculating the cross product would have made him "cry" on a test), and activated τ_1 to complete the task. P9 said the 0's on the right side of the equation and the cross product triggered the notion of a dot product; his suggestion to activate τ_1 confirms this. P7* also explicitly addressed features of the problem: "I see some pattern here. That pattern is these coefficients, they have these vectors," and then activated τ_1 to complete the task and concluded this takes less "time," is "elegant," and is "outstanding."

Other participants (P3, P8) tried to take the features of the problem into account to produce a technique, but could not do so. P8 was unable to activate knowledge that would bypass the need to calculate the cross product, but her comments suggest she may have been motivated by the presence of decimals and the fact that the entries of the cross product vectors were coefficients in the equations. Indeed, regarding the cross product, P8 said she "didn't actually want to calculate it. [She] just wanted to see how it looks. In the end, it will give [her] a vector." In any case, the inclusion of the cross product symbol did indicate to P3 and P8 that they should operate outside of the norm: they hesitated to do any calculations and instead tried to recall theoretical knowledge about cross products.

Other participants (P2, P4, P5, P6) commented on features of the problem but still chose to operate within the norm, unable to activate anything else when asked if they could think of an approach that wouldn't involve calculating the cross product or parametric solutions. For P5, it was "right off the bat" that she "noticed that... the solutions [the vectors in the cross product], this is the coefficients of each variable." P2 pointed out the numbers were unusual: "Usually on the test, don't put, er, fractions." P6: "it's really hitting me with a big decimal... I like fractions a lot more [laughs]." And P4: "I do not like the numbers I'm seeing [laughs]." He also "realized that the two vectors that were given, the cross product, are the two rows here that I have." Nevertheless, P2, P4, P5, and P6 were not triggered by these features to search for approaches different from their norm of calculating cross products and solving linear systems.

5.3.3.4 Summary: in response to non-normative features of the problem, some students successfully and spontaneously activated knowledge that is not normative in LA1; other students also reacted to these features but were limited by norms

Students had strong reactions to features of the problem (Section 5.3.3.3) and this is what propelled some of them to spontaneously try to act outside of their norm (see Sections 5.3.3.2.2 and 5.3.3.2.3). Three of the participants (P1, P7*, P9) succeeded to do so; their comments and activity suggested they operated outside of the norm because the features of the problem were such that this norm was overtly unpalatable—and P1, P7*, P9 had knowledge that could help them get out of that entanglement.

The rest of the participants were not able to activate knowledge that is not normative in LA1 tasks. Two participants (P3, P8) wanted to, but could not. Four participants (P2, P4, P5, P6) were satisfied activating their usual knowledge (calculating the cross product and plugging its components in or finding the parametric solutions of the system), despite having explicitly reacted to the 'unusual' features of the problem. They were unable to activate any other knowledge, despite being prompted to think of approaches that wouldn't involve calculating the cross product or finding the parametric solutions (P6 acknowledged that if I'm asking that, there likely was another option). One participant (P10) was unsure even of the normative technique she had suggested ($[\tau_4, \theta_{52}]$ is normative in that it's a combination of techniques needed to perform LA1 tasks that are most similar-looking to Problem 3). P10's other comments (e.g., when she thought there was an issue because the cross product has 3 components, w_1, w_2, w_3, while there are only 2 equations) suggest her grasp of the normative knowledge was weak, so it is not surprising that she struggled when given Problem 3—a problem that does not directly correspond to any of the normative tasks in LA1.

These seven participants were unable to activate knowledge other than calculating cross products and row-reducing the augmented matrix to find parametric solutions, in spite of the fact that the features of the problem made this knowledge inefficient (and, indeed, all participants who activated it did so inefficiently: they all ended their activity with rounding errors). At the same time, these participants all seemed aware of the distinctive features of the problem. But it seems that their norms as Students in LA1 prevented them from acting on this awareness. P3, P4, P5, and P6's attempts to talk about theoretical knowledge from LA1 showed all they had was fragmented knowledge; they drew these fragments from surface-level features of tasks they'd done in LA1. This makes sense, as LA1 students do not need to know the mathematics that frames the techniques they use.

When P2 was asked if he can think of a way of doing Problem 3 without finding all the solutions, he said: "I'm not sure if there is another way. To do it. [...] Usually, when uh. I have to find like an answer. In vector form. I would solve it uh, in this way." "This way," for P2, was to row-reduce the augmented matrix to find the parametric solutions. In LA1, tasks that involve linear systems *are* usually accomplished via row-reduction, and, of course, this does always work; and tasks students are required to perform do not make this technique so unpalatable that students are encouraged to activate any other knowledge. Altogether, I infer that the norm that row-reducing augmented matrix is a cure-all, together with the norm that students need only operate at the surface-level when they perform tasks in LA1, impinged on participants' ability to mobilize knowledge that could have allowed them to do Problem 3 more effectively.

5.4 LA1 Problem 4

The following was the fourth problem presented to the 10 LA1 students in the TBI:

Find a non-trivial solution of the following system:

$$\begin{array}{rcrcrcl} -5.2x & + & 2y & + & \pi z & = & 0 \\ 4x & - & 1.3y & + & 4z & = & 0 \end{array}$$

5.4.1 Reference model for LA1 Problem 4

Problem 4 is a task of type "find a non-trivial solution of a homogeneous linear system," but my focus is on the narrower task type t: "find a non-trivial solution of a homogeneous linear system of 2 equations in 3 unknowns in $\mathbb{R}^3$." In LA1, the normative technique would be to row-reduce the augmented matrix to find the general solution of the system and then pick a value for the parameter to find a non-trivial solution. I focus on t because I designed Problem 4 as a follow-up to Problem 3 in the TBI on purpose: if students activated, for Problem 3, a technique that was not normative (i.e., if they activated knowledge other than computing the cross product or row-reduction), would they spontaneously transfer knowledge from Problem 3 to Problem 4? And if not, would students be able to transfer any knowledge when prompted?

My reference model for Problem 4 is therefore centered on the task of type t.

The reference model for Problem 3 captures some of the theoretical knowledge relevant to t. Two homogeneous linear equations in 3 unknowns, in $\mathbb{R}^3$, correspond to two planes $\mathcal{P}$ that go through the origin. I refer by θ_1 to the technologies in the reference model for Problem 3 that include and relate planes $\mathcal{P}$ that go through the origin to their equations, the intersection of a pair of planes $\mathcal{P}$ to its algebraic representation, as well as the notions of orthogonality and dot and cross products. I refer by Θ_1 to the discourse that frames these technologies: the algebraic, geometric, and logical discourse that gives them authority, along with the view that linear algebra and Euclidean geometry are built axiomatically from our physical reality.

A technique produced by $[\theta_1, \Theta_1]$ is to find a vector that generates the intersection of planes $\mathcal{P}$ that go through the origin.

If the normals of planes are scalar multiples of one another, then they are parallel, and so the planes overlap; in this case, any point on one plane is on the other as well, so a technique for finding a non-trivial solution of the equation is to plug values for two unknowns into one of the equations and solve for the third. I refer to this technique by $\tau_{1-\text{overlap}}$ and its framing discourse by $\theta_{1-\text{overlap}}$. This is praxeology $[t; \tau_{1-\text{overlap}}; \theta_1, \theta_{1-\text{overlap}}; \Theta_1]$.

If the normals of the planes are not scalar multiples of one another, then the planes are not parallel. The discourse in $[\theta_1, \Theta_1]$ produces a technique for finding a non-zero vector in their intersection: since the normals n_1 and n_2 are not parallel, their cross product is non-zero, and the discourse in θ_1 shows this cross product is in the intersection. So $\mathrm{n}_1 \times \mathrm{n}_2$ is a non-trivial solution of the system. I will denote this technique by $\tau_{1-\text{line}}$ and the related praxeology is then $[t; \tau_{1-\text{line}}; \theta_1; \Theta_1]$.

Task t can be accomplished without recourse to the geometric representation of the equations. In this case, the reference model for Problem 2 provides the needed theoretical discourse. Equations of the form $A\mathrm{x} = 0$ (of the type discussed in Problem 2) have at least one solution ($\mathrm{x} = 0$). The theoretical discourse in the reference model for Problem 2 shows that consistent systems of linear equations have either one solution or infinitely many. In the case that there are more unknowns than equations (say, m unknowns and n equations), then A has more columns than rows. Since $RREF(A)$ has at most n leading ones, there are $m - n$ free variables. The general solution therefore involves parameters, which can have any value, so the system has infinitely many solutions. This is the case for a task of type t (2 equations, 3 unknowns). I refer by $[\theta_2, \Theta_2]$ to this discourse, in conjunction with the theoretical discourse from the reference model for Problem 2 (about augmented matrices, row operations, and linear systems and their potential number of solutions).

A technique produced by $[\theta_2, \Theta_2]$ is τ_2: to row-reduce the augmented matrix for the system to find the parametric solutions of the system. To accomplish task t, where the goal is to find a non-trivial solution, τ_2 concludes with plugging in a non-zero value for the parameter. This praxeology is $[t; \tau_2; \theta_2; \Theta_2]$.

My reference model for tasks of type t therefore consists of the following praxeologies:

- $[t; \tau_{1-\text{overlap}}; \theta_1, \theta_{1-\text{overlap}}; \Theta_1]$,
- $[t; \tau_{1-\text{line}}; \theta_1; \Theta_1]$, and
- $[t; \tau_2; \theta_2; \Theta_2]$.

5.4.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 4

The task in Problem 4 resembles a task normally given on final exams in LA1: to find a basis for the solution space of a homogeneous linear system (usually of 3 equations in 4-7 unknowns). This is the only LA1 task I found on past midterm and final exams that related to homogeneous linear systems.

I discussed the LA1 task "to find a basis for the solution space of a homogeneous linear system" in the model of knowledge to be learned that is relevant to Problem 3 (Section 5.3.2); I will denote it here as I did there, by t_4. This task normally had the coefficient matrix already in RREF, so students' technique was mostly in finding the parametric equations corresponding to this RREF and using these equations to find a basis for the solution space (possibly by explicitly writing the parametric equations in vector form, but this is not required). τ_4 involves these steps: row-reduce the augmented matrix, find the general solution in terms of parameters, possibly express the parametric solutions in vector form, and use the (vector form of the) parametric solutions to identify a basis for the solution space.

No theoretical discourse is needed for task t_4, though the theoretical discourse about the geometric representation of this system (discussed in my reference model for Problem 3, in Section 5.3.1) is part of the knowledge to be taught in the course. The expression "trivial solution" is also part of the discourse in the knowledge to be taught in LA1.

In addition to t_4, another related task on LA1 exams is that of using Gauss-Jordan elimination to find all the solutions of a linear system; I will denote this task by t_5. I discussed what students have to know to accomplish tasks of type t_5 in my model of knowledge to be learned for Problem 2 (see Section 5.2.2; specifically, see the row about "t_2" in Table 5.3). The technique τ_5 is to row-reduce the augmented matrix; if a row of the form $[0 \cdots 0|a]$ is found, where $a \neq 0$, the conclusion is that there are no solutions, and otherwise the technique is to write one of the reduced forms of the augmented matrix in the form of equations, and use these to solve the system.

Finally, there are other types of tasks in LA1 where the normative technique is to (produce and) solve a homogeneous linear system and activate $[t_{4,\text{nb}}, \tau_{4,\text{nb}}]$, where I write $t_{4,\text{nb}}$ (instead of t_4) to mean that these tasks involve solving a homogeneous linear system but do not involve finding a basis for the solution space (the subscript *nb* is for 'no basis'). I addressed these in my models of knowledge to be learned that is relevant to Problem 2 (in Section 5.2.2) and Problem 3 (in Section 5.3.2). I mention them here but refer the reader to these sections for more information:

- to find intersections of lines and/or planes in $\mathbb{R}^3$ (Section 5.3.2); and
- to check if a set of vectors is linearly independent (Section 5.2.2).

The model of knowledge to be learned in LA1 to perform tasks of the type in Problem 4 thus consists of the praxeologies $[t_4; \tau_4]$, $[t_{4,\text{nb}}; \tau_{4,\text{nb}}]$, and $[t_5; \tau_5]$, where the techniques $\tau_4, \tau_{4,\text{nb}}$, and τ_5 mostly consist of the same activity: row-reducing an augmented matrix so as to find a general solution.

5.4.3 Knowledge LA1 students activated in response to Problem 4

Table 5.9 (on p.161) summarizes the paths of participants' activity as they worked on Problem 4. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step. This table has a slightly different layout compared to the tables of students' activity in other problems; this is mainly to accommodate students' responses to prompts I planned to give students at the end of Problem 4. There were three types of prompts. One, which I will call P-T1 (prompt of type 1), was to ask students (who hadn't commented about this) whether they see any similarity (or differences) between Problems 3 and 4. Another prompt (P-T2) was to ask students if they know of any geometric relation between a cross product u × v and the vector u, and u × v and the vector v. Yet another prompt (P-T3) was to tell students that u × v is orthogonal to both u and v.

These prompts were designed to check whether students could mobilize the knowledge that a cross product u × v is orthogonal to u and v. Problem 4 was given immediately after Problem 3; both problems involve a homogeneous linear system of two equations in three unknowns, with non-integer coefficients. In Problem 3, students are asked to show that u × v is a solution of the system, where the first, second, and third components of the vectors u and v are the coefficients of x, y, z, respectively, in the first and second equations. In Problem 4, students are asked to find a non-trivial solution. I wondered whether students who mobilized the orthogonality of cross products in Problem 3 would mobilize the knowledge from Problem 3 and say that a non-trivial solution for the system in Problem 4 is such a cross product; and if they did, would this be their spontaneous reaction to Problem 4 or would they still engage in row-reducing augmented matrices, the technique normative to the LA1 task that *looks* like Problem 4 (to find the basis of a homogeneous linear system, which requires students to find its solution space). This question, in the TBI, therefore targets the case of participants P1, P7*, and P9. I also wondered whether students who hadn't brought up or used the orthogonality property of cross products (in Problem 3) would be able to mobilize it if I told them about it.

Table 5.9: Paths of LA1 Students' Activity in Problem 4

Practical block $[t, \tau]$		Step 1 Participant	Step 2 Participant	Step 3 Participant	Type of engagement with $[t, \tau]$	Response to P-T1	Response to P-T2	Response to P-T3
τ_4: row-reduce augmented matrix	to perform $t_{4,nb}$ (find parametric solutions, PS) and (on prompt) plug value (specifically, 1) into parameter to find non-trivial solution (NTS)	P1			enacts (completes $t_{4,nb}$ using incorrect augmented matrix, finds NTS on prompt)			
		P2			enacts (completes $t_{4,nb}$, finds NTS)			
		P9			enacts (completes $t_{4,nb}$, finds NTS on prompt)			
		P4			enacts (completes $t_{4,nb}$ via combination of τ_4 and algebraic manipulation of original equations, finds NTS on prompt)			
		P6			partially enacts (uses combination of τ_4 and algebraic manipulation of original equations), does not want to finish because of non-integer numbers, describes expected result (PS)			
		P8			partially enacts (reduces augmented matrix partially), and on prompt describes the rest of what she would do (find PS, plug value into parameter to find NTS); uses 3.14 instead of π			
		P7*			describes (τ_4, find PS, plug value into parameter to find NTS)			
			P3		partially enact (wrote augmented matrix and did one row operation), describe the rest of τ_4, describe expected result (an equation in which x is expressed in terms of y and z), and on prompt, describe how to use this to find a NTS			
	not sure what to do with expected result because of incorrect rules	P5			partially enacts (reduces augmented matrix partially), hesitates because of non-integer numbers, describes expected result (row of type $[0 \cdots a \vert 0]$ where $a \neq 0$), gets stuck (expected row means $a = 0$ to P5, but problem statement suggests system has solutions)			
	no goal identified, used afterwards in Step 2 approach	P10			enacts (reduces augmented matrix nearly to RREF)			
produce new equation(s) from the given ones so as to solve the system	find components of $(-5.2, 2, \pi) \times (4, -1.3, 4)$ so as to produce a third equation	P3			partially enacts (finds components of cross product), describes goal (to find a third equation)			
	use combination of row operations on augmented matrix and algebraic operations on original equations to produce new system of equations, use incorrect algebra to solve the latter and find a single solution to the system		P10		enacts (finds $z = 8$, describes how to use this to solve for x in an equation P10 produced in x and z, describes how to use this to find the value of y); uses approximations			
plug a value (specifically, 0) into one unknown and solve the equations for the other unknowns (possibly by inspection)			P1		suggests and gives incorrect example (one equation instead of system, incorrect solution suggested: if $3x + 2y = 0$ then take $x = \frac{2}{3}, y = -\frac{3}{2}$)			
			P5		suggests (considered only 1 equation instead of system, suggests inspecting $ax + by = 0$, has $(-b, a)$ as solution)			
			P6		suggests			
use determinants, but no concrete suggestion			P2		thinks there is a way to accomplish something using determinants, does not know what			
mobilize cross product (CP) orthogonality and compute CP			P7*		describes (if a and b are vectors of coefficients in equations 1 and 2, respectively, then their cross product is a solution; cross product can be calculated)	P9		
find a cross or dot product to see if anything would come out of it, decide to plug into equation after finding CP components				P5	enacts, dismisses because z gets a particular value and P5 expects z to act like a parameter, on prompt from interviewer (how can you check what you found?) plugs solution she found into equation, does not get 0	P5		
Problem 3 is a verification problem, Problem 4 is a solving problem; compute CP and plug into equations to validate						P10		
compute CP to mobilize Problem 3's $[\tau_4, \theta_{52}]$						P8		
does not mobilize CP orthogonality despite knowing it	Problem 3 is a verification problem, Problem 4 is a solving problem (used CP orthogonality in Problem 3)					P1		
	mentions CP orthogonality but cannot mobilize						P2	
	mentions CP orthogonality but ignores						P4	
	tries to mobilize CP, no mathematical foundation for technique							P4
	concludes that plugging in the CP components in Problem 4 should work because it worked in Problem 3							P5
	given P-T3, cannot mobilize CP orthogonality							P3
								P8
								P10

no prompt for P6 because he had brought up cross product orthogonality in Problem 3 and was unable to use it then

prompt type 1 (P-T1): do you see any similarity between problems 3 and 4?

prompt type 2 (P-T2): in problem 3, do you see a geometric relation between u × v and u? u × v and v?

prompt type 3 (P-T3): referring to problem 3: u × v is orthogonal to u and u × v is orthogonal to v, does that help (to think of another approach)

5.4.3.1 Practical blocks of students' activity in response to Problem 4

All students spontaneously used the normative LA1 technique for problems involving homogeneous equations, τ_4, to engage with Problem 4 (see Section 5.4.3.1.1). Asked if they could think of an approach other than τ_4, some students suggested approaches other than τ_4 (see Section 5.4.3.1.2). In the end, to complete Problem 4 successfully, eight participants (all but P5, P10) mobilized τ_4 and could not mobilize anything else, and three participants (P5, P7*, P9) mobilized Problem 3 (see Section 5.4.3.1.3); among these, only P7* and P9 were able to mobilize the orthogonality property of cross product, which they'd already brought up and used in Problem 3. Indeed, in reaction to prompts P-T1, P-T2, and P-T3, most students were not able to mobilize Problem 3 and the orthogonality property of cross products to suggest an approach other than τ_4 for Problem 4 (see Section 5.4.3.1.4). I had given these prompts to try and get students to mobilize the orthogonality property of cross products but only one student was triggered to do so, and this was a student who had already mobilized it in Problem 3 (see Section 5.4.3.1.4.1). In response to prompts P-T1, P-T2, and P-T3, most students either couldn't make any suggestion, engaged with superficial features of the prompts, or reactivated their LA1 norms (see Section 5.4.3.1.4.2).

5.4.3.1.1 All students spontaneously used the normative LA1 technique for problems involving homogeneous equations, τ_4, to engage with Problem 4. From Table 5.9, I see that 9 of the 10 participants spontaneously engaged in row-reducing the augmented matrix (τ_4). The remaining participant (P3) did so as well after I intervened to put a stop to her spontaneous approach, where she used an approach (with no accurate mathematical basis) so as to produce a third equation; P3 believed a system of 2 equations in 3 unknowns needs a third equation to be solved (I hypothesize P3 would have then engaged in τ_4 to row-reduce the augmented matrix of that new system). So all the participants mobilized τ_4 spontaneously or nearly so. Eight (P1, P2, P3, P4, P6, P7*, P8, P9) used τ_4 with the goal of finding the parametric solutions of the system; half of them spontaneously mentioned plugging in a value for the parameter to find a non-trivial solution, and half did so when I asked how they'd find a non-trivial solution (after they had found their parametric solutions). One participant (P5) used τ_4, but as she described what she expected to happen, got stuck: she expected a row of the type $\left[0 \;\; \cdots \;\; 0 \;\; 16 \,\middle|\, 0\right]$ to appear, which (for P5) would mean $16 = 0$, so the system would have no solutions, but, for P5, this contradicted the suggestion from the problem statement that there is a (non-trivial) solution. (P5 had also brought up this incorrect rule in Problem 2; I discuss it on p.117). Finally, P10 activated τ_4 but it was not clear that she expected to get parametric solutions; she switched approaches after getting a REF of her augmented matrix, reverting to the original system of equations and using algebraic operations to produce a new equation (EQ1); P10 then found the RREF of her augmented matrix, used it to produce an equation (EQ2) corresponding to one of its rows, and solved the system of equations EQ1-EQ2 algebraically (where incorrect algebra led P10 to find a single solution for the system).

5.4.3.1.2 Asked if they could think of an approach other than τ_4, some students suggested approaches other than τ_4. The other approaches suggested by participants, before I brought up prompts P-T1, P-T2, and P-T3, include the following:

- to plug a value (specifically, 0) into one unknown and solve the equations for the other unknowns, possibly by inspection (P1, P5, P6)
    - this was P5's second suggested approach, after she got stuck using τ_4 and I had clarified (in response to what I thought was confusing her, at the time) that a non-trivial solution is one where "not all three [of x, y, z] are 0";
    - this was P1 and P6's second suggested approach when I asked if they could think of a way to do Problem 4 without finding all the solutions of the system; I note here that to do Problem 3, P1 had mobilized the orthogonality property of cross products, and P6 had plugged the components of the cross product into the equations;
    - in their descriptions of this approach, P1 and P5 specifically described how they'd solve an equation in 2 variables, and with integer coefficients, by inspection.
    - None of the participants addressed what would be involved in solving the system of *two* equations that would result by plugging 0 into one of the unknowns in Problem 4; if they had, they might have realized this would lead to the *trivial* solution ($x = y = z = 0$). The task was to find a non-trivial solution;
- to use determinants, but no technique actually identified (P2); I had asked P2 if he could think of an approach that wouldn't use as many calculations (as τ_4).
- to mobilize the orthogonality property of cross products and compute the cross product (P7*, who had also mobilized this property in Problem 3); I had asked P7*, after his first suggested approach, if he could think of another approach;
- to find a cross or dot product to see if anything would come out of this (P5)
    - this was P5's third suggested approach; I had asked if she could think of another one as she did not know how to proceed in her previous approaches. P5 then computed $\mathrm{n}_1 \times \mathrm{n}_2$, where n_1 was the vector of coefficients in the first equation ($\mathrm{n}_1 \cdot (x, y, z) = 0$) and n_2 the vector of coefficients in the second equation ($\mathrm{n}_2 \cdot (x, y, z) = 0$). P5 dismissed this approach because the cross product had a particular value for z, and she had expected z to be a parameter. I asked P5 how she could check what she had found, and she plugged the components of $\mathrm{n}_1 \times \mathrm{n}_2$ into one of the equations. It did not yield 0.

5.4.3.1.3 To complete Problem 4 successfully, eight participants (all but P5, P10) could only mobilize τ_4, and three participants (P5, P7*, P9) mobilized Problem 3. Apart from P5, whom I had asked if she could think of a new approach because she was stuck with τ_4, and P10, who changed approaches spontaneously and finished her approach to Problem 4 without stopping, the rest of the participants completed Problem 4 successfully using τ_4 (and in the case of P4 and P6, reverting early on to the equations corresponding to the matrix they'd partially reduced). Participants either completed Problem 4 by enacting τ_4 in full (P1, P2, P4, and P9 found parametric solutions and plugged 1 into the parameter to find a non-trivial solution); or they activated τ_4 partially and then described the rest of what they would do (P3, P6, P8 said they'd find parametric solutions and plug 1 into the parameter to find a non-trivial solution); and P7*, as per his usual in his TBI, described the entirety of his approach without enacting any of it (he'd row-reduce, find parametric solutions, and plug 1 into the parameter to find a non-trivial solution).

The two other approaches participants mobilized to successfully complete Problem 4 reflected their prior engagement with Problem 3. P7*, asked if he could think of a second approach, said he "think[s] there's some direct relationship with [Problem 3]" and then brought up the orthogonality property of the cross product he had used in Problem 3 ($n_1 \times n_2$, where n_1 was the vector of coefficients in the first equation, $n_1 \cdot (x, y, z) = 0$, and n_2 the vector of coefficients in the second equation); P9 brought it up after I asked if he saw any similarity between Problems 3 and 4. P7* and P9 were two of the three participants who both brought up and mobilized this property in Problem 3; P7* explicitly acknowledged he only thought of this approach because of Problem 3, and this is clearly the case for P9, who only brought the approach up when asked to compare Problems 3 and 4.

P5, meanwhile, didn't complete her previous approaches (τ_4, plugging 0 into the equations and solving the remaining system) and eventually decided to compute the cross product. She initially decided this approach was incorrect (the cross product had a value for z and she expected z to be a parameter), but after I asked how she could check what she found, she plugged the cross product components into an equation, did not get 0, and eventually (after being prompted with P-T3), decided that the cross product should have worked, that she must have made a calculation error, because it had worked in Problem 3. P5's approach to Problem 3 was to calculate the cross product and plug it into the equations to verify it satisfied them; she said it was from Problem 3 that she got the idea to try out the cross product in Problem 4.

5.4.3.1.4 In reaction to prompts P-T1, P-T2, and P-T3, most students were not able to mobilize Problem 3 and the orthogonality property of cross products to suggest an approach other than τ_4 for Problem 4. When students finished suggesting approaches to Problem 4, I said I wanted to return to Problem 3 and gave them one or more of the following prompts:

P-T1 do you see any similarity between problems 3 and 4?

P-T2 in Problem 3, do you see a geometric relation between $(29, -9, 3.2) \times (11, 2.1397, 41)$ and $(29, -9, 3.2)$? $(29, -9, 3.2) \times (11, 2.1397, 41)$ and $(11, 2.1397, 41)$?

P-T3 (29,-9,3.2)$\times(11, 2.1397, 41)$ is orthogonal to $(29, -9, 3.2)$ and (29,-9,3.2)$\times(11, 2.1397, 41)$ is orthogonal to $(11, 2.1397, 41)$; does that help (to think of another approach)?

I gave these prompts in the order indicated by their numbering, sometimes skipping P-T1 and/or P-T2, depending on participants' activity in Problems 3 and 4 (e.g., as P4 started to use row-reduction in his spontaneous reaction to Problem 4, he said "I just go through the way I did it last time," which I took to mean that he already had Problem 3 in mind when approaching Problem 4, so I skipped prompt P-T1 and directly gave P-T2).

5.4.3.1.4.1 Prompts P-T1, P-T2, and P-T3 were designed to get students to mobilize the orthogonality property of cross products; most students did not do so. Only P9 was triggered by one of these prompts to mobilize the orthogonality property; P9 had already mobilized this property in Problem 3, and was triggered to do so when I gave prompt P-T1.

P2 and P4 brought up the orthogonality property when I gave prompt P-T2 but were unable to mobilize it; I stated the orthogonality property again for P4 but he still did not mobilize it.

The participants to whom I gave prompt P-T3 were unable to mobilize it (P3, P4, P5, P8, P10). P8 and P10 said that for vectors to be orthogonal, their dot product must be zero, but were unable to mobilize this.

I did not remind P1 of the orthogonality property; he had brought it up and mobilized it in Problem 3, though when I gave prompt P-T1, he did not bring it up (unlike P9, who did). I did not give P6 any of the prompts; in Problem 3, he had brought up the orthogonality property and was unable to use it. I did not give P7* any of these prompts either; when I first asked if he could think of a second approach (his first was to use τ_4), he had already suggested to mobilize the orthogonality property of the cross product.

5.4.3.1.4.2 In response to prompts P-T1, P-T2, and P-T3, most students either couldn't make any suggestion, engaged with superficial features of the prompts, or reactivated their LA1 norms P9 was the only student who was prompted by P-T1 to mobilize the orthogonality property of cross products. Some students did not offer any approach in response to these prompts (P1, P2, P3). The other students (P4, P5, P8, and P10) kept engaging with normative LA1 techniques in response to these prompts. (I remind the reader that P6 and P7* did not receive any of these prompts, so they are not included in the current discussion.)

When given prompt P-T1, P1 classified Problems 3 and 4: Problem 3 is a verification problem (the solution was given and the task was to verify it) while Problem 4 is a solving problem (the task is to find a solution). P1 had mobilized the orthogonality of the cross product in Problem 3 but did not bring it up here. P2, in response to P-T2, mentioned the orthogonality of cross products and said there must be a way to do the problem other than row-reducing the augmented matrix, but that he doesn't know what that would be. To do Problem 4, P2 was only able to mobilize row-reduction; to do Problem 3, he was only able to check if the cross product was equal to one of the parametric solutions he'd found by row-reducing the augmented matrix. When I gave P-T3 to P3, I asked if that tells her anything about the cross product (in Problem 3) and the system, and her response was as follows: "I remember that I heard about it [laughs], but I don't remember what it is... Um, if it is. [pause] There is, like, no. Like, no. [pause] The trivial solution, like no. [pause] No, no, I don't remember exactly what it is." When P3 was given Problem 3, she was stuck most of the time, saying she was trying to recall theory from LA1. She had calculated the components of the cross product but dismissed their relevance to the problem, reverting to trying to remember theoretical knowledge from LA1. Prompt P-T3 gave her the knowledge that she couldn't recall, but she was unable to mobilize it.

Given prompt P-T2, P4 brought up the orthogonality property of cross products but ignored it, so I proceeded to give P-T3. P4 responded by saying that the dot products (of u $\times$ v with u and also with v, where u and v are the coefficient vectors in Problem 4) would be 0, and asked if I was trying to "hint [him] to another way of solving the problem." I said I was. P4 then proceeded to enact a mathematically unsound technique that, ultimately, circled back to τ_4. He calculated the cross product and found it would be (approximately) $(-375, -26, 161)$. He then described the rest of what he would do: take the dot product of $(29x, -9y, 3.2z)$ (equation 1 in Problem 3 was $29x - 9y + 3.2z = 0$) with this cross product and the dot product of $(11x, 2.y, 41z)$ (from the second equation, $11x + 2.1397y + 41z = 0$; P4 wrote 2. instead of 2.1397) with the cross product. He said he'd get "a system of equations and then probably reach this parameter or probably maybe a multiple of" the vector generating the parametric solutions he had found when he solved the system in Problem 3. He mentioned using an augmented matrix and getting "another version of our, the solution space they give us in the problem." He wrote

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ -14 \\ 2 \end{pmatrix},$$

where the vector of coefficients of t is 2 times the vector generating the parametric solutions he had found in Problem 3. He concluded "this [would] be [his] solution," saying he'd expect to then notice that the solutions found by solving this 'new' linear system are related to the solutions he had originally found: "and then, "oh!," this is also related to this one."

P5 had mobilized her approach from Problem 3 to do Problem 4 before I prompted her to think back to Problem 3, but after her two other approaches hadn't panned out. P5 mobilized the cross product "just to see" what would happen; she explained, later, that it was a test-taking strategy ("when I took my linear algebra tests, if I didn't know how to solve a problem, I would just like go, to like, just some of like, the definitions in that class, like, cross product, dot product or like, big ones. So I would just like, see if anything would come out of it"). In Problem 3 she had computed the cross product and plugged its components into the equations to confirm it's a solution of the system; this strategy worked. She computed the cross product in Problem 4 but was initially confused because the z-component of the cross product had a given value, and she had expected z to be a parameter (this might be explained by participants P1, P2, P4, P6, P7*, P8, and P9's prediction, at the start of Problem 4, that the solutions would involve one parameter because it's a system of 2 equations in 3 unknowns; P5 seems to have tried to allude to this as well, but her explanation was less clear than others'). After I asked P5 how she could check what she did, she plugged the components of the cross product (as she had in Problem 3) into one of the equations, but (because of an earlier calculation error) the equation was not satisfied. I then gave P5 prompts P-T1 and P-T3 (in this order). In response to P-T1, P5 said she got her idea (to compute the cross product) for Problem 4 from Problem 3. In response to P-T3, P5 said that the cross product she calculated (in Problem 4) should have given a solution, that it could have been a math error on her part. I asked what made her think this. She said it's because the cross product in Problem 3 did give her 0 when she plugged it into the equations there, "so [she] just think[s] there's a math error somewhere on [her] page.” She did not address the orthogonality property I had brought up.

P10’s initial response to P-T1 was to classify the problems like P1 had: in Problem 4, the task was “to find the solution,” whereas in Problem 3, the task was “just to” show that what was given is indeed a solution. She then suggested to plug the cross product components she’d previously found (in Problem 3) into the equations to verify that she’d get 0. I asked what she’d do if she didn’t get 0. She said she vaguely remembers that the cross product would be “the normal.” And so I asked what she meant: she remembered that “using the cross product, you can find the normal” but could not say what a normal would be. I then gave her prompt P-T3 and asked if this helps in any way to do the problem. She paused, asked me to reconfirm what would be orthogonal to what, and eventually said: “there’s something else, there’s cross product or dot product. I’m not sure which one but it would be zero. I think the cross product would be 0...” Shortly thereafter, she said it’s the dot product that would be zero, but wasn’t sure. She tried to say what she’d do next; tried “to remember what... to do [...] with this [pointed at the definition of orthogonality I had given her] and this [pointed at the system].” She could not remember.

Given prompt P-T1, P8 suggested a technique for solving Problem 3 that she had suggested but dismissed while engaging with Problem 3 because she thought she should be using theory (that she could not recall) to perform this task (see Table 5.8 on p.147). She suggested to mobilize the Problem 3 technique $[\tau_4, \theta_{52}]$, wherein the strategy is to find the parametric solutions of the system, compute the cross product, and check if a particular value for the parameter would yield the cross product. I then gave P8 prompt P-T3. Would she be able to mobilize the orthogonality property of the cross product? She had tried and failed, in Problem 3, to recall theory she thought might be relevant. P8 paused when I gave prompt P-T3. Then, she laughed. “I hadn’t even thought about orthogonality because when I see equations, [I put] that away!” She then stayed quiet for some time, and eventually said she’s trying to find the relation between x, y, z in the system and her sketch of two vectors (arrows) v_1 and v_2 and a vector (arrow) w orthogonal to both (all with same initial point).

Participants P4, P5, P8, and P10’s attempts at tackling Problem 3—and explanations for their attempts—after being prompted with P-T3 help to identify the knowledge they were missing and which may have blocked them (as well as P2, P3, and P6, all of whom were unable to mobilize the orthogonality property of cross products despite being aware of it) from mobilizing the orthogonality property of cross products. P4 activated superficial features of the prompt and of Problem 3, so as to circle back to a norm from LA1: use the cross product in Problem 3, take its dot product with vectors made up of the *terms* in the equations (that is, the vectors $(29x, -9y, 3.2z), (11x, 2.y, 41z)$), and get a new system of equations and solve it using τ_4 (the normative technique). He expected this to produce the same solution space as the one he’d previously found in Problem 3. P5, meanwhile, used prompts P-T1 and P-T3 to gain confidence in her suggestion to calculate the cross product made up of the ‘coefficient vectors’ in Problem 4 (to find a non-trivial solution of the system); after P-T3, she concluded this approach should have worked because it worked in Problem 3 (and made no reference to the orthogonality property stated in P-T3). P10’s responses to P-T1 and P-T3 show her knowledge from LA1 is limited to the surface. She recalled words ("normal," orthogonal) but did not know what they meant. She knew that from a technical perspective, finding a cross product produces a normal (whatever this may be); and she knew that orthogonal means that either the cross or dot product is 0, but was not sure which. P8, given prompt P-T1, suggested a circuitous technique for Problem 3 (compute the cross product, find the parametric solutions of the system, and find a parameter that generates the cross product) that is a combination of norms from LA1 (compute cross products when you see them, row-reduce augmented matrices when you see homogeneous linear systems). P8's explanation was most revealing: when she sees equations, she puts away orthogonality. And she was unable to use orthogonality when it was brought back out by P-T3.

5.4.3.2 Summary: students row-reduce augmented matrices when they see homogeneous linear systems and cannot mobilize much more than this

There was a clear distinction between students' capacity to activate augmented matrix reduction and calculate cross products and their (in)capacity to activate geometric knowledge related to linear systems and cross products. In this section, I will refer to the activity and explanations triggered in students by Problem 4, in addition to those students had in Problem 3, to argue this distinction is the result of algebraic and geometric knowledge being disjoint units of knowledge in what students are expected to learn in LA1; further, the privilege given to row reduction (as knowledge to be learned) encourages students not to acquire more than a surface-level grasp of other knowledge that relates to linear systems.

Students spontaneously and accurately activated practical knowledge about augmented matrix reduction: upon reading Problem 4, the initial response of several students (P1, P4, P6, P7*) was to predict there would be infinitely many solutions and that there would be one parameter. They also had some of the knowledge that justifies this: there were 2 homogeneous equations and 3 unknowns, so, as P9 put it when he got to his parametric solutions, there would be no "constraints" on one of the unknowns. Apart from these predictions, 9 of the 10 participants spontaneously and immediately row-reduced the augmented matrix of the system to perform the task in Problem 4; the one student who didn't do so (P3) instead thought she had to produce a third equation because there were 3 unknowns, and when I told her it wasn't necessary to have a third equation, she reverted to row-reducing the augmented matrix of the system.

Row-reduction is not an *inappropriate* technique for finding a non-trivial solution of a system of two equations in three unknowns. But I do want to underscore that students' activation of this technique had a normative quality; they did not necessary activate it from the perspective that it would allow them to produce a non-trivial solution. Indeed, one student (P2) stopped, midway through his row-reduction, to wonder how he would use his results. He expected to find parameters and wasn't sure, in advance, how this would be useful: "I'm not sure if this is the way to do it because uh [pause] you can get a parameter [...] I'm not sure, once I obtain the parameters, I'm not sure like uh... What do I do?" It could be P2 had assumed, upon starting his row-reduction, he would find a unique (and non-trivial) solution; but his pause, mid-way through row reduction, to say he's "not sure this is the way to do it" (and his capacity to predict he'd have a parameter) suggests he hadn't necessarily thought of whether row-reduction would work when he started to activate it.

P2's pause and hesitation, midway through his row reduction, suggests he had engaged in row reduction because that's what he was used to doing when he saw homogeneous linear systems. In LA1, tasks that involve such systems always require students to either find all the solutions (e.g., to find the basis of the solution space) or to determine the number of solutions it would have (e.g., to determine if a set of vectors is linearly independent). In light of other students' comments and activity throughout Problem 4, it seems that this norm developed tunnel vision in participants. I say this in part because of some students' spontaneous prediction (upon reading the problem statement) that the system would have 1 parameter and students' choice to find *all* the solutions of the system so as to find *one* solution. I also recall P2, P4, and P8's suggestions to find the parametric solutions of the system in Problem 3 so as to check that the given cross product was one of these solutions. But it's two examples in particular (involving 5 participants in Problem 4) that highlight the tunnel vision triggered in students by homogeneous linear systems. In Problem 4, P5 and P9 had engaged in row-reducing the augmented matrix and, afterwards, also computed the cross product u × v, where u and v are the vectors made up of the coefficients in equations 1 and 2, respectively (I choose to not call them normals of the planes to avoid suggesting students knew these were normals). When they found the cross product components to have specific values, their immediate reaction was to say this was a contradiction: they had expected z to "be" a parameter. They did resolve this confusion, but that they even had this reaction suggests students hyperfocus on the results found via row reduction. The other example of students' tunnel-focus on row reduction is from the activity of P1, P7*, and P9, who were the only ones, after a prompt from the interviewer, to mobilize the orthogonality property of cross products to do Problem 3. Upon receiving Problem 4, all three reverted to row-reducing the augmented matrix; P7* only recalled his approach from Problem 3 after I asked if he could think of an alternative approach, P9 only did so after I asked if he saw any similarity between Problems 3 and 4, and P1 did not bring up his knowledge from Problem 3 when I gave him the same prompt I had given P9.

Row-reducing linear systems does *work*, and a signification portion of LA1 tasks can be accomplished using only this technique. This may encourage Students, whose goal is to succeed in their tasks (and not to acquire linear algebra knowledge, which is the task of the Learner), to develop tunnel vision that accounts for the cure-all technique of row-reduction but not for other knowledge about linear systems. Through students' engagement with Problems 3 and 4, it seems that the geometry of linear systems is peripheral vision that is lost; I make the case for this claim in the coming paragraphs. And this is the issue with students having tunnel vision: anything other than row reduction falls to the periphery. Participants either didn't have other knowledge related to linear systems or were unable to mobilize it. They struggled to remember geometric knowledge and most could not mobilize it. I will address, first, the case of the two students who *were* able to mobilize geometric knowledge (P7* and P9), and then the case of those who were not able to do so.

In both Problems 3 and 4, P7* and P9 spontaneously row-reduced the augmented matrices of the linear systems; in Problem 4, P7* activated the orthogonality property of cross products only when asked if he could think of a different approach, and P9 activated it when asked if he saw any similarity between Problems 3 and 4. It was the design of Problem 3 that led P7* and P9 to mobilize their knowledge in non-normative ways - at least, the design of the problem was part of what triggered this activity. I hypothesize that the other factor—the factor that enabled P7* and P9 to behave differently from the rest of the participants—is the difference in their experience or positioning in linear algebra courses. When I asked P9 how he knew, in Problem 4, that the cross product would lead to a *non-trivial* solution (that is, that the cross product wouldn't be the zero vector), P9 had said he knew this from *high school* (not from LA1): he knew a cross product of vectors is zero if and only if the vectors are parallel or identical. P7*, meanwhile, behaved differently from how the other participants did. Even though this knowledge was often fragmented, he consistently invoked a larger collection of knowledge from LA1 than the other participants had; he enthusiastically tried to identify, without prompt, various techniques through which to complete the TBI problems, and, in extreme contrast with the other participants, rarely engaged with arithmetic; he expressed acute interest in the problems and some of the techniques he conjured ("outstanding," "elegant"). Given these aspects of P7*'s activity, I infer his positioning throughout his tenure as a LA1 student to have included that of a Learner. I hypothesize P7*'s broader knowledge and lesser focus on arithmetic may be explained by this positioning; it may, additionally, be explained by a previous experience in university-level mathematics studies: he had begun an undergraduate degree in applied mathematics at a different university, where he completed analysis, advanced algebra, and advanced geometry courses, and began courses on ordinary and partial differential equations as well as abstract algebra.

Before I address the case of the participants who did not mobilize geometric knowledge in Problem 4, I outline the geometric knowledge shared by P7* and P9. I wondered if, in Problem 3, participants who mobilized the orthogonality property of cross products (P1, P7*, P9) had only the algebraic knowledge needed to mobilize this property. First, they did know that an equation of the form

$$ax + by + cz = 0$$

can be represented using a dot product:

$$(a, b, c) \cdot (x, y, z) = 0.$$

They also knew the orthogonality property of cross products and that vectors u and v are orthogonal if $\mathrm{u} \cdot \mathrm{v} = 0$. I wondered if P7* and P9 had the geometric knowledge that frames these algebraic representations.

P7* and P9's descriptions of the geometry were not entirely coherent, but did have traces of appropriate geometric knowledge. When I asked P9 why the cross product he found in Problem 4 contradicts what he had found earlier (that the system has parametric solutions), he actually turned to the geometry of the situation to make sense of it:

> If I do the cross product, I get... Oh no! Okay. Okay. Okay. Okay. No, no, no, okay. So I will get... No no, okay. I will get a line, which is this, and z is equal to t and then I replace the parameter so - no no, okay. Okay. So it's whatever point on this line, so I get the first point or the closest point so. Okay, yes, it works. Okay.

The line P9 referred to is "the line perpendicular to both the vectors. The vector perpendicular to this vector and this vector"; but P9 did not relate "this vector and this vector" to the equations, nor to any planes, so I don't know if he knew these vectors would be normals of the planes represented by the equations. P7* also eventually brought up a line perpendicular to two vectors, and points on this line, when I asked what the solution set (obtained via row reduction) would look like graphically: "let me think of it [pause] so a and b forms a plane [draws vectors a and b with coinciding initial point and the plane formed by the (linear combinations of) the vectors], and w is going to be a vector that is perpendicular or normal to it [draws w with initial point coinciding with those of a and b, w perpendicular to a, b]." P7* also eventually spoke of a line related to this w:

> If we, you know, fix the starting point of w as the starting point of a and b, then the w is just going to be any point it's just going to be on this line [drew a line going through the shared initial point of a,b, and w, and parallel to w]. Okay. Yes. Which means that the endpoints of w [are] just gonna be on this line, it's just going to be a line. It's a line. Yes. This is a line. Yes. [pause] Okay, I think I think this is correct answer, okay.

Neither P7* nor P9 made specific references to Problems 3 and 4 in these explanations. They had a concept image that links the cross product of two vectors with a line parallel to the cross product and orthogonal to the two vectors; presumably, they associated such a line somehow with the solutions of a system of two linear equations in three unknowns. P9 also drew on geometric knowledge to explain why the cross product of the vectors (made up of the coefficients in equations 1 and 2, respectively) in Problem 4 wouldn't be the zero vector: the vectors weren't parallel. But P7* and P9's explanations did not account for the geometry of the homogeneous linear systems in Problems 3 and 4. Indeed, P7*'s sketch of vectors a and b that generate a plane rather reflects vector equations of planes (equations of the form $(x, y, z) = t_1\mathrm{v}_1 + t_2\mathrm{v}_2 + \mathrm{c}$, which are constructed on the basis of two non-collinear vectors v_i ($i = 1, 2$) parallel to a plane and a point c on the plane); his talk of a vector w orthogonal to a and b, and a line parallel to w (and going through its initial point, a point on the plane) is reminiscent of two types of LA1 practice problems that relate cross products with planes ("find a point-normal equation of a plane, given its vector equation" and "find the equation of a line perpendicular to a plane and going through a given point"). Regardless of the source of P7*'s concept image, the geometric knowledge he (and P9) proposed only attended to the cross product, and not to the equations themselves.

Before I continue to other participants' knowledge about the geometry of homogeneous linear systems, I wish to attend to the other geometric explanation P7* offered when I asked what the solution set of the system would look like graphically. Midway through his explanation (about the vectors a and b and the vector w perpendicular to them), P7* deviated and thought again about the form ("$x = (\ ) + (\ )s$") of the general solution of the system. The mishmash of concepts that ensued suggests P7* had a library of surface-level knowledge that wasn't founded in the mathematics that underlies either algebraic or geometric representations of vectors, lines, and planes:

> This is very tricky, very, very tricky. I think, I think this is gonna be, so for example, I have an s [the parameter]. I, I've, I've read this somewhere that it's gonna be two parallel lines. And the result is just gonna be this whole

> line parallel to the line that is going through this initial point of s [drew two parallel lines and a point on one of them]. I think it's very, like, this and that reminds me of, you know, of the parameter equation of a line, which is something like you know, $3+2t$. Yes, I may be wrong, but I do remember this thing. I think that this is just, kind of, these two parallel lines [...] they may be useful here. But these have no [...] relationship between a and b here [from his initial sketch] [pause] let's put it this way, when we are adding here, this is a vector s, right? Now, when we are adding vector [*sic*], as we're adding, we're adding another vector is just gonna be, it's just gonna, I'm thinking of the transformation, is it going to be parallel? Is it gonna be, you know, shifting [underlines column of constants in $x = (\ ) + (\ )s$] ? Or is it gonna be rotating [underlines column of coefficients of s in $x = (\ ) + (\ )s$]? I'm thinking that this one may be rotating. This one is going to be rotating, but this one is going to be shifting. So it's going to be rotating this s? No, no, no, it's going to be rotating this vector, and then do the shift. So rotating first shape later, it depends on what this s is. So s is going to just, can't be which rotate in this one. That's what I'm thinking of. Okay.

(This was the third problem in a row in which P7* brought up transformations.) This description showed P7* was confused about the meaning of the objects in the expression $x = (\ ) + (\ )s$. For example, he first spoke of the parameter s as a point ("the result is just gonna be this whole line parallel to the line that is going through this initial point of s") and then he spoke of "rotating" s; and even though he noticed this may be off—"[s]o it's going to be rotating this s? No, no, no, it's going to be rotating this vector [not clear to which vector he referred here]"—he persisted in his interpretation of the equation $x = (\ ) + (\ )s$ as a representation of transformations.

The rest of the participants could not mobilize geometric knowledge in Problem 4, even once they observed or were told that u×v is orthogonal to both u and v (where u and v are the vectors in Problem 3), nor did they call upon the algebraic representations that P1, P7*, and P9 had mobilized in Problem 3; what they were all missing, specifically, was the representation of $ax+by+cz = 0$ as $(a, b, c)\cdot(x, y, z) = 0$. I expect students who know how point-normal equations (such as $ax + by + cz = 0$) are constructed from the visual representation of planes would be equipped to spontaneously view an equation such as $ax + by + cz = 0$ in dot-product form; after all, it's the result of taking the dot product of vectors (x, y, z) parallel to the plane with a normal (a, b, c) of the plane. P7* and P9's explanations of geometric elements of Problems 3 and 4 put to question whether they had this theoretical knowledge, and their comments in Problem 3 suggest they had mobilized the orthogonality property of cross products mainly because the problem included a cross product, the constants in the linear system were 0, and the scalars in the cross product vectors were repeated in the system (P9: "when I first looked, I thought zero, there's something about it and then what I thought further, I said 'oh, yes; dot product, cross product'"; and P7*: "I see some pattern here. That pattern is these coefficients, they have these vectors").

In the remainder of this section, I address comments other participants made that help to clarify students' position relative to geometric knowledge in LA1.

P6 knew he did not have geometric knowledge: "stuff like cross product and dot product and like, point to a plane, that kind of stuff... Was... I, if I could turn it into just like a numerical thing? I would do that." P6 seems to view this as a personal choice, but in light of the LA1 tasks students are *expected* to perform in relation to linear systems and cross products, I surmise it's rather the heavy institutional weight on computations in LA1 that steered P6 away from the need to have geometric knowledge. P5 and P10's comments support this: P5 remembered (in Problem 3) "problems that [she] did at the end of the term" involved cross products and the notion of perpendicularity, and in Problem 4, P10 recalled cross products produce something called a "normal," but did not know what a normal is and could not contextualize this memory. P10's recollection of a link between "cross product" and "normal" is likely from experience she had practicing LA1 problems in which the task is to find the normal (or point-normal equation) of a plane, given vectors parallel to the plane (P10 made no mention of planes). P10's knowledge was limited to surface features of this type of problem: she knew that calculating a cross product would produce something called a "normal." P5's knowledge also seems restricted to surface-level features of LA1 problems, as all she had was a fragile grasp of the geometry of cross products: she knew that cross products have something to do with vectors that are "perpendicular *or parallel*" (my emphasis; she did not know whether cross products have something to do with perpendicularity or whether they had something to do with parallel vectors) and she said she's "confusing a bunch of different ideas together."

The geometric knowledge that LA1 tasks require students to have, when it comes to linear systems, is superficial. They *need* to have the associations that link certain words with certain procedures (e.g., such as P10's association of "normal" with "cross product") or certain words with certain algebraic expressions (e.g., such as the association of the word "plane" with equations of the form $ax + by + cz = d$). But they do not need to know the theoretical knowledge that creates these associations. P8's reaction to prompt P-T3, wherein I told her that the cross product in Problem 3 is orthogonal to the vectors of which it's made, is a symptom of the disconnect between the knowledge students are expected to learn in LA1: "I hadn't even thought about orthogonality because when I see equations, [I put] that away!" In P8's (LA1) experience, one thing (homogeneous linear systems in 3 unknowns) has nothing to do with the other (orthogonality). As she tried to find a link, her focus was on finding a link between x, y, z (in $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$) and a sketch she made of a vector w orthogonal to two vectors v_1 and v_2. In terms of the reference model, what was missing for P8 (and for the other students) was the construction of equations (for planes) using a normal (a, b, c) of a plane, vectors (x, y, z) parallel to the plane, and the orthogonality of the normal with vectors parallel to this plane.

Participants did generally know, on surface, that linear systems are related to lines and planes in 2 or 3 dimensions. But this association was superficial and in some cases even led to confusion about the algebra. For example, consider this mashup of algebra and geometry from P5 (she was talking about the augmented matrix of the system in Problem 4):

> I know that there's only two - there's only two, like, lines - two rows. And so... that will make like any other additional rows, like all 0 0 0. 0. It's like the only - so these two are the only ones that matter. I think... it's going to be 2d. And then... So z wouldn't make sense in like a 2d space. [Because]

> this is only x and y, so z can be like... it's like not important as a value. And so it would have to cancel out.

Another symptom of the confusion that might be triggered in students when they attempt to merge algebra and geometry is P3's fuddled description of her initial attempt at Problem 4. Her spontaneous activity was to compute the cross product $(-5.2, 2, \pi) \times (4, -1.3, 4)$. Before doing this, she had asked me to clarify what a non-trivial solution is (I explained what a trivial solution is, and then said that a non-trivial solution is a solution different from the trivial one). P3 then said: "oh, so what I have to do is the same with the previous one, I guess," and set to compute the cross product. After P3 did calculations to find components of the cross product, I asked her to explain what she was doing: "I just did the... Uh, cross product, uh vector. To find... the third. Equation. [pause] Which is an intersection." I asked why the cross product would give "the intersection" and P3 initially said she "ha[s] no idea" and then: "if I want to find the three variables I, I need, and there's three variable – three equations. And that is why I tried to find another equation to solve this problem." This description brings to mind P3's struggle to describe the geometric representation of the linear system and cross product in Problem 3: her description suggested the cross product is related to a third plane (see Section 5.3.3.2.2). Equations, intersections, cross products—P3 seemed to know, on surface, that there is a link between them, but was missing the knowledge to substantiate that link.

P4 also knew that the linear systems in Problems 3 and 4, geometrically, were related to the intersection of planes; for instance, in Problem 3, he explained that

> this is a cross product. And it is the same vector as the solution. That means *the solution could be the normal of a plane* [emphasis added], for example. And these are planes, now that I think about it, and so the intersection of those planes. [...] So this is the intersection of a plane, which is a line, the intersection of a plane, it's going to be a line, and this, the cross product of these two is parallel to the equation of the line. So this is just a geometric interpretation of my solution.

I won't put too much stock in P4's claim that a solution of an equation could be the normal of the plane (represented by the equation), in case it was a slip of the tongue. But his activity and explanations in Problem 4 revealed his knowledge was skimmed from the surface of LA1 tasks ("a lot of my knowledge was gained from the past exam questions") and was not rooted in any theoretic (algebraic or geometric) knowledge. Indeed, in response to prompt P-T3 (where I pointed out that the cross product in Problem 3 is orthogonal to the vectors of which it was made, as P4 had brought up this property himself in response to P-T2 but ignored it), the activity P4 suggested had no substance. He suggested to take the dot product of vectors $(a_i x, b_i y, c_i z)$ (taken from the Problem 4 equations $a_i x + b_i y + c_i z = 0$, $i = 1, 2$) with $(a_1, b_1, c_1) \times (a_2, b_2, c_2)$. He hypothesized that doing so would produce a system equivalent to the system in Problem 4; their solution spaces would be the same. The choice to invoke vectors of the form $(a_i x, b_i y, c_i z)$ shows P4's knowledge (about vectors, dot products, cross products, equations, lines, planes) was superficial; it was not founded in any algebraic and geometric knowledge that can substantiate the association P4 had between linear systems, planes, and equations.

From participants' activity and explanations in Problems 3 and 4, I can conclude that they know of a technique that always works to solve a linear system: row-reducing its augmented matrix. They also know how to compute a cross product. They know what it means that two vectors are orthogonal. They associate equations like those in Problems 3 and 4 with planes. Some know such planes might have a line as their intersection. Students' activity and explanations show these are surface-level units of knowledge left over from their experience performing LA1 tasks. Such tasks overemphasize (and over-privilege) row reduction of augmented matrices as a technique for anything related to (homogeneous) linear systems: students do not need to mobilize any knowledge other than row-reduction, and their experience in LA1 developed in them an instinct to spontaneously pop an augmented matrix and row-reduce it when they see a linear system. The algebraic and geometric discourse that substantiates the correspondence between linear systems and their geometric representations is peripheral, and even when students have some of the relevant geometric discourse, it is limited.

5.5 LA1 Problem 5

The following was the fifth problem presented to the 10 LA1 students in the TBI:

> Given $k \in \mathbb{R}$, the vectors $(-k, 1, 1), (-1, 1, k)$, and $(1, 0, 1)$ form a parallelepiped of volume 0. Find the values of k for which the vectors are linearly independent.

5.5.1 Reference model for LA1 Problem 5

Problem 5 is a task of type t: "determine the conditions under which a given set of three vectors in $\mathbb{R}^3$ is linearly independent," where "conditions" refers to the value(s) of an unknown in the components of the given vectors.

A set of vectors $S = \{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$ in a vector space V is said to be linearly independent if the trivial linear combination is the only linear combination of S that produces the zero vector $\mathbf{0}$: that is, if $c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_k\mathrm{v}_k = \mathbf{0}$ implies that $c_i = 0\ \forall i = 1, \ldots, k$. Otherwise, S is said to be linearly dependent. This stems from the conception of linear dependence of a set of vectors as the possibility of expressing one vector as a linear combination of the others.

If V is finite-dimensional of dimension n, then any set of more than n vectors is linearly dependent. A set S of one vector v is linearly independent if and only if v is not the zero vector. And a set of two vectors is linearly independent if and only if neither vector is a scalar multiple of the other. It is due to the last three properties that I selected the feature in task t as I did: the task concerns sets of *three* vectors in $\mathbb{R}^3$, a vector space of dimension 3.

Depending on the given vectors, a task of type t may be performed by inspection or by activating the definition of linear independence. For instance, if $S = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is in $\mathbb{R}^3$, the technique is to check the conditions under which the truth of the equation

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = \mathbf{0}$$

would imply that $c_1 = c_2 = c_3 = 0$ (where $c_1, c_2, c_3 \in \mathbb{R}$). If the vectors v_i are expressed in terms of their components, that is, if

$$\begin{aligned} \mathrm{v}_1 &= (v_{11}, v_{12}, v_{12}) \\ \mathrm{v}_2 &= (v_{21}, v_{22}, v_{23}) \\ \text{and } \mathrm{v}_3 &= (v_{31}, v_{32}, v_{33}) \end{aligned}$$

then the above equation corresponds to a system of linear equations:

$$\begin{array}{ccccccc} c_1 v_{11} & + & c_2 v_{21} & + & c_3 v_{31} & = & 0 \\ c_1 v_{12} & + & c_2 v_{22} & + & c_3 v_{32} & = & 0 \\ c_1 v_{13} & + & c_2 v_{23} & + & c_3 v_{33} & = & 0 \end{array}$$

If this system has only one solution (where $c_1 = c_2 = c_3 = 0$), then the vectors are linearly independent. One technique for verifying this is to use Gauss-Jordan elimination (we'll refer to this by τ_1) and the other (τ_2) is to check whether the determinant of the coefficient matrix is non-zero; if it is, then the system has a unique solution.

Problem 5 can be performed using τ_1 as follows: the vectors $(-k, 1, 1), (-1, 1, k)$ and $(1, 0, 1)$ are linearly independent only for the values of k for which

$$c_1(-k, 1, 1) + c_2(-1, 1, k) + c_3(1, 0, 1) = (0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. That is, the vectors are linearly independent only for the values of k for which

$$\begin{array}{ccccccc} -kc_1 & - & c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & kc_2 & + & c_3 & = & 0 \end{array} \tag{5.2}$$

has only zero as the value for the coefficients. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -k & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & k & 1 & 0 \end{array}\right]$$

If $k = 1$, then the reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case, the solutions of the system have the form $(-1, 1, 0)t$ ($t \in \mathbb{R}$). Indeed, if $c_1 = -t, c_2 = t, c_3 = 0$ for some $t \in \mathbb{R}$, then (recalling this is the case in which $k = 1$)

$$-t(-k, 1, 1)+t(-1, 1, k)+0(1, 0, 1) = -t(-1, 1, 1)+t(-1, 1, 1) = (t-t, -t+t, -t+t) = (0, 0, 0).$$

If $k \neq 1$, then the reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{k-1} & 0 \\ 0 & 1 & \frac{1}{k-1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case, the solutions of the system have the form $(\frac{1}{k-1}, -\frac{1}{k-1}, 1)t$ ($t \in \mathbb{R}$). Indeed, if the coefficients (c_1, c_2, c_3) have this form, then:

$$\frac{t}{k-1}(-k,1,1) - \frac{t}{k-1}(-1,1,k) + t(1,0,1) = (\frac{-kt+t}{k-1} + t, \frac{t-t}{k-1}, \frac{t-kt}{k-1} + t) = (0,0,0)$$

Since t can have any real value, this shows that no matter the value of k, there exist non-trivial linear combinations of $(-k,1,1), (-1,1,k)$ and $(1,0,1)$ that produce the zero vector. So the vectors are not linearly independent for any value of k.

As previously stated, another way to check if the linear system in (5.2) has only $(c_1, c_2, c_3) = (0,0,0)$ as a solution is to check if the determinant of the coefficient matrix is non-zero; this is technique τ_2. The determinant can be calculated using θ_{21}, a cofactor expansion along any row or column, and/or θ_{23}, properties that relate row operations with determinants (as discussed in the reference model for Problem 2). In the case of the coefficient matrix in 5.2, these calculations would lead to the result that the determinant is 0. Alternatively, the formula for the volume of a parallelepiped could be mobilized to conclude that the determinant of the coefficient matrix is 0. Indeed, the volume of a parallelepiped generated by the vectors $\mathrm{u} = (u_1, u_2, u_3), \mathrm{v} = (v_1, v_2, v_3), \mathrm{w} = (w_1, w_2, w_3)$ is the absolute value of

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

We'll denote this last technology by θ_{23}. Mobilizing it for Problem 5, which states that the volume of the parallelepiped formed by $(-k,1,1), (-1,1,k)$ and $(1,0,1)$ is 0, leads to the result that the absolute value of

$$\det \begin{bmatrix} -k & 1 & 1 \\ -1 & 1 & k \\ 1 & 0 & 1 \end{bmatrix}$$

is 0. Since $\det A = \det A^t$, it follows that the determinant of the coefficient matrix of the linear system in (5.2) is 0, so $(c_1, c_2, c_3) = (0,0,0)$ is *not* its only solution; the vectors are therefore linearly dependent for any value of k.

The theoretical discourse that produces both techniques is the same as that outlined in the reference model for Problem 2 in Section 5.2.1 (see Table 5.2 on p.101). I will denote the theoretical discourse specific to Gauss-Jordan elimination by $[\theta_1, \Theta_1]$ and the theoretical discourse specific to the determinant of the coefficient matrix by $[\theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}; \Theta_2]$ (where θ_{24} is the theorem stating that if $\det A = 0$, then $A\mathrm{x} = \mathbf{0}$ has infinitely many solutions).

When $V = \mathbb{R}^3$, linear independence has a geometric representation that builds on the construction of $\mathbb{R}^3$ as a vector space from the definition of geometric vectors (arrows). For instance, two vectors v_1 and v_2 are defined to be parallel if they are scalar multiples of one another. This definition reflects the notion that if two arrows (geometric vectors) are parallel, then their direction is the same (or directly opposite) while the length of one is a multiple of the other (and, in turn, the definition of length in vector spaces agrees with this construction). A set of parallel vectors is linearly dependent because there is a non-trivial combination of the vectors that produces the zero vector (for instance, if $v_1 = 5v_2$, then $-5v_1 + 5v_2 = \mathbf{0}$).

A set of three vectors in $\mathbb{R}^3$ is linearly independent if and only if the vectors are non-coplanar (I denote this technology by θ_{31}). Vectors are said to be coplanar (parallel to the same plane) if they lie on the same plane when positioned so their initial points overlap. This technology produces a third technique for performing tasks of type t: if three vectors in $\mathbb{R}^3$ are known to be (non-)coplanar (under certain conditions), then they are linearly (in)dependent (under those conditions). I denote this technique by τ_3.

One last notion pertinent to Problem 5 is that of a parallelepiped. A parallelepiped is the shape generated by three vectors in $\mathbb{R}^3$. (More precisely, it is the shape generated by all linear combinations of three vectors, where the coefficients are real numbers between 0 and 1, inclusively, and when the vectors are positioned so their initial points coincide with the origin.) If such a shape has volume 0, then the three vectors must be coplanar (otherwise, the parallelepiped would have non-zero height and its volume would be larger than 0). I denote this technology by θ_{32}; this, together with θ_{31} and τ_3, shows that the vectors given in Problem 5 are linearly dependent for all values of k; indeed, since the given vectors form a parallelepiped of volume 0 for any $k \in \mathbb{R}$, they are coplanar for all values of k, and hence are not linearly independent for any value of k.

I denote by Θ_3 the theory that gives authority to the technologies that produce τ_3: Θ_3 is the algebraic, geometric, and logical discourse that frames the technologies that relate algebraic constructs (such as vectors, linear independence, linear combinations, parallelepipeds) with geometric constructs (such as vectors, linear independence, linear combinations, parallelepipeds), as well as the view that the axioms on which linear algebra and Euclidean geometry are founded reflect human perception of physical reality.

My reference model for tasks of type t therefore consists of the following praxeologies:

- $[t; \tau_1; \theta_1; \Theta_1]$: using the algebraic definition of linear independence together with the component form of vectors to produce a homogeneous linear system (whose number of solutions determines whether the vectors are linearly independent or not), and using Gauss-Jordan (or Gaussian) elimination to determine whether the system has a unique solution or not, and using this to determine whether it is only the trivial linear combination that produces the zero vector;

- $[t; \tau_2; \theta_{21}/\theta_{22}/\theta_{23}, \theta_{24}; \Theta_2]$: using the algebraic definition of linear independence together with the component form of vectors to produce a homogeneous linear system (whose number of solutions determines whether the vectors are linearly independent or not), and finding the determinant of the coefficient matrix (using either knowledge about determinant calculations or a formula for the volume of a parallelepiped) to determine whether the system has a unique solution or not, and using this to determine whether it is only the trivial linear combination that produces the zero vector;

- $[t; \tau_3; \theta_{31}, \theta_{32}; \Theta_3]$: using geometric knowledge about the volume of the parallelepiped formed by three vectors to conclude whether three vectors are coplanar, and using knowledge about the geometric interpretation of linear independence in $\mathbb{R}^3$, to determine whether three vectors are linearly independent.

5.5.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 5

In LA1, the task of the type "determine the conditions under which a given set of three vectors in $\mathbb{R}^3$ is linearly independent" is not normative, though a task of similar type did occur on one final exam problem in the exams to which I had access. That task was to find the conditions under which three vectors in $\mathbb{R}^3$ would be parallel to the same plane (that is, coplanar). There too, the vectors were given in component form and some of the components were expressed in terms of an unknown k. I will denote this task by t_3.

More normative tasks in LA1 when it comes to the linear independence of vectors are those of type t_4, to determine whether a given set of three vectors $\mathrm{v}_1 = (v_{11}, v_{12}, v_{13}), \mathrm{v}_2 = (v_{21}, v_{22}, v_{23}), \mathrm{v}_3 = (v_{31}, v_{32}, v_{33})$ is linearly independent (as in 2 past exam problems) or to show that a given set of three vectors is linearly independent (as in 1 past exam problem); the task of type "to determine if..." is also the only one recommended to students on the list of practice problems associated with the textbook section about linear independence.

There are two normative techniques for accomplishing t_4 in LA1. One is to set up an augmented matrix where the first column is the vector v_1, the second column is v_2, the third is v_3, and the column to the right of the bar is made up of 0's; then, this matrix is reduced so as to determine if the RREF of the coefficient matrix is I_3 or not. If it is, then the set of vectors is linearly independent; otherwise, the set is linearly dependent. It is up to instructors' discretion as to how many marks (if any at all) are allocated towards a statement of the type "$\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ are linearly independent only if $c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = 0$ implies that $c_1 = c_2 = c_3 = 0$." It is also up to instructors' discretion as to how many marks (if any at all) are allocated towards showing that the augmented matrix is constructed from this initial equation ($c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = 0$). I will refer to this technique by τ_{41}. The second normative technique for accomplishing t_4 is τ_{42} and builds on the same initial equation (as in my reference model). The focus instead is on the determinant of the matrix whose first column is the vector v_1, second column is v_2, and third is v_3; the technique is to compute this determinant. If the determinant is non-zero, then the vectors are linearly independent; if the determinant is zero, then the vectors are linearly dependent. Again, it is up to instructors' discretion as to whether students have to justify any part of this technique.

We're now equipped to describe the technique for the less normative task t_3, which appeared in a problem on one past final exam: "to find the conditions under which three vectors in $\mathbb{R}^3$ are parallel to the same plane." The vectors $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ are given in component form, and some of the components are expressed in terms of an unknown k. To do this task, students would have to activate the technology θ_3: a set of vectors in $\mathbb{R}^3$ are parallel to the same plane only if they are linearly dependent. One this technology is activated, the task is similar to t_4: to find the conditions under which three vectors in $\mathbb{R}^3$ are linearly dependent. Both τ_{41} and τ_{42} can be adapted here. Adapting τ_{42}, for instance, might look like this: find the determinant of the matrix whose first column is the vector v_1, second column is v_2, and third is v_3. This determinant would be an algebraic expression in terms of an unknown. For the vectors to be linearly dependent, the determinant must be equal to 0. The technique is then to find the values of the unknown for which the determinant equals 0. I will denote this technique by τ_3.

Another normative type of task in LA1 that is related to the question of linear independence of vectors is t_5: to show a set of n vectors in $\mathbb{R}^n$ (where $n = 2, 3$) is a basis for $\mathbb{R}^n$. One of the normative techniques for t_5 in LA1 involves showing the set of vectors is linearly independent (using either τ_{41} or τ_{42}) and then activating the knowledge that a linearly independent set of n vectors in an n-dimensional vector space is a basis for the space. I found only one past exam problem to have this task, but I consider this task as normative (as opposed to t_3) because one of the course outline's recommended problems (in the textbook section about bases) is a task of type t_5.

Finally, the only type of task in LA1 that has to do with parallelepipeds is t_6: to find the volume of the parallelepiped determined by three vectors in $\mathbb{R}^3$, given the endpoints of these vectors (and the endpoints, as per the usual in LA1, are single-digit integers). This task appeared in two past LA1 exams as well as in one of the course outline's recommended problems (in the textbook section about cross products; here, the vectors were given in terms of their components, and not in terms of their endpoints). The technique for task t_6 is to first find the vectors in terms of their components, using the formula

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A),$$

and then to find the volume of the parallelepiped (e.g., generated by the vectors $\mathrm{u} = (u_1, u_2, u_3), \mathrm{v} = (v_1, v_2, v_3), \mathrm{w} = (w_1, w_2, w_3)$) using the theorem stating that the volume is the absolute value of

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

In summary, the praxeologies LA1 students are expected to have relative to constructs pertinent to Problem 5 include:

- $[t_3; \tau_3]$: to use row-reduction or determinants to determine the conditions under which three vectors in $\mathbb{R}^3$ are coplanar (this occurred in 1 exam problem);
- $[t_4; \tau_{41}/\tau_{42}]$: to use row-reduction or determinants to determine whether or to show that a given set of three vectors in $\mathbb{R}^3$ is linearly independent (this occurred in 3 exam problems and appeared in the list of recommended practice problems);
- $[t_5; \tau_{41}/\tau_{42}]$: to show a set of n vectors in $\mathbb{R}^n$ (where $n = 2, 3$) is a basis for $\mathbb{R}^n$ by showing they are linearly independent and activating the knowledge that a set of n linearly independent vectors in an n-dimensional vector space is a basis for that space (this occurred in 1 exam problem and appeared in the list of recommended practice problems);
- $[t_6; \tau_6]$: to use the formula for the volume of a parallelepiped formed by 3 vectors in $\mathbb{R}^3$ to find its volume (this occurred in 2 exam problems and appeared in the list of recommended practice problems).

The geometric interpretation of linear independence of vectors in $\mathbb{R}^3$, as described in my reference model for Problem 5, is part of the knowledge to be taught in LA1. However, in light of the heavier focus in LA1 on the algebraic techniques pertinent to linear independence (techniques that circle back to the LA1 norm of row-reducing matrices or calculating their determinant), I wondered whether and how students would activate their knowledge about parallelepipeds and their volumes to do Problem 5. Would they consider the geometric implications of a parallelepiped having volume 0 (that is, that it must have no height, so the 3 vectors of which it's formed must be coplanar)? If so, would they be able to use this knowledge to make an inference about the linear independence of the vectors? Would they only (be able to) activate the formula for the volume of such an object? Or would they ignore the information about the volume of the parallelepiped, and only activate τ_{41} or τ_{42} to do the task?

5.5.3 Knowledge LA1 students activated in response to Problem 5

Table 5.10 (on p.183) summarizes the paths of participants' activity as they worked on Problem 5. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step.

It will be useful for the discussion below to recall that the volume of a parallelepiped formed by three vectors $\mathrm{u} = (u_1, u_2, u_3), \mathrm{v} = (v_1, v_2, v_3), \mathrm{w} = (w_1, w_2, w_3)$ is the absolute value of

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}. \tag{5.3}$$

I will refer by v_i $(i = 1, 2, 3)$ to the vectors given in Problem 5. They are $\mathrm{v}_1 = (-k, 1, 1), \mathrm{v}_2 = (-1, 1, k)$, and $\mathrm{v}_3 = (1, 0, 1)$.

Students generally attempted to activate knowledge about the volume of a parallelepiped (formed by three vectors) to do Problem 5. Two students (P1 and P10) mobilized the formula in (5.3) (as their first activity when engaged with Problem 5) and the given information (that the volume is zero) to deduce

$$\det \begin{bmatrix} -k & 1 & 1 \\ -1 & 1 & k \\ 1 & 0 & 1 \end{bmatrix}$$

is zero, but did not know what to make of this and got stuck. It is not clear if P10 understood the problem statement to mean this determinant would equal 0 *for all* k; in P1's case, I surmise he did not know this, because he later came back to this determinant, calculated it, found it was 0, and concluded "[the determinant does] not help. Because you just have 0 equals 0." For three students (P2, P8, P9), it was clear they did *not* immediately infer, from the problem statement, that the three vectors form a parallelepiped of volume 0 *for all* k, as they had mobilized τ_6 and used the formula in (5.3) to calculate the volume of the parallelepiped and got stuck upon finding it to equal 0. They had expected to find an expression in terms of k which they could then set equal to 0 (since the volume ought to equal 0). But they could not establish such an equation. Three other students

(P4, P6, and P5) did *not* immediately activate formula (5.3); they focused, instead, on the geometric implication of the parallelepiped having volume 0. They knew the vectors must be "on" the same plane; I do not use the term "coplanar" because not all participants had such a lucid interpretation of the situation (P1, P6, and P9 visualized the vectors to form a parallelogram). The implication the vectors are coplanar (or form a parallelogram) was not sufficient for many participants to complete the task, and they alternated between the geometric and algebraic representations to do so. For example, for P6, it was contradictory that linearly independent vectors in 3-space would form a 2-dimensional shape.

Apart from participants' focus on geometric and/or algebraic representations of a parallelepiped having volume 0, participants (P1, P2, P4, P9, P10) also mobilized algebraic knowledge they related with the notion of linear independence (in the form of τ_{41} and/or τ_{42}). Two participants (P3, P5) only mobilized (algebraic) knowledge that was not relevant to the task; their activity seemed motivated by a goal of establishing equations in which k could be isolated.

Table 5.10: Paths of LA1 Students' Activity in Problem 5

Practical block $[t, \tau]$	Participant's engagement with $[t, \tau]$: Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	
infer that since volume is 0, determinant of matrix (where R_i is v_i) is 0, get stuck	P1								enacts
	P10								describes
activate τ_6 (use formula to find the volume of parallelepiped (scalar triple product (STP) or determinant form, where determinant is of matrix where R_i is v_i) so as to set it equal to 0 (volume is 0) and solve for k to infer the value(s) of k for which the vectors are LI)	P8								enacts, gets stuck: finds STP is 0, stuck because there is no k to isolate
	P9								partially enacts: finds STP is 0, concludes volume is 0 for any k (briefly says this is not so for $k = 1$ because in that case, two of the vectors are equal to one another, but then reconsiders and concludes volume is 0 no matter what)
		P6							enacts, doubts: finds two identical rows; states again that if volume is 0, one vector must be LD, wonders whether there are values of k for which they are LI; says the two vectors (that led to identical rows) must be collinear. Double-checks for calculation errors.
				P4					determines that no matter what value k has, volume is 0 so vectors are on the same plane, so there are no values of k for which they are LI
				P6					determines that for any value of k, the determinant is 0, so for any k the vectors are LD
					P1				partially enacts: finds determinant is equal to 0 and says "that will not help because you just have 0 equals 0"
give formulaic interpretation of a prism having volume 0	P5								from volume formula, height or length or base must be 0
give geometric interpretation of a parallelepiped having volume 0	P4								vectors must be on the same plane
	P6								says vectors form a 2-dim. parallelogram in 3-space, but thinks this is a contradiction: LI vectors in 3-space can't form 2-dimensional shape
		P2							says vectors are coplanar, tries to infer whether this means they are LI or LD; struggles; I give definition of linear independence and then P2 concludes coplanar means LD.
		P3							partially enacts: draws 3-dim. Cartesian graph, plots points (0,1,0), (-1,0,0), (-1,0,1), (1,0,1) after I say that if a parallelepiped has volume 0 then it's flat. Describes what seem to be the points in her sketch. Seems to consider what the points could be for different values of k.
			P4						vectors must be on the same plane
			P6						for the vectors to be LI, parallelepiped must have volume greater than 0; one vector must be scalar multiple of one of the other vectors
			P8						vectors are on the same plane so they are LD, there does not exist k for which they are LI
				P9					two vectors must be overlapping; so no value of k for which the vectors are LI; after I draw the possibility of 3 vectors where none are overlapping but volume is 0, P9 says "they are coplanar" so belong to the same subspace so they can be written in terms of each other so they are not LI
				P10					I asked if P10 could do this, P10 is unable to mobilize ("I have no idea")
						P1			describes: "It's gonna be the xy plane. No, it's like a part of... Volume is 0, so it's gonna be just like this," draws parallelogram
								P7*	enacts: object has volume 0 so 3-dim. vectors are on same plane. Confused because $v_1 + v_3$ is not parallel to v_2; realizes linear combination could have coeff. other than 1; concludes 3 vectors "on a plane" are LD
give geometric interpretation of 3 vectors in $\mathbb{R}^3$ being LI	P7*								describes: $v_1 + v_2$ "should not be on the same line" as v_3 (should not be "parallel")
$[t_4, \tau_{42}]$ (calculate the determinant of the matrix made up of the given vectors)	P2								enacts, doubts: finds determinant is 0, wonders if determinant 0 means vectors are all LI or all LD
		P4							enacts, doubts: finds determinant is 0, double-checks, wonders if this means k can have any value, wonders if he should be using STP.
use cross products to create equations in which k can be isolated	P3								enacts: finds the cross product of two of the given vectors, use its components as coeff. to produce the equation $(k-1)x-(-k^2+1)y+(-k+1)z = 0$ and concludes $k - 1 + (-k + 1) = 0$ (may have plugged the coeff. of $(1, 0, 1)$ to get this); starts over with the cross product of another pair of vectors, but this time adds up the cross product components and sets them equal to 0.
$[t_4, \tau_{41}]$ (row-reduce a matrix)		P1							partially enacts: writes $-kk_4 + k_4 + k_4 = 0$ and augmented matrix where coeff. matrix has v_i as R_i (incorrect matrix for τ_{41}), and constant terms all 0.
		P10							suggests and then dismisses: isn't sure, but considers row-reducing the matrix where R_i is v_i (incorrect matrix for τ_{41})
		P7*							suggests: writes the matrix A where C_i is v_i, and writes $Ax = 0$ (where 0 is the 3×1 zero matrix)
			P7*						partially enacts: reduces the coeff. matrix to a form in which $A_{21} = A_{31} = 0$ and identifies a value of k for which $A_{32} = 0$ and $A_{33} \neq 0$
			P9						partially enacts: row-reduces the matrix where column i is vector v_i (with $k = 5$) and gets a row of 0's. Concludes in this case vectors are LD. Writes the matrix where C_i is vector v_i (with $k = 0$) and says vectors are LI in this case.
				P7*					partially enacts: writes augmented matrix made up of matrix obtained in Step 3 and a right-most zero column.
				P1					enacts: row-reduces (incorrect) matrix from Step 2, eventually gets two identical rows and concludes there are infinitely many solutions so "for LI I have multiple case." Stuck.
						P4			wants to row-reduce augmented matrix where C_i is v_i and right-most column is the zero vector; expects to find no solutions, to confirm what he found with the determinant (no solution would mean no k for which the vectors are LI) and gives incorrect rules about what the results of row reduction would mean for linear independence
						P7*			enacts: sets to find values of k for which the augmented matrix from Step 4 is consistent; checks cases $k = 1, -1, 2$ and finds conclusions to contradict his previous result that the vectors are LD when $k = 2$ and LI otherwise.
check if one vector is a linear combination of the others: by inspection, particular values for k		P8							suggests: describes writing one vector as a linear combination of the other two, given a value for k (e.g., $k = 0$); does not conclude anything about values of k but does say $(1, 0, 1)$ is not a linear combination of the other two because of the 0 in the second component.
		P9							enacts: observes that if $k = 1$ then the first two vectors are equal, in which case they are LD and one of them is "redundant"; says vectors are LD when $k = 0$ and LI when $k > 1$
			P1						says no combination of $(-1, 1, k), (1, 0, 1)$ would give $(-k, 1, 1)$; when asked why, says "oh no, never mind" but backtracks on this as well. Tries to find useful linear combinations in the case that $k = 0$. Stuck.
					P4				checks $k = 1$, sees two vectors are identical and thinks that'd always be the case; tries $k = 5$ and sees it isn't
check if one vector is a linear combination of the others: knows relevance, does not mobilize			P10						states vectors are LI if they are not linear combinations of one another. Stuck.
check if one vector is a linear combination of the others: for values of k found in a previous step			P5						plugs $k = -2, -1, 0$ into the 3 vectors and considers whether the set of vectors (corresponding to each k) is LI or LD; says the vectors with $k = 0$ are LD because two sum up to give the third; says the vectors with $k = 1$ are LI (note that 2 vectors are identical here); says the vectors with $k = -2$ are LI (checked by inspection, adding/subtracting the vectors)
check if one vector is a linear combination of the others: check if two vectors add up to a scalar multiple of the third vector					P7*				enacts: finds $v_1 + v_2$ to check if it's a scalar multiple of v_3, dismisses because $v_1 + v_2$ has 0 as a component and the corresponding component for v_3 is not 0; similarly with $v_2 + v_3$ and v_1; finds $v_1 + v_3 = (1-k, 1, 2)$ and, since $v_2 = (-1, 1, k)$, solves $\frac{1-k}{-1} = \frac{1}{1} = \frac{2}{k}$ to find values of k for which the vectors are LD. Concludes vectors are LD if $k = 2$ and LI otherwise.
							P7*		stuck, then partially enacts: finds values of k for which $v_1 + v_3$ is not proportional to v_2 so as to find values of k for which the vectors are LI. Wonders whether proportionality would mean vectors are LD.
use dot products to create equations in which k can be isolated		P5							takes dot product of each pair of the given vectors; sets each dot product equal to 0 and also sets each dot product equal to each other. Gets 3 values of k (0,1,-2) by solving some of the equations, is reminded of how a 3×3 matrix can have 3 eigenvalues.

LI: linearly independent; LD: linearly dependent; STP: scalar triple product; $v_1 = (-k, 1, 1), v_2 = (-1, 1, k), v_3 = (1, 0, 1)$.

Table 5.10 shows participants' zigzagged engagement with Problem 5: they alternated between activating (or trying to activate) algebraic and/or geometric knowledge about linear independence and/or the volume of a parallelepiped. Some students spontaneously alternated between algebraic and geometric representations, and in some cases, I prompted students to alternate when they got stuck with one representation or the other. Participants P2, P4, P6, P7*, P8, and P9 completed the task successfully in this way; P1, P3, P5, and P10 did not, despite interventions designed to help them get unstuck. I discuss participants' activity relative to each attempted technique first.

5.5.3.1 Students produced equations with the aim to isolate k (P1, P3, P5, P6, P8, P9, P10).

Two produced equations that were irrelevant to the task (P3, P5). Others (P1, P6, P8, P9, P10) activated a formula for the volume of the parallelepiped and, given the information that the volume is 0, set the expression for the volume equal to 0 (or, in the case of P10, only suggested to do so).

For all these participants, the choice to produce an equation in which to solve for k was spontaneous—that is, I had not intervened to prompt students in this direction. Students either engaged in this activity as their initial response to Problem 5 (P1, P3, P8, P9, P10) or turned to it after having considered the implications of a parallelepiped having volume 0 in terms of the formula for a rectangular prism (P5) or in terms of the geometric vectors that form the parallelepiped in $\mathbb{R}^3$ (P6).

The two students (P3, P5) who produced equations that were irrelevant to the task were unable to mobilize other LA1 knowledge productively and did not complete the task in Problem 5. I discuss their cases in Section 5.5.3.1.1.

In Section 5.5.3.1.2, I discuss the activity of P1, P6, P8, P9, and P10 as they produced an equation of the form "volume of the parallelepiped equals 0." P10 stated this equation but did not mobilize it; P8 and P9 activated the formula for the volume of a parallelepiped, found the volume is 0, and did not use this to conclude anything about the linear independence of the vectors; P1 decided the equation was not helpful because it led to the equation $0 = 0$; and P6 interpreted the equation's solution accurately after having also inferred that a parallelepiped of volume 0 is a parallelogram, but went back and forth between the equation and his geometric interpretation before settling on the conclusion that the vectors are linearly dependent for all k.

5.5.3.1.1 Students who produced equations irrelevant to the given task were also unable to mobilize other LA1 knowledge productively (P3, P5). P3's initial and spontaneous reaction to Problem 5 was to produce an equation in k. P3 mobilized technical knowledge from LA1 (how to calculate cross products) to produce equations that had no relevance to Problem 5 (see Table 5.10 for more information on these equations). P3 was not the only participant to produce equations in which k could be solved but which had no value in the task at hand; after P5's initial formulaic interpretation of the parallelepiped having volume 0, she used dot products to create equations in which k could be isolated; she had assumed the parallelepiped to be a right rectangular prism and decided to find the values of k for which the vectors are orthogonal. Her calculations were correct but irrelevant to the task. P5 then used the values she had found for k to check if one of the given vectors could be expressed as a linear combination of the others (for those values of k); this part of P5's activity showed she knew the LA1 definition of linearly independent vectors. Nevertheless, she was unable to do the task by inspection: for instance, she did not notice that when $k = 1$, two of the vectors were identical, even though she had written out the three given vectors in the case that $k = 1$. P5 was unable to mobilize any other knowledge to do the task.

P3's knowledge was more limited than P5's: after she said that she "forgot how to verify that the vectors are linearly independent or not," I gave her the LA1 definition of linear independence (a finite set of two or more vectors is linearly independent if none of the vectors can be written as a linear combination of the others), but P3 did not mobilize it. As stated above, P3 produced equations (irrelevant to the task) in which k could be isolated; her justification for doing so was an incoherent mutation of LA1 knowledge ("the cross product is the intersection of these two systems, which means this equation is the intersection of these two planes"). I therefore prompted P3 to focus instead on the information about the volume of the parallelepiped, to see if she would mobilize any geometric knowledge. She did not. Instead, she sketched a Cartesian graph with axes x, y, z and plotted a few points (with components 0, 1, and -1). P3 did not mobilize any other knowledge to do the task.

5.5.3.1.2 Among the students who activated algebraic knowledge about volumes spontaneously (P1, P6, P8, P9, P10), only two (P6, P8) spontaneously activated a geometric interpretation of a parallelepiped having volume 0. Among the students whose equation was based in a formula for the volume of a parallelepiped, three eventually completed the task (P6, P8, P9) and two did not (P1, P10). P1 and P10's initial reaction to the problem was to reference the LA1 formula for the volume of a parallelepiped but they did not (at least, at this initial stage) activate the formula to calculate the volume; instead, for P1 and P10, the volume formula may have influenced how they activated (surface-level features of) the normative LA1 technique τ_{41} for task t_4 of determining whether vectors are linearly independent. P6, P8, and P9 did activate the LA1 formula to calculate the volume but struggled with the result that the volume is 0.

For all the participants who mobilized the formula for the volume of a parallelepiped, what seemed to produce their technique (based on the comments they made as they found the volume to be 0, which I will get to shortly) was the knowledge that if the volume of a parallelepiped is 0, then the expression for the volume (given by the LA1 formula) must equal 0; in this case, k could be isolated and part of the task ("find the value(s) of k for which...") completed. It's possible some of the students interpreted the problem statement to mean that vectors $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ form a parallelepiped of volume 0 *only for some* k.

Among these participants, for whom the choice to activate a LA1 formula for the volume of the parallelepiped was spontaneous, only two (P6, P8) chose, without a prompt on my part, to activate a geometric interpretation of a parallelepiped having volume 0. For P6, the initial reaction to Problem 5 was to visualize a parallelepiped of volume 0 (his interpretation was that two of the vectors must overlap, perhaps with one longer than the other, and so that the vectors formed a parallelogram); P8 chose to visualize the vectors as a third step in her approach to Problem 5; her interpretation was that the vectors are on the same plane. The others did not spontaneously attempt to visualize the situation. I prompted P9 to do so after his third failed attempt to tackle the problem: like P6, he also visualized two of the vectors to overlap. For P1, my prompt came after his fifth failed attempt at the problem; like P6, he also visualized a parallelogram. P10 was unable to visualize the situation ("I have no idea"). I will return to participants' geometric interpretations in Section 5.5.3.6, but are outlining what and how P1, P6, P8, P9, and P10 activated their geometric knowledge here so as to contrast it with what and how they activated LA1 algebraic(-formulaic) knowledge about the volume of a parallelepiped.

I will discuss the case of P1 and P10 first and that of the other participants second.

For P1 and P10, engaging the volume formula was a spontaneous reaction and ultimately a springboard for activating surface-level features of τ_{41}. P1 and P10 started out by inferring that since the volume of the parallelepiped formed by $\mathrm{v}_1, \mathrm{v}_2$, and v_3 is 0, the determinant of the matrix (where R_i is v_i) is zero. (P1 knew the formula; P10 had said she could not remember it so I offered it.) P1 and P10 both got stuck at this stage. P1 couldn't "find the link [between the volume being 0 and] the question," which was about the linear independence of the vectors that were given.

P1 and P10 proceeded to use the matrix from the volume formula to activate a different technique for the task. P10 suggested to row-reduce the matrix (where R_i is v_i) and did not follow through with her suggestion. P1 wrote the corresponding *augmented* matrix, with constant terms 0, and, like P10, did not follow through with row-reducing it.

P10's suggestion to row-reduce the matrix where R_i is v_i seems based in surface-level features of the formula I had shown her (for the volume of a parallelepiped) as well as surface-level features of the LA1 technique τ_{41} for checking whether three vectors are linearly independent. The discourse that produces τ_{41} leads to an augmented matrix where C_i (not R_i) is v_i (for $i = 1, 2, 3$) and where the entries in the right-most column are all 0. P10's suggestion to row-reduce the (non-augmented) matrix (identical to the one in the volume formula) reflects the surface-level feature of τ_{41} wherein a matrix made up of the given vectors gets row-reduced.

P10 did not explain why row-reducing the matrix would work or help to solve the problem. When I asked what she thought would happen if she reduced the matrix, she said she doesn't "think that would work" and did not make any other suggestions. P10 did not clarify what she had in mind when she suggested to row-reduce the matrix: I don't know if she thought of τ_{41} as a technique for performing t_4 (determining if a set of vectors is linearly independent), or if she was suggesting to row-reduce a matrix made up of the vectors she'd been given because, for instance, row-reducing a matrix features in many of the techniques relevant to LA1 tasks.

P1's suggested matrix was augmented (and so, marginally closer in appearance than P10's to the one in τ_{41}), but, like P10's, mimicked the matrix in the volume formula rather than the one that would occur when τ_{41} is activated. P1's matrix and the justification he gave for it suggest P1 had fused the volume formula with a surface-level grasp of the discourse that produces τ_{41}: he had written the equation

$$-kk_4 + k_4 + k_4 = 0$$

to demonstrate an implication of the equation $k_1v_1 + k_2v_2 = 0$, which he had written alongside the descriptor "$k_{1,2} = 0$." This equation and descriptor were the "one thing" P1 said he remembered about linear independence when he returned to Probem 5 at the end of the interview. P1 said this equation would lead to the same approach he'd already used, and after a minute pause in which he said he was "confused," P1 decided "it should be a different k for each one," and this equation would then lead to the augmented matrix he used. The coefficients of the k_4's in P1's equation are the components of the vector $(-k, 1, 1)$. His equation incorrectly calques the equation in the question at the crux of τ_{41}: do there exist a, b, c, not all zero, for which the equation

$$a(-k, 1, 1) + b(-1, 1, k) + c(1, 0, 1) = (0, 0, 0) \tag{5.4}$$

is true? If yes, then any one of the vectors can be expressed as a linear combination of the others, and so they are linearly dependent. Otherwise, the vectors are linearly independent. In LA1, equation (5.4) and its related discourse are part of the technique to be taught in LA1, as per examples in the course textbook. But students are not required to produce the discourse attached to equation (5.4); at most, some instructors may expect students to produce the equation (if the task is t_4: to determine if a set of vectors is linearly independent) for partial marks, but this is up to the discretion of each instructor and so is not a norm of the LA1 institution.

That P1 wrote the equations he did ($k_1v_1 + k_2v_2 = 0$, $-k(k_4) + k_4 + k_4 = 0$) shows he knew the augmented matrix in τ_{41} captures *some* homogeneous equation. But P1's equations show he did not know the reasoning that produces τ_{41} and relates its results to linear independence. After P1 worked with the augmented matrix he produced, he asked for the definition of linear independence; after reading this definition, P1 verified, by inspection, whether one of the given vectors was a linear combination of the other two; and after this, P1 returned to and row-reduced the matrix he had initially produced. Throughout this activity, P1 did not activate knowledge that relates the definition of linear independence to τ_{41}. P1's homogeneous equation, then, shows that despite intent to justify the augmented matrix he'd formed, P1 was still activating only a figment of τ_{41}: an operation involving the vectors is set equal to 0.

For P1 and P10, the observation that the determinant (of the matrix made up of the given vectors) is zero was a surface-level remark; they knew that since the volume is 0, the expression in the formula for the volume should equal 0. But they did not make any substantive inference from this observation or demonstrate they had more than a surface-level grasp of that volume formula (and related equation) throughout the remainder of their engagement with Problem 5. After P1's suggestion to row-reduce a matrix, he asked to see the definition of linear independence and proceeded to check, by inspection, if one of the vectors was a linear combination of the other two (for some cases of k); after her suggestion to row-reduce a matrix, I asked P10 what it would take for the 3 vectors to be linearly independent, and she knew this means "they're not linear combination[s] of each other" but said she "[doesn't] know how to solve this." Since P10 was stuck, I asked her what the vectors might look like, if they form a shape that has volume 0. P10 had "no idea." P1's (second) response to this question was to draw a parallelogram. (This supplanted his first response: the "xy plane.") P10 did not activate anything else for Problem 5. P1, before I had prompted him to visualize the vectors, had actually returned to row-reducing the matrix (discussed above) and found there would be "infinitely many solutions" (because his reduction led to a matrix in which two rows were identical) but got stuck (he was "confused with the problem"). After this, P1 decided to calculate the determinant (same as the one in the volume formula) and found it to equal 0: "that will not help, because you just have 0 equals 0." His conclusion: "I... can't find anything."

P1 did not interpret the "infinitely-many solutions" result of his row-reduction (to make a conclusion about the linear independence of the given vectors) and dismissed the technique in which he calculated the determinant ("that will not help, because you just have 0 equals 0"). This suggests his mobilization of the volume formula and related equation ('it equals 0') was akin to P3 and P5's activating their LA1 knowledge to produce *an equation* (any equation) so as to isolate k; P1's surface-level grasp of knowledge about equivalent equations, equations of the form $0 = 0$, linear independence, and parallelepipeds did not equip P1 to mobilize this equation (or the volume formula) in a more productive way. (This is different from the cases of P6, P8, and P9, who also seemed, like P1, P3, and P5, to have initially expected the task to be to isolate k in some equation, but *were* able to mobilize a geometric interpretation of the scenario, even if only to compensate for the fact that k could not be isolated—and not necessarily because they privileged a geometric interpretation as more suitable than an algebraic one for a problem about a parallelepiped of volume 0.) It's not clear what P10 perceived the task to be, because she did not vocalize much or offer enough suggestions, but that all she *did* offer was surface-level features of either a formula and, potentially, a LA1 technique (τ_{41}) shows her LA1 knowledge, like that of P1, did not suffice to perform the task in Problem 5.

P8 and P9 created an equation so as to find the value(s) of k for which the volume of the parallelepiped is 0. P8 and P9 created an equation in which an expression for the volume of the parallelepiped was set equal to 0. The expression P8 and P9 activated for the volume of the parallelepiped had the form of a scalar triple product: a dot product of one of the vectors (say, v_1) with the cross product of the other two $(\mathrm{v}_2, \mathrm{v}_3)$. From a geometric perspective, for this scalar triple product to equal 0, v_1 must be orthogonal to a vector orthogonal to both v_2 and v_3; that is, the three vectors must be coplanar. I highlight this property because, unlike the determinant formulation for the volume of a parallelepiped (as in equation (5.3)), the scalar triple product formulation readily lends itself to a geometric interpretation.

For P8 and P9, this activity was their initial and spontaneous reaction to Problem 5. They didn't infer, from the problem statement, that the volume would be 0 for *any* k. P8 got stuck when she found the scalar triple product to be 0 ("oh my god, not good"). She did not know how to interpret this because it didn't "give [her] any k." P9 did know how to interpret this: he surmised the volume of the parallelepiped is 0 for any k. Neither

P8 nor P9 made any conclusion, at this point, about the values of k for which the given vectors are linearly independent.

After this initial reaction to Problem 5, P8 and P9 tried to activate other algebraic knowledge but it did not suffice to complete the task. P8 and P9 eventually reflected back on their initial equation when they proceeded to a geometric interpretation of the scenario. P8 had drawn a sketch in which 3 arrows had a coinciding initial point and were coplanar, and a fourth arrow that was orthogonal to the rest of the lot. For P8, this explained what she had found earlier about the volume: "it's all here in the same plane and that's why it's zero." (She then activated knowledge she recalled from her LA1 textbook: "there was a graph or something in the book that [showed that] if they lie in the same plane they are dependent.") When I asked P9 if he could use the fact that the parallelepiped has volume 0, he had a narrow perspective on the characteristics of vectors that form a parallelepiped of volume 0:

> It has volume zero because two of them are [pause] oh no [pause] okay... So here, it's zero because two of them overlap. So you don't have a volume. Therefore I have to... But there is no value, because whatever the value of k, the volume is zero, therefore they overlap, two of them overlap all the time, therefore, there is no value for which they are [linearly independent]—so, okay, I believe that's my answer.

P8 and P9 were able to complete the task once they'd made a geometric interpretation of the volume being 0; but that's not the point I wish to highlight here. My focus, here, is on P8 and P9's handling of the equations they had initially proposed as an approach for Problem 5. Neither P8 nor P9 had concluded anything about the linear independence of the vectors after they'd calculated the volume and found it to be 0; P9 had only concluded that the volume is 0 for any k (and P8 was stumped). Further, they did not spontaneously make any geometric interpretation to make sense of what they had found by activating the formula for the *volume of a parallelepiped*. P8 turned to a geometric interpretation only once she'd exhausted other avenues, and P9 turned to it only because I had prompted him to do so. P8 and P9's initial reaction to Problem 5 was guided by their knowledge of a LA1 formula (for the volume of a parallelepiped) and, perhaps, the norm (from LA1 and its prerequisite mathematics courses) of solving equations when a task is to find the value(s) of an unknown.

P6 mobilized the LA1 formula for the volume of a parallelepiped to cope with the contradiction he perceived between the problem statement and the conclusion he'd reached with his geometric interpretation. P6 expected there to be values of k for which the vectors are linearly independent. In his initial and spontaneous reaction to Problem 5, he had inferred the vectors form a "2-dimensional parallelogram" in 3-space. While P6's visualization was restrictive, he did surmise, from this, that the vectors must be linearly dependent: "how can I have a 2-dimensional parallelogram in 3d space with three vectors that are linearly independent?" Indeed, he had taken the problem statement to mean there *are* values of k for which the vectors are linearly independent. It's this confusion that prompted P6 to mobilize an equation to perform the task: "I guess I can just take the determinant, and then set it equal to zero to find a parallelepiped of volume zero, [to find] what value k might be. So I guess I can do that.

Let's see. Gotta start somewhere."

But P6 found the volume to be zero for any k. He stated again that "the question kind of implies that there are" values of k for which the vectors are linearly independent; he had "trouble with this idea of how three vectors [could] be linearly independent and [form] a parallelepiped of volume zero." In LA1 and in pre- and co-requisite mathematics courses, the norm is that problems involving unknowns *have* solutions, unless a problem normatively *can* "have no solutions" (such as the task to solve a given linear system); otherwise, the norm is that the problem statement includes the phrase *if any* (P4 alluded to this: "I would like it if it was [an] 'if any' question"). The assumption, from the wording of Problem 5, that there *are* values of k for which the vectors are linearly independent brings to mind students' reactions to Problem 1, where they assumed the matrices are invertible, based on what was normative in LA1, and, similarly, that the equation *has* a solution; again, this contrasts with students' readiness in Problem 6 to accept that a quadratic equation may have no solution, knowledge that is part of what students are expected to learn in high-school algebra. I am highlighting, here, the normative quality of students' interpretation of tasks, and how it is not, in the current case, based in the mathematics at stake.

After repeating his first two steps twice (visualize the vectors, check the values of k for which the volume equals 0), P6 did eventually gain confidence in his conclusion that the vectors are linearly dependent for all k: "I'm not sure. But I guess I'm starting to think that maybe what I did is okay." When I asked why he thinks this, he referred to "the determinant equation": "that determinant is zero, so for all values of k, all real values of k, there's no solution where [the vectors] aren't linearly dependent." To answer my question about why he thinks "what [he] did is okay," P6's final reference was to the determinant being 0—not to his geometric interpretation. Despite his initial and spontaneous inclination to examine the geometry of the situation, his algebraic formulations were key to building his confidence in the conclusion he had reached.

5.5.3.2 Some students mobilized τ_{41} (P1, P4, P7*, P9, P10) or τ_{42} (P2, P4) to find values of k for which the vectors are linearly independent.

For one student (P2), the spontaneous reaction to Problem 5 was to activate τ_{42}, a LA1 technique for task t_4: to determine whether vectors are linearly independent. Other students (P1, P4, P7*, P9, P10) turned to a LA1 technique for this task (either τ_{41} or τ_{42}) later in their approach to Problem 5. P7*'s initial reaction was to give a geometric interpretation of three vectors in $\mathbb{R}^3$ being linearly independent: he said $\mathrm{v}_1 + \mathrm{v}_2$ "should not be on the same line" as v_3, that the sum of two vectors should not be parallel to the third vector. After this initial reaction, P7* activated τ_{41}; I identify four stages in P7*'s mobilization of τ_{41} and note them as steps 2, 3, 4, and 6 of his engagement with Problem 5. P1, P4, P9, and P10's initial reactions had to do with the volume of the parallelepiped: for P4, the volume being 0 meant the vectors are on the same plane, but he did not make any inference from this about the vectors' linear independence; P1 and P10 had brought up the formula for the volume of a parallelepiped and said the expression must equal 0, but did not activate this any further; and P9 had also brought this formula up, and after calculating the volume using this formula, concluded the volume is 0 for any k. For

P4, the next approach to Problem 5 was to activate τ_{42}; τ_{41} was P1 and P10's second approach as well. P9 brought up τ_{41} as a third approach; his second was to check, by inspection, whether one vector is a linear combination of the others for particular values of k. Among his observations was that the vectors are linearly independent when $k > 1$. He activated τ_{41} when I asked how he knew this.

Below, I first describe P2 and P4's engagement with τ_{42}, then P1, P4, P7*, P9, and P10's engagement with τ_{41}. Throughout, I will attend to their activity as well as to the comments they made to determine the theoretical blocks that had prompted these participants to activate τ_{41} and/or τ_{42}.

5.5.3.2.1 Students who mobilized τ_{42} were able to use it to complete the task (P2, P4). Upon reading Problem 5, P2 said he "forgot if" a "determinant" being zero means vectors "are dependent." For him, the question of vectors being linearly independent had "something to do with determinants so [...] the way to do [the problem] [was] to write the vectors in matrix form" and then to "find the determinant of this" matrix. He then calculated the determinant and found it to be 0. Since he had found that "the determinant equals 0," he wasn't "sure if [he] did something wrong." After a pause, he proposed an explanation: "oh, it's right. Equals 0. So for all values of k, the vectors are linearly independent." But he wasn't "sure": "I'm not sure if when the determinant equals zero, they are dependent or independent. I forgot which one. If for determinant equals zero, they are independent, I would [say that] for any value of k, the vectors are independent. And [if] for determinant equal zero they are dependent, I would [say] there are no values [of] k" for which the vectors are linearly independent.

For P2, linear independence "[had] something to do with determinants" and the matter of whether a determinant is 0 or non-zero would determine whether vectors are linearly dependent or independent. But he forgot which result (zero, non-zero) implies which conclusion (dependent, independent). Since he was stuck, I pointed out that he hadn't used the information from the first part of the problem statement—the volume of the parallelepiped formed by the vectors is 0. For P2, this meant the vectors are "coplan[ar]." But again, he couldn't remember which of the two possible conclusions (dependent or independent) is implied by this result: "[I] imagine that if they are coplanar, they are on the same plane. [This] means one cannot depend on the other, because... Imagining the 3D world, if they are in the same plane [pause] I forgot if they are independent or dependent."

In short, P2 knew linear (in)dependence had "something to do with determinants" and with the question of whether a vector "depend[s] on the other," but did not know, between linear dependence and independence, which was which. I therefore offered to show him a definition of linear independence (the definition to be taught in LA1: a finite set of vectors is said to be linearly independent if none of the vectors in the set can be expressed as a linear combination of the others). P2 read the definition and concluded the vectors are dependent because, if they are on the same plane, "they can be expressed as a combination of the others."

P4 knew, by the time he mobilized τ_{42}, that the vectors are "on the same plane." Upon reading Problem 5, he said "cross product of three" ("I directly think of that") but, before mobilizing this, he first gave his geometric interpretation of the volume of the parallelepiped being 0: “of volume zero, okay [laughs]. And now everything’s shattered [laughs]. Of volume zero. Ah okay, so they’re all on the same plane.” For P4, the vectors being “on the same plane” “mean[t] that [he will] use a scalar triple product.” He decided to find “the scalar triple product such that it is not zero, and then k would be anything other than” whatever would yield a scalar triple product that is 0. P4 looked up an expression for the scalar triple product on the internet (“I forget, the scalar triple product was the determinant of... putting them on top of each other? [...] Ah, yeah yeah, correct. [laughs] It’s the... the cross product of two times the dot product of the other. But you can also do it as the determinant of [...]”) and decided to calculate the scalar triple product “the determinant way.” P4 found the scalar triple product to be zero:

> Okay, that’s weird. Because the scalar triple product, no matter what k is, is zero. Which is weird to me, because I would expect it to be... a number. I don’t know. Scalar triple product. Yeah, so I - I would probably do... Determinant of this such that it is not zero, and then solve - solve for k, but k is gonna cancel out. So does that mean k belongs to $\mathbb{R}$? Because any va—no [pause] so no matter what value I would put in, I’m gonna have zero is equal to zero. That makes things a bit complicated.

Like P2, P4 knew that whether the vectors are linearly independent or not depends on whether the scalar triple product is zero or non-zero. P2 did not remember which result leads to which conclusion but knew that the scalar triple product being zero meant that, *for all k*, the vectors are linearly (in)dependent. P4 wanted to find the values of k “such that [the scalar triple product] is not zero” and so intended to solve an equation of the form “an expression in k equals 0” but had found that “no matter what value [he] would put in, [he would get that] zero is equal to zero.” For P4, this result was a sign to reconsider the suitability of the knowledge he had activated: “let me make sure now that this truly means that I should use a scalar triple product.”

P4 then revisited his geometric interpretation and concluded, again, the vectors “must be on the same plane.” To him, this “justifie[d] [his] use for the scalar triple product” yet again. (I specify that he concluded this “yet again” because he had made a similar comment the first time he observed the vectors are “on the same plane”: “[this] means that I’m gonna use a scalar triple product.”) P4’s comments help to explain what the “justif[ication]” was for him (that is, his comments bring to light the theoretical block that was producing P4’s activity in Problem 5): “directly, when I think of vectors on the same plane, I think of two things, the scalar triple product or linear dependence.” P4’s justification for bringing up the scalar triple product is that he knows there is a relation between vectors being “on the same plane,” the scalar triple product of the vectors, and the linear dependence of these vectors. He “knows that [if] the scalar triple product [is] not zero, it means [the vectors are] not on the same plane. If it is zero, that means that they are in the same plane and the linear combination... So the linear dependence means they are on the same plane.” That “the scalar triple product is zero” implies “they lie on the same plane” is something P4 knows “as a fact, [from] studying.” And “as for the linear dependence,” he “remember[s] [that] in class, [they] related the scalar product to the linear dependence part.” He also had a geometric interpretation of these notions: “to visualize linear dependence of vectors, [he] would visualize them being on the same plane. That’s why [he] connects those two together.” This visualization came from “lectures”:

the "professor showed pictures of vectors being on the same plane or not being on the same plane; dependence, independence."

P4 had only a surface-level grasp of the knowledge of a relation between a scalar triple product of vectors, whether the vectors are coplanar, and whether they are linearly independent. He did not mobilize the *geometric* knowledge that relates the scalar triple product of three vectors to their linear (in)dependence: the absolute value of this scalar triple product is the volume of the parallelepiped they form, and this volume (being 0 or not) indicates whether the vectors are coplanar, and therefore whether they are linearly dependent. It's possible P4 knew the volume is the absolute value of the scalar triple product (as a formula among others from LA1), but he did not mobilize it to progress in his path toward examining the linear independence of vectors (that form a parallelepiped of volume 0). I infer this from P4's activity: after he inferred that the vectors are "on the same plane" because they form a prism of volume 0 (as per pre-LA1 formulaic and geometric knowledge about prisms: "there's no width, there's no height - that means they're all in the same plane"), he proceeded to calculate their scalar triple product to determine the value(s) of k for which it is (not) zero. And the comments P4 made about τ_{41} (which I discuss in 5.5.3.2.2) suggest he did not have the *algebraic* knowledge that relates the scalar triple product of three vectors to their linear independence: the discourse that produces both τ_{41} and τ_{42}.

In P4's second return to the scalar triple product, he wanted to check whether he could get a non-zero value for the scalar triple product: he knew that for values of k for which the scalar triple product isn't 0, the vectors would be linearly independent ("that's how I would expect it to be"). Given that he got "0 is equal to 0," he said he "need[s] to do some logic to decide" whether the vectors are "always on the same plane or never on the same plane." He eventually concluded that, in the current case, the vectors "are on the same plane": "no matter what values k are, I would always get zero, which would imply that no matter what k is, they're always on the same plane." He paused and concluded this "mean[s] that there are no values of k for which" the vectors are linearly independent. He chuckled as said he'd have "like[d] it if it was an 'if any' question."

P4 brought up the vectors being on the same plane on several occasions, but it was only when he activated this in conjunction with τ_{42} (calculate the scalar triple product, check whether it's zero or non-zero) that he completed the given task. Upon reading Problem 5, he knew that since the vectors form a parallelepiped of volume 0, they must be on the same plane "because there's no width, there's no height." But what he had thought of "directly" upon reading the problem was the scalar triple product. He knew there is a relation between vectors being on the same plane, their scalar triple product (being zero or non-zero), and their linear (in)dependence. He knew, "as a fact, [from] studying," that a zero scalar triple product implies vectors "lie on the same plane," and he recalls his "professor show[ing] pictures of vectors being on the same plane or not" in the context of the question of linear dependence. But P4 did not mobilize this knowledge as a direct means to complete the task. Instead, the normative LA1 knowledge P4 was activating had him calculate a scalar triple product (to determine the linear independence of given vectors) in spite of the observation he'd already made about the vectors lying "on the same plane." For P4, the vectors being on the same plane was not, initially, a justification for the vectors being linearly dependent. Instead, *twice* in a row, it was a justification for calculating a scalar triple product. For P4, the conceptual association between "scalar triple product," "linear dependence of vectors," and "coplanar vectors" was primarily a guide toward which algebraic technique to activate.

P2 and P4's mobilization of τ_{42} and comments they made as they activated it reveal the effects LA1 norms, relative to constructs with algebraic and geometric representations (be it the notion of linear independence, of a parallelepiped, or of a scalar triple product), can have on students' practices. Students may know *of* a relation between terms (e.g., linear dependence, scalar triple product, coplanar) and, given a task and an interviewer's prompt designed to encourage them to act on this relation, *may* mobilize conceptual relations and geometric representations: P2 did so (and was able to reach a conclusion about linear independence once he'd reviewed the definition) but P4 did not. Additionally, from P2 and P4's activity and related comments, it follows that the practice they developed in LA1 (and, likely, prerequisite mathematics courses) privileges calculations over activating geometric representations of concepts. Even though he knew the geometric representation of linearly dependent vectors (they are coplanar), P4 did not mobilize it to do the task. And P2 was able to perform the task by activating this representation but only once I had prompted him to do so; further, he did not believe he would get full marks if he submitted such an "analysis" for grades but did think his calculations would award him full marks, and, further yet again, he said this "analysis" would not have convinced him if he had been working on his own ("usually, I do calculations").

5.5.3.2.2 Students were unable to complete the task via τ_{41} (P1, P4, P7*, P9, P10). None of the participants activated τ_{41} in their initial reaction to Problem 5. P1 and P10 turned to it after they got stuck—they had initially said that since the volume of the parallelepiped is 0, the determinant of the matrix (where R_i is v_i) is 0, but they did not activate this any further. P7* activated τ_{41} in steps 2, 3, 4, and 6 of his engagement with Problem 5. P9 turned to τ_{41} as a third step in his engagement with Problem 5: initially, he had activated τ_6 (use the LA1 formula for the volume of a parallelepiped to calculate the volume) and had concluded the volume is 0 for any k, but had made no conclusions about the linear independence of the vectors. When I prompted him to address the vectors' linear independence, he turned to checking, by inspection, if one vector is a linear combination of the others. Among his observations was that if $k > 1$, the vectors are linearly independent. I asked P9 how he knew this and that's when he activated τ_{41}. P4, finally, turned to τ_{41} after he had already concluded, from τ_{42}, that the vectors are linearly dependent for all k.

P1 and P10's mobilization of τ_{41} involved a matrix that loosely reflected the matrix that should have been activated in this technique. I remind the reader of the discourse that produces τ_{41}, as it would apply to Problem 5: the given vectors are linearly independent if and only if the only values of a, b, c for which

$$a(-k, 1, 1) + b(-1, 1, k) + c(1, 0, 1) = (0, 0, 0) \tag{5.5}$$

holds are $a = b = c = 0$. Equating the corresponding components produces a linear system of three equations in the unknowns a, b, and c. The augmented matrix of this system has, as column C_1, the vector $(-k, 1, 1)$ (which I denote by v_1), vector $(-1, 1, k)$ (v_2) as C_2, $(1, 0, 1)$ (v_3) as C_3, and the zero vector as its column of constants. If the reduced row echelon form of this matrix has the form $[I_3|0]$, then the system has a unique solution

($a = b = c = 0$) and the vectors are linearly independent. Otherwise, the system has infinitely many solutions; specifically, it includes non-trivial solutions and so the vectors can be expressed as linear combinations of one another.

P1 and P10 mobilized a matrix that is superficially similar to the one appropriate for τ_{41}: P1's matrix was augmented, the entries in the right-most column all 0, but the vectors corresponded to the rows of the matrix rather than its columns. P10's matrix was not augmented; it was a 3×3 matrix in which R_i was v_i. Perhaps P1 and P10's matrices were mimickeries of the matrix in the formula for the volume of a parallelepiped, which they had activated in their initial response to Problem 5: that matrix has v_i as R_i.

In addition to his matrix, P1 also wrote the equation $-kk_4 + k_4 + k_4 = 0$. This equation incorrectly reflects the equation that produces τ_{41}. The components of the vector $(-k, 1, 1)$ are coefficients of some unknown (k_4) and the equation is homogeneous.

Neither P1 nor P10 mobilized their matrices any further at this stage. P10 was "not sure what to do to go about the problem" and did not "know if [she] should just solve the matrix"; when I asked her to clarify what she meant, she said: "solving as in [finding its] reduced echelon form." I asked what she thought would happen if she did that, and she said she "[doesn't] think that would work." She did not mobilize τ_{41} beyond her suggestion to reduce the matrix where R_i is v_i. P1, for his part, asked for a definition of linear independence (I gave the definition on a piece of paper: a finite set of vectors is said to be linearly independent if none of the vectors in the set can be expressed as a linear combination of the others) and proceeded to check, by inspection, whether two of the vectors could combine to generate the third.

P1 returned to τ_{41} after he got stuck checking by inspection whether some of the vectors are linear combinations of the others. He wondered about the case $k = 0$; when I asked about the case that $k \neq 0$, he jumped ship and revisited τ_{41}: "I'm just going to try to solve this." He proceeded to reduce the augmented matrix he had previously written. Eventually, he produced a matrix in which rows 1 and 2 were identical and concluded "it's clear [there are] infinitely many solutions for this problem, because you have the same row here and here." I asked P1 what this tells him. He said that "for linear independence [there are] probably multiple cases." But then, he paused. He was "confused with the problem." He did not resolve this confusion and changed course again: he mobilized the formula for the volume of a parallelepiped with the intention of finding the value(s) of k for which that volume is 0 (as discussed in Section 5.5.3.1.2). P1 made no further comments about τ_{41}.

P4 and P9 mobilized matrices that were appropriate for τ_{41}. Prior to mobilizing τ_{41}, P9 had been checking by inspection whether one vector could be written as a linear combination of the others. He claimed the vectors are linearly independent when $k > 1$ and I asked how he knew this. That's when P9 activated τ_{41}. He did so to examine whether the vectors are linearly independent if $k = 5$: his matrix had vector v_i (with $k = 5$) as column i. He row-reduced this matrix and found a row made up entirely of 0's. He concluded that, in this case, the vectors are linearly dependent. Why? "When you row-reduce, you should have a diagonal with values and then the rest is zero." I presume he meant that when vectors are linearly independent, "when you row-reduce," you get a diagonal matrix (where no row is entirely made up of 0's). He then wrote the matrix corresponding to the case in which $k = 0$ and, without reducing the matrix, said the vectors are linearly independent in this case.

Before P9 proceeded to the case $k = 0$, I had asked questions to gauge whether P9 had the discourse that produces τ_{41}. Did he have the knowledge that relates linear independence to τ_{41}, or was he activating the rule "a row of zero in an augmented matrix implies there are infinitely many solutions," and another rule, "infinitely many solutions means vectors are linearly dependent"? P9 knew a row of 0's implies vectors "are linearly dependent because you can write one vector in terms of the others." I asked P9 how this works. He "forgot how to explain it"; "in this case, you can write the third, the last one, in terms of..." He paused and eventually said he "forgot about this one." After P9 said the vectors are linearly independent when $k = 0$, I asked how he knows this. "Oh no. I forgot linearly independent - how I used to do it. I wish to reduce. If I have a diagonal, then it is linearly independent." He remembered "the first example": "1, 0, 0, and then 0, 1, 0, and 0, 0, 1" (the 3×3 identity matrix). He said "these three vectors are linearly independent" and continued: "but how do I do it? I don't remember." Ultimately, P9 did not explain how he knew that the case "infinitely many solutions" means that "you can write one vector in terms of the others." His last attempt to do so was this explanation:

> When you have three linearly independent vectors, when you put them in a matrix, to see what their span is, you want to see that the only thing that this span contains is 000. Therefore, in this case, I don't have it: because [of the] last row of zeros, I have infinitely many solutions. Therefore it is not independent.

P4 activated τ_{41} in his last attempted engagement with Problem 5. He had already concluded, after mobilizing τ_{42}, that the vectors are not linearly independent for any k. After he had done this, he decided to check, by inspection, whether the vectors can be expressed as linear combinations of one another. He started with the case in which $k = 1$, observed two vectors are identical, briefly thought this would always be the case no matter the value of k, then realized it wasn't so in the case $k = 5$. After writing out the matrix corresponding to the case in which $k = 5$, he said that "because [he] know[s] that it could also relate to linear independence, [he's going to] try to solve it as [$A\mathrm{x} = \mathrm{b}$] with the b column equal to zero." So he decided to "augment the matrix with the b column [equal to] zero." Which matrix? P4: "Sorry. I'm going to make a linear combination, sorry." This comment brings P4's engagement with τ_{41} closer to the discourse that produces it than other participants' mobilization of the technique; none of the participants demonstrated discourse any more explicit than P4's claim of "mak[ing] a linear combination."

The matrix P4 wrote had vector v_i as C_i and the zero vector as its right-most vector. He did a few row operations, including one in which he multiplied a row by $\frac{1}{k}$. (He did not address the possibility that $k = 0$.) He then decided against this last operation and stopped this activity: "No! I don't know." P4 had "difficulty doing the augmented matrix with the k."

P4's goal in activating this technique, which I recognize as the LA1 technique τ_{41}, was to "try to solve for k"; he "expect[ed] there to be no solution, [based on his] previous analysis." He knew that "if there is no solution, that means that there is never a k for which they are linearly independent. They're always dependent. They're always on the same plane." He knew that if there is only "a trivial solution for the linear combination, it means no coefficient will [make it so] they are not on the same plane, they're not independent." This explanation, together with P4's previous claim of "mak[ing] a linear combination," suggests P4 knew of the left side of equation (5.5) but not of its right side; he knew that the linear independence of vectors has something to do with the linear combinations of these vectors, but he did not have the knowledge that completes the link between "linear independence" and "linear combinations." He tried an example: "let's say I had a solution with, for example, 2 1 3, unique solution, but that means there is a solution for them being independent. But if there is no solution for k, that means there is no value for which I can make them..." By "solution," P4 meant "a solution of the system $A\mathrm{x} = \mathrm{b}$," but he could not explain where this equation came from. (In other words: still, P4 made no reference to equation 5.5.)

P4 "[could not] explain [where the equation $A\mathrm{x} = \mathrm{b}$ comes from] because [he hadn't] grasped the concept of $A\mathrm{x} = \mathrm{b}$ enough to be able to explain [this]; [he] just [knew] that when [he's] solving an augmented matrix like [the one had written], [he's] tackling [an equation of the form $A\mathrm{x} = \mathrm{b}$]." When I asked P4 a last time what the connection was between his augmented matrix and linear independence, P4 seemed to understand my question: "between solving this and linear independence? I just know that when I want to find the linear combination of a system, I want to solve the vectors as columns and... with the zero matrix." He said he "think[s] there's something [he] lack[ed] in the course; [he] didn't really understand why [he was] doing [what he was doing]. [He] would just categorize the problems as much as possible." (Problems from "past exams," for instance.)

P4 said that if he had to submit something for Problem 5 to get grades, one of his methods would have sufficed. He knew it would suffice to explain that "for any value of k in $\mathbb{R}$, the scalar triple product is always zero, which means that the system is always linearly dependent." But, "to prove it to [himself]," to convince himself the vectors are dependent no matter what, he would want to see τ_{41} all the way through. He would "expect for there to be no solution to this system. And if there isn't, then [he] would say that there's no value of k. If there was a solution, [he] would be a bit confused, because it wouldn't work with [his previous] hypothesis."

Like P1, P4, P9, and P10, P7*'s mobilization of τ_{41} was rickety. Initially, P7* said the vectors would "form the kernel of $\mathbb{R}^3$."[19] Prompted to clarify what he means, P7* said the "algebraic approach" would be to "use these three vectors as the column vectors," wrote the matrix A where column i is vector v_i, and wrote

$$A\mathrm{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

P7* did not attempt to solve the equation but did describe his expectations:

[19]The term "kernel" is not used in LA1 nor in its co- and pre-requisite courses; I distinguish P7* from the other participants because of his previous educational experience, which includes higher-level mathematics courses (including courses in analysis, advanced algebra, ordinary and partial differential equations, abstract algebra, and advanced geometry) taken in an unfinished degree in applied mathematics.

> Since they are linearly independent, this linear system will have only the trivial solution, because they cannot be expressed by one combination, so if I do Gaussian elimination on the augmented matrix, I will see that the last row would become something like 0 0 0, and 0. But here, there's something with k. There will be a formula in k. For example, $-2+k$. And to make [pause] no, wait a minute, wait a minute, oh! This will not work because it will have no solution. Yes, no solution. No solution; no solution means that this one is not going to be zero. But here, this one is zero, just give me a second, let me fix it.

P7*'s next move clarifies what issue P7* perceived to need "fixing." He reduced the *coefficient* matrix to the following:

$$\begin{pmatrix} 1 & k & 1 \\ 0 & 1-k & -1 \\ 0 & k^2-1 & k+1 \end{pmatrix}$$

What he wanted, here, was for k^2-1 to be zero and for $k+1$ to be non-zero: "we want to make sure that the last row becomes something that's 0, 0, and then something not 0." He expected to find the condition(s) (on k) such that the coefficient matrix would reduce to "a beautiful identity matrix." This would ensure the vectors are linearly independent. Prompted to find the values of k for which the reduced row echelon form would be an identity matrix, P7* proceeded by focusing on the third row of the matrix. He found the third row could reduce to $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ if $k=1$ and deduced that in this case, the matrix would reduce to the identity matrix. I pointed out to P7* that rows 2 and 3 of the matrix would be $\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 2 \end{pmatrix}$, respectively, if $k=1$. P7* paused and concluded: "Perfect. Yeah. So I only need one row to tell us it's going to be linearly independent, but I have two rows; even better!"

P7*'s certainty was temporary; he knew that to solve a linear system, the coefficient matrix does not suffice. The augmented matrix is necessary:

> Wait a minute [pause] no, no, I think I need to fix that. So, actually, I need to use the augmented matrix, I need to put another column here to figure out if there is a linear combination between these three matrices [columns], I cannot just do the massage on this matrix. Yes. But the final answer will be very close to [the one obtained by reducing the coefficient matrix], I just need to add an all-zero column on the right side [to the matrix found previously] because it's a homogeneous system.

P7* produced this augmented matrix, paused, and said: "that's the thing I have just figured out, it will be all 0's here, I cannot use this one to directly figure out if it's linear dependent or not. I cannot have something like 0 0 0 and a number, right?" He wrote the row 0 0 0 1 underneath the matrix.

At this point, P7* abandoned τ_{41} ("let's just forget about this matrix") and opted to revive his initial approach, wherein he sought to find values of k for which the sum of two vectors would be a scalar multiple of a third vector. This led P7* to conclude the vectors are linearly dependent when $k=2$ and linearly independent otherwise.

I asked P7* what would give him the most marks if he had this question on an exam (his last approach, or what he had done with the matrices?). His answer circled the reason for which he struggled to mobilize τ_{41}:

> I would definitely use [the approach wherein the vectors were found to be linearly independent when $k \neq 2$]. Gaussian elimination is always easy to do, but you have to understand its structural meanings before going through it, otherwise, it's going to just be nonsense."

P7*'s difficulty mobilizing τ_{41} suggests he did not know why the technique works—why a coefficient matrix having an identity matrix as its reduced row echelon form means the columns are linearly independent vectors. P7* used relevant words to describe the goal of the technique ("we need to put [a zero column] here to figure out if there is a linear combination between the [three columns]") but the inappropriate use of the qualifier ("if there is" a linear combination of the vectors), together with how he mobilized τ_{41}, point to a weak grasp of the mathematical theoretical knowledge underlying the technique.

P7* continued: "we want to create a row that looks like 0 0 0 1. However, in a homogeneous system, the very right column is going to be all zero entries." In the absence of knowledge that ties the objective of τ_{41} to the definition of linear independence, P7* activated rules about what the matrix or its rows ought to look like. Earlier, when he worked the coefficient matrix, he knew he wanted it to reduce to an identity matrix. Now, he wanted to "create a row that looks like 0 0 0 1." His knowledge of τ_{41} depended on surface-level features of matrices and he struggled to mobilize their "structural meanings."

P7*'s last attempt to mobilize τ_{41} came after he mentioned his "0 0 0 1" objective. He suddenly switched goals: "if I want to use Gaussian elimination to check linear dependence, I need this matrix to be consistent." For P7*, his previous finding, wherein a reduced form of the coefficient matrix, with $k = 1$, had the rows $\begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 2 \end{pmatrix}$, was of an "inconsistent" matrix. For P7*, these two rows meant the "third variable [in the system presumably represented by his initial augmented matrix] won't be unique" because its value is both -1 and 2. After P7* said this, he caught and corrected himself, having seemingly remembered that -1 and 2, here, are coefficients of that variable, and that the augmented matrix corresponded to a homogeneous linear system: "oh! It's zero. Yes, yes. Sorry. So these two [rows] both give us that the third variable is zero." He then (correctly) deduced the second variable would be a free variable, so "the answer is not unique but [the vectors are] linearly dependent because [the system] is consistent."

Finally, P7* checked what the augmented matrix would look like if $k = 1$ and found that one row would be made up entirely of 0's. 0, 0, 0, 0. "This is something that I love, right? The last row becomes all zero entries. So for sure, the third variable becomes the free variable, and there are infinitely many solutions, but it is still linearly dependent." (P7* had previously concluded the vectors are linearly independent whenever $k \neq 2$.) P7* then made more computations to examine the matrix in the case that $k \neq -1, 1$. Throughout, he was concerned with "making sure there are solutions," "making sure the matrix is consistent." No matter what, the system would be consistent; it is homogeneous.

From the rule *if the augmented matrix is consistent then the column vectors are linearly dependent*, P7* concluded the vectors are linearly dependent no matter the value of k. This contradicted his previous finding that the vectors are linearly independent when $k \neq 2$: " I'm not sure. I am not sure where this went wrong, I have these conflicting answers. But I think I have done it correctly. I think this one is definitely correct. Things must happen. Mistakes must happen in that proportional thing [checking whether $v_1 + v_3$ is proportional to v_2]."

P7*'s struggle was marked by a lack of knowledge that relates the linear system captured by his augmented matrix and the definition of linear independence. Without this theoretical knowledge (theoretical in the ATD sense of "theory that produces and justifies technique"), P7*'s technique was instead driven by rules about surface-level properties of a matrix whose columns are the vectors in question: if it reduces to an identity matrix, then the vectors are linearly (in)dependent; if it reduces to a matrix with a row of 0's, then the vectors are linearly (in)dependent; if the "matrix" is consistent (that is, if it does not reduce to a matrix with a row of type $0\ 0\ \cdots\ 0 \mid t$, where $t \neq 0$), then the vectors are linearly independent.

P6 did not mobilize τ_{41} but did mention it toward the end of his engagement with Problem 5. P6's attempt to explain why τ_{41} works helps to highlight the difficulty underlying participants' mobilization of τ_{41}. P6 had established the vectors are linearly dependent for any k but fussed over whether his approach constituted a proof of the matter. Once he'd established that it did, he continued: "while I was calculating the determinant, I noticed that if I was going to do just linear independence, then I would just start off doing the row reduction." He knew that the only difference between row-reducing a matrix and calculating its determinant "is that he could do column operations." He did not know, off the top of his head, why row reduction would work to check linear independence (I asked): "I'm not sure honestly, that's just what I was taught." But he continued:

> Why would that work? I can try to think about it for a second [pause] Why would row reduction work? [pause] Because row reduction is basically just linear combination. Right? So what you're doing is you're taking the three vectors or whatever. And you're just adding a scalar and then subtracting them or adding them to each other. *You're trying to figure out a way to make sure that... There's some, some, some combination where, you know, if I can get it to where it's, you know, is it... Gauss-Jordan is [for finding] the RREF? So if I get into Gauss-Jordan [if the RREF is I], then obviously, there are no, there are no scalars that I can multiply these by where they would all equal 0.* [Emphasis added.]

P6's description of τ_{41} brings to mind P7*'s: "we need to use the augmented matrix [...] to figure out if there is a linear combination between these three [columns]." The notion of linear combination appears in both P6 and P7*'s description of the technique, but only P6 managed to make a statement that nears the relevant one: the goal of τ_{41} is to examine, somehow, linear combinations of the vectors that produce the zero vector.

Part of the struggle may well be inherent to the mathematics at stake: qualifiers are at the core of the question targeted by τ_{41}: "do there exist values of a, b, c, different from

0, such that $a\mathrm{v}_1 + b\mathrm{v}_2 + c\mathrm{v}_3 = (0, 0, 0)$?" Logical quantifiers are known to pose difficulties to students even in advanced university mathematics courses (Chellougui, 2009; Harel & Sowder, 2007; J. Selden & Selden, 1995). Apart from the lack of institutional *need* to know why τ_{41} works, the difficulty inherent to the mathematics at stake may underlie P6's struggle in his attempt to explain τ_{41}. He knew that row-reducing the augmented matrix at stake in τ_{41} corresponds to finding "scalars, a, b, c, such that they [sic] are equal to some matrix—some other vector, $\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3$. When we're checking for linear independence, they're all equal to zero." But he struggled to explain τ_{41} further: his explanation was that if the coefficient matrix reduced to an identity matrix, then "there are no scalars that I can multiply [the vectors] by where they would all equal 0." It's possible P6 struggled to mobilize logical quantifiers. None of the other participants who used τ_{41} mobilized a description as close to the relevant one.

The LA1 norm relative to τ_{41} has students produce a matrix made up of given vectors, reduce the matrix, and make an appropriate deduction from a row echelon form of the matrix to conclude whether given vectors are linearly (in)dependent. The norm does not require students to justify the matrix they produce. Students do not need to explain the relevance of their matrix to the task of determining if given vectors are linearly independent. The norm also does not require students to justify the conclusion they reach (about the linear independence of given vectors). This norm allows students to activate only surface-level features of τ_{41}. For P1 and P10, this meant that a few weeks after the end of the semester in which they passed LA1, they produced a matrix similar to the appropriate one on a surface-level, but inappropriate for the given task. They knew their matrix ought to be made up of the vectors, but did not know the vectors ought to make up the columns of the matrix, rather than its rows. P4 and P9 did position their vectors appropriately. But P9's interpretation of the row echelon form he had found was dubious: he was activating the rule "row of 0's" means "infinitely many solutions," but could not justify this rule. P4 had "difficulty with doing the augmented matrix with the k," and his description of what he hoped to find ("no solution to this system") was inappropriate (the system was homogeneous - it would have a solution, no matter what). For P4 and P9, the LA1 norm was insufficient for them to apply τ_{41} productively to a task where the vectors differed from the norm: vectors in $\mathbb{R}^3$ with an unknown component (k), as opposed to vectors in $\mathbb{R}^3$ with components that are known single-digit integers. P6 did not mobilize τ_{41} but did mention the technique and attempt to explain why it is valid for checking linear independence: the difficulty he seemed to have in putting things to word suggests that part of the difficulty at stake here has to do with mobilizing logical quantifiers, a concept core to the theory that produces τ_{41}.

In Section 5.5.3.3.3, I look to an element of P1, P4, and P9's activity (as they engaged with Problem 5) that might have prompted them (but did not) to mobilize the definition of linear dependence to apply τ_{41} in an accurate and productive way to perform the task in Problem 5. Prior to mobilizing τ_{41}, P1, P4, and P9 had all decided to check, by inspection, whether one of the vectors was a linear combination of the others. After I take this part of P1, P4, and P9's activity into account, I re-examine (in Section 5.5.3.3.3) how the LA1 norm relative to τ_{41} supported and inhibited P1, P4, and P9's activity as they engaged with Problem 5.

5.5.3.3 Some students inspected the vectors to check if they're linearly dependent (P1, P4, P5, P8, P9, P10).

This was either a second, third, or fifth approach for the participants who activated the definition of linear independence. They tried to activate the definition of linear independence directly by inspecting, either for specific values of k or in general, whether one of the given vectors was a linear combination of the other two. Below, I start by describing the context in which each participant opted to activate the definition of linear independence as an approach for Problem 5. I then describe what students did as they activated it, and I conclude by addressing what students' follow-up activity was after they stopped inspecting the vectors: I do this to examine whether students made anything of the definition of linear independence beyond a direct or case-specific application (such as considering the three vectors in the case that $k = 1$).

5.5.3.3.1 For most students (P1, P4, P8, P9, P10), inspecting whether one of the vectors was a linear combination of the others was a salve for their uncertainty and perhaps a way to get insight into the given vectors or into the problem. P5 was an exception: inspecting whether one of the vectors was a linear combination of the others was not a salve for her uncertainty and was not about gaining insight into the problem. It was the only part of P5's activity of which she seemed certain and built on what she had found in her earlier engagement with the problem. Unlike P1, P4, P8, P9, and P10, P5 could not propose any other knowledge for examining the linear independence of the given vectors.

P5's initial and spontaneous technique in Problem 5 was to take the dot product of each pair of the given vectors and set each equal to zero (she had thought, initially, that the edges forming a parallelepiped could not be orthogonal, and I had shown her a kleenex box to show this was false; her follow-up reaction was that she could assume the vectors are orthogonal). As P5 continued, she eventually also set the pairs of dot products equal to each other. From her manipulations, P5 found various values for k: $k = 1, k = 0, k = -2$. This made her think of "things [she] kn[e]w": having found three values for k "remind[ed]" her of "how a three by three matrix would have three eigenvalues." At this point, I interjected to remind P5 the goal of the task was to find values of k for which the vectors are linearly independent. P5 then used the values she had found for k, along with the definition of linear independence, to check, by inspection, whether one of the vectors was a linear combination of the others.

It's not clear if P5's goal, from the get-go, was to identify values of k and then plug them into the vectors to check, case by case, whether the vectors are linearly independent. P5's activity and comments seemed to have no relation to the task, and my intervention (reminding her of the task), after P5 had come to associate k with the values $1, 0$, and -2, led to P5 plugging these values in to the vectors. It's not clear whether P5 would have done this without my intervention, but what is clear is that this is the only knowledge P5 activated in relation to the notion of linear independence. When I asked P5 if she'd do anything differently on an exam, she said she couldn't think of anything else. I asked if she would get full marks for her work: "I'm not confident in my answer. So probably not." I asked what she thought was missing and she said she "[doesn't] think [she's] necessarily missing anything, but" she had "started off by assuming the parallelepiped [had edges that were at 90 degree angles with respect to one another]." She didn't "think [her] approach [was] necessarily 100% correct."

What set P5's approach apart from that of the rest was that, for her, checking the linear independence of the vectors by inspection built on what she had previously activated, and what she had previously activated had no relevance to the task. She had found values for k and decided to check, by inspection, whether the vectors are linearly independent in those cases. And she did not activate any other notion or technique relevant to the given task. Other participants' trajectories prior to activating the definition of linear independence suggests it as a salve for uncertainty they had relative to a previous approach (sometimes spontaneously, sometimes in response to a question I'd asked, usually to bring their attention back to the task: identifying values of k for which the vectors are linearly independent).

P8 and P9 brought this definition up as a second step in their engagement with Problem 5; their first was to use a formula to calculate the volume of the parallelepiped formed by the vectors, but they had found this to equal 0 and did not use this to consider the linear independence of the vectors. P8 had gotten stuck when she found the volume to be 0, so I asked if she had thought of a way to do the problem without solving the equation "scalar triple product equals 0." P9 had said the volume is 0 for any k and declared that was "[his] answer." Since he hadn't addressed the question (for what values of k are the vectors linearly independent?), I prompted him to answer it. These were the interventions that had prompted P8 and P9 to activate the definition of linear independence.

For P1 and P10, the definition of linear independence came up in the third step of their engagement with Problem 5. Both had begun the problem by observing that the formula for the volume of a parallelepiped, here, would equal 0; they had then abandoned this observation and brought up (an incorrect version of) τ_{41} (a LA1 technique for determining whether vectors are linearly independent). After P1 had written his augmented matrix and an equation to which he related this matrix, he asked for the definition of linear independence. I gave him the LA1 definition on a piece of paper. After P10 had written her matrix and suggested to row-reduce it, she said she wasn't sure this approach would work. She was stuck; and so, I asked her what it would take for the three vectors to be linearly independent. These were the prompts that led P1 and P10 to address the definition of linear independence.

P4 decided to activate the definition of linear independence after he had already completed the task successfully (through τ_{42}). Even though he had completed the task, the comments he made at the end of his engagement with Problem 5 show he was not convinced by what he had found: to convince himself that the vectors are linearly dependent no matter the value of k, P4 wanted to activate τ_{41} to completion. τ_{41} was the technique P4 activated *after* he checked, by inspection, whether one of the vectors could be expressed as a linear combination of the others (in the cases $k = 1$ and $k = 5$). P4 had activated this latter approach immediately after he had completed the task successfully via τ_{42}. I therefore surmise that when P4 activated the definition of linear independence, he, like P1, P9, and P10, was attempting to salve uncertainty relative to his mobilization, toward Problem 5, of LA1 techniques associated with the notions of volumes and linear independence.

5.5.3.3.2 Participants checked by inspection whether one vector was a linear combination of the others, sometimes without assigning a value to k, and otherwise assigning small integer values (mostly 0, 1). P10 was an exception; she knew that for vectors to be linearly independent, they must not be linear combinations of one another. But she did not mobilize this.

Two participants (P1, P8) tried to check, by inspection, whether one of the vectors was a linear combination of the others without assigning a particular value to k. P8's attempt at this was explicitly written out on paper, whereas P1 only said he was trying to do this. P8 had initially tried to orally describe the notion of a vector being a linear combination of others, and when I asked her to clarify, she wrote out the equation

$$\begin{pmatrix} -k \\ 1 \\ 1 \end{pmatrix} = a \begin{pmatrix} -1 \\ 1 \\ k \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and then wrote that "$k = -a + b$ [sic]." She explained that, in general, she'd find a value for k using a and b. I asked how she'd do this, but her response targeted something different. She responded that it wouldn't be possible to write $(1, 0, 1)$ as a linear combination of the other two because of the 0; it would have to equal 1 plus a multiplied by 1, but that's not possible "unless everything is 0." P1 said he thinks no combination of $(-1, 1, k), (1, 0, 1)$ would give $(-k, 1, 1)$; when asked why, he said "oh no, never mind"—and proceeded to assign a value to k.

P1 tried to find pertinent linear combinations in the case $k = 0$; it's not clear which combinations P1 attempted as he did not describe them. He couldn't "see if [the vectors are] independent or not" and asked for a "practical example for linear independence." I wrote down the vectors $(1, 2)$ and $(3, 4)$ and said these vectors are linearly independent of each other because it's not possible to write one as a linear combination of the other; one is not a scalar multiple of the other. P1 returned to the case $k = 0$ and said he still wasn't sure if "zero works or not." (At this point, I asked: "what if k isn't 0?" P1 returned to τ_{41}, which he had started to activate before he saw the definition of linear independence.)

Like P1, P9 and P4 also assigned value(s) for k and checked, for these cases, if a vector is a linear combination of other vectors. P9 started with $k = 1$. He observed that, in this case, two of the vectors are equal. He said these vectors are "linearly dependent." One of them is "redundant." He then said, without justifying his claims, that the vectors are linearly dependent when $k = 0$ "and when k is equal to more than one, [the vectors are] linearly independent." I asked P9 how he knew that if $k > 1$, the vectors are linearly independent, and he proceeded to mobilize τ_{41} to the case in which $k = 5$. P4, like P9, also verified $k = 1$ first ("I picked $k = 1$ to try to compare"—he had already determined the vectors are linearly dependent for all k). He saw two of the vectors were identical when $k = 1$. At first, he thought this would be the case for any value of k, but he realized this might be a "coincidence" and decided to assign a different value ($k = 5$) to check;

he saw none of the vectors were equal to one another in this case.

Unlike P1, P9, and P4, whose choice of values to assign to k (0, 1, 5) seemed driven by a desire to be able to examine whether one vector was a linear combination of the others, P5's choice of values for k showed this step was a continuation of her previous engagement with Problem 5. In that previous activity, P5 had created equations in k (that were irrelevant to the task) and, upon solving them, found $-2, -1$, and 0 as values for k. These are the values she assigned to k when she checked whether one of the vectors is a linear combination of the others. She crossed out the option in which $k = 0$: she said these vectors are linearly dependent because two of them added up to give the third. She said the vectors are linearly *in*dependent when $k = 1$: in this case, the vectors were

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Unlike P4 and P9, P5 did not say anything about two of these vectors being identical. For the case in which $k = 2$, P5 wrote out the vectors and "did the math in [her] head": "for example, if I was taking the two... subtracting the second vector, and then adding the third vector, that would make this equal to two. But then, the second row wouldn't be equal to the one. So none of the combinations work." She determined the vectors are linearly independent when $k = 2$ despite having checked only a handful of linear combinations. When I asked P5, at the end of her engagement with Problem 5, whether she would get full marks for her work, she said she did not "think [she's] missing anything"; she only pointed out, as a potential issue, that she had assumed the parallelepiped would have orthogonal edges, and that this may not always be the case.

5.5.3.3.3 Participants' subsequent activity built on their exploration of linear combinations in a way that mimics LA1 norms but does not account for the mathematics at stake (P1, P4, P8, P9). These students activated τ_{41} when they could not determine whether the vectors are linearly independent by inspection, but their activation of τ_{41} did not always reflect the relevant mathematics. Of the six participants who activated the definition of linear independence, three (P1, P4, and P9) proceeded to τ_{41}, a technique for determining whether vectors are linearly independent. The technique builds on the definition of linear dependence that P1, P4, and P9 had been activating: a finite set of vectors $\{\mathrm{v}_i\}$ $(i = 1, ..., n)$ is linearly dependent if any one of the vectors can be expressed as a linear combination of the other vectors. Without loss of generality, suppose

$$\mathrm{v}_1 = a_2\mathrm{v}_2 + \cdots + a_n\mathrm{v}_n$$

for some a_i ($a_i \neq 0$ for some i). Then

$$\mathrm{v}_1 - a_2\mathrm{v}_2 - \cdots - a_n\mathrm{v}_n = 0$$

and there is a non-trivial linear combination of the vectors that produces the zero vector. Conversely, if there exist a_i, not all zero, for which

$$a_1\mathrm{v}_1 + \cdots + a_n\mathrm{v}_n = 0,$$

then, since $a_j \neq 0$ for some $j \in 1, ..., n$,

$$\mathrm{v}_j = \sum_{i\in\{1,...,n\}\setminus\{j\}} \frac{-a_i}{a_j}\mathrm{v}_i$$

so $\{\mathrm{v}_i\}_{i=1}^n$ is linearly dependent.

The discourse outlined above is knowledge to be taught in LA1 but students are not required to learn it. P1, P4, and P9's mobilization of τ_{41} (discussed in Section 5.5.3.2.2) suggests they did not know the role played by the definition of linear independence in the validity and suitability of τ_{41} as a technique for determining whether vectors are linearly independent. In any case, if they did know the role played by that definition, they did not mobilize it: I discuss the case of each of these participants in the following paragraphs.

P1 activated a matrix that does not match that which would result by applying the discourse that links linear independence to τ_{41}; the matrix he brought up was the same as the matrix in the formula for the volume of a parallelepiped formed by three vectors, which P1 had activated in his initial reaction to Problem 5. Both matrices (in the formula for the volume and the one that results in an application of τ_{41}) are made up of the given vectors: in the case of the volume formula, the vectors form the rows of the matrix, whereas in the case of τ_{41}, the vectors form the columns of the matrix. Further, P1's interpretation of what he had found via τ_{41} shows a focus on surface-level features of τ_{41} that are not rooted in the notion of linear independence. Indeed, when he found his original matrix to reduce to a matrix with two identical rows, he said: "it's clear we're going to have infinitely many solutions for this problem because I have the same row here and here." (Recall the rules I had witnessed students activating in Problem 2: "row of 0's means infinitely many solutions" and "proportional rows lead to a row of 0's."). P1 deduced that "for linear independence, I have multiple cases": the vectors are linearly independent for several values of k. But, given P1's setup of his original matrix, this finding rather means there exist non-trivial linear combinations of the vectors $(-1, 1, -k), (1, 0, 1), (k, 1, 1)$ which produce the zero vector. This means these vectors (which are not the ones given in Problem 5) are linearly dependent.

P1's engagement with τ_{41} skated along surface-level features of τ_{41}: he produced a matrix made up of the given vectors and found it reduces to a matrix with a row of 0's, so the "row of 0's" rule implies "infinitely many solutions." The inaccuracy of these surface-level features (relative to accurate start and endpoints for τ_{41}) further shows P1 was not mobilizing any relation between τ_{41} and the definition of linear independence, even though he had just engaged it with prior to activating τ_{41}. P1's transition from exploring linear combinations of the given vectors to activating τ_{41} suggests that when it comes to the normative LA1 relation between τ_{41} and the notion of linear dependence, what this relation develops in students is the practice of activating τ_{41} when a linear dependence relation between vectors is not obvious (e.g., such as the case of two vectors that are (not) scalar multiples of one another). τ_{41} is merely one of the steps students can take after having inspected potential linear dependence.

Such a practice may not seem inherently unproductive, but P4's activity brings out its potential failings. After P4 checked whether one of the given vectors was a linear combination of the others in the cases $k = 1, 5$, he proceeded to τ_{41}. Unlike P1, P4 activated a matrix appropriate for τ_{41}. He struggled to reduce the matrix because of the entries with the unknown (k). P4's explanations of what he expected to find by activating τ_{41} suggests that he, like P1, was focused on surface-level features of the technique. He knew, from his previous mobilization of τ_{42}, the vectors are not linearly independent for any k; given this "previous analysis," he expected to find "no solution" by activating τ_{41}. "What I would want is no solution. If there is no solution, that means that there is never a k for which they are linearly independent. They're always dependent. They're always on the same plane." What P4 was expecting belonged to a surface-level feature of τ_{41}: reducing an augmented matrix amounts to determining if a linear system has a unique solution, infinitely many solutions, or no solutions. But reducing the matrix appropriate to τ_{41} amounts to determining if a linear system has a unique solution or infinitely many solutions. There is no option in which there are "no solutions"; after all, the augmented matrix corresponds to a homogeneous linear system. Further, any solutions are values of coefficients which form a linear combination of the given vectors that equals the zero vector. The solutions are not values of k.

τ_{41} can reveal information about values of k for which the vectors are linearly independent; but a surface-level grasp of τ_{41}, free of the discourse connecting the technique to the notion of linear independence, makes it difficult to apply τ_{41} productively to Problem 5. Indeed, in the case of P4's matrix, no matter the value of k, the reduced row echelon form corresponds to a linear system with at least one free variable; this implies there are non-trivial linear combinations of $\mathrm{v}_1, \mathrm{v}_2$ and v_3 that produce the zero vector. If the vectors I had given had been designed differently, it could have happened that for some values of k, $\mathrm{v}_1, \mathrm{v}_2$ and v_3 would produce the zero vector only by taking their trivial linear combination. But P4's expectations from τ_{41} focused on surface-level features of the technique: he seemed to refer to the solutions of the linear system corresponding to the matrix he intended to reduce. These expectations did not match up with how τ_{41} could help to perform the task at hand.

It is possible that when P4 said that "what [he] would want is no solution," he did not consistently mean to use "solution" in the sense of a "solution of the linear system that corresponds to his augmented matrix." (I specify that he did not *consistently* mean this because he did explicitly say this was his intention: when he said "if there is no solution for k," I asked "a solution of what?" and he clarified he meant "a solution of the system $A\mathrm{x} = \mathrm{b}$.") Students at this level of post-secondary mathematics courses do not always use mathematical terms consistently. P4 expected his activation of τ_{41} to lead to the discovery that no value of k could produce vectors that are linearly independent. And he struggled to activate τ_{41} as soon as he attempted to deal with the entry that involved k; he did an operation involving that entry which would not have led him closer to resolving the task, decided against that operation, and ended his attempt to mobilize τ_{41}: "No! I don't know." This, along with what he said he hoped would happen by applying τ_{41}, suggests two possibilities: the first is that P4 did not know how to reduce a matrix with unknown entries,[20] and the second is that P4 did not know what to aim for in applying τ_{41} to the current task.

A lack of know-why relative to τ_{41} would make it difficult to adapt the normative LA1 application of τ_{41} to Problem 5—especially if the norm relative to τ_{41} allows students to merely activate surface-level features of the technique. P4 confirmed he did not have the knowledge that relates τ_{41} to a definition of linear dependence (the "know-why"); he

[20]Though there is a normative LA1 task that involves such matrices: the task to find values of an unknown entry in an augmented matrix such that the related linear system has either one solution, infinitely many solutions, or no solutions.

activated τ_{41} because he "just kn[e]w" it's the thing to do. This was the norm in LA1:

I What does [your augmented matrix] have to do with linear independence? What's the connection?

P4 Between solving this and linear independence?

I Yeah.

P4 I just know, as a fact, that when I want to find the linear combination of a system, I want to solve [an equation of the form $A\mathrm{x} = \mathrm{b}$ with] the vectors as [the] columns [of A] and with the zero matrix [as b].

I Okay.

P4 Yeah. I think something I lacked in the course was that I didn't really understand why I was doing [what I was doing].

P4 felt he did not know why he used the techniques he used in LA1; from the theoretical perspective that students who pass a course do so by adhering to the norms of a course, P4's perception makes sense. For instance, the norm in LA1 relative to τ_{41} does not require students to know why it works. Students need only know to use it given the normative task to verify if so-and-so(-and-so) are linearly independent vectors. The incoherence in P4's justification for his approach ("when I want to find the linear combination of a system," "if I only have a trivial solution for the linear combination, it means no coefficient will give me that they are not on the same plane, they're not independent") supports P4's perception.

P9 transitioned from activating the definition of linear dependence to mobilizing τ_{41} after I asked how he knew the vectors to be linearly independent when $k > 1$. He applied τ_{41} to the case of $k = 5$. His justification was incoherent: "to prove they are linear independent, you [need to show you] cannot write these three—[you need to show] these three cannot write a vector that belongs to their span. So if you put over here a, b, c, it becomes $(-5, 1, 1)(-1, 1, 5)(1, 0, 1)$, so you'd have to reduce it." Here's what he wrote (verbatim):

$$\left[a \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} b \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

Underneath this, he wrote the matrix whose columns are the above vectors (respectively). He was activating the notion of linear combination which relates linear dependence to τ_{41}. But he seemed to be activating only a surface-level feature of this relation: his explanation ("these three cannot write a vector that belongs to their span") was both inaccurate (by definition, vectors in the span of a set of vectors are vectors that "can be written" in terms of those vectors) and was not about the notion of linear dependence. When P9 found his matrix reduced to a matrix with a row of 0's, he said the vectors are linearly dependent "because you can write one vector in terms of the other." This is accurate, but P9 struggled to explain why this is so: "I forgot how to explain it [...] in this case you can write the third the last one in terms of or [pauses] I forgot about this one." What he did remember was a mashup of the notions of linear independence, of the span of a set of vectors, and of LA1 techniques that regularly have students "put [vectors] in a matrix": "when you have three linearly independent vectors, when you put them in a matrix, to see what their span is, you want to see that the only thing this span contains is 000." I suspect this mashup is a result of a normative LA1 task taught shortly after students are exposed to the notion of linear independence: to determine whether a set of 3 given vectors is a basis for $\mathbb{R}^3$. This task has students determine whether a given set of vectors is linearly independent and whether its span is all of $\mathbb{R}^3$.

In Section 5.5.3.2.2, I had already examined how participants had activated τ_{41}. P1 and P10's application of τ_{41} was divorced from what it would mean for $\mathrm{v}_1, \mathrm{v}_2$, and v_3 to be linearly independent; they knew their matrix ought to be made up of these vectors, but used these vectors as the *rows* of their matrix (when these vectors should have been its columns). A surface-level grasp of τ_{41} would make it difficult to remember how to position the vectors. P4 and P9 positioned their vectors appropriately but struggled to explain why the results of τ_{41} would tell them anything about linear independence. Recall the LA1 norm relative to τ_{41}: students need to produce an appropriate augmented matrix whose entries are always known small integers; students need to reduce this matrix; and students need to decide whether the reduced row echelon form implies the vectors are linearly (in)dependent. P9 shared the rules for making this last decision: "if I have a diagonal then it is linear independent" and a "row of zeroes" means "we have infinitely many solutions" and therefore the vectors are "not independent." Students know τ_{41} works for checking whether vectors are linearly independent, and they can form rules that operate on surface-level features of τ_{41} to help them apply τ_{41} successfully. Students' application of τ_{41} to Problem 5 suggests what they know of the relation between "linear dependence of vectors" and τ_{41} is this: τ_{41} works to check if vectors are linearly dependent.

P1, P4, and P9's transition from activating the definition of linear independence to τ_{41} confirms students do not have more knowledge about this relation. They had tried to check, by inspection, if one vector could be written as a linear combination of other vectors. This was trivial in one case (e.g., with $k = 1$, where both P4 and P9 observed two vectors are identical). But they could not, by inspection, make a general conclusion. And so they activated τ_{41}. I had thought that, perhaps, given P1, P4, and P9's transition from mobilizing a definition of linear independence to mobilizing τ_{41}, they might be able to make (at least partially) explicit the relation between the definition and the technique. This could have helped to guide their application of τ_{41} to a task that differed from the LA1 norm. But they did not activate any such relation. The LA1 norms related to linear independence allow students to operate only surface-level features of τ_{41}. This made it difficult for them to adapt τ_{41} to Problem 5, which did not adhere to the LA1 norms for the task of examining the linear independence of a set of vectors.[21]

Reflecting on P2 and P4's capacity to activate τ_{42} successfully and contrasting it with P4, P1, and P9's struggle to mobilize τ_{41} for Problem 5, I note that τ_{42} leads to a result to which a normative LA1 rules applies: applying τ_{42} to Problem 5 leads to the result that the determinant of a matrix formed by the given vectors is 0. The LA1 rule for τ_{42} is that a determinant equal to 0 implies the vectors are linearly dependent. This rule applies to Problem 5 accurately; there's no need for further discourse. τ_{41} does not transfer as neatly; a surface-level application of τ_{41} such as P1 and P4's (whose comments suggested

[21] In particular, the LA1 norm whereby the components of vectors are always known.

they expected the solutions of the system corresponding to the augmented matrix to be *values of* k) would rather lead either to the incorrect conclusion (such as P1's) that there are several values of k for which the vectors are linearly independent or to the incorrect expectation (such as P4's) that the system has no solutions (because he knew the vectors are not linearly independent for any k).

P8's follow-up activity (after inspecting the vectors' potential for linear independence) diverged from P1, P4, and P9's focus on algebraic representation: she spontaneously mobilized a geometric interpretation. She made two sketches. One was of 4 arrows with coinciding initial point, where 3 vectors were parallel to the same plane and the other vector was orthogonal to the 3 coplanar vectors. The second sketch seemed to be of a parallelogram and was crossed out. P8 explained "there was a graph in the book that [depicted vectors that] lie in the same plane [and it was stated] they are dependent." She concluded that "there doesn't exist any k that [would] make [the vectors] independent." P8 had mobilized knowledge she recalled from the textbook but she did not trace it back to the definition of linear dependence she had just activated. Indeed, when I asked how she knew that vectors being in the same plane means they are linearly dependent, she said: "there was a graph, which I barely remember... Something [about how] if they are in the same plane, then I have three coordinates here but the third one actually doesn't exist in this one plane." I asked her to clarify this last part. "For example, here, I can move it here, here, but not upwards, because that is everything I have: the plane. So they will be expressed—one of them will be a linear combination of the [other] two." I asked how she would know one of them would be a linear combination of the other two. I wanted to see if she would use her sketch to demonstrate how one geometric vector might be a linear combination of the other two geometric vectors. But P8's responses derailed from the geometry at stake and rerouted onto normative technical LA1 knowledge: "I guess it's because again, using these vectors, if I would put them in rows, as equations, I will end up with a... row of zeroes. And basically, I will get a parameter and I will have to express the third one using the other two." P8 was "not sure about [any]thing," though, and said: "I'm pretty sure I remember the graphs. Seeing that if they lie in the same plane, they are dependent. I'm not sure of my explanations."

P8 spontaneously retrieved a graph wherein three vectors are coplanar to answer a question about the linear dependence of a set of vectors. This suggests LA1 norms related to linear independence inculcated her with the knowledge that linear dependence of vectors has a geometric representation. But, prompted to explain how she knew that if vectors are coplanar, then one could be expressed as a linear combination of the others, P8 did not return to her sketch. And she stopped talking about the geometry at stake. Instead, she leaped to τ_{41}. This raises doubt as to whether P8 knew why coplanar vectors could be expressed as linear combinations of one another; in any case, even if she would have summoned this knowledge given the appropriate combination of prompts, she did *not* summon it. Asked how she knew that, given coplanar vectors, one vector would be a linear combination of the other two, P8 did not mobilize the sketch she had summoned from memory from LA1. She referred to τ_{41} instead—an algebraic technique for determining the linear dependence of a set of vectors.

P8 gave a pockmarked explanation of the relation between τ_{41} and the notion of linear dependence. Her explanation leapt from "parameters" to one vector being a linear combination of the others: "I will get a parameter and I will have to express the third one using the other two." From this, it's not clear P8 knows that there being a parameter means that, given the equation

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 = (0, 0, 0),$$

one of the coefficients can be assigned a parameter and the other coefficients might be expressed in terms of this parameter. Since a parameter could have any real value, and in particular, non-zero values, there would then be a non-trivial linear combination of $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ that produces $(0, 0, 0)$. That P8 omitted this discourse from her explanation ("basically I will get a parameter and I will have to express the third one using the other two") brings to question whether P8 had the knowledge that relates τ_{41} to the definition of linear dependence. P8 seemed hesitant as she explained the relation of τ_{41} to linear dependence so I asked her about this hesitation: "I'm pretty sure I remember the graphs. Seeing that if they lie in the same plane, they are dependent. I'm not sure of my explanations." P8's confidence lay in what she recognized as a recollection from LA1 (graphs she had seen in the textbook). What she was not confident in was explanations she gave in an attempt to relate the different pieces of knowledge she did recollect from LA1 (a definition, a sketch, and τ_{41}).

P8's lack of confidence in her explanations of the concepts underlying τ_{41} and the sketch she made, in addition to her return to algebraic technique in response to my question ("how would you know one of them would be a linear combination of the other two?"), which I had asked in response to P8's description of the geometry at stake, brings to mind P2's comment that he "usually [does] calculations" when I asked what would convince him of his conclusion (that the vectors are linearly dependent) if he had been working on his own. Between his explanation of the geometry (the vectors are coplanar so one is a linear combination of the other two) and his mobilization of τ_{42}, he said he would be convinced by the latter but not the former: he "usually [does] calculations." Further, P2's model of what's expected of students in LA1 included τ_{42} but not the geometric representation: the calculations, he said, would grant him full marks on a submission in the course, but the explanation of the geometry would not.

What P1, P4, P8, and P9 mobilized after they engaged with the definition of linear dependence operated on surface-level features of the mathematics at stake; they either did not or struggled to relate their follow-up activity (be it τ_{41} or a geometric representation) with the definition of linear dependence. P5 and P10, meanwhile, did not mobilize any other knowledge after this stage. P10 knew vectors are linearly independent if they cannot be expressed as linear combinations of one another but did not mobilize this in any way; since she seemed stuck, I reminded her the vectors form a shape that has volume 0 and I asked what that would look like. P10 had "no idea." P5 had already deduced (in her initial and spontaneous reaction to Problem 5) that either the "base," "height," or "length" of the parallelepiped must be 0 (she seemed to be mobilizing a formula for the volume of a parallelepiped) but did not make any further geometric interpretations; and when she finished activating the definition of linear independence, I asked if she could think of any other approach and she did not.

The activity of those who continued after they engaged with the definition of linear independence reflects LA1 norms: they couldn't check, by inspection, whether three vectors were linearly independent, so they brought up τ_{41} (P1, P4, P8, P9) or a textbook sketch that related the notion of linearly dependent vectors with the property of being coplanar (P8). But this subsequent activity and participants' related comments and explanations show students struggled to inform this activity by the mathematics at stake. Their activity was persistently marked by an absence of knowledge that relates the definition of linear dependence with knowledge normatively related to it in LA1.

5.5.3.4 One student (P7*) mobilized a narrow definition of linear dependence to examine the linear dependence of the vectors.

P7*'s spontaneous reaction to Problem 5 was to mobilize a geometric interpretation of three vectors being linearly dependent: the sum of two of the vectors must be parallel to the third vector. P7* later mobilized this same narrow definition of linear dependence as he activated an algebraic technique for the task: he checked whether $\mathrm{v}_i + \mathrm{v}_j$ would be a scalar multiple of v_k (where $i \neq j \neq k$ and $i, j, k \in \{1, 2, 3\}$). I discuss P7*'s activity in each case in the following subsections.

P7*'s spontaneous reaction to Problem 5 was a geometric interpretation based on a narrow definition of linear dependence. Upon reading the problem statement, he said:

> First thing is, you know, I know that these two—let's say this is v_1, v_2, and v_3; the summation of v_1 and v_2 should not be on the same line of v_3, these should not be parallel, which means they should not be on the same line, otherwise, they will be linearly dependent.

P7* did not build on this definition at this point; as per his fashion with other problems in the interview, he promptly moved on and suggested a new approach: τ_{41}.

After P7* engaged with τ_{41} in steps 2, 3, and 4 of his activity, steps 5 and 7 of his activity were to mobilize the algebraic interpretation of his narrow definition of linear dependence. He found the components of $\mathrm{v}_i+\mathrm{v}_j$ ($i \neq j; i, j \in \{1, 2, 3\}$) and checked whether the vector was a scalar multiple of v_k ($k \neq i, j; k \in \{1, 2, 3\}$). He started with $\mathrm{v}_1 + \mathrm{v}_2$, which is $(-k - 1, 2, k + 1)$; given that $\mathrm{v}_3 = (1, 0, 1)$ had 0 for its second component and $\mathrm{v}_1 + \mathrm{v}_2$ did not, P7* found this troublesome: "if they are on the same line, then these coefficients should be proportional. I think I maybe made some mistake here. Let's do the other way because I am having trouble with this zero." He calculated the components of $\mathrm{v}_2 + \mathrm{v}_3$. But this, too, had a component that was 0 (and v_1 had none).

"Forget about it, I think a more promising way to do that is $\mathrm{v}_1 + \mathrm{v}_3$." The 0-issue disappeared; he found $\mathrm{v}_1 + \mathrm{v}_3 = (1 - k, 1, 2)$, and he knew $\mathrm{v}_2 = (-1, 1, k)$. "That's promising. They have to be proportional to be linearly dependent." The other values of k would be those for which the vectors are linearly independent, he explained. He set up the proportion:

$$\frac{1-k}{-1} = \frac{1}{1}$$

So $1 - k = -1$, and so $k = 2$. He checked: with $k = 2, \mathrm{v}_1 + \mathrm{v}_3 = (-1, 1, 2)$. This worked out well given that $\mathrm{v}_2 = (-1, 1, k)$. He concluded the vectors are linearly dependent in this case and linearly independent when $k \neq 2$.

This was step 5 of P7*'s engagement with Problem 5. After he found this, I asked whether this approach or his approach using Gaussian elimination would yield him more marks on a test; P7*'s response to this question ended with a return to τ_{41}. This last attempt at τ_{41} led P7* to a result that contradicted what he had found in step 5: he found the vectors to be linearly dependent *for any k*. This prompted P7* to revisit his proportionality scenario.

He knew he couldn't "do the proportional thing" with the combinations of vectors that involved one vector with a 0-component and another vector with no 0-component. So he refocused on $v_1 + v_3$. He exclaimed. He realized he had found that if $k = 2$, the vectors are dependent, and had *assumed* this to imply the vectors are independent when $k \neq 2$. He thought this assumption to be the culprit. ("We cannot get this conclusion. This conclusion is incorrect, it's too strong, I only know that $k = 2$ belongs to the dependent condition.") So he checked the proportionality again, this time with the assumption that $k \neq 2$:

$$\frac{1-k}{-1} \neq 1, \frac{1-k}{-1} \neq \frac{2}{k}, \frac{1}{1} \neq \frac{2}{k}.$$

At first, he said he expected these conditions to lead to a value of k for which the vectors are linearly independent, but then seemed unsure of the matter as he considered concrete examples of pairs of vectors that are linearly dependent because they are proportional ($(1,2,3)$ and $(2,4,6)$) and pairs that aren't proportional ($(3,4,6)$ and $(1,2,3)$) and hence are linearly independent.

> So, these vectors, they are not linearly independent. They are - wait a minute. Yes. No. No, I'm kind of getting dizzy [figuratively]. Just a second. Give me a second. I think I'm, I'm. Yes. So, if they are proportional, that means, it means these two - these two vectors they are parallel, which means they are linearly dependent.

Given the amount of time P7* had spent on the problem at this point (nearly 30 minutes), I gave, at this point, a last prompt to bring his attention to an aspect of the problem he had ignored up to this point: that the parallelepiped formed by the vectors had volume 0. By considering the geometric implications of a parallelepiped of volume 0, P7* realized the three vectors "are on the same plane," and since they are in $\mathbb{R}^3$ but on the same plane, cannot form all of $\mathbb{R}^3$; from this "decrease in dimension," he deduced the vectors must be linearly dependent. This jostled him to review his previous and narrowed definition of linear dependence:

> But, but, but you know, it looks very promising that if this summation vector is not parallel to this one, then they are linearly *in*dependent. So I have no idea. [pause] Oh, wait a minute. Wait a minute. I... No. No. [pause] Oh, wait a minute. I know where I got it wrong. Yes. In this linearity condition with v_1 and v_2, the coefficients are both 1. Yes. So this is a parallelogram. Yes. Yes. But actually, it could be anywhere, right? It could be forming everywhere, because the coefficients can be [any] number.

It was only upon considering the geometry of a parallelepiped of volume 0 that P7* eventually realized he had been mobilizing a narrow definition of linear combinations: such combinations needn't have coefficients equal to 1. Notice that P7* had activated τ_{41} several times before and during his activation of that narrow definition. This suggests that, as seemed to be the case in participants' practice when they mobilized augmented matrices and row operations (in Problems 2 and 5, for instance), P7*'s use of τ_{41} was an instance of throwing Gauss-Jordan elimination at a problem to fix it, without knowing (or, at best, without mobilizing) the mathematics at stake in the technique.

5.5.3.5 Some students (P5, P6, and possibly P4) turned to the base-height-length rectangular prism volume formula to conceptualize a parallelepiped of volume 0.

In the next section, I address students' mobilization of the geometry at stake in Problem 5. In the current section, I address knowledge mobilized by P5 and P6 (and possibly P4) and which sits on the outskirts of the geometry at stake: if a parallelepiped has volume 0, then its base, height, or length must be zero. P5 and P6 and, possibly, P4 recalled the base-height-length formula for the volume of a rectangular prism. Among these students, P5 did not mobilize any geometric interpretation of vectors forming a parallelepiped of volume 0.

The reference to the volume formula was (P4 and) P5's initial and spontaneous reaction to Problem 5. At first glance, it might come across as an observation of the geometry at stake, but P5's subsequent comments show this was not the case for her, and may or may not have been so for P4. Indeed, P4's initial reaction was this: "of volume zero - okay, so they're all on the same plane. Yeah. Because there's no width, there's no height, that means they're all in the same plane." Consider, now, P5's initial reaction to Problem 5: "that will make either the length width or height zero... I think. The volume would be base times width times—length times width times height. And to make it 0, one of them would have to be zero." P5 was drawing from a volume formula from previous mathematics courses: the volume of a rectangular prism is its "length times [its] width times [its] height." It's not clear whether, initially, P4 had this formula in mind when he said "there's no width, there's no height" or if he had a flattened object in mind when he visualized a parallelepiped of volume 0. In any case, P4's comments later in his engagement with Problem 5 show he was visualizing the parallelepiped and not exclusively relying on a formula: he sketched a cube ("for the sake of simplicity") and described what it might look like with (non-)zero volume.

P6 brought up similar formulaic knowledge, but did not seem to put as much stock in it as P5 had:

> You can definitely have a two-dimensional line or object in a three dimensional space. Right? That's not something I'm unsure about. Though maybe I should be... because you know, volume, it's length times width times height.

P6's continued monologue includes imagery that goes beyond this formula: "obviously, the height of some objects is not constant throughout their, their, their... you know, their objects." (P6 struggled with the terminology; I surmise he was referring to irregular shapes.) Given this description, and given P6's explicit visualization of a 2-dimensional object (he had sketched a parallelepiped generated by 3 vectors), I infer P6 was relying on a geometric interpretation of a parallelepiped having volume 0, rather than any formula he may have known, from previous mathematics courses, about the volume of a rectangular prism.

P5's reasoning focused on formulaic knowledge. She was activating the multiplication property of zero: a product is 0 if and only if one of its factors is 0. For her, "base," "height," "width," and "length" were traces of formulaic knowledge; she did not mobilize the geometry at stake. Unlike P4, who mobilized his deduction about the dimensions of the parallelepiped to infer the vectors are coplanar, P5 rather moved on to mobilize algebraic knowledge that had no relevance to the given task.

5.5.3.6 Students struggled to complete the task by activating geometric representations of a shape of volume 0 and of linear independence in $\mathbb{R}^3$.

Most participants did not activate a geometric representation spontaneously or as an immediate response to Problem 5 (P1, P2, P3, P5, P7*, P8, P9, P10); one participant (P4) did activate a geometric representation, saying the vectors are on the same plane, but did not mobilize knowledge that coplanar vectors are linearly dependent, mobilizing instead his algebraic knowledge about linear independence; and one participant (P6) mobilized both a geometric representation of a shape of volume 0 and a geometric representation of linearly independent vectors (knowing they must not be coplanar) but was thrown off by what he perceived the task to be (P6). Based on participants' activation of the geometry at stake, I identify their grasp of the geometry as either null (P3, P5, P10), restricted (P1, P6, P9), or accurate (P2, P4, P7*, P8). Given the information that the vectors formed a parallelepiped of volume 0, P2, P4, P7*, and P8 described the vectors as coplanar (or as being "on the same plane"); P1, P6, and P9 figured the parallelepiped to be a parallelogram—for P6 and P9, this was related to two of the vectors being collinear (P6, P9); P3 and P5 made no geometric interpretation despite having skirted around the geometry at stake; and P10 had "no idea" what a parallelepiped of volume 0 might look like.

Activating a geometric representation was not the spontaneous nor immediate reaction of most participants when they read Problem 5. Indeed, only P4 and P6 spontaneously and immediately considered the geometric representation of a parallelepiped of volume 0. P8 turned to this, spontaneously, only after she had struggled to mobilize algebraic representations from LA1 (she had used a formula to calculate the volume of the parallelepiped formed by the given vectors, and was stumped when the expression produced—"0"—did not involve a k she could isolate). P5 brought up the base-height-length formula for the volume of a rectangular prism and deduced one of the prism's dimensions would be 0 (since a product is 0 only if one of its factors is 0), but P5 did not make any further inference from this. Instead, she continued by treating the parallelepiped as a right rectangular prism; indeed, when her engagement with Problem 5 had ended, she said her approach (wherein she had set dot products of pairs of the given vectors equal to 0, so as to find values of k for which the vectors are orthogonal) was based on the assumption that the edges of the parallelepiped are orthogonal to one another ("I started off by assuming the parallelepiped, like, it's not necessarily 90, but I assumed it was 90"). None of the other participants spontaneously brought up a geometric representation in their attempt to complete the given task; they brought one up only when I prompted them to do so.

Among the only two students (P4, P6) who did activate a geometric representation spontaneously and immediately in reaction to Problem 5, neither was able to rely on this representation to complete the problem. Both needed the algebraic representations they activated in response to their interpretation of the geometry.

P6 had inferred the vectors to form a parallelogram. He thought two of the vectors must be collinear and this view did not change when I suggested the term "coplanar":

P6: There's collinearity. Is that the right word I'm using, by the way? Sort of, not really.

I: Coplanar.

P6: Coplanar. Yeah, right. Coplanar for planes and collinear for lines. Yeah, that makes sense. One of them is either superimposed on top of the other or [is] some scalar multiple [of the other].

His association "coplanar for planes and collinear for lines" was based in the common root of the words; but P6's description ("one of them is either superimposed on top of the other or [is] some scalar multiple [of the other]") showed he did not know what it means for vectors to be coplanar. If anything, his imagery was more informed by a (literal) surface-level take on parallelepipeds (a 2-dimensional equivalent is a parallelogram) and did not take into account the vectors at stake:

$$(-k, 1, 1), (-1, 1, k), \text{ and } (1, 0, 1).$$

None of these are scalar multiples of the other (except in the case $k = 1$, of which P6 made no mention). P6's geometric misrepresentation aside, he correctly inferred, from the vectors forming a 2-dimensional shape, that they are not linearly independent.

P6 found this to contradict the task: he knew linearly independent vectors in $\mathbb{R}^3$ could not form a 2-dimensional shape. He turned to algebraic representations to address this contradiction. He activated τ_6, the LA1 technique (formula) for the task of calculating the volume of a parallelepiped: "I guess I can just take the determinant, and then set it equal to zero to find a parallelepiped of volume zero, [find] what value k might be." He found the volume to equal 0. He flipped back and forth one more time between his geometric and algebraic representations of a parallelepiped with volume 0. He knew he "didn't do any of [the] math wrong." He "just [wasn't] sure the math [he had done] was enough to *prove*" (his emphasis) that "there [are] no values for k [for which the vectors are] linearly independent." He wanted me to confirm his conclusion: "you're making me really unsure, you have to tell me if I am getting this right, or just totally wrong." In return, I asked what it would take to convince him. He explained:

> I'm not sure. But I guess I'm starting to think that maybe what I did is okay. Because in the determinant equation, I got the $1k$ plus $1k$ plus 1. So obviously, for any value of k, this will still go to zero. That means—or like one of these, you can still subtract this from this or this from this. So one of these rows will be all zeros, which will mean that determinant is zero. So

> for all values of k, all real values of k (I guess k is a real number), there's no solution where they aren't linearly dependent. So no, I think I did okay, actually.

What convinced P6 of the conclusion he had reached was the result of his algebraic manipulations. Indeed, when I pointed out he had his calculation of the determinant and an explanation of the geometry, he prioritized his algebraic manipulations in his response:

> It's just not a proof, right? I didn't prove that for every single—I mean, maybe I did, and I just don't understand that I did. But I just don't think I proved that for every single value of k... No, no, no, I did, right. Because you get the same equation. So no matter what value you plug in for k, it doesn't matter.

P6's grasp of the geometry at stake was not sufficient to convince him the vectors are linearly dependent for any k; he needed the algebraic representation he mobilized to gain confidence in the conclusion he had reached. When he decided he *had* proved the vectors are linearly dependent for any k, he twice referred to the algebra and did not mention the geometry.

P4, who had inferred the vectors must be coplanar from the get-go, did not conjure the relation coplanar-linearly dependent. He mobilized τ_{42} instead. This is the LA1 technique of calculating the determinant of a matrix made up of given vectors to determine if the vectors are linearly independent. From τ_{42}, he found the determinant of the matrix formed by the vectors to equal 0. This made him doubt the suitability of τ_{42} for the given task:

> That's weird. Because the scalar triple product, no matter what k is, is zero. Which is weird to me, because I would expect it to be... a number. I don't know. Scalar triple product. Yeah, so I would probably [find the] determinant of this such that it is not zero, and then solve for k. But k is not gonna cancel out. So does that mean k belongs to $\mathbb{R}$? Because any va—no. [pause] So no matter what value I would put in, I'm gonna have zero is equal to zero. That makes things a bit complicated. [pause] Okay, let me make sure now this truly means that I should use a scalar triple product.

To "make sure" he "should use a scalar triple product," P4 revisited his geometric representation of a parallelepiped of volume 0. He sketched a cube ("I'm just gonna take it as a cube for the sake of simplicity") and considered the implications, for the vectors of which it's made, of the cube having volume zero:

> If it has volume zero, they must also all be on the same plane. I'm just thinking, if I were to minimize this, is it gonna keep on going and still be... [There will be] some sort of volume until they're completely aligned on each other. So the vectors, to form a parallelepiped of volume zero, they must be on the same plane. So I think it justifies my use for the scalar triple product.

P4 did not activate a geometric representation so as to use it directly to perform the task; he activated it because it belonged to a basket of facts about linear independence he had acquired by "studying" in LA1. "Directly, when I think of vectors on the same plane, I think of two things, the scalar triple product or linear dependence." He had acquired the geometric association from lectures in which the "professor showed pictures of vectors being on the same plane or not being on the same plane; dependence, independence." But P4 did not mobilize this association to complete the task; instead, every time he brought up the vectors being coplanar, he concluded this "justifie[d]" mobilizing τ_{42}.

Upon return to τ_{42}, P4 concluded the vectors must be linearly dependent for any k:

> I have 0 is equal to 0. Which would mean that... I'm trying to figure out if this means that they are always linearl - uh, on the same plane or they're never on the same plane. [Recall P4 had already established, twice, the vectors are coplanar.] That's what I'm trying to figure out and I need to probably do some logic to be able to decide which is which. Okay, so, if the scalar triple product is equal to zero, then they are on the same plane.

P4 did not explicitly relate his interpretation of the algebra (the vectors are coplanar) with what he had already deduced from the geometry (the vectors are coplanar). And he knew linearly dependent vectors are coplanar: "to visualize linear dependence of vectors, I would visualize them being on the same plane." Even if he didn't know coplanar vectors are linearly dependent, comments he made toward the end of his engagement with Problem 5 suggest he knew, from the scalar triple product being 0, that the vectors are linearly dependent. But, to be certain, P4 went on and mobilized more algebraic representations; the remarks he made at the end of his engagement with Problem 5 (which I address toward the end of this paragraph) suggest he was not convinced by what he had initially mobilized. As a fifth step in his engagement with Problem 5, he activated the definition of linear dependence and checked, by inspection, whether one of the vectors is a linear combination of the other two (when $k = 1, 5$). As a sixth and last attempt Problem 5, he started to activate τ_{41} but got stuck with the row operations. He activated these approaches because he knew they also had something to do with linear independence: "from [τ_{42}], I deduced there would be no values of k for [which the vectors are] linearly independent, but because I know that it could also relate to linear independence, I'm gonna try to [do the task via τ_{41}]." P4 knew τ_{42} sufficed but wanted to do the task via τ_{41} to be convinced of the conclusion he had reached. Indeed, when I asked what he would need to show to get full marks for this problem if he had to submit it for grades, he said one of the methods would "be enough," "but just to fully prove [his] point," he would do "both." To convince himself the vectors are linearly dependent no matter the value of k, he "would probably try [τ_{41}] again." Whatever the case, it did not suffice to know the vectors are coplanar and that coplanar vectors are linearly dependent. Calculations were necessary.

A comment P2 made at the end of his engagement with Problem 5 made explicit the normative LA1 fervor: "usually, I do calculations." His spontaneous and initial response to Problem 5 was to activate τ_{42}. He found the determinant (of the matrix made up of the given vectors) is 0 and knew this meant the vectors are either linearly dependent for all k or linearly independent for all k. He didn't know which. I pointed out he hadn't used the information from the first sentence (about the volume being 0) and asked if it told him anything. He said: "it means they are… coplanar." And again, he knew this meant the vectors are either linearly dependent for all k or linearly independent for all k:

> Imagine if they are coplanar. They are on the same plane. This means that

> one cannot depend on the other, because... Imagining the 3D world. If they are in the same plane... [pause] I forgot if they are independent or dependent.

P2 took me up on the offer to see the definition of linear independence (a finite set of two or more vectors is linearly independent if none of the vectors can be written as a linear combination of the others). Once he read it, he came to a conclusion, albeit cautiously:

> They are on the same plane. It means they are dependent [pause] because... They can be expressed as a combination of the others. But... with three vectors [pause] I'm [wondering] if the determinant way is the correct way or... if I should try to write [...] one as a combination of the [other] two. [pause] Honestly, I would do it like this. Yeah. Because I remember that [the vectors being] independent or dependent has to do with the determinant. And this is the easiest way to do it. Because I get zero, so I get a straightforward answer without having to analyse it.

It's not clear if P2 was referring to the geometry at stake or to the matter of writing one vector as a linear combination of the others when he said "to analyse it." When I asked him if thinking about the volume being 0 is something that would convince him his conclusion was valid, he said: "if the volume is 0, it means they are like coplanar. If they are coplanar, it means they are dependent." But, asked if he'd get full marks if he wrote this on an exam, he said: "No, I don't think so." And asked if it'd convince him if he were working on his own: "No. Usually, I do calculations."

P8 had also started her approach to Problem 5 with calculations. She considered a geometric interpretation only after she hit a wall in her calculations. She had first used a formula for the volume of a parallelepiped with the hopes of identifying the value(s) of k for which the volume would equal 0. But her manipulations produced an expression in which there was no k to isolate. Next, she tried and failed to determine, by inspection, whether one vector could be expressed as a linear combination of the other vectors. Finally, she drew a sketch in which 3 arrows had coinciding initial point and were coplanar, and a fourth arrow that was orthogonal to the rest of the lot. To P8, this explained what she had found earlier: "it's all here in the same plane and that's why it's zero." I asked P8 if this told her anything about the linear independence of the vectors and she recalled that "there was a graph or something in the book that [showed that] if they lie in the same plane they are dependent." P8 then concluded: "I guess there doesn't exist any k that will make it independent."

For P8, the validity of her conclusion came from the authority of the course textbook and not from knowledge of the underlying mathematics: "I'm pretty sure I remember the graphs. Seeing that if they lie in the same plane, they are dependent. I'm not sure of my explanations." P8 was able, with the help of a prompt in which I asked her to clarify her explanations, to partially justify why vectors that "lie in the same plane" are linearly dependent: initially, she said that "if they are in the same plane, then I have three coordinates here but the third one actually doesn't exist in this one plane." I asked P8 to explain what she meant about the "third one" not existing in the plane. She said: "for example, here, I can move it here, here, but not upwards, because that is everything I have: the plane. So they will be expressed - one of them will be a linear combination of the [other] two." In response to a prompt in which I asked P8 how she knew one vector would be a linear combination of the other two if they're on the same plane, P8 did not refer to the geometry at stake; she brought up the calculations involved in τ_{41} instead.

Apart from P2, P6, and P8, P7* and P9 are the other participants who mobilized the knowledge that coplanar vectors are linearly dependent. I discuss P7*'s case first.

P7* addressed the parallelepiped having volume 0 only after I prompted him to do so, 25 minutes into his engagement with Problem 5. P7* said "this means the vectors are on the same plane. I asked if this tells him anything about the linear dependence of independence of the vectors, and he paused. Since "they are three-dimensional vectors on the same plane," they "are definitely linearly dependent." "Because they cannot form $\mathbb{R}^3$; they form only $\mathbb{R}^2$, a two-dimensional space." P7* got stuck at this point because his earlier technique had seemed "promising" but yielded a different conclusion: "it looks promising that if the sum of two of the vectors is not parallel to the third, then they are linearly independent." P7* soon realized he had incorrectly assumed linear combinations of vectors can only have 1 as coefficients ("the coefficients can be [any] number") and concluded the vectors are indeed linearly dependent.

P7*'s explanation was that by taking varied linear combinations of v_1 and v_2, it would be possible to get v_3. I asked how he knew this. "There are two ways to look at it. The first way is that, by changing the coefficients [of v_1 and v_2], their sum could be any[thing]." The second way is this: "we select two directions: the direction of v_3 and the direction that is normal to v_3. I then try to figure out the projection of v_1 and v_2 on these two [directions]. I just need to massage this coefficient so that the two projected vectors on the normal direction cancel out." P7* juxtaposed accurate and inappropriate mathematics related to the problem at stake.

P9 also brought up a geometric representation only once I had prompted him to do so; this was his fourth and last attempt at Problem 5. (His initial approach was to calculate the volume using a formula; he determined the volume is 0 for any k but made no conclusion about the linear independence of the vectors; when I asked if he could make a conclusion about this, he tried to check, by inspection, whether one of the vectors was a linear combination of the others (for particular values of k); after he made a general claim about the linear dependence of the vectors when $k > 1$, I asked how he knew this and he tried to activate τ_{41} with $k = 5$ and then $k = 0$.) Like P6, P9's inference from the parallelepiped having volume 0 was that two of the vectors must overlap: "it has volume zero because two of them are [pause] oh no [pause] Okay... so here, it's zero because two of them are overlapping." He concluded from this that there are "no values of k for which they are linearly independent." He repeated several times that two of the vectors must overlap for the parallelepiped to have volume 0; when I drew three vectors that aren't overlapping but still form a parallelepiped of volume zero, P9 said "they are coplanar," and explained this means "we can write them [pause] in terms of each other. So they are not linearly independent."

When I asked P1 how he would visualize a parallelepiped of volume 0, he said: "it's going to be xy plane—no, it's like a part of... The volume is 0, so it's gonna be just like this." He drew a parallelogram. I asked if this helped him in any way and he said it did not. In his engagement with Problem 5, P1 had attempted five other approaches, all of which had been based in algebraic manipulations (τ_{41}, checking if one vector is a linear combination of the others by inspection, τ_6). He was unable to complete the task via any of these techniques and said that on an exam he would "just pass the question."

P3, P5, and P10, finally, had little to no grasp of the geometry at stake. P5 had brought up a formula for the volume of a rectangular prism to deduce one of the parallelepiped's dimensions must be 0; but this clearly did not translate into any accurate geometric interpretation, given her assumption that the parallelepiped was a right rectangular prism (this assumption had produced P5's technique for Problem 5: she tried to find the values of k for which the vectors are orthogonal). When I prompted P3 to use the fact that the volume of the parallelepiped is 0, she was unable to mobilize any relevant knowledge. Initially, she said: "do you mean that these are... [That these] can be considered as a line, not a plane?" And then: "I have no idea." I asked P3 if she knew what a parallelepiped is. She didn't. I showed her a sketch of a parallelepiped formed by 3 vectors, returned to Problem 5, and said: "you're told here that these three vectors, they form a parallelepiped of volume 0." P3 wondered if this meant they form "a dot." She then sketched a Cartesian graph and plotted some points. I interjected to say that if a parallelepiped with volume 0 must be flat. P3 took this as a hint about how she should approach the task: "so you mean that I do not really need to calculate?" But she was unable to activate any of this. She "[tried] to imagine how the plane looked like" but "there [we]re too many possibilities." She had "no idea" what it meant for the parallelepiped to be "flat." She briefly considered an irrelevant geometric property: "I think I'd like to.. find the distance between these three points." But she stopped. ("I have no idea.") P10 similarly had "no idea" what to make of the parallelepiped having volume 0 when I asked if she could use this.

Participants' ability to mobilize the geometry at stake to complete the task ranged along three scales: their ability to make an inference about the vectors' positioning relative to one another, given that the vectors form a parallelepiped of volume 0; their ability to make an inference about the linear dependence of vectors, given that the vectors are coplanar (or "on the same plane"); and how much they prioritized geometry as an approach to tackle the given task - that is, whether they chose to address the statement about the volume of the parallelepiped. No participant was at the top of all three scales; that is, even if a participant was able to immediately infer, from the vectors forming a parallelepiped of volume 0, that the vectors "belong" to the same plane and are thus linearly dependent, this participant chose to consult their algebraic knowledge as well to confirm what they had found geometrically (e.g., P6). And if a participant was confident in the conclusion they found by activating the geometry at stake, they inferred the vectors are coplanar (from the parallelepiped having volume 0) only after having first tried (and struggled) to activate their algebraic knowledge (e.g., P8). The LA1 norm to calculate at all costs came at the expense of these participants' ability to mobilize or trust what they mobilized using geometric representations.

5.5.3.7 Summary: students' response to Problem 5 revealed a dearth of geometric knowledge of objects in 3-space.

In designing Problem 5, I expected students to struggle to mobilize the normative and algebraic LA1 technique for verifying the linear independence of vectors because it would differ from the way it typically unfolds in LA1 tasks about linear independence (as these tasks always involve vectors with known components). I expected the information about a parallelepiped of volume 0 to be an invitation to explore an alternative approach, which would turn out to lead to a two-step resolution to the problem. As it turns out, most students opted to activate algebraic techniques, and most of these students did struggle; in spite of this, all but one (P8) turned to a geometric interpretation of a parallelepiped of volume 0 only after I had prompted them to do so; among these students, P2, P7*, P9 were able to complete the problem by focusing on the parallelepiped having volume 0, while P1, P3, and P10 were not.

The number and type of vectors involved—three vectors in $\mathbb{R}^3$—seems to impinge on students' capacity to mobilize a geometric interpretation either of linear independence or of a parallelepiped having volume 0. Several students knew that if *two* vectors are parallel or scalar multiples of one another, then they are linearly dependent. But the involvement of a third vector muddied the waters. P6, for instance, inferred that a parallelepiped of volume 0 must be a parallelogram as "one of the [vectors]" must be "superimposed on top of the other or [be] some scalar multiple [of the other]," a notion P6 associated with linear dependence. (This does not take into account the components of the given vectors, from which it is clear that no vector is a scalar multiple of another vector.) A similar effect on P9: he also inferred the parallelepiped "has volume zero because two of [the vectors] [...] overlap" and deduced the vectors are linearly dependent for any k. P1 also inferred the parallelepiped must be a parallelogram but was unable to make a conclusion about the linear dependence of the vectors. Three students were unable to summon a picture of a parallelepiped of volume 0: P3 asked if I meant "that the [vectors] can be considered as a line," rather than "a plane"; P5 made a deduction based on a primary school formula for the volume of parallelepiped, but proposed no visual interpretation; and P10 had "no idea" what to make of such a volume. P7* spent 25 minutes under guise that three vectors are linearly independent if the sum of two of the vectors was parallel to (or a scalar multiple of) the third; this seems a juxtaposition of the notion of linear combination and that of linear dependence of *two* vectors, wherein two vectors are scalar multiples of one another.

The mathematics at stake in Problem 5 lends itself to algebraic and geometric knowledge related to linear independence of vectors in $\mathbb{R}^3$ as well as algebraic and geometric knowledge about parallelepipeds and their volume. Unlike normative LA1 tasks, which are paired with a normative technique (or two), the task in Problem 5 rather points to constructs that belong to several and unrelated LA1 tasks (see the LA1 praxeologies I identified as relevant to Problem 5 in Section 5.5.2). This seems to be reflected in the variety of knowledge participants tried to activate. But the framing of the problem, as one about vectors that form a parallelepiped of volume 0, can ostensibly be expected to stir up a focus on the geometry at stake. In spite of this, only two students (P4, P6) spontaneously and immediately activated a geometric interpretation about vectors forming a parallelepiped of volume 0, and neither student had sufficient knowledge or trust in this knowledge to complete the problem without recourse to algebraic LA1 techniques. I gather that the LA1 norm to indulge algebraic technique and place geometric knowledge firmly in the realm of knowledge to be taught (but not of that to be learned) hindered students' ability to mobilize the type of knowledge most appropriate for the problem at stake.

5.6 LA1 Problem 6

The following was the sixth problem presented to 9 of the 10 LA1 students[22] in the TBI:

Solve the following system of equations:

$$\begin{array}{rcrcrcl} x^2 & + & x & + & 1 & = & 0 \\ 2x^2 & + & 4x & - & 6 & = & 0 \end{array}$$

5.6.1 Reference model for LA1 Problem 6

Problem 6 is a task of type T, to "solve a system of two quadratic equations in one variable with real coefficients." More precisely, though, the task t is to "find common real root(s) of two real quadratic polynomials." I did not specify, in the problem statement, whether the task is to find real or complex solutions, because students in LA1 are not required to have taken courses in which complex numbers are knowledge to be taught. Given this context, the task was (albeit implicitly) to find real solutions of the given system.

A quadratic polynomial $ax^2 + bx + c$ has 2 roots in $\mathbb{C}$ and either 0, 1, or 2 of these are in $\mathbb{R}$. The number of real roots is determined by the discriminant of the quadratic, $D = b^2 - 4ac$: if D is positive, the quadratic has two distinct real roots $\left(\frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a}\right)$; if D is 0, the quadric has two equal real roots $\left(\frac{-b}{2a}\right)$; and if D is negative, the quadratic has no real roots.

The quadratic $x^2 + x + 1$ has negative discriminant (-3) so the first equation in Problem 6 has no real solution; hence, in the context of these TBIs, this knowledge suffices to deduce the system has no solution. In any case, the discriminant of the quadratic in the second equation is positive, so both its roots are real, whereas those of the first quadratic are both complex numbers, so there are no common solutions to the two equations. I denote by τ_1 the technique in which t is completed by calculating the discriminant(s) of the quadratic(s) in Problem 6. I denote by θ_{11} the knowledge that the solutions of a system of equations consist of all values that satisfy all equations in the system, and by θ_{12} any algebraic technology (such as discriminants of quadratics or factoring of quadratics) used to find the solution(s) of a quadratic equation.

From a graphical perspective, the task to find common real roots of two quadratic polynomials $a_ix^2 + b_ix + c_i$ $(i = 1, 2)$ is akin to finding the points of intersection of the

[22] Due to time constraints unrelated to the TBI, P8 was only able to do Problems 1 - 5.

parabolas $y_i = a_i x^2 + b_i x + c_i$ ($i = 1, 2$) that are also on the x-axis (as $y_i = 0$ for $i = 1, 2$). Since the graph of the first quadratic is a parabola that does not intersect the x-axis, the parabolas do not intersect along the x-axis. The system has no solution. Various technologies can be used to approximate the graphs of the two quadratics in Problem 6 or to determine that one of these does not intersect the x-axis: their discriminant, a formula for the vertex of a parabola (i.e., its x-coordinate is determined by the coefficients of the parabola), and other knowledge that relates coefficients of a quadratic to information about the parabola it represents (e.g., the sign of a in $ax^2 + bx + c$ determines the orientation of the parabola). I denote this family of technologies by θ_{22}, without denoting any technology more specifically as students' use of this knowledge is not the target of my investigation; I denote by τ_2 the technique of completing t by attending to the graphical perspective of the system of equations in Problem 6, and I denote by θ_{21} the knowledge that a solution to a system of equations corresponds to a point of intersection of the graphs of the equations.

I denote by Θ the algebraic and logical discourse that frames the concepts of quadratic polynomials and of systems of equations.

The reference model for activity through which to complete Problem 6 is summarized by the praxeological models $[t; \tau_1; \theta_{11}, \theta_{12}; \Theta]$ and $[t; \tau_2; \theta_{21}, \theta_{22}; \Theta]$.

5.6.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 6

There is no LA1 task of the type in Problem 6 (t), but the notion of systems of equations is at the core of the linear algebra course. The concept of what constitutes a solution to a system of equations is knowledge to be taught in LA1. In LA1, knowledge to be taught and learned involves linear equations in more than one unknown. In Problem 6, the equations are quadratic and involve one unknown. LA1 students are familiar with quadratic equations in one unknown from prerequisite mathematics courses taken either in high-school or at university. In those prerequisite mathematics courses, students learn techniques for solving such equations (including the quadratic formula based in the notion of discriminant, discussed in Section 5.6.1) and learn that such equations have either 0, 1, or 2 solutions. I denote by τ_1^{HS} any technique students are usually expected to learn in high-school algebra for solving quadratic equations (I use the subscript 1 to note the alignment between τ_1 in my reference model for this problem and τ_1^{HS}), and I note that students are not expected, in high-school algebra (nor in LA1), to solve systems of quadratic equations. Through τ_1^{HS}, it can be determined that the first equation in the given system has no solutions and that the second equation has two solutions.

Knowledge to be taught in LA1 includes θ_{11}^{KtbT} (notation I use to reflect θ_{11} from my reference model for Problem 6, but also to acknowledge this technology is mainly knowledge to be taught ("KtbT") in LA1, as I explain next): the definition of a solution of a *system* of equations as an element that solves *each* equation in the system. θ_{11}^{KtbT}, together with τ_{HS}, show that the system in Problem 6 has no solution. I do note that students are not expected to learn that if one equation in the system *given in a task* has no solution, then the system has no solution. Strange as this claim may seem, I make it on the basis of the types of equations given in LA1 tasks: linear equations which always have solutions (as they are never of type $0 = a$ where $a \neq 0$). In that sense, systems in LA1 fail to have solutions not because one of their equations fails to have a solution, but because different equations (with solutions) fail to have *common* solutions. Additionally, considering the linear-system-solving tasks given in LA1 midterm and final exams (addressed in Section 5.2.2), and in light of participants' comments during their engagement with Problem 2 of the TBI, the conclusion that a system has no solution is intrinsically connected with the "no-solution" result indicated by the LA1 technique for solving linear systems: when row-reducing an augmented matrix produces a row corresponding to a false equation of type $0 = a$, where $a \neq 0$. When such an augmented matrix is reduced to a matrix with such a row, students can immediately conclude the original system has no solution, and are *not* required to produce discourse of the following type: "since the system corresponding to this row echelon form includes a false equation, this system has no solution; and since the original system is equivalent to a system with no solution, the original system also has no solution."

Other LA1 knowledge pertinent to t has to do with the graphical representation of systems of equations and their solutions, and this is often related in LA1 to the question of how many solutions a linear system may have. Knowledge to be learned includes the theorem that systems of *linear* equations have either no solutions, 1 solution, or infinitely many solutions (for example, in one type of task that recurs in the exams to which I had access, students are asked to determine conditions under which a given linear system has one solution, infinitely many solutions, or none). A constructive algebraic proof for a theorem about the possible (number of) solutions of linear systems is knowledge to be taught and is included in the course textbook, and the results of this theorem are supported by graphical discourse about solutions of linear systems of two equations in two unknowns in $\mathbb{R}^2$ or of three equations in three unknowns in $\mathbb{R}^3$: it is to be taught that the solutions to a system of 2 equations in 2 unknowns in $\mathbb{R}^2$ correspond to the points of intersection of the lines that are graphs of these equations, it is to be taught (or, rather, reviewed, from high-school algebra) that 2 lines (in 2-space) can either overlap and hence have infinitely many intersection points, be parallel and distinct and thus have no intersection points, or have different slopes and so have a unique point of intersection; and a 9-part diagram in the course textbook shows the different ways in which 3 planes in 3-space might intersect, demonstrating that either they have a unique point of intersection, intersect along a line (and so have infinitely many points in common), intersect along a plane (and so have infinitely many points in common), or not have any point at which all 3 planes intersect (and so have 0 points in common).

The graphical representations of linear systems (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and their solutions are knowledge to be taught in LA1, but it's harder to argue they are knowledge to be learned[23]. It is possible that problems target this knowledge in assignments associated with the textbook sections that include the graphical representations of linear systems. But among the 116 midterm and final exam problems to which I had access, only two

[23] I remind the reader that by "knowledge to be learned" and "knowledge students are expected to learn" I refer to knowledge students need so as to complete tasks on midterm and/or final exams; I do not include the knowledge students are expected to wield to do assignments because these are worth only 10% of a student's final grade, students can access any resource of their choosing to complete assignments, and students submit only the final result of a task, meaning that the way students reach their results is not part of knowledge they are expected to demonstrate.

problems (parts (a) and (b) of the same problem on one final exam) involved the correspondence between linear systems and their graphical representations: the first task was to find the equation of a plane that passes through a given point and a line $\mathcal{L}$ that is defined as the intersection of two planes given in point-normal form ($ax + by + cz = d$), and the second task was to find the coordinates of the intersection of $\mathcal{L}$ and a plane $\mathcal{P}$, where $\mathcal{P}$ is also given in point-normal form. Core to the completion of these tasks is θ_{21}^{KtbT} (to reflect θ_{21} from my reference model for Problem 6)[24]: the knowledge that a solution to a system of equations corresponds to a point of intersection of the graphs of the equations. Technology θ_{21}^{HS} can be used to complete t together with any high-school technique for producing graphs of quadratic equations (which I will denote by τ_2^{HS} to reflect τ_2 from my reference model and as a reminder that I refer to usual high-school knowledge).

In LA1, students are to learn Gaussian and Gauss-Jordan elimination—or, more broadly, row-reduction of augmented matrices—as techniques for solving linear systems. The theory that produces these techiques is knowledge to be taught: an augmented matrix captures the coefficients and constants of a linear system, and each elementary row operation corresponds to an algebraic operation for producing equivalent equations (as discussed in the reference model for Problem 2 in Section 5.2.1). The theory that produces Gaussian and Gauss-Jordan elimination (e.g., including the notion of equivalent equations) is not knowledge to be learned; no assessment requires students to have this knowledge. Students are to learn how to interpret row echelon forms of an augmented matrix to make a deduction about the solutions to a given linear system.

I denote by τ_3 the technique in which the augmented matrix of the system in Problem 6 is reduced via row-operations to solve the system—the technique for solving systems of equations in LA1. Such a reduction would yield the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

This corresponds to the system

$$\begin{array}{rcrcr} x^2 & & + & 5 & = & 0 \\ & x & - & 4 & = & 0 \end{array}$$

As the first equation corresponds to $x^2 = -5$, which has no (real) solution, while the second equation is solved if $x = 4$, this system has no solution. I note this argument does require θ_{11}^{KtbT}: the knowledge that a solution to a system is a value that satisfies each equation in the system.

In sum, the knowledge to be learned (and taught) in LA1 through which Problem 6 can be completed is summarized by the praxeologies $[t; \tau_1^{HS}; \theta_{11}^{KtbT}]$, $[t; \tau_2^{HS}; \theta_{21}^{KtbT}]$, and $[t; \tau_3; \theta_{11}^{KtbT}]$.

[24]As before, I use the superscript "KtbT," acronym for "knowledge to be taught," to underscore this technology belongs to knowledge to be taught but not to knowledge to be learned in LA1.

5.6.3 Knowledge LA1 students activated in response to Problem 6

Table 5.11 (on p.228) summarizes the paths of participants' activity as they worked on Problem 6[25]. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step.

Throughout this section, I will refer by EQ1 and EQ2 to equation 1 ($x^2+x+1=0$) and equation 2 ($2x^2+4x-6=0$) of the system, respectively. The calculations students made toward solving these equations were generally accurate. I do not specify results students found when these were accurate; when they were not, I indicate as much in Table 5.11 by writing that a technique was enacted "inaccurately" by a given participant. I only specify inaccurate results a participant found if the inaccuracy was due to a misconception (e.g., as in P9's inappropriate interpretation of his augmented matrices in Step 1), rather than a miscalculation (e.g., as in P10's incorrect factorization of the quadratic in EQ2, in Step 4).

Recall that EQ1 has no real solutions and EQ2 has 2 distinct real solutions. In the context of my interviews, as participants will not have necessarily been exposed to complex numbers, I expected that students who found EQ1 to have no solutions would deduce that the system has no solutions, even without examining EQ2, based on θ_{11}^{KtbL}, the principle that any solution of a system must satisfy all equations of which it consists, even if exam questions did not explicitly require this technology, perhaps because of a deduction that if students know what constitutes a solution of an equation, then they also know what constitutes a solution of a *system* of equations. For the purpose of this discussion, which aims to elicit participants' knowledge about systems of equations and the LA1 techniques they learned for solving (linear) systems, I focus participants' actions toward solving Problem 6 in Table 5.12: in this table, I categorize participants' actions according to whether they did or did not deduce the system has no solutions after having found either that EQ1 has no (real) solutions or that solutions they found (for EQ2 or for an equation formed by adding or subtracting corresponding sides of one equation from the other) did not satisfy one of the equations. I also indicate instances in which participants deduced, from such findings, that their approach was incorrect.

[25]Due to time constraints unrelated to the TBI, P8 was not able to do Problems 6 - 8.

Table 5.11: Paths of LA1 Students' Activity in Problem 6

Practical block $[t, \tau]$	Type of engagement with $[t, \tau]$: Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	
solve EQ1 (discriminant)	P1								enacts, deduces system has no real solutions
	P4								enacts, does not deduce system has no solutions
			P5						enacts, does not deduce the system has no solutions
solve EQ2 (factoring) and check whether its solutions satisfy EQ1	P3								enacts, does not deduce the system has no solutions
		P6							enacts, does not initially deduce system has no solutions; suggests it after a prompt from interviewer
solve EQ2 (factoring)		P4							enacts, says the solutions of EQ2 are the solutions of the system
		P5							enacts
				P10					enacts incorrectly
solve the system by solving both equations (discriminant, quadratic formula)	P7*								enacts, deduces system has no real solutions
		P2							starts to enact (inappropriate technique)
		P9							enacts, deduces system has no solutions
			P2						partially enacts: solves EQ1 and EQ2 but does not deduce system has no solutions
solve the system by HS system-solving technique (substitution or addition/elimination) and verify if values found solve the system		P3							enacts, does not deduce system has no solutions
		P7*							partially enacts: finds a value for x and says it's already known this value does not solve the equations
		P10							partially enacts: finds a value for x but does not verify whether it is a solution
						P2			partially enacts: finds a value for x but does not verify whether it is a solution
add or subtract 'equations' to reduce the system to a single equation and check if its solution(s) correspond to the solutions of either EQ1 or EQ2	P5								partially enacts: finds an equation and abandons
	P6								enacts, does not deduce the system has no solutions
	P10								partially enacts: finds an equation but struggles to solve it
				P4					enacts, deduces technique is incorrect
				P5					enacts, does not deduce system has no solutions
					P5				enacts, deduces there are no values of x that solve both EQ2 and the new equation
							P2		enacts, deduces technique is incorrect
substitute values for the unknown in EQ1 to check whether it has solutions					P2				enacts, deduces EQ1 has a solution because small value (0.1) of x makes quadratic have a value close to 0 (0.99)
interpret the equations and their solutions graphically			P7*						describes accurately and describes what expected graphs would be if the system were not homogeneous
use Gauss-Jordan elimination	P2								starts to enact: writes augmented matrix
	P9								enacts incorrectly several times: first, interprets the RREF correctly ($x^2 = -9, x = 4$); second, interprets the RREF incorrectly (expresses x^2, x in terms of a parameter: $x^2 = -9t, x = 4t, x = t$); third, interprets the RREF incorrectly ($\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$, $x = -3$ and $x = 2$); fouth, interprets the RREF incorrectly twice ($\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix}$; first deduces $x = 3, x = 4$, then changes to $x^2 = 3t, x = 4t, x = t$)
			P3						enacts, deduces technique is wrong because values for x, x^2 are contradictory
			P4						enacts incorrectly first, correctly second, and dismisses the approach: interprets RREF in terms of unknowns that are not part of the system ($x - z = 0, y + 2z = 0$), then realizes the system is in terms of powers of x, and dismisses the approach
			P10						enacts, finds two values for x, finds they do not solve the equations, does not deduce the system has no solutions
				P2					enacts, finds a negative value for x^2, says this "is impossible," deduces the system has no solutions after being prompted to make a conclusion about the system
verify and reflect on results found in previous steps								P2	discusses: says results found in Steps 1, 3, 4 are "impossible" and those from Step 7 do not solve EQ1, does not deduce system has no solutions.

EQ1: equation 1 of the given system ($x^2 + x + 1 = 0$); EQ2: equation 2 of the given system ($2x^2 + 4x - 6 = 0$); HS: high-school.

5.6.3.1 All students uncovered inconsistencies in the system; most struggled to conclude the system has no solutions.

All students had sufficient knowledge of how to solve quadratic equations to solve them accurately, but most did not have sufficient knowledge of systems of equations to conclude the system had no solution. As a first step in the analysis of participants' engagement with Problem 6, I organize a description of their activity in terms of whether and how they came to deduce the system has no solution. I begin with Table 5.12, which categorizes the deductions student made after gaining one of the following pieces of knowledge: that EQ1 has no solution, that the solutions of EQ2 do not solve EQ1, or that the solutions of equivalent equations (produced by various algebraic techniques) do not satisfy EQ1 and/or EQ2. I categorize participants according to whether, after gaining such knowledge, they did *not* deduce the system has no solutions, whether they deduced their approach must have been incorrect, whether they deduced the system has no solutions but only after prompts from the interviewer (prompts attending to the matter that solutions of a system must solve all equations), or whether they deduced, on their own, that the system has no solutions.

Table 5.12: Students' deductions about the system from the solutions they found (Problem 6)

	Participants who, after gaining K, ...			
Knowledge K gained about the solutions of equation 1 (EQ1) and/or equation 2 (EQ2)	did *not* deduce the system has no solutions.	deduced their approach was incorrect.	deduced the system has no solutions only after a prompt from the interviewer (that solutions of a system must solve both equations).	deduced the system has no solutions.
EQ1 has no solutions	P4 (Step 1) P5 (Step 3)			P1 (Step 1)
The solutions of EQ2 do not solve EQ1 (plugging values into EQ1, or solving EQ2 and finding EQ1 has no solutions).	P2 (Step 3) P3 (Step 1) P6 (Step 2)		P6 (Step 2)	P7* (Step 1) P9 (Step 2)
The solutions found by creating a new equation (by adding/subtracting corresponding sides of EQ1 and EQ2) or via HS or LA1 system-solving technique (HS substitution, HS elimination/addition, Gauss-Jordan elimination) do not satisfy EQ1 and/or EQ2.	P3 (Step 2) P5 (Step 4) P6 (Step 1) P10 (Step 3)	Gauss-Jordan elimination: P3 (Step 3) P4 (Step 3) Subtracting corresponding sides of EQ1 from EQ2 to create a new equation: P2 (Step 7) P4 (Step 4)	P2 (Step 4, but then went on with four more approaches; P2's comments in Steps 5-8 confirm he was not convinced the system had no solution) P4 (Step 4)	P5 (Step 5)

Of the nine participants who attempted Problem 6, all had the evidence needed, from their computations, to conclude the system has no solutions. However, only three (P1, P7*, P9) came to this conclusion on their own and as soon as the evidence appeared; two

(P4, P6) found two pieces of evidence, did not deduce the system has no solutions, and finally did after a prompt from the interviewer; one (P5) came to this conclusion after having found three pieces of evidence; one (P2) seems to have come to this conclusion after a prompt from the interviewer, but did not seem convinced—he decided to tackle the problem via more approaches; and two (P3, P10) did not come to any conclusion about the system. Finally, at some point in their engagement with Problem 6, three students (P2, P3, P4) found their contradictory results (of the type $x = a$ and $x = b$, where $a \neq b$) to mean their *approach* was incorrect.

The participants who completed Problem 6 correctly immediately upon finding evidence that the system has no solutions did so either in Step 1 or 2 of their approach. P1 found the quadratic in EQ1 has a negative discriminant, deduced it has only imaginary solutions, and concluded the system has "no real solutions." (P1 also commented this is "not material for MATH 204" because in that course, "we don't care about complex numbers.") P7* similarly found this quadratic to have negative discriminant; he also found solutions to EQ2, which he said "of course, do not hold for the first equation"; he concluded the system does not have real solutions. He did ask if he "need[s] to take a look at [imaginary] solutions"; after I said it's up to "whatever [he] think[s]," P7* said he didn't think it necessary. P1 and P7* both came to the "no solution" conclusion in Step 1 of their approach; P9's first step was a series of incorrect attempts at Gauss-Jordan elimination—incorrect by virtue of interpreting the RREF of his augmented matrix inappropriately (e.g., introducing unknowns apart from x and ascribing parameters where there was nothing to parameterize). After P9 abandoned this approach, he used high-school techniques for solving EQ1 and EQ2 individually; he found the values of the two distinct solutions for EQ2 and found that EQ1 has only imaginary solutions. He concluded the system has no solutions.

P6 circled the same conclusion as P1, P7*, P9, but only seemed certain of it after a prompt I gave him. Initially, P6 subtracted corresponding sides of EQ1 from EQ2 and produced a new quadratic equation; he found its solutions do not solve EQ1, but made no conclusion from this about the system. His next approach was to solve EQ2; again, he found values that did not solve EQ1. He took this to mean the following: "if I draw the graph of a parabola, a parabola either won't have a y-intercept or an x-intercept, I guess, right? Because this is saying where y is equal to zero, or y- will have two y-intercepts, right? Or it will have one if the y-axis is tangent to the parabola. [The quadratic in EQ2] definitely has two [roots]... But if I plug them into the first one, then the roots are not the same. Okay, so they won't be zero at the same x values." Nevertheless, P6 said he felt frustrated: "maybe these are all just tricky questions, but I feel like I keep on like arriving at the answer that there's no answer. And it's slightly frustrating." Why frustrating? "Because I feel uncertain about my answers. But I also like solving things. And I'm not sure I'm doing a very good job." I asked what it would "take to convince [him] there are no values [of x that make both equations true]" and, at first, he said "better knowledge of linear algebra." Ultimately, though, he was "reasonably certain [of his] answer" but the wording of Problem 6 ("solve") "ma[de him] think there is [a solution]"; "whenever I read a question that says, find this, do that, do this, I assume that it can be found or it can be solved." He said "[he] guesses the solution is that there are no solutions," and as I prepared to move on to Problem 7, he asked: "was I far off?"

P4, like P6, only concluded the system has no solutions after I prompted him in this direction. His spontaneous reaction to the problem was to consider Gauss-Jordan elimination, but “according to [his] knowledge, the linear algebra only works for first order polynomials” (he wasn’t “sure,” though). Instead, he opted to solve EQ1 by finding its discriminant. He found it to be negative and deduced EQ1 has no solutions. P4 then moved on to solve EQ2; he found its solutions to be -3 and 1 and concluded these are the “solutions to the system.” He said “these are two completely independent equations” and decided to “augment” the system and use Gauss-Jordan elimination after all. He wanted to see if his “suggestion at the beginning was wrong - that only first order polynomials work [with Gauss-Jordan elimination].” Once P4 found the RREF of the augmented matrix of the system, he interpreted it to correspond to the following equations:

$$\begin{array}{rcrcrcl} x & & & - & z & = & 0 \\ & & y & + & 2z & = & 0 \end{array}$$

and wrote the parametric equations $x = t$, $y = -2t$, $z = t$ as a general solution to the system. P4 noticed the issue shortly thereafter: “ooh my God! These aren’t even y and z. I was assuming y and z.” But he did not correct his interpretation of the RREF. Instead, he concluded that “this is wrong”—that is, the method of Gauss-Jordan elimination is altogether “wrong” for the problem at stake. He said he would “go” with his earlier conclusion that the solutions to the system are 1 and -3. I asked if these values make both equations true, and P4 said: “yes.”

P4’s last attempt at Problem 6 followed this question; he subtracted the sides of EQ1 from the corresponding sides of EQ2 and produced the equation $x^2 + 3x - 5 = 0$. He solved it and found values of x different from the ones he had found earlier. P4 said he didn’t know why the new values of x differed from the previous ones and plugged them into EQ1; they did not satisfy it. P4 deduced his approach was “wrong,” but he couldn’t say why. “It’s not something I can explain,” he said.

P4 struggled with the system itself: “what really disturbs me was that these two [equations] are grouped together. This one has solution, the second one, and the first one does not.” He proposed, as a conclusion of sorts, that “the first one [has] no solution and for the second one, [the] solutions [are] -3 and 1”—so he “would put that [...] -3 and 1 are the solutions of the system of equations.” I pointed out he said “of the system,” and P4 responded that “the problem is that if you’re saying system, that means the two are related, and these [-3 and 1] should be values that work for both of them.” I pointed out, finally, that “the task is to find the values of x that make both equations true,” and only then did P4 deduce that there aren’t any:

> Oh, yeah! So, yeah, thank you. Because I would say that there’s no solution for the system of equations, mainly due to the first one not having any solution. Yeah, because I just remembered that the definition of system was that the solution would be fitting to all of them, not just to one. So since the first one does not have any solution, then the system of equations doesn’t have any solution.

For P5, the inconsistency between EQ1 and EQ2 did not suffice to deduce the system has solutions; it took more evidence to reach this conclusion. In Step 2 of her engagement with Problem 6, P5 found the distinct solutions of EQ2, and in Step 3, she noted, by observation, the quadratic in EQ1 could not be factored and was not satisfied by 1, one of the solutions of EQ2; she also computed the discriminant of the quadratic in EQ1, found it to be negative, and concluded EQ1 has no solutions. Nevertheless, P5 made no conclusion about the system of equations at this point. She went on to a fourth attempt at the problem, this time using an equation she had found in her first approach ($3x^2+5x-5=0$, found by adding corresponding sides of EQ1 and EQ2) and found it was not satisfied by the solutions of EQ2. P5 then made a last and fifth attempt: she subtracted the sides of EQ1 from the corresponding sides of EQ2 (producing $x^2+3x-7=0$), solved this new equation, found its solutions do not satisfy EQ2, and she finally concluded the system has no solutions. To be convinced the system has no solutions, if she were doing this at home, P5 would use a graphing calculator to confirm there is no intersection.

P2's conclusions were ambiguous and his attempts at Problem 6 numerous. After his initial ditched attempt to use Gauss-Jordan elimination, he started to solve the system by solving each equation; but his approach was to isolate the constant and factorize the left side of an equation (where the expression had the form $ax^2+bx, a \neq 0, b \neq 0$), and he abandoned this promptly. Next, he found the discriminant of EQ1 was negative and found EQ2 has 1 as a solution; he said "this is impossible because if $x=1$, the first equation is impossible." But he did not conclude, from this, that the system has no solutions. Instead, he gave Gauss-Jordan elimination another go.

P2's Gauss-Jordan elimination led to the RREF of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix},$$

He knew this corresponded to the equations $x=4$ and $x^2=-5$. Again, he used the term "impossible" to qualify the latter result. I asked what this tells him about the system and P2 said it has no solutions. Nevertheless, for P2, for Gauss-Jordan elimination to imply a system has no solutions, the RREF obtained must follow a certain dress-code: "you should get 0 0 and then a [non-zero] number. And because I got this [1 0 -5 in first row, 1 0 4 in second], that means that it is not impossible [that is, it is not without solutions]. So there is an answer [a solution]... but what [is it]?" Even though P2 had suggested the system might not have a solution, he does not seem to have been convinced of this: instead, he was not sure the Gauss-Jordan approach was appropriate ("I'm not sure I can use a matrix") and went on to tackle the problem with other techniques.

In Step 5, P2 estimated that EQ1 *should* have a solution because, after plugging -0.1 as a value for x, found the quadratic was approximately 0.99; this is in spite of P2's earlier finding that EQ1 has no solutions (in light of the negative discriminant). He went on, unprompted, to solve the system using the high-school method of addition (multiplying both sides of one equation by a constant and adding corresponding sides to the second equation, so as to produce an equation with less unknowns); he found $x=4$ but said nothing of this, continuing instead onto a seventh approach to Problem 6, setting the quadratics of EQ1 and EQ2 equal to one another and solving this equation. He found two values for x, but was "certain this is wrong because if I substitute this into the first equation, then there's no way to get 0." In a last and eighth attempt at Problem 6, P2 reflected on the various "impossible" results he found in his attempts but was sure "there's an answer": "if this was on a test, then I would write it's impossible; I know there's an answer" and, after again referring to an "impossible" result he found ($x^2 = -5$, from Gauss-Jordan elimination), reiterated that he is "sure there's an answer." I infer that by "answer," P2 indeed meant a value that solves the system: he expected "the range of the answer to be between -1 and 0." P2's final comments were an acknowledgement that his assessment may be incorrect: after I asked why he was sure there is an answer, he paused and eventually said that "it might be impossible because [he] used $x = -0.1$ and got 0.99" in EQ1 (Step 5) and re-observed that the solutions found for the equation in Step 7 do not solve the first equation in the system.

Finally, P3 and P10 did not deduce the system has no solutions at any point in spite of having found evidence that shows the system has no solutions.

P3's spontaneous reaction to the problem was to solve EQ2; upon finding its solutions do not satisfy EQ1, P3 sought another approach ("I'm trying to find if there is any other way that I can solve this problem"). P3 then isolated x^2 in one equation, substituted this expression for x^2 $(-x-1)$ in the other equation, solved the new equation, found $x = 4$, checked if this solved the system. It did not. A third approach: P3 substituted x^2 by t in both equations, wrote the augmented matrix for the system, reduced it to its RREF, and interpreted it to mean that $t = -5$ and $x = 4$. P3 knew this is a contradiction, but assigned blame to the wrong perpetrator: "if this is the right approach, [the value found for] t should be [the value of x^2] but it is not." P3 concluded the *approach* was "wrong" because, if $x = 4$, then "t should be 16." This was P3's last attempt at Problem 6. She said that "[the graphs of $x^2+x+1 = y$, $2x^2+4x-6 = y$] do have an intersection": "I feel like they do; they *should* have an intersection." She sketched concave-up parabolas with vertices above the x-axis and, after I asked why she thinks the graphs are like this, made a new sketch of a concave-up parabola crossing the x-axis at -3 and 1 (thus representing $2x^2 + 4x - 6 = y$), but did not address what the graph of $x^2 + x + 1 = y$ might look like and how these sketches relate to the problem—that is, that the interest, in Problem 6, is in points of intersection of the parabolas along the x-axis.

P10 also found evidence of the system being inconsistent. In some of her approaches, she either struggled with the algebra (in Step 1, she tried to factorize a quadratic over the integers, and then abandoned the approach) or solved one equation but did not verify if it satisfied the other equation (Step 2). In Step 3, P10 used Gauss-Jordan elimination and found the RREF of the augmented matrix to correspond to the equations $x^2 + 5 = 0$ and $x = 4$, which she interpreted to mean $x^2 = \sqrt{5}$ and $x = 4$. P10 paused twice as she explained what this meant. She was not sure because of the "two separate answers" and eventually said neither "seem to fit the equation[s]." P10's last attempt at Problem 6 was to solve EQ2; she did so incorrectly, finding values of x that do not solve it, but did not say anything about the solutions she found.

All participants found evidence of inconsistencies in the system they were given, but most participants struggled to come to a conclusion about the system. Participants' continued attempts to hack away at the problem suggest they were *aware* these inconsistencies were, in effect, inconsistent. Their reluctance or inability to conclude, from this, that the system has no solutions highlight gaps in their knowledge about systems of equations and what it means to "solve" them. I discuss this in Section 5.6.3.2. Parallel to participants' bereft knowledge about what is implied by "solving" a system was a lack of autonomy in evaluating the validity of system-solving techniques they had used in LA1. This lack of autonomy presented in students as a lack of confidence in their approach or an inability to evaluate the validity of their approach (e.g., deducing, from contradictory values found for x, that their approach must be incorrect). This lack of autonomy points to an absence of knowledge about the algebraic and logical foundations of system-solving techniques students are to learn in LA1; in Section 5.6.3.3, I discuss comments students made that suggest students' lack of autonomy may be supported by LA1 norms that allow students to operate exclusively along surface-level features of problems they are to solve.

5.6.3.2 For many students, that the system might have no solution was an unpalatable possibility.

Apart from P1, P7*, and P9, who concluded on their own the system has no solutions as soon as they found evidence to that effect, the rest of the participants struggled with this evidence.

P5 came to this conclusion on her own, but only in her fifth attempt at the problem and after having found three pieces of evidence of inconsistencies in the system.

P4 and P6 explicitly acknowledged the results they found (on more than one occasion) were problematic—in the sense of inconsistency between the equations—but only concluded the system has no solution after I pointed out that a solution to the system must solve every equation in the system. P4's comments tipped me off that he needed this clarification: "what really disturbs me was that these two are grouped together. This one has solution, the second one, and the first one does not." He continued: "the problem is that if you're saying system, that means the two [equations] are related, and [the solutions of EQ2] should be values that work for both of them."

P2 had also found several results that each pointed to an inconsistency between the equations, and suggested the system had no solution after I asked what one of his results ($x = 4, x^2 = -5$) told him about the system. But P2 was not convinced the system had no solution: he expected EQ1 to have a solution because plugging in a value for x (0.1) produced 0.99, which he estimated to be close to 0. (This is in spite of having found, earlier, that the discriminant of EQ1 is negative.)

P3 and P10, finally, did not make comments that suggested suspicion to the possibility that the system has no solutions. Instead, P3 blamed her inconsistent results on her approaches, which she presumed faulty. P10 did not offer much reasoning about her results; in her penultimate step, she noted that two values she found for x did not "fit" the equations, and finally tried but failed to solve EQ2.

Participants were either reluctant or unable to conclude the system had no solution despite ample evidence to that effect. In the next section, I discuss the (LA1 and TBI-related) institutional elements that may explain the unpalatability of this possibility.

5.6.3.3 Students did not have agency over system-solving techniques and results thereof; they may have also had a weak sense of agency over these matters.

In this section, I investigate the hesitation of six students (P2, P3, P4, P5, P6, P10) as they struggled to make a conclusion about the system after they found inconsistencies in the system.

Several students suspected their inconsistent results were due to incorrect technique but offered no justification as to why their technique might be incorrect, other than that it led to an inconsistent result. P2, P3, P4 made comments to this effect; P2 and P4 fished for more and more approaches every time they came to an inconsistent result—all essentially based in the same concept of producing equivalent equations, but not always producing equivalent *systems* (e.g. as they reduced the given system to a single equation); P6 was frustrated that, yet again in this interview, he had found inconsistent results and doubted the accuracy of his approaches; P5 concluded the system had no solutions conditionally ("if I did the math right, the x values for this one don't make the [quadratic in the] first equation equal to zero") and only after seeing three pieces of evidence to this effect—to be convinced of this conclusion, if she had been doing this at home, she would use a graphing calculator to see that the graphs have no intersection.

5.6.3.3.1 Students were positioned as participants who had struggled with previous problems in the TBI; this may have weakened their sense of agency relative to LA1 knowledge. By "sense of agency," I refer to the meaning suggested by Sierpinska et al. (2008): to have a sense of agency is to believe one can make something happen. The TBI problems were designed to bring out what students did and did not know in relation to LA1 knowledge, and by the time P6 reached Problem 6, for instance, he indicated his sense of agency had taken a hit:

> Maybe these are all just tricky questions, but I feel like I keep on like arriving at the answer that there's no answer. And it's slightly frustrating. Because I feel uncertain about my answers. But I also like solving things. And I'm not sure I'm doing a very good job.

P6's comments, in addition to those made by others (e.g., P4's claim that Problem 6 is "a very peculiar question") attest to his position as participant in a study that presented him with problems that brought to light limitations of knowledge that had sufficed to pass a course he had recently taken. This contrasted with his position as someone who enjoys to solve problems; on this and other occasions in the interview, P6 said he enjoyed to read books about material in certain domains of mathematics and had actually brought one with him to the interview, to ask me a question about content in that book. A concern with "doing a very good job" was part of what P6 took into consideration as he tackled the interview problems. This may have undermined his confidence in the knowledge he had. Even after P6 concluded the system has no solution and as I was about to give him the next problem, he sought reassurance: "was I far off?" His knowledge of the mathematics at stake was insufficient face and, to appease a resulting lacking sense of agency, he appealed to my authority as the interviewer.

P6's comments—making explicit the frustration and uncertainty he felt in tackling problems that challenged LA1 norms—suggest that the hesitation with which students grappled, as they struggled to make a decisive conclusion about a system that ostensibly had no solution, may have been reinforced by a lost sense of agency.

5.6.3.3.2 Several students' take on augmented matrices and elementary row operations neglected the mathematics at stake. The notion of augmented matrices was mobilized incorrectly by two students: P9, who spontaneously concluded the system has no solution after he found EQ1 to have a negative discriminant and EQ2 to have two real solutions, and P4, who only concluded the system had no solution after I said (following his fourth attempt at Problem 6) that "the task is to find the values of x that make both equations true" (P4 had already found that EQ1 had no (real) solution). Two other students (P2 and P3) mobilized augmented matrices and elementary row operations appropriately but failed to conclude the system has no solution (despite having found a result pointing to this conclusion); for P3, this was due to uncertainty as to whether the technique *was* appropriate, and for P2, this was because he prioritized a LA1 rule about systems with no solutions (a rule accurate for *linear* systems but not others) over the inconsistent result he found.

P9's spontaneous reaction to Problem 6 was an incorrect application of row reduction to the system. Initially, he interpreted the reduced row echelon form (RREF) of the coefficient matrix appropriately: $x^2 = -9$, $x = 4$. But then, he paused. Eventually, he ammended these equations to $x^2 = -9t, x = 4t, x = t$. He did not explain why.

Next, P9 restarted the row-reduction process from scratch; this time, he found the RREF to be

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix},$$

and deduced that $x = -3$ and $x = 2$. He then started over—again—this time reducing to

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix}$$

and said the answer is $x = 3$, $x = 4$, and "the third one is equal to t," writing $x^2 = 3t, x = 4t, x = t$.

P9 was unable to land on an accurate interpretation of the RREF of the augmented matrix. He used the entries as coefficients, sometimes of the appropriate unknown and othertimes not, and he introduced a parameter as a result of this confusion: in his last attempt, he treated the third column entries as coefficients of x, in which case x would have been a free variable and hence assigned the parameter t. In contradiction with this (already incorrect) interpretation, P9 also used the second column entries as coefficients of x. P9 abandoned this approach and opted to calculate the discriminants of the quadratics instead.

When P4 used row reduction, he interpreted the reduced row echelon form of the augmented matrix as a system of equations in unknowns x, y, and z, as per the LA1 norm for coefficient matrices with three columns. From his equations, he obtained the parametric equations $x = t, y = -2t, z = t$ and said “this doesn't work in any way to [his] quadratic way [sic] of approaching a problem.” He added he wouldn't “trust” this approach. As he kept talking, he realized that “of course [he] wouldn't [trust it]” because what he was solving was the equation $x^2 + x + 1 = 0$, so the unknowns “aren't even y and z.” He concluded this approach was “wrong” and made no attempt to correct it (by interpreting the RREF in terms of the appropriate unknown). Instead, he said he would stick with the solutions he found by factoring the quadratic in EQ2: values of x which he insisted, at that point, to be solutions to the *system*.

P4 seemed primed to doubt the suitability of row reduction as a technique for solving a system of two quadratic equations. His spontaneous reaction to Problem 6 was this monologue:

> I want to use a quadratic formula, but I'm thinking maybe there's a better way to go about this. I would think about augmenting it. According to my knowledge, linear systems work and linear algebra only work for the first order polynomials. I think. I'm not sure... So I will actually, first let's start with the quadratic equation.”

And he did. After his first three attemps at Problem 6, P4 decided to give the augmented matrix approach a try to “see if [his] suggestion at the beginning was wrong - that only first order polynomials work.” He did not mobilize algebraic theory that underpins augmented matrices and elementary operations to assess whether these tools are appropriate; to check their suitability, he planned to wield the result they would yield.

After P4 realized he had misinterpreted the RREF of the augmented matrix of the system, he did not correct his interpretation. He rejected the approach altogether. This, together with the comment that he was “not sure” as to whether an augmented matrix would “work” in this context, constitute a failure to mobilize mathematical theory at stake in Gauss-Jordan elimination: what the entries in an augmented matrix represent, first, and second, what elementary row operations represent relative to the original system. P9 did not voice suspicion as to the suitability of Gauss-Jordan elimination as an approach for the problem at hand, but his inability to identify what the matrix entries represent (relative to the given system) and his subsequent abandonment of the technique are similarily a failure to mobilize mathematical theory at stake in this technique.

P3 mobized row reduction accurately and found the RREF of the augmented matrix to correspond to the equations $x^2 = -5$ and $x = 4$. She knew this is contradictory, as x^2 is 16 if x is 4. But for P3, this did not mean the system has no solution. Instead, it meant the technique was inappropriate.

P2 also mobilized row reduction accurately, but his comments about Gauss-Jordan elimination suggest that a driving force behind his conviction that the system *has* solutions (despite all evidence he found to the contrary) was surface-level knowledge that normally sufficed in LA1. When he reflected on the RREF he found in Step 4,

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix},$$

he fixated on a rule about what the RREF of an augmented matrix must look like for a system to have no solutions: "you should get that if it is impossible [if there are no solutions], you should get 0 0 and then a [non-zero] number. And because I got this [1 0 -5 in first row, 1 0 4 in second], that means that it is not impossible. So there is an answer... but what [is it]." P2's knowledge about systems with no solutions was a surface-level property of augmented matrices: *for a system to have no solutions, the RREF of the augmented matrix must include a row in which all the entries except for the right-most entry are zero.* This rule works in LA1 as systems are linear in that course; the issue is that the norm P2 had developed in LA1 - just like the norm developed by P4 and P9 - did not enable him to use *other* knowledge about system-solving toward the task in Problem 6. P2 knew that "if you had 0 0, then a [non-zero] number, it is impossible because $0x^2 + 0x$ cannot be equal to a number other than 0." This is true! But P2 expected this to be the only scenario in which a quadratic system might have no solution.

P2's long search for a resolution to Problem 6 (amounting to 8 different attempts) stemmed, in part, from him prioritizing a surface-level rule about Gauss-Jordan elimination over algebraic and logical concepts that justify augmented matrices and elementary row operations. As a consequence, for P2, that the RREF produced the equation $x^2 = -5$ did *not* mean that the original system corresponded to a system with no solutions. Ultimately, he was "not sure I can use a matrix" for the problem at hand.

The surface-level rule P2 activated is accurate for linear systems and, as such, is not problematic in LA1. But the institutional norm in LA1 does not present students with instances in which to engage with the algebra and logic that justify and produce the notions of augmented matrices and elementary row operations, concepts at the core of the course. For P2, P3, P4, and P9, not mobilizing the underlying theory was an obstacle in Problem 6. P2 made king of surface-level rules at the expense of what he had found (that the RREF of the augmented matrix corresponded to a system with the equation $x^2 = -5$). He would not conclude the system had no solution because the rule about the *no solution row-type* was not satisfied. The algebra and logic underpinning Gauss-Jordan elimination were not knowledge to be mobilized; this resulted in four participants' inability to adapt the technique for a system of two quadratic equations.

5.6.3.3.3 Participants did not activate algebraic and logical theory that underpin system-solving techniques taught in high-school and LA1. In addition to P2, P3, P4, and P9's failure to mobilize algebra and logic at the basis of Gauss-Jordan elimination, other system-solving attempts were rife with instances in which they failed to activate the same algebra and logic underlying the concept of equivalent systems.

In Step 2 of her approach to Problem 6, P3 used the high-school technique of substitution to solve the system: she isolated x^2 in one equation, substituted the expression for x^2 $(-x - 1)$ in the other equation, solved that equation, found that $x = 4$, checked if this solves the system, found that it does not - and did not say, as a conclusion, that the system has no solution, even after Step 3 of her approach, in which she used Gauss-Jordan elimination, found contradictory values for x and x^2, and deduced the technique was not suitable. P3, like other participants who found inconsistent values for x, did not have an explanation for the results she was finding.

Six of nine participants struggled to make something of the contradictory values they found for x via system-solving techniques; this reflects an inability to mobilize the notion of equivalent systems. The notion of equivalent systems was absent from students' knowledge in another type of instance: some students 'reduced' the system they'd been given to a *single* equation. The steps involved in such 'reductions' were one of three options:

- Some students equated the quadratics in EQ1 and EQ2, justifying it on the basis that both quadratics must equal 0.
- Other students added the left sides of the equations to one another and their right sides as well to produce a new equation.
- Other students subtracted the left side of one equation from the left side of the other equation, and did similarly with the right sides.

In all three cases, students did not mobilize the notion of equivalent *systems*. This was rectified as students knew to verify whether the values they found for x satisfied the two original equations—and discovered they did not (P2 in Step 7, P3 in Step 2, P4 in Step 4, P5 in Step 4, P6 in Step 1, and P10 in Step 3).

What students did not seem to know was that in producing a new equation, they had produced an equation that was equivalent to *one* of the equations in the original system. When P2 used the *reduce-a-system-to-a-single-equation* technique, he found two possible values for x and was "certain" what he did was "wrong" because the values did not satisfy EQ1; but he had no other explanation as to why the technique he used might have been "wrong." For P4 (in Step 4), that the 'reduction' technique produced a value for x that didn't satisfy EQ1 and EQ2 also meant the approach was wrong. And it is. A system of two equations does not correspond to a single equation. But P4 did not know what the issue was: "I was trying to subtract the first one from the second to form a new solution, to see if it makes any difference. And it does [as the solutions of the new equation are different the solutions of EQ2, and they don't solve EQ1]. But it's not something I can explain." When students use Gauss-Jordan elimination, they unwittingly produce equivalent systems throughout the resolution process. It is not necessary to know the concept of equivalent systems to pass LA1.

5.6.3.3.4 Summary: most students lacked the agency and sense of agency needed to conclude the system had no solutions. I distinguish, as per Sierpinska et al. (2008), between *sense of agency* and *agency*: to have agency is to "*actually* [make] things happen." In the context of solving a problem in mathematics, to have agency is to have the mathematical knowledge needed to settle the problem. In this context, a student may have a *sense* of agency (a sense they can make things happen) without actually having any agency. I view students' hesitation vis-à-vis the inconsistencies they found as the reaction wrought by lacking *agency*, accompanied in at least some students by a lacking *sense* of agency triggered by their struggle to adapt norms that worked in LA1 to interview tasks.

Apart from the three students who immediately deduced the system had no solution once they found EQ1 to have no (real) solution and EQ2 to have two real solutions, the other six participants struggled, even though they found similar evidence in one or more of their attempts. P5 needed to find evidence of no solution to the system three times before accepting this conclusion; P2 was indecisive as to whether the system had no solution; P4 and P6 concluded there was no solution only after I specified, in a prompt, that a solution would satisfy both equations (P4 thanked me after this prompt and said he had forgotten this was what was meant by "solutions of a system"); and P3 and P10 made no explicit comment suggesting there might be no solution.

In light of P4 and P6's comments about the problem - "peculiar" (per P4) and "tricky" (per P6) - and the feelings these problems might have triggered in them (e.g., P6's frustration), I infer that a diminished sense of agency played a role in at least some students' hesitation to conclude the system has no solution. I hypothesize that such diminished sense of agency was brought on, in part, by the design of the interview problems (to elicit what students do and do not know relative to LA1 knowledge students are expected to learn), and by their position as *student* or *study participant* (one with less authority over the knowledge at stake than a teacher or interviewer).

Participants' potentially diminished *sense* of agency notwithstanding, it remains that their knowledge from LA1 was bereft: first, it did not suffice for six of nine students to voluntarily mobilize knowledge of what constitutes a solution of a system of equations, and second, several students did not mobilize any of the algebra and logic on which LA1 system-solving techniques are based to evaluate the suitability of their techniques, even as they suggested their technique might be incorrect (as P2 and P4 did, for example, in reference to both Gauss-Jordan elimination as well as the reduce-a-system-to-an-equation technique). I therefore surmise that any potentially diminished sense of agency does not fully account for the hesitation engendered in students by the system to be solved in Problem 6. Several participants did not have the *agency* needed to conclude the system had no solution (indeed, P4 explicitly acknowledged having forgotten that a solution of a system must satisfy all its equations); and the norms in LA1 for solving systems did not provide students with the agency needed to evaluate the suitability of their system-solving techniques.

5.7 LA1 Problem 7

The following was the seventh problem presented to 9 of the 10 LA1 students[26] in the TBI:

Determine the number of solutions of this system of equations:

$$(x, y) = (1, 3) + t(1, 5)$$

$$(x, y) = (3, 7) + r(-2, 1)$$

[26]Due to time constraints unrelated to the TBI, P8 was only able to do Problems 1 - 5.

5.7.1 Reference model for LA1 Problem 7

The task t in Problem 7 is to find the number of solutions of a system of two vector equations in $\mathbb{R}^2$. This is a task of type T: to find the number of solutions of a system of two linear equations. Linear systems have either no solutions, one solution, or infinitely many solutions; indeed, if there are two distinct solutions, then these two can be used to construct infinitely many distinct solutions. Graphically, the solution to a linear system in $\mathbb{R}^2$ corresponds to the set of intersection points of the lines that are graphs of the linear equations. Two linear equations in $\mathbb{R}^2$ correspond to two lines: two lines may be parallel and distinct (in which case the system has no solution); they may overlap (in which case there are infinitely many solutions); or they may be non-parallel and then intersect at exactly one point (in which case the system has one solution).

The linear equations in Problem 7 are given in vector form. An equation of type

$$(x, y) = (x_0, y_0) + t(v_1, v_2), t \in \mathbb{R} \tag{5.6}$$

is defined as the equation of a line. The following paragraphs summarize the geometry at the basis of the component definitions of vector, scalar multiplication of vectors, and vector addition, which together justify this definition of vector equation of a line.

A vector v $= (v_1, v_2) \in \mathbb{R}^2$ is defined geometrically as the arrow with the origin $(0, 0)$ as its initial point and the point (v_1, v_2) as its terminal point; any arrow of the same length and with the same direction, regardless of the placement of its initial point in 2-space, is said to be equivalent to v. The zero vector is a vector whose terminal point is also its initial point; in component form, this is $(0, 0)$. The sum of u + v two vectors u $= (u_1, u_2)$ and v $= (v_1, v_2)$ is defined as $(u_1 + v_1, u_2 + v_2)$. This definition corresponds to the geometric definition of vector addition: with the initial point of v placed atop the terminal point of u, u + v is defined as the arrow whose initial point is that of u and terminal point is that of v (see Figure 5.1).

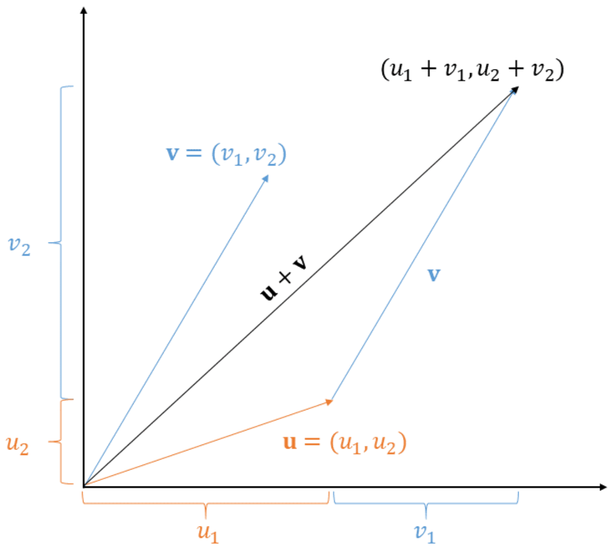


Figure 5.1: Vector addition in $\mathbb{R}^2$

A negative of a vector v $= (v_1, v_2)$, geometrically, is a vector such that the sum of the two forms the zero vector. The negative −v of vector v is unique and −v $= (-v_1, -v_2)$.

The scalar multiple $k\text{v}$ ($k \in \mathbb{R}$) is defined as the vector (kv_1, kv_2); given concepts from Euclidean geometry (e.g., proportionality of edges in similar triangles), this corresponds to the geometric definition of $k\text{v}$, the vector whose length is $|k|$ times the length of v and whose direction is the same as that of v if k is positive and the opposite of the direction of v if k is negative. Two vectors are said to be parallel if they are scalar multiples of one another.

In light of the above definitions, and given the reasoning captured in Figure 5.2, equation 5.6 is called the vector equation of the line parallel to (v_1, v_2) and going through the point (x_0, y_0).

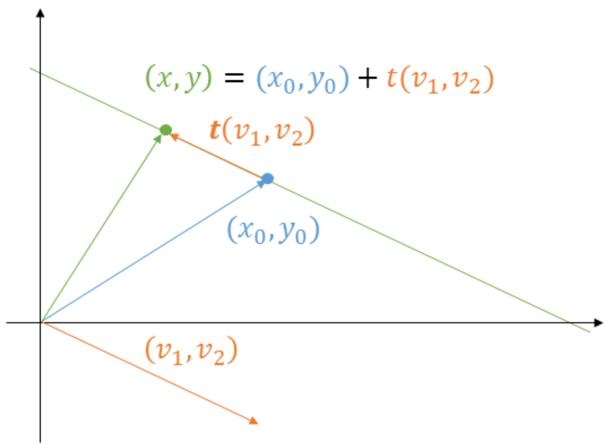


Figure 5.2: Graph of the line $\ell : (x, y) = (x_0, y_0) + t(v_1, v_2), t \in \mathbb{R}$

The exposition so far, along with the algebra, logic, and geometry that frames it, makes up the theory Θ that frames task t (to find the number of solutions of a system of two vector equations in $\mathbb{R}^2$).

The solutions of the equations in Problem 7 form two lines: one line parallel to $(1, 5)$ and the other to $(-2, 1)$. Since these vectors are not scalar multiples of one another, they are not parallel; hence, the lines are not parallel, and must intersect at only one point. Hence, the system of equations has one solution. I will denote by τ_1 the way in which I used the definitions of vector equation of a line (θ_{11}), of parallel vectors (θ_{12}), and the postulate that two non-parallel lines in $\mathbb{R}^2$ intersect at one point (θ_{13}). τ_1 is one technique through which task t can be performed.

t can also be accomplished by solving the system algebraically. Given the definitions of scalar multiplication and vector addition (θ_{21}), the system of equations in Problem 7 is as follows:

$$(x, y) = (1 + t, 3 + 5t)$$
$$(x, y) = (3 - 2r, 7 + r)$$

By definition of equality between vectors (θ_{22}), it follows (x, y) is a solution if and only if its x-coordinate has the form $1+t$ and $3-2r$ for some $t, r \in \mathbb{R}$ and its y-coordinate has the form $3 + 5t$ and $7 + r$ for the same t, r. Now, $1 + t = 3 - 2r$ and $3 + 5t = 7 + r$ if and only if $t = \frac{10}{11}$ and $r = \frac{6}{11}$. The ordered pair (x, y) produced by these values for t and

r is therefore the only solution to the system - and so, the system has only one solution. I denote this second technique by τ_2. I denote by Θ_2 the algebraic and logical discourse that frames the techniques for solving linear equations in $\mathbb{R}^2$.

Another technique through which to perform t is to find point-normal equations corresponding to the system. I added this technique and its underlying theory to my reference model after two participants suggested it in response to Problem 7; it is a circuitous approach to the problem that renders it into one perhaps more familiar to students from LA1, as they are typically given systems of linear equations in point-normal form (i.e., $ax + by = c$) rather than in vector form.

A point-normal equation for a line ℓ in $\mathbb{R}^2$ is based in the notion of orthogonality. Two vectors are said to be orthogonal if their dot product is 0. A normal of a line is a vector orthogonal to the line. Suppose ℓ contains a point (x_0, y_0). Suppose $\mathrm{n} = (a, b)$ is a normal of the line. If $(x, y) \in \ell$, then $(x - x_0, y - y_0)$, the vector with initial point (x_0, y_0) and terminal point (x, y), is parallel to the line. As such, it is orthogonal to n: $\mathrm{n} \cdot (x - x_0, y - y_0) = 0$. Conversely, any point (x, y) that satisfies the equation $\mathrm{n} \cdot (x - x_0, y - y_0) = 0$ must be a point in ℓ. This is the reasoning for which

$$(a, b) \cdot (x - x_0, y - y_0) = 0,$$

that is,

$$a(x - x_0) + b(y - y_0) = 0,$$

is said to be an equation for ℓ; it is called a point-normal equation of the line. I denote by θ_3 the exposition through which I produced this point-normal equation.

I denote by τ_3 the technique in which the vector equations in Problem 7 are represented via point-normal equations and the system is solved using these equations instead. As the line $(x, y) = (1, 3) + t(1, 5)$ is parallel to $(1, 5)$ (knowledge deduced from θ_{11}), a normal (a vector orthogonal) of this line is $(-5, 1)$ (since $(1, 5) \cdot (-5, 1) = 1(-5) + 5(1) = 0)$). I denote by τ_{31} know-how for finding a normal of a line given a vector parallel to that line. The line contains the point $(1, 3)$, so a point normal equation for the line is

$$-5(x - 1) + (y - 3) = 0,$$

that is, $-5x + y = -2$. I denote by τ_{32} the know-how in which the point-normal equation of a line is produced given the formula produced by θ_3, a normal for the line, and a point on the line. Using τ_{31} and τ_{32} again, it can be shown a point-normal equation for $(x, y) = r(3, 7) + (-2, 1)$ is $x + 2y = 17$. The task is then to determine the number of solutions of the system

$$\begin{array}{rcrcr} -5x & + & y & = & -2 \\ x & + & 2y & = & 17 \end{array}$$

Any of the techniques discussed in the reference model for Problem 2 shows this system has one solution; I denote this family of techniques by τ_{33}. In sum, τ_3 corresponds to the combination $\{\tau_{31}, \tau_{32}, \tau_{33}\}$ as outlined above.

The reference model for activity through which to complete Problem 7 is summarized by these praxeological models:

- $[t; \tau_1; \theta_{11}, \theta_{12}, \theta_{13}; \Theta]$;
- $[t; \tau_2; \theta_{21}, \theta_{22}; \Theta, \Theta_2]$; and
- $[t; \tau_3; \theta_3, \theta_{11}; \Theta, \Theta_2]$.

5.7.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 7

The type of task in which the number of solutions of a linear system is at stake appears in 5 of the past exams to which I had access to from 2014 to 2019 and in 2022: 2 of 7 final exams and 3 of 5 midterm exams. In all 5 problems that focused on the number of solutions of a linear system, the linear system was in 3 equations in the unknowns x, y, and z, and some of the coefficients and/or constants were either expressions in an unknown k (for example, $k(k-1), k+1, k^2-1, 2k+3, 2k, k$) or the integers 1, 2, or 3. In all 5 of these problems, the task was to find the values of k for which the system has no solution, exactly one solution, or infinitely many solutions. The technique for this task is to use Gauss-Jordan elimination on the augmented matrix of the system. For the values of k for which the reduced row echelon form (RREF) of the coefficient matrix is the identity matrix, the system would have one solution; for the values of k such that the RREF of the coefficient matrix has a row of 0's, the question of whether the system has no solution or infinitely many would be answered by the value of the right-most entry in that row, but in the augmented matrix (if that entry is non-zero, then he system has no solution; otherwise, the system has infinitely many).

The task in Problem 7 differs from these final and midterm exam problems in two ways. First, the task here is *not* to find conditions under which a given system has such-and-such number of solutions; the task is rather to determine the number of solutions. Second, the format in which the linear system is given is different. In the LA1 exam problems, the system is given in the form of point-normal equations ($ax+by+cz=d$), whereas in Problem 7, the system is in the form of vector equations ($(x, y) = (x_0, y_0) + t(v_1, v_2)$). In the next few paragraphs, I discuss the knowledge students are expected to learn in LA1 that would be pertinent to the task at hand.

In LA1, knowledge to be learned about the number of solutions of systems of linear equations includes the theorem that such systems have either no solutions, 1 solution, or infinitely many solutions. The graphical representation of these options (in $\mathbb{R}^2$ or $\mathbb{R}^3$) are knowledge to be taught: linear equations of the form $ax+by=c$ correspond to lines, and, in $\mathbb{R}^2$, lines can either overlap (and so have infinitely many points of intersection), be parallel and distinct (and so have no points of intersection), or not be parallel (and so have only one point of intersection); and, in $\mathbb{R}^3$, linear equations of the form $ax+by+cz=d$ correspond to planes, which can either overlap (and so have infinitely many points of intersection), intersect along a line (and so have infinitely many points of intersection), be parallel and distinct or pairwise intersect along 3 distinct lines (and so have no point of intersection), or be such that they have exactly one point of intersection (if two planes intersect along a line and the third plane intersects that line at a single point).

The correspondence between graphical and algebraic representations of linear systems appears as knowledge to be learned in 3 problems on 2 of the 13 exams to which I had access (2 of the problems occur as parts (a) and (b) of the same problem on 1 exam). In one of these problems, the task is to find the point-normal form of a plane given a normal (a, b, c) for this plane and a point (x_0, y_0, z_0) on this plane. The knowledge needed to complete this task is the following: that if (a, b, c) is a normal of a plane, then a point-normal form of that plane is $ax + by + cz = d$, where d is the value found by plugging the coordinates of (x_0, y_0, z_0) into the expression $ax + by + cz$. In Section 5.6.2, I discuss the other two problems (parts (a) and (b) of one exam problem) in which the correspondence between graphical and algebraic representations of linear systems shows up as knowledge to be learned: the first task was to find the equation of a plane that passes through a given point and a line ℓ that is defined as the intersection of two planes given in point-normal form ($ax + by + cz = d$), and the second task was to find the coordinates of the intersection of ℓ and a plane $\mathcal{P}$, where $\mathcal{P}$ is also given in point-normal form. Core to the completion of these tasks is the knowledge that a solution to a system of equations corresponds to a point of intersection of the graphs of the equations. Considering the one feature shared by the few exam problems that involve the correspondence between graphical and algebraic representations of linear systems, I conclude that the knowledge to be learned here includes the point-normal forms of planes in $\mathbb{R}^3$.

The knowledge students are to learn in LA1 about the number of solutions of linear systems has less to do with graphical representations and consists, mainly, of a technique through which to determine this number: to solve the system. In the final exam problems in which the number of solutions of a system was at stake, students could use Gauss-Jordan elimination (or Gaussian combined with computations directly involving equations) to complete the task. In Problem 7, this technique cannot be implemented directly. To implement this technique, students have two options.

One option through which to activate the usual system-solving technique is to activate what they are expected to learn, in LA1, about addition, scalar multiplication, and equality in $\mathbb{R}^2$ (and which mimics what they are to learn about matrix algebra): that is, θ_{21} and θ_{22} from my reference model. This is needed to obtain, from the given system, the following system instead:

$$\begin{array}{rcrcr} x & = & 1 & + & t \\ x & = & 3 & - & 2r \\ y & = & 3 & + & 5t \\ y & = & 7 & + & r \end{array}$$

This leads to mobilizing τ_2 from my reference model. Students who mobilize their knowledge of vector addition, scalar multiplication, and equality to produce these equations needn't recognize that the vector equations they were given are graphically represented by lines; it suffices to mobilize system-solving techniques (from high-school, such as substitution and/or addition) to solve the above system and find it has a unique solution.

A second option through which to activate the usual system-solving technique is to find point-normal equations for the lines, as in $\tau_3 = \{\tau_{31}, \tau_{32}, \tau_{33}\}$ in my reference model. I note that students are never tasked on midterm and final exams to find point-normal equations of lines given their vector equation in $\mathbb{R}^2$—the task completed by mobilizing $\{\tau_{31}, \tau_{32}\}$, which is nevertheless a mobilization of knowledge to be taught (θ_3 in my reference model), and which I will therefore denote by $\{\tau_{31}, \tau_{32}\}^{KtbT}$. The two exam problems that *do* require students to know the anatomy of a vector equation for a line are in the form of the following task: to find parametric equations of a line passing through a given point P and perpendicular to a plane (given in point-normal form). To do this, students need to recognize, from the point-normal equation of the plane, a normal of that plane; students need to know this normal is perpendicular to this plane; students need to know this normal is parallel to the line they seek; and students need to know they can use this normal as vector v and the given point P in the vector equation $\mathrm{x} = P + t\mathrm{v}$ for the line (as P is a point on the line and v a vector parallel to the line).

To perform τ_3 toward Problem 7 (that is, to find point-normal equations for the lines so as to then solve the system of these equations), students must know the anatomy of a vector equation (i.e., its definition, denoted by θ_{11} in my reference model for this problem), which is knowledge to be learned so as to do only 2 of the 116 exam problems to which I had access (and which I therefore denote by θ_{11}^{KtbT}, as it is predominantly "knowledge to be taught" in LA1); they must also know that a normal of a line is a vector perpendicular to it (knowledge that is expected of students in a slightly greater number of exam problems); and, to find a normal for the given lines, they must know to produce—by observation—a vector perpendicular to a given vector (observation, as a technique for completing tasks, is not typically expected of students in LA1). Once point-normal equations for the lines are produced, students can then use τ_{33}: any of the techniques to be learned in LA1 for solving systems of linear equations (in point-normal form).

A last technique through which to complete Problem 7, using knowledge to be learned in LA1, mobilizes knowledge of vector equations. While it is not *necessary* to recognize that Problem 7 features vector equations of lines, this is the knowledge through which the task can be completed most efficiently—as in τ_1 from my reference model: the technique wherein it is noted that the lines are not parallel and therefore intersect at only one point. My focus in this paragraph is on the geometry captured by vector equations of lines (or planes) and which is needed so as to mobilize τ_1: that such equations indicate a point on a line (or plane) and a vector parallel to a line (or two non-collinear vectors parallel to a plane)[27]. This is knowledge that is ideal to have been learned in LA1 in the context of 5 problems (on 4 final exams) among the past exams to which I had access. I specify "ideal to have been learned" because 3 of these problems can also be completed without knowledge of vector equations. For example, one of the problems is to find the distance between a point and a plane; the plane here is given in terms of a point it contains and a line it contains (given in the form of parametric equations). Hence, to find the distance between the given point and plane, students must first find an equation for the plane. To this end, students must engage with the given parametric equations of the line. If a student knows vector equations of lines, they can recognize from the parametric equations a vector parallel to the line and a point on the line and complete the task more promptly. That said, if a student does not know vector equations of lines, they can still complete the task: they can, instead, use the parametric equations to find coordinates of two points on the line and then use these to find a vector parallel to the

[27]The reasoning behind the geometric correspondence between lines and these two types of equations, as delineated in the reference model for this problem (in Section 5.7.1), is knowledge to be taught in LA1, but not to be learned.

line. Given that among the 5 problems that could involve vector equations of lines or planes, 3 can be completed without knowledge of the anatomy of a vector equation (i.e., that it involves a point on a line/plane and a vector parallel to the line/plane), I conclude that only 2 problems (among the 116 past final and midterm exam problems to which I had access) required students to mobilize knowledge of vector equations of lines or planes.

Given the rarity with which students need to know the definition of vector equations of lines (denoted by θ_{11} in my reference model) to complete tasks on exams, I denote it by θ_{11}^{KtbT} in the model of knowledge to be learned to underscore that its inclusion in LA1 is mainly as knowledge to be taught. Hence, one praxeological model of the way in which knowledge to be learned (or taught) in LA1 can be mobilized to complete Problem 7 is $[t; \tau_1; \theta_{11}^{KtbT}, \theta_{12}, \theta_{13}]$ (where the latter two technologies refer to the knowledge that vectors are parallel if and only if they are scalar multiples of another and the knowledge that non-parallel lines in $\mathbb{R}^2$ intersect at exactly one point)[28]. The other two praxeologies that model activity through which to complete Problem 7 attend to the technique students are more accustomed to wielding in LA1 for determining the number of solutions of a linear system: actually solving the system. This includes $[t; \tau_2; \theta_{21}, \theta_{22}]$, which does not require knowledge of the correspondence between the algebra of a vectors equation and the geometry of the line it represents, and $[t; \{\tau_{31}, \tau_{32}\}^{KtbT}, \tau_{33}; \theta_{11}^{KtbT}]$, which does require knowledge of this correspondence (as captured by θ_{11}^{KtbT}).

5.7.3 Knowledge LA1 students activated in response to Problem 7

Table 5.13 (on p.249) summarizes the paths of participants' activity as they worked on Problem 7[29]. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step.

Throughout this section, I will refer by EQ1 and EQ2 to $(x, y) = (1, 3) + t(1, 5)$ and $(x, y) = (3, 7) + r(-2, 1)$, respectively.

Nearly all students completed Problem 7. P1 and P7* mobilized the direction vectors of the two lines (that is, vectors parallel to the lines, as indicated in EQ1 and EQ2) and observed that, since they are not scalar multiples of one another, the lines are not parallel and must therefore intersect at only one point; this approach corresponds to technique τ_1 in my reference model (Section 5.7.1). The rest of the students who completed Problem 7 did so through an approach that corresponds to τ_2 in my reference model. This was the approach P2, P3, P4, P5, P6, and P9 mobilized: they searched for values of r and t that would produce the same ordered pair (x, y). Their approach essentially consisted of finding the vectors on the right-hand sides of EQ1 and EQ2 ($(1+t, 3+5t)$ and $(3-2r, 7+r)$, respectively), equating the corresponding components ($1+t = 3-2r, 3+5t = 7+r$), and

[28]I do not address the extent to which θ_{12} and θ_{13} are knowledge to be learned in LA1 as their relevance to t follows only if θ_{11}^{KtbT} is known, and my interest in Problem 7, a problem about systems of linear equations in vector form, is more in the acquisition of θ_{11} than that of the other technologies.

[29]Due to time constraints unrelated to the TBI, P8 was not able to do Problems 6 - 8.

solving this last system. P3, P4, P5, P6 found unique values for r and t in this way, then plugged them back into EQ1 and EQ2 to confirm the same ordered pair (x, y) is produced by these values of r and t. When P9 found a unique value for t, he deduced r would have a unique value and so that x and y would as well, and concluded the system would have one solution. Due to a missing negative sign in P2's approach, he found values for r and t that produced the same value for x but not for y; nevertheless, P2 believed this technique should have produced the same ordered pair in both equations (in part because the same value for x was produced in this way). P10 was unable to complete the problem and struggled to understand what was meant by a "solution of the system" (the clarification that this would be a "pair (x, y) that makes both equations true" did not seem to help).

Table 5.13: Paths of LA1 Students' Activity in Problem 7

Practical block $[t, \tau]$		Type of engagement with $[t, \tau]$: Step 1	Step 2	Step 3	Step 4	
Find values of r, t that produce the same ordered pair (x, y) by...	solving the system by observation	P1				attemps to enact, dismisses: says that the values $t = 0, r = 1$ produce one solution, dismisses once I ask for clarification and P1 finds these values produce different ordered pairs (x, y)
	solving the system obtained by equating the expressions for x and for y in EQ1 and EQ2 ($[\tau_2; \theta_{21}, \theta_{22}]$)	P2				enacts partially (inattention error in calculation): finds values of r, t that produce the same value for x in both equations but different value for y, cannot find error (missing negative sign) but believes technique should have produced the same pair (x, y) in both equations.
		P3 P5				enacts and verifies that the values found for r, t produce the same ordered pair (x, y) in EQ1 and in EQ2
		P6				enacts, finds values of r, t, wonders how to figure out the number of solutions of a system, figures this system has only one, and verifies that the values found for r, t produce the same ordered pair (x, y) in EQ1 and in EQ2
		P9				enacts: finds value of t, deduces it would fix the value of r, and there would be only one ordered pair (x, y) that solves the system
			P2			prompted to think of another approach, starts to enact, then realizes approach would be the same as previous
			P3			prompted to think of another approach, starts to enact, says the numbers obtained will be the same as before
			P4			enacts and verifies that the values found for r, t produce the same ordered pair (x, y) in EQ1 and in EQ2, deduces there is only one ordered pair (x, y) that solves the system
	solve a new system implied by the system (but not equivalent to it) to find values of r, t that produce the same ordered pair (x, y) in EQ1 and EQ2		P1			enacts: gets the system $x + y = 4 + 6t, x + y = 10 - r$, solves it, finds $r = 6 - 6t$, sets $t = 1$ and gets $r = 0$; plugs these values of r, t into the equations and find they do not produce the same ordered pair (x, y); verifies calculations (all are correct), does not know why the technique didn't work
Find a way to find the number of solutions without solving the system		P4				Notes that $(1, 5)$ and $(-2, 1)$ (in EQ1 and EQ2, respectively) are not scalar multiples of one another and then says "of course, because [(1, 3) and (3, 7)] are different"; says he's "trying to think of any indicator of what the number [of solutions to the system] could be, without solving it, but do[es]n't think there is [such an indicator]."
Find the number of intersection points of the lines by checking if they are parallel, overlap, or neither, by...	finding point-normal equations of the lines and solving the system (τ_3)	P7*				suggests
				P1		suggests: recognizes in the vector-equations the "direction vector" and a "point," says it would "be easier" (to find the intersection) if the equations are in the form $x + by = c$
	checking if the vectors $(1 + t, 3 + 5t), (3 - 2r, 7 + r)$ are proportional to determine if the lines are parallel		P7*			suggests, then doubts: says that for $(1 + t, 3 + 5t)$ to be proportional to $(3 - 2r, 7 + r)$, $\frac{1+t}{3-2r}$ must equal $\frac{3+5t}{7+r}$; proposes to determine the values for which the denominators equal 0 and investigate the proportions for other values of r; dismisses approach after I ask how he knows the lines being parallel implies this proportionality.
	checking if values of r, t, with $r = t$, produce the same points (x, y)			P4		inspects: plugs values into r, t ($r = t = 2, r = t = 3, r = t = 4$) to check if the same ordered pair (x, y) is produced and so to check if the lines are "the same"
	comparing the direction vectors of the lines			P7*		enacts: attempts to explain how the lines are graphically constructed by the vectors in the equations and says that since $(1, 5)$ and $(-2, 1)$ are not "proportional" the lines are not parallel and hence intersect at only one point
					P1	enacts: makes a sketch of the lines by plotting two points on each line, then notes the direction vectors of the lines are not colinear so the lines are not parallel, so there is only one point of intersection; observes this is all the question had asked for, that there was no need to actually find the intersection point.
Add up the vectors on the right-hand side of each equation (mobilize θ_{21})		P10				enacts
Make a graph			P10			enact partially and incorrectly in response to a prompt: I ask what EQ1 looks like graphically because P10 is stuck; P10 plots points $(1, 3)$ and $(1, 9)$, says t is "the size of the vector"
Sets the right-hand sides of the equations equal to one another				P10		starts to enact and gets stuck: writes $(1, 3)+t(1, 5) = (3, 7)+r(-2, 1)$ and eventually says: "I don't think it'll work"; "wouldn't know how to solve" the equation $1 + t = 3 - 2r$

EQ1: $(x, y) = (1, 3) + t(1, 5)$; EQ2: $(x, y) = (3, 7) + r(-2, 1)$.

5.7.3.1 One student (P10) did not have the knowledge needed to know what the task was

P10's attempts and comments indicated she did not have sufficient knowledge to grasp the task. Her spontaneous reaction was to add up the vectors on the right-hand sides of both equations. She knew these are $(1+t, 3+5t)$ and $(3-2r, 7+r)$. But she got stuck here; this is when she said she didn't know what was meant by "solution of the system." After I said that a solution to the system would be a pair (x, y) that makes both equations true, P10 paused and eventually asked if this was "like translation." She explained: "translation is pretty similar to something like this but I'm not sure like how to do it." After this, I prompted P10 to consider the graphs of the equations. She drew a Cartesian plane and plotted the points $(1, 3)$ and $(1, 9)$. She struggled to articulate the role of t in EQ1 ("t means that it could be... the size of it, usually, of the vector") but knew of a relation between scalar multiplication of vectors and the length of a vector (she drew an arrow when she said that if t was "a higher number it [the vector] would be bigger"). After this, P10 spontaneously said: "I'm not sure if it'll work, but maybe equating them together—I'm not sure." She wrote the equation $(1, 3) + t(1, 5) = (3, 7) + r(-2, 1)$ and got stuck again. "I don't think it'll work," she said. She "wouldn't know how to solve" the equation $1 + t = 3 - 2r$.

5.7.3.2 Only two students (P1, P7*) mobilized reasoning based in the geometry of vector equations, and one (P4) did so in retrospect at the end of his interview

Only one student *spontaneously* attempted to complete Problem 7 via reasoning that would not require a procedural resolution of the problem. This was P4. He noted that in the equations

$$\begin{array}{rcrcr} (x, y) & = & (1, 3) & + & t(1, 5) \\ (x, y) & = & (3, 7) & + & r(-2, 1) \end{array}$$

the vectors $(1, 5)$ and $(-2, 1)$ are not scalar multiples of one another and said he was "trying to think of any indicator of what the number [of solutions to the system] could be, without solving the system," but he made nothing else of the two vectors not being scalar multiples of one another and ultimately said he doesn't "think there is" an indicator of the number of solutions; he then went on to mobilize τ_2 and solved the system. After he had done so, he noted that "if [he] think[s] about it geometrically," a line would "only intersect a line once. And that's it." After I asked if this was the case "no matter what," P4 corrected himself: "maybe I need to determine if these are the same line. But they aren't the same line. Right?"

P4 did not mobilize, at this point, the geometric knowledge needed to ascertain whether the vector equations represented the same line. He did not return to $(1, 5)$ and $(-2, 1)$ to reason about the lines being distinct; instead, he opted to plug values into r, t such that $r = t$ (e.g., such as $r = t = 2$) to check whether these produced the same ordered pairs (x, y). This procedural approach would not have worked even if the lines were the same (e.g., the lines $(x, y) = r(3, 4)$ and $(x, y) = t(6, 8)$, where $r, t \in \mathbb{R}$, are the same, but $r = 2$ and $t = 2$ produce different points).

Given that P4 did not mobilize $(1, 5)$ and $(-2, 1)$ appropriately, at this point, I might have deduced his knowledge about vector equations (that is, that they correspond to lines) was only superficial. P4 ended up completing Problem 7 via procedural knowledge, finding the values of r and t that produced the same ordered pair (x, y) in both equations. But P4's comments at the end of his interview reveal that what drove P4's actions, in Problem 7, was not insufficient knowledge about the geometry of vector equations. No; what drove P4's decisions on what to mobilize was what he *thought would have been expected of him in LA1.* Indeed, just after P4 had finished Problem 7 and just before I gave him Problem 8, he said: "I always find simpler solutions in the end." When P4 finished his engagement with Problem 8, I referred to the comment he had made and asked if he meant that in general, or just in the interview. P4's response was revealing:

> Um [pause] I don't - I don't think I could remember a time when... maybe today I was just rushing into it. Yeah [pause] yeah, probably. Because, yeah. Also today, I think the questions are aimed to see what I think about; the first thing, second thing I think about. So *maybe I directly tried to solve it using my linear algebra knowledge, instead of looking at it from a more... logical point of view* [emphasis added]. Like, for example, now, I never ever saw a problem like [Problem 8]. But *if I were in the linear algebra [LA1]... mindset* [emphasis added], I would probably try to find that intersection using... some, like the point-normal equation or whatever. So maybe I just, today, I kind of went in before thinking—for example, determine the number of solutions. I think I could have hypothesized that 'oh okay, these aren't the same line, so they must meet somewhere, and only 1 - 1, 1 place,' so I could have said that without having to solve everything.

P4's comments show that, at least as he addressed the question I had just posed, he answered from the position of a participant in a study about LA1: "today, I think the questions are aimed to see what I think about; the first thing, second thing I think about." And he distinguished between "[his] linear algebra knowledge" (from LA1) and "a more... logical point of view." This distinction already came up in P4's engagement with Problem 8, where he qualified one of his suggested approaches as one that would have been expected of him in LA1, and the other as a "problem solving" type (I discuss this in Section 5.8.3). P4 distinguished knowledge that is "logical" or borne in a problem-solving behavior from knowledge that is *expected of students in LA1.* I will return to this broader aspect of P4's perspective in a later discussion of participants' positioning in their interview. For now, I highlight that for P4, the observation that the two lines in Problem 7 are *distinct* and so must intersect was *not* knowledge that would have been expected of students in LA1 to solve Problem 7.

Apart from P1, P4, and P7*, who all mobilized the graphical representation of vector equations eventually but not spontaneously, the other participants' comments suggest they would not have had the knowledge to do so, whatever they may have thought was expected of them. In the following paragraphs, I first address comments made by P2, P3, P5, P6, P9, and P10 and which attest to their lack of knowledge of the theory that relates vector equations to lines; I finish this section with a discussion of P1 and P7*'s mobilization of this knowledge.

When I asked P2 if he could imagine what one of the equations would be like on a graph, he said he couldn't.

When I asked P3 if she could think of a way to do the problem without doing calculations, she said she had "no idea."

I asked P5 what the first equation looks like graphically; at first, she said it would be a line, and when I asked what makes her say that, she said "it looks like it's following the $y = mx + b$ [pattern]. She started to draw an x-axis and a y-axis, added two points along the y-axis as she said "you put up 1, 3 [as in the point $(1, 3)$], and then... 1, 5 [as in the point $(1, 5)$]"—and paused. She said "t doesn't make sense" and that the graph "wouldn't be a line." Instead, she said she thinks "it would just be a point because t has an actual value." (For instance, if $t = 1$, then $(x, y) = (1, 3) + (1, 5) = (2, 8)$ and this is a point). P5 explained this wasn't "like $y = mx + b$, where x can keep changing"; "the way [she] looked at the equation, the t had an actual value."

P6 voiced uncertainty as to whether τ_2 (finding the values of t and r that generate the same ordered pair (x, y)) sufficed to find the number of solutions of the system, and wondered out loud how one could find the number of solutions (even as he engaged with τ_2) but did not allude to the graphs of the vector equations as knowledge that it potentially relevant to this task. P6 knew these equations represented lines - indeed, upon reading Problem 7, he asked: "this is a parametric equation of a line, right?" There were two reasons for which he thought these equations represent lines. First: there are "two variables." Second: "you know, it's a formula, I guess, [that] I memorize." He tried to justify why it would be a line but all he could mobilize was the knowledge that the equations were of a two-dimensional object: "if this was like x, y, z, it still might be a line, but then there's a chance it's a plane but because it's just x, y, it's a line, it's definitely two-dimensional." P6's mention of the notion of linear independence in reference to $(1, 3)$ and $(3, 7)$—rather than $(1, 5)$ and $(-2, 1)$—confirms that P6 did not have the geometric knowledge that justifies why vector equations represent lines.

After P9 completed the problem via τ_2, I asked if he could think of any other approach. "Nope." P9 had mentioned, while engaging with τ_2, that he was looking for an "intersection," so I referred back to this and asked what he meant. He explained, referring to the terms on the right-hand sides of EQ1 and EQ2, that "this is a vector and another vector so [when] you add them you will get one vector and over here [it's] the same, so you will have the intersection of these two vectors." I take this explanation of the term "intersection" as evidence of P9's lack of knowledge of the relation between vector equations and their graphs.

When P10 said "these are coordinates of the graph," I asked her to show what she meant, and she made the sketch in Figure 5.3 as she said one point was $(1, 3)$, the other $(1, 5)$, and that t "could be the size of the vector," in that it determines "how big the vector is."

P2, P3, P4, P5, P6, P9, and P10's grasp of the geometry of vector equations did not suffice to complete Problem 7 by reasoning about the lines captured by these equations. P4 knew the lines are distinct (though he failed to mention they are not parallel), but did not mobilize this until later on in the interview, when, to explain what he meant when he said "always find[s] simpler solutions in the end," he spoke about what he thought was expected of him (in LA1) versus what seemed more "logical" to him. Only P1 and P7*

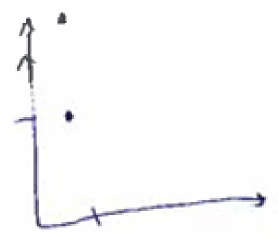


Figure 5.3: P10's sketch of the equation $(x, y) = (1, 3) + t(1, 5)$

had sufficient knowledge to mobilize the geometry of vector equations in full, and, as it did for P4, this reasoning took back stage to the procedural approaches they suggested first and voluntarily.

P1's first two approaches had to do with finding values of r and t that would produce the same ordered pair (x, y) in EQ1 and EQ2. P1 failed to complete Problem 7 in these attempts (I discuss these in Section 5.7.3.3). His third approach came after I prompted him to to think of the problem "from another direction" and to "visualize" what the first equation represents. P1 said it was a "line" and knew what to look for: there would be one solution to the system if the lines intersect at one point and infinitely many if the lines overlap. He even referred, at this point, to one of the equations as he said: "this is a direction vector and this is one point." Still, he did not opt to reason with this knowledge, sticking instead to the procedural route. It would "be easier [to find the intersection of the two lines] if I find the $x + by = c$ [form] of each equation." He sketched the two lines (he used the vector equations to find two points on each line) but did not activate the procedure he proposed because he couldn't "remember how to find [a] normal vector" to the line.

P1 struggled in his attempts to tackle Problem 7 procedurally. This was the context in which P1 finally opted to reason that the lines intersect at exactly one point because their direction vectors are not parallel. P1 appreciated this reasoning: "I just realized that it was very easy because the question is not asking for the solution, it was asking for the number of solutions. So I just had to see if the two direction vectors are collinear or not." I note that P1 only considered the graphical representation of the equations after I prompted him to do so.

P7* attended to the graphical representation of the equations spontaneously - that is, without any prompt on my part - but his initial attempt was incorrect. Step 2 of his engagement with Problem 7 started with this stream of thought:

> Let's regard $(1, 3)$ as a vector. $(1, 5)$—no, no, no, no. t is vector. No. I'm trying to use the vector that is rotating and shifting. No, it's not going to work here. Going to be too complicated. But let's take a look. [pause] The number of solutions. It's not asking us to find the solutions. So there should be a more elegant way to figure out the number. [pause] Yes, there are two lines, right. So the other possibilities are that the two lines are parallel, they have no solution, or they are not parallel, which means they're going to have

one solution. And all I need to figure out is whether they are parallel or not.

P7* had the right idea graphically, but initially proposed an incorrect take on vector equations in his attempt to "figure out [...] whether [the lines] are parallel or not." P7* thought that for the lines to be parallel, the vectors on the right-hand sides of EQ1 and EQ2—that is, $(1+t, 3+5t)$ and $(3-2r, 7+r)$— must be scalar multiples of one another (he called this "proportional"). His suggestion was then to determine the values of r and t for which

$$\frac{1+t}{3-2r} = \frac{3+5t}{7+r}$$

is satisfied. After I asked P7* what made him believe that the lines being parallel would imply this, he discarded his suggestion:

> It's just an instinct, but if you think about it, it may be wrong, and it's very likely to be wrong. Yes, I should not. Maybe I should, you know, prove a lemma. But it's going to make this more complicated. I'm not sure. I'm really not sure if they're proportional. So this method may not work.

As P7* harkened back to his initial procedural suggestion ("This method may not work. So I think that the promising way is to transform it into the Cartesian equation"), I tried to prompt his attention back to the geometry at stake: "how do you know these are lines?" P7* said this was "taught in the lectures," but continued: "let me think about it." This was the context in which P7* started to make the two sketches in Figure 5.4.

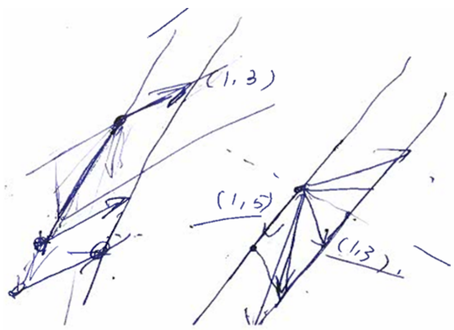


Figure 5.4: P7*'s sketch of the equation $(x, y) = (1, 3) + t(1, 5)$

The right-most sketch and comments P7* made imply he knew the line is parallel to $(1, 5)$ and is then fixed in 2-space by $(1, 3)$: "t scales $(1, 5)$ and by adding the other thing [presumably, $(1, 3)$] [...] that comes to the parallel thing." Other comments P7* made indicated he did not spontaneously mobilize this knowledge before I asked how he knew that the equations corresponded to lines. For instance, P7* said: "what troubled [him] [...] [was] that $(1, 5)$ and $(1, 3)$ actually share the same coordinate, but [he] do[es]n't think that will be a very big problem." This suggests that, at least initially, P7* was not mobilizing the knowledge that $(1, 3)$ was a point on the line and $(1, 5)$ a vector parallel to the line. When I asked P7* to explain how he used these two vectors to get the line in his sketch, he said:

> There are many ways to look at it. I think the clear is the way it is. This is the tangent direction. And let's take the normal direction. $(1, 3)$ is the only

vector to contribute to the projected vector on the normal direction, which means there's always going to be a constant distance between this line to this line, which means this is parallel.

I presume P7* was referring to the second sketch in this explanation: whatever the vector $(1,3)$ may be (he "fixed the initial points of the two vectors to be the same"), the *length* of the component of $(1,3)$ orthogonal to $(1,5)$ (the "normal direction" to which P7* referred) was the "constant distance" between the two lines (in the second sketch). P7*'s description seemed to focus on explaining why the line (represented by EQ1) is parallel to $(1,5)$. P7*'s sketches and comments attest to his capacity to mobilize the linear combination $(1,3)+t(1,5)$ and the "geometric meaning of the summation of two vectors" (as he put it) to explain why $(x,y)=(1,3)+t(1,5)$ was the equation of a line parallel to the vector $(1,5)$.

Among all participants, P1 and P7* were the only ones able to complete Problem 7 by mobilizing graphical knowledge about vector equations. P4 mentioned this approach but did not mobilize it in full: he said $(1,5)$ and $(-2,1)$ were not scalar multiples of one another, and said also the lines were "distinct," but did not justify the latter claim by the former; his other approaches toward Problem 7, which I discuss further below, suggest P4 did not completely grasp the geometry at stake in vector equations. Putting aside P4's capacity to mobilize the geometry at stake, though, what is clear, from his comments, is that he did not view this geometry as the knowledge *that would have been expected of students in LA1.* Apart from P4's perceptions, what is also evident is that P1 and P7* mobilized their geometric knowledge only after I prompted them in this direction. P1 had struggled to activate procedural knowledge and I had asked if he could visualize one of the equations. He could, and still he proposed to activate procedural knowledge, albeit different from the procedure he proposed initially. It was only after he struggled with his second suggested procedure that P1 finally mobilized the knowledge he had about the graphs of vector equations. P7*, meanwhile, did spontaneously draw on the knowledge that vector equations correspond to lines, and did suggest to check whether the lines are parallel. But his initial attempt at checking this involved an incorrect procedure (check if $(1+t,3+5t)$ is "proportional" to $(3-2r,7+r)$), a procedure likely inspired by the knowledge that lines are parallel when their direction vectors are parallel, but it was only when I asked P7* what made him think the vector equations are lines that he retrieved accurate knowledge. In light of the interventions that led P1, P4, and P7* to mobilize graphical knowledge, I infer it was not norms from LA1 that prompted them to do so. P4's comments show that, for him, at least, mobilizing graphical knowledge is not knowledge students are expected to wield in LA1.

5.7.3.3 Summary: LA1 norms related to vector equations contribute to a practice in which students only mobilize algebraic-procedural knowledge about such equations and linear systems

τ_2 is the technique whereby the number of solutions of the system is found by finding the values of r and t that produce the same ordered pairs (x,y). This was P2, P3, P5, P6, and P9's spontaneous reaction to Problem 7 and was also the only approach they knew and attempted to mobilize. After P2, P3, P5, and P9 completed the problem, I asked if they could think of another approach. P9 said he couldn't; P5 said she could have changed the order in which she had done some calculations, but knew this was still the same approach; P2 and P3 started to engage with an apparently-new approach but quickly noticed either that it was equivalent to their first (P2) or that it would yield the same values for r and t (P3); when I asked P2 if he could imagine what the equations represented graphically, he could not, and when I asked what he would do if he were doing this problem at home, he said he would not do anything differently.

P6 was not certain τ_2 sufficed to find the number of solutions to the system but struggled to mobilize other knowledge. As he started τ_2, he voiced uncertainty ("A number of solutions. I guess it's just finding *a* solution. Is there more than one solution? Probably..."), and as he neared the end of the techique, he still voiced uncertainty. Indeed, when he found a value for t, he said: "so that's one solution, I can check.... How do you find the number of solutions of a system of equations?" He paused. "I guess there's just one solution."

Given P6's wording—"I *guess* there's just one solution" [emphasis added]—I asked P6 what made him think there is just one solution. He explained that "we have two equations and two unknowns so there's no independent variable." And "as long as they're linearly independent," P6 said that "it" would be "collinear or linearly independent." It wasn't clear what P6 was referring to by "it" so I asked for clarification. His explanations were murky but seem related to the potential reduced row echelon form of an augmented matrix and the use of Gauss-Jordan elimination in determining the linear independence of vectors: "we can't reduce a row to zeros or I can't have some linear combination of this that sums up to zero." P6 was not certain as to whether there might be "more than one value you could plug into t and r" so that the same ordered pair (x, y) is produced. At this point, he decided to find the value of r corresponding to the value he had found for t, and then confirmed these two values produced the same ordered pair (x, y) in both equations.

P6 said that if he were to hand his work in on an exam, he thinks he "would have to explain some reasoning." P6 tried to mobilize the notion of linear independence as a line of "reasoning" to support the conclusion that the system had exactly one solution, but he struggled. He had "some theorem" in mind but couldn't articulate anything about it ("that's, like, okay, you know, if you have, like, these things and this and there's like, like—I feel like the piece that I'm missing is the way to prove that they are linearly independent, because I'm not sure how to do that with this, because I feel like I could"). I asked P6 to clarify which vectors he referred to when he spoke about linear independence and he pointed to $(1, 3)$ and $(3, 7)$ in EQ1 and EQ2, respectively. He may have also pointed to the other vectors, but even if he had, he was pointing at one too many vectors for the notion of linear independence to be relevant to the problem. Indeed, in the equations

$$\begin{aligned}(x, y) &= (1, 3) + t(1, 5)\\(x, y) &= (3, 7) + r(-2, 1)\end{aligned}$$

only the linear independence of $(1, 5)$ and $(-2, 1)$ is relevant to the number of solutions of the system. But P6 was paying closer attention to the other pair of vectors (the ones that correspond to *points* on the lines, rather than vectors parallel to the lines):

> what I was thinking is that you can set t and r to zero and then row-reduce. And obviously, they're not collinear, because three [from (1,3) in EQ1] and

> seven [from (3,7) in EQ2] are prime numbers and they're in the same column. So there's no scalar. I guess you could multiply by seven over three. It's not... it's not linearly dependent. I don't think. Almost 100 percent certain.

P6 said he was "questioning [his] life" as he engaged with Problem 7. I wasn't confirming whether his reasoning was accurate or not, and while P6 understood these were the conditions of the interview, statements I was making (such as "what do you think?") reminded him of instances in which he would respond similarly to his friends when they studied together. By the end of his engagement with Problem 7, P6 said he "think[s] there's a solution" and that he was "having a good time." P6's attempt to mobilize the concept of linear independence to corroborate a result he found through τ_2 is an example of a student trying *to use mathematics to validate* their knowledge. At the same time, the emotional signals from P6 (which I have noted previously, e.g., as in my analysis of his engagement with Problem 6) are instances in which P6 sought *validation from the interviewer*, that is, from someone with authority over the mathematics at stake.

P6 was able to mobilize technical knowledge about linear systems that was normatively used in LA1 - that is, the algebraic operations needed to solve a linear system and to check that a solution found is indeed a solution. He solved the system and knew how to verify the solution he found: he plugged the values he found for r and t into EQ1 and EQ2 and found they produced the same ordered pair (x, y) in both equations: $\left(\frac{21}{11}, \frac{83}{11}\right)$. But he did not have sufficient agency over the theory that produces this technique (τ_2) to know that it ensured that $\left(\frac{21}{11}, \frac{83}{11}\right)$ would be the only ordered pair that would satisfy both equations. In absence of this knowledge, he tried to activate the concept of linear independence of vectors but did not do so appropriately. In the absence of sufficient agency over the mathematics at stake, P6 tried to appeal to my authority over the mathematics. Nevertheless, I emphasize P6's keen wish to *validate*, as well as his attempt to mobilize mathematics (and not only someone else's authority) to validate his work: these contrast with previous research that found students at this stage of post-secondary mathematics showed a "lack of interest in verifying the validity of a solution to a mathematical problem" (Sierpinska et al., 2008).

P6 was not the only student to attempt and fail to activate knowledge apart from τ_2. For P4, τ_2 was not the spontaneous reaction to Problem 7 (unlike P6, for whom it was) but, like P6, it constituted the only knowledge he activated to complete the problem—at least until he returned to the problem at the end of the interview in response to a question I asked about a comment he had made (that he "always find[s] simpler solutions in the end"). P4's spontaneous reaction was to look for a way to find the number of solutions of the system without actually solving it. He noted that $(1, 5)$ and $(-2, 1)$ (from $(x, y) = (1, 3) + t(1, 5)$ and $(x, y) = (3, 7) + r(-2, 1)$) are not scalar multiples of one another, but did not make any inference from this about the problem: he said he was "trying to think of any indicator of what the number [of solutions to the system] could be, without solving it, but," despite the observation he had made about the two vectors, still didn't "think there is [such an indicator]." In this first response to the task, P4 did not draw on the knowledge that relates lines to vector equations, even though he knew these equations had lines as their graphs. Indeed, P4's last attempt at Problem 7—after he had already determined (via τ_2) the system has one solution—was to figure out whether the lines are parallel and overlapping, parallel and distinct, or not parallel. But he still did not mobilize the geometric knowledge needed to do this. His approach consisted of checking whether the same values of r and t (e.g., if $r = t = 2$) would produce the same points (x, y). P4 mobilized geometric knowledge of vector equations only after a prompt, at the end of the interview, to clarify what he meant when he off-handedly said that he "always find[s] simpler solutions in the end." P4's activity corroborates his perception that, in his practice, the "simpler solutions"—in the case of Problem 7, one based in the geometry underling vector equations—take back-stage to techniques that are normative in LA1. P4's geometric knowledge of vector equations only became available for mobilization when his objective was to show problem-solving capacity; indeed, when he first considered to find "of any indicator of what the number [of solutions to the system] could be, without solving it," he said he did not "think there is [such an indicator]."

Unlike P4, P1 and P7* mobilized their knowledge that EQ1 and EQ2 are equations of lines that are parallel to the vectors $(1, 5)$ and $(-2, 1)$ earlier on in their engagement with Problem 7, even if not spontaneously so. They completed the problem soon after they brought up this knowledge. But mobilizing it was not their spontaneous reaction to Problem 7. Their earlier reactions were procedural in nature.

In his first two attempts at the problem, P1 tried and failed to find values of r and t that would produce the same ordered pair (x, y). He tried to do so, initially, by observation: he suggested, at first, that $t = 0, r = 1$ would produce one solution, and dismissed this after my request for clarification prompted him to discover that these values of r and t produced different ordered pairs (x, y). (It seems that, at first, P1 misunderstood the task: "Sounds to me like it's already solved. And like t could be anything." Since "(x, y)" was isolated in both equations, each equation was already "solved.") P1's second attempt was to solve a system implied by the original system: he found the parametric equations of both lines ($x = 1 + t, y = 3 + 5t$ and $x = 3 - 2r, y = 7 + r$) and used these to produce a new system of equations:

$$\begin{array}{ccccccc} x & + & y & = & (1+t) & + & (3+5t) \\ x & + & y & = & (3-2r) & + & (7+r) \end{array}$$

Unbeknownst to P1, the new system was not *equivalent* to the original system. The values of r and t he found in this way, therefore, did not produce the same ordered pairs (x, y) through EQ1 and EQ2. P1 did not know why. He tried to find a mistake in his calculations, but there were none. He did not attend to the logical implications he (had not) employed in his technique and which constitute the theoretical underbelly of equivalent systems (this brings to mind similar lapses wherein students created non-equivalent systems as they engaged with Problem 6).

P1's third suggestion for how to tackle Problem 7 was the same as P7*'s spontaneously suggested approach: to find corresponding equations in the form $ax + by = c$ and solve the system in that way (this corresponds to technique τ_3 in my reference model). What prompted this suggestion from P1 was that I asked if he could visualize what EQ1 represents. He knew it was a line (indeed, he knew it was a line passing through the point $(1, 3)$ and with $(1, 5)$ as "direction vector") and his spontaneous reaction to this realization was this: "Oh! So I find the intersection of the lines." He said they would either have one point of intersection or infinitely many.

Despite P1's knowledge about vector equations, his spontaneous reaction to the task of finding the number of intersection points of two lines was to mobilize the norm from LA1: the norm of solving systems of equations of the form $ax + by = c$. I assert P1 was activating this norm because of a comment he made: he said he wanted to think "of a way [to] put [the system] in an augmented matrix to [have] a familiar thing to solve." The format in which the system was given was "not a familiar way to solve this problem - [students had not done] this in [LA1]." This brings to mind the comment P5 first made in response to Problem 7: "right off the bat, I don't recognize a problem like this." A wish for "familiarity" may also be what prompted P7*'s spontaneous reaction, which P7* believed he "would use [...] in an exam because it is what popped up into [his] mind first."

Neither P1 nor P7* actually mobilized an approach akin to τ_3, but P1's comments, together with this approach being P7*'s spontaneous suggestion *and* the fact that both P1 and P7* actually knew about direction vectors, show an instance in which the norm from LA1 delayed a reaction that is most aligned with the mathematics at stake. This is similarly the case for P4, who said his initial reaction to Problem 7 was shaped by what he thought would be expected of him in LA1: "if I were in the linear algebra [LA1]... mindset, I would probably try to find that intersection using... some, like the point-normal equation or whatever." P1, P4, and P7* did have sufficient grasp of graphical representations of vector equations to eventually operate beyond LA1 norms for linear systems, but the rest of the participants did not—indeed, even when prompted to consider the graphical representations of the given equations, they could not.

The comments P5 made upon receiving Problem 7 suggest what may have contributed to participants' relatively uniform activity in response to this problem:

> Right off the bat, I don't recognize a problem like this. So... I would probably do... [pause] Just because it's a system, the (x, y) are equal to each other, so I would probably make these two equations equal to each other.

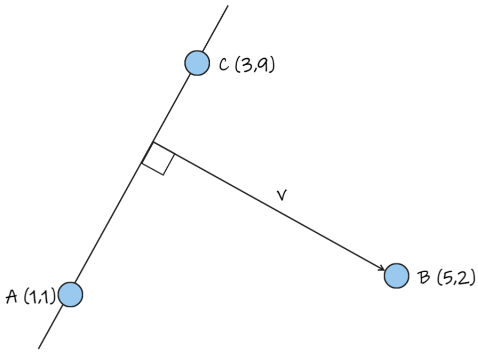
P1 also spoke of rendering the problem into a more "familiar" one: he suggested to rewrite the equations so their form is more familiar ($x + by = c$). P1 didn't do this because he didn't know *how* to do it, not because of a lack of will. P7*'s spontaneous reaction to the problem was also to rewrite the equations in this format. Apart from P1, P7*, and P10, all other participants were able to render the problem into a more familiar one by equating the corresponding components in EQ1 and EQ2. This led to a system of linear equations (in the form $ar + bt = c$), and participants knew how to tackle this.

Apart from students' potential intent to render the problem into one more familiar to them, the comments they made and inability to mobilize other knowledge (as discussed in this section and the previous, where I examined what students knew of the geometry of vector equations) show that the norms they developed in LA1 had only equipped them to (voluntarily) mobilize procedural knowledge about vector equations and linear equations. I apply this to P1, P4, and P7* as well: even though they were eventually *able* to mobilize graphical knowledge of vector equations and linear system, their choice to do so did not come spontaneously. P1 made this choice after I prompted him to consider the geometry of one equation and after he struggled with his procedural algebraic approach, and P7* mobilized what he knew of direction vectors only after I asked how he knew the vector equations corresponded to lines. P4 mobilized this knowledge only in retrospect, at the end of the interview, when I prompted him to explain what he meant when he said he "always finds simpler solutions in the end." Geometric knowledge about vector equations was, indeed, what was "taught in the lectures," as P7* put it. The knowledge to be taught, which P7* described also as knowledge actually taught, was not knowledge students are normally required to learn in LA1. It took further prompts from an interviewer—in the context of a study—to prompt P7* to "think about" geometric knowledge.

5.8 LA1 Problem 8

The following was the eighth problem presented to 9 of the 10 LA1 students[30] in the TBI:

> Find the length of the vector $\vec{v}$, which has B as terminal point and is orthogonal to the line that goes through the points A and C.



5.8.1 Reference model for LA1 Problem 8

The task t in Problem 8 is to find the length of a vector $\vec{v}$ that is identified in terms of its terminal point $B(5, 2)$ and its orthogonality to a given line. The line is given by the property that it passes through the points $A(1, 1)$ and $C(3, 9)$. Implicitly, the task is to find the distance between a point and a line in $\mathbb{R}^2$. A variety of geometric and algebraic knowledge can be mobilized to complete task t. I will suggest six techniques.

Theory about orthogonal decompositions in inner product spaces is at the basis of three of the techniques through this problem can be solved. Given a finite-dimensional subspace W of an inner product space V, $W^\perp$ is the subspace of all vectors orthogonal to the vectors in W. Suppose $y \in V$. There exist unique vectors $u \in W$ and $x \in W^\perp$ such that $y = u + z$ (Friedberg et al., 2015). In $\mathbb{R}^2$ (with the standard inner product - that is, the dot product), this implies that, given a vector y, and the subspace W of $\mathbb{R}^2$ spanned by a non-zero vector a, there exist unique vectors $u \in W$ and $x \in W^\perp$ such that $y = u+x$. In other words, there exist a unique vector u parallel to a and a unique vector x orthogonal to a for which

$$y = u + x. \tag{5.7}$$

[30]Due to time constraints unrelated to the TBI, P8 was only able to do Problems 1 - 5.

Figure 5.5 represents the notion of orthogonal decomposition.

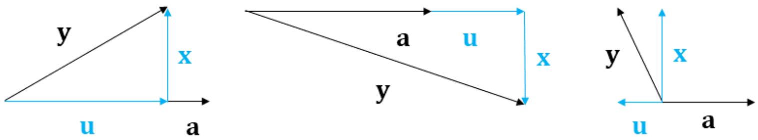


Figure 5.5: Orthogonal decomposition of a vector $y \in \mathbb{R}^2$ in terms of its orthogonal projection u in span$\{a\}$ and a vector x orthogonal to u

Since $u \in \text{span}\{a\}$, $u = ka$ for some $k \in \mathbb{R}$. The value of k is known. From Equation 5.7, I have

$$y = ka + x.$$

Since x is orthogonal to a (that is, their dot product is 0), and from properties relating norms and dot products, it follows that

$$\begin{array}{rcccc} y \cdot a & = & (ka + x) & \cdot & a \\ & = & ka \cdot a & + & x \cdot a \\ & = & ka \cdot a & + & 0 \\ & = & k\|a\| & & \\ \Rightarrow \quad k & = & \frac{y \cdot a}{\|a\|^2} & & \end{array}$$

It then follows that

$$u = \frac{y \cdot a}{\|a\|^2} a.$$

u is called the orthogonal projection of y onto a and denoted by $\text{proj}_a y$; x (such that $u + x = y$) is called the orthogonal component of y along a. In summary, the following identities give the orthogonal decomposition of y relative to a:

$$u = \text{proj}_a y = \frac{y \cdot a}{\|a\|^2} a \tag{5.8}$$

$$x = y - u = y - \frac{y \cdot a}{\|a\|^2} a \tag{5.9}$$

Applying the concept of orthogonal decomposition to Problem 8, it follows $\overrightarrow{AB}$ has a unique orthogonal projection u on $\overrightarrow{AC}$ and there is a unique vector x orthogonal to u such that

$$u + x = \overrightarrow{AB}.$$

Given the uniqueness of x, it follows that $\vec{v} = x$. So a first technique (τ_1) for completing Problem 8 consists of using identity (5.9) to find the components of $\vec{v}$:

$$\vec{v} = x = \overrightarrow{AB} - \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AC}\|^2} \overrightarrow{AC},$$

and the length of $\vec{v} = (v_1, v_2)$ can then be found using the definition of length:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

A second technique (τ_2) also uses the notion of orthogonal decomposition. The vector $\vec{v}$ is the orthogonal projection of $\overrightarrow{BC}$ (alternatively, $\overrightarrow{BA}$) onto a normal of the line through A and C. Any vector orthogonal to $\overrightarrow{AC}$ is a normal n for this line; since $\overrightarrow{AC} = (3-1, 9-1) = (2, 8)$, one such option is $n = (4, -1)$. Applying identity (5.8) to this situation, I can find $\vec{v}$ and then its length.

A third technique (τ_3) for completing Problem 8 starts by noting that the length of $\vec{v}$ is the distance between the point B and the line passing through A and C. A formula for the distance D between a point (x_0, y_0) and a line $ax + by + c = 0$ in $\mathbb{R}^2$ is produced through τ_2:

$$D = \frac{|ax_0 + by_0 + c|}{\|n\|},$$

where $n = (a, b)$ is the normal of the line given by the equation $ax + by + c = 0$.

To mobilize this technique, a point-normal equation for the line through A and C is needed. A normal for this line is any vector orthogonal to $\overrightarrow{AC} = (2, 8)$ (for instance, $(4, -1)$), and either A or C can then be used as a point on the line. For example, if I use the normal $(4, -1)$ and the point $A(1, 1)$, I have the equation

$$4(x-1) - 1(y-1) = 0,$$

so the line in question is given by the equation $4x - y - 3 = 0$ and so the distance between $B(5, 2)$ and this line is

$$D = \frac{|4(5) - 2 - 3|}{\sqrt{4^2 + (-1)^2}}.$$

A fourth technique (τ_4) for completing Problem 8 starts by noting that $\|\vec{v}\|$ is the height (relative to base $\overrightarrow{AC}$) of the parallelogram formed by $\overrightarrow{AC}$ and $\overrightarrow{AB}$. The area $\mathcal{A}$ of a parallelogram formed by vectors $\mathrm{u} = (u_1, u_2)$ and $\mathrm{v} = (v_1, v_2)$ is defined as the product of its height h and base b, but is also given by the formula

$$\mathcal{A} = \left|\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|.$$

Applying both formulas to the area of the parallelogram formed by $\overrightarrow{AC} = (2, 8)$ and $\overrightarrow{AB} = (4, 1)$, I have an equation in which $\|\vec{v}\|$ can be isolated:

$$\|\vec{v}\|\|(2, 8)\| = \left|\det \begin{bmatrix} 2 & 8 \\ 4 & 1 \end{bmatrix}\right|.$$

The formula for the area of a parallelogram formed by two vectors $\mathrm{u} = (u_1, u_2)$ and $\mathrm{v} = (v_1, v_2)$ in $\mathbb{R}^2$ is produced by an identity for the area $\mathcal{A}$ of a parallelogram formed by two vectors in $\mathbb{R}^3$: the area is the length of their cross product. Embedding $\mathrm{u} = (u_1, u_2)$ and $\mathrm{v} = (v_1, v_2)$ in $\mathbb{R}^3$ using, for instance, the map

$$(x, y) \mapsto (x, y, 0),$$

we see that the area of the parallelogram formed by u and v is the length of the cross product of $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$:

$$\|(u_1, u_2, 0) \times (v_1, v_2, 0)\| = \left\|\left(\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}, 0, 0\right)\right\| = \left|\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right|.$$

The formula for the area of a parallelogram in $\mathbb{R}^3$ results from the definition of dot product of vectors in terms of their length and the angle θ between them ($\mathrm{u}\cdot\mathrm{v} = \|\mathrm{u}\|\|\mathrm{v}\|\cos\theta$) and Lagrange's identity ($\|\mathrm{u}\times\mathrm{v}\|^2 = \|\mathrm{u}\|^2\|\mathrm{v}\|^2 - (\mathrm{u}\cdot\mathrm{v})^2$). These identities, together with the theory that frames them, constitute the technology that produces τ_4.

While τ_i ($i = 1, 2, 3, 4$) operate on the geometric notions of orthogonality, distance, and area, and all produce either the components of $\vec{v}$ or its length, a last family of techniques consists of producing systems of equations in which the unknowns are the coordinates of the initial point $D(x_0, y_0)$ of $\vec{v}$ when it is positioned, as in Problem 8, such that its terminal point is $B(5, 2)$. Once the initial point of $\vec{v}$ is found, the vector itself and its length can be found as well. I will describe two techniques (τ_5, τ_6) in which the aim is to find the coordinates of D, though combinations of tools used in either can form other techniques in this same family (producing a system of equations whose only solution is D).

By τ_5, I refer to the production of equations that capture geometric properties of the situation other than equations of the lines whose intersection is D. For example, a pair of equations can be produced by focusing on orthogonal vectors that involve $D(x_0, y_0)$ as an endpoint. Since $\overrightarrow{AC} = (2, 8)$ is orthogonal to $\overrightarrow{DB} = (5 - x_0, 2 - y_0)$, I have that

$$2(5 - x_0) + 8(2 - y_0) = 0,$$

from which it follows that

$$x_0 = 13 - 4y_0.$$

Similarly, since $\overrightarrow{AD} = (x_0 - 1, y_0 - 1)$ is orthogonal to $\overrightarrow{DB} = (5 - x_0, 2 - y_0)$, I have that

$$(x_0 - 1)(5 - x_0) + (y_0 - 1)(2 - y_0) = 0.$$

Substituting the expression for x_0 previously obtained ($x_0 = 13 - 4y_0$) into this new equation produces an equation only in y_0. Once this equation is solved, the value for y_0 can be used to find the value of x_0. Other equations can be produced as a result of the properties of the geometry at stake; for instance, since ADB is a right-angle triangle, the Pythagorean theorem can be also be used to produce an equation in the unknowns x and y (the coordinates of the point D).

A last technique τ_6 consists of finding the point of intersection of two lines: the line ℓ_1 passing through $A(1, 1)$ and $C(3, 9)$, and the line ℓ_2 passing through $D(x_0, y_0)$ and $B(5, 2)$. The point-normal equations for the lines can be found as follows. Since ℓ_1 is parallel to $\overrightarrow{AC} = (2, 8)$, a normal for ℓ_1 is $(4, -1)$. Given this normal for ℓ_1 and knowing the line passes through $A(1, 1)$, it follows that an equation for ℓ_1 is

$$4x - y = 3.$$

Since ℓ_2 is orthogonal to ℓ_1, which, in turn, is parallel to $(2, 8)$, it follows that $(2, 8)$ is a normal for ℓ_2. Since this line passes through $(5, 2)$, it follows that an equation for ℓ_2 is

$$2x + 8y = 26.$$

The point $D(x_0, y_0)$ is the solution to the system formed by the equations for ℓ_1 and ℓ_2.

The six techniques I suggest can be categorized according to the task they each accomplish:

τ_1, τ_2 the task (t_1) is to find an appropriate vector in the orthogonal decomposition of one vector (either $\overrightarrow{AB}$ or $\overrightarrow{CB}$) relative to another vector ($\overrightarrow{AC}$);

τ_3 the task (t_3) is to find the distance between a point and a line;

τ_4 the task (t_4) is to find the height of a parallelogram produced by two of the vectors at stake;

τ_5 the task (t_5) is to find the initial point of $\vec{v}$, a point that is the solution of a system of equations produced as a result of the orthogonality of $\vec{v}$ and the line through points A and C;

τ_6 the task (t_6) is to find the initial point of $\vec{v}$, a point that is the intersection of two lines.

The theory and technology that frame these techniques follow the thread of those discussed in the reference models for Problems 3 and 7 (see Sections 5.3.1 and 5.7.1, respectively): the algebraic, geometric, and logical discourse that define $\mathbb{R}^2$ as an inner product space, along with the view that the axioms that underpin linear algebra and Euclidean geometry are founded in our physical reality. Technique τ_6, wherein the task is turned into a matter of solving a linear system, is additionally framed by the correspondence between the algebra and geometry of linear systems in $\mathbb{R}^2$ and the theory that underpins the resolution of linear systems (as discussed in the reference model for Problem 2 in Section 5.2.1).

5.8.2 Knowledge to be learned in LA1 to perform tasks of the type in Problem 8

The task in Problem 8 is open ended in that it does not prescribe a technique and various techniques are available for completing it. The concepts and formulas mobilized in τ_i (from my reference model for Problem 8) appear as knowledge to be learned in LA1: orthogonal projections, orthogonal decompositions, the formula for distance between a point and a line in $\mathbb{R}^2$ (or that for distance between a point and a plane in $\mathbb{R}^3$), areas of triangles formed by two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and to find a point that is the solution of a system of equations. Problem 8 departs significantly from tasks in LA1 that target this knowledge, however: LA1 tasks implicitly prescribe the technique to be used because the tasks are not open-ended—that is, they ostensibly leave room for only one approach. Given the nature of the mathematics at stake, it's possible more than one type of approach would do, but given the norms in LA1, students are expected to activate the technique to which a task hints at or explicitly requires. I illustrate what I mean by describing the LA1 tasks from past final and midterm exams that targeted the notions of orthogonal decomposition, distance, area, and intersection of lines and/or planes.

I remind the reader that the LA1 tasks to which I refer (to discuss knowledge to be learned) are from 6 final exams and 4 midterm exams from the years 2014 to 2019.

5.8.2.1 Knowledge to be learned about orthogonal decompositions

In LA1, tasks in which orthogonal decompositions are at stake occurred on 4 of the (6 final) exams to which I had access. In two of these, the task explicitly required students to "find the orthogonal projection" of a given vector u on v, with both vectors in $\mathbb{R}^3$ and with single-digit integer components. On one exam, the task was to "find vectors w_1 and w_2 so that $v = w_1 + w_2$," where $v \in \mathbb{R}^3$ is given (with single-digit integer components), "and such that w_1 is parallel to a given vector $u \in \mathbb{R}^3$ and w_2 is orthogonal to u." This task does not include the terms "orthogonal projection" or "orthogonal component," but the description and notation for w_1 and w_2 corresponds directly to how these concepts appear in the knowledge to be taught in LA1. I will refer to the task targeted in these exams by t_7: the task is to find one or two vectors in the orthogonal decomposition of one vector relative to another vector in $\mathbb{R}^3$.

Finally, on one exam, there is a task in which students must mobilize knowledge about orthogonal projections, but the task is a blip less explicit. The task is to "find w_1, w_2 so that $v = w_1 + w_2$," where $v \in \mathbb{R}^3$ is given (with single-digit integer components), "and w_1 is parallel to a given line ℓ [given by parametric equations] and w_2 is perpendicular to ℓ." This differs from how the concept of orthogonal decomposition is to be taught in LA1 in that w_1 is defined as being parallel to a *line*, rather than a given vector. Students are expected to recognize the task is to find an orthogonal decomposition of v relative to a vector parallel to ℓ. Given the notation (w_1, w_2) and the description ($v = w_1 + w_2$, where w_1 is parallel to a given object and w_2 is perpendicular to that object), the task is still closer in appearance to t_7 than to any other task to be taught or learned in LA1. For the way in which this task was phrased in this exam, students must also know how to find a vector parallel to a line; since the line is given in terms of parametric equations, students could either mobilize knowledge about vector equations (as discussed in the knowledge to be learned about vector equations, in Section 5.7.2) or find two points on the line to find a vector parallel to this line.

To perform the task t_7, students are expected to know the formulas for the orthogonal projection and orthogonal component of a vector in $\mathbb{R}^3$ relative to another vector (these are the same as those given in my reference model). To mobilize these formulas, students must know how to compute dot products using the component form of a vector, how to compute the length of a vector, and how to multiply a vector by a scalar.

5.8.2.2 Knowledge to be learned about distance between objects in 2 or 3-space

In LA1, tasks in which students are instructed to find a distance between objects occurred on 2 of the (6 final) exams to which I had access. In these exams, the task was to find the distance between a given point (with single-integer components) and a line in $\mathbb{R}^2$ given in the form $Ax = By + C$ (where A, B, C are single-digit integers). The technique is then to rewrite the line's equation in the form $ax + by + c = 0$ so as to deploy the formula for the distance between a point (x_0, y_0) and a line $ax + by + c = 0$:

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{5.10}$$

In light of these exam problems, what students are expected to learn about distance is how to use a formula for the distance between a point and a line in $\mathbb{R}^2$. The tasks in which students had to find the distance between a point and line already had the point and line in a format nearly ready to be used in the formula, though students are expected to know to rewrite $Ax = By + C$ in the format $ax + by + c = 0$ to use the formula correctly. The distance formula itself does not require students to use any algebra past what they are expected to have learned in high school. Under no circumstances are students expected to recognize, as they might in Problem 8, that a given task is equivalent to finding the distance between a point and a line or plane.

5.8.2.3 Knowledge to be learned about areas of parallelograms in $\mathbb{R}^i$ $(i = 2, 3)$

In LA1, tasks in which the area of a parallelogram is at stake occurred in 4 of the (6 final) exams to which I had access. In all four exams, the task was to find the area of a triangle given its vertices (points in $\mathbb{R}^3$ with components that are all single-digit integers). The "find the triangle's area" task also appears in the list of recommended problems from the LA1 textbook section about cross products; this is the section that includes the theorem which states that the area of a parallelogram formed by vectors $u, v \in \mathbb{R}^3$ is the length of their cross product. This section also includes an example of the "find the triangle's area task." As per the example in the textbook section, to complete the task in LA1, it suffices to find the area of the parallelogram formed by two (non-collinear) vectors (formed by the vertices) and halve this area. To find the area of the parallelogram, students must first know that, if A is the point (x_A, y_A) and B is (x_B, y_B), then $\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$, and they can then mobilize the theorem which states that the area of a parallelogram formed by $u, v \in \mathbb{R}^3$ is $\|u \times v\|$. To use this theorem, students must know how to find the cross product of two vectors and how to find the length of a vector.

The task to find the area of a triangle in $\mathbb{R}^3$ given by its vertices can be performed using other knowledge (e.g., orthogonal decomposition to find the height relative to a given base, or distance between a vertex and its opposing edge, again with the goal of finding a height relative to a given based). I presume any instructor grading such a submission would grant marks for alternative techniques that use knowledge to be taught in LA1 (such as orthogonal decompositions and distance formulas). But students are not *expected* to view the task in any way other than the normative one in LA1; given that the "find the triangle's area" task is in the textbook section in which the cross-product formula for the area of a parallelogram in $\mathbb{R}^3$ is given, I infer the knowledge students are *expected* to learn to perform such a task amounts to using the cross-product formula.

5.8.2.4 Knowledge to be learned about intersections of lines and/or planes $\mathbb{R}^i$ $(i = 2, 3)$

In LA1, tasks in which students are expected to find the intersection of lines and/or planes occurred in only one of the (6 final) exams to which I had access. The task in this exam was to find the coordinates of the intersection of a line ℓ and a plane $\mathcal{P}$; the line was defined as the intersection of two planes in point-normal form ($a_i x + b_i y + c_i z = d_i$, $i = 1, 2$) and $\mathcal{P}$ was also given in point-normal form ($a_3 x + b_3 y + c_3 z = d_3$). One technique is to solve the system formed by the three equations. The formulation of the equations (as $ax + by + cz = d$ for some $a, b, c, d \in \mathbb{N}$) mimics that of equations given in the LA1 task of solving a system of linear equations (discussed in the model of knowledge to be learned to perform tasks of the type in Problem 2; see Section 5.2.2). Gauss-Jordan elimination is the normative LA1 technique to this end.

5.8.2.5 In LA1, tasks that involve geometric objects explicitly or nearly-explicitly dictate the technique students are expected to use.

In light of the tasks students are expected to perform and which involve lengths, distances, and areas in $\mathbb{R}^i$ ($i = 2, 3$), I conclude students are not expected to know *when* to deploy these tools and related formulas; they are only expected to know how to use the formulas at stake and are only required to use them on command. This captures what students are expected to learn relative to techniques τ_i ($i = 1, \ldots, 4$) in my reference model for Problem 8.

The task of finding the intersection point of lines and/or planes is not typically phrased as such in LA1; only one exam question among the 116 I examined explicitly required students to find the intersection of a line and plane. But the task in that question does mimic the normative LA1 task of solving a system of three linear equations in $\mathbb{R}^3$ (I discuss this normative task in the model of knowledge to be learned to perform tasks such as Problem 2; see Section 5.2.2). Indeed, the formulation of the problem—wherein the line is described as "the intersection of two planes," both of which are given in point-normal form ($ax + by + cz = d$), and wherein the third plane is given in point-normal form as well—is a surface-level feature that makes the problem similar to the task of solving a system of three equations. Even if this exam problem does not dictate the technique to be used, the representation chosen for the objects at stake makes the task amenable to a technique students are usually instructed to use: to solve a system of linear equations.

I conclude the knowledge students are expected to learn in LA1 suffices to perform Problem 8 *if* they recognize the task to be any one of t_i ($i = 1, 3, 4, 5, 6$) from my reference model for this problem. LA1 students are not normally expected to complete open-ended problems, and they are not normally expected to know *which* knowledge about geometry in $\mathbb{R}^i$ ($i = 2, 3$) to mobilize.

5.8.3 Knowledge LA1 students activated in response to Problem 8

Table 5.14 (on p.269) summarizes the paths of participants' activity as they worked on Problem 8[31]. As before, Step 1 refers to the activity a participant spontaneously engaged in upon reading the problem statement; I group students according to Step 1 and color-code the groups to help trace students' paths thereafter. I categorize a student's activity in a new step if they presented it as such; if I prompted for another approach and a participant described one that is essentially equivalent, I still categorized it as a new step.

Throughout this section, I refer by D ($D(x, y)$) to the initial point of $\vec{v}$ when the vector is placed such that its terminal point is $B(5, 2)$; I also refer by ℓ_1 to the line that passes through $A(1, 1)$ and $C(3, 9)$ and by ℓ_2 to the line that passes through B and is orthogonal to ℓ_1.

[31]Due to time constraints unrelated to the TBI, P8 did not do Problems 6 - 8.

Five students (P1, P2, P4, P7*, P9) (essentially) completed Problem 7; one (P6) suggested an appropriate technique but did not give sufficient detail to suggest he could complete it; and three (P3, P5, P10) did not complete the task nor take any steps appropriate for completing the task.

Among those who essentially completed the task, only P4 and P9 found the length of $\vec{v}$, but the others had found sufficient information and described the remaining steps sufficiently to indicate they knew how to complete the task. Indeed, P1 and P2 had found the coordinates of D and said the remaining steps are to find $\vec{v}$ using its initial point D and terminal point B and then to calculate the length of $\vec{v}$; and among the various correct approaches P7* had suggested, all that was missing was the execution of the calculations he had described.

P6 had suggested an appropriate technique for solving the task—create two equations so as to find the coordinates of D, and while he did produce one equation in the unknowns x, y, he struggled to produce a second one.

P3, P5, and P10 did not complete the task. P3 and P5 mobilized formulas from LA1 in inappropriate ways (e.g., P5 created an equation by setting $\overrightarrow{AC} \cdot \vec{v}$ equal to the distance between the initial and terminal points of $\vec{v}$, and P3 produced the equation $(a, b) \cdot (5, 2) = 0$, where $(5, 2)$ are the components of B and (a, b) was defined such that it was in no way a vector orthogonal to B). Finally, P10 did not know what to do. She considered searching for the coordinates of D but said she did not know how to do so, and she knew how to find the components of $\overrightarrow{AC}$ but said she did not know how this would be useful to the task.

Table 5.14: Paths of LA1 Students' Activity in Problem 8

Practical block $[t, \tau]$		Type of engagement with $[t, \tau]$: Step 1	Step 2	Step 3	Step 4	
find D by creating and solving an equation or a system of equations that correspond to the given scenario, find the length of $\vec{v}$ by calculating $\|\overrightarrow{DB}\|$	use the components of A and C as coefficients of unknowns in an equation of the form $ax+by+c=0$ to represent line(s)	P3				enacts partially: produces equations $3x+9y+c=0$ and $x+y+d=0$ with the aim to find ℓ_1, reduces the augmented matrix for the system $3x+9y=0$, $x+y=0$, then, in response to a prompt about how she got this matrix, reduces the augmented matrix for her original system, loses sight of her goal ("what was it for?")
	produce the equation $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$ (EQ1: $8x+2y=0$), then say a second equation is needed to solve for two unknowns, and produce a second equation EQ2 (τ_5 and/or τ_6)	P1				enacts partially, describes the rest: finds D (calculation error leads to incorrect x coordinate) (as 'EQ2,' produces parametric equations for ℓ_1) and describes the rest (find $\|\overrightarrow{DB}\|$)
		P2				enacts partially, describes the rest: finds D (applies Pythagorean theorem to triangle ADC to get EQ2: $(x-1)^2+(y-1)^2+(x-5)^2+(y-2)^2=17$) and describes the rest (find $\|\overrightarrow{DB}\|$)
		P6				enacts partially, gets stuck: produces EQ1, suggests to apply Pythagorean theorem to triangle ADC to get EQ2 but does not express it in terms of x, y (only in terms of names of the triangle's edges)
	find equations for ℓ_1 and ℓ_2 (τ_6)		P9			describes: use direction vector $\overrightarrow{AC}=(2,8)$ of ℓ_1 to find parametric equations for ℓ_1 and ℓ_2, find intersection D, find $\|\overrightarrow{DB}\|$
	produce an equation using the dot-product definition of orthogonality but with vectors that are not orthogonal		P3			partially enacts: writes $(a,b)\cdot(5,2)=0$ and $5a+2b=0$; from P3's sketch, (a,b) seems to be the initial point of $\vec{v}$ (terminal point $B(5,2)$).
	no clearly suggested equation		P4			suggests to find D, starts to enact (finds $\overrightarrow{AC}=(2,8)$), decides to mobilize his first suggested approach
find the area of the parallelogram formed by $\overrightarrow{AC}$ and $\overrightarrow{AB}$ so as to find the height of the triangle ABC, as this is $\|\vec{v}\|$ (τ_4)		P4, P7*				suggests: find $\frac{1}{2}\|\overrightarrow{AC}\times\overrightarrow{AB}\|$, as it and $\frac{1}{2}bh$ are both the area of the triangle formed by $\overrightarrow{AC}$ and $\overrightarrow{AB}$ is , where b is the base ($\|\overrightarrow{AC}\|$, known) and $h=\|\vec{v}\|$ the height
				P4		enacts (uses LA1 determinant formula for area of a parallelogram in $\mathbb{R}^2$)
adapt the LA1 "area of a triangle is half the area of a parallelogram" technique				P5		draws the parallelogram formed by $\overrightarrow{AB}$ and $2\overrightarrow{BC}$ (describes the latter as the vector obtained by going 2 units along negative x-direction, 7 units along positive y-direction (as in $\overrightarrow{BC}=(2,7)$)); finds distance between B and C; says $\|\vec{v}\|$ is half the length of the diagonal of the new parallelogram, the length of which she says she can find using the length of $2\overrightarrow{BC}$; says this strategy is similar to a LA1 midterm question she did correctly, where the task was to find the area of a triangle (or parallelogram)
use knowledge about orthogonal projections	calculate length of an orthogonal projection irrelevant to the given problem: $\text{proj}_{\overrightarrow{AC}}B$	P9				enacts: finds parametric equations for the line through A and C, then finds $\|\text{proj}_{\overrightarrow{AC}}B\|$ and says this is $\|\vec{v}\|$
	find $\text{proj}_{\overrightarrow{AC}}\overrightarrow{CB}$, $\|\overrightarrow{CB}\|$, and use the Pythagoean theorem to find $\|\vec{v}\|$ (length of the third edge of the triangle) (τ_1)			P7*		describes
use formula $\frac{\|ax_0+by_0+c\|}{\sqrt{a^2+b^2+c^2}}$ for the distance between a point (x_0,y_0) and a line $ax+by+c=0$ to find the distance between B and ℓ_1, as this is $\|\vec{v}\|$ (τ_3)			P7*			enacts partially, describes the rest: finds equation in the form $ax+by+c=0$ for the needed line, explains how to use the distance formula
					P4	suggests (does not find any equation for ℓ_1)
mobilize trigonometric formulas and formulas about right triangles to use $\theta=\angle BAC$ and $\|\overrightarrow{AB}\|$ to find $\|\vec{v}\|$ (viewing the latter as the edge opposite to θ in the right triangle BAC)					P7*	describes
use the dot product definition of orthogonality between vectors and a formula for the distance between points		P5				partially enacts: writes $\begin{bmatrix}2\\8\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\end{bmatrix}=0$, where $(2,8)=\overrightarrow{AC}$, $(v_1,v_2)=\vec{v}$, calculates distance between A and B, and after I hint to find the initial point of $\vec{v}$, writes $2v_1+8v_2=\sqrt{(5+x)^2+(2+y)^2}$ where, per P5, the right-hand side is the distance between the initial and terminal points of $\vec{v}$
try to find a technique by recreating the sketch so the vectors at stake have their initial points at the origin			P5			sketches a line segment between points $(0,0)$ and $(2,8)$ and a line segment perpendicular to the first and with endpoint $(4,1)$
stuck		P10				doesn't remember formula for length of a vector, and once she receives it, is quiet; considers to find $\overrightarrow{AC}$ but doesn't know what she'd use it for; considers to find the initial point D of $\vec{v}$, but is "not sure how to do that"; asks if D is the origin, in which case she'd do "B minus the origin" and find the length of this vector, but doesn't think D is the origin; is not sure what to do; writes $CA=(2,8)$.
create a system of equations by applying the Pythagorean theorem to triangles ADB, CDB				P3		writes: $\alpha^2=e^2+v^2, \beta^2=f^2+v^2$ ($\alpha=\|\overrightarrow{AB}\|$, $\beta=\|\overrightarrow{CB}\|$, $e=\|\overrightarrow{AD}\|$, $f=\|\overrightarrow{DC}\|$) and then $\alpha^2-e^2=\beta^2-f^2$.

$A(1,1), B(5,2), C(3,9)$; $D(x,y)$: the initial point of $\vec{v}$ when it is placed such that its terminal point is B; ℓ_1: the line passing through A and C; ℓ_2: the line passing through B and orthogonal to ℓ_1.

5.8.3.1 Two students (P4, P7*) self-proclaimed to act differently from LA1 student norms as they spontaneously and successfully mobilized geometry from LA1 and high-school mathematics courses

P4 and P7* were the only two students who recognized the suitability of various geometry-centric technologies to be learned in LA1: between the two of them, the technologies that came up were formulas for areas of parallelograms and triangles in $\mathbb{R}^2$ and $\mathbb{R}^3$, the notion that systems of equations can be used to find an intersection point of lines in $\mathbb{R}^2$, formulas for vectors in an orthogonal decomposition, a formula for the distance between a point and a line in $\mathbb{R}^2$, and trigonometric formulas. While in typical LA1 tasks (such as to find the area of a parallelogram, to find the orthogonal projection of one vector onto another, etc.), this knowledge has the function of a technique (use a given formula to perform a certain task), P4 and P7* were able to use this knowledge in the function of technologies: knowledge needed to perform a certain technique (they needed, for instance, LA1 formulas for areas of parallelograms in 2- or 3-space to find the height of a given triangle). In this section, I first describe how P4 and P7* mobilized this knowledge in the function of technologies, and then describe opinions P4 and P7* shared about the techniques they suggested for Problem 8 and techniques they perceived would be expected or typical of students in LA1.

5.8.3.1.1 P4 and P7* suggested to mobilize cross products to use the area of triangle ABC (in $\mathbb{R}^2$) to find its height $\|\vec{v}\|$ Both recognized, upon receiving Problem 8, that the length of $\vec{v}$ is the height of triangle ABC (relative to base AC) and that a pair of formulas for the area of a triangle could be wielded to find the height of this triangle. One of the formulas P4 and P7* suggested was that the area of a triangle with base b and height h is $\frac{bh}{2}$. The other formula they suggested was $\frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{AB}\|$. They thought $\|\overrightarrow{AC} \times \overrightarrow{AB}\|$ to be the area of the parallelogram formed by the two vectors. This would have been accurate if the vectors were in $\mathbb{R}^3$. P4 and P7* did not spontaneously realize their mistake, but both did eventually resolve it. I denote by τ_4' the technique whereby P4 and P7* aimed to find the height of triangle ABC. This notation refers to τ_4 from my reference model from Problem 8: the technique whereby the identified task (t_4) is to find the height of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

5.8.3.1.1.1 To mobilize τ_4', P7* reasoned geometrically to adapt the cross product formula for parallelogram areas in $\mathbb{R}^3$ to a parallelogram in $\mathbb{R}^2$, whereas P4 mobilized the LA1 formula for the area of parallelograms in $\mathbb{R}^2$ (that results from the reasoning P7* used). P4 did not try to use the cross product formula he'd initially proposed for the area of a triangle, but rather used a corollary of that formula. He did not address the discrepancy between his original suggestion and what he used. P7* did not notice an issue with the cross product formula he'd initially proposed until I prompted him to do the calculations he suggested; eventually, he used the geometry at stake to adapt the cross product formula for parallelogram areas in $\mathbb{R}^3$ to parallelograms in $\mathbb{R}^2$.

Despite what P4 had said about the area of a parallelogram being the length of a cross product (a theorem true in $\mathbb{R}^3$), in practice, he mobilized a LA1 formula for the area of a parallelogram in $\mathbb{R}^2$ (he "remember[ed] it, [he] just - [he] kn[e]w it's a formula"), though he didn't address the discrepancy between the formula he'd used and the claim he'd made (about cross products). He found the components of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then got to calculating the area of the parallelogram they form: "is the area of the parallelogram they form [what I get] when I put them above each other? The determinant, that's what I put them on top of each other." What P4 wrote matches with a corollary of the theorem (to be taught and learned in LA1) about cross products and parallelogram areas in $\mathbb{R}^3$. The corollary states the area of a parallelogram formed by vectors (u_1, u_2) and (v_1, v_2) in $\mathbb{R}^2$ is the absolute value of

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

P7* did not attempt any of the calculations he'd described for τ_4', but he did bring up the cross product formula up again later on in his engagement with Problem 8. At that point, I prodded P7* to recognize the issue: I pointed out that cross products are in $\mathbb{R}^3$. This did not get P7* to notice an issue (he first insisted that the *norm* of the cross product would be the area of the paralellogram, and after I asked which two vectors he'd use in the cross product, he still didn't notice an issue as he focused on which choice of vectors would be suitable, rather than the nature of the vectors). So I prompted him to show what his calculations would look like. This worked. P7* noticed a problem after he wrote the following:

$$\begin{vmatrix} 4 & 2 & e_1 \\ 1 & 8 & e_2 \\ & & \end{vmatrix}$$

"Wait, wait. Why is it not working here? Wait. [pause] I'm not sure where it's going, which part is going wrong, but we are not [getting] a square matrix here. So I cannot calculate the determinant." P7* didn't notice the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is not defined; he rather noticed he could not use a formula for cross products that involves a determinant. He knew determinants "can" only be "calculate[d]" for square matrices.

P7* abandoned the suggestion to use the area of the triangle ABC at this point. He moved on to make his last suggested approach, and when he finished it, he asked: "do you know which part had gone wrong with this cross product thing?" I gave him the definition of cross product. This triggered in P7* a resolution: assign 0 as a third coordinate to the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$. P7* also had an explanation for this based on the mathematics at stake: "because this triangle is on a plane, which means there's no third dimension, which means they share, they share the same third coordinate, and I just assume it's zero. You may also say that it's going to be 1 1 [as in, assign 1 as the third coordinate for both vectors], but I think it will not change the answer. Maybe? I'm not sure. But I don't think it [would], according to the characteristics of the geometry."

P7*'s eventual and mathematics-based reconciliation of the cross product formula for triangle areas in $\mathbb{R}^3$ with the situation at stake (a triangle in $\mathbb{R}^2$) contrasts with P4's "I know it's a formula" justification for the formula he used. What P4 used is a corollary of the cross product formula for areas of parallelograms in $\mathbb{R}^3$. P7* had actually provided the explanation that produces the corollary P4 used. I acknowledge this contrast between the nature of P4 and P7*'s explanations and note that P7*'s background includes higher-level and abstract mathematics courses from an undergraduate mathematics degree he had started at a university in another country.

5.8.3.1.2 P4 and P7* knew $\|\vec{v}\|$ is the distance between B and the line ℓ_1 through A and C. τ_3 is the technique I identified in my reference model for Problem 8 wherein the length of $\vec{v}$ is viewed as the distance between B and ℓ_1. P4 brought τ_3 up spontaneously after he had completed the problem via τ_4' (τ_3 was his third suggested approach), and P7* enacted τ_3 partially as his second suggestion of a technique for Problem 8.

P7*'s description of τ_3 sufficed to indicate he could have finished the problem in this way: he had produced a point-normal equation for ℓ_1, that is, an equation of form $ax + by + cz = d$, and knew the coefficients a, b, c of x, y, z and the coordinates of B were those needed to use the distance formula.

P4's description was not as thorough as P7*'s. He suggested to find a point-normal equation for ℓ_1. He knew an equation of this form is needed to deploy the formula for distance between a point and a line in $\mathbb{R}^2$. It's not clear whether P4 had sufficient knowledge to produce a point-normal equation for ℓ_1. When P4 brought up τ_3, he first said: "let me try it. I don't really remember." He then said, referring to ℓ_1, that it would "be $t(2, 8)$" (where $(2, 8)$ are the components of $\overrightarrow{AC}$). "I don't remember if it's going to be plus 5 2," he said as he wrote $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, as in the coordinates of point B, "is it? I'm - I'm not sure. Is this a formula for this line?" It was after this that P4 said he "could do the point-normal equation," "plug in x, y as $5, 2$," which are the coordinates of B, "and then use the distance formula: $\frac{ax_0+bx_0+c}{\sqrt{a^2+b^2+c^2}}$." Given P4 had relied on surface-level features of vector equations when he first attempted an equation for ℓ_1, it's not clear if he would have similarly depended on (easy-to-forget or misuse) surface-level features to produce a point-normal equation or if he had theoretical knowledge through which to produce point-normal equations.

5.8.3.1.3 P4 knew the initial point of $\vec{v}$ is the intersection of two pertinent lines. He said one approach to Problem 8 would be to find this intersection point, use it to find the components of $\vec{v}$, and then use the latter to find $\|\vec{v}\|$. P4 did not explicitly describe how to do this, but he did mention the point is an intersection which could be found "using... some, like, the point-normal equation or whatever." P4 said this was the technique he would use "if [he] were in the linear algebra [LA1] mindset." P4 said he "think[s]" this is "what [teachers] would expect [students] to do in this [LA1] course." I infer, from the comments about "intersection," "the linear algebra mindset," and what would be expected of students in LA1, that P4 had in mind the normative LA1 task wherein a point of intersection of two lines in $\mathbb{R}^i$ ($i = 2, 3$) is found by solving the system of equations corresponding to the lines. In my reference model for Problem 8, I refer to this task by t_6 and its related technique τ_6.

5.8.3.1.4 P7* suggested to use trigonometric formulas so as to use $\angle CAB$ and edge AB to find the height $\|\vec{v}\|$ of triangle CAB P7* denoted $\angle CAB$ by θ and the height of triangle CAB (relative to edge AC) by h. He brought up various formulas and described how to use them together to find $\|\vec{v}\|$. One of the formulas he brought up was the cross product formula for areas of parallelograms (in $\mathbb{R}^3$), "the geometric meaning" of which he had "learned [...] in this course [LA1]." The other formulas P7* mobilized are formulas he had "learned [...] before, in junior high school and senior high school."

The first formula he brought up was

$$\sin\theta = \frac{h}{AB}, \tag{5.11}$$

where he claimed the norm of AB and $\sin\theta$ are known. He said $\sin\theta$ can be found in "two ways."

One way to find $\sin\theta$, he said, was through dot products:

$$\cos\theta = \frac{a \cdot b}{|a \cdot b|}$$

This incorrect formulation is produced from the definition of dot product of two vectors a and b with angle θ between them:

$$a \cdot b = \cos\theta \|a\| \|b\|$$

P7* said he was referring by a and b to $\overrightarrow{AC}$ and $\overrightarrow{AB}$, respectively. P7* then suggested to "use the relationship between $\sin\theta$ and $\cos\theta$ [...] which means [$\sin\theta$] is $\sqrt{1-\cos\theta^2}$."

P7*'s second way for finding $\sin\theta$ was to use the area "S" of triangle ABC. He claimed S would equal "$\frac{1}{2}\overrightarrow{AC}\overrightarrow{AB}\sin\theta$" (a formula I presume he'd drawn from the cross product formula for the area of a parallelogram formed by vectors $a, b \in \mathbb{R}^3$, along with the identity

$$a \times b = (\|a\|\|b\| \sin\theta)n,$$

where n is a unit vector orthogonal to both a and b in the same direction as their cross product). Since $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are both known here, so are their lengths and their cross product, so $\sin\theta$ could be found as well.

Once $\sin\theta$ is found, P7* explained, equation (5.11) could then be used to find h (that is, $\|\vec{v}\|$).

Some prompts I gave P7* made him realize he could not calculate $\overrightarrow{AB} \times \overrightarrow{AC}$. At first, he proposed to abandon this approach to finding $\sin\theta$. "There are two ways to find out the θ." The first way he mentioned would again depend on the area of the triangle, but the second is the "law of cosine," which indeed would complete the task of finding $\sin\theta$ if applied to triangle ABC, as all its side-lengths are known.

After P7* resolved how to find $\sin\theta$ without using the area of ABC, he asked if I "kn[e]w which part had gone wrong with [that] cross product thing." I gave P7* the definition of cross product and this triggered in him a resolution to the issue (which I discuss in Section 5.8.3.1.1.1, in the context of P7*'s mobilization of τ_4' to find $\|\vec{v}\|$).

5.8.3.1.5 P4 and P7*'s comments shed light on their perception of what is expected of LA1 students and of what they value, as problem-solvers, in a solution to a problem. Upon receiving Problem 8, P7* said there are "so many ways to do" it and he had "no idea where to start"; P4 decided—"the first thing I'm thinking about"—to "do this with high school geometry." He paused and, before explaining how he'd use high school geometry, said: "could I know the area of this?" A pause. "Oh yes! I could." P4's spontaneous reaction to the problem was also the first suggestion P7* gave: they said the length of $\vec{v}$ is the height of triangle ABC and knew they could find the triangle's area (using Euclidean geometry and LA1 knowledge) and then use this to find the height of the triangle.

At this point, the two participants had only described this approach and did not embark on any calculations. P7* went on to suggest a second idea he had (as he did throughout the interview, suggesting approaches one after the other and with no explicit prompt to do so) and P4 wondered: "is there another [way], a better way? Perhaps."

While P7* said, about his first suggestion, that he "[thinks] most people would do that" approach (presumably, referring to LA1 students), P4 placed his bet elsewhere. Indeed, his second proposition was to find the initial point of $\vec{v}$:

> I could try to find this point [the initial point of $\vec{v}$ in the given image]. And then, then I know this vector and [I can] try to find the norm of the vector, *which I think [is] what they would expect us to do in this course.* [emphasis added]

(Apart from P4 and P7*'s various suggestions for how to complete Problem 8, the only apt technique suggested by the other participants was to use equations to find the initial point of $\vec{v}$; I discuss this in Section 5.8.3.2.) P4 found the components of $\overrightarrow{AC}$ and then decided against this equations-based approach: "nah, I'm gonna try *my own* solution" (emphasis added). P4 distinguished between his perception of what is expected of students in LA1 and solutions he rather perceived as "[his] own."

For P4, simplicity was one character of solutions he perceived as "[his] own." Indeed, in between his engagement with Problems 7 and 8, P4 said he "always find[s] simpler solutions in the end." When I referred back to this comment at the end of P4's interview, and asked if he meant it in general or "just today" (during the interview), P4 explained:

> Today, I think the questions are aimed to see what I think about; the first thing, second thing I think about. So *maybe I directly tried to solve it using my linear algebra knowledge, instead of looking at it from a more... logical point of view.* Like, for example, now, I never ever saw a problem like [Problem 8]. But *if I were in the linear algebra [LA1]... mindset*, I would probably try to find that intersection using.. some, like, the point-normal equation or whatever. [emphases added]

P4 perceived the solutions he proposed—solutions based in the geometry at stake (height of a triangle, in particular, but also distance between a point and a plane)—a more "logical point of view" for the given problem than creating equations to find an intersection point, an approach he perceived to be more aligned with LA1 norms.

Other comments P4 made reinforce the distinctions he perceived between what's expected of LA1 students (equation solving) and mathematics over which he felt he had authority ("my own") and which he deemed more "logical" for certain problems:

> Also! Sometimes it does happen, like in the exam. I don't know if you know this type of question, in calculus, in [differential single-variable calculus], they would give us a system of equations, and then [we have to] find a and b such that the function is continuous and differentiable for example. So I always—like *in the midterm and when studying, I will always solve it algebraically and I always find a solution like that.* In the final, at the end of the day, [I] played it very well. I remember it was—I think $-ax+b$ maybe—no. *I just remember that trying to solve it algebraically was not working, I would end up with like arc sine or whatever. And I couldn't actually reach somewhere. But I think - I thought about it geometrically, and then I got to [something].* So this would be my sine x graph. Oh, sorry! Yeah, it was a piecewise function. And then, the second part was linear. And so no matter what, this wouldn't be differentiable, because this is a corner, I have a linear meeting a curve and the, the continuous it will always be, because no matter what a is, this only affects the... the wavelength of the function, but it will always meet here and it will always be continuous. So sometimes, like, under [laughs] *maybe under pressure, I find maybe another way to look at the problem. But... usually I, yeah, usually I think I fall into the mistake of... robust way of thinking about it*, you know what I mean? [emphases added]

I asked P4 what makes something "robust." His answer: "proof, rigorous math, algebra." P4's perception was that LA1 student norms for solving problems are to "always" use algebra. On "the midterm" and "when studying," he would "always solve [...] algebraically" and "we [students?] would always find a solution like that." It would take an out-of-the-ordinary situation—being "under pressure"—for him to change tactics: for instance, when he was unable to solve a problem algebraically on a final exam, or on the occasion of this interview. After all, P4's perception was that if he "were in the [LA1] mindset," he would have tackled Problem 8 by using equations to find the intersection point (of ℓ_1 and the line through B and parallel to $\vec{v}$). That P4 perceived the LA1 "mindset" to correspond to this strategy shows what he did *not* perceive to be expected of LA1 students: to use various formulas to be learned in LA1 (such as distance formulas or formulas for areas of triangles) so as to produce techniques that are his "own."

Additionally, P4 said that to "always solve [...] algebraically," is to "[fall] into a mistake." Another comment P4 made reflects a similar opinion. When I asked P4 which approach he'd submit for grades, he said both (using the distance formula and using triangle areas) are "correct," and when I followed with another question (what if he were just doing the problem "for himself?"), P4 stated a preference:

> I like this one, because this is more logical than, because that one depends on my knowledge of linear algebra. This one is.. problem solving, you know what I mean? This is the first time I've ever seen the problem this way.

P4 contrasted between a logical, problem-solving approach, and one based on "[his] knowledge of linear algebra." This brings to mind the distinction between techniques that reflect students' norms from LA1 and techniques that are more appropriate or efficient for a given problem. As a LA1 student, P4 did not perceive that students are expected to problem-solve, that is, to produce solutions that are "[their] own." As "this is the first time [P4 has] ever seen the [distance] problem this way," I infer P4 meant, by solutions that are "[his] own," solutions that are not replicas of techniques for problems students are used to seeing in LA1. Recall the comments P4 made in response to other problems in the interview: when he saw Problem 2 (to solve a linear system), he said that "the second [he] see[s] $A\mathrm{x} = \mathrm{b}$, oh ok, REF. [He knows he] need[s] to reach REF and start[s] solving toward that." Various other comments P4 made throughout the interview, along with his activity in the interview, indicated that, when faced with a problem students typically have to solve in LA1, P4 spontaneously activates LA1 norms. But, given an open problem such as Problem 8, one of a type he'd never seen ("this is the first time I've ever seen the problem this way"), P4 did not have a ready-made technique to fall back on and had to produce an approach of "[his] own."

At the same time, however, P4's comment opposing "robust way[s] of thinking" (when referring to "proof, rigorous math, algebra") with "find[ing] another way to look at [a] problem" suggests he does not know that "his" ways of solving are "robust." He did not have sufficient authority over the technologies he used to know his use of LA1 technologies was robust (e.g., he was "not entirely sure that [...] [he] could have picked any [pair of vectors among $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}$] to find the area of the parallelogram, but [he thought] that is true").

In sum: P4 sensed some techniques are "better" ("is there another, a better way?"). Given an open problem that does not prescribe a technique neither implicitly nor explicitly (as LA1 problems do), P4 assessed potential approaches according to criteria that, while vaguely defined (e.g., "logical"), ranked *what's expected of LA1 students (solving equations)* lower in a certain sense: he preferred to activate his "own" approach, and he said that throughout the interview, he had tried to solve the given problems using his LA1 knowledge, "instead of looking at [them] from a more... logical point of view," and he perceived that automatically activating algebraic approaches is a mistake ("usually I think I fall into the mistake of... robust [algebraic] way[s] of thinking"). Nevertheless, P4's grasp of the theory was insufficient to lend him the knowledge that geometry-based approaches were as "robust" as the "rigorous [...] algebra" that is normative in LA1.

P7* also shared his perceptions of LA1 (student and examiner) norms and how these relate to his problem-solving practices. As he said as soon as he finished reading Problem 8, he had thought of various ways through which to complete the task ("I'm thinking I have so many ways to do that. I have no idea where to start.") He claimed later that he "use[d] the first one [areas of triangles] because it [would] be" what he "think[s] most people [would] do."

P7*'s criteria for what to submit on a LA1 test differed from P4's (whose solutions, per what he said earlier in the interview, would be those similar to what he would have practiced on past exams). If P7* were to submit a solution to Problem 8 on a test, he "wouldn't write all of" the approaches he had proposed in the interview. He "would just use the one that requires the least wording, because it [would] be much easier, and it [would] make sure it's going to be completely correct." He would "not write [...] complicated things." Additionally, P7* was aware of markers' expectations that students demonstrate LA1 knowledge (as opposed to knowledge from other mathematics courses): referring to his trigonometry-based resolution of Problem 8, he said that "if [he were to] use the rule of cosines, maybe the marker [would say], 'oh, [it] is not allowed to use [this] here.' [He's] not sure. So [he would] not use it."

P7*'s criteria for what to submit for grades in LA1 differed from his criteria for being convinced of the validity of a solution. To be convinced a result he finds is accurate, P7* enjoys replicability: "if I have time, I use different methods and figure out if the answers are the same." He recognized this is not expected in LA1:

> Actually, in the midterm of 204, there [was] one question of finding the size, find an area of the parallelogram. I [did], I kind of [did] a discussion, because there can be two different kinds of scenarios. But there's only one scenario that was actually marked, you know, that was marked and the other. Well, the marker did not say that I was wrong, but he just did not care about it.

The institutional conditions under which LA1 markers grade are as follows: in a typical fall semester, there can be over 400 students registered in several sections of LA1 (e.g., there might be 75 registered in a section with teacher A, 80 with teacher B, etc.). Common marking is an institutional strategy in courses such as LA1 (that is, mathematics courses prerequisite for various university programs). The strategy is such that teachers share in the grading of midterm and final exams by splitting grading by question (e.g., teacher A grades questions 1-3 for all 400 students, teacher B questions 4-5 for all students, etc.). Teachers have 5 business days to finish grading midterm and final exams. Given the conditions and constraints in which teachers grade LA1 (and other math courses at this level, such as differential and integral single-variable calculus), experiences such as the one P7* described are likely common: should a student submit more than one solution to a problem, the marker only grades one. Students are not expected to demonstrate nor rewarded (with marks) for demonstrating a breadth of knowledge in response to any given question on an exam.

Apart P7*'s comments about what is not expected of LA1 students, other comments he made shed light on what he favours in a solution to a problem. For example, after he proposed to use the projection of $\overrightarrow{CB}$ on $\overrightarrow{AC}$, he said: "oh, it is no longer elegant, but I could use the Pythagorean way to do that." Indeed, if I denote the initial point of $\vec{v}$ by D, then the length of $\text{proj}_{\overrightarrow{AC}}\overrightarrow{CB}$ is the length of DC. Since the length of CB can be found (as B and C are known), the Pythagorean theorem can be used to find the length of DB. As P7* suspected (he repeated a second time that "there should be [...] a more elegant projection way," but he was "not sure"), the Pythagorean theorem was unnecessary here as $\text{proj}_{\overrightarrow{AC}}\overrightarrow{CB}$ can be used to directly find $\vec{v}$, as

$$\text{proj}_{\overrightarrow{AC}}\overrightarrow{CB} + \vec{v} = \overrightarrow{CB}.$$

Even though P7* suspected orthogonal projections could be used more "elegant[ly]" in Problem 8, his next suggestion had nothing to do with these: "ah, yes, there is another way to do that, but I think it's very close to the first method." He then described how he would use trigonometry. Again, as he described this last suggestion, he spoke of elegance and simplicity: he said this was "the simplest way" and "more elegant" than his first suggestion (of using the area of triangle ABC to find its height $\|\vec{v}\|$).

P7* had referred to the notion of "elegance" previously in his interview. I hypothesize his concern with elegance, an oft-cherished quality in varied fields in mathematics, has less to do with his experience in LA1 than it does with his experience in higher-level mathematics courses in a mathematics major he had started in his home country before emigrating to study abroad.

5.8.3.2 Most students attempted to complete the task by producing equations whose solutions they expected to lead to the initial point of $\vec{v}$ (P1, P2, P3, P5, P6, P9).

Six of the nine participants tried to tackle Problem 8 by producing equations whose solution they expected to be the components of the initial point D of $\vec{v}$ (when the vector is placed such that its terminal point is $B(5,2)$): P1, P2, P3, P5, P6, and P9. For P1, P2, P3, and P6, this was the spontaneous reaction. This technique was P9's second approach, so not his spontaneous reaction, but it was the only one through which he managed to complete the task. For P5, finding D was not a spontaneous reaction; but a comment she had made prompted me to bring her attention to that point—and this, in turn, prompted P5 to produce an equation that featured its unknown coordinates (x, y) in an equation.

To produce equations, these students mobilized knowledge from LA1 and/or high-school geometry. This consisted of one or more of the following: the dot-product definition of orthogonality, know-how for producing parametric, point-normal, vector equations of lines, know-how for calculating distance between points or the length of a segment given its endpoints (be they expressed in terms of constants or unknowns), and the Pythagorean theorem. P1, P2, and P9 produced a system whose solution was indeed the initial point of $\vec{v}$; P6 proposed a system of two equations whose solution would have been this point, but his second equation was not expressed in terms of the unknown coordinates of D (as his first equation was) and he did not complete the task; and P3 and P5 produced equations that paraded surface-level grasp of relevant knowledge but did not correspond to the objects at stake in Problem 8.

Among these participants (P1, P2, P3, P5, P6, P9), none completed Problem 8 successfully through any other technique—that is, through techniques not based in the LA1 norm of solving (linear) systems so as to find a point (or line or plane). P1 and P2 did not attempt to do so when I asked if they could suggest any other approach; P9 had previously attempted another approach, but unsuccessfully; P6 did not attempt any other approach and was out of time to do so, though he insisted his grasp of "geometry" was lacking; and P3 and P5's approaches did not correspond to the mathematical objects at hand.

5.8.3.2.1 P1, P2, P6, and P9 produced systems of equations that represented the mathematical objects at stake. P1, P2, and P6's spontaneous reaction to Problem 8 was to mobilize the dot-product definition of orthogonality to produce an equation that captures the orthogonality of $\vec{v}$ to ℓ_1[32]. P1, P2, and P6 knew $\vec{v}$ is orthogonal to $\overrightarrow{AC}$; they found, given the endpoints of $\overrightarrow{AC}$, that $\overrightarrow{AC} = (2, 8)$, and after having assigned unknown components (x, y) (using x, y or other symbols for the unknowns) to the initial point of $\vec{v}$, they expressed $\vec{v}$ as $(5-x, 2-y)$. P1, P2, and P6 said the dot product of $\vec{v}$ and $\overrightarrow{AC}$ is zero, and thus produced the equation $2(5-x)+8(2-y) = 0$. After producing this equation, all three participants knew they needed another equation: P6 said he "need[s] to figure out one more equation, so [as to] relate these"; P2 paused and said "[he has] two unknowns here, [so he] should get two equations "; and P1 said "[he's] going to need more equations, because there are two variables."

To produce a second equation, and given the orthogonality of segments AD and DB, P2 and P6 appealed to the Pythagorean theorem. P2's initial notation for this theorem's application was dubious (he wrote $AD^2 + DB^2 = AB^2$), but he applied it appropriately to the *lengths* of the vectors at stake ($\overrightarrow{AD}$, etc.) and produced the following equation:

$$(x-1)^2 + (y-1)^2 + (5-x)^2 + (y-2)^2 = 17$$

P2 knew how to proceed from here: he said he'd use his first equation to express x in terms of y, substitute this into the second equation, solve for y, and then use this to solve for x. P6, meanwhile, wrote

$$a^2 + b^2 = 17,$$

and specified that 17 was the magnitude of $\overrightarrow{AB}$ and b the magnitude of $\vec{v}$. But P6 did not relate a and b to the vectors with endpoint D—that is,$\overrightarrow{AD}$ and $\overrightarrow{DB}$, as P2 had, and which would have enabled him to produce a second equation in x and y, and he got stuck. Perhaps P6 would have overcome this hurdle, and perhaps he wouldn't have; at this point, P6 proceeded to talk about his struggles with geometry and did not return to the task at hand.

To establish a second relation between x and y, P1 produced parametric equations for ℓ_1 and substituted the expressions for x and y (in terms of the parameter t) into his first equation to solve for t. He then identified the ordered pair (x, y) produced by this value of t. He knew this would be the initial point of $\vec{v}$.

P9 used the orthogonality of $\overrightarrow{AC} = (2, 8)$ and $\vec{v}$ differently; he used it to find a direction vector for ℓ_2 (the line parallel to $\vec{v}$ and passing through B). P9 knew that since $(2, 8) \cdot (-4, 1) = -8 + 8 = 0$, $(-4, 1)$ is orthogonal to $\overrightarrow{AC}$ and thus is a direction vector for ℓ_2. He said $(-4, 1)$ can then be used, along with point B, to form a vector equation for ℓ_2. Similarly, he knew $\overrightarrow{AC} = (2, 8)$ is a direction vector for ℓ_1 and so that it could be used to produce a vector equation for ℓ_1. He knew D (the initial point of $\vec{v}$) is the intersection point of ℓ_1 and ℓ_2 and so that it could be found by solving the system of two vector equations. P9 did not solve this system but his activity in Problem 7 involved successfully solving a system of two vector equations, so I infer he had the knowledge needed to do so.

[32] I refer to the line passing through $A(1, 1)$ and $C(3, 9)$ by ℓ_1 for clarity; students did not assign a name to this line.

5.8.3.2.2 P3 and P5 produced equations that drew from surface-level features of normative LA1 knowledge but did not correspond to the geometric objects at stake. For both P3 and P5, producing equations was the spontaneous reaction to Problem 8. Both, however, produced equations that failed to capture the geometric objects at stake, even if they drew on surface-level features of LA1 knowledge about orthogonality, lines, and distance. I discuss P5's equation first and P3's second.

Upon reading Problem 8, P5 said the following: "I know that it's orthogonal, it's going to be a cross product of these two vectors." As she said this, she pointed to $\overrightarrow{AC}$ and $\vec{v}$ in the image and I corrected her to say she meant dot product. P5 wrote the equation

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

and then paused for one minute. I asked what she was thinking and, as she started to answer—"I was trying to think about what, geometrically, I could do with this"—she corrected the $\times$ symbol to $\cdot$, said she "could find the length of this," wrote $2v_1 + 8v_2 = 0$, and said the distance between A and B is $\sqrt{4^2 + 1^2} = \sqrt{5}$. She knew this from the "distance formula between the two points." She then said she was "trying to think if there's anything geometrically [she] can do to find the length of it because [AB is] the hypotenuse of the right triangle," and "[she] just [doesn't] know the length" and paused. P5 was referring to one of the edges of triangle ADB other than its hypotenuse (I use D to refer to the initial point of $\vec{v}$ in the image), so I gave a hint to see how P5 would proceed: "to find the length of a vector, you need the initial and terminal points." I pointed out the initial and terminal points of $\vec{v}$.

The hint I gave prompted P5 to assign (x, y) as coordinates to the initial point of $\vec{v}$. She said she would "use a distance formula" and wrote the following equation:

$$2v_1 + 8v_2 = \sqrt{(5 + x)^2 + (2 + y)^2}$$

On the left of P5's equation is the dot product $\overrightarrow{AC} \cdot \vec{v}$. On the right is the length of $\vec{v}$ (with an error in the coefficients of x, y, which ought to be -1). P5 did not pursue this equation any further nor make any further comments about it. The equation seems a continuation of what P5 said as she wrote the equation $2v_1 + 8v_2 = 0$: that she "could find the length of this." This comment matches up with what P5 wrote on the image in Problem 8 (see Figure 5.6 on p.281). Even considering the difficulties of mathematics students at this level with the equal symbol (Kieran, 1981; Knuth et al., 2006), P5's equation, comment, and the traces she left on the sketch in Problem 8 seem to assign ill geometric meaning to the expression $2v_1 + 8v_2$ (the dot product of $\overrightarrow{AC}$ and $\vec{v}$): that it corresponds to the length of $\vec{v}$.

The first two of P3's attempts at Problem 8 reflected the LA1 technique of solving a linear system to identify a geometric object (such as a point, a vector, a line, or a plane). As I outline below, however, both attempts floundered as a result of insufficient knowledge of geometric concepts.

P3's spontaneous reaction to Problem 8 was an attempt to recall knowledge about orthogonality: "I don't remember exactly, but if it's orthogonal, I think... Like... There

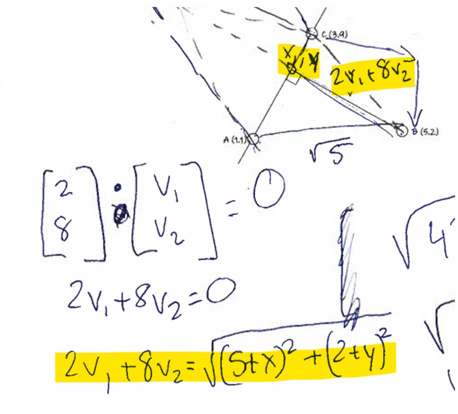


Figure 5.6: P5's additions to the sketch in Problem 8 (with my highlights)

is some kind of condition about it, so maybe I would try to... Find that first." She continued:

> [I can find a] line or a plane which go through these two points and then I would like to set arbitrary points which names a and b and it goes through these two. [pause] Or maybe the a and b go here and it goes to like this [see Figure 5.7]. So since they are orthogonal I think... I can find this point. So after then, yeah, it would be easier to get the distance from here to here.

P3 initially added (a, b) as an arbitrary point on the line ℓ_2[33] and may have later decided to place it as the initial point of $\vec{v}$:

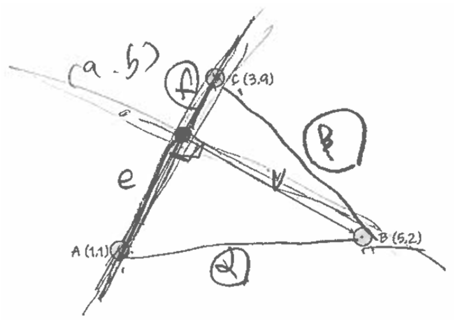


Figure 5.7: P3's additions to the sketch in Problem 8

Given P3's comments about not "remember[ing] exactly" some "condition" related to orthogonality, I offered her the definition at this point (two vectors are said to be orthogonal if their dot product is zero). After she read this definition, P3 asked for the definition of "dot product." I wrote an example to illustrate it:

$$(1, 2, 3) \cdot (4, 5, 6) = 1(4) + 2(5) + 3(6) = 4 + 10 + 18 = 32.$$

After a brief pause, P3 asked if "this [was] about the distance," but it was not clear if she was talking about the definitions I had given her or Problem 8, so I asked if she was

[33] P3 did not assign a name for the line passing through B and parallel to $\vec{v}$; I refer to it as such for clarity.

talking about the definition of orthogonality. P3 then dismissed her questioning ("Ah no, no, I mean. Wait, no. No no, it's ok.") and proceeded to solve a system she produced (so as to find a "line or plane"):

$$\begin{array}{ccccccc} 3a & + & 9b & + & c & = & 0 \\ a & + & b & + & d & = & 0 \end{array}$$

Given that the coordinates of $A(3, 9)$ and $C(1, 1)$ are the coefficients of the unknowns a, b, I speculate P3 intended to find the line through the points A and C, though regardless of the intention, these equations do not correspond to any line relevant to the given task. Above these equations, P3 had written $ax + by + c = 0$. P3 seems to have known that such equations capture something at stake - perhaps lines - but from her choice of coefficients, I infer her knowledge of such equations was superficial at best. (That is, she knew equations of this format have something to do with "line[s] or plane[s].")

P3 started by reducing the following matrix:

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

I asked from where she got this matrix. P3 had "no idea," she "just tried [something]," and shortly after corrected the matrix:

$$\left[\begin{array}{cc|c} 3 & 9 & -c \\ 1 & 1 & -d \end{array}\right]$$

When P3 had found the reduced row echelon form of her augmented matrix, she paused. I asked what she was thinking, and she said she "was thinking... what was it for?" P3 had lost sight of her goal, which she had originally said was to find a "line or a plane which go through" a given pair of points.

P3's first attempt at Problem 8 was to produce a system of equations to identify a point and then a vector: indeed, P3 had said, toward the start of her engagement with this technique, that "it would be easier to get the distance from here to here" once (a, b) is found, so her goal seems to have been to find the initial point of $\vec{v}$ and then find its distance from the terminal point of the vector. But the equations P3 produced were unrelated to the situation at hand and she could not make anything of the reduced row echelon form of the system.

After P3 wondered what her first attempt was "for," she abandoned her first approach as she started on a new equation:

$$(a, b) \cdot (5, 2) = 0$$

Based on P3's sketch, (a, b) was either the initial point of $\vec{v}$ (when placed such that its terminal point is $B(5, 2)$) or some other point along ℓ_2. Neither choice of (a, b) is orthogonal to $(5, 2)$. This is why I intervened and asked to what (a, b) referred; P3's answer confirmed it was necessary to clarify to P3 what the problem statement said. I pointed at $\vec{v}$ and $\overrightarrow{AC}$ in the image and said that their being orthogonal meant *their* dot product is 0. P3 paused, then said: "I... have no idea." She laughed as she proceeded to suggest a last strategy that mobilized high school knowledge (the Pythagorean theorem).

In P3's last attempt, she produced an equation that did actually correspond to the objects at stake, but she was unable to render the equation into a form useful for the task. She denoted the edges AB and CB of triangle ABC by α and β, respectively, and the edges AD and DC by e and f, respectively. She then applied the Pythagorean theorem to the right-angle triangles ADB and CDB and obtained this system of equations:

$$\begin{array}{rcrcr} \alpha^2 & = & e^2 & + & v^2 \\ \beta^2 & = & f^2 & + & v^2 \end{array}$$

From here, she obtained the equation $\alpha^2 - e^2 = \beta^2 - f^2$. In this equation, α and β are known as they are the lengths of vectors with known endpoints (though P3 did not indicate she knew this); but e and f are both unknown, so this equation in two unknowns is insufficient to complete the task.

When it came to the LA1 technique of solving linear systems to identify geometric objects, P3's mobilization was a surface-level spin on the mathematical objects involved: her first approach consisted of equations of the form $ax + by + c = 0$ with coefficients calqued from coordinates of A and C, and her second approach consisted of an equation of the form "dot product of an unknown vector with B equals zero," where the unknown vector ought'nt be orthogonal to B. She did not have sufficient knowledge about algebra and geometry of lines in $\mathbb{R}^2$ (e.g., recall P3's request for the definition of dot product), orthogonality, and vectors (e.g., recall her misuse of the vectors $A(1, 1)$, $C(3, 9)$, and $B(5, 2)$) to inform her use of the technique. P3's engagement with the technique of producing a linear system to identify a geometric object, starting with her suggestion to find a "line or a *plane*" (emphasis added), when the objects at stake are in $\mathbb{R}^2$, to the equations she produced (having only a superficial basis in the given situation, in that they had the form of equations recognizable from LA1 and involved coefficients of points in the sketch), to the comment that she had "no idea" how she got her first augmented matrix, to the realization she wasn't sure what her goal was in her first attempt ("what was it for?"), and to the determination that she had "no idea" after I pointed out that, in this situation, $\overrightarrow{AC}$ and $\vec{v}$ have dot product 0, suggests P3's choice of technique was not based in the mathematics at stake so much as it reflected the LA1 norm to solve linear systems to complete tasks that involve lines, planes, or points.

5.8.3.2.3 Students who attempted to complete Problem 8 by producing and solving equations did not produce other techniques that were appropriate for the task. While P1, P2, P6, and P9 differed from P3 and P5 in their ability to mobilize algebraic and geometric concepts (such as orthogonality, equations of lines, and the Pythagorean theorem), they had two traits in common. First, they all aimed to complete the task by producing equations so as to activate the LA1 norm of solving systems of equations. Second, they were united in their inability to mobilize other conceptualizations of $\vec{v}$—in particular, concepts from LA1 that allow for a direct computation of its length: first, the conceptualization that $\|\vec{v}\|$ is the distance between B and ℓ_1, and second, the conceptualization of $\vec{v}$ as the component of $\overrightarrow{AB}$ orthogonal to $\overrightarrow{AC}$. I highlight these conceptualizations for the convenience of the techniques they afford, though there is a variety of techniques through which the task can be completed other than solving equations that correspond to the given situation.

P9's spontaneous reaction to Problem 8 was to recall that he "had a formula for the magnitude of the perpendicular projection" and then to produce the parametric equations for ℓ_1. P9 then said he forgot the formula for orthogonal projection so I gave it to him. "So I just apply the formula." P9 recognized that's all the task required; but he calculated an orthogonal projection that was irrelevant to the task. He claimed the length of $\vec{v}$ is $\|\text{proj}_{\overrightarrow{AC}} B\|$. But $B(5, 2)$ is the vector with the origin as its initial point and $(5, 2)$ as its terminal point; not only is the projection of B onto $\overrightarrow{AC}$ not $\vec{v}$, it is a vector wholly irrelevant to the task.

I asked P9 how he knew what he had calculated was the length of $\vec{v}$—to see if justifying his technique would make him notice the error—but it did not: instead of addressing the way in which he applied the formula, he referred to the authority of the textbook as validation. When I asked, at the end of his engagement with Problem 8 (after he had proposed to complete the task by solving the system of equations representing ℓ_1 and ℓ_2 to find their point of intersection), what it would take to convince him his answer is correct, P9 said "the [orthogonal projection] formula" would be it, "because while studying it in the book, it explains that [...] the vector you find is mainly to get this." But he used this formula to target a vector irrelevant to the task. To validate the suitability of orthogonal projections to the task, P9 did not appeal to the geometry at hand; he appealed to the authority of the textbook.

Apart from P9, P5 was the only participant among those who produced equations who had broached an alternative that wouldn't involve solving equations. (P6 had mentioned "maybe there's something I could do [with] this other triangle here and do some fun stuff," but was not any more specific than this.) After P5 produced an equation irrelevant to the task, a question I asked about this equation—how did P5 know that $\overrightarrow{AC}$ is $(2, 8)$?—seems to have prompted P5 to try a different path. I include P5's answer as it seems to have prefaced her second approach:

> I don't know if that's like 100% how to find it, but I just did the distance. So, like, I drew this and I moved this one down and I kind of did that in my head and, like, moved this one to $(0, 0)$.

P5's second approach was to "shift the points over so A is at the origin." She redrew the sketch such that the point $A(1, 1)$ was translated to the origin (and so, the point $B(5, 2)$ was translated to $(4, 1)$ and $C(3, 9)$ to $(2, 8)$). She concluded this "didn't really help."

P5 was stuck so I asked what else she knew about the initial point of $\vec{v}$. P5 then offered a third approach: "I guess I could make this a quadrilateral and then this would be the diagonal." See Figure 5.8 for the quadrilateral to which P5 refers. For clarity, I refer by E to the top right vertex of the quadrilateral (that is, the point obtained by adding $\overrightarrow{BC}$ to C) and by F to the top left vertex (that is, the point obtained by adding $2\overrightarrow{BC}$ to A). P5 similarly described how to find the coordinates of E; as C is 2 units to the left of B and 7 units above it, she could trace the same trajectory, starting from C, to get to E. P5 explained her objective: "I would find the vector length [of $\overrightarrow{BF}$] and then

I'd halve it." She incorrectly deduced, from $\overrightarrow{BC}$ being half of $\overrightarrow{BE}$, that the length of $\vec{v}$ would be half that of $\overrightarrow{BF}$. The assumption at the basis of this error is that $\vec{v}$ aligns with the diagonal of P5's quadrilateral: indeed, at the start of P5's engagement with this technique, she said "I could make this a quadrilateral and then this would be the diagonal"; as she continued, she said "this is doubled. Okay. Oh, got it. Okay. So this is this point, and this is this point. Now it has more sense. And then... and then that's the diagonal." Notice, finally, that P5's sketch has the diagonal of the quadrilateral overlap with $\vec{v}$.

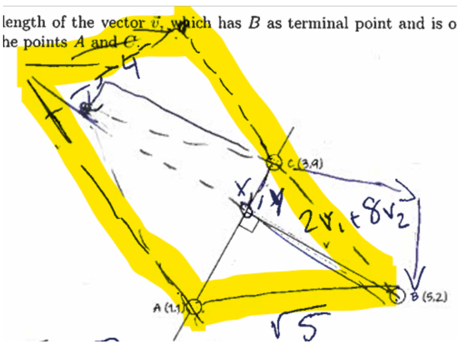


Figure 5.8: P5's quadrilateral in Problem 8 (with my highlights)

A comment P5 made clarifies what seems to have inspired the suggestion that $\vec{v}$ is half the diagonal $\overrightarrow{BF}$. There was a "similar [problem] on [the] midterm that [she] got right." P5 had "used a similar strategy" for that problem. The problem to which P5 refers stated that 3 parallelograms have 3 common vertices $(1,2)$, $(2,3)$, and $(1,1)$[34]. The first task was to find the coordinates of the fourth vertex of each parallelogram; the second task was to find the length of the longest diagonal amongst the six diagonals of these parallelograms. The technique needed to complete both tasks is precisely that suggested by P5. P5 had found surface-level features of Problem 8 similar to surface-level features of this midterm problem: both specified three points A, B, and C in 2-space, and there was a goal to find the length of a vector similar to (half) a diagonal. For P5, this made the problems "similar"; but P5's suggestion did not correspond to the mathematical objects at stake. P5 did say she "[doesn't] know if this is a mathematical way to solve it"; in the absence of mathematical knowledge on which to base her suggestions, P5 relied on surface-level features of problems she had to complete to pass LA1.

Like P9, P5 did not have recourse to knowledge that could produce a technique that would not involve solving equations. And nor did P1, P2, P3, and P6: when I asked P1 and P2 if they could suggest any other approaches, they said they could not. P3 suggested three approaches, all three of which aimed to produce equations whose solutions she expected to be relevant. P6's other suggestions were vague: "maybe there's something I could do, like this other triangle here, and do some fun stuff. [...] maybe the two triangle things would be easier for me to think about." I asked what other triangle "thing" he'd use; he said there's an "equilateral triangle" in reference to the triangle ABC, but this triangle is not equilateral. He continued: "honestly, I don't know all the manipulations

[34] I had access to the midterm test in question and were thus able to identify the "similar" problem to which P5 and other students had referred during their engagement with Problem 8.

you can do with triangles, but I know that probably starting from here with two side lengths and the information of the points you can like figure something out." P6's vague descriptions do not specify an alternate technique through which to find $\|\vec{v}\|$.

Solving systems is a norm in LA1. As discussed in the model of knowledge to be learned in LA1 relative to solving systems of equations (to perform tasks such as that in Problem 2) (Section 5.2.2), a variety of tasks in LA1, which I denoted by t_5 in Section 5.2.2, are of the type "to solve a linear system so as to accomplish a different LA1 type of task" (such as to find intersections of lines and/or planes, to determine linear independence of vectors, etc.). This explains P4's perception that in the "LA1 mindset," the approach to Problem 8 would be to produce equations to find a point of intersection. This also explains P6's perception that using knowledge about triangles is not "really the linear [LA1] way of solving" Problem 8:

> Maybe there's something I could do [with] this other triangle here and do some fun stuff. [...] Maybe the two triangle things would be easier for me to think about. But I know that that's not like, I don't feel like, that's not really the linear way of solving it. But it might be a way to solve it.

Given the normative quality of t_5 in LA1, I underscore the contrast between P1, P2, P3, P5, P6, and P9's ability to recognize equations *can* be used to identify a geometric object and their inability to mobilize LA1 knowledge(-to-be-learned) that does not involve solving equations. In P3 and P5's case, the inability to mobilize other LA1 knowledge broached on their ability to produce appropriate equations. In light of the norms relative to t_5, this is not surprising: the tasks in which students are responsible for producing an appropriate system of equations (listed in Section 5.2.2 as LA1 tasks I identify as being of type t_5) can be completed by calquing integers from a given set of vectors into a matrix and reducing this matrix (recall some students' comments about "putting" vectors given to them as "rows" or "columns" in a matrix they produced to perform certain tasks, as in Problem 5). This norm implies that students *can* complete tasks of type t_5 without knowing the mathematical reasoning for which the systems they produce are relevant; in turn, P3 and P5 were unable to produce appropriate systems when the usual technique (putting vectors as rows or columns of a matrix) was irrelevant.

5.8.3.3 Three students (P3, P5, P10) could not mobilize LA1 knowledge in a way that contributed toward completing the task.

The first two of P3's attempts were to produce equations that were false. P3 knew the task could be completed by finding the initial point of $\vec{v}$, but lacked knowledge to produce a system of equations whose solution was indeed that point. P3's third attempt was to use the Pythagorean theorem twice, again producing an equation—one whose solution P3 might have hoped to be the length of $\vec{v}$—but the number of unknowns (2) was too large for this single equation to be useful.

P5's first attempt, like P3's, involved equations whose solutions P5 might have hoped to be relevant, but the equations did not correspond in any useful way to the geometry at stake. P5's second and third attempts focused on the geometry (e.g., in the second step of her engagement with Problem 8, she shifted the points in the given sketch so the initial point of $\overrightarrow{AB}$ was now the origin, instead of A). P5's second step failed to trigger any technique; it led only to her third step, in which she drew a quadrilateral and made the false claim that its diagonal was twice the length of $\vec{v}$.

P3 and P5's attempts were both inappropriate for the given task and both reflected surface-level features of tasks they had experienced in LA1. P10, meanwhile, did not make any concrete suggestions for how to tackle the task. Her immediate reaction to Problem 8 was that she did not remember the formula for the length of a vector. I gave it to her, and she went quiet. Every contribution P10 made, from here on, was only in response to a prompt in which I asked what she was thinking. First, she said she considered finding $\overrightarrow{AC}$, but did not know what she'd use it for. Then, she said she considered to find the initial point (D) of $\vec{v}$, but was "not sure how to do that." She asked if D is the origin (if it were, she said she would do "B minus the origin" and find the length of this vector). But she didn't think D is the origin. She wasn't sure what to do. Her engagement with Problem 8 finished with this: $CA = (2, 8)$.

Without implicit or explicit instruction for how to tackle an open-ended problem, P3, P5, and P10 were unable to mobilize LA1 technologies to produce representations that corresponded to the mathematics at stake. P10 considered finding D but had no idea how to do so; P3 and P5 knew this point could be found with an appropriate (system of) equation(s) but could not produce an apt system.

P3, P5, and P10's engagement with Problem 8 contrasts, first, with P1, P2, P6, and P9, who were able to mobilize at least one LA1 technology to act out the technique of solving a system of equations to find points in 2-space: orthogonality in $\mathbb{R}^2$ (P1, P2, P6), equations of lines in $\mathbb{R}^2$ (P1, P9), and lengths of vectors (as P2 did so as to use the Pythagorean theorem). P3, P5, and P10's responses to this open-ended problem contrasts most with P4 and P7*'s flexible use of LA1 technologies, as they mobilized various conceptions of $\vec{v}$ (as the height of a triangle, as a vector related to an orthogonal projection, as the distance between a point and a line, and as a vector whose initial point is an intersection of two lines) to produce various techniques for the task.

P3 and P5 did not lack the technologies other participants mobilized, however. After all, it is P3's knowledge of orthogonality that produced the equation $(a, b) \cdot (5, 2) = 0$, but perhaps an inability to define vectors well (e.g., (a, b)) that presented a problem. And P5 mobilized not only the orthogonality of $\overrightarrow{AC}$ and $\vec{v} = (v_1, v_2)$ appropriately, having produced the equation $2v_1 + 8v_2 = 0$, but also the formula for distance between points relatively appropriately, having produced the expression $\sqrt{(5+x)^(2+y)^2}$ in reference to the distance between the initial and terminal points ((x, y) and $(5, 2)$, respectively) of $\vec{v}$. But P5 then created a mutation—the equation $2v_1 + 8v_2\sqrt{(5+x)^(2+y)^2}$. P3 and P5 were unable to use LA1 technologies without instruction for *how* to use them. In LA1, tasks that involve orthogonality or distances either explicitly or implicitly (by norms) tell students how to use these, in that all LA1 tasks come with an associated normative technique. LA1 students are not required to recognize where a technology is appropriate nor to design a technique of "[their] own" (to borrow from P4's expression). This (lack of) requirement shows in P3, P5, and P10's inability to use technologies to produce a technique, even as they showed signs of knowing what these technologies are.

5.8.3.4 Summary: the open-ended nature of the task highlights the limitations of norms established by knowledge to be learned in LA1 and points to the affordances of tasks that do not prescribe techniques

At the end of Section 5.8.2, I noted that LA1 students are not normally expected to complete open-ended problems, and they are not normally expected to know *which* knowledge about geometry in $\mathbb{R}^i$ ($i = 2, 3$) to mobilize. The knowledge students are expected to learn in LA1 suffices to perform Problem 8 *if* they recognize the task to be any one of t_i ($i = 1, 3, 4, 5, 6$) from my reference model for this problem. Of the 9 participants who attempted Problem 8, 5 mobilized sufficient knowledge to indicate they could complete the task. Of these 5, 2 students activated a variety of technologies from knowledge to be learned in LA1 (P4 attending to algebraic and geometric concepts from LA1 as he activated $\tau_1, \tau_3, \tau_4, \tau_6$ and P7* focusing on several notions from LA1 and high-school geometry as he activated τ_1, τ_3, τ_4 and trigonometric formulas), used these technologies to produce several techniques, and made comments that showed a preference for certain approaches over others. The 3 other students who showed capacity to complete the problem suggested one viable technique: to find the coordinates of the initial point D of $\vec{v}$ by solving a system of equations produced by algebraic representations of the objects at stake (τ_5 and/or τ_6). These 3 students, like P4 and P7*, combined technologies from knowledge to be learned in LA1 and high-school geometry to produce equations for these two lines: the dot-product definition of orthogonality (P1, P2, P9), notion of a normal as a vector orthogonal to a line (P9), parametric equations for lines (P1, P9), length of vectors in $\mathbb{R}^2$ (P2), and the Pythagorean theorem (P2). Of the 4 students unable to mobilize sufficient knowledge to indicate they could complete the task, there was 1 who suggested the beginnings of τ_5: P6 produced one equation using the dot-product definition of orthogonality, and knew a second equation was needed so as to find the coordinates of D, but did not make concrete suggestions as to how it might be produced. The activity of the remaining 3 students also targeted τ_5, but failed to produce any appropriate algebraic representations.

Except for P7*, all students attempted τ_5 or τ_6—to use algebraic representations of the objects at stake to produce a system of equations whose solution is D—and among these, only one student (P4) was able to mobilize a technique in which the target was not to solve a system of equations. This can be explained by the heavy weight ascribed to system-solving tasks in knowledge to be learned in LA1: 45% of the exam tasks to which I had access required system-solving as a technique to complete the task, and 26% of the exam tasks were to solve a linear system so as to accomplish a task whose objective is not strictly to find the solution of that system (e.g., to determine if a set of vectors is linearly independent or to find the intersection of planes) (see Table 5.3 in Section 5.2.2); additionally, system-solving techniques and tasks occurred in each of these midterm and final exams. No other technique nor technology to be learned in LA1 stands up to this standard of solving systems to as to accomplish a task: formulas for orthogonal decompositions, for example, were the expectation for 3.5% of the exam tasks, needed in 4 of the 6 final exams I considered, and were only ever needed in a task type that normatively requires formulas for orthogonal decompositions (unlike system-solving techniques, which are normatively required for tasks other than those of type "to solve a system"). The case for orthogonal decompositions is similarly the case for distance formulas (1.7% of exam tasks, 2 of 6 final exams, only needed for tasks that explicitly call for these formulas given an objective *to find a distance*), as well as for formulas for areas of parallelograms

(3.5% of exam tasks, 4 of 6 final exams, only needed for tasks that explicitly call to find an area of a parallelogram or triangle).

It's no surprise, then, that the techniques students produced mainly aimed to solve a system of equations. This is what's usually done in LA1. P4 even made a comment to this effect:

> I could try to find [the initial point of $\vec{v}$]. And then, then I know this vector and [I can] try to find the norm of the vector, which I think [is] what they would expect us to do in this course.

But to produce equations relevant to the given task, students did need to mobilize other knowledge to be learned in LA1, knowledge that is rarely to be learned but also to be activated only when students are instructed to do so, and this challenge blocked 4 of the 8 students who had hoped to complete the task via a system of equations.

The success of 5 of the 9 participants to mobilize technologies from LA1 (and, for P2 and P9, perhaps knowledge about vector equations reignited by their engagement with Problem 7) so as to produce techniques for a problem of a type they'd never encountered, however, does point to what LA1 students could potentially achieve, given the opportunity. The open-ended task did not prescribe a technique, neither explicitly nor by similarity to a normative LA1 task type. Even considering the case of P1, P2, and P9, who were limited to the relatively more standard τ_5/τ_6, students who had sufficiently grasped technologies that are to be learned in LA1 (albeit not as much as system-solving techniques) were able to *produce* a technique; this is in stark contrast to the standard that students may perform tasks in LA1 by reproducing surface-level features of available solutions of similar tasks.

I distinguish between the knowledge P1, P2, P9 mobilized (to fulfill τ_5/τ_6) and the knowledge P4 and P7* mobilized to produce their various techniques ($\tau_1, \tau_3, \tau_4, \tau_6$). P1, P2, and P9 turned to technologies that target objects whose relevance to Problem 8 was made explicit by the problem statement: orthogonality and lines. In contrast, P4 and P7* recognized the relevance of objects not explicitly mentioned in the task: a parallelogram and its height, an orthogonal projection[35], and a distance between a point and a line. I noted, in the model of knowledge to be learned that can be used to complete Problem 8, that these technologies become relevant only if students are able to view the task in terms different from those explicitly stated (to find a distance; to find the length of an orthogonal component; to find the height of a parallelogram).

It is not surprising that P7* had reached for technologies not obviously relevant to the task; this was his behavior throughout the TBI, usually offering a minimum of 4 approaches (be they relevant or not) for each task. But P4's activity, together with his comments about what he believed would have been a typical LA1 technique (τ_6, system-solving to find a point) versus what he viewed as a technique better suited for the task at hand, point at what a task like Problem 8 can achieve: give permission to students to engage with the mathematics targeted by knowledge to be learned in LA1. This is different from P4's engagement with Problem 7, for example, which lent itself more

[35]P9 had also mentioned the notion of orthogonal projections, but the projection he proposed had no relevance to the given task.

obviously to a standard LA1 task, and for which P4 activated a non-standard approach only after I asked, at the end of the TBI, what he meant when he said that he "always finds simpler solutions" (Section 5.7.3.2). This prompt had given P4 "permission" to mobilize an approach that he felt would not have been expected in LA1. But Problem 8 did not correspond to any LA1 task as Problem 7 had (in that Problem 7 was, essentially, the LA1 task of determining the number of solutions of a linear system, and the representation of objects in Problem 7 lent itself directly to mobilizing standard system-solving techniques from LA1). But, beyond t_5 (or t_6), there was no standard task after which to calque a technique for Problem 8; and even for τ_5 and τ_6, students had to make their own choice as to which technology to wield to produce appropriate algebraic representations. P3, P5, P6, and P10's struggle to produce appropriate algebraic representations show the limitations of the knowledge students are usually required to learn in LA1. The knowledge students are required to learn in LA1 does not give students permission to engage with mathematics in the way Problem 8 does: students are expected to demonstrate mastery of a certain set of techniques, and it suffices in LA1 to operate along surface-level features of these techniques. P4's activity shows the potential of tasks that do not prescribe a technique to prompt students to engage with technologies (to be learned in LA1) in ways that go beyond their superficial features (i.e., formulas).

5.9 Analysis of students' positioning across TBI problems

The goal in analyzing participants' positioning is to examine evolutions in students' positioning between those of Student (during their tenure as LA1 students) and Learner (during their tenure as LA1 students or during their participation in the TBI). Remember the conception of Students and Learners: a Student's activity is guided by the motivation to get a certain grade in their course, whereas a Learner's activity is guided by a cognitive motivation to develop understanding.

Hardy's (2009a) and Broley's (2020) studies of students' models of knowledge to be learned in Calculus and Analysis courses show that tasks that are normal—normal in the institution in which they're administered—don't allow for a Learner disposition as they enable imitative strategies and knowledge acquisition limited to surface-level features through which the 'usual' tasks can be completed. Such tasks do not provide opportunities for Learner activity even among students who may have an interest in gaining understanding or are enthusiastic about mathematics.

Looking at participants' positioning throughout their interview can reveal triggers that can change students' activity from that belonging to a Student to one of a Learner. I look to two potential sources for such triggers: the students and the problems. Are there specific problems where students are more likely to make this switch? Problems where students do not make this switch at all? What are the features of the former and latter problems? And who are the students who switch from Student to Learner positions—are there features that seem characteristic to them but not to others, or not necessarily? Ultimately, by looking at positioning, I want to look at the experiences the TBI problems afforded participants, and use this to get a sense for what students are afforded by problems that are 'normal' in LA1.

In Section 3.2.2, I discussed the four positions elaborated in (Sierpinska et al., 2008; Hardy, 2009a; Broley, 2020). Student, Learner, Client, and Person; in my analysis of students' TBI activity, I did not identify instances alluding to students having positioned themselves as Clients or Persons in LA1. This is not to say they had not occupied these positions, but that their activity and comments did not give a chance to examine students' potential experience in either position. This brings up another point: the results of this analysis, addressing features of Learner and Student positions as experienced by the TBI participants, are not comprehensive. If a new student engages with these TBI problems tomorrow in a similar interview environment, they may exhibit altogether new behavior. Different tasks may trigger different activity or comments as well.

What this analysis does reveal, however, is a collection of (71) behaviors that characterize students' positioning as Student or Learner; these behaviors describe features of students' TBI activity and/or comments *in relation* to knowledge to be taught (KtbT), knowledge to be learned (KtbL), or knowledge that may have actually been taught (KT) (if suggested by a student's comment) in LA1. The general guideline for qualifying a behavior as that of a Student or Learner[36] was this: if the behavior corresponds only the

[36] I discuss this guideline in more detail, along with all aspects of the methodology for analyzing

needs of the KtbL in LA1 (as in, to receive a passing grade), it is that of a Student; if the behavior extends beyond these needs, it is that of a Learner.

I classified these 71 behaviors into (25) 'position properties': 9 properties indicative of Learner positioning and 16 indicative of Student positioning. These properties justify the sense in which a given behavior indicates a Student or Learner position in the LA1 institution. Any two behaviors that indicate a certain position for the same reason were therefore classified by the same position property; in turn, the position properties are not necessarily mutually exclusive (e.g. there can be overlap between 'surface-level grasp of KtbL' and 'failure to use KtbT that is not KtbL') and the choice to classify a behavior according to one property or another (or both) depended on the instance being analysed. For the purposes of this analysis, I omit a discussion of all 71 behaviors I identified in students' activity and comments, but give an example of one property indicative of the Student position along with instances of behavior which share in this property, and do similarly for one property indicative of the Learner position.

7 of the behaviors exhibited by participants shared in the property that they showcased a 'compartmentalization of knowledge by KtbL tasks'; this is one of the 16 properties I identified as indicative of Student positioning. I give examples to illustrate each behavior (B) sharing in this property:

B applies technique for LA1 task with similar surface-level features but different in substance

example P5 had found surface-level features of Problem 8 similar to surface-level features of a LA1 midterm problem: both involved three points A, B, and C (in 2-space), and there was a goal to find the length of a vector similar to (half) a diagonal. For P5, this made the problems "similar"; but P5's suggestion did not correspond to the mathematical objects at stake.

B categorizes certain technologies as cues to mobilize certain LA1 KtbL techniques

example P2's justification for using the normative technique τ_{42} (calculating a determinant of a matrix consisting of 3 vectors to check if these vectors are linearly independent) in Problem 5, saying this: "I remember that [the vectors being] independent or dependent has to do with the determinant. And this is the easiest way to do it. Because I get zero, so I get a straightforward answer without having to analyse it."

B claims to have done past final exams/categorized knowledge by LA1 tasks as a strategy for gaining LA1 knowledge

example This comment from P4: "I recognize from memory that when I tried to set up the - a system equations like this one, I would recognize that I was trying to find the intersection of two planes, which is a line. And this - this is based on ex - exactly just one question that I solved. [inaudible] I remember that question. I don't think that yeah, other than the lecture information. A lot, a lot of my knowledge was gained from the past exam questions. So like one,

students' positioning, in Section 4.3.3

one question I - I tackled was - would always be in the back of my head when approaching another problem. Especially because it's an intro course and the questions are similar to each other."

B engagement with task (mobilized knowledge, technique) is conditioned by superficially-similar routine task from LA1 KtbL

example P9's activity while engaged with Problem 1: after finding M_3 is not invertible, he suggested to multiply by C^{-1} so as to isolate it and then find its inverse C. P9's engagement with Problem 1 was conditioned by the normative LA1 task for solving matrix equations, and the know-how for that task did not suffice for Problem 1.

B struggles to identify objective of a task that is not a normative LA1 task

example In response to Problem 3, P3 was unable to suggest any approach upon reading the problem and even after I reworded the question: she had initially tried to recall knowledge that would relate the cross product with the system, then thought the system was missing an equation (specifically, $0x + 0y + 0z = 0$), and then rewrote the first equation in the form of a matrix equation ($A\mathrm{x} = 0$). After I reworded the problem for P3, she said "she forgot how to handle this" and tried to recall what the "teacher" had said: "I remember like during the class. The teacher was like, keep talking about like, after this calculation it should not be 0 or [pause] oh, it's about independency." P3 was stuck so I gave her the definition of cross product; she said that "[she's] trying to remember... The way that [she] solved this kind of question before." P3's activity and comments reveal that she could not identify the objective of the task.

B tries to produce technique on the basis of experience with LA1 task involving similar surface-level features

example P5, in response to Problem 3, remembered surface-level features of LA1 tasks involving cross products: "from like, doing problems in the past that had me, like, jig [sic], which thing would make like the cross product zero, and like, the numbers were like, you just like flip... like a number. Or, like, put it like plug in numbers that are like similar. I don't know how to, like, explain it! Like it would be... Like I would have... Like, I'm just trying to remember like past homework. Like, we'd have like, two equations like this. And it'd be like, what would... I don't know if it was exactly like a similar question. It's like, find two vectors that would like... that are like *perpendicular* [emphasis added] or parallel. or whatever. I don't know. I think I'm like confusing a bunch of different ideas together." Asked what made her say the word perpendicular, P5 said "those are a lot of the problems that [she] did at the end of the term. And this is kind of around the time [she] remember[ed] like doing... things like that."

B suggests to convert task to its usual appearance in LA1 so as to use normative LA1 KtbL

example P1, P2, P3, P5, P6, and P9's mobilizing in Problem 7 of the normative LA1 row-reduction technique so as to find the number of solutions of a system of two vector equations in $\mathbb{R}^2$.

Meanwhile, 3 of the behaviors exhibited by participants shared in the property that they showcased a 'use of KtbT/KT/KtbL combination'; this is one of the 9 properties I identified as indicative of Learner positioning. I give examples to illustrate each behavior (B) sharing in this property:

B mobilizes LA1 KtbT/KT/KtbL directly pertinent to the given task instead of KtbL pertinent to task that involves a similar feature

example P1, in response to Problem 3, *started* to do the necessary computations (wrote the expressions needed to calculate the components of the given cross product) but stopped short of doing any calculations; writing these expressions out seems to have prompted him to think of another approach, as he instead activated τ_1 (cross product property of orthogonality) to complete the task.

B seeks to mobilize knowledge that is directly pertinent to the given task rather than to the similar LA1 task

example P4, in response to Problem 8 and after suggesting a first approach (τ_4), P4 wondered: "is there another [way], a better way? Perhaps." P4 considered the task in terms of t_1 and t_2: "I could try to find this point [the initial point of $\vec{v}$ in the given image]. And then, then I know this vector and [I can] try to find the norm of the vector, *which I think [is] what they would expect us to do in this course.* [emphasis added]." P4 found the components of $\overrightarrow{AC}$ and then decided against this equations-based approach: "nah, I'm gonna try *my own* solution" [emphasis added]. P4 distinguished between his perception of what is expected of students in LA1 and solutions he rather perceived as "[his] own" and "perhaps" a "better way" for the given task.

B tries to reason about a situation using LA1 knowledge other than KtbL directly associated with normative task

example P8, in response to Problem 5, mobilized a combination of algebraic and geometric knowledge from LA1 after being confused about the result of her calculations; her geometric knowledge helped her interpret the abnormal algebraic result.

My analysis of students' positioning will attend mainly to the position properties that I found to qualify their activity and comments as characteristic of Student or Learner positioning. I will organize this analysis around a selection of charts that help to answer the following guiding questions: are there specific problems where students are more or not at all likely to switch from a Student to Learner position? What are the features of the former and latter problems? Who are the students who switch from Student to Learner positions—are there features that seem characteristic to them but not to others, or not necessarily? The charts I created for this analysis show the distribution of positions and their properties across problems and, in turn, help to draw links between features of problems (when considered in relation to LA1 knowledge) and the behaviors they can afford to students.

Before I get to the charts, two declarations. First, I decided to omit P7* from the analysis of students' positioning because his previous educational experience includes participation in several higher-level university mathematics courses, as well as participation as a tutor for adult students in prerequisite mathematics courses; this experience significantly departs from that of the other participants. Second, I remind the reader that one participant (P8) was unable to attempt Problems 6-8 due to time constraints unrelated to the TBI. This shows in some of the charts below. This does not pose issue for the purposes of the analysis of students' positioning so I still included the data from P8's TBI for this analysis.

5.9.1 Task features and the positions to which they are amenable

I attend to Figures 5.9, 5.10, 5.11, and 5.12 to identify features of tasks that make them more or less amenable to Student or Learner positions.

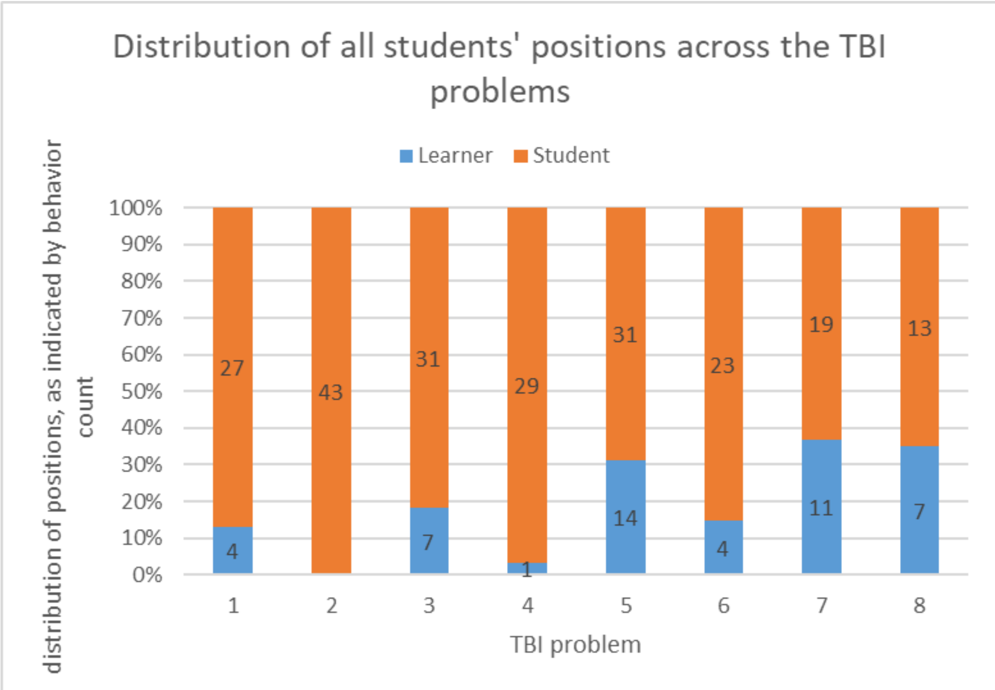


Figure 5.9: Distribution of Learner and Student positions across the TBI problems, as indicated by the count of behaviors indicating each position

From Figure 5.10, I note that, with the exception of one participant in Problem 6 whose activity exclusively displayed Learner position properties, all participants displayed Student position properties in response to every problem. Student behaviors characterize the bulk of behaviors identified for each problem, with the smallest share being 63% of identified behaviors for Problem 7 and the greatest being 100% of identified behaviors for Problem 2 (see Figure 5.9). None of the problems triggered Learner behavior in *all* participants: Problem 2 ranks the 'lowest' with 0 participants responding with Learner behavior and Problems 5 (attempted by all students) and 8 (attempted by all students except for P8) each having 5 distinct participants show Learner behavior (Problem 5 triggered this behavior in P2, P4, P6, P8, and P9; Problem 8 triggered it in P1, P2, P4, P6, and P9). I start off by addressing the positions and position properties afforded by each TBI problem, in order from the problem that failed to encourage any Learner positioning

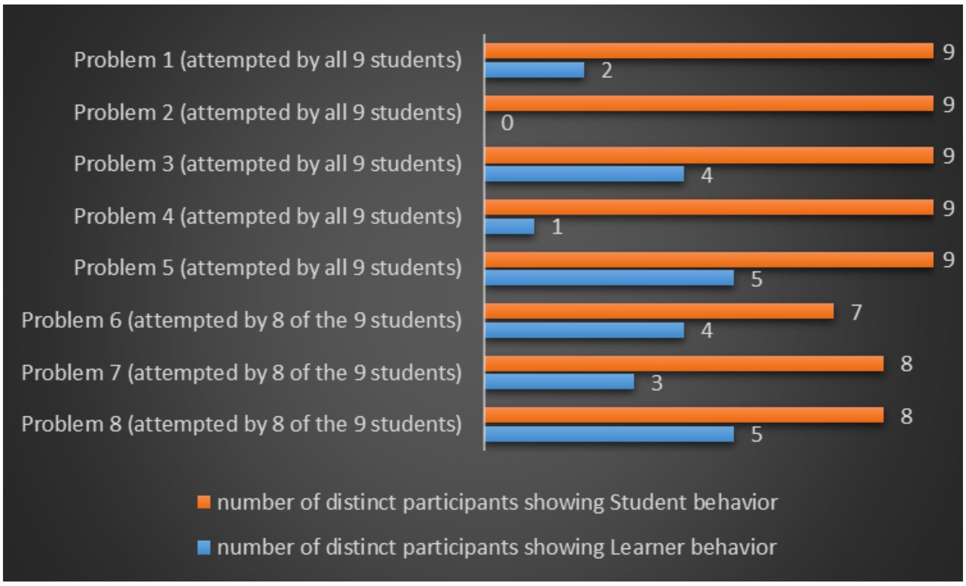


Figure 5.10: Count of distinct participants showing Student or Learner behavior in response to each TBI problem

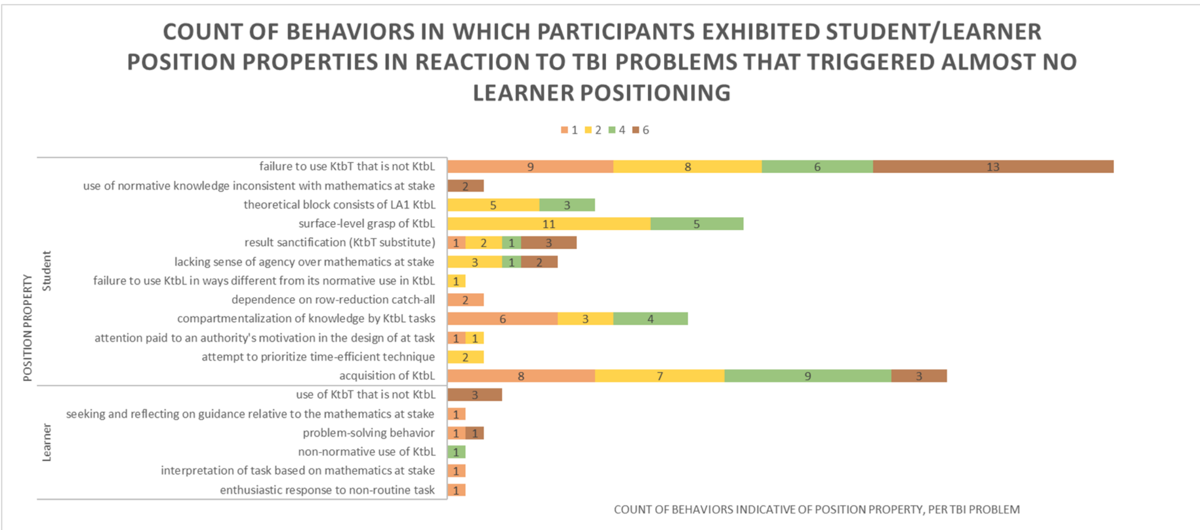


Figure 5.11: Behavior count of position properties for TBI problems that do not encourage learner position

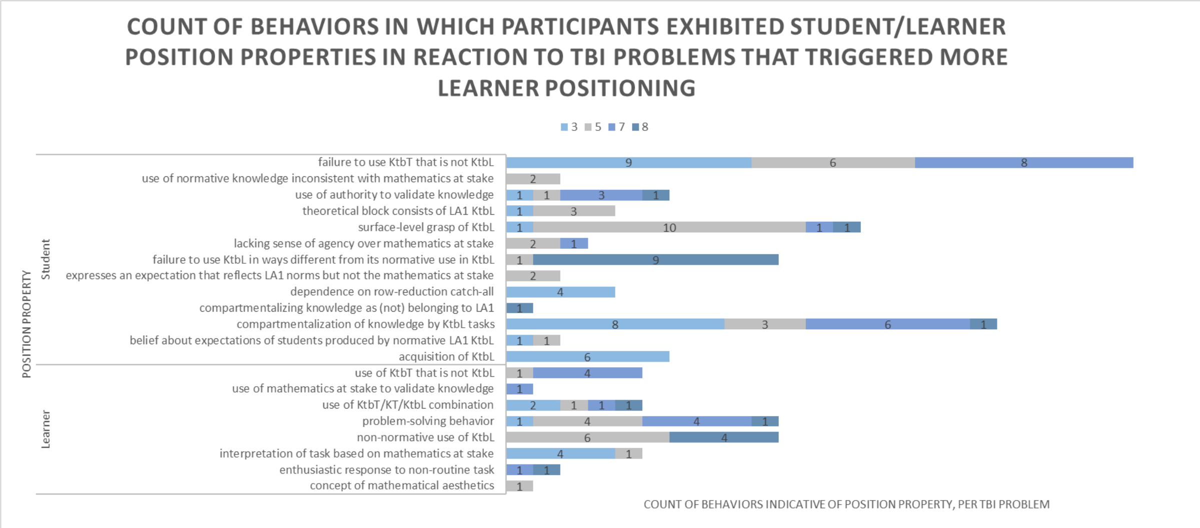


Figure 5.12: Behavior count of position properties for TBI problems that can encourage learner position

to the problems that proved most amenable to encouraging Learner positioning.

Problem 2 did not trigger any Learner positioning in participants. This is the problem that most closely resembled a normative LA1 task: to solve a linear system of less than a handful of equations in less than a handful of variables with small integer scalars. Its main distinguishing factor from the normative ask was the assertion that the coefficient matrix is invertible, and this factor did not detract from the applicability of the technique most usually used in LA1 for this task. Problem 2, from students' perspective, was the routine LA1 task, and this showed in their behavior. From Figure 5.11, I see that the behaviors triggered in students, from most counted to least, belong to 12 of the 16 position properties I had identified for the Student position: surface-level grasp of KtbL, failure to use KtbT (knowledge to be taught) that is not KtbL, acquisition of KtbL, theoretical block consists of LA1 KtbL (in that students produce or justify their techniques on the basis that these belong to KtbL in LA1), lacking sense of agency over the mathematics at stake, compartmentalization of knowledge by KtbL tasks, result sanctification (a KtbT substitute, in that results obtained from applying a technique take the justifying and technique-producing role of mathematics in the KtbT), attempt to prioritize time-efficient technique (a consideration useful for exams), failure to use KtbL in ways different from its normative use in KtbL, and attention paid to an authority's motivation in the design of a task.

Problem 4 saw only one instance of Learner behavior (a non-normative use of KtbL by P9, who mobilized his non-normative use of KtbL from Problem 3 to complete Problem 4); this accounts for 3% of the behaviors identified in the activity triggered in students by this problem. Problem 4, like Problem 2, resembles a normative LA1 task: while the problem statement is to find *one non-trivial solution* to a homogeneous linear system of 2 equations in 3 unknowns, and while the scalars include non-integer and irrational numbers, students' activity seemed mostly triggered by the normative LA1 task resembling this one—to find solutions to a homogeneous linear system of 2 equations in 3 unknowns. From Figure 5.11, I see that the behaviors triggered in students, from most counted to least, belong to 7 (but mostly 5) of the 16 position properties I had identified for the Student position: acquisition of KtbL (attested to by students' mobilization of the technique usually used for the normative task similar to Problem 4), failure to use KtbT that is not KtbL, surface-level grasp of KtbL, compartmentalization of knowledge by KtbL tasks, theoretical block consisting of LA1 KtbL, result sanctification (KtbT substitute), and lacking sense of agency over mathematics at stake.

Problem 1 allowed for 4 instances of Learner behavior: 3 in P6 and 1 in P3. None of these behaviors indicated P3 and P6 had behaved as Learners in LA1, but rather indicated the problem had triggered in P3 and P6 behavior that would befit a Learner position: seeking and reflecting on guidance relative to the mathematics at stake, problem-solving behavior, interpretation of a task based on the mathematics at stake, and, once the task was no longer perceived as routine, an enthusiastic response to a non-routine task. Otherwise, 87% of the behaviors identified for Problem 1 corresponded to a Student positioning. From Figure 5.11, I see that the behaviors triggered in students, from most counted to least, belong to 6 of the 16 Student position properties: a failure to use KtbT that is not KtbL, acquisition of KtbL, compartmentalization of knowledge by KtbL tasks, dependence on row-reduction catch-all (in that a substantial portion of LA1 tasks can be solved by row-reducing an appropriate matrix), result sanctification, and attention paid to an authority's motivation in the design of a task. Like Problems 2 and 4, the surface-level appearance of Problem 1 as a highly routinized LA1 task is the feature that triggered behaviors that show Students' satisfaction, during LA1, in strictly acquiring the minimal knowledge needed to complete a routine task.

The scales start to tip with Problem 6, though—like with the other TBI problems—students' behaviors still mostly point to their having occupied the position of a Student during their participation in LA1. There were 4 instances of Learner behavior, each from a different participant (P1, P5, P6, and P9): 2 of the students used KtbT that is not KtbL (they used the knowledge that a solution to a system is a value that satisfies *each* equation in the system; no KtbL task explicitly requires students to mobilize this knowledge, and indeed 6 students failed to use it) and one exhibited problem-solving behavior. The remainder of the behaviors identified (85%) indicate Student positioning and, from most counted to least, belong to 5 of the 16 properties of the Student position: failure to use KtbT that is not KtbL, result sanctification (KtbT substitute), acquisition of KtbL, and a lacking sense of agency over the mathematics at stake. In the case of Problem 6, what mostly triggered the Learner position was the need to mobilize knowledge that teachers likely take for granted that students know (i.e., that a solution to a system is a value that satisfies all of its equations) and which they never require students to use in graded tasks. While students knew Problem 6 was not routine given the unusual characteristic that the equations were quadratic and not linear, their behaviors showed that the knowledge they had acquired in LA1 for solving systems of equations was not supported by any of the corresponding KtbT; it was conditioned by the routine.

For the remainder of the problems, I name the behaviors (and not only the position properties) that had led me to classify a student in the Learner position; I do this to help illustrate how these problems were amenable to encouraging Learner positioning. The list of all identified behaviors, classified by position property (for the Student and

Learner positions), is in Appendix B[37].

Problem 3 saw Learner behavior account for 18% of identified behaviors; these were attributable to 4 of 9 participants. I identified three Learner position properties in relation with Problem 3. The property 'interpretation of task based on mathematics at stake' showed in of all 4 participants who behaved as Learners here in that they had reevaluated the objective of the task after identifying a property of a mathematical object at stake (namely, a geometric property of cross products), after having initially responded to the task as if it is a superficially-similar routine task from LA1 KtbL. For two students, this behavior-position property combination preceded the next one: the property of using a combination of KtbT, KT (knowledge actually taught, according to students' comments), and KtbL showed through their mobilization of (LA1) KtbT/KT/KtbL directly pertinent to the task instead of KtbL pertinent to a task that involves a similar feature (e.g., such as to calculate a cross product whenever it is a feature of a task, or to find all solutions of a system whenever a system's solutions are at stake). One student failed to mobilize knowledge directly pertinent to the task (namely, the orthogonality of cross products), but did display behavior of the problem-solving property of a Learner: she *sought* a mathematical property intrinsic to the task and which may have an advantage over already-identified pertinent knowledge.

Considering the behavior-positioning property pairings that showed the Learner position in students' responses to Problem 3, I note the potential of Problem 3 to trigger Learner behavior in students (e.g., by prompting them to question whether their normative knowledge is the most suitable for the task), as well as to elicit students' positioning as Learners as participants in LA1 (e.g., as in the case of 2 participants who *had* the less-usually-needed knowledge about cross products). Two features were responsible for triggering Learner behaviors: one, the non-routine task of showing an object, expressed in what students usually treat as an operation (cross product), is a solution of a system; and two, the non-integer nature of the scalars, which seem to have made unpalatable the 'usual' task to calculate cross products when they appear.

Problem 5 saw Learner behavior account for 31% of identified behaviors; these were attributable to 5 of 9 participants. I identified 6 Learner position properties in relation with Problem 3; I will discuss the two that were exhibited by more than one student at a time.

The property 'non-normative use of KtbL' showed, in different ways, in of all 5 participants who behaved as Learners in response to Problem 5. The most prevalent behavior (related to this property) was to mobilize a mathematical property of a task component that is not needed in LA1 tasks involving that component (a geometric visualization of a parallelepiped of volume 0; LA1 tasks involving parallelepiped volume only call for a volume formula); as some students struggled to mobilize the property accurately, I see this behavior as an example of the task's *potential* to trigger Learner behavior (by encouraging students to seek knowledge they don't normally need to use).

[37]This list is not comprehensive in that a new student may do the same problems I had proposed in the TBI interview and new behaviors, position properties, or positions may be identified. I also believe the list of behaviors and position properties can be refined, but its current state suffices to explore the positions triggered by certain types of task features.

Also in students' responses to Problem 5, the property 'problem-solving behavior' showed most often in the behavior to mobilize a variety of techniques pertinent for a task (and compare/reflect on the results obtained). Namely, students alternated between geometric and algebraic representations. I attribute this behavior to two features of the task. One is the non-routine combination of geometric and algebraic objects in a single task; students had no guideline as to what knowledge to use and when. The second feature is in the choice of mathematical objects (vectors involving an unknown entry), which made it difficult to use the routine technique (for determining linear independence) for various reasons.

Problem 7 saw Learner behavior account for 37% of identified behaviors; these were attributable to 3 of 8 participants. I identified 5 Learner position properties in relation with Problem 3; I will discuss the two that were exhibited by more than one student at a time.

The property 'problem-solving behavior' showed, in different ways, in 2 of the 3 participants who behaved as Learners in response to Problem 7. Both had sought to mobilize knowledge that is directly pertinent to the given task rather than to the similar LA1 task—the similar one, namely, being to find the general solution of a homogeneous linear system. Among these two students, one had exhibited another problem-solving behavior: he evaluated techniques based on criteria that are not required in LA1 KtbL ("simpler," "more logical"). In light of this, and given that the other student's problem-solving behavior came about after a struggle to use a routine technique, given non-routine algebraic expressions, I surmise the feature of Problem 7 that made it amenable to triggering Learner positioning was this: while it was possible to tackle the task (find the number of solutions of a system of 2 vector equations) by substituting it by a routine one (solve a corresponding system of equations in real numbers), students are not used to making this conversion and it is computationally unpalatable.

The two students who engaged in problem-solving behavior in response to Problem 7 also shared in the position property 'use of KtbT that is not KtbL.' This showed in two behaviors. First, after a struggle with normative LA1 knowledge, they mobilized LA1 KtbT that is not LA1 KtbL. This confirms the role of a struggle to adapt normative LA1 knowledge in prompting students to problem-solve, and, in this case, seek KtbT. The non-routine nature of the task created a need for this behavior to come about. This attests to the potential of Problem 7 to trigger Learner behavior. The other behavior (that I classified as 'use of KtbT that is not KtbL') attests the potential of Problem 7 to elicit knowledge students *had* but did not originally mobilize: KtbT that is not KtbL. The behavior is this: the students mobilized knowledge that is KtbT in LA1 and only rarely KtbL. This showed these students had acquired this knowledge in LA1 even if it was rarely KtbL; I take this as a sign of their having occupied a Learner position in LA1[38].

Problem 8 saw Learner behavior account for 35% of identified behaviors; these were attributable to 5 of 8 participants. I identified 4 Learner position properties in relation

[38]Note this does not mean these students *exclusively* occupied the Learner position in LA1. Indeed, there is plenty of evidence they had mostly positioned themselves as Students—but there is evidence their experience included at least some Learner positioning.

with Problem 3. I discuss the one that was exhibited by more than one student at a time: a non-normative use of KtbL. This showed in students through a behavior of identifying and mobilizing relevant technologies from LA1 and prerequisite math courses through which to complete an open-ended task. I attribute this behavior to the non-routine and open-ended nature of the task. It was non-routine in that it did not correspond to any usual LA1 task. By 'open-ended,' I mean that it was amenable to various components of praxeologies that make up KtbL in LA1. This combination forced students to be in charge of two aspects in producing a technique: as there was no implicit nor explicit assignment of technique (or technology), it was up to students to identify an appropriate one (among the varied technique or technologies that are KtbL); and, whichever technique or technology students opted for, it was up to them to mobilize more knowledge to build a complete technique (e.g., if a student opted to find the length of $\vec{v}$ by first solving some system to find its initial point, then the student needed to wield more knowledge to produce an appropriate system). The features of Problem 8 that made it amenable to triggering Learner positioning also highlight the prevalence of the Student position in LA1: most students struggled to mobilize sufficient knowledge to produce a complete technique (e.g., 7 of 8 students failed to use KtbL in ways different from its normative use in KtbL).

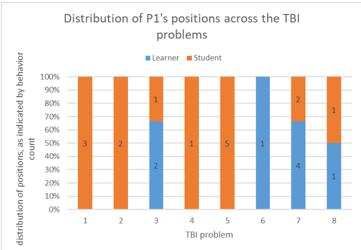
The patterns of positions, position properties, and behaviors triggered in participants by the TBI problems can be a starting point for the design of tasks that, so long as they are not institutionally routinized, could prompt students to engage in activity that is more driven by mathematics than other considerations. Based on the few instances of Learner position triggered by Problems 1, 2, and 4, and given the features that mark these problems apart from the rest, I make this first conjecture: if a task resembles nearly identically to a routine task from KtbL, students almost exclusively mobilize what they had learned for that routine task. Tasks that elicited cases of Learner positioning did so by prompting students either to mobilize knowledge they had gained as Learners, or to try to mobilize knowledge other than routines from KtbL. Comparing tasks that elicited significantly more cases of Learner positioning than others with those that elicited fewer of these (e.g., Problems 5, 7, and 8 vs Problems 3 and 6), I note the former included more non-routine components, were less amenable to routine techniques, or did not correspond to any routine at all. For students who responded as Learners (even if not exclusively), these features were involved (even if not always sufficient[39]) when this positioning was triggered.

5.9.2 Observations about Students who were (not) resistant to exhibiting Learner behavior in response to non-routine tasks

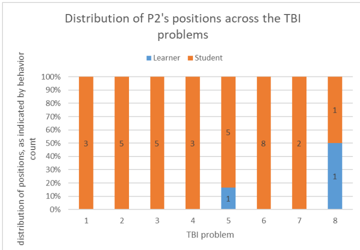
All participants[40] presented as Students, both in terms of their activity and comments they made about their experience in LA1. In that sense, I can refer to all my TBI students

[39]This is in reference to the fact that one student, P4, only mobilized KtbT *after* I, as the interviewer, gave him 'permission' to do so when I asked what he had meant, at some point during his interview, when he said he always finds "simpler" solutions in the end.

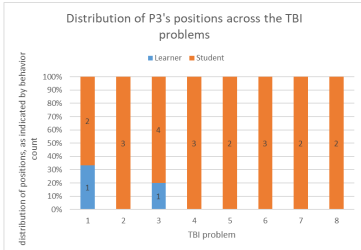
[40]With the exception of P7*, whose positioning, as I previously explained, is omitted from the current analysis.



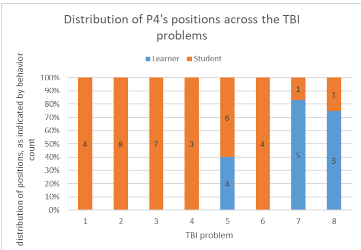
(a) P1



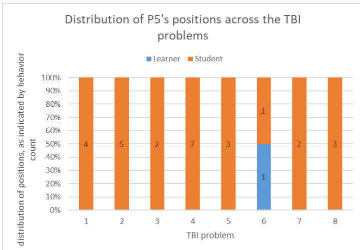
(b) P2



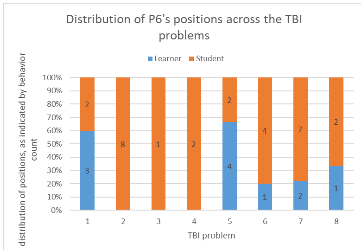
(c) P3



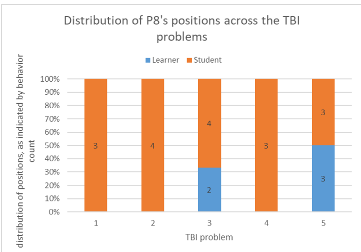
(d) P4



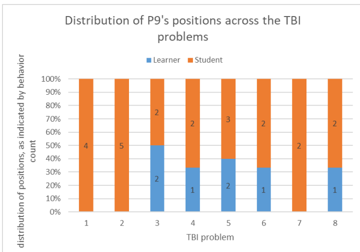
(e) P5



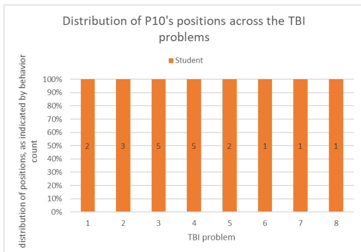
(f) P6



(g) P8



(h) P9



(i) P10

Figure 5.13: Each participant's distribution of Learner and Student positions across the TBI problems, as indicated by the count of behaviors indicating each position

as Students. That said, some (P1, P4, P6, P8, and P9[41]) exhibited potential to behave as Learners and in some cases evidence that they *had* behaved as Learners in LA1. In contrast, P2, P3, P5, and P10 seemed staunchly committed to the Student position. The position properties that qualify their commitment show a high attachment to KtbL norms in LA1. To discuss Students' varying levels of resistance (or amenability) to the Learner position, I start (in Section 5.9.2.1) from the one in whom I detected no Learner behavior and end with those in whom I detected more (even if limited) mobility toward the Learner position. I follow (in Section 5.9.2.2) with a discussion of the implications of KtbL norms in LA1 for the positions its students are encouraged to occupy.

5.9.2.1 Case-by-case discussion of students' Student-to-Learner mobility

One student's comments and activity (those of P10) exclusively related to the Student position. P10's activity and comments reflected several properties of the Student position: acquisition of KtbL, compartmentalization of knowledge by KtbL tasks (i.e., in that she associated certain knowledge with certain KtbL tasks, rather than any broader mathematical knowledge), a dependence on row-reduction as a 'catch-all' technique (in that this was the only suggestion she had for certain tasks, even though she was unable to mobilize it to complete them and despite prompts in which I asked if she could think of any other approaches), a failure to use KtbL in ways different from its normative use in KtbL, a failure to use KtbT that is not KtbL, a surface-level grasp of KtbL, a theoretical block consisting of KtbL (in that her justification for the validity of KtbL was that it was KtbL—that is, knowledge that is usual for LA1 Students), and a use of authority to validate knowledge (this is related to the last item—when asked how she normally checks if something is a solution of a system, P10 responded that she was "not sure," that maybe "grades do it").

I identified 1-2 Learner behaviors in the activity of P2, P3, and P5 (each). Their comments and activity otherwise pointed to their having occupied only a Student position in LA1. Their behaviors all shared in Student properties that characterized P10's behavior: acquisition of KtbL, compartmentalization of knowledge by KtbL tasks, a failure to use KtbL in ways different from its normative use in KtbL, a failure to use KtbT that is not KtbL, a surface-level grasp of KtbL, and a theoretical block consisting of KtbL. The three Students' behavior also shared in the property 'result sanctification (KtbT substitute)': they relied on the results they reached to determine whether their technique was valid. Results were a substitute for KtbT in the sense that KtbT, in LA1, includes the knowledge that validates the techniques Students used. P2's behavior indicated a few other properties of the Student position as well (e.g., his use of normative knowledge was inconsistent, on one occasion, with the mathematics at stake in a task).

I detected two Learner behaviors in P2's activity, both cases of a non-normative use of KtbL. In Problem 5, he mobilized a mathematical property of a component that is not needed in LA1 tasks involving that component (he knew that if three vectors form a parallelepiped of volume 0, they are coplanar. In Problem 8, he identified and mobilized relevant technologies from LA1 and prerequisite math courses through which to complete

[41]Though I keep in mind P9's comment about having learned some LA1-type linear algebra in high school, which implies he had a trajectory at least slightly different from that of the rest of the participants, when it comes to acquisition of LA1 knowledge.

the open-ended task (he produced an appropriate system of equations whose solution was the initial point of $\vec{v}$). I detected one case of Learner behavior in P5's activity: in Problem 6, after a struggle with normative LA1 knowledge, she mobilized LA1 KtbT that is not LA1 KtbL—the knowledge that a solution of a system is a value satisfying all equations in the system. In these cases of Learner behavior, what P2 and P5 mobilized may seem like just a scrape beyond what is usually needed to do final exam tasks, but other participants failed to mobilize what they had in these cases. This suggests that even if P2 and P5's position in LA1 was mainly that of a Student, they did occasionally step into the Learner position, and some of the TBI tasks had features that prompted them to mobilize knowledge they had gained from that position.

P3's instances of Learner behavior did not attest to her having occupied a Learner position during LA1; they were rather instances of her potential to act as a Learner under certain circumstances. P3 tried to interpret a task based on the mathematics at stake on two occasions. The first was in response to Problem 1; upon finding that one of the matrices on the left-hand side of the equation had no inverse, P3 reevaluated the objective of the task. She realized that C may or may not be invertible (but did not know how to determine this, nor what to conclude in the case that it is not invertible). The second occasion P3 tried to interpret the task based on the mathematics at stake (and not only on norms from LA1) was in Problem 3, where P3 took the cross product symbol to mean that she should be using some theoretical knowledge from LA1 instead of computations. P3 did not have non-KtbL LA1 knowledge from which to draw what she seemed to be looking for. That said, P3's activity showed the tasks' potential to drive Students to seek non-normative knowledge.

The Students who displayed a greater number of Learner behaviors (7-12) were P1, P4, P6, and P9; I predict, based on P8's activity in Problems 1 - 5 (where I counted 5 instances of Learner behavior), that she would have as well, given the time to attempt Problems 6 - 8. Altogether, their behaviors corresponded to these properties of the Learner position:

- concept of mathematical aesthetics (P4)
- enthusiastic response to non-routine task (P6)
- interpretation of task based on mathematics at stake (P1, P4, P8, P9)
- non-normative use of KtbL (P1, P4, P6, P8, P9)
- problem-solving behavior (P1, P4, P6, P8, P9)
- seeking and reflecting on guidance relative to the mathematics at stake (P6)
- use of KtbT that is not KtbL (P1, P4, P6, P9)
- use of KtbT/KT/KtbL combination (P1, P4, P8, P9)
- use of mathematics at stake to validate knowledge (P6)

Among these, three reveal the students had also occupied a Learner position during LA1 (and/or previous mathematics courses):

- a non-normative use of KtbL:
  - P9 in response to Problem 4
  - P4, P6, P8, and P9 in response to Problem 5
  - P1, P4, and P9 in response to Problem 8
- use of KtbT that is not KtbL:
  - P6 in response to Problem 5
  - P2 and P9 in response to Problem 6
  - P1 and P4 in response to Problem 7
- use of KtbT/KT/KtbL combination:
  - P1 and P9 in response to Problem 3
  - P8 in response to Problem 5
  - P4 in response to Problems 7 and 8

P9 had mentioned having learned some linear algebra in high school. There was no chance to ask follow-up questions about this during the interview. It's possible that his greater mobilization of LA1 knowledge is due to this additional experience as a linear algebra student.

In contrast, consider the Student position properties exhibited by these students' behaviors:

- acquisition of KtbL (P1, P4, P6, P8, P9)
- attempt to prioritize time-efficient technique (P1, P4)
- attention paid to an authority's motivation in the design of at task (P4, P8)
- belief about expectations of students produced by normative LA1 KtbL (P9)
- compartmentalization of knowledge by KtbL tasks (P1, P4, P6, P8, P9)
- compartmentalizing knowledge as (not) belonging to LA1 (P6)
- dependence on row-reduction catch-all (P4, P8, P9)
- expresses an expectation that reflects LA1 norms but not the mathematics at stake (P4)
- failure to use KtbL in ways different from its normative use in KtbL (P1, P4, P6, P9)
- failure to use KtbT that is not KtbL (P1, P4, P6, P8, P9)
- lacking sense of agency over mathematics at stake (P4, P6, P8, P9)
- result sanctification (KtbT substitute) (P4, P6)

- surface-level grasp of KtbL (P1, P4, P6, P9)
- theoretical block consists of LA1 KtbL (P4, P8)
- use of authority to validate knowledge (P6, P8, P9)
- use of normative knowledge inconsistent with mathematics at stake (P1, P6)

Even if the cases of Learner behavior were sparse, especially in comparison to evidence attesting to students having Studented in LA1, these cases show some students may still Learn even as their primary tactic is to Student. That said, Learner behaviors were concentrated in the less LA1-routine TBI problems and some occurred in response to prompts they received from myself but would not normally receive in the context of LA1. This, together with the near-to-total absence of Learner instances from TBI problems that matched more routinized tasks in LA1, suggests students are not given institutional opportunities or permission to behave as Learners in LA1 and mobilize knowledge they may acquire through this position: the norm is that KtbL be routinized knowledge.

5.9.2.2 Implications of KtbL norms in LA1 for the positions its students are encouraged to occupy

It is not surprising, given the institutional rules for LA1, that all students' activity and comments attested to their having occupied the Student position in LA1. Given the requirement to pass LA1 to gain entry into many university programs, the stakes are high—higher even than for other university courses, where failure to pass has its repercussions but does not block a student from starting their chosen degree. From an institutional perspective, every student has motivation to Student in any course that rates a student's performance in some way or another—even if only in the bare-bones distinction between a "pass" and a "fail." In contrast, by the definition of the Learner position as a cognitively-oriented one, a motivation to Learn is not institutionally-guaranteed.

TBI students' restricted mobility between the Student and Learner positions show that, in LA1, there is a wide gap between the behaviors that support these positions. The Student and Learner positions have distinct objectives; that said, in operationalizing the positioning framework, I came to distinguish between these objectives and the behaviors they engender (in a sense similar to the ATD premise that activity consists of a practical block driven by some theoretical block). Through this distinction, I realized that the Student and Learner positions needn't automatically engender contrasting behaviors. Since the Student position is defined by the objective to obtain a certain grade in a course, it is course norms that determine what a Student must achieve to obtain this grade. The course norms, therefore, determine the behavior engendered by a Student position. In the case of LA1, I found that course norms (namely, routinization of knowledge) engender behaviors that block students from Learner behavior in two ways: one, students might fail to acquire mathematical praxeologies altogether, and, two, students might just opt not to mobilize mathematical praxeologies when they do have them. In theory (as this has yet to be investigated), there may be a mathematics course—where the Student position exists due to some institutional rule—with norms such that the behavior needed to Student is identical to the behavior that allows a student to Learn.

Chapter 6

Discussion and Conclusions

This final chapter is organized into three parts. I first discuss the results from Chapter 5 in relation to each of the three research questions. Second, I discuss how this research contributes to the body of research on the teaching and learning of linear algebra: its originality, how it sheds light on issues from a different perspective of those used in the literature (it not only contributes to the understanding of issues previously identified, it brings to light issues—or points to sources of issues—that had not been previously identified). Third, I give final remarks in the form of main conclusions, limitations of this work, and potential avenues for future work.

6.1 Results in relation to the research questions

This section is split into three parts. In each of these parts, I discuss my results and analyses (presented in Chapter 5) as I consider them in relation to each of the research questions:

> What are the praxeologies students are expected to form when considering problems posed in LA1 final exams?
>
> What is the nature of the knowledge LA1 students mobilize when they solve linear algebra tasks? What kind of (mathematical or non-mathematical) praxeologies do they activate?
>
> Research on the learning of calculus has modeled students' practices and found them to consist of routinizing techniques and building non-mathematical praxeologies; are these practices replicated in linear algebra?

6.1.1 Discussion of results and analysis in relation to the first research question: what are the praxeologies expected when considering problems posed in LA1 final exams?

The praxeologies expected when considering problems posed in LA1 final exams are marked by a routinized or normative quality. I use "routinized" and "normative" and variants on their root words interchangeably. Tasks are routine (normal, normative, etc.) in that there is a limited set of task types that occur in LA1 final exams and the variety of tasks that occur in a final exam is stable from one semester to the next. Another praxeological element that is routinized is the way in which technologies are to be used: the types of tasks that are routinized limit what students need to know about each technology.

A quick reminder on how I define expressions such as "what students need to know," "knowledge to be learned," and "what's expected of students." The definition stems from this study's theoretical framing on mathematics learning institutions. The LA1 institution belongs to a system of several institutions. Overarching it: stakeholders at broader societal levels (mathematicians involved in the design of courses both historically and more recently, education ministries, various industries, etc.); the University, where LA1 plays the role of a mathematics course prerequisite to many programs in various departments, and which coordinates administrative and academic rules; the Mathematics and Statistics Department, which, at this University, is charged with coordinating the LA1 institution. Students, as members of the LA1 institution, are also members of the University institution; their membership is defined by the objectives to get a certain grade in their courses (including LA1) and to obtain a type of degree. Other objectives students may have are to get into a certain career path or to develop understanding of a certain domain of human activity. But the objectives the LA1 institution and related University institutions regulate - that is, through rules and strategies for upholding rules - are the grades and degrees students can obtain. Strategies include, for example, a course examiner in charge of creating uniform course assessments for all students registered in LA1 (regardless of the instructor they have), a grading scheme placing most of the weight of a student's final course grade on the final exam, and common grading of midterm and final exams such that exam problems are partitioned among instructors, who each grade their assigned problems for all (e.g. 400) students registered in the course. Given these strategies, and the norms developed relative to these strategies, what students "need to know" to pass their course is determined by the problems on the exams and the minimum of knowledge needed to produce solutions to these problems.

The norm in the LA1 institution is that midterm and final exams are stable throughout the years. There is a limited set of task types in each exam and students know what knowledge they can expect to have to demonstrate. Even if an exam might occasionally include one or two tasks that do not belong to the regular types, the bulk of an exam grade is built from the normal task types[1].

One routinized element of the praxeologies that make up KtbL is their tasks. This was suggested by my preliminary analysis of 4 midterm and 6 final exams given between 2014 and 2019 and confirmed by the analysis of KtbL that relates to my TBI tasks (the latter corresponds to 75% of the tasks in the exams to which I had access). The nature of these tasks is such that they can be completed using the same technique, so the knowledge students need to acquire is restricted to that sufficient to recognize a task type and then administer its normative LA1 technique. For the sake of this discussion, I categorize TBI tasks into two types: those that can be configured as a linear-system-solving task,

[1]Over the last decade, and perhaps before, midterm and final exams from previous terms are made available to students for studying purposes. This has contributed to the normalization of tasks: students are given insight into what to expect, one more explicit than that they may infer from what they shown or told in class. This puts pressure on the instructor in charge of producing exams to remain consistent—to remain routine. Today, this normalization is strengthened by the existence of sites that have inventories of exams, organized by subject and by institution, with solutions geared not necessarily to the construction of mathematical praxeologies, but rather to what needs to be written to pass the exam.

and those that cannot.

Tasks specific to the resolution of a linear system implicitly or explicitly instruct which technique to use, always to find all solutions of a linear system: Gauss-Jordan elimination (this is implicit when other techniques are not applicable, say, if a coefficient matrix is not square), Cramer's rule, or multiplication by the inverse of the coefficient matrix. A significant portion of routine tasks targeting other LA1 constructs (e.g., bases, linear independence, intersection of loci) are normatively associated with row-reduction technique for solving linear system or using determinants to determine whether a (homogeneous) system has one or infinitely-many solutions. These tasks are designed in such a way that they are always amenable to being completed by reducing an appropriate augmented matrix or finding an appropriate determinant. The matrices at stake can be produced by copy-pasting sets of scalars in the task. Some tasks (e.g., about finding the intersection of loci) already involve a linear system, in which case the augmented matrix can be lifted straight from there; others (e.g., to determine the linear independence of vectors) involve vectors in $\mathbb{R}^n$ which can then be used as either rows or columns in a determinant or as columns in an augmented matrix. Either way, it is not necessary to have the mathematical theoretical discourse that connects the original task with the choice to reduce a matrix or calculate a determinant, nor the discourse connecting the original task with the result obtained by augmented-matrix reduction or determinant-calculation.

Tasks that can be configured as linear-system-solving have a normative technique, then: copy-paste scalars from the task in the order in which they appear into a matrix or determinant, and reduce said matrix or calculate said determinant to determine if it's (non-)zero. Scalars in these tasks are always single-digit integers and matrices usually involve 3-4 rows and columns. Scalars in determinant tasks often involve many 0's and are therefore amenable to using a single cofactor expansion and then using mnemonic devices for determinant calculations. Given the (number and type) of other final exam tasks (and time available to complete the exams), there is no need to develop optimal efficiency (in terms of number of steps or amount of time needed) when using these techniques.

The tail-end of these techniques is similarly routinized: the objectives of tasks are amenable to associating a fixed set of rules with the results of matrix reduction and determinant calculation. An augmented matrix reduction that ends with a reduced row echelon form involving an identity matrix: there is a single solution to the task. A reduction that leads the left-hand side of the augmented matrix to having a row where all entries are 0: there are either infinitely many solutions (if the right-hand side of the matrix also has a 0) to the task or none at all (if the right-hand side of the matrix has a non-zero entry in that row). The end-tail of the determinant-wielding technique is the same: if the determinant is 0, there is a limited number of conclusions students need to know (e.g., if the task is to find the number of solutions of a system, a zero determinant implies infinitely many solutions; if the task is to determine if vectors are linearly independent, a zero determinant implies linear dependence and a non-zero determinant independence). These fixed rules are amenable to making accurate conclusions for any LA1 task that can be configured as a linear-system-solving task. I had not caught onto this in my analysis of KtbL, but students' activity and comments in response to the TBI tasks revealed this was a possibility afforded by the tasks normative in KtbL.

In short, tasks that can be configured as linear-system-solving tasks are amenable to techniques that are routinized from the start to the end of their implementation. Tasks that cannot be configured as linear-system-solving tasks are amenable to routinization of a different part of their praxeologies—the technologies they involve: these are matrix equation tasks, tasks about the area of a triangle or parallelogram or volume of a parallelepiped, distance and orthogonal decomposition tasks, and point-normal equation tasks.

Each of these tasks requires students to invoke a limited number of properties (usually of arithmetic or algebraic nature) of one technology in particular. To complete matrix equation tasks, it suffices to know how to find the inverse of a 2×2 or 3×3 matrix and how to multiply matrices. To complete area and volume tasks, it suffices to know formulas for the area of a parallelepiped in 2 or 3-space (which, in turn, require students to know formulas for cross products, determinants, and/or norms of vectors); distance and orthogonal decomposition tasks usually instruct students to find a distance or orthogonal decomposition, at which point it suffices to know the formulas for each and how to calculate the constructs within (dot products, vector norms, etc.), and point-normal equation tasks at most require students to recognize the role of the coefficients in a point-normal equation (so as to know where to place the entries of (a, b, c), a vector declared in the task statement to be a "normal" of some plane) and that plugging coordinates of a point into an expression of the form $ax + by + cz$ would yield the value of d in the point-normal equation $ax + by + cz = d$. Knowledge about dot and cross products does not require more than their formulaic definitions (i.e., how to calculate them); among the 116 exam tasks I studied, only 2 required the orthogonality property of cross products and only 1 required the knowledge that two vectors are orthogonal.

Additionally, KtbL delimits the use of technologies to routine situations. A task either explicitly instructs which technology to use or this is implicitly communicated as the task is of a type that appears (as an example or recommended practice problem) in a textbook section about a given technology. This is similarly the case for technologies pertinent for tasks that can be configured as a linear system task. In solve-a-linear-system tasks, these technologies include augmented matrices, row operations, (reduced) row echelon forms, parametric equations, Cramer's rule, and formulas for calculating determinants. Technologies are similarly routinized in tasks that can be configured as a linear system task. For example, the routine task to determine the linear independence of a given set of vectors requires students to recognize the term "linear (in)dependence" and associate it with the usual row-reduction or determinant techniques and their possible results. KtbL about linear (in)dependence delimits the use of the concept of linear (in)dependence to routine situations.

Students are instructed explicitly (by task instructions) or implicitly (by course norms) as to when and how to use certain LA1 technologies. Other LA1 technologies don't necessarily need to be *used* and rather need to be known nominally, for example because they are the end-goal of a task. My analysis of TBI students' activity and comments revealed how these technologies are routinized. One example illustrates this well: in KtbL tasks, solutions of linear systems appear mainly through the form they take in a reduced row echelon form of an augmented matrix. They do not usually appear or need to be interpreted in other ways—solutions of linear systems are always the end-goal of a task, and never a starting point from which to complete some other task. It suffices to know “solutions of linear systems” as the end-result of a row reduction (or Cramer’s rule calculation). What this routinization means, for example, is that it is not *necessary* (in the grade-getting sense), in LA1, to know the definition of a solution of a system of equations: something that satisfies each equation in a system.

Considered in toto, KtbL praxeologies in LA1 are routinized. Tasks are routine in that it is always the same limited number of task types that recur and the associated techniques (from start to end of their implementation) work no matter what. Technologies too are routinized: it suffices to know a limited number of properties of any KtbL technology, and the properties to know are usually in formulaic or algebraic form.

A last comment about the answer to my first research question—what are the praxeologies expected when considering problems posed in LA1 final exams?—is about how I came to this answer. To determine the praxeologies that characterize KtbL, in line with the methodologies used by Hardy (2009a) and Broley (2020), I attended to tasks in past midterm and final exams and the course textbook to determine the knowledge expected of students for completing these tasks. Placing exam tasks in relation to one another from one semester to the next, as well as in relation to tasks that appear as solved examples and practice problems recommended on the course outline, highlighted the routinization of KtbL praxeologies.

In addition to this approach, I found the characterization of how KtbL can be routinized to benefit from the analysis of TBI students’ activity and comments. While the purpose of that analysis was to address my second research objective (to determine the nature of the knowledge LA1 students mobilize when they solve linear algebra tasks), it helped to reveal other ways in which the norms of KtbL tasks enable routinization of knowledge. For example, in analyzing knowledge needed to complete the routine task to solve a linear system, my focus was in the technique expected to be activated in LA1 to solve a system. I did not pay attention to the knowledge students are expected to have about *solutions* of linear systems. One reason for which I did not pay attention to this is a methodological limitation: I did not have access to graded exams, which could indicate what instructors expect from their students to know about the solutions of the linear systems they solve. This limitation, however, has an alternative solution to analysing graded exams—one which may actually reveal more than instructors’ marking norms, when it comes to the knowledge students are expected to have about solutions of linear systems. Indeed, students’ comments and activity in response to several problems in the TBI indicated that several students struggled to mobilize the definition of *solution of a system*. At the same time, students’ comments showed they associated solutions of a system with entries appearing on the right-hand side of reduced row-echelon forms of matrices. Considering this is the sort of “solutions” in routine LA1 tasks about linear system solutions, it’s not surprising that students don’t actually need to know what a “solution of a system” is. But my focus, in modeling the knowledge to be learned about linear systems, was on the techniques used to *get to* solutions of system. This did not suffice to bring about an observation about what students (do not need to) know about solutions of the systems they solve. It was the analysis of students’ mobilized praxeologies that brought about this inference about the knowledge students are expected to learn.

6.1.2 Discussion of results and analysis in relation to the second research question: what is the nature of the knowledge LA1 students mobilize when they solve linear algebra tasks? What kind of praxeologies (mathematical or non-mathematical) do they activate?

I call a praxeology *mathematical* if its practical and theoretical blocks are rooted in the mathematics at stake in a sense that can be generally agreed-upon among members of the scholarly linear algebra institution. I call a praxeology *non-mathematical* if one of its components is not rooted in this mathematics; this does not mean any component of the praxeology is *absent* as, indeed, a praxeology is just a model of a human activity occurring in an institution, and any such activity, per My theoretical framework, consists of practical and theoretical blocks and their respective components (task, technique; technology, theory). A praxeology is therefore non-mathematical if any of its components (task, technique, technology, theory) is rooted in something other than the mathematics targeted by the activity; for instance, Hardy (2009b) reports on how students' theoretical blocks can be a blend of social, cognitive, and didactic norms.

What students mobilized as they engaged with the TBI tasks was largely determined by and delimited to what normally suffices to complete KtbL tasks. This applies to both their practical and theoretical blocks: their know-how corresponded to the know-how necessary and sufficient to complete KtbL tasks, and comments students made showed that KtbL norms were what produced their mobilized activity. Both practical and theoretical blocks came at the expense of the mathematics at stake in the TBI tasks and students struggled to complete most of them. The praxeologies students activated were largely non-mathematical: upon recognizing the technologies explicit in a task, students retrieved the routinized aspects of praxeologies normally associated with these technologies and then mobilized their LA1 routine. It's not that the routines had no footing in (linear algebra) mathematics whatsoever; but there were inconsistencies between these routines and the mathematical components or objectives of the TBI tasks, and students were mostly unable to breach these inconsistencies.

Throughout this section, unless otherwise indicated, when I refer to "students" I mean all participants except for P7*. P7* had taken and started to take various higher-level university math courses (which he called analysis, ordinary and partial differential equations, advanced algebra, abstract algebra, and advanced geometry) as part of an applied mathematics degree at a different institution and was also a tutor for adult students in prerequisite mathematics courses. P7*'s comments and activity throughout the interview did indicate his knowledge mainly reflected that acquired through LA1, but his behavior set him sufficiently apart from other participants that, in light of his experience, warrants addressing P7*'s participation separately from a discussion of findings I made from the other participants. Given P7* had nevertheless taken LA1 in the previous semester like all other participants had, and given that the knowledge he mobilized did reflect and was constricted by KtbL norms, his comments and activity help to answer my broader questions about the impact of these norms on students' praxeologies.

To characterize students' praxeologies as (non-)mathematical, I organize the discussion by their practical and theoretical blocks and then reflect on the praxeologies as a whole in a summary at the end of this section.

6.1.2.1 Students' practical blocks

The techniques students mobilized reflected what they would normally do in LA1 given a task that involves a certain technology; students' activity showed how these norms can also restrict what students *can* or *do* mobilize to complete a task. Depending on the TBI task, and given the nature of the mathematics involved, the norms students mobilized were not the most suitable for the given task (usually involving an unnecessarily computational load) but did suffice for some students to complete tasks (Problems 3, 4, 7); in other cases, the norms failed to suffice as students' mobilization of these norms were limited to surface-level features of techniques and technologies (Problem 2, 5, 6, 8); and in one case, a normative praxeology from KtbL disabled students from completing a certain task or even identifying its objective according to the mathematics at stake (Problem 1). The techniques students had routinized in LA1 are mathematical, even if routine and even if mobilized along surface-level features, when they are mobilized toward a routine LA1 situation. But the TBI students' struggles illustrated how the operationalization of techniques along exclusively routine and surface-level features can result in students mobilizing non-mathematical practical blocks.

I split this discussion into two parts: first, I discuss students' approaches to the tasks that could be configured as a linear-system-solving task, and second, tasks that could not.

6.1.2.1.1 Whenever possible, students activated the LA1 system-solving task and its routinized techniques of reducing an augmented matrices or calculating a determinant to check if it is 0 Students found a way to mobilize the normative LA1 system-solving task and techniques toward every TBI task. Some students attempted to substitute Problem 1 by such a task and, while they abandoned the attempt, did not note that the system produced would not have been linear. What students mobilized toward Problems 3 and 4 was mostly row reduction. Students attempted to substitute Problem 5 by the normative "determine if these vectors are linearly (in)dependent" LA1 task and its associated row-reduction or determinant-calculation techniques. Students attempted to complete Problems 6 and 7 using the row-reduction technique. Most students' approaches toward Problem 8 swiveled around producing a system of equations whose solution would be the desired objective of the task.

Students' choice to mobilize row reduction or determinant-calculation toward this variety of tasks reflects the LA1 norm to activate these techniques in various situations (as discussed in Section 5.2.2). At face-value, it would seem that students learn from this LA1 norm that various linear algebra problems can be resolved via system-solving techniques. Students' activity in response to Problem 8, for example, shows this is indeed the case for some students (as they produced equations reflective of the given task); but students' activity and comments in response to other problems show students may not even perceive these techniques in relation to a specific linear system (e.g., as in Problem 5, where students' comments and activity betray their ignorance of the homogeneous linear system that connects the normative LA1 techniques for determining linear independence with the definition of independence). The irrelevant equations and systems produced by others (e.g., P3 and P5) in response to Problems 5 and 8 also show that substituting by linear-system-solving tasks can also be a knee-jerk reaction or a way to side-step insufficient grasp of other knowledge pertinent to a task. In mimicking a heavily routinized mathematical technique, students can develop a technique that is non-mathematical: they produce equations—any equations, regardless of their footing the task at stake—so as to complete a task by solving equations—or they bypass these equations altogether to manipulate some box of numbers.

It's possible students produce equations without realizing their irrelevance; but this, contrasted with their knowledge that various problems *can be represented by systems of equations*, highlights the way in which the routinization of linear-system-solving tasks and techniques in KtbL, along with their heavy weight assigned to this routinization on final exams, can work against students' acquisition of other KtbL. In turn, this means that while LA1 students may develop the knowledge that various linear algebra problems can be resolved by linear-system-solving techniques, they fail to acquire the knowledge needed to mobilize these tasks and techniques when an explicit linear system is not provided.

Students' mobilization of row reduction techniques in several of the TBI tasks shows varied ways in which they might use these techniques in ways that split them from the mathematics at stake. One such way has to do with the efficiency that row reduction is intended to bring toward solving systems in linear algebra. In Problem 2, students' mobilization of row-reduction (across the board) went against the purpose of this technique as one for optimizing efficiency, especially systems that cannot be solved by observation. Students did not consider the specifics of *this* linear system before mobilizing row reduction. Additionally, students were unable to make efficient choices of row operations, favoring instead routinized aspects of Gaussian elimination (get a leading 1 in the top row first; a leading 1 in the next row second; or get 0's in as many entries as possible, without paying attention to the effect of one operation on previously-achieved 0's or leading 1's); they were nowhere near a row echelon form of the augmented matrix even after 10-15 minutes of this activity. Throughout the TBI, students converted their task to a decisively inefficient linear-system-solving task when other LA1 technologies would have afforded a decisively more efficient approach; this was not always as a result of lacking other knowledge, as P4's activity in response to Problem 7 attests. His initial approach to the task was to convert it to one that could be solved via row reduction; he only retroactively solved it using other knowledge at the end of his interview when I had given him permission to do so by asking him to clarify what he meant, earlier in the interview, when he said he always finds simpler solutions in the end. Per P4, he had mobilized the row-reduction approach because he believed that's what was expected of him in a LA1 context—even as he knew of another approach more efficient for the task. Students' row-reduction technique is mathematical(ly efficient) in only one sense: as long as they execute their row operations accurately, they likely do eventually solve the linear system they are tasked to solve. Students do not mobilize the technique efficiently nor with the aim to achieve efficiency—on the contrary, their comments show their concern with the burden of time and risk of calculation errors it brings.

Students' concern with efficiency when it comes to row-reductions and determinant calculations, as well as their inability to mobilize these efficiently, are important. To my knowledge, studies of students' practices in Calculus, for example, do not document a concern with efficiency. I attribute this concern to the nature of linear algebra tasks but also the heavy weight ascribed in final exams to row-reduction and determinant calculation techniques; I therefore view this as a missed opportunity to use KtbL to encourage in students, at minimum, an appreciation for the objectives behind the mathematical techniques they are expected to mobilize. But it's also a missed opportunity to develop in students knowledge for how to use row reduction efficiently: the choices students made in my TBI as they row-reduced reflects Maciejewski & Star's (2019) finding that even within the routinized activity of row reduction, there is sufficient freedom for students to mobilize varied criteria for what they perceive to be appropriate to reach a solution. My participants used superficial criteria (e.g., get 0's and 1's) in rigid ways that led them to undo their progress toward a solution. Based on findings from an intervention designed to help undergraduate Calculus students use procedures flexibly, Maciejewski & Star (2016) posit that flexible procedural knowledge can be taught.

Students' struggle to mobilize row-reduction and determinant-calculation techniques efficiently may relate with numeracy difficulties, but also to their surface-level grasp of the techniques and related technology. This brings us to discuss further ways in which students' mobilization of mathematical techniques can be non-mathematical. I have found that students can struggle to remember how to interpret results of routinized techniques—recall students' comments about what reduced row echelon forms (RREF) and (non-)zero determinants can say about the number of solutions of a linear system; rather than interpreting RREF in terms of the system of equations they represent, students have rules for interpreting 3 non-mutually exclusive types of RREF as conclusions of "one solution," "no solution," or "infinitely-many solutions."

Rules about surface-level features, while accurate for normative LA1 tasks, can lead to further non-mathematical mobilization of mathematical technique. Students struggled to interpret results of routinized techniques when applied to non-routine tasks. For instance, students struggled to conclude what the RREF of the augmented matrix in Problem 5 meant about linear independence, as the non-routine inclusion of the unknown k in the vector entries required knowledge that is not needed when students solve the LA1 task for determining linear independence (this task always ascribes known values to all vector entries). This came up again in Problem 6 as students struggled to find the system of equations corresponding to the RREF of the augmented matrix of a system of *quadratic* equations. Some students, after finding in Problem 6 either that one equation had no solution or that the solutions of one equation did not satisfy the other, even failed to mobilize the knowledge that a solution of a system is a value satisfying each equation—they kept looking for an alternative technique; one student acknowledged having forgotten this and another had expected the "no solution" conclusion to correspond to a certain type of RREF. While students' row-reduction technique is mathematical in that they generally know which row operations to do and how to do them, the technique can be non-mathematical when they are unaware of what the boxes of numbers actually represent.

When it comes to techniques for tasks that can be configured as linear-systems-solving tasks, while the techniques students mobilize have mathematical components (e.g., row reduction of an augmented martrix), they include components that rather reflect norms of LA1 that, due to the routinization of tasks and techniques in LA1, are accurate in the routinized LA1 context but not necessarily so in linear algebra in general. Students' conception of equation solving tasks can also be normative and non-mathematical. For example, students' notion of what it means to solve an equation or a system is defined by course norms rather than the task's broader meaning in algebra as a domain of mathematics. Indeed, recall students' expectation in Problem 1 that to solve the matrix equation for the matrix C is to isolate C. In Problem 5, students' expectation was that there *would* be values of k for which the vectors are linearly independent; this seems to reflect the norm in LA1 system-solving tasks that when a system involves unknowns, there are values of this unknown for which the system has solutions. In Problem 6, students readily accepted that one of the quadratic equations had no solution—this was on the basis of its discriminant being negative; but students did not expect to come to a "no solution" conclusion for linear system tasks, short of evidence to this effect in the form of a certain type of row in a RREF. Students' perception of the task "to solve an equation" depended on norms from courses that target the given type of equation.

Whenever possible, students activated the LA1 system-solving task and its routinized techniques of reducing an augmented matrices or calculating a determinant to check if it is 0. While students' mobilized techniques did include mathematical components, these were restricted to the few routines needed for normal LA1 tasks and this limitation seems to be the source for the otherwise non-mathematical components of students' practical blocks.

6.1.2.1.2 Students struggled to complete tasks using LA1 technologies other than those normally involved in system-solving tasks In my discussion of the praxeologies expected of students when considering final exam tasks in LA1, I noted that KtbL delimits the use of technologies to routine situations: a task either explicitly instructs which technology to use or this is implicitly communicated as the task is of a type that appears (as an example or recommended practice problem) in a textbook section about a given technology. My students struggled to mobilize technologies belonging to tasks that in LA1 are not normally to be configured in terms of a linear system.

Students' knowledge about matrix algebra was restricted to the routine needed to solve LA1 matrix equations: find inverses and multiply by inverses. Even though most students did note that the right-hand side of the equation was an identity matrix, and even though most students spoke about C and its inverse, they did not mobilize knowledge from a different KtbL task (to determine if a matrix is invertible)—that a matrix may have no inverse—to reconsider that "to solve a matrix equation" may not be synonymous with "to isolate a matrix," as the KtbL norm suggests for the task to solve a matrix equation. Two students attempted to consider this but only after a prompt I made to this effect, and they were unable to mobilize knowledge through which to investigate this possibility.

Students' knowledge about cross products was conditioned by the norms in LA1 to calculate cross and dot products when they are needed (e.g., in area-finding tasks or orthogonal decomposition formulas) and most students did not mobilize the orthogonality property of cross products and dot-product definition of orthogonality when it would have provided a most suitable technique to complete Problems 3 and 4.

When it came to mathematical components in Problem 5—a parallelepiped formed by 3 vectors in $\mathbb{R}^3$, its volume of 0, and linear independence—most students' activity fixated on the algebraic representations normally associated with linear independence in KtbL (especially those of row reduction or determinant calculation) and a few on algebraic representations normally associated with parallelepiped volume in KtbL. Students struggled to activate a geometric representation of a parallelepiped of volume 0, confirming their knowledge of parallelepipeds was conditioned by the formula volume of the object privileged in LA1 parallelepiped tasks and not by the definition of the object.

Students struggled to conclude in Problem 6, be it due to a lack of agency or a lacking sense of agency, that the system had no solution despite having found either that one equation had no solution or that the solutions of one equation did not satisfy the other. This situation was not typical of LA1 tasks, where linear equations given in tasks always have solutions (equations of the form $a = b$ where $a \neq b$ are never part of an original system), and students seemed to be searching for an algebraic representation similar to that usually associated with the "no solution" conclusion in LA1: an equation of form $0 = a$, where $a \neq 0$.

Most students were able to complete Problem 7, but their techniques were conditioned by the usual LA1 technique for solving linear systems: they used the vector equations to produce equations that could be solved via high-school system solving techniques or row-reduction. Students mostly did not mobilize the geometric reasoning to be taught in LA1 to explain the sense in which vector equations capture lines in $\mathbb{R}^n$ ($n = 2, 3$).

Finally, in response to Problem 8, most students attempted to complete the task by producing (a system of) equations whose solution they expected to be the initial point of the vector whose length they sought to find. This would have allowed students to mobilize the technique they usually use to find lengths of vectors (and this was the stated task in Problem 8: find the length of a vector). Among the nine students who attempted Problem 8, three were unable to produce any relevant equation and four others were able to mobilize 1-2 technologies from LA1 (along with the Pythagorean Theorem) to produce 1-2 equations representative of the given situation. Two students (one of whom is P7*, the student whose activity is mostly omitted from this discussion due to his significantly different experience in mathematics courses) did recognize the relevance of LA1 technologies not centred on linear systems (e.g., distance between two points, orthogonal decomposition) and which LA1 students normally use only when explicitly instructed to do so.

When it came to LA1 technologies other than those normally involved in system-solving tasks, students struggled to use them in the absence of instruction to use them or instruction for how to use them. This suggests that the praxeologies students have in connection to these technologies can be restricted to their surface-level features in routinized LA1 situations. This shows, for example, in how some students used dot and cross product calculations accurately to produce equations irrelevant to the tasks they were given. This is not to say all students are doomed to mobilizing non-mathematical praxeologies (indeed, most students were able to produce algebraic representations relevant to the given geometric situation): but the KtbL in LA1 does allow for students to develop praxeologies limited to formulas denoted by some name ("dot product," "cross product," "volume of parallelepiped").

My students' responses to the TBI tasks give evidence that a few students did mobilize mathematical praxeologies related to technologies that are not usually (in LA1) 'linear system technologies', this happened when non-routine components of the TBI tasks triggered in these students a Learner positioning or in response to a prompt from the interviewer: P1 and P9 in response to the inclusion in Problem 3 of a cross product in the statement of what is otherwise a routine task; P9's mobilization toward Problem 4 of the technique he produced from Problem 3 after I prompted to compare the two tasks; P6 and P8's spontaneous mobilization of a geometric interpretation of a parallelepiped having volume 0—though P6's was inaccurate—and P2 and P4's accurate geometric interpretation when prompted to give one; P1 and P9's knowledge in Problem 6 that if two equations making up a system are not satisfied by the same values, then the system has no solution; P1 and P4's mobilization in Problem 7 of the geometry at stake in vector equations; and P4's mobilization in Problem 8 of various technologies not restricted to the norm of solving tasks by finding an appropriate linear system (conceiving of the length of the vector as the height of a parallelogram whose area could be found with LA1 knowledge, conceiving of the length of this vector as a distance between a point and a line, conceiving of the unknown initial point of the vector as an intersection point of two lines).

While some of my students were triggered by non-routine elements of tasks to mobilize mathematical praxeologies related to technologies that are not usually (in LA1) 'linear system technologies', and while certain students (those having exhibited almost exclusively Student positioning) mobilized, on several occasions, non-mathematical praxeologies related to these technologies, most students simply *didn't* mobilize praxeologies related to these technologies unless they appeared in a routinized form. When a routinized form appeared, some students activated practical blocks that were exclusively reflective of the mathematics at stake in the task, and which I would therefore call mathematical practical blocks (e.g., as when P6 computed the cross product in Problem 3 and plugged its components into both equations to confirm it is a solution, or when P1 and P9 observed it is unnecessary to compute the cross product as the task can be completed using the orthogonality property of the construct). Other students activated practical blocks with mathematical and non-mathematical characteristics (e.g., as when P2 and P4 computed the cross product in Problem 3 *and* found all solutions of the linear system, therefore combining two normative techniques, so as to show the cross product is one of the infinitely-many solutions). Some students activated practical blocks that exclusively reflected what students would normally do in LA1 and not what they assessed to be necessary to complete the task, in which case I call these non-mathematical practical blocks (e.g. as when P3 and P10 were triggered by the cross product in Problem 3 to compute this cross product—which is what is done in KtbL when cross products appear—but made no suggestion as to how to use what they had computed).

Overall, what students mobilized most in response to each TBI problems was in the realm of system-solving tasks and their usual technique. I can't make a single claim about the nature of *all* students' praxeologies in relation to technologies that are not usually (in LA1) related to linear system tasks and techniques, but I can say this: these praxeologies ranged from non-mathematical to mathematical and combinations thereof, and those who did mobilize exclusively non-mathematical praxeologies (for technologies not usually connected to linear system tasks or techniques) were also those who exclusively exhibited Student positioning, while mobilization of mathematical praxeologies (for these technologies) occurred when non-routine aspects of tasks triggered a Learner positioning in a student. This range in non-to-mathematical praxeologies occurred when the nature of the mathematics at stake afforded students the flexibility to use normative knowledge abnormally; but, given a task where the nature of the mathematics at stake did not allow for this (Problem 1), the routine from LA1 built around this mathematics (matrix equations) blocked students altogether from mobilizing a mathematical perception of the task: students' perception remained normative, treating the task as one in which a matrix must be isolated rather than one in which the task is to solve an equation.

6.1.2.2 Students' theoretical blocks

Students' comments and activity showed their theoretical blocks were a blend of didactic and mathematical norms. The technology they used to produce their techniques was the knowledge routinized in KtbL (normative tasks and normal ways of using linear algebra technologies in LA1 KtbL) and the theory giving legitimacy to this technology (that is, the routinized knowledge) was a mixture of surface-level features of the mathematics at stake and of didactic norms.

Students' mobilization of KtbL technologies (row reduction, linear (in)dependence, cross products, matrices and matrix equations, solutions of systems of equations, etc.) was often stripped of knowledge that isn't needed to complete the tasks in which these technologies usually appear. For example, the norm to row-reduce in LA1 (or use system-solving techniques from high-school algebra) showed in students' techniques for solving a system with invertible coefficient matrix (Problem 2), for showing a given object solved a system (Problem 3), for finding one non-trivial solution of a system (Problem 4), for investigating the linear independence of 3 vectors forming a parallelepiped of volume 0 (Problem 5), for solving a system of 2 quadratic equations (Problem 6), for determining the number of solutions of a linear system in $\mathbb{R}^2$ where both equations were expressed in vector form, and for finding the length of a vector with unknown initial point but other known properties (Problem 8). All students showed a tendency to prioritize techniques that are usually used in KtbL for tasks that can be configured as a linear-system-solving task; at the same time, students struggled to produce mathematical techniques. For example, in response to Problem 7, one participant used a high-school linear system solving technique (one corresponding to an elementary row operation students learn in LA1) and produced a system that was implied by the original system but not *equivalent* to the original system. The participant did not know why the solutions he found did not satisfy the original system; all to which he had recourse was to check for calculation errors. The notion of equivalent systems was missing from this participant's praxeology and from that of many other students.

Students' struggle to adapt techniques from KtbL to unusual situations (e.g., as when students struggled to interpret the implication of the RREF for the linear independence of the given vectors, or when students struggled to interpret the results of their system-solving in Problem 6) show that their choice to wield these techniques wasn't led so much by knowledge of their mathematical suitability for any given task (e.g., students' interpretations of the RREF in Problem 5 showed they lacked awareness of the reasoning that produces row-reduction as a technique for determining independence), as much as by the fact that this is what they usually did in LA1 in response to certain triggers.

Comments students made confirm they used routinized KtbL as a guide for what knowledge to mobilize. For example, in response to Problem 3, when I asked P2 if he could think of a way to show the cross product was a solution of the system without first finding all the system's solutions, he said:

> I'm not sure if there is another way. To do it. [...] Usually, when uh. I have to find like an answer. In vector form. I would solve it uh, in this way and I would find the. A vector in a parameter form and then I choose th—choose the parameter based on what the question wants. Yeah. So I'm not sure that there's another way.

The justification for his choice of a technique was in the "usual." Other students also referred to LA1 tasks as a guide for what techniques to use, instead of referring to the underlying mathematics as justification:

- P5, Problem 3: "From like, doing problems in the past that had me, like, jig [sic], which thing would make like the cross product zero, and like, the numbers were like, you just like flip... like, a number. Or, like, put it, like, plug in numbers that are, like, similar. I don't know how to, like, explain it! Like, it would be... Like, I would have... Like, I'm just trying to remember like past homework. Like, I would have like, two equations like this. And it'd be, like, what would... I don't know if it was exactly, like, a similar question."
- P4, Problem 3: "I recognize from memory that when I tried to set up the—a system of equations like this one, I would recognize that I was trying to find the intersection of two planes, which is a line. And this—this is based on exactly just one question that I solved. [inaudible] I remember that question. I don't think that—yeah, other than the lecture information, a lot, a lot of my knowledge was gained from the past exam questions. So like one, one question I tackled would always be in the back of my head when approaching another problem. Especially because it's an intro course and the questions are similar to each other."
- P4, Problem 8: "I could try to find [the initial point of $\vec{v}$]. And then, then I know this vector and [I could] try to find the norm of the vector, which I think [is] what they would expect us to do in this course."

In writing "[c]omments students made confirm they saw the routinized KtbL as a guide for what knowledge to mobilize" (above), I include another implication: the routinized KtbL also acts as a guide, for students, regarding what knowledge *not* to mobilize. Here are some examples:

- P9, Problem 3: referring to the non-normative technique he had produced to complete the task (using the orthogonality property of cross products), P9's comments suggest he viewed mobilizing this knowledge as an extra—not what's expected in LA1, but acceptable as *an embellishment* appended to what's expected on an exam: "the only thing [I would] do is calculate [the cross product] and then replace [as in, plug it in to the equations] here and [I would] get zero eventually. [...] I can also write the analysis that this is because of [the orthogonality property of the cross product]."

- P2, Problem 5: "usually, I do calculations." Referring to different ways (that had come up in his response to Problem 5) through which to determine the linear independence of the three vectors, he said: "I'm [wondering] if the determinant way is the correct way or... if I should try to write [...] one as a combination of the [other] two. [pause] Honestly, I would do it like this [using the routinized "determinant way"]. Yeah. Because I remember that [the vectors being] independent or dependent has to do with the determinant. And this is the easiest way to do it. Because I get zero, so I get a straightforward answer without having to analyse it [using knowledge about the parallelepiped, formed by these vectors, having volume 0]."

- P4, Problem 5: P4 knew linearly dependent vectors are coplanar, knew the vectors forming a parallelepiped of volume 0 must be coplanar, but prioritized two routinized techniques (τ_{41} and τ_{42})—two variations of the same technique type—to determine the linear dependence of the given vectors: "from [τ_{42}], I deduced there would be no values of k for [which the vectors are] linearly independent, but because I know that it could also relate to linear independence, I'm gonna try to [do the task via τ_{41}].")

- P4, Problem 7: his original technique was to express the vector equations in an algebraic representation amenable to LA1/high-school system-solving techniques. At the end of the TBI, when I asked him to clarify a comment he had made at a different point in the interview (about always finding "simpler" solutions in the end), P4 said: "if I were in the linear algebra [LA1]... mindset I would probably try to find that intersection [in Problem 8] using... some, like the point-normal equation or whatever. So maybe I just, today, I kind of went in before thinking—for example, determine the number of solutions. I think I could have hypothesized that 'oh okay, these aren't the same line, so they must meet somewhere, and only 1 - 1, 1 place,' so I could have said that without having to solve everything." P7 had LA1 knowledge through which to solve the problem in a "simpler" way, but only mobilized it after receiving permission to do so; he distinguished this approach from what he would use in a "linear algebra mindset."

- P4's general comment about what he mobilizes: "today, I think the questions are aimed to see what I think about; the first thing, second thing I think about. So *maybe I directly tried to solve [problems] using my linear algebra knowledge, instead of looking at it from a more... logical point of view.* Like, for example, now, I never ever saw a problem like [Problem 8]. But *if I were in the linear algebra [LA1]... mindset*, I would probably try to find that intersection using... some, like the point-normal equation or whatever" [emphases added].

Throughout the TBI, students' *choice* of what to mobilize was justified by *LA1 norms* and their activity suggested they could not have mobilized mathematical discourse that would explain the relevance of a technique for a given task. Some students' comments confirm they looked to what was usual in LA1 as a source for the techniques they used: in this sense, norms from KtbL made up the technology that produced the techniques they mobilized.

Comments made by two students address the theory that gives legitimacy to didactic norms as technologies. One student recounted doing past final exams because he knew problems in introductory mathematics courses tend to be similar to one another. Another student referred to "grades" as what would confirm to her that her work was correct. The objective of a Student's membership in a course is to obtain a certain grade in the course: this objective forms the theory whereby didactic norms indicate what is legitimate, appropriate, and expected in the course.

Apart from didactic components of students' theoretical blocks, these blocks did include mathematical norms for some students. Asked what they expected to happen if they proceeded with their techniques (e.g., in Problem 2), students referred to the mathematics that normally appeared in their experience: the reduced row echelon form of their augmented matrix would have one of three forms, each of which would then correspond to one of three possibilities for the number of solutions a linear system could have. Students mobilized explanations for these correspondences that, while incomplete and occasionally incorrect, showed an attempt to justify using the mathematics at stake: students brought up, for example, that a row with a certain pattern in its entries would correspond to an equation of the form $0 = a$ where $a \neq 0$, which is then a false equation. Students struggled but nevertheless attempted to justify the relevance of the normative LA1 technique for verifying linear independence, to mobilize the mathematical meaning of vectors being linearly (in)dependent. This justification was not part of KtbL in the course, but some students' attempt to mobilize such justification shows their theoretical blocks can include mathematical norms from LA1.

One student's activity and comments underscore how students' non-mathematical theoretical blocks (and the didactic norms informing them) can regulate their activity in opposition to mathematical praxeologies they may have. In reaction to Problem 7, P4 originally substituted the task by a routine system-solving task and used a normative system-solving technique to complete it. Toward the end of the interview, I asked P4 to clarify a comment he had made earlier on—something to the effect of always finding a "simpler solution." To illustrate his point, P4 said he could have completed Problem 7 by saying that the two vector equations represent lines that are not parallel (he knew this based on the direction vectors in the equations), so they must intersect at one point, and the system must therefore have one solution; he explained he had opted for the approach he had originally given because that's what he would usually do in LA1. Even if a student *knows* of a non-normative way to use mathematics from a course to complete a task, and even if they believe this way to be more suitable (e.g., by virtue of it being "simpler," say, by consisting of far fewer steps), they might opt for the normative approach. It's what they believe is what's expected of them—and this belief is informed by the norms of the course.

This last comment shows that students may *have* mathematical praxeologies but *mobilize* non-mathematical ones by virtue of their models of what they are expected to learn in a course. This highlights the relevance of attending to what students *mobilize*, and not only the less conspicuous knowledge they may *have*: apart from the methodological difficulty (though not impossibility) of ascertaining what knowledge a person does and does not have (e.g., I would easily have missed P4's geometric knowledge of vector equations had I not asked the question that prompted him to reveal it), examining what students mobilize in opposition to what they do not mobilize can help to characterize how course norms regulate students' (non-)mathematical activity. I view what the knowledge students mobilize and the knowledge students have to form what Chevallard (1985) terms "knowledge actually learned" in a course: it is a mixture of the knowledge students acquired in a course and the knowledge they choose to mobilize from what they have acquired.

6.1.2.3 Instances in which students mobilized mathematical praxeologies

A few instances in the TBI had students mobilize mathematical praxeologies. This happened with students whose activity in the TBI suggested they had occupied a Learner position alongside a predominantly Student position when in LA1. The trigger for these students to mobilize a mathematical praxeology seems to have been permission or encouragement to deviate from mobilizing the routinized praxeologies of KtbL; this permission came from either non-routine components of tasks or from the interviewer, that is, from the authority in the TBI situation.

One task feature that prompted students to mobilize mathematical praxeologies was when mobilizing routinized knowledge was perceived as decidedly inconvenient for completing a task, and a task included familiar components that are not usually coupled in routine tasks. For example, Problem 3 had two non-routine components in what otherwise resembled a routine task to solve a homogeneous system of two equations in three unknowns: the scalars were unusually irrational and a cross product appeared in the statement. (By (non-)routine, I mean in comparison with tasks in KtbL.) The apparent inconvenience of calculating with scalars different from integers seemed motivation enough for P1 and P9 to mobilize an approach different from the computational routine they had spontaneously first suggested; P1 said this approach would have made him "cry." What P1 and P9 mobilized consisted of a mathematical praxeology: they activated knowledge about technologies intrinsic to the task to produce a technique for completing the task. The perceived computational load sufficed for P1 and P9 to reflect on components intrinsic to the task (e.g., P1 mentioned noticing that the scalars in the cross products were the scalars in the linear system) and to then choose to mobilize (non-computational) properties of the components intrinsic to the task.

Another example of the task feature of "non-routine combinations of familiar mathematical components" is in Problem 6. This was a task belonging to a same global task type as a routine LA1 task: to solve a system of equations. But this was not the local task type from LA1 as the equations were quadratic and not linear. The task type and task components were therefore familiar to students and enabled them to mobilize knowledge acquired in more than one mathematics course. P1 and P9 mobilized mathematical praxeologies: they solved the quadratic equations (using knowledge from high-school algebra directly pertinent to the resolution of quadratic equations) and deduced from their finding that the system had no solutions.

It's worth noting that comments P1 and P9 made on these occasions suggested their previous mathematics experience may have extended beyond that of my other participants; P9 mentioned having learned some linear algebra in high-school, and P1 knew about imaginary numbers. This might help to explain their ability to mobilize knowledge others did not mobilize, but still, their choice to mobilize it was enabled or encouraged (in short, permitted) by features of the tasks they were proposed and perhaps the situation they were in. P9, for example, said that on an exam, the resolution of Problem 3 via the orthogonality property of cross products would be an extra, and not something he would submit in isolation; he would primarily do calculations, and possibly add the "analysis" based on the other mathematical properties as an explanation for the result obtained computationally.

P9's comment addresses the third feature that might be needed for students to choose to mobilize mathematical praxeologies (as opposed to exclusively routinized ones): contextual permission to do so. Problem 7, like Problems 3 and 6, involved a non-routine combination of familiar mathematical components, and triggered in P1 and P4 a mathematical praxeology. That said, P4 did not mobilize it in full at first, privileging first an approach more in line with a routinized technique in LA1: isolate unknowns using algebraic system-solving computations. P4 mobilized the mathematical praxeology only after receiving permission from the interviewer to do so. In this case, the permission came in the form of a question: P4 had said, at a different point in his interview, that he always finds simpler solutions in the end, and I asked him (at the end of the interview) to clarify this comment. He referred back to Problem 7 to illustrate his point, and that's when he mobilized a mathematical praxeology to complete Problem 7. He perceived this as a more "logical" approach to the task than the one he had originally proposed, which, in turn, he perceived to reflect "the linear algebra [LA1]... mindset" (or what I would call a LA1 norm). I take this perception a reflection of P4's mathematical theoretical block in this instance: his technique was produced by a reflection on mathematical properties he perceived most pertinent ("logical") to the given task.

I identified two other instances in which participants mobilized mathematical praxeologies. One is P1's mobilization of the geometry of vector equations in Problem 7, after a struggle mobilizing rather more routinized (computational) knowledge, and this observation on mathematical properties intrinsic to the task: "I just realized that it was very easy because the question is not asking for the solution, it was asking for the number of solutions. So I just had to see if the two direction vectors are collinear or not."

Another instance is P4's mobilization in Problem 8 of various technologies that were not made explicit in the task as they usually are in LA1 tasks that call for their use. Two task features encouraged this behavior. First, Problem 8 is a non-routine task, so even when students mobilized a technique reflective of the most heavily routinized LA1 task (to complete tasks by solving linear systems), there was freedom and need to identify how to operate within this routine. Second, it is an open-ended LA1 task, in that various technologies from LA1 can be used to produce a technique through which to complete it. P4 made a comment alluding to the way in which such features give permission to mobilize praxeologies driven by the (LA1) mathematics intrinsic to a task and not by attention to didactic norms in LA1:

> I could try to find this point [the initial point of $\vec{v}$ in the given image]. And then, then I know this vector and [I can] try to find the norm of the vector, *which I think [is] what they would expect us to do in this course.* [emphasis

added]

P4 decided against the equations-based approach: "nah, I'm gonna try *my own* solution" [emphasis added]. The open-endedness of the task encouraged P4 to engage in (mathematical) problem-solving behavior—a behavior that centres mathematical properties of components intrinsic to a task as a way to produce technique.

Reflecting on the above, what seems to be key to students' mobilization of mathematical praxeologies—apart from having sufficient mathematical knowledge to do so—is permission to do so. One feature with potential to communicate this permission is an overtly non-routine combination of technologies from KtbL and possibly the inclusion of features that render routine approaches especially unpalatable to students accustomed to these routines. Nevertheless, several students' comments (P4's, P9's, and others made by P2) suggest how even such features may fail to prompt students to mobilize mathematical praxeologies without explicit permission from an authority: in the context of a didactic institution, Students are primed to identify what's normally expected of them in this institution. In LA1, what's normative to KtbL is the routinization of certain techniques and technologies. Students learn the lesson: they are expected to mobilize these routines. Short of a change to these norms, students may need interference from an authority—not just to entertain the non-routine (e.g., as P3 and P6 had attempted to do in Problem 1 when my questions prompted them to consider that Problem 1 was not the normative matrix equation task), but to choose to mobilize it.

6.1.2.4 P7*'s praxeologies

P7*'s behavior throughout the TBI was unusual in two ways. In response to almost every problem, P7* suggested a variety of activities through which to tackle the problem, though these activities were rarely complete (in that they consisted of a concept P7* thought might be pertinent) and frequently irrelevant to the mathematics intrinsic to the task. Another way in which P7*'s behavior was set apart from that of the other participants was that he did not engage in any of the calculations he proposed unless I prompted him to do so. (I had given such a prompt on at least one equation to investigate whether the calculations would lead P7* to notice an inconsistency in a proposition he had made.) In light of the extreme difference between P7*'s behavior and that of the rest of my participants, and given the significant gap between P7*'s prior mathematical education and that of my other participants, I surmise the behaviors P7* exhibited to reflect his pre-LA1 experience rather than the impact of LA1 on his mathematical activity.

I rather infer the impact of LA1 on P7*'s mathematical activity from the observation that the knowledge he mobilized, like that of the other participants, was similarly restrained by norms from KtbL. For instance, when engaged with Problem 1, P7*'s perception of the task was restricted to the usual matrix equation task in KtbL, with all his techniques aiming to *isolate* C; the main distinction between P7*'s activity in this task and that of others was his awareness that the normative isolation technique did not apply because two of the matrices were unknown and therefore potentially not invertible. In a similar vein, P7* struggled to recall mathematical theoretical knowledge about linear system technologies (e.g., row reduction, determinants, augmented matrices and their reduced row echelon form) as his knowledge here was restricted to their surface-level features.

P7* was one of the few students able to step outside the norms when a task ostensibly deviated from the usual. The variety of praxeologies he suggested to mobilize, even if often described superficially or even irrelevant, enabled P7* to sidestep the limitations of normative LA1 knowledge (e.g., as in Problems 3 and 4 when he mobilized cross and dot product properties students do not normally need to use). This, along with the enthusiasm he expressed in response to some of the techniques he contrived in response to non-routine triggers (exclamations that a technique was "elegant" or "outstanding"), attest to P7* having occupied a Learner position throughout his tenure as a LA1 student.

P7* having occupied a position of Learner contrasts with the surface-level grasp he displayed relative to knowledge that is normative in KtbL. But this contrast may point to the opportunities missed in LA1 as a result of the norm to routinize tasks and technologies in the course. P7*'s comments attest to the missed opportunities: he believed the normative computation (turning Problem 7 into a usual linear system LA1 task) he had spontaneously suggested to be what he would do on an exam because it is what popped up in his mind first. On a different occasion, P7* commented on how he had once submitted two solutions toward an exam problem, and the marker had graded only one of them. This is not surprising, given the constraints on LA1 instructors' marking: an instructor has to grade hundreds of submissions made for each of the 2-4 questions they may be assigned, and grading usually has to be completed within 5 days. The mixture of rules, strategies, and norms regulating what students' activity in a course, together with the heavy routinization that characterizes KtbL in LA1, can restrict the knowledge mobilized even by Learners to surface-level features of LA1 praxeologies.

6.1.2.5 Summary

Students activate the standard task/technique (system-solving and its 2 related techniques) whenever possible (not whenever necessary) and routinize knowledge about technologies. Knowledge students mobilize is restricted by KtbL norms. Knowledge students mobilize is also inconsistent with or inappropriate for the mathematics at stake, and attempts to complete task using norms revealed a superficial grasp of technologies often absent of mathematical meaning other than their corresponding formulas. Students whose activity or comments showed evidence of their having occupied both Learner and Student positions in LA1 were, occasionally, able to mobilize knowledge to be taught or mobilize KtbL in non-normative ways, but this came either after a failure of a normative use of KtbL, in response to an ostensibly non-normative feature in a task, or after prompts from the interviewer. The usual LA1 experience does not give these opportunities, and students whose comments and activity suggested their position during LA1 was exclusively that of a Student were unable to mobilize knowledge similarly, given these opportunities.

The knowledge LA1 students mobilize when they solve linear algebra tasks is driven in large part by the didactic, social, and mathematical norms from the LA1 institution and algebra courses preceding it. Students' practical blocks reflect norms established in routinized components of KtbL, and I found students' mobilization of these norms can go contrary to the mathematics at stake in a task. Meanwhile, students' theoretical blocks are shaped by the status of LA1 as a didactic institution of which students are members because of academic and administrative rules that make of it a prerequisite mathematics course for many university programs. As Students driven by the objective to get a certain grade in LA1, and given the mechanisms that regulate students' grades in LA1, students are encouraged to have practical blocks produced by the norms that characterize knowledge to be learned. If norms that characterize KtbL are the technologies Students are expected to use as source for their techniques (e.g., in the sense that mimicking a normative technique can be a way to produce a technique), it is by virtue of the theory (in the ATD sense of the word) that KtbL is what students need to mobilize to pass the course.

Students' comments confirm they mobilize what they believe to be expected of them in the course, possibly even to the detriment of what they *could* otherwise mobilize. Some students' comments confirm they look to normative assessment tasks (i.e., KtbL) to identify what's expected of them.

Norms that characterize KtbL are such that students activate non-mathematical praxeologies when they solve linear algebra tasks. The routinization of techniques and technologies that characterizes norms in KtbL allow students to operate along surface-level features of these praxeological components. To complete tasks in KtbL, it suffices to recognize technologies nominally or by their usual algebraic representation, and then to mimic a routine. These norms show in students' non-mathematical praxeologies in that their activity is conditioned by and delimited to the routines that work for KtbL tasks. Students' comments showed they were activating what they "usually" would given a certain cue from a LA1 task, and exhibited a lacking sense of agency when the usual failed to apply.

Apart from one task (Problem 1), where students' perception of its objective was shackled to a normative task and they were altogether unable to mobilize anything but norms related to that task, not all students were prohibited by LA1 norms from completing non-routine tasks. Three students—whose comments and activity suggested they had exclusively occupied the position of Students during their tenure as LA1 students—were unable to complete or even mobilize mathematical knowledge toward most of the TBI tasks. But others were able to mobilize some relevant mathematics (albeit often limited to its surface-level features), even though their comments had also painted them as having mostly occupied Student positioning in LA1. This was in part due to the nature of linear algebra and the TBI tasks: many were amenable to linear-system solving techniques, which constitute a significant portion of KtbL.

On occasion, some students paid attention to the mathematics intrinsic to a task instead of exclusively linking it with a routine task, and mobilized KtbL in non-normative ways or mobilized knowledge that extends into knowledge to be taught in LA1. This came either after a failure to use KtbL in a usual way, in response to an ostensibly non-normative feature of a task (e.g. such as inclusion of technologies that do not usually appear together in LA1 tasks), or in response to prompts from an interviewer. The usual LA1 experience does not give students these opportunities, however, and these opportunities fell flat on students whose comments and activity suggested their position during LA1 was exclusively that of a Student. These instances can serve, however, as a starting point for design of tasks that can shift norms in courses like LA1 to encourage students to mobilize more mathematical praxeologies.

6.1.3 Research on the learning of calculus has modeled students' practices and found them to consist of routinizing techniques and building non-mathematical praxeologies; are these practices replicated in linear algebra?

Twenty years' worth of research on the learning of calculus has modeled students' practices and found them to consist of routinizing techniques and building non-mathematical praxeologies (discussed in Sections 2.1 and 2.2 of the literature review). The question that sparked the research in this dissertation was based in a reflection on this body of research: the finding of students' routinization of techniques and building of non-mathematical praxeologies at the post-secondary level is based on research in the learning of *one* domain in mathematics—calculus. Given the great many epistemological and cognitive difficulties known to characterize the learning of calculus, I wondered how the institutional emphasis on routine tasks stacks up in the mixture of cognitive, epistemological, and institutional features that can explain students' disengagement from mathematical components intrinsic to calculus tasks. Given a course in a different domain of mathematics but characterized by similar institutional features, do students' non-mathematical praxeologies persist?

The institutional conditions and constraints that typically regulate post-secondary college calculus courses (Hardy, 2009a) are the same as those that usually regulate a post-secondary introductory linear algebra course on vectors and matrices (LA1). Based on my experience in teaching such a course, and based on a preliminary analysis of 10 recent LA1 midterm and final exams given over the span of 5 years, I inferred that knowledge students are expected to learn (in the ATD sense) in LA1 was also characterized by routinization. This made of LA1 a suitable target for my question.

While there appears to be little to no research done from the institutional perspective on (introductory or other) linear algebra learning[2], anecdotally and in deduction from the popular use of a selection of introductory linear algebra textbooks, it seems that the institutional features and nature of knowledge to be learned in LA1 are typical of introductory linear algebra courses in post-secondary institutions not only in Canada but in other countries as well. I expect the course to follow similar standard across different institutions (at least in North America) due to its role as one of three post-secondary mathematics courses that are prerequisite for many university programs; the standardization is suggested, for example, by the role played by the Ministry of Education and Higher Education of Quebec, which issues institutional standards (such as number of instruction hours per semester and course objectives) for courses such as college calculus and introductory linear algebra (e.g., recall Figure 3.1). With these considerations in mind, in the remainder of this discussion, I will refer by "introductory linear algebra courses" (or introductory linear algebra) to courses similar to LA1.

[2]The only studies I found are those of Arsac & Behaj (1998) and De Vleeschouwer (2010), who also used the ATD framework; their research objectives do not relate to mine.

My analysis of knowledge to be learned in introductory linear algebra confirmed that it is characterized by routinization. Descriptions of the routinization afforded by textbook and final exam tasks in calculus courses (e.g., Hardy, 2009a; Lithner, 2004) describe a process wherein tasks can be identified by surface-level features and their techniques routinized, in that the steps can be mimicked so as to complete a given task. I found that some of the tasks normative in introductory linear algebra lend themselves to this type of routinization. Another praxeological component of activity in introductory linear algebra is routinized: a collection of technologies, each of which appears in tasks that either implicitly (by course norms) or explicitly instruct students to activate the technology. Each technology is to be mobilized in only one way throughout the course (e.g., mobilization of cross products always amounts to calculating them).

Praxeologies that model activity in the discipline of mathematics have components that are amenable to routinization in variable ways, and routines are part of mathematics, but this is not all there is to a mathematical praxeology. That said, KtbL in introductory linear algebra courses, like KtbL in college calculus courses, is characterized by routinization. Research (referenced in Section 2.2.2) on task classifications in calculus courses address the message sent to students by the emphasis on routinization in their learning environment: reliance on surface-level features is enabled and imitative strategies encouraged. Research (discussed in Section 2.2.3) on the practices students have been observed to enact in response to (non-)routine calculus tasks confirms students follow on the message they are given by their learning environment. They learn a restricted set of procedures, recognize tasks by their surface-level features and not by mathematical components intrinsic to these tasks (e.g., therefore identifying limit-finding tasks by the type of function involved rather than by a type of indetermination), and do not learn the mathematical justifications that produce the procedures they mimic. Students are limited by these practices, easily forget routines they had memorized for short-term use, and struggle to adapt to different task types or even to recognize a task is of a different task type when it visually resembles a routinized task.

My study replicates these findings. The KtbL in introductory linear algebra sends a similar message to students about the nature of the praxeologies they are expected to build. The message is well-received; students' activity in the TBI was conditioned by the knowledge routinized in KtbL and overwhelmingly delimited to its surface-level features (boxes of numbers, formulas, etc.). TBI students who managed, in a few rare instances, to escape the shackles of these conditions and limitations, showed how students' activity might be delimited not only by the shortage of mathematical knowledge enabled by norms of routinization, but also by how these norms shape their perception of what they are expected to exhibit: they made comments to the effect that they would not submit the non-routine techniques they had proposed (in the interview) for grades, or at least would submit them only as additional justification to complement a routinized technique. This illustrates one way in which course norms can contribute to student beliefs that may be counterproductive to their learning in later levels, a phenomenon addressed by Raman (2004), Schoenfeld (1989), and Tall (1992).

My research follows in the steps of (Barbé et al., 2005; Bergé, 2008; Brandes, 2017; Broley, 2020; Hardy, 2009a) in their use of Chevallard's (1985, 1999) ATD framework to study the teaching and learning of calculus by examining knowledge along different stages of didactic transposition in the calculus course institution. I propose to view the 'last' stage of didactic transposition, that is, that of the 'knowledge actually learned by students,' not in terms of the knowledge students *have*, as treated in previous research, but in terms of the knowledge students *mobilize*. This does not do away with the knowledge a student may or may not have. I rather shift the focus to what students actually activate in response to certain prompts: this is a mixture of knowledge cognitively available to a student and the knowledge they put to action. In my analysis of the praxeologies students activated, I was at times able to decipher, from students' activity and comments, knowledge they did or did not have, and contrast it with knowledge they chose to mobilize. This helped to characterize the mechanisms that regulated their (non-)mathematical praxeologies. My paradigmatic example is that of the participant who mobilized a mathematical and non-routine praxeology only after he received 'permission,' from the interviewer, to use knowledge different from what he believed would be expected of him in introductory linear algebra.

My research also builds on the positioning framework first proposed by Sierpinska et al. (2008) and later elaborated upon by Hardy (2009a) and Broley (2020). I sought to identify, on the basis of students' comments and activity, the position(s) they had occupied during their tenure in the linear algebra course. One aim was to examine the positions encouraged by the norms of the course, the practices enabled by these positions, and to compare these with previously identified positions: those of Learner, Student, Client, and Person (Sierpinska et al., 2008). The positioning framework elaborated so far proposed definitions for each position. The positions are defined by the objective a student can have as a member in a post-secondary mathematics course. I contribute to the framework with a proposed operationalization of these definitions: classify instances of a participant's activity in terms of behaviors that put the activity in relation with course norms, and, by reflecting on whether this behavior contributes to the objectives of a Student, Learner, Client, Person, or possibly as-of-yet unelaborated position, identify this contribution as a property of that position.

The operationalization I propose conforms with the ATD perspective that human activity consists of a practical block (here, the behavior) and a theoretical block (that produces the practical); as in the ATD framework, I emphasize that praxeologies that model activities of a Student (Learner, Client, etc.) are institutional; the institutional norms of any given course may lead Students to have activity further or closer to that of a Client or that of a Learner, for example. My operationalization of the positioning framework lent itself to my second objective: to identify features of task that may trigger students to switch positions (relative to the introductory linear algebra institution).

The positions I identified in my students were those of Learners and Students. This is not to say that these are the only positions available to students in the linear algebra course—indeed, an additional participant may have revealed another positioning, and I may have found additional positionings with further iterations of my operationalization. I found that in the introductory linear algebra institution, Student positioning leans on mobilization of non-mathematical praxeologies and Learner positioning encourages mobilization of mathematical praxeologies; a course with didactic and mathematical norms different from the introductory linear algebra institution may have its Student and Learner positions qualified differently. All students (with the exception of P7*) displayed signs of having predominantly acted as Students in their linear algebra course. A few displayed signs of having occupied, to a limited extent, a Learner position, and some did not display such signs, but were triggered by features of the TBI tasks to abandon practices they had developed from a Student position and instead behave in ways that reflect a Learner positioning. The analysis of students' positions helped to show how the routinization that characterizes the didactic and mathematical norms of introductory linear algebra, when adhered to as an ideal Student might (ideal in that they act upon the messages sent by norms of knowledge to be learned), can act as a stopper for mobilizing mathematical praxeologies.

Just as there is a vast body of research on epistemological, conceptual, and cognitive sources for students' difficulties in the learning of calculus, there is a body of research on such sources for students' difficulties in the learning of linear algebra (albeit not much when it comes to linear systems and their solutions, per Stewart et al. (2019), technologies that made up a large part of the praxeologies I investigated in this research). The research on students' learning in both domains seeks to explain students' difficulties as well as the absence of certain mathematics from their practices. The replication of the findings from research on the learning of calculus—that is, that the institutional norms that emphasize routinization in college calculus and introductory linear algebra courses enable and encourage students to mobilize non-mathematical praxeologies—highlights the power of institutional routinization to enable and encourage students to operate along surface-level features of the mathematics targeted by an intended curriculum. Routinization of knowledge to be learned fails to give students the opportunity (and the permission, if students are inclined) to engage mathematics past its surface-level features—let alone to engage with mathematics that is documented as a source of conceptual or cognitive difficulties.

6.2 Contributions of this work to research on linear algebra education

My study contributes to research on linear algebra education in several ways. Its main contribution is in the affordances it demonstrates of an institutional perspective that is rarely (if ever) used in linear algebra education research. The findings I made from this perspective complement some of those made on topics that are popular in research on the learning of linear algebra. Without going into a comprehensive account of each finding, I give a few examples of how my results complement research made from cognitive or conceptual perspectives: research about students' reasoning about linear combinations of vectors, students' coordination of geometry and algebra, students' geometric reasoning, and students' difficulties with the structural, unifying, generalizing processes that qualify linear algebra. My study also contributes to the research through its focus on students' reasoning in the topics of linear systems and their solutions; my review of the literature confirms few papers have investigated this topic.

In 2001, Dorier and Sierpinska noted the common claim in "discussions about the teaching and learning of linear algebra that linear algebra courses are badly designed and badly taught, and that no matter how it is taught, linear algebra remains a cognitively and conceptually difficult subject." In spite of the common assumption that linear algebra courses are “badly designed,” “badly taught,” and which sparked a reform of linear algebra courses globally, Dorier’s seminal (2000) work on the state of research on linear algebra education shows the institutional perspective was nearly absent (indeed, only one of the works was noted to have adopted it) as interest was rather in examining the cognitive and conceptual sources of students’ difficulties. Stewart et al.’s more recent (2019) review of developments in research on linear algebra education spanning the previous decade shows the trend from the 1990s persisted, with much of the research still attending to conceptual and cognitive sources of students’ difficulties as well as controlled teaching experiments. Stewart et al. (2019) point out the ample room for analysis of linear algebra textbooks and assignments with a focus on their content and how they are used by instructors and students. The institutional perspective I propose extends this focus to the institutional mechanisms that regulate students’ (non-)mathematical activity in linear algebra courses.

A first contribution of my work to research on linear algebra education is in the affordances brought by the institutional perspective. The framework helped to identify institutional mechanisms that enable and encourage students to exclusively mobilize surface-level features, usually of algebraic form, of linear algebra concepts and techniques. These mechanisms include the objectives of students’ membership in introductory linear algebra courses—courses that are prerequisite for many university programs and which are, as a result, an obligation students must fulfil rather than a course they elect to take (as is the case even of students heading into mathematics programs: given the status of introductory linear algebra as a prerequisite requirement, students are likely to occupy the position of Students whose objective is to get a certain grade in the course. Another mechanism includes assessment rules and norms, which make it such that a student’s grade is essentially determined by their performance on a final exam, an assessment that usually includes the same tasks from one semester to the next and which is graded by all course instructors. These mechanisms in particular, along with others as well (e.g., limited number of hours in which to cover a substantial curriculum, students’ other potential obligations), encourage and enable students to limit what they learn to mobilizing surface-level features of concepts and techniques that suffice to complete the tasks that usually appear on exams.

My study is small in scale but the non-mathematical praxeologies I found students to mobilize, and my investigation of institutional mechanisms regulating these, complement findings made in other studies. For example, the topic of linear combinations of vectors is one of the most addressed in linear algebra education research (Stewart et al., 2019) and is usually examined via cognitive and conceptual frameworks (e.g., APOS, Tall’s three worlds, Sierpinska’s modes of thinking, as referenced in Section 2.3.1). One such study (Stewart & Thomas, 2010) found students lack visual representations of linear dependence and prefer algebraic (computational) methods to problems such as that of determining if a set is linearly independent, and used APOS and Tall’s three worlds to examine how an emphasis on matrix processes may not help students understand the concept. Another study (Dogan-Dunlap, 2010) used Sierpinska’s modes of thinking to analyse student homework solutions and found students privileged algebraic and arithmetic thinking modes (over a geometric mode) to explain whether sets of vectors are linearly independent and to develop conjectures about linearly independent sets in $\mathbb{R}^3$. The findings made by Stewart & Thomas (2010) and Dogan-Dunlap (2010) are replicated in this study. While students generally knew that the linear dependence of two vectors in

2- or 3-space meant they are parallel, they struggled to mobilize a geometric interpretation of linear dependence of three vectors; I also found that what students did mobilize in relation to linear independence was one of two algebraic techniques for determining linear independence that are routinized in the linear algebra course. Using the institutional perspective, I traced these observations to the lack of need and opportunity, in tasks students are expected to solve (to pass the course), to mobilize geometric knowledge about linear independence, and its contrast with the normative linear independent task which is completed by row reduction or determinant calculations (i.e., matrix processes, or the arithmetic and algebraic thinking modes).

Students' use of geometry and their coordination of algebra and geometry are topics often addressed in linear algebra education research. Reflecting on papers that attended to the role of geometry in students' understanding of eigenvalues and eigenvectors, linear independence, linear transformations, and some other linear algebra concepts, Stewart et al. (2019) note that while "some studies claimed that geometry (in some cases together with the help of appropriate software) enhanced students' learning experiences, some results showed that students performed better in more routine algebraic questions" and maintain that the implications of geometry in linear algebra courses deserve further investigation, for instance, with more systematic investigations of how geometric modes can be coordinated with algebraic ones to enhance students' learning. Based on my findings, that students perform "better in more routine algebraic questions" merits further investigation as well. My results show the potential implications of institutional emphasis on routine algebraic modes. First, students' better performance in routine algebra tasks can lack in substance as routinization allows students to operate on superficial knowledge, and this knowledge does not transfer well (if at all) to less routine tasks. Second, the routinization of algebraic technique in both algebraic and geometric contexts can enable students to exclusively mobilize algebraic representations of so-called 'geometric' concepts.

In this study, I mostly omitted the topic of vector spaces from the tasks I presented in the interviews, but one of the tasks nevertheless offers an implication of institutional emphasis on routinization of algebraic modes for students' structural thinking (Dorier & Sierpinska, 2001) or difficulties with unifying and generalizing processes (Dorier, 2000a). I recount the main finding from this task in the next paragraph and follow this with how this finding offers an implication of the institutional emphasis on routinization for the structural thinking that students may (fail to) build.

Problem 1 of the TBI was to solve a matrix equation of form $ABC = I$ for some matrix C. This visually resembled a routine matrix algebra task in the introductory linear algebra course. The routine task in the course is actually a task to isolate a specified matrix given an equation, but is usually stated using the expression "to solve an equation." The task in Problem 1, however, was not a task to isolate a matrix, and was indeed the broader mathematical task of solving an equation: determining whether it has solutions (and if it does, identifying them). But students' matrix algebra knowledge was so conditioned by the routine task that they did not recognize the broader task at hand. This was in spite of the fact that students recognized two things: first, that one of the matrices to the left of C was not invertible, and second, that the I in the equation as an identity matrix. Students persisted in trying to concoct ways to isolate C.

Students' perception in Problem 1 was entirely conditioned by the routine matrix-isolation task from their course and what they mobilized was restricted to the normative technique for this task. This was, ostensibly, a task about the structure of matrix algebra—specifically, about identity matrices having a role akin to that of 1 in the real numbers when it comes to multiplicative inverses. Considering the knowledge to be taught about matrix algebra in introductory linear algebra, as indicated by the textbook section where the normative "solve a matrix equation (i.e. isolate a matrix)" task appears, there is an intent to teach about structural elements of matrix spaces (even if the term 'space' only appears in the last chapter of knowledge to be taught in the course: vector spaces). But this does not make it to the knowledge students are expected to learn about matrices. This brings to mind one of the first objectives in the 1990s boom of research on linear algebra education: to investigate students' difficulties with the structural mode in linear algebra. Dorier et al. (2000) elaborates on the epistemological sources for these difficulties and proposed, as a potential salve for these difficulties, "meta-level activities." These include explicit discourse from the teacher about the unifying and generalizing significance of axiomatic structure and its methodological affordances. Such discourse may help students acknowledge that axiomatic structure is a thing of importance to mathematicians, and some students may even be convinced of its significance, but this, on its own, is unlikely to turn the axiomatic structure of linear algebra into "part of the 'cognitive furnishing' of the students' minds" (Dorier & Sierpinska, 2001). Without mathematical problems that give substance to this discourse, though, its potential is limited. My finding from Problem 1 gives an example of how a routinized treatment of algebraic objects in introductory linear algebra can be a missed opportunity to prepare students, via the tasks that are offered, to start to develop structural modes of thinking.

Finally, one of the main contributions of this study to research on the teaching and learning of linear algebra is in the elaboration of the knowledge students mobilize about linear systems and their solutions in a certain (and likely common) type of institutional context. Few papers investigate students' reasoning in the topics of linear systems and their solutions (Stewart et al., 2019). Due to the nature of many topics in introductory linear algebra, these topics tend to extend onto other topics in the course. This allowed us to design various types of tasks that helped to qualify students' reasoning on the topics of linear systems and their solutions. My results (discussed in Section 6.1.2) elaborate on the nature of (technical and theoretical) knowledge students can develop about linear systems and their solutions in institutional contexts that privilege routinization of technical knowledge.

6.3 Final remarks

In this last section, I close with the main conclusions of this research, remark on its limitations, and propose avenues for future work.

6.3.1 Main conclusions

The aim of this thesis was to sharpen the focus on the effect of institutional routinization in students' learning of mathematics. As prior research on routinization was done near-exclusively in the context of calculus courses, I decided to investigate the linear algebra context. I adapted a framework and methodology from a body of research that investigated routinization using an anthropological and institutional lens.

In analyzing the data to extract the knowledge students mobilized, I stacked it against the knowledge they had been expected to learn so as to draw out the nature of their praxeologies. This traced the absence of mathematical praxeologies from what students mobilized to two aspects of the institutionalized routinization of knowledge. One, the routines, along with the nature of the routinized knowledge (e.g., boxes of numbers), allowed students to strip mathematical justifications from routines and mathematical meaning from routinized technologies. Second, students' comments, along with one case in which a student opted for a non-mathematical praxeology instead of a mathematical praxeology, showed students mobilized what they believed was expected of them, and that they looked to routinized knowledge to determine what's expected of them.

I also analysed the data to identify positions students had occupied during their course; to this end, I proposed an operationalization of the positioning framework described by Sierpinska et al. (2008), Hardy (2009a) and Broley (2020). In reflecting on this operationalization and a posterior analysis of what it revealed about students' positions, I realized that the institutional mechanisms that regulate the impact of routinization on the praxeologies students mobilize similarly regulate the impact of (institutional) routinization on students' positioning: the course norm of routinization determines the activity that best serves the Student position. This aligns with the ATD view of activity as consisting of a practical and theoretical block: the tasks Students opt for (e.g., to learn the routines needed to do the usual exam tasks) and the techniques they use (e.g., to do past exams, to look for solution templates in solved examples) are produced by the reasoning that to pass their course, they have to satisfy what's normally expected in that course. Just as routines are par for the course in mathematics, the Student position may also be such in didactic institutions. Courses that emphasize routinization, however, propel Students to behave in ways drastically different from those that lend themselves to a Learner behavior, as routinization, together with an aim to get a certain grade in course, may propel students to focus exclusively on the routines needed to get these grades.

This last remark highlights the importance, in the context of research on the effects of course norms on the knowledge students mobilize, of coordinating an analysis of the didactic transposition of knowledge with an analysis of other institutional mechanisms that regulate students' activity.

6.3.2 Limitations and avenues for future work

This study has some limitations. Some are due to time constraints and others to the nature of the task-based interview methodology. Limitations due to the former include the small scale of the study—an eleventh participant may have exhibited activity I have not observed in the first ten participants. That said, the inferences I made on the basis of this study's participants attest to the existence of certain types of practices in linear algebra students, and the existence of these practices replicate findings from earlier studies that used the same framework and similar methodology. Another limitation due to time constraints is that I was unable to analyse instructors' grading of students' exam submissions; this would have helped to inform the models of knowledge to be learned, as I took this knowledge to be that which is needed for students to pass their course. A limitation due to the nature of the task-based interview methodology is the potential impact of earlier TBI tasks on students' engagement with later TBI tasks. I addressed this in my analysis of students' engagement with Problem 6, where I noted, on the basis of one student's comments, that students' struggle with earlier TBI tasks might have contributed to undermining the sense of agency they had relative to their linear algebra knowledge. I had noted this limitation to acknowledge its potential role in students' hesitation to state a conclusion for Problem 6.

One potential avenue for future work is a continuation of the current comparison of findings about students' practices in calculus with students' practices in linear algebra. Broley (2020) found the routinization in earlier calculus courses to be an obstacle in students' engagement, in a real analysis course, with tasks that resemble earlier calculus tasks. This work belongs to a body of research about difficulties students experience in the transition through tertiary mathematics studies. One of the original objectives of this doctoral thesis was to examine, as I had for LA1, students' practices in a second-year linear algebra course that is part of a university mathematics degree (I called it LA2). I eventually delimited the research objectives to those addressed in this dissertation, but an initial analysis of students' engagement with one TBI task (designed according to a guideline similar to the one I used for the LA1 TBI) revealed a potential persistence of institutional routinization and its effects on students' (non-)mathematical activity. A future study could focus on how routinization norms from the earlier linear algebra course (LA1) and norms from a mathematics-degree linear algebra course regulate what students mobilize as they progress in their study of linear algebra.

I envision the elaboration of the positioning framework as another direction for future work, mainly with the purpose to build understanding of how knowledge to be learned in a course, together with students' positions, as institutionally-relative mechanisms, regulate students' engagement with mathematics in their courses.

One avenue could be to continue developing the positioning framework in the context of courses where knowledge to be learned is routinized, but expand this to courses that range across students' mathematics studies (e.g., including courses at the end of an undergraduate degree or in mathematics graduate degrees). This could help to examine the role of routinization in courses where the institutional positions to which students are disposed may vary—for example, courses students elect to take, or courses at a level of mathematics studies where students' previous mathematics learning is not exclusively characterized by routinization.

More generally, the positioning framework could stand to be elaborated or refined by conducting studies similar to the current one but in mathematics courses with different institutional norms. Courses where knowledge to be learned is not exclusively characterized by routinization could help to investigate what behaviors would characterize the Student, Learner, Client, Person (or other student positions) in such a context. For example, I found the institutional norms regulating LA1 to call for Student behavior that is far from and even at odds with Learner behavior; are there course norms that encourage Student behavior that is close to Learner behavior? Or perhaps closer to behavior of other positions—e.g., Client or Mathematician-in-Training?

My use of the ATD-IAD framework to examine the effect of institutional routinization on students' activity revealed that routinization does not always block students from acquiring mathematical praxeologies, but it can block them from mobilizing these. Institutional norms can discourage students from mobilizing mathematics intrinsic to a task when it is not what students believe they are expected to do in their course—even if that mathematics belongs to knowledge to be taught or knowledge actually taught in the course. To determine what might encourage students to mobilize mathematics intrinsic to a task, I propose further investigation of the interplay of knowledge and positioning as institutionally-relative mechanisms regulating students' activity.

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Appendix A

TBI protocol

Table of Contents

How to begin the TBI with each participant:

1. Say: "In this interview, I will ask you to solve some problems. I am not here to judge or evaluate you. The purpose is not to check whether your work is correct. What I want to see is how you solve the problems and why you approach the problems the way you do. For that, I will need you to think out loud as you solve the problems. That means I'll need you to explain what you are doing and why you are doing it. Is that ok?"
2. Give them the consent form and time to read it and ask questions.
3. If they sign the consent form, ask if they are ready to start the interview; when they are, start the audio-recording.
4. Give the first problem.

General considerations for interventions

1. **If they do not know a definition or formula**, give it to them.
2. **If they are quiet for 1 minute**, remind them to think out loud.
3. **If they ask a direct question**, keep in mind the goal to see how they do a problem and the reason they give for doing it that way.
   a. Turn the question back to them.
   b. If it's a request for validation of what they are doing (e.g., "should I solve by Gauss-Jordan or By Cramer?"), say "do what you think is most appropriate" and when they finish, ask "why did you choose that method" and "would you have done that on an exam?"
4. **If they're stuck**
   a. on something that requires knowledge not taught in LA1/251 but which we expect them to know from previous courses (e.g., finding solutions of a quadratic equation), provide the information needed to proceed. Otherwise,
   b. Ask what they would have done if they were stuck on this problem on an assignment or exam.
   c. If they are stuck on something that requires knowledge taught in LA1, give a series of increasingly directive hints (without saying what to do) without giving the answer away;

   d. If they are still stuck, suggest moving on to another problem and returning to this one if there is time after they've attempted all the other problems.

5. **If they're going in a wrong or overly time-consuming direction**,
   a. Do not let them go on for more than 10 minutes.
   b. If what they are trying to do is not clear, ask for clarification.
   c. Ask what it is they're hoping will happen.
   d. Acknowledge what they are trying to do and ask if they can think of another approach. If they cannot, suggest moving on to another problem and returning to this one if there is time after they've attempted all the other problems.
6. **If you must improvise**, keep in mind the goal is to see how the participants solve the problems and why they do what they do as they solve.
7. **While they are attempting a problem**,
   a. E.g., For LA1 participants, we know Problem 3 has connection to Problem 2 so ask about "now you've done Problem 3, what do you think about Problem 2?"
   b. Ask about how they solve problems for themselves and for exams. E.g., for LA1 participants, given a set of vectors that are linearly independent, what do *you* need, as an individual, to be convinced that they are or aren't linearly independent?
8. **After they attempt a problem, time permitting,**
   a. Ask follow-up questions about *anything* the participant had produced:
      i. If the participant's reasoning or process was unclear at some point and was not explained (e.g., it was not worth interrupting them at that moment), ask them to clarify it.
      ii. "Is this something you would have done on an exam?"
      iii. "Would you have received full marks for this?"
      iv. "If you were solving this problem just for yourself, would you have been convinced by what you did?"
      v. "Did you think of any other approaches to the problem?"
         1. If they did, "why did you use this approach instead of that one in the end?"
         2. If they did not, and there is time, ask if they can think of another approach. Otherwise, suggest moving on to another problem and returning to this one if there is time after all other problems have been attempted.
9. **Once all problems had been attempted and problem-specific follow-up questions asked,**
   a. Ask follow-up questions you were unable to ask about specific problems.
   b. If they exhibited strong emotions at any point, ask them to explain how they felt during the interview or about some of the problems.
   c. If a participant had mentioned, while solving the problems, that they would use the computer to accomplish some tasks, offer a computer and ask if they can show what they would have done.
10. **Once all follow-up questions have been asked or 2 hours have passed (whichever comes first), end the interview, say the following, and stop the audio-recording thereafter:**
    "Research from past 20 years about calculus courses suggests that the types of problems students have to do to pass calculus might encourage students to develop practices that aren't based in mathematics; for example, the problems encourage students to learn routines for how to compute certain types of limits, and students can get by with these routines without knowing why they work. It's not the students' fault, it's just the type of learning that is encouraged by the design of the course. We want to find out if students have similar practices in linear algebra, or if the nature of the math in linear algebra might foster different types of practices. We're not interested

in students' role in shaping their learning - our focus is rather on the institution. We want to find out how these courses' designs might shape students' learning."

LA1: Problem-specific considerations for interventions

Problem 1 (LA1)

Solve the following equation for C.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} BC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1. If the participant seems stuck from the start, ask "what are you thinking?" and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: "this times A times that times that times B times C is equal to that," pointing at the appropriate part of the equation while saying this, "you need to find the matrix C that makes this equation true."
    b. If they are still stuck, ask: "What do you think you're stuck with? Why do you think you don't understand the question?"
        i. If they do not get the equation (for example, they think C is the identity matrix), say: "A, B, and C are all matrices here. To the left of the equal sign, there's a product of all these matrices; to the right of the equal sign, there's just that matrix [point at it]. You have to figure out what C has to be for this equation to be true."
        ii. If they still do not know how to start, give this hint: "Let's think about an equation with real numbers... For example, 1/5 is a solution of 5x = 1 because 5(1/5) is 1."
        iii. If they need another hint, ask: "Let's say you had something like this [write all matrix algebra on a piece of paper]: [2 1   3 2]A = [5 6   7 8], then to solve for A, we can use the fact that [2 –1   -3 2][2 1   3 2] = [1 0   0 1]."
        iv. If they are still stuck, move on to another problem.
2. If the participant is not stuck, intervene according to the following options:
    a. The participant asks if they must isolate C.
        i. Say: "The goal is to solve for C – so do whatever you think makes sense."
    b. The participant successively multiplies by inverses of the matrices to the left of C.
        i. The participant says they do not remember how to find the inverse of a matrix.
            1. Give a piece of paper that has the course textbook's theorem about inverses of 2x2 matrices.
        ii. The participant seems stuck or unsure about multiplying by the inverse of A (or B).
            1. Say: "What are you thinking?"
                a. If they are aware that A may not be invertible, say: "Ok, I see. Do whatever you think is most appropriate."
                    i. If they proceed with the assumption that A is invertible, then, when they finish, say: "You used the inverse of A over there – but what if A isn't invertible?"

iii. The participant multiplies by inverses but on a side of the equation that is inappropriate.
   1. As soon as this happens, say: "remember that multiplication is not commutative."
      a. If they say they don't know what I mean, say: "never mind."

iv. The participant multiplies by the inverse of [2 3   4 6] but does not notice at any point that this inverse does not exist – or they make an error in the determinant and so believe the inverse exists.
   1. Do not say anything until they are done. When they finish, say: "I have a question. This matrix [point at the matrix [2 3   4 6] in the problem statement] doesn't have an inverse, but you multiplied by its inverse - can you fix that?"
      a. If they are still stuck after 5 minutes, say: "the matrix to the right of the equation is an identity matrix. Does this help?"

c. The participant expresses A, B, and C as matrices with unknown entries (for example, $C = [c_{11}\ c_{12}\ \ c_{21}\ c_{22}]$ and creates a linear system from the given matrix equation.
   i. Stop them after 10 minutes and ask: "what do you hope will happen as you keep going?"
   ii. Follow up with: "I see where you're going. Can you think of another way to approach this problem?"
   iii. If they cannot think of another approach, give the hints for the scenario in which the participant is stuck from the get-go.

Problem 2 (LA1)

The coefficient matrix below is invertible. Solve the system:

$$\begin{bmatrix} 9 & 16 & 3 & 4 \\ 5 & 6 & 0 & 8 \\ -2 & 3 & 0 & 4 \\ 3 & 6 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 2 \\ -3 \end{bmatrix}$$

1. If the participant seems stuck from the start, ask "what are you thinking?" and intervene according to the appropriate scenario:
   a. If they do not understand the question, say: "This equation between matrices is like a system of linear equations. [Point at the coefficient matrix] This is the coefficient matrix of that system. You're told it has an inverse. You need to find the values of w, x, y, z for which this equation is true."
   b. If they are still stuck, ask: "What do you think you're stuck with? Why do you think you don't understand the question?"
      i. If they need the definition of "invertible," give the definition from the course textbook (have a piece of paper ready with the definition already on it).
      ii. If they still do not know how to start, give this hint: "If you were to multiply these two matrices out [point at the matrices on the left of the equation], you'd have this matrix: [write out the product on a piece of paper]. With the equation between these matrices, do you see how this is a "system"? A system of linear equations. [Point at the entries in equivalent positions on either side of the equal sign.] There's an equation between this and that, this and that, this and

that... So the coefficient matrix of this linear system is that [point at the coefficient matrix].”

iii. If they need another hint, say: "We can rewrite this equation like this: Ax = b [write “Ax = b” on a piece of paper], where A is the coefficient matrix [point at the coefficient matrix], x is the matrix of unknowns [point at it in the problem statement], and b is the matrix of constant terms [point at it in the problem statement]. Does this help?”

iv. If they are still stuck, move on to another problem.

2. If the participant is not stuck, intervene according to the following options:
    a. The participant uses Gauss-Jordan elimination.
        i. The participant is stuck or is not yet done at the 10-minute mark (whichever comes first).
            1. Say: “Let’s pause here. What do you expect will happen as you keep going?”
            2. Then say: “Ok. Have you thought of a way to solve this problem without doing as many calculations?”
                a. If they have, ask why they picked this approach instead.
                b. If they haven’t, say: “the coefficient matrix is invertible; what does this tell us?”
        ii. The participant finishes within less than 10 minutes.
            1. Say: “Have you thought of a way to solve this problem without doing as many calculations?”
                a. If they have, ask why they picked this approach instead.
                b. If they haven’t, say: “The coefficient matrix is invertible. What does this tell us?”
    b. The participant uses a technique to find the inverse of the coefficient matrix.
        i. The participant is stuck or is not yet done at the 10-minute mark (whichever comes first).
            1. Say: “Let’s pause here. What do you expect will happen as you keep going?”
            2. Then say: “Ok. Have you thought of a way to solve this problem without doing as many calculations?”
                a. If they have, ask why they picked this approach instead.
                b. If they haven’t, say: “the coefficient matrix is invertible; what does this tell us, other than the fact it has an inverse?”
        ii. The participant finishes within less than 10 minutes.
            1. Say: “Have you thought of a way to solve this problem without doing as many calculations?”
                a. If they have, ask why they picked this approach instead.
                b. If they haven’t, say: “The coefficient matrix is invertible. What does this tell us?”
    c. The participant uses Cramer’s rule.
        i. The participant is computing the determinant of the coefficient matrix and is stuck or not yet done at the 10-minute mark (whichever comes first).

        1. Say: “Let’s pause here. What do you expect will happen as you keep going?”
    d. The participant observes (-1, 0, 0, 0) is a solution.
        i. The participant does not address whether this is the only solution.
            1. Say: “are there other solutions?”

Problem 3 (LA1)

Show that $(w_1, w_2, w_3) = (29, -9, 3.2) \times (11, 2.1397, 41)$ is a solution of the following system.

$$\begin{array}{rcrcrcl} 29x & - & 9y & + & 3.2z & = & 0 \\ 11x & + & 2.1397y & + & 41z & = & 0 \end{array}$$

1. If the participant seems stuck from the start, ask “what are you thinking?” and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: “This [point at the cross product in the problem statement] is a cross product of the two vectors – the first vector has components 29, -9, 3.2, and the second vector has components 11, 2.1397, and 41. You have to show that their cross product makes both equations true.”
    b. If they are still stuck, ask: “What do you think you’re stuck with? Why do you think you don’t understand the question?”
        i. If they need a definition, give it (have a piece of paper ready with a copy of the course textbook’s definitions of cross product).
        ii. If they still do not know how to start, give this hint: "The cross product is some vector – we can call it (w_1, w_2, w_3) because it has three components. I mean that if we compute the cross product, the result is a vector with three components. That vector makes both equations true – your job is to show this.”
        iii. If they are still stuck, move on to another problem.
2. If the participant is not stuck, intervene according to the following options:
    a. The participant decides to compute the cross product.
        i. The participant does not recall the definition of the cross product or has an incorrect definition.
            1. Say: “Here's the definition of cross product.” and give it (have a piece of paper ready with a copy of the course textbook’s definitions of cross product).
        ii. The participant asks if they can use approximations of the numbers.
            1. Say: “Do whatever you think is most appropriate.”
        iii. The participant computes the dot product using the correct definition, plugs the result into the two equations, and at least one is false.
            1. If they notice one of them is false, do not intervene unless they do not try to fix the issue.
            2. If they do not notice one of them is false, say: “wait, this doesn’t solve that equation; look, the equation is false when you plug in what you found.”

iv. The participant computes the cross product using the correct definition, plugs the result into the two equations, both are true, and the work is correct throughout.
    1. Say: "Have you thought of a way to solve this problem without computing the cross product?
        a. If "yes," ask why they picked this approach instead.
        b. If "no," ask if they can think of a geometric relation between the cross product (29, -9, 3.2) x (11, 2.1397, 41) and the vectors (29, -9, 3.2) and (11, 2.1397, 41).
            i. If they cannot, move on to Problem 4. Once they have finished Problem 4, go back to this problem (Problem 3) and say: "these vectors are orthogonal."[1]

b. The participant uses Gauss-Jordan elimination.
    i. The participant asks if they can use approximations of the numbers.
        1. Say: "Do whatever you think is most appropriate."
    ii. The participant is stuck.
        1. Say: "What do you think will happen when you finish this process?"
    iii. The participant finishes the process.
        1. If the participant does not already address the following, say: "So what does this tell you about the cross product? Is it a solution of the system?"
        2. After they've addressed the previous question: "Could you have solved this problem without finding *all* the solutions of the system?"
            a. If "yes," ask why they picked this approach instead.

c. The participant observes that the equations can be expressed as $(29, -9, 3.2) \cdot (x, y, z) = 0$ and $(11, 2.1397, 41) \cdot (x, y, z) = 0$ and the cross product is orthogonal to both $(29, -9, 3.2)$ and $(11, 2.1397, 41)$, so it is a solution to the system.
    i. Say: "Ok. Can you describe what this all looks like on a graph?"

Problem 4 (LA1)

Find a non-trivial solution of the following system:

$$\begin{array}{rcrcrcl} -5.2x & + & 2y & + & \pi z & = & 0 \\ 4x & - & 1.3y & + & 4z & = & 0 \end{array}$$

1. If the participant seems stuck from the start, ask "what are you thinking?" and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: "When x, y, z are all 0, that's one solution of the system, do you see that? [Wait for response.] The triple (0,0,0) is called a "trivial" solution of the system – each component is 0. A solution whose components aren't *all* 0 is called a "non-trivial" solution. You have to find a solution that isn't trivial. A solution whose components aren't all 0.

[1] We only say this after they have accomplished Problem 4 because if we tell them this before they do Problem 4, it may influence how they see Problem 4.

   b. If they are still stuck, ask: “What do you think you’re stuck with? Why do you think you don’t understand the question?”
      i. If they are still stuck, move on to another problem.
2. If the participant is not stuck, intervene according to the following options:
   a. The participant uses Gauss-Jordan elimination.
      i. The participant asks if they can use approximations of the numbers.
         1. Say: “do whatever you think is most appropriate.”
      ii. The participant stops the approach.
         1. If they stop because they are stuck, first say: “what do you think will happen once you’ve finished the process?”
         2. Once they’ve answered the previous question, or if they completed the Gauss-Jordan elimination, say: “can you think of a way to solve this problem without finding *all* the solutions of the system?”
            a. If “yes,” ask why they picked this approach instead.
            b. If “no,” ask if they can find any similarities between this problem and Problem 2.
         3. Once the above questions have been addressed, and if the similarity with Problem 2 hasn’t been discussed, ask if they can find any similarities between this problem and Problem 2.
   b. The participant uses the cross product of the normals of the planes.
      i. The participant computes the cross product and says the result is a non-trivial solution but does not explain why.
         1. Say: “why is this a solution?”
         2. If their answer involves plugging the cross product into the equations, say: “why did you compute the cross product? What made you think it would work?”
      ii. The participant says the cross product is a non-trivial solution but does not explain why.
         1. Say: “why do you think it is a solution?”

Problem 5 (LA1)

Given $k \in \mathbb{R}$, the vectors $(-k, 1, 1), (-1, 1, k)$, and $(1, 0, 1)$ form a parallelepiped of volume 0. Find the values of k for which the vectors are linearly independent.

1. If the participant seems stuck from the start, ask “what are you thinking?” and intervene according to the appropriate scenario:
   a. If they do not understand the question, say: “You’re given three vectors here [point at the vectors] and you have to figure out what value (or values) k must have for the vectors to be linearly independent. You’re given the vectors themselves, and you’re also told that they form a parallelepiped whose volume is 0.
   b. If they are still stuck, ask: “What do you think you’re stuck with? Why do you think you don’t understand the question?”
      i. If they do not know what a parallelepiped is, say: “a parallelepiped is a shape that looks like this [give a piece of paper that has a ready-made drawing of a parallelepiped on it]. The parallelepiped in the drawing is formed by these

vectors [give another piece of paper with the same parallelepiped on it, this time with vectors drawn over three edges that share a vertex, and the initial point of each vector is that vertex].

        ii. If they need a definition, give it (have a piece of paper ready with a copy of the course textbook's definition of linear independence and formula for the volume of a parallelepiped).
        v. If they still do not know how to start, give this hint: "what does it take for these vectors to be linearly independent? If you think about what linear independence means?"
        vi. If they need another hint, ask: "Let's think about the other piece of information first. You know they form a parallelepiped of volume 0. What would that look like, a parallelepiped of volume 0?"
        vii. If they are still stuck, move on to another problem.

2. If the participant is not stuck, intervene according to the following options:
    a. The participant writes the homogeneous equation whose solution indicates whether the given vectors are linearly independent.
        i. The participant is stuck or not yet done at the 10-minute mark (whichever comes first), say: "Let's pause here. What do you expect will happen as you keep going?"
        ii. After they have answered the previous question, or if they finished solving the homogeneous equation, say: "Have you thought of a way to solve the problem without solving this equation?"
            1. If "yes," say: "why did you pick this approach instead?"
            2. If "no," say: "the parallelepiped has volume 0 – is that something you can use?"
    b. The participant uses the formula for the volume of the parallelepiped and the fact that the volume is 0.
        i. The participant is stuck or not yet done at the 10-minute mark (whichever comes first), say: "Let's pause here. What do you expect will happen as you keep going?"
        ii. After they have answered the previous question, or if they finished solving the equation, say: "Have you thought of a way to solve the problem without solving this equation?"
            1. If "yes," say: "why did you pick this approach instead?"
            2. If "no," say: "what does it mean, geometrically, that the volume of the parallelepiped is 0? What does that shape look like?"
    c. The participant says that since the parallelepiped has volume 0, the vectors are parallel to the same plane.
        i. The participant is stuck.
            1. Say: "Ok, that's true. Does this tell you anything about linear independence?"
            2. If they cannot answer the previous question, say: "what does it take for these vectors to be linearly independent? If you think about what linear independence means?"

ii. The participant says that since the vectors are parallel to the same plane, they are linearly dependent.
    1. Say: "Ok. Can you explain that a bit more?"

Problem 6 (LA1)

Solve the following system of equations:

$$\begin{array}{rcrcrcl} x^2 & + & x & + & 1 & = & 0 \\ 2x^2 & + & 4x & - & 6 & = & 0 \end{array}$$

1. If the participant seems stuck from the start, ask "what are you thinking?" and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: "this is a system of equations. You have to solve it. What values of x make both equations true?"
    b. If they are still stuck, ask: "What do you think you're stuck with? Why do you think you don't understand the question?"
        i. If they are still stuck, move on to another problem.
2. If the participant is not stuck, intervene according to the following options:
    a. The participant notices the equations aren't linear and asks if they can use Gauss-Jordan elimination to solve the system.
        i. Say: "do whatever you think is most appropriate."
    b. The participant uses Gauss-Jordan elimination (or elementary row operations).
        i. The participant is stuck interpreting the reduced augmented matrix.
            1. Say: "what are the equations that correspond to what you found?"
            2. Then: "Have you thought of a way to solve the system without using row operations?"
                a. If "yes," ask why they picked this approach instead.
                b. If "no," say: "this equation [point at the first equation] only involves one variable; and this equation [point at the second equation] also involves only that variable. Does that help you think of another approach?"
    c. The participant solves each equation.
        i. The participant is stuck because they do not know how to solve a quadratic equation.
            1. Give the quadratic formula on a piece of paper (have it ready-made).

Problem 7 (LA1)

Under what condition does the following system have infinitely many solutions?

$$\begin{array}{rcrcrcc} x & - & 3y & + & kz & = & k+5 \\ 2x & - & (k+8)y & - & 4z & = & 6 \\ (k-1)x & + & (7-k)y & + & 6z & = & -9 \end{array}$$

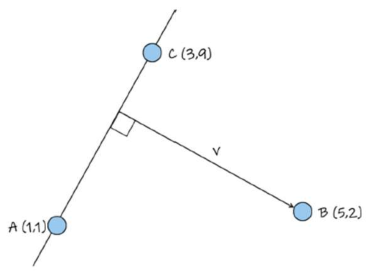
1. If the participant seems stuck from the start, ask "what are you thinking?" and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: "what value (or values) does k need to have so the system would have infinitely many solutions?

b. If they are still stuck, ask: “What do you think you’re stuck with? Why do you think you don’t understand the question?”
    i. If they still do not know how to start, give this hint: "Let’s say k is 1. Then the system would look like this: [take a piece of paper and write out what the system would be]. How would you check if the system has infinitely many solutions, in this case?”
    ii. If they are still stuck, move on to another problem.

2. If the participant is not stuck, intervene according to the following options:
    a. The participant reduces the augmented matrix of the system.
        i. If they are stuck/are not yet done at the 10-minute mark (whichever comes first), ask: “Let’s pause here. What do you expect will happen as you keep going?”
        ii. Then say: “Let’s put aside, for a moment, the value of k for which this system has infinitely many solutions. What does it take, graphically, for a system like this to have infinitely many solutions?”
    b. The participant determines the condition under which the first two equations are equivalent to one another
        i. If they describe this as “the equations are multiples of each other,” say: “can you clarify what you mean by that?”
        ii. If they find that k must be –2 for the first two equations to be equivalent and are stuck (for 5 minutes) trying to take the same approach with equations 1 and 3 or 2 and 3: “with the first two equations, what happens if k is –2?”
            1. Then ask: “what if k is not –2?”

Problem 8 (LA1)

Find the length of the vector $\vec{v}$, which has B as terminal point and is orthogonal to the line that goes through the points A and C.



1. If the participant seems stuck from the start, ask “what are you thinking?” and intervene according to the appropriate scenario:
    a. If they do not understand the question, say: “This is the vector v [point at the vector]: its initial point is there [point at it] and its terminal point is there [point at it]. You have to find its length. All you know about v is this: its terminal point has those coordinates [point at the coordinates of point B] and v is orthogonal to this line [point at the line through the points A and C].”

b. If they are still stuck, ask: “What do you think you’re stuck with? Why do you think you don’t understand the question?”
    i. If they need a definition, give it (have a piece of paper ready with a copy of the course textbook’s definitions of orthogonal and of length of a vector).
    ii. If they still do not know how to start, give this hint: "To find the length of a vector, you need to know its initial and terminal points.”
    iii. If they need another hint, ask: "You’re missing the coordinates of the initial point of v, right? But what *do* you know about that point?”
    iv. If they are still stuck, move on to another problem.

2. If the participant is not stuck, intervene according to the following options:
    a. The participant assigns unknown coordinates to the initial point of v and produces equations whose solutions are the coordinates of that point (or indicates they wish to do so).
        i. If the participant does not know how to algebraically represent that vector v is orthogonal to the line through A and C, but indicates they want to do so, say: “two vectors are orthogonal if their dot product is zero.”
        ii. If the participant gets stuck trying to solve a quadratic equation, give the quadratic formula on a piece of paper (have it ready-made).
        iii. After the participant is done, say: "did any other approach come to mind, other than using this formula?”
            1. If “yes,” ask: “why did you pick this approach?”
            2. If “no,” and there is time, ask: “can you think of any other approach?”
    b. The participant uses the formula for the distance between a point and a line (or indicates they wish to do so).
        i. If the participant does not recall the formula for the distance between a point and a line, give it to them (have a piece of paper with the formula from the course textbook).
        ii. After the participant is done, say: "did any other approach come to mind, other than using this formula?”
            1. If “yes,” ask: “why did you pick this approach?”
            2. If “no,” and there is time, ask: “can you think of any other approach?”
    c. The participant uses the formula for the component of vector AB (or CB) orthogonal to vector AC (or indicates they wishes to do so).
        i. If they do not recall the formula for the component of a vector orthogonal to another vector, give it to them (have a piece of paper with the formula from the course textbook).
        ii. After the participant is done, say: "did any other approach come to mind, other than using this formula?”
            1. If “yes,” ask: “why did you pick this approach?”
            2. If “no,” and there is time, ask: “can you think of any other approach?”
    d. The participant embeds the objects in R^3 and uses formulas for the area of parallelograms to find the length of vector v (treating it as the height of the parallelogram formed by vectors AC and AB).

i. If they do not recall the formula for area of parallelograms in R^3, give it to them (have a piece of paper with the formula from the course textbook).
ii. After the participant is done, say: "did any other approach come to mind, other than using this formula?”
    1. If “yes,” ask: “why did you pick this approach?”
    2. If “no,” and there is time, ask: “can you think of any other approach?”

Appendix B

Identified behaviors, classified by position properties of the Student and Learner positions

PP: position property
B: behavior

Behaviors classified by property of the Student position:

PP acquisition of KtbL

- **B** re-runs a normative LA1 technique
- **B** spontaneous reaction is a normative LA1 technique

PP attempt to prioritize time-efficient technique

- **B** compares steps involved in similar techniques to assess which would take less exam time
- **B** values comparing techniques to determine which would take less time

PP attention paid to an authority's motivation in the design of at task

- **B** interprets elements of task to have the function of a hint to the problem-solver

PP belief about expectations of students produced by normative LA1 KtbL

- **B** believes students are expected to demonstrate calculations, not use concepts

PP compartmentalization of knowledge by KtbL tasks

- **B** applies technique for LA1 task with similar surface-level features but different in substance
- **B** categorizes certain technologies as cues to mobilize certain LA1 KtbL techniques

- **B** claims to have done past final exams/categorized knowledge by LA1 tasks as a strategy for gaining LA1 knowledge
- **B** engagement with task (mobilized knowledge, technique) is conditioned by superficially-similar routine task from LA1 KtbL
- **B** struggles to identify objective of a task that is not a normative LA1 task
- **B** tries to produce technique on the basis of experience with LA1 task involving similar surface-level features
- **B** suggests to convert task to its usual appearance in LA1 so as to use normative LA1 KtbL

PP compartmentalizing knowledge as (not) belonging to LA1

- **B** categorises knowledge as LA1 knowledge and not-LA1 knowledge

PP dependence on row-reduction catch-all

- **B** all attempts are to activate the standard LA1 task of solving a system of equations
- **B** struggles with other LA1 technologies, activates the standard LA1 task of solving a system of equations

PP expresses an expectation that reflects LA1 norms but not the mathematics at stake

- **B** expresses an expectation that reflects LA1 norms but not the mathematics at stake

PP failure to use KtbL in ways different from its normative use in KtbL

- **B** all attempts are to activate the standard LA1 task of solving a system of equations
- **B** fails to use technique from one normative LA1 task to answer a question about a different normative LA1 task
- **B** unable to mobilize LA1 knowledge toward a task that is not similar to any normative LA1 task
- **B** unable to mobilize normative LA1 KtbL for a non-normative task

PP lacking sense of agency over mathematics at stake

- **B** asks for validation
- **B** expresses a feeling of not understanding reasoning behind LA1 techniques
- **B** expresses frustration at uncertainty over validity of knowledge
- **B** expresses lack of confidence in knowledge students are not usually required to use
- **B** expresses lack of confidence in mathematical knowledge and requests validation from interviewer
- **B** wants to validate results but only knows how to do so for some types of results

PP result sanctification (KtbT substitute)

- **B** depends on results to assess suitability or validity of technique
- **B** persists with a normative technique even after becoming aware a condition for the technique is not met
- **B** validates a technique based on knowledge that its end-result is what is sought

PP surface-level grasp of KtbL

- **B** associates a task with KtbL to which it has no connection because surface-level features are similar
- **B** fails to reflect on mathematical properties intrinsic to a situation
- **B** has theoretical block that normally holds in LA1 but is not founded in mathematical properties intrinsic to the situation
- **B** mobilizes/remembers LA1 KtbL incorrectly
- **B** struggles/unable to remember/mobilize more than surface-level features

PP theoretical block consists of LA1 KtbL

- **B** engagement with task (mobilized knowledge, technique) is conditioned by superficially-similar routine task from LA1 KtbL
- **B** identifies knowledge to learn by doing past final exams
- **B** looks to KtbL to produce technique
- **B** uses a technique or validates it based on its status as a norm in LA1
- **B** uses KtbL technology to produce technique that has no basis in/relevance to situation

PP use of authority to validate knowledge

- **B** appeals to an authority and not to mathematics at stake for validation
- **B** expresses lack of confidence in mathematical knowledge and requests validation from interviewer
- **B** validates knowledge on basis that it was KtbL/KT/KtbT in LA1

PP use of normative knowledge inconsistent with mathematics at stake

- **B** applies surface-level rule about normative LA1 technique to a situation in which the mathematics at stake is different
- **B** expresses an expectation that reflects LA1 norms but not the mathematics at stake

PP failure to use KtbT that is not KtbL

- **B** depends on results to assess suitability or validity of technique
- **B** does not complete task because of failure/inability to mobilize LA1 KtbT that is not KtbL
- **B** does not complete task initially because of failure to mobilize LA1 KtbT that is not KtbL

- **B** knowledge about a technology is restricted to the formula(s) needed in LA1 tasks that involve that technology
- **B** mobilizes incorrect technique for solving linear systems and unable to determine why result is inconsistent with expectations
- **B** struggles/unable to complete task using knowledge other than normative LA1 KtbL (e.g., theoretical KtbT, using registers in a way different from their routine KtbL use, etc.)
- **B** struggles/unable to remember/mobilize more than surface-level features
- **B** technique driven by personal preferences (e.g., habits) or course norms and are not based in mathematical properties intrinsic to a situation
- **B** unable to determine whether a normative LA1 technique can be used in a non-normative scenario
- **B** unable to mobilize any LA1 knowledge (coherently, appropriately, or at all) toward a non-routine task
- **B** unable to mobilize normative LA1 KtbL for a non-normative task
- **B** unable to mobilize normative LA1 technique accurately for a non-normative task
- **B** validates knowledge on basis that it was KtbL/KT/KtbT in LA1

Behaviors classified by property of the Learner position:

PP concept of mathematical aesthetics

- **B** evaluates techniques based on criteria that are not required in LA1 KtbL ("simpler," "more logical," "elegant")

PP enthusiastic response to non-routine task

- **B** expresses emotional response toward and interest in non-routine (relative to LA1 KtbL) aspects of TBI task

PP interpretation of task based on mathematics at stake

- **B** reevaluates objective of task by attending to a property of a mathematical object at stake, after having initially focused on surface-level features of more routinized KtbL

PP non-normative use of KtbL

- **B** identifies and mobilizes relevant technologies from LA1 and prerequisite math courses through which to complete an open-ended task
- **B** mobilizes mathematical property of component that is not needed in LA1 tasks involving that component
- **B** mobilizes normative LA1 technique accurately for a non-normative task

- **B** uses a technique produced in response to a previous non-routine TBI task

PP problem-solving behavior

- **B** evaluates techniques based on criteria that are not required in LA1 KtbL ("simpler," "more logical," "elegant")
- **B** investigate a mathematical property so as to use it to complete a task
- **B** mobilizes knowledge from a different mathematics course
- **B** mobilizes variety of techniques pertinent for a task (and compares/reflects on the results obtained)
- **B** seeks mathematical property intrinsic to a task component that may have an advantage over already-identified pertinent knowledge
- **B** seeks to mobilize knowledge that is directly pertinent to the given task rather than to the similar LA1 task

PP seeking and reflecting on guidance relative to the mathematics at stake

- **B** seeks and tries to build on help to overcome an obstacle

PP use of KtbT/KT/KtbL combination

- **B** seeks to mobilize knowledge that is directly pertinent to the given task rather than to the similar LA1 task
- **B** tries to reason about a situation using LA1 knowledge other than KtbL directly associated with normative task
- **B** mobilizes LA1 KtbT/KT/KtbL directly pertinent to the given task instead of KtbL pertinent to task that involves a similar feature

PP use of mathematics at stake to validate knowledge

- **B** attempts to verify a result that is does not need to be verified in LA1 KtbL

PP use of KtbT that is not KtbL

- **B** after struggle with normative LA1 knowledge, mobilizes LA1 KtbT that is not LA1 KtbL
- **B** mobilizes knowledge that is KtbT in LA1 and only rarely KtbL
- **B** spontaneous reaction includes LA1 KtbT that is not LA1 KtbL
- **B** validates knowledge on basis of mathematics at stake in KtbT that is rarely or never KtbL