Social Interaction and Cultural Activities: A Theoretical and Empirical Analysis of Friendship Networks

Siming Xie

A THESIS
IN
THE DEPARTMENT
OF
ECONOMIS

Presented in Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy (Economics) at
Concordia University
Montreal, Quebec, Canada

October 2024

CONCORDIA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

This is to certify that the thesis prepared

By: **Siming Xie**

Entitled: **Social Interaction and Cultural Activities:
A Theoretical and Empirical Analysis of Friendship Networks**

and submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy (Economics)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________________ Chair
Dr. Wei Sun

______________________________ External Examiner
Dr. Vincent Boucher

______________________________ Examiner
Dr. Szilvia Pápai

______________________________ Examiner
Dr. Jan Victor Dee

______________________________ Examiner
Dr. Julius Owusu

______________________________ Thesis Supervisor
Dr. Ming Li

Approved by: ______________________________
Dr. Christian Sigouin
Graduate Program Director of Department of Economics

December 6, 2024 ______________________________
Dr. Pascale Sicotte
Dean of the Faculty of Art and Science

Abstract

Social Interaction and Cultural Activities: An Economic Analysis on Friendship Networks

Siming Xie, PhD
Concordia University, 2024

This thesis consists of three chapters on social networks, focusing on cultural activities and social interactions.

Chapter 2 introduces a model of friendship formation that explores whether cultural activities can help reduce segregation across racial groups. The model incorporates a bias in the matching process, where individuals engaged in the same activities are more likely to form friendships. This framework aligns with empirical findings, including increased friendships among club members and reduced segregation when students from different racial backgrounds participate equally in cultural activities.

Chapter 3 presents an empirical analysis of club participation, examining how individual decisions are shaped by social norms. The findings indicate that white students participating in basketball and black students in baseball are influenced by the size of their racial group, whereas this effect is not observed for the reverse combinations. Using a Linear-in-Mean (LIM) model, the study reveals that previous research may overestimate peer effects on individual decisions, mainly due to the endogeneity of friendship networks. The results suggest that cultural clubs play a role in fostering social cohesion across racial groups.

Chapter 4 presents a model that examines how social norms influence students' decisions to participate in clubs, highlighting a trade-off between the costs of participation and the benefits of social engagement. The model predicts that as school size increases, participation rates decline, but a higher proportion of devotee members are committed to the club. Moreover, social engagement has a stronger influence on participation decisions in larger schools. Empirical data support these predictions.

Acknowledgment

This doctoral journey has been long and challenging, and I am deeply grateful to those who have supported me along the way.

First and foremost, I would like to express my heartfelt gratitude to my supervisor, Dr. Ming Li, for his unwavering support, patience, and guidance throughout this process. The difficulties I encountered during my PhD studies were greater than I had ever anticipated, and the journey took longer than expected. Through every obstacle, Dr. Li provided not only academic insight but also encouragement during the most difficult moments. His patience and belief in me have been invaluable, and I am truly grateful for his support.

I am also sincerely thankful to my committee members, including Dr. Wei Sun (defense chair), Dr. Vincent Boucher (external examiner), and Dr. Szilvia Pápai, Dr. Jan Victor Dee, and Dr. Julius Owusu (examiners), for their constructive feedback and thought-provoking discussions. In particular, I extend my deep appreciation to Dr. Jan Victor Dee, whose guidance and insights were instrumental during the research process.

I would also like to express my gratitude to the professors and faculty members I have met throughout this program for their valuable discussions and for fostering a sense of academic community that made this journey more fulfilling. I deeply appreciate the encouragement and support from my peers and friends, whose companionship made this journey memorable.

My deepest thanks to Anna. Her strength, compassion, and quiet encouragement have stayed with me through every step of this journey. I am truly grateful for her presence in my life.

To all who have been part of this journey—thank you.

Dedication

In memory of my grandparents,
who filled my life with love and joy.

To KimDou, my quiet supporter, my source of joy,
and my constant companion throughout this journey.

Contribution of Authors

All the chapters are solo-authored.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Cultural Activities and Racial Segregation: A Network Formation Model

Social networks play a crucial role in shaping behaviors and facilitating the exchange of values and information within societies. One key characteristic of these networks is homophily, the tendency of individuals to connect with others who share similar traits, such as race, age, or gender. This tendency often reinforces social segregation, which can influence various societal outcomes, including the diffusion of knowledge and the establishment of collective norms.

While fixed traits like race have been extensively studied in the context of network formation, the flexible nature of cultural activities presents a distinct opportunity for fostering cross-group connections. This chapter explores the role of cultural activities in reducing social segregation across racial groups.

The primary objective of Chapter 2 is to develop a model that captures key aspects of network structures involving multi-dimensional social activities, while remaining tractable enough to allow for closed-form solutions. In this model, each agent possesses two dimensions of traits: the first is fixed, such as race, while the second is flexible, such as club participation. The model incorporates a biased matching process, where agents who participate in the same cultural activities meet each other with a probability higher than their fraction in the overall population. Agents aim to maximize their individual payoffs, and their strategies collectively define the network formation game.

The analysis focuses on stationary equilibria, where individuals' behavior is time-invariant and the inflows of agents in the matching process balance the outflows. These equilibria assume that the initial market configuration aligns with equilibrium behavior, ensuring that the inflows and outflows of each type of agent are balanced.

The model successfully replicates several empirical findings: (i) the number of social connections for club members increases with both the relative size of the student's racial group and the popularity of the club within the school; (ii) students participating in clubs have more social connections than those who do not; and (iii) social segregation reaches its lowest levels when students from different racial groups participate in cultural activities at similar rates.

1.2 Empirical Findings in Peer Effects and Friendship Connections

Chapter Three presents empirical findings regarding students' club participation decisions and their friendship connections.

In this analysis, I separate each student's friends by race and employ a Linear In Mean (LIM) model to estimate the spillover effects from different types of peers. Focusing on basketball participation, I find that, for white students, the spillover effects from their Black friends are greater than those from their white friends. These results suggest that prior studies may have overestimated peer effects on individual decisions, primarily due to the endogeneity of friendship networks.

The reflection problem arises from two main sources: (1) the spillover effects of an individual's activities on their friends, and (2) the endogenous network structure, in which individuals tend to form connections with those who are similar to themselves. To address these issues, I use peers' health conditions and energy levels as instrumental variables (IV). While the application of IV effectively addresses the first issue, it does not fully account for the influence of endogenous network formation.

This chapter also presents additional empirical findings. First, I observe that white students in basketball and Black students in baseball are strongly influenced by the size of their racial group, but this pattern does not hold for the reverse combinations.

Compared to non-members, club members are more likely to form connections with other members within the club and are also more likely to build cross-racial friendships. This finding suggests that cultural clubs foster social cohesion across different racial groups in society.

The data further indicate that participation rates are significantly negatively correlated with school size, mainly due to capacity limitations in clubs at larger schools. At the same time, a stronger clustering effect is observed among club members in larger schools—these students are more likely to form connections within the club rather than random connections across the school.

Additionally, I classify club members into two groups: devotees, who are intrinsically motivated by the sport and participate in only one club, and connectors, who participate in multiple clubs with the aim of building more social connections. On average, connectors have more friendships than devotees and non-members, and their homophily index is higher than that of devotees but similar to that of non-members. As school size increases, the participation rate among connectors declines at a faster rate than that of devotees.

1.3 Benefits from Social Engagement: A model and Empirical Analysis

Chapter Four develops a model that examines how prevailing social norms influence students' decision-making processes regarding extracurricular club participation. The model focuses on the tradeoff between the effort and time required to engage in a club and the social benefits derived from forming connections within the club. Based on prevailing social norms, students form expectations about their potential social payoffs, which in turn shape their individual decisions about whether or not to join a club.

In this model, I assume that students choose to join a club based on the expected social benefits, driven by the likelihood of forming connections with peers who share similar traits, such as race or shared hobbies. Each student aims to maximize their social payoff by optimally distributing their effort between making friends within the club and outside it. The total effort, E, is divided between these two channels, and the number of friendships is determined by a concave function of effort, which captures diminishing returns on increased effort.

The social payoff from joining a club, V, is modeled as a function of the benefits per connection within the club, v_m, and outside the club, v_s. Students allocate their effort to maximize the social payoff. By solving the optimization problem, I derive the equilibrium allocation of effort and the corresponding social benefits. I then estimate the net benefit of club participation, ΔV, as the difference in payoffs between joining and not joining the club.

The model also incorporates the cost of participation, which is influenced by students' individual talents, assumed to be randomly distributed. The participation decision is shaped by a tradeoff between the benefits from forming social connections and the costs associated with club participation. Using this framework, I estimate the equilibrium participation rate, which is the point where the additional payoff from joining the club equals the participation cost. This equilibrium is influenced by the size of the racial group, the participation rate of other students, and the benefits of forming connections.

Specifically, the model predicts that the steady-state participation rate decreases as school size increases, while the proportion of devoted members within clubs rises. Empirical data support the model, confirming that social benefits significantly influence students' club participation. The benefits of each connection are greater in larger schools, both for connections within clubs and those across the school, indicating that students' participation decisions are increasingly driven by social payoffs as school size grows.

The analysis in this chapter provides insights into the social dynamics within schools, showing how club participation fosters cross-group interactions and promotes social cohesion. By estimating the social benefits from friendship connections, this model contributes to a deeper understanding of the role that cultural and sports clubs play in shaping students' social networks.

Chapter 2

Cultural Activities and Racial Segregation: A Network Formation Model

2.1 Introduction

In social networks, individuals with similar traits tend to form connections at a higher rate than those with different characteristics, a phenomenon known as *homophily*. This tendency has been observed across various dimensions, including race, age, and occupation, and is a key driver of social segregation. Segregation within a society significantly impacts information diffusion, learning efficiency, and the speed at which consensus is reached among its members.

This research begins with the observation that individuals who share personal hobbies are more likely to form social connections, potentially reducing segregation along racial lines. Unlike immutable characteristics such as race or religion, personal interests and participation in cultural activities are flexible traits that can transcend these predetermined categories. Since individuals are embedded in multidimensional social networks, where they engage in different types of relationships across various contexts, it is crucial to explore the interactions in these multiple dimensions. Instead of focusing solely on fixed characteristics, this study shifts attention to traits that can be altered at a relatively low cost.

This chapter examines the role of cultural activities in shaping friendship networks in a society. Cultural activities, such as joining clubs, provide a channel through which individuals can forge new connections. From the empirical analysis, I find that club members are more likely to form connections with each other within the club and they tend to have more connections than non-members.[1] Therefore, a key feature of my model is that individuals with shared hobbies have a higher probability of forming connections.

This chapter explores the role of cultural activities in shaping friendship networks within a society. Activities such as club participation serve as avenues for individuals to establish new connections. The data indicate that club members are more likely to form friendships with other members and tend to have more connections overall compared to non-members. Building on this observation, the model developed in this study incorporates a key feature: individuals who share common hobbies have a higher probability of forming connections. This framework allows for a detailed analysis of how cultural activities influence the dynamics of network formation.

Currarini, Jackson, and Pin (2009) investigate the patterns of friendship networks in U.S. high schools, where racial homophily among students is prevalent. They developed a model that includes biases in both preferences and the meeting process, explaining empirical observations of racial homophily. Building on their work, this study extends the analysis by considering interactions between homophily effects in a two-dimensional model. To simplify the analysis, the model initially focuses on two racial groups and one personal interest, such as basketball, which individuals either like or dislike. The model introduces matching bias toward same-hobby friendships and a preference bias favoring same-race connections.

The objective is to develop a model that can capture some important aspects of network structures with multi-dimensional social activities while remaining tractable enough to yield closed-form solutions. The analysis focuses on stationary equilibria, where individuals' behavior is time-invariant and the inflows of agents in the matching process balance the outflows. These equilibria assume that the initial market configuration aligns with equilibrium behavior, ensuring that the

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[1] This study uses data from the National Longitudinal Study of Adolescent to Adult Health (Add Health), a nationally representative longitudinal survey of adolescents in grades 7-12 during the 1994-95 school year in the United States.

inflows and outflows of each type of agent are matched.

Numerous studies demonstrate that individuals' behaviors are influenced by their peers: for instance, when adopting new technology, smoking, engaging in criminal activities, going to college, etc. Social segregation enables groups to maintain distinct behaviors, norms, and cultures. At the same time, agents in social networks not only choose their actions but often also have substantial control over whom they interact with. The joint determination of actions and social connections is a growing area of research, as it provides a more complete understanding of how behaviors and networks co-evolve.

Unlike most behaviors, participation in cultural activities presents a distinct opportunity for individuals to form new social ties at a relatively low cost, which in turn affects the structure of social networks. The analysis suggests that when a sport or cultural activity becomes popular among all students, racial segregation in schools decreases. However, in some schools, significant differences in the popularity of activities among racial groups persist, leading to higher levels of segregation.

My analysis show that, on average, students who join sports clubs such as basketball or baseball have more friendship connections than those who do not. Additionally, the number of connections increases with both the relative size of their racial group and the percentage of their friends who join the club. This model helps to explain these patterns of friendship formation in the context of cultural activities.

This study makes three primary contributions. First, it integrates the analysis of multilayer networks with welfare considerations in network formation, providing insights into the interdependencies between different network layers. The model provides a welfare-based approach to understanding the structuring process in multidimensional networks. Ignoring these interdependencies can lead to erroneous conclusions about network formation.

Second, the model highlights the interaction between cultural activities and network structure. Unlike other behaviors, cultural activities not only affect individual choices but also make individuals build new connections and in turn reshape the social networks.

Third, the model provides policy implications, particularly regarding the role of cultural activities in reducing social segregation. Encouraging shared hobbies among different racial groups can reduce segregation, especially for minority groups. Social coordination plays a key role in this process. Shared hobbies are a striking example of strategic complementarity, where the incentive to participate in an activity increases as more individuals in one's social network choose the same activity. By fostering participation in shared activities, new cross-group connections can form, potentially disrupting segregated networks and leading to a new equilibrium with reduced segregation.

The remainder of this chapter is organized as follows: Section 2.2 reviews the related literature. Section 2.3 introduces the homophily index and presents the empirical observations from the data. In Section 2.4, I introduce the baseline model, which assumes agents have a biased preference toward same-race connections, and demonstrate that the weighted homophily index is minimized when cultural activities are equally popular across racial groups. The advanced model, incorporating biased preference toward both same-race and same-hobby connections, is then presented. Section 2.5 discusses the empirical context and shows consistency between the theoretical model and empirical observations. Section 2.6 concludes. Proofs and derivations are provided in Appendix.

2.2 Related Literature

In this chapter, I develop a mechanism of network formation in which agents with shared personal hobbies have a higher probability of meeting one another, and it explores the role that common cultural activities play in reducing racial segregation within a society. Jackson (2014) and Jackson et al. (2017) review how social network structures—such as connections, clusters, and centrality—shape economic behaviors by influencing information flow, partnerships, and decision-making.

Economists have provided theoretical predictions on homophily, as seen in the works of Currarini et al. (2009), Currarini et al. (2016), Bramoullé et al. (2012), and Iijima and Kamada (2017). There is supporting empirical evidence, including studies by Fong and Isajiw (2000), Baerveldt et al.

(2004), and Bramoullé and Rogers (2009). Boucher (2020) finds that racial homophily is primarily driven by the impact of students' observable characteristics o n payoffs, rather than by network effects. Jackson et al. (2023) examine the dynamics of friendship networks, finding that homophily based on certain traits remains relatively stable over time. Karimi et al. (2018) develop a social network model to evaluate how homophilic and heterophilic behaviors affect minorities within social networks. Currarini et al. (2016) study the formation of homophilous social networks, and how agents' preferences and their meeting opportunities determine the observed mix of in-group and cross-group ties. In their model, agents search in the matching pool for their own racial group when the absolute group size is large enough, but search to population when their own type is a small group. The anticipation of future interaction is also at the heart of Baccara and Yariv (2013), where homophilous peer groups form in connected intervals along the preferences dimension.

At the same time, the analysis of multidimensional social networks has gained attention within the network science community. Existing studies on network dynamics have often considered the role of similarities across various dimensions, but these dimensions are typically estimated separately or controlled for one another. Block and Grund (2014) demonstrate that while homophily across various dimensions tends to have positive main effects, the interaction between these effects is often negative. Atkisson et al. (2020) suggest that in multidimensional social networks, social ties are not necessarily formed because two agents are optimal partners for each other in a specific domain, but because they are connected in a more influential or significant layer of the network. To the best of my knowledge, this research is the first to propose a model in which agents exhibit two-dimensional homophilous behavior.

The co-evolution of links and actions in coordination games has been explored from various perspectives. Ballester et al. (2006) show that their model results in "nested split graphs," which represent networks with a strict hierarchical structure. Hiller (2017) and Baetz (2015) model the problem of network formation as a simultaneous game, where agents decide jointly on actions and links. Ely (2002) examines a model in which players' strategies and locations evolve simultaneously, with only efficient strategies persisting. Badev (2021) proposes a model that captures the interplay between individual incentives, local peer effects, and social welfare, applying it to the study of adolescent smoking behavior.

2.3 Background: Patterns of Friendship Networks and Homophily

In social contexts, individuals often have the freedom to choose their personal hobbies and decide whether to participate in activities, such as joining a club. These cultural activities may bridge predetermined societal divisions, including racial segregation. However, it is important to recognize that actions and social connections co-evolve through strategic complementarity. Individuals may select hobbies based on their friends' interests and decide whether to join a club by considering its racial composition.

2.3.1 Patterns of Friendship Networks

An example is depicted as a graph in Figure 2.1, using data from two U.S. middle schools. In these graphs, students are represented as nodes, with colors corresponding to their racial groups. The left graph illustrates racial homophily: white students predominantly form friendships with other white students, while black students tend to connect with other black students. In contrast, the right graph shows no significant clustering by race, but instead by grade level. The students interviewed in the second school are in either grade 7 or grade 8, and they primarily form friendships within their own grade.

In the friendship network depicted in the left graph, 52% of the students are white and 41% are black. The right graph represents a network where 46% of the students are white and 28% are black. In both schools, white students constitute the majority, and black students represent the largest minority. Despite these demographic similarities, the patterns of network formation are markedly different.

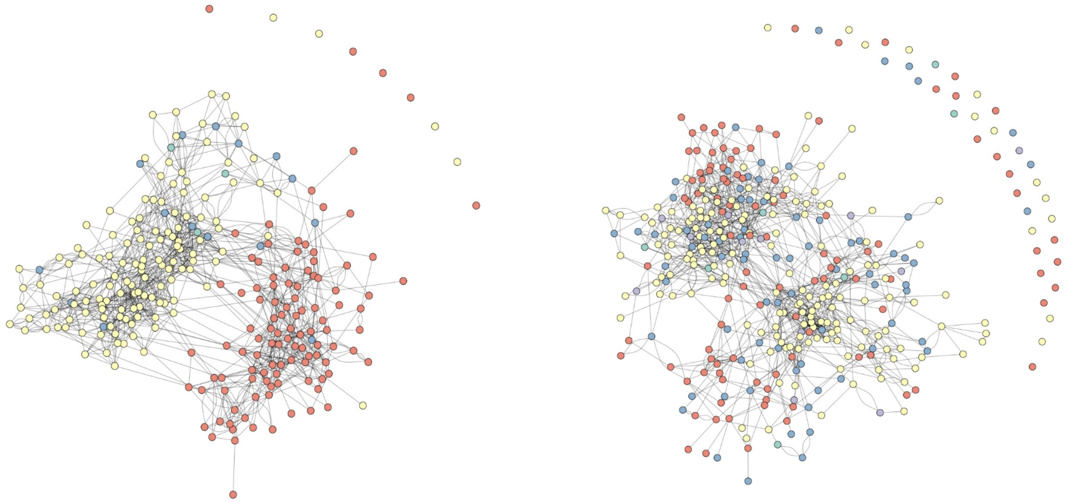

Figure 2.1: Illustrative Friendship Networks

Friendship networks in two U.S. middle schools. Students are coded by race: yellow nodes for white; orange nodes for black; green nodes for Asian; purple nodes for Indian; blue nodes for others.

Next, we turn to the composition of cultural activities in these schools. In the first school, 15% of white students and 45% of black students joined the basketball club, while in the second school, 31% of white students and 35% of black students participated. As noted in previous sections, the co-evolution of friendships and activities is complex. It is not immediately clear whether a less segregated network structure leads to similar hobbies among racial groups, or if cultural activities themselves reduce racial segregation. Further analysis is required to uncover the mechanisms driving this phenomenon.

2.3.2 Measuring Homophily

In this section, I provide the definition of homophily index, which will later be used in the matching model presented in Section 4.

Consider a society composed of K distinct types of agents. Let N_g, where $g \in \{1, ...K\}$, denote the number of type g agents in the population, and let $w_g = \frac{N_g}{N}$ represent the proportion of type g agents in the overall population, where $N = \sum_{g=1}^{K} N_g$ is the total population size. Let s_g denote the average number of friendship connections that agents of type g form with others of the same type, and let d_g represent the average number of connections that type g agents form with individuals of different types. The homophily index measures the proportion of connections formed within the same type.

Definition of Homophily. Homophily index for type g is defined as:

$$H_g = \frac{s_g}{s_g + d_g},$$

This index captures the extent to which individuals form connections within their own type relative to their total number of connections.

Definition of Inbreeding homophily. Inbreeding homophily presents when the tendency to form connections within one's own type exceeds the group's proportion in the population. Formally, inbreeding homophily present when $H_g > w_g$, where w_g represents the proportion of group g in the population. This implies that agents are more likely to form friendships within their own group than would be expected under random mixing.

The homophily index has been widely used in previous studies on social networks. In this chapter, I introduce the concept of *weighted homophily*, which accounts for the homophily level within groups that are further subdivided into subgroups. A detailed explanation of weighted homophily is provided in Section 2.4.3.

2.4 Model

This section presents a mechanism of network formation, primarily based on the framework introduced by Currarini et al. (2009)(CJP). In line with their study, the model assumes that agents derive higher payoffs from forming connections with others who share the same racial group, with diminishing marginal returns as the number of friendships increases. Agents participate in a matching process each period, paying a fixed cost to establish new connections, and they exit the process when the marginal payoff from forming a connection falls below this cost.

Building on the foundation of CJP, this model introduces a key enhancement: agents possess two-dimensional characteristics. Specifically, in addition to racial group identity, agents are characterized by participation in cultural activities, such as shared personal hobbies. This modification allows for an investigation into whether matching biased on one dimension (e.g., cultural activities) can mitigate the homophily index on the other dimension (e.g., racial group).

In the model, each agent derives utility from forming connections, but the utility function exhibits diminishing returns as the number of friendships increases. During each period, agents pay a fixed cost to build a new connection and remain in the matching process until the marginal payoff turns negative. A key assumption is that agents prefer connections with those similar to themselves, gaining more utility from forming same-characteristic connections, whether based on race or participation in the same cultural activities. Consequently, agents in the majority group tend to stay longer in the matching process, building more connections by meeting same-type agents with a higher probability.

This model incorporates a biased matching process, where agents with the same personal hobbies—interpreted as participating in the same cultural activities—meet each other with a probability higher than their fraction in the overall population. The agents aim to maximize their individual payoffs, and their strategies collectively constitute the network formation game.

I consider two models to represent agents' biased preferences for same-type connections. The *baseline model* assumes agents have a preference bias only toward within-race connections, gaining higher utility when connecting with others from their racial group. This simplified model allows us to derive key results, though it does not capture all empirical observations. To address this limitation, I introduce the *advanced model*, where agents exhibit preference biases toward both race and hobbies.

In this model, each agent is characterized by two dimensions. The first dimension represents an inherent trait, such as race, language, or religion, while the second dimension corresponds to an optional characteristic, such as participation in cultural activities or personal hobbies. For clarity, the first dimension is referred to as "race," and the second as "hobbies." The second dimension is modeled as a binary variable H, where $H = 1$ indicates that an agent participates in a cultural activity, and $H = 0$ otherwise.

In the network formation process, there exist matching bias in the second dimension (i.e., club members to members) and biased preference in the first dimension (i.e., white to white). In the advanced model, the biased preference extended on both dimensions.

The network formation process incorporates two forms of bias: *matching bias* in the second dimension, where agents with the same traits are more likely to form connections (e.g., club members connecting with other members), and *biased preference* in the first dimension, where individuals receive a higher payoff for connecting with others of the same racial group (e.g., White-to-White connections). In an extended version of the model, biased preferences apply to both dimensions.

The society consists of two racial groups, denoted as i and racial group j. where each racial group is subdivided into two subgroups: those who join a club ($H = 1$) and those who do not ($H = 0$). As a result, there are four distinct agent types in the population, represented as $g \in G = \{i0, i1, j0, j1\}$. To simplify notation, the racial subgroups are defined as $i = \{i0, i1\}$ and j is $j = \{j0, j1\}$ corresponding to agents of racial types i and type j, respectively. Similarly, the subgroup of agents who participate in cultural activities ($H = 1$) is denoted as $\mathbf{1} = \{i1, j1\}$, hose who do not participate ($H = 0$) are denoted as $\mathbf{0} = \{i0, j0\}$.

A more general framework, where agents can belong to a finite set of types $T = \{1, 2..., K\}$ is introduced in Section 2.4.6.

Let h_i denote the fraction of agents with the hobby in group i,

$$h_i = \frac{N_{i1}}{N_i} = \frac{N_{i1}}{N_{i0} + N_{i1}},$$

where N_{i1} is the number of type-$i1$ agents and N_{i0} is the number of type-$i0$ agents. Similarly, h_j represents the fraction of agents with the hobby in group j, such that $h_j = N_{j1}/\left(N_{j0} + N_{j1}\right)$, where N_{j1} and N_{j0} are the number of agents of type $j1$ and $j0$, respectively. The population proportions of the four agent types are denoted as

$$W_g = \left(w_{i0}, w_{i1}, w_{j0}, w_{j1}\right) = \left((1 - h_i)w_i, h_i w_i, h_j w_j, (1 - h_j)w_j\right),$$

where w_i is the proportion of agents in racial group i, and $w_j = 1 - w_i$.

In the following section, I introduce a baseline model in which agents exhibit a preference for connections within their own racial groups. In the extended model, I add the assumption that type-1 agents also prefer to form connections with others of the same type due to shared personal hobbies. The primary distinction between the two models lies in the differences in the agents' payoff functions.

2.4.1 Baseline: Biased Preference on Racial Groups

Setting of the Baseline Model

In this section, I begin by defining the agents' payoffs. By solving the agents' optimization problem, I derive the number of friendships, $t_g(q)$, as a function of the matching probability, q. Next, I introduce the matching process and define the matching probability, $q(M)$ based on the the stock of agents of each types M_g. Here, M_g represents the cumulative presence of type g agents in the matching process, determined by the number of periods they participate. Finally, I define the steady-state equilibrium as a triple (t, M, q) where the inflows of agents for each group g matches the outflow.

Agents and Payoffs. The payoff function of an agent of type g, with s_g being the number of same-type friends and d_g being the number of different-type friends, is represented by $U_g(s_g, d_g)$. The biased utility function for an agent of type g is defined as:

$$U_g(s_g, d_g) = (s_g + \gamma d_g)^\alpha,$$

where $\gamma \in (0, 1)$ represents the biased preference, indicating that agents of type i gain $1 - \gamma$

more utility from forming same-type connections than from cross-type connections. The parameter $\alpha \in (0, 1)$ captures the diminishing returns to connections, reflecting the idea that as the number of connections increases, the marginal utility of additional connections decreases.

To ensure that the utility function U is well-defined, I allow agents to form fractional friendship connections, following the assumptions in Currarini et al. (2009) (CJP). Under this formulation, U is a continuous function that is strictly increasing in the number of friendships, with continuous first- and second-order partial derivatives. Furthermore, the utility function exhibits diminishing marginal returns to additional friendships, meaning that the increase in utility decreases as the number of same-type (s_g) and different-type (d_g) friends grows larger.

Diminishing returns to friendships are defined as:

$$s_g U_s(as_g, ad_g) + d_g U_d(as_g, ad_g) < s_g U_s(s_g, d_g) + d_g U_d(s_g, d_g)$$

for all (s_g, d_g) and $a > 1$, where U_s and U_d denote the partial derivatives of U with respect to s_g and d_g, respectively. This condition implies that the more connections an agent has already established, the lower the marginal utility of forming additional connections. In this model, an agent must pay a fixed cost c to form each connection. As a result, when the marginal utility of a new connection falls below the cost, the agent stops forming new connections.

The goal of this study is to analyze the relationship between cultural activities and racial segregation. Thus, we interpret s_g and d_g as the number of connections within and across *racial groups*, respectively. For example, for an agent of type $g = i0$ (a member of racial group i who does not participate in cultural activities), the within-race connections s_g represent connections with other members of *racial group* $i = \{i0, i1\}$; while cross-race connections d_g represent connections with members of *racial group* $j = \cup \{j0, j1\}$.

Matching Process. For agents of type g, there is an inflow of agents of mass N_g entering the matching process per unit of time. For each unit of time spent in the matching pool, an agent forms one new connection and incurs an opportunity cost $c > 0$. Since the utility function U exhibits diminishing returns to friendships, agents will leave the matching process once their expected utility equals the cost of forming a new connection.

Let M_g denote the stock of type-g agents in the matching pool. If agents of type g remain in the matching process for t_g units of time, the steady-state stock of type-g agents in the matching pool is given by $M_g = t_g N_g$. Therefore, the relative stock of agents of a given type increases either through a higher inflow per unit of time or by remaining in the matching process longer.

The mix of friendships that agents form depends on the stocks M_g of each type $g \in G$. These probabilities are endogenous to the equilibrium and determined by the steady-state stocks of agents in the matching process. The matching process is described by a function $f : \mathbb{R}^4 \to [0,1]^{4\times 4}$, where $\mathbf{q}^{4\times 4} = f\left(M_{i0}, M_{i1}, M_{j0}, M_{j1}\right)$ represents the matching probabilities as a function of the stocks of agents in the matching pool, and each element $q_{g,g'}$ represents the probability that an agent of type g meets an agent of type g' per unit of time, with $g, g' \in G = \{i0, i1, j0, j1\}$. For an agent of type g, the matching probabilities are given by a row in the matrix $\mathbf{q}_g = (q_{g,i0}, q_{g,i1}, q_{g,j0}, q_{g,j1})$, and the row sum $\sum\limits_{g' \in G} q_{g,g'} = 1$.

Given the matching probabilities $\mathbf{q}$, the utility function of an agent of type g can be expressed as:

$$U_g = (s_g + \gamma d_g)^{\alpha} = (q_{g,s} t_g + \gamma q_{g,d} t_g)^{\alpha}. \tag{2.1}$$

Here, $q_{g,s}$ represents the probability of forming a *same-race* connection, while $q_{g,d}$ is the probability of forming a *cross-race* connection for an agent of type g. For instance, for an agent of type $g \in \{i0, i1\}$, the same-race group is $s = \{i0 \cup i1\}$ and the cross-race group is $d = \{j0 \cup j1\}$.

In steady-state equilibrium, the inflow of agents equals the outflow, and the stocks of each agent type remain constant. Consequently, an agent of type g faces the same matching probabilities in each period and forms one new connection per period. Thus, the number of within-race connections, s_g, determined by the product of the matching probability for same-race agents $q_{g,s}$ and the total time spent in the matching process: $s_g = q_{g,s} t_g$. Similarly, the number of cross-race connections is $d_g = q_{g,d} t_g = (1 - q_{g,s}) t_g$. Finally, since an agent forms one new connection per period, the total number of connections is equal to the total time in the matching process, i.e., $s_g + d_g = t_g$.

During the matching process, each agent forms exactly one new friendship per period, regardless of the type of agent they meet. Agents continue to form additional friendships in each period until the marginal utility of forming another connection falls below the fixed cost, c. Consequently, agents of type g chooses the optimal number of periods to stay in the matching process by solving the following optimization problem:

$$\max_{t_g} U\left(t_g, q_{g,s}, q_{g,d}\right) - ct_g \tag{2.2}$$

For an agent of type $g \in G$, the optimal number of connections t_g, given the matching probability $\mathbf{q}_g$, satisfies the first-order condition:

$$q_{g,s}U_s\left(s_g, d_g\right) + q_{g,d}U_d\left(s_g, d_g\right) = c$$

Solving this equation, we derive the number of connections for agents of type g, t_g, as a function of their matching probability $\mathbf{q}_g$:

$$t_g(\mathbf{q}_g) = (\frac{\alpha}{c})^{\frac{1}{1-\alpha}}(q_{g,s} + \gamma q_{g,d})^{\frac{\alpha}{1-\alpha}} \tag{2.3}$$

From (2.3) and the utility function (2.1), it follows that the utility of a type-g agent increases linearly with the total number of connections they form:

$$U_g = \frac{c}{\alpha} \cdot t_g.$$

Consequently, the agent's total net payoff, accounting for the cost of forming all connections, is given by:

$$U_g - ct_g = \frac{c(1-\alpha)}{\alpha} \cdot t_g.$$

This result indicates that, holding α and c fixed, the overall payoff from friendship connections increases with the total number of connections, suggesting that agents derive higher net benefits from maintaining a larger social network.

Metrics of Matching Probabilities. The model introduces a matching bias for agents who participate in cultural activities, resulting in a higher probability of meeting one another. This bias depends only on participation status and is independent of racial identity. Consequently, agents with the same participation status (either participants or non-participants) share the same matching probabilities, regardless of race. As a result, the full matching matrix $\mathbf{q}^{4\times 4}$ can be reduced to a simplified $\mathbf{q}^{2\times 4}$ matrix. The first row represents the meeting probabilities for non-participants (type **0**), and the second row corresponds to the meeting probabilities for participants (type **1**).

The matching process introduces a bias based on agents' participation status in cultural activities, represented by the probabilities $q_{1,1}$, $q_{1,0}$, $q_{0,1}$, and $q_{0,0}$. For instance, $q_{1,0}$ represents the probability that an agent who participates in the activity meets an agent who does not. In the biased matching process, agents who participate in the activity meet each other with a probability $q_{1,1}$ exceeds their relative proportion in society. The probability $q_{1,1}$ increases with the strength of matching bias, while the other three matching probabilities adjust accordingly.

The matching probabilities are defined as:

$$\begin{aligned} q_{1,1} &= \left(\tfrac{M_1}{M_1+M_0}\right)^b, \\ q_{1,0} &= 1 - q_{1,1}, \\ q_{0,1} &= \tfrac{M_1}{M_0} q_{1,0}, \\ q_{0,0} &= 1 - q_{0,1}, \end{aligned} \tag{2.4}$$

where $b \in [0,1]$ represents the strength of matching bias. A lower value of b indicates a stronger bias, meaning that agents are more likely to meet others with the same participation status. Specifically, when $b = 1$, the matching is *unbiased*, while when $b = 0$, the matching process is *extremely biased*. The implications of these two extreme cases are discussed in Section 2.4.4.

In particular, the equation $q_{0,1} = \frac{M_1}{M_0} q_{1,0}$ in (2.4) follows from the symmetry of the matching

Table 2.1: Matching Probability for Different Types of Agents

A. *Matching Bias On The Dimension Of Hobbies*

	Meeting Type 0	Meeting Type 1
Agents of Type 0	$q_{0,0}$	$q_{0,1}$
Agents of Type 1	$q_{1,0}$	$q_{1,1}$

B. *Matching Probability Is Unbiased On Racial Groups*

	Meeting Type 0		Meeting Type 1	
	i_0	j_0	i_1	j_1
Agents of Type 0	$\frac{M_{i0}}{M_0}q_{0,0}$	$\frac{M_{j0}}{M_0}q_{0,0}$	$\frac{M_{i1}}{M_1}q_{0,1}$	$\frac{M_{j1}}{M_1}q_{0,1}$
Agents of Type 1	$\frac{M_{i0}}{M_0}q_{1,0}$	$\frac{M_{j0}}{M_0}q_{1,0}$	$\frac{M_{i1}}{M_1}q_{1,1}$	$\frac{M_{j1}}{M_1}q_{1,1}$

process: the total number of friendships that type-1 agents form with type-0 agents must equal the total number of friendships that type-0 agents form with type-1 agents. This condition is captured by the equation $M_0 q_{0,1} = M_1 q_{1,0}$.

In this model, the four matching probabilities defined (2.4) are illustrated in Table 2.1, Panel A. Since the model assumes no bias along the racial dimension, the probability of meeting agents from a particular racial group depends solely on their relative proportions within each participation subgroup. For example, given that the probability of agents with hobbies meeting those without hobbies is $q_{1,0}$, the probability of meeting agents of type $i0$ is given by:

$$q_{1,i0} = \frac{M_{i0}}{M_0} q_{1,0},$$

where $\frac{M_{i0}}{M_0}$ denotes the relative stock of type $i0$ agents within subgroup $\mathbf{0}$. These disaggregated probabilities are shown in Table 2.1, Panel B.

The complete set of matching probabilities for the $\mathbf{q}^{2\times4}$ matrix is summarized in Table 2.2. In this representation, the matching probabilities are independent of the agent's racial identity. Therefore, the full matching matrix $\mathbf{q}^{4\times4}$ can be simplified to a $\mathbf{q}^{2\times4}$ matrix.

Table 2.2: Matching Probability Matrix

	Meeting Type 0		Meeting Type 1	
	i_0	j_0	i_1	j_1
Agents of Type 0	$q_{0,i_0} = q_{i_0,i_0}$ $= q_{j_0,i_0}$	$q_{0,j_0} = q_{i_0,j_0}$ $= q_{j_0,j_0}$	$q_{0,i_1} = q_{i_0,i_1}$ $= q_{j_0,i_1}$	$q_{0,j_1} = q_{i_0,j_1}$ $= q_{j_0,j_1}$
Agents of Type 1	$q_{1,i_0} = q_{i_1,i_0}$ $= q_{j_1,i_0}$	$q_{1,j_0} = q_{i_1,j_0}$ $= q_{j_1,j_0}$	$q_{1,i_1} = q_{i_1,i_1}$ $= q_{j_1,i_1}$	$q_{1,j_1} = q_{i_1,j_1}$ $= q_{j_1,j_1}$

Steady-State Equilibrium

A steady-state equilibrium of the system, for a given set of inflows N_g and a utility function U, is defined as a specification of strategies for each type g, where the resulting stocks of agents in the search process lead to matching probabilities $\mathbf{q}_g$ that justify the strategies, and where the outflow of agents equals the inflow. A steady-state equilibrium is represented as a triple $(t_g, M_g, \mathbf{q}_g)$ for agents in each group $g \in G = \{i0.i1, j0, j1\}$ that satisfies the following conditions, and there is at least one g such that $t_g > 0$:

I. Agents optimize their payoff given meeting probabilities: The number of connections t_g solves the utility maximization problem given by (2.1) for each type g, subject to the matching probability $\mathbf{q}_g$.

II. Strategies determine stocks: The stock of type-g agents in the matching pool is determined by $M_g = N_g \cdot t_g$.

III. Stock determine matching probability: $\mathbf{q}^{2\times4} = f\left(M_{i0}, M_{i1}, M_{j0}, M_{j1}\right)$; and they are homogenous with respect to race.

As described in the model, the number of connections formed by type-g agents, t_g, is a function of the matching probability $\mathbf{q}_g$; the matching probability $\mathbf{q}_g$ are functions of the stocks $\boldsymbol{M}^{1\times4}$ in the matching pool; and the stock of type-g agents, M_g, is a function of the number of connections t_g. A steady-state equilibrium is reached when the triple $(t_g, M_g, \mathbf{q}_g)$ for all $g \in G$, satisfy these conditions simultaneously.

Agents from majority racial groups benefit from a *feedback effect*. When an agent belongs to a larger group, they have a higher probability of meeting others from the same racial group, which enables them to form more advantageous connections. As agents from this group remain in the matching pool for longer periods, the stock of type-g agents, M_g, increases in proportion to their number of connections, t_g. This increase in stock further raises the probability of meeting agents from the same group, thereby reinforcing the process. The feedback loop operates as follows: as t_g increases, M_g rises, which in turn increases the matching probability $q_{\cdot,g}$, leading to a further increase t_g. This positive feedback mechanism applies to any $g \in G$, where larger groups inherently gain advantages in forming connections due to their higher initial probability of meeting same-type agents.

As in the CJP model, if agents' utility functions are unbiased between within-race and cross-race connections, then all agents in my model will also spend the same amount of time in the matching pool. This implies that they will form the same number of connections.

Proposition 2.1: *If utility function is type-neutral, such that $\gamma = 1$, then for any matching bias and any steady-state equilibrium, all types of agents form the same total number of friendships.*

If agents' preferences are type-neutral, then the marginal utility of forming a new connection is independent of their type and depends only on how long they have already spent in the matching process. Therefore, all agents, regardless of their type, spend the same amount of time in the matching process and form the same number of friendships. Moreover, the matching probability for each type of friend corresponds to the relative size of that type in the population, such that:

$$q_{g,g'} = w_{g'}$$

for all g and $g' \in G$.

2.4.2 Homophily Index and Weighted Homophily

Matching Probability and Homophily Index

In this model, the homophily index for the four types of agents can be represented using the matching probabilities. As defined in Section 3, the homophily index H_g is given by:

$$H_g = \frac{s_g}{s_g + d_g},$$

where $s_g = q_{g,s}t_g$ and $d_g = (1 - q_{g,s})t_g$ represent the number of within-race and cross-race connections, respectively. Therefore, the homophily index for agents of any group g is equivalent to their matching probability with same-race friends:

$$H_g = q_{g,s}.$$

The specific homophily indices for each group are as follows:

$$\begin{cases} \text{Homophily of type i1} & H_{i1} = q_{1,i}, \\ \text{Homophily of type i0} & H_{i0} = q_{0,i}, \\ \text{Homophily of type j1} & H_{j1} = q_{1,j} = 1 - q_{1,i}, \\ \text{Homophily of type j0} & H_{j0} = q_{0,j} = 1 - q_{0,i}. \end{cases} \tag{2.5}$$

Here, $q_{1,i}$ represents the probability that agents with hobbies (type **1**) meeting agents from group **i**, where $q_{1,i} = q_{1,i0} + q_{1,i1}$; and $q_{0,i}$ is the probability that agents without hobbies (type **0**) meet agents from group **i**, where $q_{0,i} = q_{0,i0} + q_{0,i1}$.

Similarly, the matching probabilities for agents from racial group j are given by the relations $q_{1,j} = 1 - q_{1,i}$ and $q_{0,j} = 1 - q_{0,i}$.

Weighted Homophily

Within this framework, agents are characterized along two dimensions: a primary characteristic, such as racial identity, which is associated with observed patterns of segregation, and a secondary characteristic that differentiates agents within the same primary group, such as participation in cultural activities. This secondary characteristic allows us to identify heterogeneity within each primary group and analyze the variation in friendship connections formed by agents of the same racial background.

For example, some agents from one racial group may participate in a cultural activity that is predominantly associated with another racial group, making them more integrated into that group's social network. In contrast, other agents from the same racial group who do not participate in the activity may remain more socially segregated.

To accurately measure social segregation, it is crucial to distinguish the connection patterns within these subgroups. Ignoring secondary characteristics would obscure the diversity in connection patterns within each primary group, preventing a comprehensive understanding of how individuals build social networks and how participation in activities influences overall integration.

To evaluate social segregation between racial groups, it is essential to assess the overall connection structure for all agents within a racial group. I introduce the *Weighted Homophily Index* to measure the general level of homophily for all agents of type i, which includes both subgroups $i0$ and $i1$.

Definition of Weighted Homophily:

The *Weighted Homophily* index is defined as the proportion of within-type connections formed by all type-i agents relative to the total number of connections formed by them:

$$\begin{aligned} WH_i &\equiv \frac{N_{i0}s_{i0} + N_{i1}s_{i1}}{N_{i0}\left(s_{i0} + d_{i0}\right) + N_{i0}\left(s_{i1} + d_{i1}\right)} \\ &= \frac{w_{i0}s_{i0} + w_{i1}s_{i1}}{w_{i0}t_{i0} + w_{i1}t_{i1}}, \end{aligned}$$

where $w_{i0} = w_i(1 - h_i)$ and $w_{i1} = w_i h_i$ are the relative size of group $i0$ and $i1$, respectively. In this context, type i refers to racial groups. By dividing racial group i into two subgroups, I can assess the proportion of agents in each group who share the same hobby and examine the relationship between racial segregation and the prevalence of cultural activities.

Proposition 2.2: *For any matching process and steady-state equilibrium, if agents of one race-type $h = \{0, 1\}$ are homophilous such that $H_{ih} > w_i$, then the agents of other race j and the same trait h must be heterophilous such that $H_{jh} < w_j$.*

Proof. From (2.4), we know that:

$$H_{jh} = 1 - H_{ih} \text{ and } w_j = 1 - w_i.$$

This implies that if $H_{ih} > w_i$, then

$$H_{jh} = 1 - H_{ih} < 1 - w_i = w_j$$

for any $h = \{0, 1\}$. □

2.4.3 Model Simplification and the Existence of the Steady-State Equilibrium

During the empirical analysis, I found that setting $\alpha = \frac{1}{2}$ allows for flexibility in adjusting γ and b to fit the data. To validate this specification, I estimated the model under two scenarios: one where $\alpha = \frac{1}{2}$ and another where α is treated as a free parameter. The results show that the estimated values for the number of connections are very similar across both cases, indicating that fixing $\alpha = \frac{1}{2}$ does not significantly alter the model's explanatory power. The detailed results of the parameter estimates are provided in Section 2.5.2.

To enhance the tractability of the model, I set the utility curvature parameter $\alpha = \frac{1}{2}$. This specification simplifies the structure of the model while preserving its core characteristics. Setting

$\alpha = \frac{1}{2}$ does not restrict the generality of the results, as it reduces the nonlinearity of the utility function, thereby facilitating both analytical derivations and empirical estimation.

By setting $\alpha = \frac{1}{2}$, the resulting utility function exhibits a linear relationship between the number of friendships t_g and the matching probability vector $\mathbf{q}_g$:

$$t_g = \frac{1}{4c^2}\left(q_{g,s} + \gamma q_{g,d}\right)$$

In this framework, the overall number of friendships in the society is captured by the homogeneous cost c, while the discrepancies between racial groups are explained by the preference bias γ and matching bias b. Increasing the number of parameters can make the model easier to fit to the data, but it also introduces the risk of overfitting.

Given these findings, I adopt the simplified linear model (2.4.3) for the remainder of the analysis. This approach will be used to establish the following theoretical properties and proofs.

Matching probability $q_{1,i}$ and $q_{0,i}$

As described in (2.5), the homophily index H_g for each group $g \in G$ can be expressed in terms of the matching probabilities $q_{1,i}$ and $q_{0,i}$. Given H_g, we can derive the number of friendships $t_g(H_g)$ and the corresponding stocks of agents $M_g(t_g)$.

Based on the matching process, the matching probabilities $q_{1,i}$ and $q_{0,i}$ are defined as:

$$\begin{aligned} q_{1,i} &= \frac{M_{i1}}{M_1} q_{11} + \frac{M_{i0}}{M_0} q_{10}, \\ q_{0,i} &= \frac{M_{i1}}{M_1} q_{01} + \frac{M_{i0}}{M_0} q_{00}, \end{aligned} \tag{2.6}$$

where M_1 and M_0 are the total stocks of participating and non-participating agents, respectively. The homophily index H_g for each group g can then be obtained from (2.5), and the number of friendships is given by:

$$t_g = \frac{1}{4c^2}\left(\gamma + (1-\gamma)\, H_g\right).$$

The total stocks for participants and non-participants are:

$$\begin{aligned} M_1 = M_{i1} + M_{j1} &= N_{i1}t_{i1} + N_{j1}t_{j1}, \\ M_0 = M_{i0} + M_{j0} &= N_{i0}t_{i0} + N_{j0}t_{j0}, \end{aligned}$$

where N_g denotes the population size of group g, and the time spent in the matching process differs by group. Importantly, both M_1 and M_0 depend endogenously on the equilibrium connection intensities t_g.

By substituting these expressions for M_1 and M_0 into (2.6), we obtain the following system:

$$\begin{aligned} q_{1,i} &= f_1(q_{1,i}, q_{0,i}), \\ q_{0,i} &= f_2(q_{1,i}, q_{0,i}). \end{aligned}$$

Existence of the Steady-State Equilibrium

Conditions (I) and (II) of the steady-state equilibrium are satisfied by construction:

(I) t_g solves the utility maximization problem subject to the matching probability $\mathbf{q}_g$, and

(II) the stock of agents is defined as $M_g = N_g \cdot t_g$.

Therefore, to establish the existence of a steady-state equilibrium, it remains to verify that condition (III) holds—namely, that the matching probability matrix satisfies:

$$\mathbf{q}^{2\times 4} = f\left(M_{i0}, M_{i1}, M_{j0}, M_{j1}\right).$$

Given the matching probabilities $q_{1,i}$ and $q_{0,i}$, the corresponding number of friendships $t_g(q_{g,i})$ and stocks $M_g(q_{1,i}, q_{0,i})$ are are determined endogenously. Thus, the existence of a steady-state equilibrium is equivalent to finding a fixed point for the system of equations in (2.6).

Let $K = (0, 1)^2$ denote the two-dimensional space of matching probabilities. Since $(q_{1,i}, q_{0,i}) \in K$ is a compact and convex set, and the system (2.6) defines a continuous function $f : K \to K$, by the Brouwer Fixed Point Theorem, there exists at least one fixed point $(q_{1,i}, q_{0,i})$ such that

$$f(q_{1,i}, q_{0,i}) = \begin{bmatrix} f_1(q_{1,i}, q_{0,i}) \\ f_2(q_{1,i}, q_{0,i}) \end{bmatrix}.$$

This fixed point guarantees the existence of a steady-state equilibrium under the given conditions.

2.4.4 Results of the Baseline Model

In this section, I first present the matching probabilities for two special cases: the *unbiased* matching process and the *extremely biased* matching process, denoted by q^{ub} and q^{eb}, respectively. Then I describe how matching probabilities vary with the matching bias b and the relative popularity ratio r.

Special Case: Unbiased Matching ($b = 1$)

The first case is the unbiased matching process. Under the condition $b = 1$, the matching probability becomes:

$$q_{1,1} = \left(\frac{M_1}{M_1 + M_0} \right)^b = \frac{M_1}{M_1 + M_0}.$$

This means that the matching probabilities are proportional to the relative stocks of agents in the society. From the general matching process described in (2.4), we have:

$$q_{g,g'} = \frac{M_{g'}}{\sum\limits_{g' \in G} M_{g'}}$$

Table 2.3: Unbiased Matching for $b = 1$

	Meeting Type 0		Meeting Type 1	
	i_0	j_0	i_1	j_1
Agents of Type 0	$\frac{M_{i0}}{\sum M_g}$	$\frac{M_{j0}}{\sum M_g}$	$\frac{M_{i1}}{\sum M_g}$	$\frac{M_{j1}}{\sum M_g}$
Agents of Type 1	$\frac{M_{i0}}{\sum M_g}$	$\frac{M_{j0}}{\sum M_g}$	$\frac{M_{i1}}{\sum M_g}$	$\frac{M_{j1}}{\sum M_g}$

for all $g' \in G$, implying that the matching probability for any other group is directly proportional to its relative stock in the matching pool. In this scenario, the matching process is unbiased, and all agents share the same matching probability. Table 2.3 illustrates the matching probabilities in this case.

Proposition 2.3: *In the unbiased matching process ($b = 1$), the meeting probabilities for all four types of agents are equal and correspond to their relative stocks in society. Consequently, the homophily index for agents within the same racial group is the same, i.e., $H_{i0} = H_{i1}$ and $H_{j0} = H_{j1}$. Additionally, agents from the same racial group form the same number of connections, i.e., $t_{i0} = t_{i1}$ and $t_{j0} = t_{j1}$.*

When $b = 1$, the equilibrium meeting probabilities for an agent of racial group i are given by:

$$q^*_{1,i} = q^*_{0,i} = \frac{M_{i0} + M_{i1}}{\sum M_g}.$$

Since $M_g = w_g t_g$ and by substituting t_g by:

$$t_{i0} = t_{i1} = \frac{1}{4c^2}\left(\gamma + (1-\gamma)q_{\cdot,i}\right).$$

We can derive the relationship between meeting probabilities as:

$$q_{\cdot,i} = \frac{w_i\left[\gamma + (1-\gamma)q_{\cdot,i}\right]}{w_i\left[\gamma + (1-\gamma)q_{\cdot,i}\right] + (1-w_i)\left[1 - (1-\gamma)q_{\cdot,i}\right]}.$$

Solving this equation, the unique solution for $q_{\cdot,i} \in [0, 1]$ is:

$$q^{*ub}_{1,i} = q^{*ub}_{0,i} \quad = \quad \frac{2w_i(1-\gamma) - 1 + \sqrt{(1-2w_i)^2 + 4\gamma^2 w_i(1-w_i)}}{2(1-\gamma)(2w_i - 1)}. \tag{2.7}$$

The corresponding meeting probabilities for agents in racial group j are given by:

$$q^{*ub}_{1,j} = q^{*ub}_{0,j} = 1 - q^*_{0,i}.$$

Finally, the optimal number of connections for each group are:

$$\begin{aligned} t^*_{i0} &= t^*_{i1} = \frac{1}{4c^2}\left(\gamma + (1-\gamma)q^*_{\cdot,i}\right), \\ t^*_{j0} &= t^*_{j1} = \frac{1}{4c^2}\left(1 + \gamma - (1-\gamma)q^*_{\cdot,i}\right). \end{aligned} \tag{2.8}$$

Claim 1: *In an unbiased matching process, the majority group tends to exhibit inbreeding homophily , while the minority group exhibits heterophily.*

When $w_i > w_j$, compared to agents of type j, agents of type i have a higher probability of meeting same-type friends and thus form more connections. As a result of this feedback effect, the matching probability for type-i agent in the steady-state equilibrium exceeds the relative size of their group, i.e. $q_{\cdot,i} > w_i$. In contrast, for group j, the matching probability satisfies $q_{\cdot,j} > w_j$.

Special Case: Extreme Matching Bias ($b = 0$)

When $b = 0$, the system exhibits extreme matching bias, where agents with personal hobbies meet only others who share the same hobbies, with no cross-type connections between type-1 and type-0. From (2.4), we obtain:

$$q_{1,1} = (\frac{M_1}{M_1 + M_0})^b = 1.$$

In this scenario, type-1 agents only interact with other type-1 agents, and type-0 agents only interact with other type-0 agents. The matching probabilities for this case are given in Table 2.3.

Table 2.4: Extremely Biased Matching for $b = 0$

	Meeting Type 0		Meeting Type 1	
	i_0	j_0	i_1	j_1
Agents of Type 0	$\frac{M_{i0}}{\sum M_0}$	$\frac{M_{j0}}{\sum M_0}$	0	0
Agents of Type 1	0	0	$\frac{M_{i1}}{\sum M_1}$	$\frac{M_{j1}}{\sum M_1}$

1. *When* $h_i = h_j$:

   The matching probabilities simplify to:

   $$q^*_{0,i} = q^*_{1,i} = \frac{M_{i0}}{M_0} = \frac{M_{i1}}{M_1}.$$

   Following the same steps as in the unbiased matching process ($b = 1$), we obtain the steady-state matching probabilities are the same with those under unbiased matching in (2.7):

   $$q^*_{\cdot,i} = q^{*ub}_{\cdot,i}.$$

   This indicates that the steady-state matching probabilities is independent of h_i nor h_j.

2. *If* $h_i \neq h_j$,

   Then the matching probabilities are given by

   $$q_{0,i} = \frac{M_{i0}}{M_0}, \quad \text{and} \quad q_{1,i} = \frac{M_{i1}}{M_1}.$$

   The steady-state matching probabilities depend on the *relative popularity ratio*,

   $$r_1^{ij} \equiv \frac{h_i}{h_j}.$$

For type-$i1$ agents, the homophily index:

$$q^{*eb}_{1,i} = q\left(\gamma, w_i, r_h\right) = \frac{(1-2\gamma)w_i r_h - (1-w_i) + \sqrt{[w_i r_h - (1-w_i)]^2 + 4\gamma^2 w_i(1-w_i) r_h}}{2(1-\gamma)[w_i r_h - (1-w_i)]}. \tag{2.9}$$

Similarly, for type-$i0$ agents, define the ratio

$$r^{ij}_0 \equiv \frac{1-h_i}{1-h_j},$$

and the homophily index is given by:

$$q^{*eb}_{0,i} = q(\gamma, w_i, r_{1-h}).$$

This function has the same form as the matching probability for type-1 agents, in (2.9), but with r^i_h replacing by r^i_{1-h}. The homophily index for agents in the opposite racial group are $q^*_{1,j} = 1 - q^*_{1,i}$, and $q^*_{0,j} = 1 - q^*_{0,i}$. In addition, the number of connections $t^*_g = \frac{1}{4c^2}\left(\gamma + (1-\gamma)q^*_{g,s}\right)$.

When $h_i = h_j$, the popularity ratio $r_h = r_{1-h} = 1$. This leads to equal matching probabilities for type-1 and type-0 agents:

$$q^{*eb}_{0,i} = q^{*eb}_{1,i} = q(\gamma, w_i, 1).$$

This represents a special case where the matching probabilities are equal, corresponding to the matching probability described by (2.7).

Figure 2.2 provides an example illustrating homophily values under two matching processes: an unbiased matching process ($b = 1$) and an extreme biased matching process ($b = 0$). In this example, $h_i > h_j$, leading to an increase in homophily for group $i1$ and $j0$ as the matching bias intensifies (from $b = 1$ to $b = 0$). Conversely, homophily for groups $i0$ and $j1$ decreases with increasing bias. In the graphs, values above the 45-degree line indicate inbreeding homophily, where the matching probability with same-race agents is higher than their proportion in the population. In addition, note that the homophily values are the same for all four groups under the unbiased matching process.

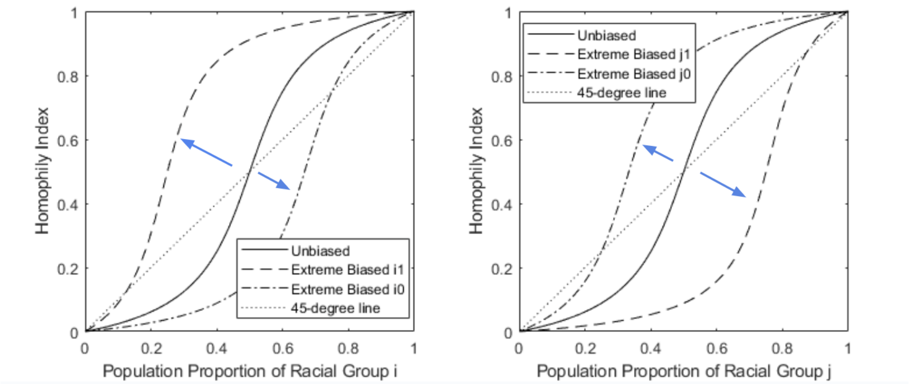

Figure 2.2: Homophily Index under Unbiased and Extreme Biased Matching Conditions

Note: These two graphs show the values of homophily under the unbiased matching process and the extreme biased matching process. The graph on the left displays homophily for racial group i, while the graph on the right shows the homophily index for group j. In this example, the parameters are set as $\alpha = 0.5$, $\gamma = 0.2$, $h_i = 0.6$, and $h_j = 0.2$.

Table 2.5: Biased Matching Process When $0 < b < 1$

	Meeting Type 0	Meeting Type 1
Agents of Type 0	$q_{0,0} > \frac{M_0}{M_1+M_0}$	$q_{0,1} > \frac{M_0}{M_1+M_0}$
Agents of Type 1	$q_{1,0} < \frac{M_0}{M_1+M_0}$	$q_{1,1} > \frac{M_1}{M_1+M_0}$

Biased Matching Process: $0 < b < 1$

In the general matching process where the matching bias parameter b lies strictly between 0 and 1, the matching probabilities exhibit distinctive characteristics. As shown in Table 2.5, the probabilities $q_{1,1}$ and $q_{0,0}$—which represent the likelihood that agents meet others with the same participation status—exceed the relative shares of participants and non-participants in the population. This reflects the presence of positive matching bias favoring within-type interactions.

Proposition 2.4: *If the popularity level of the cultural activity is the same across different racial groups, that is , $h_i = h_j$, then:*

1. *All agents have the same matching probabilities: $\mathbf{q}_{0,g} = \mathbf{q}_{1,g}$ for any $g \in G$. These probabilities are identical to those under the unbiased matching process.*

2. *Agents within the same racial group have the same number of connections: $\mathrm{t}_{i0} = \mathrm{t}_{i1}$ for any*

racial group i.

Proof. Suppose that the matching probability to same-race agents are the same for type-$i1$ and type-$i0$ agents, $q_{1,i} = q_{0,i}$. From (2.3), if the matching probabilities are equal, agents in these two subgroups will remain in the matching process for the same number of periods, implying $t_{i0} = t_{i1}$.

Under the condition $h_i = h_j$this equality of connections ensures that the ratio of stocks within and outside the cultural activity is balanced, i.e., $\frac{M_{i1}}{M_1} = \frac{M_{i0}}{M_0}$. Using equation (2.6), the homophily values for these two groups are equal and correspond to their relative stocks:

$$q_{1,i} = q_{0,i} = \frac{M_{i1}}{M_1} = \frac{M_{i0}}{M_0}.$$

Therefore, $q_{1,i} = q_{0,i}$ is an equilibrium.

□

By following the same steps used to solve q in the scenario $b = 0$ and $h_i = h_j$, we obtain the same result as the closed-form solution (2.7) derived under the unbiased matching process,

$$q^{*}_{\cdot,s} = q^{*ub}_{\cdot,s} \qquad \text{if } h_i = h_j, \text{for any } h_i,\ h_j, \text{ and } b \in (0,1)$$

which means, the homophily indices $H_g$are equivalent to those under unbiased matching process, as long as $h_i = h_j$.

In general cases where matching bias exists, agents who join cultural activities and those who do not face different types of partners and have different matching probabilities for forming new connections. However, when the cultural activity is equally prevalent among different racial groups ($h_i = h_j$), agents from different races join the activity in the same proportion. In a steady-state equilibrium, the inflow and outflow are constant in each period, so the relative stocks of agents from each race and the composition of races in the search pool are the same both within the club and outside it.

In this special scenario, agents inside and outside the club face the same matching probabilities, implying that the number of friendship connections depends only on agents' racial types and not on whether they join the club. Moreover, this results in the matching process yielding the same steady-state equilibrium as the unbiased matching process, where the matching probabilities and the number of connections depend solely on the relative sizes of each race, not on the popularity of the cultural activities.

Although we cannot obtain closed-form solutions when $0 < b < 1$ and $h_i \neq h_j$, certain properties of the matching process still hold.

Proposition 2.5: *If the fraction of agents sharing the same hobby is greater in racial group i than in racial group j (i.e., $h_i > h_j$), then, in the steady-state equilibrium:*

1. *Agents of type i club participants are more homophilous and have more friendship connections than type-i non-participants. Specifically, $H_{i1} > H_{i0} \iff q_{1,i} > q_{0,i}$ and $t_{i1} > t_{i0}$.*
2. *In contrast, in racial group j, club participants are less homophilous and have fewer friendship connections than non-participants, i.e., $H_{j1} < H_{j0} \iff q_{1,j} < q_{0,j}$ and $t_{j1} < t_{j0}$.*

The proof of this proposition is provided in the Appendix.

From the expressions for $q_{1,i}$ and $q_{0,i}$ (see 2.6), we observe that the matching probabilities for groups $i1$ and $i0$ (denoted by$H_{i1} = q_{1,i}$ and $H_{i0} = q_{0,i}$) represent weighted averages of the relative stocks. In the context where cultural activities are more popular among group i than group j, the relative proportion of students from group i, compared to group j, is higher within the club than outside the club. Consequently, students from group i who choose to participate (type $i1$) are more likely to form friendships with same-race agents than those choose not to participate (type $i0$).

Furthermore, the number of friendship connections is an increasing function of the homophily index. Proposition 2.5 also implies that when a cultural activity is more popular among group i, students who participate in the activity form more friendship connections than those who do not, i.e., $t_{i1} > t_{i0}$. Additionally, students of type $i1$ experience higher payoffs compared to those of type $i0$.

Proposition 2.6: *The homophily index is a monotonic function of the matching bias b and reaches its upper or lower bound under the extreme matching bias scenario, where $b = 0$.*

1. *When $h_i > h_j$:*

   As the matching bias increases, homophily for groups $i1$ and $j0$ increases; while homophily for groups $i0$ and $j1$ decreases.

2. *When $h_i < h_j$:*

   As the matching bias increases, homophily for groups $i1$ and $j0$ decreases; while homophily for groups $i0$ and $j1$ increases.

Proof: Detailed proofs are provided in the Appendix.

Upper Bound and Lower Bound: By Proposition 2.6, the matching probability for any group g is bounded by the matching probabilities in the two special cases: unbiased and extremely biased matching process. When $0 < b < 1$, the steady-state matching probabilities for any group g , denoted q_g^*, lie between those of the unbiased and the extremely biased process, for any given values of h_i and h_j,.

For instance, if $h_i > h_j$, we define :

1. Upper bound: $q_{1,i}^{*\text{eb}} = q(\gamma, w_i, r_h)$, which is the steady-state matching probabilities for group $i1$ in the extremely biased case.

2. Lower bound: $q_{0,i}^{*\text{eb}} = q(\gamma, w_i, r_{1-h})$ which is the steady-state matching probabilities for group $i0$ in the extremely biased case.

where the function $q(\cdot)$ represents the matching probability in the case of extreme bias, as specified by (2.9) in Section 4.4.2.

The matching probability $q_{0,i}$ and $q_{1,i}$ satisfy the inequality:

$$q_{0,i}^{*\text{eb}} < q_{0,i}^{*} < q_{i}^{*\text{ub}} < q_{1,i}^{*} < q_{1,i}^{*\text{eb}}.$$

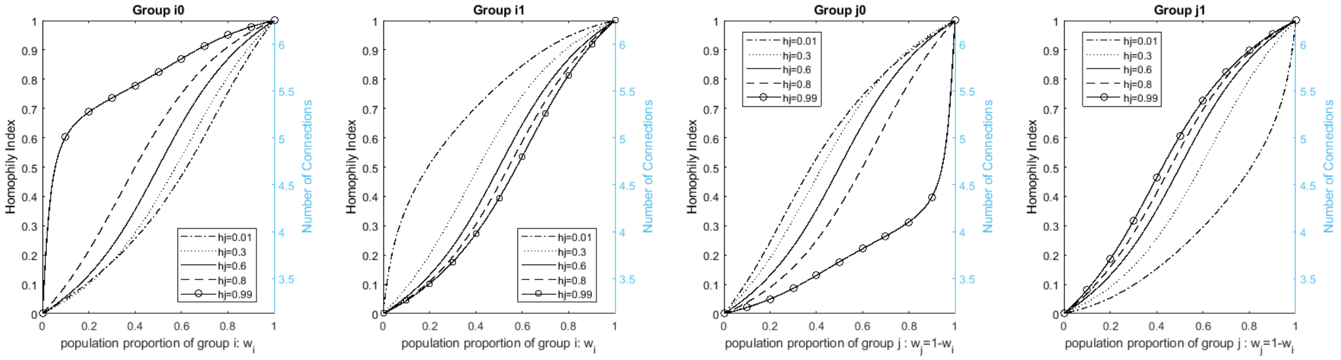

Figure 2.3: Number of connections in the general biased matching process

Note: Matching bias $b = 0.5$, and $h_i = 0.6$. The solid curve in the graph represents the case when $h_i = h_j$, so the steady-state matching probability q_g^* is equivalent to that under the unbiased matching process $q_g^{*\mathrm{ub}}$, for any group g.

Similarly, for racial group j:

$$q_{1j}^{*\mathrm{eb}} < q_{1,j}^* < q_j^{*ub} < q_{0,j}^* < q_{0j}^{*\mathrm{eb}},$$

where $q_{0,i}$ and $q_{1,i}$ correspond to the homophily indices H_{i0} and H_{i1} for racial group i, respectively; while $q_{0,j}$ and $q_{1,j}$ represent homophily index for group j.

If $h_i < h_j$, the inequalities reverse, resulting in

$$q_{0,i}^{*\mathrm{eb}} > q_{0,i}^* > q_i^{*\mathrm{ub}} > q_{1,i}^* > q_{1,i}^{*\mathrm{eb}},$$

and similarly for racial group j:

$$q_{1j}^{*\mathrm{eb}} > q_{1,j}^* > q_j^{*ub} > q_{0,j}^* > q_{0j}^{*\mathrm{eb}}.$$

One example is provided in Figure 2.3, As the proportion h_j increases, number of connections t_{j1} and t_{i0} increase, while t_{j0} and t_{i1} decrease. This demonstrates how changes in the prevalence of cultural activities among different racial groups impact the formation of social connections.

Weighted Homophily

Given the matching probability, we can calculate the weighted homophily index for group i using the following equation:

$$\begin{aligned} WH_i &= \frac{w_{i0}s_{i0} + w_{i1}s_{i1}}{w_{i0}t_{i0} + w_{i1}t_{i1}} \\ &= \frac{w_i(1-h_i)q_{i0,i}t_{i0} + w_i h_i q_{i1,i}t_{i1}}{w_i(1-h_i)t_{i0} + w_i h_i t_{i1}} \\ &= \frac{(1-h_i)q_{i0,i}t_{i0} + h_i q_{i1,i}t_{i1}}{(1-h_i)t_{i0} + h_i t_{i1}}. \end{aligned}$$

Proposition 2.7: *Weighted Homophily Index for both racial group i and j attain their minimum when the participation rate are the same in the two racial groups $h_i = h_j$.*

In the formal mathematical proof, I employ the Jacobian method to assess the effect of participation rates h_i on the homophily indices for the two groups $i1$ and $i0$, by calculating $\frac{dq_{1,i}}{dh_i}$ and $\frac{dq_{0,i}}{dh_i}$. This analysis is only tractable under the condition $h_i = h_j$, where the equations are significantly simplified. When $h_i \neq h_j$, the complexity of the matching probability prevents a closed-form solution for the Jacobian matrix. Consequently, I demonstrate that the weighted homophily index attains a local optimum at $h_i = h_j$, given any matching bias b. Further details are provided in the Appendix.

In addition, I use MATLAB to compute the matching probabilities for various parameter settings, each of which corresponds to a unique steady state. The results consistently show that the weighted homophily indices for both racial groups, i and j achieve their minimum when $h_i = h_j$. An example is illustrated in Figure 2.4, where the weighted homophily indices reach their lowest values at $h_i = h_j = 0.6$ for both racial groups i and j.

To clarify the relationship between h_i and h_j, Figure 2.5 presents the weighted homophily index across the range $h_i, h_j \in (0, 1)$, with other parameters held constant. These results confirm that the weighted homophily index indeed attains a global minimum when $h_i = h_j$.

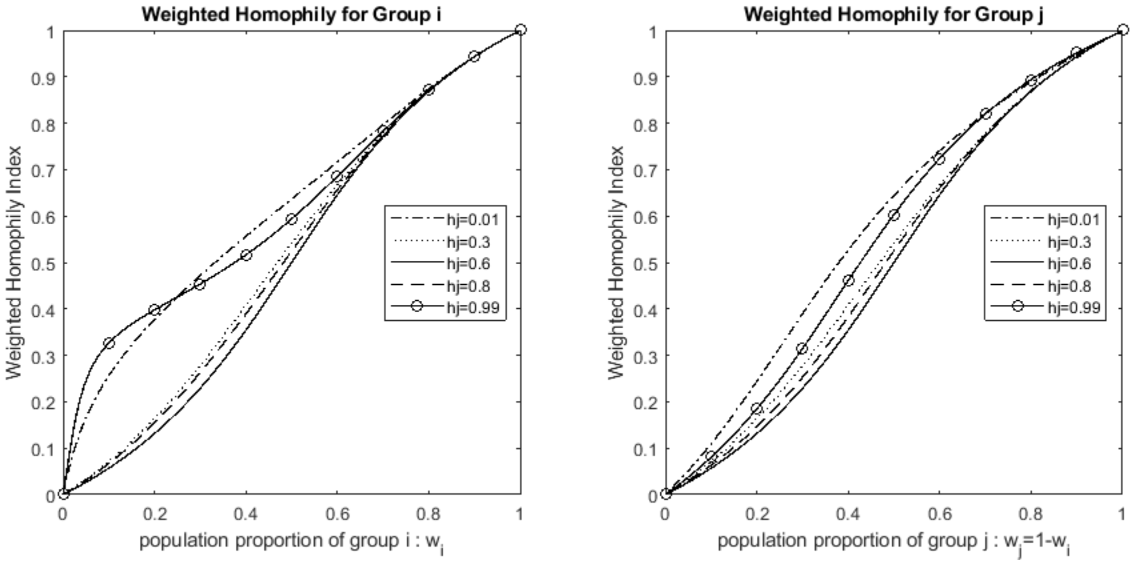

Figure 2.4: Weighted Homophily under $h_i = 0.6$

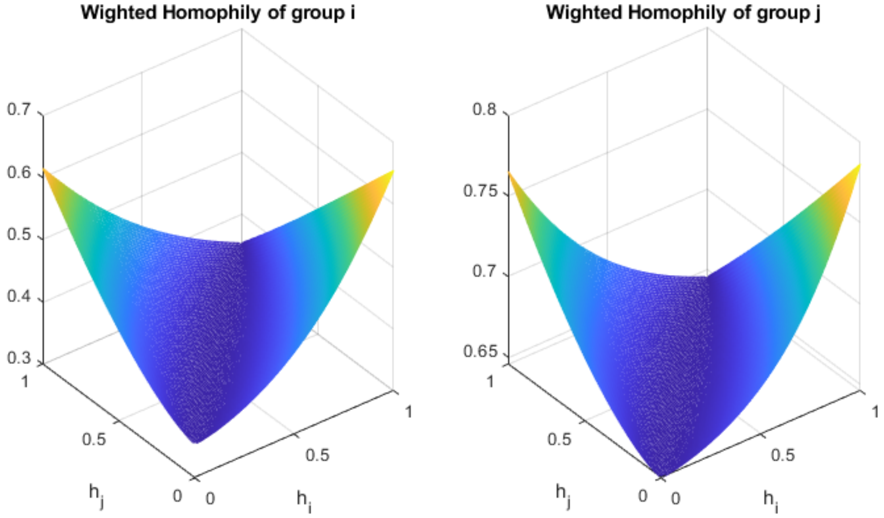

Figure 2.5: Weighted Homophily under $w_i = 0.4$

Note: In this example, the parameters are set as $\alpha = 0.5$, $\gamma = 0.5$, and $w_i = 0.4$

2.4.5 Enhanced Model: Biased Preference on Two Dimensions

I now extend the analysis to networks where agents exhibit preference biases toward both race and hobbies.

Model Setup.

We begin by specifying the utility function with two-dimensional biased preference. For agents who do not participate in the cultural activity ($g \in \{i0, j0\}$), the utility function is:

$$U_{g\in\{i0,j0\}}(s_g, d_g) = (s_g + \gamma d_g)^{\alpha}.$$

For agents who participate in the cultural activity ($g \in \{i1, j1\}$), the utility function incorporates an additional term reflecting their preference for same-hobby connections:

$$U_{g\in\{i1,j1\}}(s_g, d_g) = \left[s_g + \gamma d_g + \sigma(s_{g,1} + d_{g,1})\right]^{\alpha},$$

where $s_{g,1}$ and $d_{g,1}$ are the number of same-hobby connections within the same race and cross races, respectively; $\gamma \in (0, 1)$ denotes the preference bias toward agents from the same racial group, and $\sigma \in (0, \gamma)$ represents the preference bias toward agents who share the same hobby.

Expressing the utility functions in terms of matching probabilities and the number of connections t_g, we have:

for $g \in \{i0, j0\}$:

$$U_g(t_g, q_g) = [q_{0,s}t_g + \gamma q_{0,d}t_g]^{\alpha},$$

and for $g \in \{i1, j1\}$:

$$U_g(t_g, q_g) = [q_{1,s}t_g + \gamma q_{1,d}t_g + \sigma q_{1,1}t_g]^{\alpha}.$$

Here:

$q_{0,s}$ is the probability that an agent not participating in the cultural activity meets a same-race agent not participating;

$q_{0,d}$ is the probability that such an agent meets a cross-race agent not participating;

$q_{1,s}$ is the probability that an agent participating in the cultural activity meets a same-race agent also participating;

$q_{1,d}$ is the probability that such an agent meets a cross-race agent also participating;

$q_{1,h}$ is the probability that an agent participating in the cultural activity meets any other agent participating in the activity (regardless of race).

We then solve each agent's utility maximization problem:

$$\max_{t_g} U(t_g, q_g) - ct_g.$$

Setting the First-Order Condition by equating the marginal utility of building a new connection to the fixed cost c, we derive the optimal number of connections:

for $g \in \{i0, j0\}$:

$$t_g = (\frac{\alpha}{c})^{\frac{1}{1-\alpha}} [q_{0,s} + \gamma q_{0,d}]^{\frac{\alpha}{1-\alpha}},$$

and for $g \in \{i1, j1\}$:

$$t_g = (\frac{\alpha}{c})^{\frac{1}{1-\alpha}} [q_{1,s} + \gamma q_{1,d} + \sigma q_{1,1}]^{\frac{\alpha}{1-\alpha}}. \tag{2.10}$$

Implication of the Enhanced Model: In the baseline model, as stated in Proposition 2.5, if a cultural activity is less popular within an agent's own racial group compared to the other group, then agents who join the club are less likely to meet same-race friends and subsequently form fewer connections than those who choose not to join. Conversely, when the activity is more popular among their own race, agents who choose to join are more likely to meet same-race friends and thus build more connections than non-members.

The advanced model, however, introduces a new trade-off for agents belonging to a racial group in which the cultural activity is relatively less popular. On the one hand, participating in the activity reduces the probability of forming same-race connections, which diminishes the utility derived from within-group friendships. On the other hand, agents gain additional utility from forming connections with others who share the same hobby, regardless of racial identity. Thus, agents from both racial groups may still benefit from participating in the activity and may build more connections overall, even when the activity is less prevalent within their own racial group.

Empirical analysis in section 2.5 indicates that, on average, white students participate in basketball clubs at a lower rate than black students. Nonetheless, white students who are club members tend to build more connections than white students who do not join. In the empirical analysis, I observe that compared to non-members, club members not only form more connections with other members but also build a greater number of total friendships. This suggests that they receive a higher marginal payoff from forming connections with others who share the same hobby. Overall, the enhanced model aligns more closely with the observed data patterns and better captures the nuanced dynamics of network formation driven by both racial homophily and shared hobbies.

2.4.6 General Model with Multiple Racial Types

In a society consisting of more than two racial groups $i \in \{1, 2, ...I\}$, the model setup remains consistent with that of the two-race case. Specifically, the definition of cross-race connections extends to include agents from all other racial groups. The utility function and the optimal number of connections t_g are determined by solving the same utility maximization problem. The stocks of agents are defined as:

$$M_1 = \sum_i M_{i1}, \quad \text{and} \quad M_0 = \sum_i M_{i0}.$$

Since the matching process is unbiased with respect to race, the probability of meeting a same-race agent within the club is given by $\frac{M_{i1}}{M_1}$, and outside the club by $\frac{M_{i0}}{M_0}$.

For a more general model with multiple racial groups, we can formulate a system of equations analogous to (2.6), but extended to accommodate additional dimensions. For instance, if the society consists of three racial groups $i \in \{a, b, c\}$, we need to solve for the matching probabilities $q_{1,a}$, $q_{1,b}$, $q_{0,a}$, and $q_{0,b}$ satisfy the equations. The homophily index for racial group c can be expressed as:

$$q_{1,c} = 1 - q_{1,a} - q_{1,b} M_{i1}, \quad \text{and} \quad q_{0,c} = 1 - q_{0,a} - q_{0,b}.$$

Once the values of these four variables are obtained, we can calculate the number of connections t_g and the stocks of agents M_g for each group g. Thus, the condition for a steady-state equilibrium is to find a solution to the system of equations defined as $f : K \to K$, where K is the space $K = (0, 1)^4$ and $\left(q_{1,a}, q_{0,a}, q_{1,b}, q_{0,b}\right) \in K$. By the Brouwer Fixed Point Theorem, there exists at least one equilibrium satisfying: $f\left(q_{1,a}, q_{0,a}, q_{1,b}, q_{0,b}\right) = \left(q_{1,a}, q_{0,a}, q_{1,b}, q_{0,b}\right)$.

In a society with multiple racial groups, a racial minority group with the same population proportion as in a two-race society would tend to build more connections. To illustrate, consider a scenario where a large majority group is split into two subgroups. The relative size of each subgroup decreases, thereby lowering the probability of agents forming same-race connections and reducing the duration they spend in the matching process. Consequently, agents in the minority group—as the third racial group—benefit from the reduced stocks of competing racial groups, which increases their likelihood of forming same-race connections in the matching pool.

2.5 Data and Empirical results

2.5.1 Data Description

The National Longitudinal Study of Adolescent Health (Add Health) provides a comprehensive dataset that includes detailed social network information for a nationally representative sample of U.S. middle and high school students. In the survey, each student is asked to list up to 10 friends, capturing the structure of friendship networks within their schools. For this chapter, I utilize data

Table 2.6: Data Description

Individual-Level Statistics (63259 Students)					
	White	Black	Hispanic	Asian	other
Number of Students	38378	11001	13892	3467	16414
Proportion in the Sample(%)	46.2	13.2	16.7	4.2	19.7
Number of Connections	7.3	6.2	5.7	6.2	6.1
Avg. Homophily (%)	73.4	61.9	55.7	45.9	–
School-level Statistics (134 schools)					
Average Number of Students	226	22	33.5	3	116
Maximum Number of Students	1420	814	2230	431	584
Average Proportion (%)	55.7	4.3	6.7	0.7	21.0
Maximum Proportion (%)	85.9	77.5	93.2	29.6	42.7
Average Homophily (%)	68.9	42.5	11.5	4.7	–
Maximum Homophily (%)	87.0	82.4	94.9	71.7	–

Notes: Some students identify with more than one racial category. If a student selects Hispanic as one of their racial identities, they are classified as Hispanic, regardless of any additional racial categories selected. For students who report multiple non-Hispanic racial identities, they are categorized under the "Others" group. This classification approach is consistent with the methodology used in previous studies employing the same dataset (e.g., CJP).

from Wave 1, based on the in-school questionnaire, which provides cross-sectional information on social interactions. Since adolescents in middle schools typically have limited opportunities to interact with individuals outside of their school environment, each school can be considered an independent social unit.

For the purpose of this study, student groups are defined by two key characteristics: racial identity and club participation status. Only groups with at least five students are included in the analysis to ensure the robustness of our findings. However, school-level statistics—such as racial proportions and participation rates—are calculated using the entire student body, including smaller groups that do not meet the sample size threshold. This approach results in an analysis sample of 83152 students across 134 schools.

In this analysis, I focus on the participation rate in the basketball club, as it is identified as the most popular club among students based on the survey data. To provide a broader context, Figure

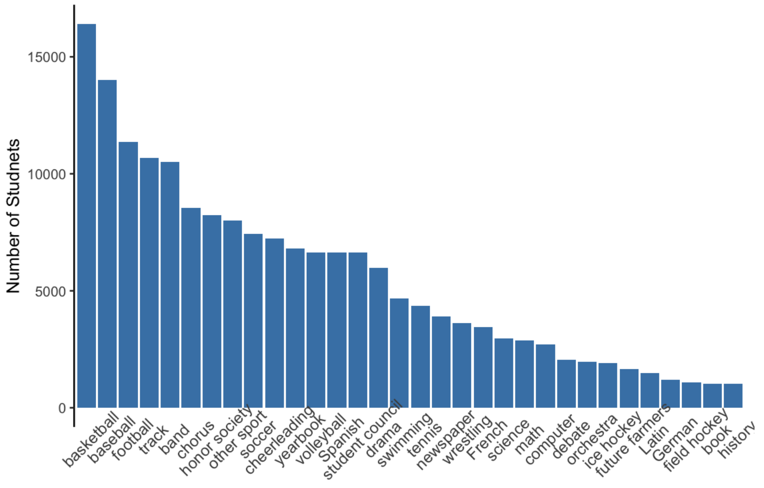

Figure 2.6: Student Participation Rates Across Clubs (Ranked by Popularity)

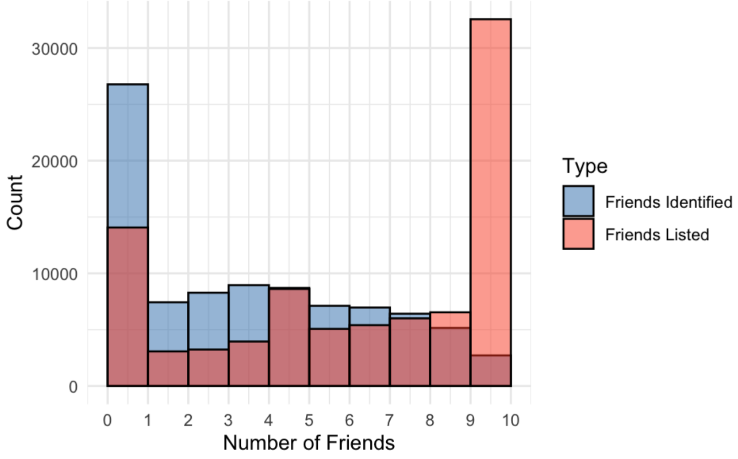

Figure 2.7: Histogram of Listed vs Identified Friends

2.6 displays the distribution of student membership across all 33 clubs included in the questionnaire, ranked from the most to the least popular.

Figure 2.7 presents the distribution of students' reported number of friendships. On average, each student listed 6.5 friends, of which 3.9 could be matched to identifiable peers in the dataset.

Compared to non-participating students, those who join clubs tend to form more friendship connections. Empirical data show that, among White students, the average number of friendships is 7.6 for basketball club members and 7.1 for non-members, and 7.7 for baseball club members compared to 7.1 for non-members. Among Black students, the pattern is similar: basketball club

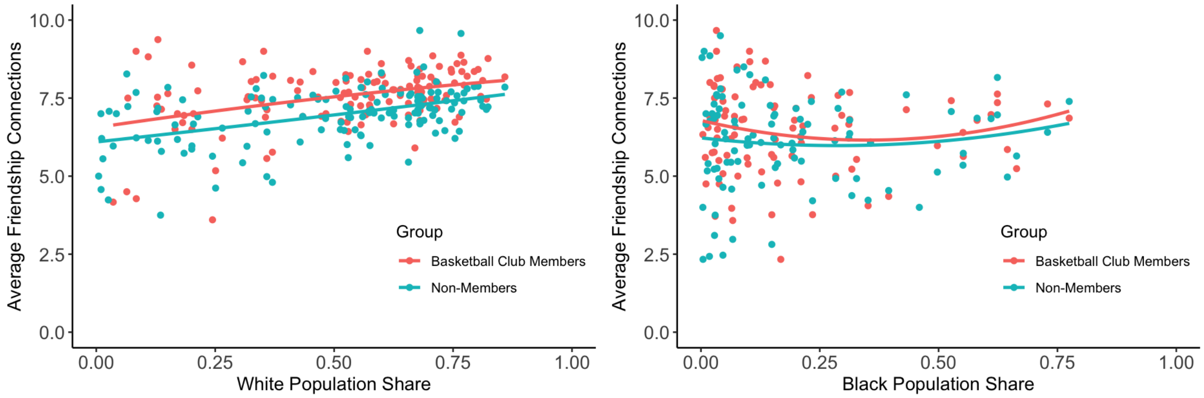

Figure 2.8: Average Number of Friendship Connections by Club Participation Status

members report 6.5 friendships on average, compared to 6.1 for non-members; for baseball, the averages are 6.9 and 6.0, respectively. These patterns are illustrated in Figure 2.8, which presents the average number of friendship connections by club participation status, separately for White and Black students.

This pattern is consistent with the model, in which club members derive higher utility from forming friendships with peers who share similar interests. As a result, they remain engaged in the matching process for a longer duration and ultimately establish more connections.

2.5.2 Fitting the Model

In this model, I assume that each school functions as an isolated society, with the matching process reaching a steady-state equilibrium. Each group within a school is defined by a pair of characteristics: race and club participation status. Race is represented as $i \in \{\text{White, Black, Hispanic, Asian, Others}\}$, and basketball club participation rate is denoted as $h \in \{0, 1\}$. I exclude groups with fewer than five students to ensure the robustness of our estimations. This filtering results in 960 groups, comprising 247 White, 161 Black, 214 Hispanic, 81 Asian, and 257 Other-race groups. Of these, 442 groups consist of club members, while 518 are non-members.

For each group within a school, I calculate the average number of friendship connections and the proportion of friends within the same racial group, corresponding to t_{ih} and $q_{ih,i}$, respectively, as described in the model. Additionally, for club members ($h = 1$), I calculate the average proportion

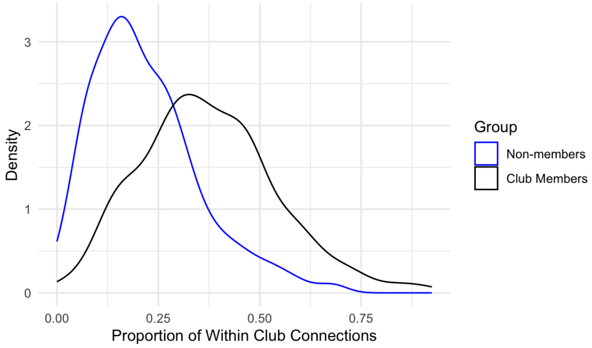

Figure 2.9: Comparison of Club-Member Friendships: Members vs. Non-Members

of friends who are also club members, denoted as $q_{1,1}$ in the model. Figure 2.9 illustrates the density distribution of the proportion of club-member friends, comparing $q_{1,1}$ for club members and $q_{0,1}$ for non-members. As shown in the figure, groups of club members tend to have a higher proportion of club-member friends than non-members. This observation is consistent with the theoretical prediction $q_{1,1} > q_{0,1}$ under biased matching ($b < 1$), thereby providing empirical support for our model.

Estimating the Matching Bias Parameter (b)

To estimate the relative stock of each group in the matching process, I compute the product of the relative group size and the average number of connections for each group. Since the total number of students in a school cancels out during the process, I do not factor in the total population N. Specifically, for club members, the relative stock is given by $M_{i1} = w_i h_i t_{i1}$, where w_i represents the relative size of racial group i, and h_i denotes the club participation rate. For non-members, the relative stock is $M_{i0} = w_i(1 - h_i)t_{01}$.

Then I compute the total stocks for club members and non-members, given by $M_1 = \sum_i M_{i1}$ and $M_0 = \sum_i M_{i0}$, respectively, and the ratio $\frac{M_{i1}}{M_1}$ and $\frac{M_{i0}}{M_0}$. Using these relative stocks and a given value of b, I calculate

$$q_{1,1}(b) = \left(\frac{M_1}{M}\right)^b, \quad \text{and} \quad q_{0,1}(b) = \frac{M_1}{M_0}\left(1 - \left(\frac{M_1}{M}\right)^b\right).$$

By (2.4) and (2.6), I estimate the matching probability to their own racial group $\hat{q}_{i1,i}(b)$ and $\hat{q}_{i0,i}(b)$

for each racial group i. The optimal value of b is the one that minimizes the sum of the squared differences between the estimated value $\hat{q}_{ih,i}(b)$ and the observed value of $q_{ih,i}$:

$$\min_b \left\{ \sum_i (\hat{q}_{i1,i}(b) - q_{i1,i})^2 + \sum_i (\hat{q}_{i0,i}(b) - q_{i0,i})^2 \right\},$$

where

$$\begin{aligned} \hat{q}_{i1,i}(b) &= q_{1,1}(b)\frac{M_{i1}}{M_1} + (1 - q_{1,1}(b))\frac{M_{i0}}{M_0}, \\ \hat{q}_{i0,i}(b) &= q_{0,1}(b)\frac{M_{i1}}{M_1} + (1 - q_{0,1}(b))\frac{M_{i0}}{M_0}. \end{aligned}$$

The fitted value of b is found to be 0.7050.

Estimating Other Parameters

Method 1: Estimating α, γ, σ, and c. Given the observed matching proportions within racial groups $q_{ih,i}$ and within clubs $q_{1,1}$, I estimate the number of connections $\hat{t}_{ih}$ using (2.10). The optimal values of the parameters are those that minimize the sum of squared differences between the estimated $\hat{t}_{ih}$ and the observed t_{ih}. In this step, there are four parameters (α, γ, σ, and c), and the results are sensitive to the initial parameter values. To address this, I use $10,000$ different starting point to find the global optimal value.

Method 2: Fixing $\alpha = 1/2$. As I state in section 4.1, setting $\alpha = 1/2$ does not lead to a loss of generality. In (2.10), the total number of connections across all students is captured by the cost parameter c. Discrepancies between racial groups are explained by the biased preference for same-race connections (γ), while differences between club members and non-members are captured by the preference bias parameter σ. Estimating α may increase the risk of overfitting; therefore, I fix $\alpha = 1/2$ and focus on estimating the optimal values of γ, σ, and c.

When comparing the optimized results of the two methods, I find that the fitted values are nearly identical. Specifically, for the 703 groups excluding the "Other" racial category, the sum of squared

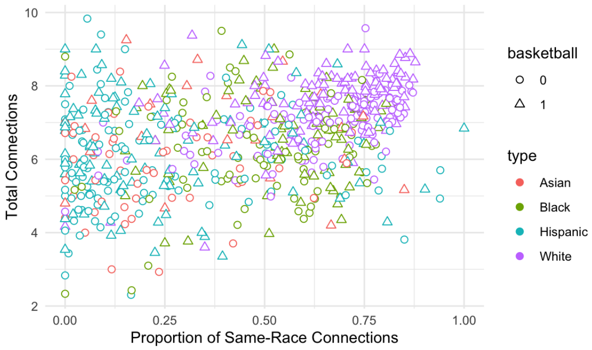

Figure 2.10: School-Level Friendship Trends by Race and Club Participation

differences between the estimates $\hat{t}^1_{ih}$ from Method 1 and $\hat{t}^2_{ih}$ from Method 2 is 0.0012. This indicates that setting $\alpha = 1/2$ does not reduce the precision of the model's fit.

Estimating Different Bias Preferences (γ_i) for Each Racial Group

The analysis reveals substantial differences in the number of connections across racial groups. Figure 2.10 illustrates the relationship between the proportion of same-race connections ($q_{ih,i}$) and the number of connections (t_{ih}), grouped by race and participation status at the school level. The previous estimation methods do not perform well in capturing these variations. Consequently, I estimate the preference bias parameter γ_i separately for each racial group. Some racial groups, for instance, build significantly more connections when students are more likely to meet same-race friends. This suggests a high preference bias toward forming same-race connections, indicating that students derive higher marginal utilities from these connections and therefore remain in the matching process longer. For racial groups where the number of connections does not significantly increase with race-based homophily, I expect a lower value for the preference-bias parameter.

Model Performance and Validation

Figure 2.11 illustrates the density distribution of observed friendship connections compared to the predicted values from the model. Since the two methods (flexible α and fixed α) yield similar fits to the data, it is challenging to differentiate between their respective curves. All three approaches show a limited ability to capture the full range of the observed distribution. Specifically,

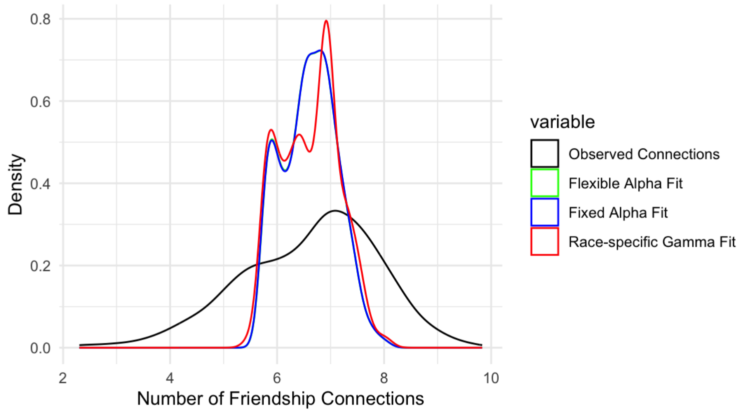

Figure 2.11: Density Distribution of Observed and Fitted Friendship Connections

The density curves corresponding to the flexible and fixed α α methods show a high degree of overlap, indicating that the fitted values are nearly identical. The estimated optimal parameters for the flexible α method are $\alpha = 0.513$, $c = 0.196$, $\gamma = 0.805$, and $\sigma = 0.159$. For the fixed $\alpha = 0.5$ method, the parameter estimates are $c = 0.186$, $\gamma = 0.795$, and $\sigma = 0.168$, suggesting that fixing αat 0.5 does not significantly alter the results. When race-specific preference biases (γ_i) are introduced, the optimal values are estimated as $c = 0.189$, $\gamma_{\text{white}} = 0.955$, $\gamma_{\text{black}} = 0.777$, $\gamma_{\text{Hispanic}} = 0.815$, $\gamma_{\text{Asian}} = 0.817$, and $\sigma = 0.177$, indicating that accounting for racial differences in preferences provides a better fit to the observed data.

Note: A discussion of this limitation and potential extensions that incorporate school-level heterogeneity is provided in the Conclusion.

the observed number of connections is widely spread across the axis, with higher densities at both the lower and upper ends. In contrast, the fitted values for all three methods are concentrated around the center of the distribution, even though the inclusion of different preference biases γ_i for each racial group slightly improves the fit to the observed data.

2.5.3 Linear Regression

The previous analysis shows that setting $\alpha = 1/2$ does not significantly reduce the performance of the model. Thus, I employ a linear regression model to analyze how the composition of connections influences the total number of friendships within each group.

I express (2.10) under the assumption that $\alpha = 1/2$ in the following linear format:

$$t_{sih} = \beta_0 + \boldsymbol{\gamma} \boldsymbol{I}_i q_{sih,i} + \sigma h q_{sih,1} + \epsilon_s + \epsilon_{sih}$$

where t_{sih} represents the average number of connections for group sih (group defined by school s, race i, and club participation status h); $\boldsymbol{\gamma}$ is a 1×4 coefficient vector, where each element γ_i

captures the impact of same-race connections for a specific racial group; $\boldsymbol{I}_i$ is a 4×1 dummy variable indicating race i for each observation; $q_{sih,i}$ denotes the proportion of same-race friends, and $q_{sih,1}$ represents the proportion of friends within the club. The variable h is a dummy indicator of club participation, taking the value 1 for members and 0 otherwise. The error term ϵ_s accounts for school-specific fixed effects, while ϵ_{sih} captures the group-level error.

Each observation in the regression corresponds to the average outcome of a racial group within a particular school. Because these group-level averages are calculated from varying group sizes, it is likely that the precision of the observed averages differs across observations. In particular, larger racial groups within schools tend to yield more reliable average values, whereas smaller groups may exhibit higher sampling variability. This implies potential heteroskedasticity in the error term, which violates the classical OLS assumption of constant variance.

To address this issue, I estimate the model using both Weighted Least Squares (WLS) and Feasible Generalized Least Squares (FGLS) in addition to the baseline OLS. In the WLS specification, I use the group size as the weight, reflecting the idea that larger groups provide more stable group-level averages and should therefore contribute more to the estimation. This approach helps improve the efficiency of the estimated coefficients by accounting for differences in measurement precision across observations.

Furthermore, to capture a more general form of heteroskedasticity that may arise from unobserved school-level factors, I apply a Feasible Generalized Least Squares (FGLS) estimator. Specifically, I allow the error variance to vary across schools—that is, all racial group observations within the same school share a common error variance, but this variance may differ across schools. This setup accommodates arbitrary differences in error variance at the school level while assuming independence across schools.

The results of the OLS, WLS, and FGLS regressions are presented in Table 2.7.

2.5.4 Weighted Homophily

Empirical evidence from this study supports Proposition 2.7: the weighted homophily index decreases as the proportion of students participating in basketball becomes more balanced between the two racial groups. The index reaches its minimum when the fraction of white students playing basketball closely matches the fraction of black students participating in the same activity.

A comparison between the weighted homophily observed in Figure 2.4 and the empirical patterns depicted in Figure 2.12 shows a strong alignment between the theoretical model and the empirical data. This indicates that the model accurately captures the impact of club participation on social network formation and homophily.

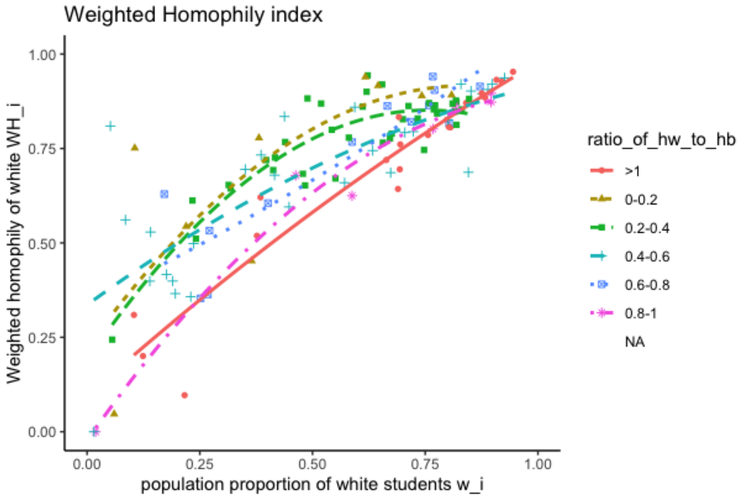

Figure 2.12: Weighted Homophily with The Ratio $\frac{h_i}{h_j}$

2.6 Conclusion

This chapter derives from the observation that individuals who participate in the same cultural activities are more likely to form social connections. Conversely, people from different racial groups often engage in different cultural activities. Understanding whether racial identity influences cultural participation or whether individuals select hobbies based on existing social connections is crucial to explaining patterns of segregation in society.

To address this, I develop a model that introduces bias in the matching process to capture several key empirical observations. My analysis demonstrates that individuals who share the same hobbies are more likely to connect with one another and, on average, have more friendships compared to those who do not participate in such activities. Additionally, the model reveals that segregation among racial groups decreases as the proportion of individuals from different races participating in the clubs becomes more balanced.

The empirical analysis in this chapter utilizes school-level data, which corresponds with the steady-state equilibrium predicted by the model. The next two chapters introduce an individual-level binary choice model, where students decide whether to join a club based on their personal characteristics, school environment, and existing friendship networks.

Discussion

In Subsection 2.5.2, Fitting the Model, one potential improvement suggested by the committee involves incorporating school-level heterogeneity into the model. In the current framework, some schools exhibit systematically higher or lower levels of friendship connections, which may reflect differences in local culture, neighborhood characteristics, or school-specific environments. As a result, the fitted values tend to be more centralized compared to the observed connections. The committee recommended allowing the cost of forming connections to vary across schools. However, there are 134 schools in the dataset, and introducing a separate cost parameter for each school leads to overfitting and a loss of model parsimony.

I attempted to allow different costs by racial group, but since the model already estimates group-specific biased preference parameters (γ) by race, this adjustment did not improve model fit or estimation accuracy.

As a potential future extension, I plan to introduce a school-specific scaling factor, α_i, based on the average number of friendships formed by students in school i. The effective cost of forming a connection in school i would then be modeled as $\alpha_i c$, where c is the baseline cost parameter to be estimated. This approach maintains a single cost parameter while allowing for some variation across schools. However, since the primary objective of this research is to examine how racial composition influences students' decisions and the formation of friendship networks, introducing school-specific parameters may confound the identification of racial preference bias, γ, as variations in α_i could absorb effects driven by racial composition. This trade-off will be addressed in future work.

	Total Number of Connections					
	OLS		WLS		GLS	
	Baseline	Fixed Effect	Baseline	Fixed Effect	Baseline	Fixed Effect
(Intercept)	5.92***		5.92***		5.89***	6.99***
	(0.09)		(0.10)		(0.06)	(0.45)
White Homophily	1.83***	1.74***	1.82***	1.72***	1.93***	1.80***
	(0.16)	(0.15)	(0.14)	(0.15)	(0.10)	(0.09)
Black Homophily	0.51*	0.62**	0.39*	0.50*	0.84***	0.73***
	(0.22)	(0.22)	(0.19)	(0.24)	(0.16)	(0.14)
Hispanic Homophily	0.07	0.33	−0.47**	0.11	0.43	0.60**
	(0.28)	(0.25)	(0.17)	(0.18)	(0.23)	(0.21)
Asian Homophily	0.36	0.67	0.52	0.80	0.84**	0.95***
	(0.38)	(0.43)	(0.30)	(0.46)	(0.30)	(0.26)
Within Club Effect	1.18***	1.14***	1.05***	1.13***	1.11***	1.35***
	(0.20)	(0.17)	(0.18)	(0.12)	(0.06)	(0.01)
R^2	0.25	0.59	0.50	0.81		
Adj. R^2	0.24	0.48	0.49	0.76		
Num. obs.	703	703	703	703	703	703
Num. groups: SCH		134		134		134
Log Likelihood					−953.62	−763.21

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 2.7: Effect of Racial and Club Homophily on Average Connections

Appendix to Chapter 2

Appendix A. Proof of Proposition 2.5

In a matching process, suppose that a triple $\left(t_g^*, M_g^*, \mathbf{q}^*{}_g\right)$ is a steady-state equilibrium, then the matching probabilities $q_{1,i}$ and $q_{0,i}$ satisfy 2.6. We rewrite it as:

$$\begin{aligned} q_{1,i} &= \frac{M_{i0}}{M_0}\left(1 - q_{11}\right) + \frac{M_{i1}}{M_1} q_{11} \\ q_{0,i} &= \frac{M_{i0}}{M_0}\left(1 - q_{01}\right) + \frac{M_{i1}}{M_1} q_{01} \end{aligned}$$

Here, both $q_{1,i}$ and $q_{0,i}$ are weighted averages of the relative stocks $\frac{M_{i0}}{M_0}$ and $\frac{M_{i1}}{M_1}$. Since $q_{11} > q_{01}$ in a biased matching process , one of the following two inequalities must hold:

$$\frac{M_{i0}}{M_0} < q_{0,i} < q_{1,i} < \frac{M_{i1}}{M_1} \quad \text{or} \quad \frac{M_{i1}}{M_1} < q_{1,i} < q_{0,i} < \frac{M_{i0}}{M_0}$$

We aim to show that when $h_i > h_j$, the inequality $q_{1,i} > q_{0,i}$ must hold.

The proof proceeds by contradiction.

Assume that under the condition $h_i > h_j$, the inequality $q_{1,i} < q_{0,i}$ holds.

We express the stocks M_g as functions of t_g, which gives:

$$\begin{aligned} \frac{M_{i1}}{M_1} &= \frac{w_i h_i N t_{i1}}{w_i h_i N t_{i1} + (1 - w_i) h_j N t_{j1}} \\ \frac{M_{i0}}{M_0} &= \frac{w_i (1 - h_i) N t_{i0}}{w_i (1 - h_i) N t_{i0} + (1 - w_i)(1 - h_j) N t_{j0}} \end{aligned}$$

Since $h_i > h_j$, it follows that:

$$
\begin{aligned}
\frac{M_{i1}}{M_1} &> \frac{w_i t_{i1}}{w_i t_{i1} + (1-w_i)t_{j1}} \\
\frac{M_{i0}}{M_0} &< \frac{w_i t_{i0}}{w_i t_{i0} + (1-w_i)t_{j0}}
\end{aligned}
$$

Next, express t_g as a function of the matching probabilities $\mathbf{q}$:

$$
\begin{aligned}
\frac{M_{i1}}{M_1} &> \frac{w_i\left[\gamma + (1-\gamma)q_{1,i}\right]}{w_i\left[\gamma + (1-\gamma)q_{1,i}\right] + (1-w_i)\left[1-(1-\gamma)q_{1,i}\right]} \\
\frac{M_{i0}}{M_0} &< \frac{w_i\left[\gamma + (1-\gamma)q_{0,i}\right]}{w_i\left[\gamma + (1-\gamma)q_{0,i}\right] + (1-w_i)\left[1-(1-\gamma)q_{0,i}\right]}
\end{aligned}
$$

Define the function:

$$
f(q) = \frac{w_i\left[\gamma + (1-\gamma)q\right]}{w_i\left[\gamma + (1-\gamma)q\right] + (1-w_i)\left[1-(1-\gamma)q\right]}
$$

Substitute $\frac{M_{i0}}{M_0}$ and $\frac{M_{i1}}{M_1}$ into the inequality $q_{1,i} < q_{0,i}$. This gives:

$$
f(q_{1,i}) < q_{1,i} < q_{0,i} < f(q_{0,i})
$$

which implies:

$$
\frac{f(q_{1,i})}{f(q_{0,i})} < \frac{q_{1,i}}{q_{0,i}}
$$

Hence:

$$
\frac{f(q_{1,i})/q_{1,i}}{f(q_{0,i})/q_{0,i}} < 1
$$

We define a logarithmic function:

$$
g(q) \;=\; = ln\left(\frac{f(q)}{q}\right) = ln\left(w_i\left[\gamma + (1-\gamma)q\right]\right) - ln\left(w_i\left[\gamma + (1-\gamma)q\right] + (1-w_i)\left[1-(1-\gamma)q\right]\right) - ln\left(q\right)
$$

The derivative of $g(q)$ is:

$$g'(q) = \frac{w_i(1-\gamma)}{w_i\left[\gamma+(1-\gamma)q\right]} - \frac{(2w_i-1)(1-\gamma)}{w_i\left[\gamma+(1-\gamma)q\right]+(1-w_i)\left[1-(1-\gamma)q\right]} - \frac{1}{q}$$

Define

$$\begin{aligned}
A(q) &= \frac{w_i(1-\gamma)}{w_i\left[\gamma+(1-\gamma)q\right]} - \frac{1}{q} \\
&= \frac{-\gamma}{\left[\gamma+(1-\gamma)q\right]q} \\
&< 0 \\
B(q) &= \frac{(2w_i-1)(1-\gamma)}{w_i\left[\gamma+(1-\gamma)q\right]+(1-w_i)\left[1-(1-\gamma)q\right]} \\
&= \frac{(2w_i-1)(1-\gamma)}{1-(1-\gamma)w_i+(2w_i-1)(1-\gamma)q}
\end{aligned}$$

If group i is majority ($w_i > \frac{1}{2}$), then $B(q) > 0$, and thus $g'(q) = A(q) - B(q) < 0$. If $w_i < \frac{1}{2}$, then condition $\gamma > \frac{1}{2}$ is necessary to ensure $g'(q) < 0$.

Given $g(q)$ is decreasing with q, and $q_{1,i} < q_{0,i}$, we get

$$\frac{g(q_{1,i})}{g(q_{0,i})} = \frac{f(q_{1,i})/q_{1,i}}{f(q_{0,i})/q_{0,i}} > 1$$

This leads to a *contradiction*.

Therefore, under the condition $h_i < h_j$, the inequality $q_{1,i} < q_{0,i}$ does not hold. Instead, the inequality $q_{1,i} > q_{0,i}$must be satisfied when $h_i > h_j$

Similarly, we can prove that $q_{1,i} < q_{0,i}$ holds under the condition $h_i < h_j$.

Appendix B. Proof of Proposition 2.6:

(Monotonicity of the Homophily Index over b)

Consider the matching probabilities on the dimension of personal hobbies, denoted as q_{11}, q_{10}, q_{01}, and q_{00}, where $b \in [0, 1]$. From (2.4), the derivatives of these probabilities with respect to b

are given as:

$$
\begin{aligned}
\frac{dq_{11}}{db} &= (\frac{M_1}{M})^b \cdot ln(\frac{M_1}{M}) < 0 \\
\frac{dq_{10}}{db} &= -\frac{dq_{11}}{db} > 0 \\
\frac{dq_{01}}{db} &= \frac{M_1}{M_0} \cdot \frac{dq_{10}}{db} > 0 \\
\frac{dq_{00}}{db} &= -\frac{dq_{10}}{db} < 0
\end{aligned}
$$

where M_1 and M_0 represent the number of individuals possessing trait 1 and trait 0 in the matching pool, respectively, so that $\frac{M_1}{M} = \frac{M_1}{M_1+M_0} < 1$.

Impacts on Homophily Index: The homophily index can be calculated using the group-specific matching probabilities. As mentioned before, the homophily index for group $i1$ is given by $H_{i1} = q_{1,i}$, and for group $i0$ is given by $H_{i0} = q_{0,i}$. Additionally, the homophily indices for the other group j are: $H_{j1} = 1 - q_{1,i}$ and $H_{j0} = 1 - q_{0,i}$

The group-specific matching probabilities are:

$$
\begin{aligned}
q_{1i} &= \frac{M_{i1}}{M_1} q_{11} + \frac{M_{i0}}{M_0} q_{10} \\
q_{0i} &= \frac{M_{i1}}{M_1} \cdot q_{01} + \frac{M_{i0}}{M_0} q_{00}
\end{aligned}
$$

The derivative of $q_{1,i}$ with respect to b is:

$$
\begin{aligned}
\frac{dq_{1,i}}{db} &= \frac{M_{i1}}{M_1} \cdot \frac{dq_{11}}{db} + \frac{M_{i0}}{M_0} \frac{dq_{10}}{db} \\
&= \left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0} \right) \frac{dq_{11}}{db}
\end{aligned}
$$

Similarly, the derivative of $q_{0,1}$is:

$$\frac{dq_{0,i}}{db} = \left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right) \cdot \frac{dq_{01}}{db}$$

Note: The lower the value of b, the higher the level of matching bias. This is because a smaller b places a greater probability on agents with similar personal hobbies meeting each other.

We now consider two cases based on the comparison of homophily indices h_i and h_j:

1. **Case 1:** $h_i > h_j$

   When $h_i > h_j$, we have $\frac{M_{i1}}{M_1} > \frac{M_{i0}}{M_0}$. Thus, it follows that:

   $$\frac{dq_{1,i}}{db} < 0 \quad \text{and} \quad \frac{dq_{0,i}}{db} > 0$$

   The homophily index for group $i1$, H_{i1}, and the homophily index for group $j0$, H_{j0}, decrease as b increases, and therefore, they increase with a higher level of matching bias. On the other hand, H_{i0} and H_{j1} increase with b and decrease with the matching bias.

2. **Case 2:** $h_i < h_j$

   When $h_i < h_j$, have $\frac{M_{i1}}{M_1} < \frac{M_{i0}}{M_0}$. In this case, the homophily indexH_{i1} and H_{j0} decrease as the matching bias increases, whileH_{i0} and H_{j1} increase with the matching bias.

3. **Lower Bound (Upper Bound) of the Homophily Index**

   Since H_g, for any $g \in G$, is a monotonic function over matching bias b, the lower bound (upper bound) of homophily for group g occurs under the extreme case of maximum matching bias, i.e. $b = 0$.

Appendix C. Impact of h_i on Homophily Index at $h_i = h_j$

In this section, we use Jacobian matrix to estimate the impact of proportion h_i on the matching probabilities $q_{1,i}$ and $q_{0,i}$, and demonstrate that q_{1i} increases while q_{0i} decreases at the point where

$h_i = h_j$.

The homophily of group $i1$ and $i0$ can are measured by matching probabilities $q_{1,i}$ and $q_{0,i}$, respectively. By the definition of the two probabilities in (2.6), we obtain the following system of equations:

$$\begin{cases} G(\mathbf{q}, h_i) = \frac{M_{i1}}{M_1} q_{11} + \frac{M_{i0}}{M_0} q_{10} - q_{1,i} = 0 \\ H(\mathbf{q}, h_i) = \frac{M_{i1}}{M_1} q_{01} + \frac{M_{i0}}{M_0} q_{00} - q_{0,i} = 0 \end{cases}$$

where $q_{11} = (\frac{M_1}{M})^b$, $q_{10} = 1 - q_{11}$, $q_{01} = \frac{M_1}{M_0} q_{10}$, and $q_{00} = 1 - q_{01}$. In this system, h_i is the only independent variable.

To explore the sensitivity of q_{1i} and q_{0i} with respect h_i, we use the Jacobian matrix of functions G and H , which contains the partial derivatives with respect to qq_{1i} and q_{0i}:

$$J = \begin{vmatrix} \frac{\partial G}{\partial q_{1i}} & \frac{\partial G}{\partial q_{0i}} \\ \frac{\partial H}{\partial q_{1i}} & \frac{\partial H}{\partial q_{0i}} \end{vmatrix}$$

The derivatives of q_{1i} and q_{0i} with respect to h_i are given by:

$$\frac{dq_{1i}}{dh_i} = -\frac{1}{J} \begin{vmatrix} \frac{\partial G}{\partial h_i} & \frac{\partial G}{\partial q_{0i}} \\ \frac{\partial H}{\partial h_i} & \frac{\partial H}{\partial q_{0i}} \end{vmatrix}, \quad \text{and} \quad \frac{dq_{0i}}{dh_i} = -\frac{1}{J} \begin{vmatrix} \frac{\partial G}{\partial q_{1i}} & \frac{\partial G}{\partial h_i} \\ \frac{\partial H}{\partial q_{1i}} & \frac{\partial H}{\partial h_i} \end{vmatrix},$$

respectively.

We now compute the partial derivatives involved in the Jacobian matrix and the above expressions.

$$
\begin{aligned}
\frac{\partial G}{\partial q_{1i}} &= \frac{\partial G}{\partial t_{i1}}\frac{\partial t_{i1}}{\partial q_{1i}} - 1 = \frac{1-\gamma}{4c^2}\frac{\partial G}{\partial t_{i1}} - 1,\\
\frac{\partial G}{\partial q_{0i}} &= \frac{\partial G}{\partial t_{i0}}\frac{\partial t_{i0}}{\partial q_{0i}} = \frac{1-\gamma}{4c^2}\frac{\partial G}{\partial t_{i0}},\\
\frac{\partial H}{\partial q_{1i}} &= \frac{\partial H}{\partial t_{i1}}\frac{\partial t_{i1}}{\partial q_{1i}} = \frac{1-\gamma}{4c^2}\frac{\partial H}{\partial t_{i1}},\\
\frac{\partial H}{\partial q_{0i}} &= \frac{\partial H}{\partial t_{i0}}\frac{\partial t_{i0}}{\partial q_{0i}} - 1 = \frac{1-\gamma}{4c^2}\frac{\partial H}{\partial t_{i0}} - 1.
\end{aligned}
$$

Let us denote:

$$b_1 \equiv \frac{\partial G}{\partial t_{i1}}, \quad b_2 \equiv \frac{\partial G}{\partial t_{i0}}, \quad b_3 \equiv \frac{\partial H}{\partial t_{i1}}, \quad b_4 \equiv \frac{\partial H}{\partial t_{i0}}.$$

Similarly, we define:

$$a_1 \equiv \frac{\partial G}{\partial h_i}, \quad a_2 \equiv \frac{\partial H}{\partial h_i}.$$

We have:

$$b_1 = \frac{\partial G}{\partial t_{i1}} = \frac{\partial}{\partial t_{i1}}\left(\frac{M_{i1}}{M_1}\right) \times q_{11} + \frac{\partial q_{11}}{\partial t_{i1}} \underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

$$b_2 = \frac{\partial G}{\partial t_{i0}} = \frac{\partial}{\partial t_{i0}}\left(\frac{M_{i0}}{M_0}\right) \times q_{10} + \frac{\partial q_{11}}{\partial t_{i0}} \underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

$$b_3 = \frac{\partial H}{\partial t_{i1}} = \frac{\partial}{\partial t_{i1}}\left(\frac{M_{i1}}{M_1}\right) \times q_{01} + \frac{\partial q_{01}}{\partial t_{i1}} \underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

$$b_4 = \frac{\partial H}{\partial t_{i0}} = \frac{\partial}{\partial t_{i0}}\left(\frac{M_{i0}}{M_0}\right) \times q_{01} + \frac{\partial q_{01}}{\partial t_{i0}} \underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

For $a_1$and a_2:

$$a_1 = \frac{\partial G}{\partial h_i} = \frac{\partial}{\partial h_i}\left(\frac{M_{i1}}{M_1}\right) \times q_{11} + \frac{\partial}{\partial h_i}\left(\frac{M_{i0}}{M_0}\right) \times q_{10} + \frac{\partial q_{11}}{\partial h_i}\underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

$$a_2 \equiv \frac{\partial H}{\partial h_i} = \frac{\partial}{\partial h_i}\left(\frac{M_{i1}}{M_1}\right) \times q_{00} + \frac{\partial}{\partial h_i}\left(\frac{M_{i0}}{M_0}\right) \times q_{00} + \frac{\partial q_{01}}{\partial h_i}\underbrace{\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)}_{=0 \text{ if } h_i = h_j}$$

Under the condition $h_i = h_j$, the terms involving $\left(\frac{M_{i1}}{M_1} - \frac{M_{i0}}{M_0}\right)$ vanish.

Next, we compute the values of b_1, b_2, b_3,and b_4:

$$\begin{aligned}
b_1 &= \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)h_i^2}{M_1^2} q_{11}, \\
b_2 &= \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)(1-h_i)^2}{M_0^2} q_{10}, \\
b_3 &= \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)h_i^2}{M_1^2} q_{01}, \\
b_4 &= \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)(1-h_i)^2}{M_0^2} q_{00}.
\end{aligned}$$

For a_1, we have:

$$\begin{aligned}
a_1 &= \frac{N w_i t_{i1} M_{j1}}{M_1^2} q_{11} - \frac{N w_i t_{i0} M_{j0}}{M_0^2} q_{10} \\
&= \frac{N w_i t_i M_{j1}}{M_1}\left[\frac{1}{M_1} q_{11} - \frac{1}{M_0}(1 - q_{11})\right] \quad \left(\text{since } \frac{M_{j1}}{M_1} = \frac{M_{j0}}{M_0} \text{ and } t_{i1} = t_{i0} = t_i\right) \\
&= \frac{M_i M_{j1}}{M_1}\left(\frac{q_{11}}{h_i M} - \frac{1 - q_{11}}{(1-h_i)M}\right) \quad (\text{since } M_1 = h_i M \text{ and } M_0 = (1-h_i)M) \\
&= \frac{M_i M_{j1}}{M_1 M}\left(\frac{q_{11}}{h_i} - \frac{1 - q_{11}}{1 - h_i}\right)
\end{aligned}$$

Similarly for a_2:

$$
\begin{aligned}
a_2 &= \frac{Nw_i t_{i1} M_{j1}}{M_1^2} q_{01} - \frac{Nw_i t_{i0} M_{j0}}{M_0^2} q_{00} \\
&= \frac{M_i M_{j1}}{M_1 M} \left(\frac{q_{01}}{h_i} - \frac{1 - q_{01}}{1 - h_i} \right)
\end{aligned}
$$

We define a constant l:

$$
l \equiv \frac{M_i M_{j1}}{M_1 M}
$$

Since $M_i < M$ and $M_{j1} < M_1$, we can get $l = \frac{M_i M_{j1}}{M_1 M} < 1$. Therefore,

$$
a_1 = l \left(\frac{q_{11}}{h_i} - \frac{1 - q_{11}}{1 - h_i} \right), \quad and \quad a_2 = l \left(\frac{q_{01}}{h_i} - \frac{1 - q_{01}}{1 - h_i} \right).
$$

The Jacobian determinant J is:

$$
J = \begin{vmatrix} \frac{\partial G}{\partial q_{1i}} & \frac{\partial G}{\partial q_{0i}} \\ \frac{\partial H}{\partial q_{1i}} & \frac{\partial H}{\partial q_{0i}} \end{vmatrix} = \left(\frac{1-\gamma}{4c^2} b_1 - 1 \right) \left(\frac{1-\gamma}{4c^2} b_4 - 1 \right) - \left(\frac{1-\gamma}{4c^2} \right)^2 b_2 b_3
$$

The derivatives of $q_{1,i}$ and $q_{0,i}$ with respect to h_i become:

$$
\begin{aligned}
\frac{dq_{1i}}{dh_i} &= -\frac{1}{J} \left[a_1 \left(\frac{1-\gamma}{4c^2} b_4 - 1 \right) - a_2 \frac{1-\gamma}{4c^2} b_2 \right], \\
\frac{dq_{0i}}{dh_i} &= -\frac{1}{J} \left[a_2 \left(\frac{1-\gamma}{4c^2} b_1 - 1 \right) - a_1 \frac{1-\gamma}{4c^2} b_3 \right].
\end{aligned}
$$

To simplify the notation, let us define:

- Denominator J as f_1,
- Numerator of $\frac{dq_{1i}}{dh_i}$ as f_2,
- Numerator of $\frac{dq_{0i}}{dh_i}$ as f_3,

We will prove that $f_1 > 0$, $f_2 < 0$, and $f_3 > 0$. Therefore,

$$\frac{dq_{1i}}{dh_i} = -\frac{f_1}{f_3} > 0, \quad and \quad \frac{dq_{0i}}{dh_i} = -\frac{f_2}{f_3} < 0.$$

Proof of the Signs of f_1, f_2, f_3 We aim to demonstrate the following inequalities to show how the matching probabilities q_{1i} and q_{01} change with respect to h_i when $h_i = h_j$:

1. $f_1 = \left(\frac{1-\gamma}{4c^2} b_1 - 1\right)\left(\frac{1-\gamma}{4c^2} b_4 - 1\right) - \left(\frac{1-\gamma}{4c^2}\right)^2 b_2 b_3 > 0$
2. $f_2 = a_1 \left(\frac{1-\gamma}{4c^2} b_4 - 1\right) - a_2 \frac{1-\gamma}{4c^2} b_2 < 0$
3. $f_3 = a_2 \left(\frac{1-\gamma}{4c^2} b_1 - 1\right) - a_1 \frac{1-\gamma}{4c^2} b_3 > 0$

Proof of (1): Showing that $f_1 > 0$

$$\begin{aligned} f_1 &= \left(\frac{1-\gamma}{4c^2} b_1 - 1\right)\left(\frac{1-\gamma}{4c^2} b_4 - 1\right) - \left(\frac{1-\gamma}{4c^2}\right)^2 b_2 b_3 \\ &= \left(\frac{1-\gamma}{4c^2}\frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)h_i^2}{M_1^2} q_{11} - 1\right)\left(\frac{1-\gamma}{4c^2}\frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)(1-h_i)^2}{M_0^2} q_{00} - 1\right) \\ &\quad - \left(\frac{1-\gamma}{4c^2}\right)^2 \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)(1-h_i)^2}{M_0^2} q_{10} \frac{1+\gamma}{4c^2}\frac{N^2 w_i(1-w_i)h_i^2}{M_1^2} q_{01} \end{aligned}$$

Since $M_1 = h_i M$ and $M_0 = (1-h_i)M$, it follows that:

$$\frac{h_i^2}{M_1^2} = \frac{(1-h_i)^2}{M_0^2} = \frac{1}{M^2}$$

Define the constant:

$$\begin{aligned} k &\equiv \frac{1-\gamma}{4c^2}\cdot\frac{1+\gamma}{4c^2}\cdot\frac{N^2 w_i(1-w_i)}{M^2} \\ &= \frac{N^2 w_i(1-w_i)(1-\gamma^2)}{16c^4M^2} \end{aligned}$$

where, $M = N\left(w_i t_i + (1-w_i)(\frac{1+\gamma}{4c^2} - t_i)\right) \geq \frac{N(1+\gamma)}{8c^2}$. This implies:

$$k \leq \frac{N^2(1-\gamma^2)/4}{N^2(1+\gamma)^2/4} < 1.$$

Substituting k, we simplify f_1:

$$\begin{aligned} f_1 &= (kq_{11}-1)(kq_{00}-1) - k^2 q_{10} k q_{01} \\ &= (kq_{11}-1)(kq_{00}-1) - k^2(1-q_{11})(1-q_{00}) \\ &= (k-1)\left[k(q_{11}+q_{00}) - (1+k)\right] \end{aligned}$$

Since $k < 1$, we have

$$q_{11} + q_{00} < 2 < \frac{1+k}{k}$$

Therefore, $f_1 > 0$

Proof of (2)**: Showing that** $f_2 < 0$

$$\begin{aligned}
f_2 &= a_1\left(\frac{1-\gamma}{4c^2}b_4-1\right)-a_2\frac{1-\gamma}{4c^2}b_2 \\
&= l\left(\frac{q_{11}}{h_i}-\frac{1-q_{11}}{1-h_i}\right)(kq_{00}-1)-l\left(\frac{q_{01}}{h_i}-\frac{1-q_{01}}{1-h_i}\right)kq_{10} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left[\left(1-h_i\right)q_{11}-h_i\left(1-q_{11}\right)\right](kq_{00}-1)-\left[\left(1-h_i\right)q_{01}-h_i\left(1-q_{01}\right)\right]kq_{00}\right\} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left(q_{11}-h_i\right)(kq_{00}-1)-\left(q_{01}-h_i\right)kq_{10}\right\} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left(h_i-q_{11}\right)+k\left(q_{11}-q_{01}\right)\left(1-h_i\right)\right\}
\end{aligned}$$

Since $q_{10}=1-q_{11}$ and $q_{00}=1-q_{01}$, and under $h_i=h_j$, we have $q_{11}=h_i^b$ and $q_{01}=\frac{h_i}{1-h_i}(1-h_i^b)$. We can write:

$$f_2 = l(1-k)\frac{h_i-h_i^b}{h_i\left(1-h_i\right)}$$

Since $l(1-k)$ is a positive constant and $h_i-h_i^b<0$, the entire expression is negative.

Therefore, $f_2<0$.

Proof of (3)**: Showing that** $f_3>0$

$$\begin{aligned}
f_3 &= a_2\left(\frac{1-\gamma}{4c^2}b_1-1\right)-a_1\frac{1-\gamma}{4c^2}b_3 \\
&= l\left(\frac{q_{01}}{h_i}-\frac{1-q_{01}}{1-h_i}\right)(kq_{11}-1)-l\left(\frac{q_{11}}{h_i}-\frac{1-q_{11}}{1-h_i}\right)kq_{01} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left[\left(1-h_i\right)q_{01}-h_i\left(1-q_{01}\right)\right](kq_{11}-1)-\left[\left(1-h_i\right)q_{11}-h_i\left(1-q_{11}\right)\right]kq_{01}\right\} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left(q_{01}-h_i\right)(kq_{11}-1)-\left(q_{11}-h_i\right)kq_{01}\right\} \\
&= \frac{l}{h_i\left(1-h_i\right)}\left\{\left(h_i-q_{01}\right)+kh_i\left(q_{01}-q_{11}\right)\right\} \\
&= l\left(1-k\right)\frac{h_i^b-h_i}{\left(1-h_i\right)^2}
\end{aligned}$$

Therefore, $f_3>0$.

Conclusion: Since $f_1 > 0$, $f_2 < 0$, and $f_3 > 0$, we have:

$$\frac{dq_{1i}}{dh_i} = -\frac{f_1}{f_2} > 0, \quad and \quad \frac{dq_{0i}}{dh_i} = -\frac{f_2}{f_3} < 0.$$

This implies:

- The probability q_{1i} (agents with hobbies meeting type i) increases with h_i.
- The probability q_{0i} (agents without hobbies meeting type i) decreases with h_i.

Implications for Homophily Indices:

- The homophily index for agents of type $i1$ is $H_{i1} = q_{1i}$, which increases with h_i.
- The homophily index for agents of type $i0$ is $H_{i0} = q_{0i}$, which decreases with h_i.
- The homophily index for agents of type $j1$ is $H_{j1} = 1 - q_{1i}$, which decreases with h_i.
- The homophily index for agents of type $j0$ is $H_{j0} = 1 - q_{0i}$, which increases with h_i.

Thus, the inbreeding homophily indices for types $i1$ and $j0$ increase with h_i, while those for types $i0$ and $j1$ decrease with h_i.

Appendix D. Weighted Homophily WH_i and WH_j Reaches Local Optimum When $h_i = h_j$

In the following proof, we fix h_j and treat h_i as a variable that influences the matching process. We demonstrate that the weighted homophily indices for both groups reach their minimum as h_i approaches to h_j.

1. Homophily Index for Racial Group i The Weighted Homophily Index is defined as

$$WH_i = \frac{(1-h_i)q_{0,i}t_{i0} + h_i q_{1,i}t_{i1}}{(1-h_i)t_{i0} + h_i t_{i1}}$$

Setting $\alpha = \frac{1}{2}$, we have:

$$t_{i0} = \frac{1}{4c^2}\left[\gamma + (1-\gamma)q_{0,i}\right]\ , t_{i1} = \frac{1}{4c^2}\left[\gamma + (1-\gamma)q_{1,i}\right]\ .$$

Substituting these into the expression for WH_i:

$$\begin{aligned} WH_i &= \frac{(1-h_i)q_{0,i}\left[\gamma + (1-\gamma)q_{0,i}\right] + h_i q_{1,i}\left[\gamma + (1-\gamma)q_{1,i}\right]}{(1-h_i)\left[\gamma + (1-\gamma)q_{0,i}\right] + h_i\left[\gamma + (1-\gamma)q_{1,i}\right]} \\ &= \frac{\gamma\left[(1-h_i)q_{0,i} + h_i q_{1,i}\right] + (1-\gamma)\left[(1-h_i)q_{0,i}^2 + h_i q_{1,i}^2\right]}{\gamma + (1-\gamma)\left[(1-h_i)q_{0,i} + h_i q_{1,i}\right]} \end{aligned}$$

Relationship Between Weighted Homophily and $E(h_i, q)$ We aim to show that the weighted homophily index WH_i reaches a local minimum when $h_i = h_j$, where h_j is the participation rate of the other group and is considered exogenous.

Define the average matching probability $E(h_i, q)$as

$$E(h_i, q) \equiv (1-h_i)q_{0,i} + h_i q_{1,i}$$

Similarly, the average of the squares is given by:

$$E(h_i, q^2) = (1-h_i)q_{0,i}^2 + h_i q_{1,i}^2$$

Since the function $f(x) = x^2$ is convex, according to Jensen's Inequality, we have

$$E(h_i, q^2) \geq \left[E(h_i, q)\right]^2$$

Thus, the weighted homophily index satisfies:

$$\begin{aligned} WH &= \frac{\gamma E(h_i, q) + (1-\gamma)E(h_i, q^2)}{\gamma + (1-\gamma)E(h_i, q)} \\ \geq &= \frac{\gamma E(h_i, q) + (1-\gamma)\left[E(h_i, q)\right]^2}{\gamma + (1-\gamma)E(h_i, q)} = E(h_i, q) \end{aligned}$$

where the equality holds if and only if $q_{0,i} = q_{1,i}$.

Showing $E(h_i, q)$ Reaches a Local Minimum at $h_i = h_j$(which implies$q_{1,i} = q_{0,i}$) We need a detailed analysis using the Jacobian method (see Appendix C).

First Derivative of $E(h_i, q)$ The derivative of $E(h_i, q)$ with respect to h_i is:

$$\begin{aligned} \frac{dE}{dh_i} &= \frac{\partial E}{\partial h_i} + \frac{\partial E}{\partial q_{1i}}\frac{dq_{1i}}{dh_i} + \frac{\partial E}{\partial q_{0i}}\frac{dq_{0i}}{dh_i} \\ &= q_{1i} - q_{0i} + h_i\frac{dq_{1i}}{dh_i} + (1-h_i)\frac{dq_{0i}}{dh_i} \end{aligned}$$

At $h_i = h_j$:

- $q_{0,i} = q_{1,i}$,
- From Appendix B, we have $\frac{dq_{1i}}{dh_i} = -\frac{f_2}{f_1}$, and $\frac{dq_{0i}}{dh_i} = -\frac{f_3}{f_1}$. Given $f_2 = l(1-k)\frac{h_i - h_i^b}{h_i(1-h_i)}$ and $f_3 = l\,(1-k)\,\frac{h_i^b - h_i}{(1-h_i)^2}$, where $l(1-k)$ is a constant positive, we can conclude $h_i f_2 + (1-h_i)f_3 = 0$.

Therefore,

$$\frac{dE(h_i,q)}{dh_i} = q_{1i} - q_{0i} + h_i\left(-\frac{f_2}{f_1}\right) + (1-h_i)\left(-\frac{f_3}{f_1}\right) = 0$$

Second Derivative of $E(h_i, q)$ Compute the second derivative:

$$\frac{d^2E(h_i, q)}{dh_i^2} = 2\left(\frac{dq_{1i}}{dh_i} - \frac{dq_{0i}}{dh_i}\right) + h_i\frac{d^2q_{1i}}{dh_i^2} + (1-h_i)\frac{d^2q_{0i}}{dh_i^2}$$

Computing $\frac{dq_{1i}}{dh_i} - \frac{dq_{0i}}{dh_i}$:

$$\frac{dq_{1i}}{dh_i} - \frac{dq_{0i}}{dh_i} = -\frac{f_2}{f_1} + \frac{f_3}{f_1} = -\frac{f_2 - f_3}{f_1}$$

From Appendix B, $h_i = h_j$:

$$f_2 = l(1-k)\frac{h_i - h_i^b}{h_i\left(1-h_i\right)}, \quad and \quad f_3 = l\left(1-k\right)\frac{h_i^b - h_i}{\left(1-h_i\right)^2},$$

where $l(1-k) > 0$.

Therefore:

$$\begin{aligned} f_2 - f_3 &= l(1-k)\left(\frac{h_i - h_i^b}{h_i\left(1-h_i\right)} - \frac{h_i^b - h_i}{\left(1-h_i\right)^2}\right) \\ &= l(1-k)\frac{h_i - h_i^b}{h_i\left(1-h_i\right)^2} \end{aligned}$$

So:

$$\frac{dq_{1i}}{dh_i} - \frac{dq_{0i}}{dh_i} = -\frac{l(1-k)\frac{h_i - h_i^b}{h_i(1-h_i)^2}}{f_1}$$

Computing $h_i\frac{d^2q_{1i}}{dh_i^2} + (1-h_i)\frac{d^2q_{0i}}{dh_i^2}$:

The second derivatives are:

$$\begin{aligned} \frac{d^2q_{1i}}{dh_i^2} &= -\left(\frac{d}{dh_i}\left(\frac{f_2}{f_1}\right)\right) = -\left(\frac{f_2'}{f_1} - \frac{f_1' f_2}{{f_1}^2}\right), \\ \frac{d^2q_{0i}}{dh_i^2} &= -\left(\frac{d}{dh_i}\left(\frac{f_3}{f_1}\right)\right) = -\left(\frac{f_3'}{f_1} - \frac{f_1' f_3}{{f_1}^2}\right). \end{aligned}$$

Adding the terms:

$$h_i\frac{d^2q_{1i}}{dh_i^2} + (1-h_i)\frac{d^2q_{0i}}{dh_i^2} = -\left(\frac{h_i f_2' + (1-h_i) f_3'}{f_1} - \frac{f_1'}{{f_1}^2}\left(h_i f_2 + (1-h_i) f_3\right)\right)$$

But from earlier, $h_i f_2 + (1 - h_i) f_3 = 0$, so:

$$h_i \frac{d^2 q_{1i}}{dh_i^2} + (1 - h_i) \frac{d^2 q_{0i}}{dh_i^2} = -\frac{h_i f_2' + (1 - h_i) f_3'}{f_1}$$

Computing $h_i f_2' + (1 - h_i) f_3'$:

$$f_2' = l(1-k) \frac{\left(h_i - h_i^b\right)(2h_i - 1) + h_i (1 - h_i)\left(1 - b h_i^{b-1}\right)}{h_i^2 (1 - h_i)^2}$$

$$f_3' = l(1-k) \frac{2\left(h_i^b - h_i\right) + (1 - h_i)\left(b h_i^{b-1} - 1\right)}{(1 - h_i)^3}$$

and

$$\begin{aligned} h_i f_2' + (1 - h_i) f_3' &= l(1-k) \frac{h_i^b - h_i}{h_i (1 - h_i)^2} \\ h_i \frac{d^2 q_{1i}}{dh_i^2} + (1 - h_i) \frac{d^2 q_{0i}}{dh_i^2} &= -l(1-k) f_1^{-1} \frac{h_i^b - h_i}{h_i (1 - h_i)^2} \end{aligned}$$

Second Derivative Simplification:

Substituting back:

$$\frac{d^2 E}{dh_i^2} = -\frac{2(f_2 - f_3) + h_i f_2' + (1 - h_i) f_3'}{f_1}$$

Using previous results:

$$\frac{d^2 E}{dh_i^2} = l(1-k) \frac{h_i^b - h_i}{h_i (1 - h_i)^2} \frac{1}{f_1}$$

Sign of the second derivative:

- Since $0 < b < 1$, $h_i^b > h_i$ for $h_i \in (0, 1)$,
- $l(1-k) > 0$,
- $f_1 > 0$.

Therefore:

$$\frac{d^2E}{dh_i^2} > 0 \quad \text{at} \quad h_i = h_j.$$

Conclusion:

$E(h_i, q)$ reaches a local minimum at $h_i = h_j$ when $0 < b < 1$.

Implications for the Weighted Homophily Index WH_i Since $E(h_i, q)$ reaches a local minimum at $h_i = h_j$ and $WH_i \geq E(h_i, q)$ with equality only when $q_{1i} = q_{0i}$, we conclude that:

The weighted homophily index for group i, WH_i, reaches a local minimum when $h_i = h_j$.

2. Weighted Homophily Index for Racial Group j The weighted homophily index for group j is defined as:

$$WH_j = \frac{(1-h_j)q_{0,i}t_{j0} + h_j q_{1,j}t_{j1}}{(1-h_j)t_{i0} + h_j t_{j1}}$$

We need to prove WH_j attains its minimum when hi approaching h_j.

Define the average matching probability for groups $j0$ and $j1$ as:

$$E_j(q) = (1-h_j)q_{0,j} + h_j q_{1,j}$$

Similar with previous proof for group i, we show that the weighted homophily index is no less than the average matching probability:

$$WH_j \geq E(q_j)$$

where the equality holds if and only if $q_{0,j} = q_{1,j}$.

From the properties of the homophily index, there exists a negative linear relationship between the matching probabilities of group i and group j, such that:

$$q_{0,j} = 1 - q_{0,i}, \quad \text{and} \quad q_{1,j} = 1 - q_{1,i}$$

Thus, we rewrite $E_j(q)$ as:

$$E_j(q) = 1 - \left[(1 - h_j)q_{0,j} + h_j q_{1,j}\right]$$

The first derivative of $E_j(q)$ with respect to h_i is:

$$\frac{dE_j}{dh_i} = -h_j \frac{dq_{1i}}{dh_i} - (1 - h_j)\frac{dq_{0i}}{dh_i}$$

From the previous proof, we know that:

$$\frac{dq_{1i}}{dh_i} = -\frac{f_2}{f_1} q_{0,i}, \quad \text{and} \quad \frac{dq_{0i}}{dh_i} = -\frac{f_3}{f_1}$$

Thus, the derivative becomes:

$$\frac{dE_j}{dh_i} = \frac{h_j f_2 + (1 - h_j) f_3}{f_1}$$

In the previous proof, we established that:

$$h_i f_2 + (1 - h_i) f_3 = 0$$

Since $h_i = h_j$ in this proof, we conclude that:

$$\frac{dE_j}{dh_i} = 0$$

Next, we compute the second derivative of $E_j(q)$ with respect to h_i:

$$\frac{d^2E_j}{dh_i^2} = -h_j\frac{d^2q_{1i}}{dh_i^2} - (1-h_j)\frac{d^2q_{0i}}{dh_i^2}$$

From the previous proof, we know:

$$h_i\frac{d^2q_{1i}}{dh_i^2} + (1-h_i)\frac{d^2q_{0i}}{dh_i^2} = -l(1-k)f_1^{-1}\frac{h_i^b - h_i}{h_i\,(1-h_i)^2}$$

Since $h_j = h_i$, we can substitute and rewrite the second derivative as:

$$\frac{d^2E_j}{dh_i^2} = l(1-k)f_1^{-1}\frac{h_i^b - h_i}{h_i\,(1-h_i)^2} > 0$$

Therefore, $E_j(q)$ reaches a local minimum at $h_i = h_j$. Since $WH_j \geq E_j(q)$ and equality only when $q_{1i} = q_{0i}$, we conclude that:

The weighted homophily index for group j, WH_j, attains a local minimum when $h_i = h_j$.

Chapter 3

Empirical Findings in Peer Effects and Friendship Connections

3.1 Introduction

In social network, the term *homophily* describes the tendency that individuals are more likely to form connections with others who share similar characteristics, such as race, personal interests, and behaviors. Additionally, *peer effects* refer to the process by which the behaviors, attitudes, and characteristics of individuals influence those around them. Because networks formed through homophily consist of individuals with aligned interests or traits, the transmission of new behaviors or the reinforcement of existing ones can occur more rapidly and effectively.

In essence, homophily sets the stage for forming networks that are not only cohesive but are also ripe environments for peer effects to take place. Within such networks, peer influences are potent because all members are initially similar, and thus more receptive to adopting shared behaviors or beliefs. This mutual receptivity amplifies the impact of peer effects, making them more pronounced than in heterogeneous groups. Thus, homophily and peer effects together create a feedback loop: homophily leads to the formation of homogeneous groups, which facilitate stronger and more efficient peer effects due to their coherence. These peer effects, in turn, reinforce the homogeneity of the group, making its structure even more robust against dissimilar influences.

This chapter explores how such dynamics influence club participation among students, posing the question: How is the decision to join a basketball club influenced by the racial composition of a school? How does it affect the friendship connections? This question is influenced by social customs: both sides have expectations about who is more likely to join the club, and these expectations are shaped by what other people have done in similar circumstances. The general point is that the stability of a convention depends on its welfare consequences for individuals. The prevailing convention emerges from the decisions of many agents, each based on their information. Under repeated interactions by numerous myopic agents, an equilibrium such as "basketball is more popular among black students" will emerge as the convention. This social norm forms common expectations before any games are played.

Consider the example of driving on the left or right side of the road. The choice of side is not crucial for social welfare; what matters is the existence of a convention where expectations and behaviors align. However, some equilibria can be suboptimal from a social welfare perspective. Kandori et al. (1993) provide a framework for understanding how conventions evolve and stabilize. It shows how repeated interactions can establish behavioral norms or conventions that, while stable societally, might not be optimal for individuals. This is the case with cultural norms in our society—they are stable but not necessarily the best state for individuals.

In the context of social conventions, particularly cultural activities, there is an opportunity to address entrenched segregation among different groups. Consider the dynamics within a basketball club as an illustration. If white students anticipate a higher presence of black students in the club, reducing their chances of forming same-race friendships, they may choose to leave the club in pursuit of higher payoffs through same-race connections. However, if the white population within the society is sufficiently large, they have more influence in the coordination and might collectively decide to participate in basketball, finding value in meeting others who share their hobby as well as their racial identity.

Many studies showed people decide to adopt some behavior based on their friends' behaviors:

for instance, when adopting new technology, smoking, engaging in criminal activities, going to college, etc. Social segregation in the network enables different groups to maintain different behaviors, norms, and cultures.

Unlike many other types of behavior, joining cultural activities offers individuals the opportunity to forge new friendship connections at relatively low cost, which can, in turn, influence the structure of social networks. By comparing friendship networks over a one-year period, the empirical data from this study indicate that students who join the same club are more likely to form new friendships with one another, thereby crossing established racial boundaries. This effect is particularly strong in large schools, where club members are twice as likely as non-members to form new friendships with other members, indicating strong cohesion among club members.

The empirical analysis begins with the application of the Linear-in-Means (LIM) model to assess peer effects, allowing us to distinguish the influence of black and white peers on students' participation decisions. We find that using instrumental variables (IV) does not fully address the reflection problem. The reflection problem consists of two components: (1) the reciprocal nature of influence, where an individual's behavior can both shape and be shaped by their peers, creating a circular causality issue, and (2) the endogenous formation of networks, where individuals with similar characteristics are more likely to form connections. While the IV approach can mitigate issues related to reciprocal influence, it cannot account for the bias introduced by network endogeneity. This limitation is particularly significant in contexts where club participation itself contributes to the formation of friendship networks, as students often form friendships predominantly with other club members. Consequently, this structure can lead to an overestimation of peer effects from fellow club members, as our results demonstrate.

To further delve into the dynamics of social interaction, I analyze the average participation rate within racial groups as an indicator of social influence, considering it an exogenous factor. I compare the participation behaviors of white and black students in baseball and basketball clubs, noting a recognized trend that baseball is more popular among white students. I find that the participation rate for black students in baseball positively correlates with their racial representation within the community, suggesting a coordination effect in their decision to join, which is similar to the pattern observed among white students in basketball. This behavior seems to be driven by the intuitive understanding that a higher presence of peers from one's own race increases the chances of forming same-race friendships within the club, thereby enhancing the overall benefits of participation.

Furthermore, our findings highlight that club members are significantly more likely to forge new friendships with fellow club participants, underscoring the presence of matching bias within club circles. Notably, the tendency towards racial homophily is less pronounced among club participants compared to non-participants, pointing to a nuanced interplay between club involvement and social connection patterns.

The remainder of this chapter is organized as follows: Section 3.2 presents the coordination game model, emphasizing the role of prevailing social norms in shaping individual decisions and the structure of social networks. Section 3.3 introduces an advanced LIM model to identify peer effects from different racial groups, along with an alternative approach that uses group averages as exogenous variables. Section 3.4 provides the empirical analysis for both methods, along with other empirical observations. These include: the differing behavior between two types of club members—devotees and connectors; comparisons between large and small schools; and the cohesive behavior observed among club members. Section 3.5 concludes.

3.2 Background and Literature Review

To provide context for our discussion on social norms and individual decision-making, the subsequent discussion employs a coordination game as an illustrative example. Specifically, we explore the scenario where white students perceive a basketball club as predominantly black, it may deter their participation due to a perceived reduction in opportunities to form same-race friendships, a factor highly valued for its social payoff. However, as the white population within a community grows, they may collectively gravitate towards the club, seeking a harmonious blend of shared interests and racial identity reinforcement.

Table 3.1: Coordination Game Between Two White Students

		White Student 2	
		Join	Not Join
White Student 1	Join	$a,\ a$	$c,\ b$
	Not Join	$b,\ c$	$d,\ d$

For instance, let's imagine a society with two white students and one student of another race. The white students face a choice: to join the basketball club or not, and they need to pay a cost of 1 to learn the skill. To simplify my model, I assume basketball players only meet fellow players, while non-players only meet other non-players. The two white students achieve a payoff only if they meet each other. If both choose not to join and meet, they gain a payoff of 2 from making a same-race friend. In contrast, if both choose to join and meet, they attain a higher payoff of 4, deriving value from making a friend who is not only of the same race but also shares their hobby.

This scenario is a classic coordination game, as illustrated in Table 3.1,where the two Nash equilibria are both players joining or both abstaining. They prefer both joining or both not joining the club over divergent choices. Because the payoff matrix is symmetric, the equilibrium (J, J) is risk dominant if and only if $a - b \geq d - c$. Conversely, equilibrium (N, N) is risk dominant if the opposite inequality holds.

The twist in this game is the uncertainty surrounding the third student's decision. If the white students believe the third student is more likely to join, their probability of meeting by not joining increases, and vice versa. The risk-dominant state is influenced by their beliefs about this third student's choice. To illustrate, let's consider two extreme scenarios presented in table 3.2. In the first scenario (table a), they believe the student of another race will definitely not join; thus, in state (N, N), a white student meets the other with a 50 probability, leading to an expected payoff of 1. Therefore, (J, J) is the risk-dominant state. In the second scenario (table c), they believe the third student will definitely join. Thus, they are certain of meeting in state (N, N) but only have a 50% chance in state (J, J) with an expected payoff of 1. In this scenario, equilibrium(N, N) is risk dominant.

Table 3.2: Payoff Matrix Between Two White Students

		White Student 2	
		Join	Not Join
White Student 1	Join	3, 3	−1, 0
	Not Join	0, −1	1, 1

(a) 3^{rd} student not join

	White Student 2	
	J	N
J	2, 2	−1, 0
N	0, −1	1.5, 1.5

(b) evenly split

	White Student 2	
	J	N
J	1, 1	−1, 0
N	0, −1	2, 2

(c) 3^{rd} student will join

Table (b) presents a payoff matrix where the third student's decision is balanced, indicating an equal likelihood of choosing either option. This table represents a hybrid of the previous two scenarios, each weighted at 50%. In this situation, (N, N), becomes the risk-dominant state, while (J, J) remains payoff dominant.

In our society, the situation is much more complex. The dynamics of coordination are influenced by two key factors: the racial composition of the student population and the total size of the population.

The proportion of white students plays a critical role in determining their collective influence on coordination outcomes. For instance, in the example featuring two white students and one student of another race, the choices made by the white students considerably shaped the outcome. In contrast, if the scenario involved two white students and ten students of other races, the influence of the white students in steering the coordination would be markedly diminished.

Society at large consists of numerous agents and is saturated with an abundance of information, making the alteration of prevailing social norms a challenging endeavor. According to the principles of stochastically stable equilibria Foster and Young (1990) and Young (2020), the stability of these norms depends on both the size of the population and the extent of information available to people when making decisions. In contexts where people have extensive information and engage in widespread interactions, the inertia can be substantial. Once an equilibrium is reached, it tends to remain stable for extended periods before a significant event causes a shift.

In contrast, middle schools provide a more personal and secure environment vital for adolescent development. Far more than mere educational institutions, they act as pivotal social platforms where students explore their interests, forge social connections, and begin to define their cultural identities.

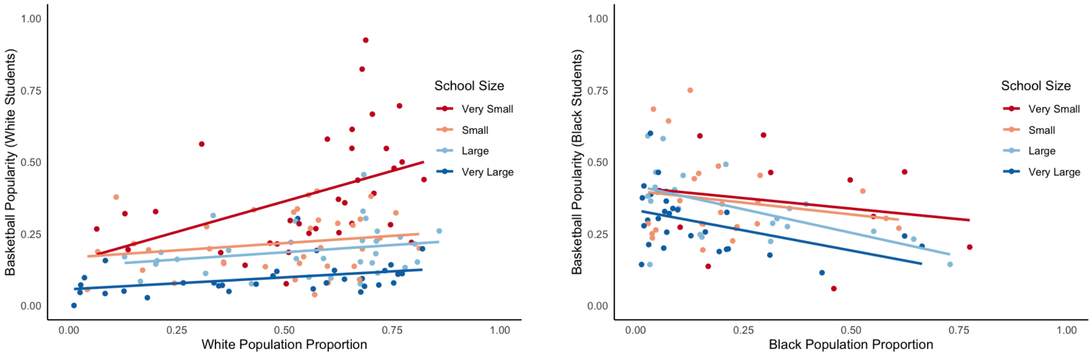

Figure 3.1: Racial Group Size v.s. Participation Rate of White and Black Students (by School Size)

Note: I categorize the schools into four size types: very small schools ($n \leq 300$), small schools ($300 < n \leq 550$), large schools ($550 < n \leq 800$), and very large schools ($n \geq 800$), where n represents the total number of students in each school.
The data show a clear negative correlation between students' participation rates and school size, which may be attributed to capacity limitations within clubs. The differing patterns of behavior between White and Black students can be explained by their social motivations. While White students tend to form more connections after joining a club, participation does not increase the total number of connections for Black students but influences the type of friends they make. When there is a sufficient presence of Black peers within the community, Black students are more likely to build same-race connections without incurring the cost of club participation.

These early social interactions are instrumental in questioning and potentially revising established social norms. Through fostering more inclusive and interconnected communities, middle schools play a significant role in the social fabric of future societies, breaking down barriers and championing diversity and integration during these critical formative years.

Empirical evidence indicates a positive correlation between the proportion of white students in a school and the popularity of basketball among this demographic. As proportion of white students in schools increases, there is a noticeable rise in the variability of basketball's popularity levels among white students across these schools, as illustrated in Figure 3.1. A similar pattern is observed among black students with respect to baseball club participation, where the relative size of the black student population in a school enhances their participation likelihood.

Jackson (2014) and Jackson et al. (2017) review how social network structures—such as connections, clusters, and centrality—shape economic behaviors by influencing information flow, partnerships, and decision-making.

Karimi et al. (2018) develop a social network model to evaluate how homophilic and heterophilic behaviors affect minorities within social networks. Boucher (2020) finds that racial homophily is primarily driven by the impact of students' observable characteristics on payoffs, rather than by network effects. Jackson et al. (2023) examine the dynamics of friendship networks, finding that homophily based on certain traits remains relatively stable over time.

Currarini et al. (2016) offer a model of homophilous social networks in which agents shift their search behavior depending on group size—searching within-group when large, and to the population when small.Baccara and Yariv (2013) emphasize the role of anticipated future interaction in forming homophilous peer groups over preferences, modeling such groups as intervals in a trait space.

In the realm of social interactions and cultural activities, particularly within the context of coordination games, the propensity for individuals with similar traits to engage and form connections plays a pivotal role. Bramoullé et al. (2012) suggest that such similarities bolster the likelihood of interaction, fostering friendship formation. This concept is particularly relevant in networks where local interactions predominate, as agents primarily interact with a proximate set of peers. However, in networks facilitating global interactions, agents have the opportunity to connect with a broader, potentially global set of individuals. Sandholm (2010) points out that while local interactions tend to decelerate the process, allowing for a greater diversity of strategies, global interactions may hasten coordination, potentially leading to homogenized behaviors. Building on a matching and search framework, Immorlica et al. (2010) study the extent to which cooperation can be sustained in equilibrium. Pin and Rogers (2015) apply a similar framework to examine immigration policy.

Block and Grund (2014) explore multidimensional homophily and show that although each dimension tends to reinforce tie formation, interactions between them can produce negative effects—highlighting trade-offs in social similarity. Ushchev and Zenou (2020) compare the local-average model (LIM) and local-aggregate model in peer influence, finding the LIM model better explains homophilous network patterns. Boucher (2016) finds that peer effects increase with the number of peers but with diminishing marginal impact, suggesting saturation in social influence.

Hiller (2017) and Baetz (2015) frame network formation as a simultaneous game where agents choose both actions and connections, allowing for a co-evolution of behavior and network. Ely (2002) models dynamic networks where strategy and location evolve together, with efficient strategies surviving over time. Badev (2021) develops a model that jointly considers individual incentives, peer influence, and welfare outcomes, applying it to adolescent smoking behavior. Bramoullé et al. (2014) further formalize strategic interactions in networks by distinguishing local complementarities and substitutabilities and showing how network topology shapes equilibrium outcomes.

Boucher and Bramoullé (2020) develop a framework for analyzing binary decisions in network settings where outcomes depend linearly on peers' choices. By characterizing equilibrium existence and uniqueness, they extend peer effect models to binary outcomes while preserving tractable and interpretable linear structures.

In understanding the mechanics of peer influence, Manski (1993) and Brock and Durlauf (2001) categorize interaction effects into endogenous effects, reflecting the reciprocal influence among peers, and exogenous effects, which include unaltered peer metrics by current behaviors. Manski's model encapsulates social interactions through anticipated outcomes from social equilibrium but acknowledges the "reflection problem"-the challenge in distinguishing influences from actual behaviors versus contextual factors due to potential linear correlations with exogenous variables.

A significant challenge, as discussed by Moffitt (2001), is differentiating correlations in outcomes caused by social interactions from those arising from correlated, unobserved variables. This dilemma is further compounded by the "exclusion bias", a concept introduced by Guryan et al. (2009) to describe the misleading negative correlation between an individual's attributes and the group's average when the individual is excluded from the peer group. This bias, however, diminishes as the group size approaches infinity. Caeyers and Fafchamps (2016) not only address this bias but also provide a method to test the existence of endogenous peer effects.

Addressing the endogeneity of peer effects, two main approaches prevail. The first one is using instrumental variables (IV) to address endogeneity. One popular IV in network economics is the exogenous characteristics of my friend's friend (e.g., Bramoullé et al. 2009; De Giorgi et al. 2010). These characteristics satisfy the exclusion restriction since they directly affect the behavior of my peers but not mine. However, this approach is not suitable for situations where individuals are partitioned into mutually exclusive groups. The other prevalent IV, which is also adopted in this chapter, utilizes the exogenous characteristics of friends as the instrumental variable (Boucher et al.

(2024)).

The second approach employs spatial autoregressive (SAR) models, as detailed by Kelejian and Prucha (1998; 2010), Lee (2004; 2007), and Drukker et al. (2013). In the SAR model, the dependent variable for a given spatial unit is explained not only by its own independent variables (exogenous inputs) but also by the values of the dependent variable for neighboring units. This is typically represented by a spatial lag term, which is a weighted average of the dependent variable values for the neighboring units. However, in the analysis of participation behavior in this study, the social interaction effect is varied with school size. There exists identification problem in the application of SAR.

Recent work in structural modeling has focused on how to identify peer effects and strategic behavior in settings with discrete outcomes and equilibrium multiplicity. Chesher and Rosen (2012) formalize the identification of simultaneous equations models for discrete choices, laying out the coherence and completeness conditions needed for estimation. Galichon and Henry (2011) extend this to the case of multiple equilibria by introducing a set-identification approach, allowing the researcher to recover bounds on parameters even when the equilibrium selection process is unknown. Similarly, Tamer (2003) proposes a method to partially identify parameters in discrete response models under incomplete information and multiple equilibria.

In the context of discrete games, Bajari et al. (2010) develop a two-step estimator that separates agents' beliefs from their structural payoffs, enabling identification without solving the full game in estimation. Aguirregabiria and Mira (2007) offer an alternative strategy using sequential estimation for dynamic discrete games, which preserves tractability while maintaining valid inference.

These tools have been applied to estimate peer effects in school and adolescent settings. Soetevent and Kooreman (2007) use maximum likelihood to estimate a discrete choice model of teen behavior with social interactions, identifying endogenous effects by exploiting variation across school-level peer groups. Nakajima (2007) models youth smoking behavior by incorporating overlapping peer groups into a structural framework, distinguishing peer effects from contextual and correlated influences. Dieye et al. (2017) study gender-based heterogeneity in peer influence on obesity, identifying structural peer effects using classroom-level variation in composition.

Lastly, Menzel (2016) develops asymptotic methods for games with many players, showing how consistent inference can still be obtained in large-network settings where equilibrium is not unique.

3.3 Peer Effects and Social Interaction

In this section, I initially analyze the impact of social interactions on white students' participation in basketball clubs to understand how peer influence shapes their decisions. Subsequently, I extend the analysis to black students, examining the two group's participation patterns in baseball and football clubs.

The analysis begins with the application of a Linear-in-Means (LIM) model to assess the peer effects on white students' participation, considering influences from both white and black peers separately. However, a challenge arises: the endogeneity of friendship networks leads to an overestimation of the influence that club friends have. While applying instrumental variables (IV) to peers' decisions helps isolate the influence of peers' decisions, eliminating the direct effect from the focal student, this approach unfortunately exacerbates the overestimation issue due to the endogenous nature of the networks.

3.3.1 LIM Model:

Linear-in-Means (LIM) model is widely used in peer-effect studies. For example, a student's decision on participating the sports club is assumed to impacted by the average participation rate of his friends.

The adjacency matrix $G = [g_{ij}]_{n\times n}$ with $g_{ij} = \{0, 1\}$ denotes individual's direct friendship connections. By definition, $g_{ij} = 1$ if and only if student nominates student j as friend; otherwise, $g_{ij} = 0$. The connection in the network is directed, which means g_{ij} and g_{ji} are potentially different, and has no self-loops $g_{ii} = 0$ for all i. Student i's neighbors is defined as $N_i = \{j|g_{ij} = 1\}$, and the row sum $d_i = \sum_{j\neq i} g_{ij}$ is the degree of connections for individual i. Finally, $\tilde{G} = [\tilde{g}_{ij}]_{n\times n}$ denotes the row-normalized adjacency matrix defined by $\tilde{g}_{ij} = g_{ij}/d_i$ for $d_i > 0$ and $\tilde{g}_{ij} = 0$ otherwise.

The Linear-in-Means Models (LIM) describe student's best-response function as:

$$y_i = \gamma \bar{y}_{-i} + \beta \mathbf{X_i} + \epsilon_i \tag{3.1}$$

where $\bar{y_{-i}} = \sum_{j=1}^{n} \tilde{g}_{ij} y_j = \frac{1}{d_i} \sum_{g_{ij}=1} y_j$ is the weighted average activity of individual i's neighbors; γ is the peer-effect parameter.

Boucher et al. (2024) study peer effects in a context in which peers do not necessarily react to the average behavior of their neighbors. Their findings reveal that in areas such as GPA, self-esteem, exercise, and study effort, individuals tend to be influenced more by peers who demonstrate higher levels of effort, providing greater positive spillovers. Conversely, in behaviors associated with negative outcomes like trouble at school, fighting, and drinking, the influential peers are those who exhibit lower effort levels. Their model also considers social clubs, highlighting that students who participate in clubs tend to exert a stronger influence on their peers who view them as friends.

However, most of these peer-effect studies set a strong assumption that the social networks are exogenous, potentially overlooking selection bias in the formation of friendships. For instance, students with higher GPAs are more likely to become friends, as are those engaged in similar activities, including club participation and study efforts. Club participation, in particular, serves as a venue for forming new connections. Empirical data indicates that students who join the clubs are indeed more likely to form new connections with fellow members.

Enhanced LIM Model

The racial group index is defined as $r \in \{w, b, o\}$, where w, b, and o represent White, Black, and Other, respectively. I estimate the peer effects exerted by peers from each racial group on each of these three racial categories. The model is specified as follows:

$$y_{irs} = \gamma_{1r} \sum_{j \in w} \tilde{g}_{ij} y_j + \gamma_{2r} \sum_{j \in b} \tilde{g}_{ij} y_j + \gamma_{3r} \sum_{j \in o} \tilde{g}_{ij} y_j + \beta X_{irs} + \epsilon_s + \epsilon_{irs} \tag{3.2}$$

where dummy variable y_{irs} represents the participation decision of individual i from racial group r in school s, with 1 indicating participation and 0 indicating non-participation. The term $\sum_{j \in r} \tilde{g}_{ij} y_j$ denotes the weighted average activity level of peers from racial group r, where $r \in \{w, b, o\}$. γ_{1r}, γ_{2r}, and γ_{3r} denote the peer effect parameters for White peers, Black peers, and peers from Other racial groups on students belonging to racial group r. $\boldsymbol{X}_i$ is a vector capturing student-specific characteristics[1]. ϵ_s represents school-level fixed effects, and ϵ_{irs} is an individual-specific error term.

The peer effect parameters, γ_{1r}, γ_{2r}, and γ_{3r} are defined as follows:

$$\begin{aligned}
\gamma_{1r} &= (\gamma_{1w}, \gamma_{1b}, \gamma_{1o}) \cdot R_i^{3\times 1}, \\
\gamma_{2r} &= (\gamma_{2w}, \gamma_{2b}, \gamma_{2o}) \cdot R_i^{3\times 1}, \\
\gamma_{3r} &= (\gamma_{3w}, \gamma_{3b}, \gamma_{3o}) \cdot R_i^{3\times 1},
\end{aligned}$$

where R_i is a dummy vector indicating the race of student i.

If peer spillovers within and across racial groups are equivalent, such that $\gamma_{1r} = \gamma_{2r} = \gamma_{3r}$, the enhanced model simplifies to the standard LIM model in (3.1). However, allowing for γ_{1r}, γ_{2r}, and γ_{3r} to differ captures potential asymmetries in peer influence due to racial group identity, thereby reflecting differential exposure to peer behaviors.

In this peer-effects analysis, I use peers' health conditions and energy levels as IVs for their participation decisions. These IVs are based on peers' exogenous characteristics and are not influenced by student i's decisions, helping to address the reflection problem by mitigating endogeneity.

The utility function for each connected individual i is defined as

$$U_{irs}(y_{irs}, y_j, \mathbf{g}) = y_{irs}\left(\gamma_{1r} \sum_{j \in w} \tilde{g}_{ij} y_j + \gamma_{2r} \sum_{j \in b} \tilde{g}_{ij} y_j + \gamma_{3r} \sum_{j \in o} \tilde{g}_{ij} + y_j \beta X_{irs} + \epsilon_s + \epsilon_{irs}\right)$$

---

[1] In this analysis, individual characteristics include gender, age, health status, energy levels, attention difficulties, academic performance, and perceived social acceptance. Additionally, a student is classified as a 'connector' if they participate in more than one sport.

which equates to the optimal response function in the enhanced LIM model 3.2 when $y_{irs} = 1$; and to 0 when $y_i = 0$.

3.3.2 Social Interaction Term

To address the problem stemming from endogenous networks, I revise the methodology by updating the variable to the average participation rate within a specific racial group. This rate is calculated as $\frac{1}{n_{rs}-1} \sum\limits_{j=1,j\neq i}^{n_{rs}} y_j$, where n_{rs} represents the size of the racial group r in school s, j denotes other students within the same group g, and the decision of the student is defined in the regression function provided in (3.3). This approach is chosen because the average activity level of a group—assuming the group size is sufficiently large—can be considered a variable unaffected by the network's structure and thus assumed to be exogenous. This adjustment allows us to better understand the influence of social norms on a student's decision to participate, by relying on a measure that remains stable regardless of individual network connections.

Student's Participation Decision is formally expressed as:

$$y_{irs} = \underbrace{\lambda \left(\frac{1}{n_{rs}-1} \sum_{\substack{j=1,j\neq i \\ j\in r,s}}^{n_{rs}} y_j \right)}_{\text{social interaction effects}} + \underbrace{\beta_1 w_{-r,s}}_{\text{effect of other race}} + \underbrace{\beta_2 X_i}_{\text{individual } char.} + \underbrace{\beta_3 Z_s}_{\text{school char.}} + \epsilon_i$$

$$\text{with } i = 1, ..., n_{rs} \quad (3.3)$$

where y_{irs} represents the participation of student i in racial group r, school s; the term $\frac{1}{n_{rs}-1} \sum_{j\neq i} y_j$ captures the social interaction effects, calculated as the average participation rate of other students within the same race r in school s, excluding student i himself; n_{rs} denotes the number of students in group rs. The error term ϵ_i is assumed to be independently and identically distributed according to a logistic distribution.

Let $w_{-r,s}$ denotes the participation in sports by other racial groups within the same school , aiming to gauge the impact of inter-racial interactions on participation; X_i encompasses individual characteristics,[2] Z_s outlines school characteristics, including factors like school size, racial composition, type, location, and region.[3]

In this study, I examine the interaction between the participation rates of students from racial groups other than one's own and the significant presence (exceeding 2%) of a specific racial group (black or white) within the school. For instance, when evaluating the participation of white students, I consider the impact of participation rates from all non-white groups, but only in scenarios where black students make up more than 2% of the school population.

Additionally, as a robustness check, I explore the interaction specifically between the participation rate of black students and their presence in the school. This approach, however, tends to produce less reliable results due to the lack of proportionate data on the black student population.

I further assess how the effect of social interactions might vary with school size by introducing an interaction term between the average participation rate and the size of the school. This analysis helps us understand if and how the influence of peer participation shifts in larger versus smaller school settings.

Let $\mathbb{X}_{irs} = \{w_{rs}, X_i, Z_s\}$ denote the set of characteristics encompassing the effect of other races, individual traits, and school attributes. Additionally, $\bar{y}_{-irs} = \frac{1}{n_{rs}-1} \sum_{j \in rs,\, j \neq i} y_j$ represents the average participation rate among students within the same racial group as student i. The probability that student i in group rs chooses to participate in clubs can be expressed as:

$$Pr(Y_{irs} = 1 | \mathbb{X}_{irs}) = \frac{1}{1 + e^{-\eta_{irs}}}$$

---

[2] In the analysis, individual characteristics include gender, age, duration of school attendance, health status, energy level, academic performance, and perceived social acceptance. I also incorporate control variables to account for each student's participation in other clubs, including basketball, baseball, football, track, other sports, and non-sport clubs. This consideration is essential as the seasonal nature of sports in US middle schools could influence students' participation decisions across different activities.

[3] At the school level, control variables encompass the type of school (public, Catholic, or private), its location (urban, suburban, or rural), and its geographical region (West, Midwest, South, or Northeast).

Here, η_{ig} is the linear predictor part of the model, defined as:

$$\eta_{irs} = \lambda \bar{y}_{-irs} + \beta \mathbb{X}_{irs}$$

This formulation allows us to estimate the likelihood of club participation for student i, factoring in both the average participation rate of their peers and a combination of other significant characteristics through a logistic regression model.

In the framework, individual choices y_{irs} are characterized as the outcome of maximizing a payoff function U, with each student's decision within group rs being:

$$y_{ig} \in \arg\max U\left(\mathbb{X}_{irs}, \mu_i^e(y_{-irs}), \epsilon_{ig}\right)$$

Here, $\mu_i^e(y_{-irs})$ represents individual i's beliefs about the behavior of other members in their group g. This model, which incorporates social interactions, is refined through assumptions specifying how $\bar{y}_{-ig}$, the average behavior of peers, is established. The payoff function is given by:

$$U_{ig}(y_{irs}, \bar{y}_{-irs}, \mathbb{X}_{irs}) = \lambda y_i \bar{y}_{-irs} + \beta y_i \mathbb{X}_{irs}$$

My model assumes that students' beliefs about their peers' choices are self-consistent: $\bar{y}_{-ig} = \mu(y_{-irs}|F_i)$. This assumption of self-consistency, commonly adopted in both theoretical and econometric literature, bridges the model by enforcing an equilibrium condition. It aligns subjective beliefs, $\mu_i^e(y_{-irs})$ with the objective conditional probabilities of others' behaviors given the information set F_i, ensuring that individual beliefs about peer behavior accurately reflect observed patterns.

Table 3.3: Data Description

	White	Black	Hispanic	Asian	Other
Number of Students	32635	7579	8860	2859	11326
Proportion in The Sample (%)	46.1	12.9	17.3	4.0	19.7
Homophily in The Sample (%)	73.4	63.6	523.0	46.9	–
Number of Connections	7.46	6.77	6.32	6.44	6.75
Club Participation Rate					
Basketball Participation (%)	17.0	27.8	14.5	15.1	22.2
Baseball Participation (%)	18.8	14.3	15.3	7.5	17.4
Football Participation (%)	11.7	15.6	13.0	8.1	14.8
Track Participation (%)	12.4	18.8	9.4	9.6	14.7

Notes: The sample consists of 63,259 students from 134 schools. This sample size is smaller than that used in Chapter 2 (which includes 83,152 students) due to missing information on certain control variables (e.g., average grade, health, and energy) for specific students.

3.4 Empirical Analysis

3.4.1 Data Description

The analysis in Chapter Two is conducted at the school level, assuming that each school reaches its steady-state equilibrium. As a result, all students from the target schools are included in the sample. In contrast, the empirical analysis in this chapter focuses on the individual level, leading to a smaller sample size. Certain observations are excluded from the sample for the following reasons: (1) missing individual characteristics for some students and (2) students reporting participation in more than ten clubs. Consequently, the sample size decreases from 83152 in Chapter Two to 63259 in this chapter (Chapter Three). Data description is provided in Table 3.3.

3.4.2 Peer Effects in LIM

The estimated peer effect coefficients for the model with different racial groups are presented in Table 3.4, while the full regression results, including some other parameters, are provided in Table 3.7 in the Appendix.

Approximately 19.7% of students reported having no friends. I assign separate school-level

Table 3.4: Peer Effects for Three Racial Groups

	White	Black	Others
λ_{1r}	3.05^{***}	1.54	0.57
	(0.44)	(1.04)	(0.50)
λ_{2r}	4.54^{***}	2.39^{***}	4.96^{***}
	(0.78)	(0.36)	(0.32)
λ_{3r}	3.02^{***}	1.77^{***}	0.73
	(0.49)	(0.42)	(0.51)

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

fixed effects (ϵ_s) for two groups within each school: students with no friends and students with at least one friend. Testing the null hypothesis $H_0 : \lambda_{1w} = \lambda_{2w}$ evaluates whether peer effects from White and Black peers are significantly different for white students. The p-value is 0.074, suggesting a marginally significant difference.

If the same fixed effect is assigned to all students within the same school, the peer effect coefficients decrease due to the inclusion of isolated students. Under this specification, the peer effect estimates are $\hat{\lambda}_{1w} = 1.42$ with standard error $\hat{\sigma}_{1w} = 0.29$ and $\hat{\lambda}_{2w} = 3.70$ with $\hat{\sigma}_{2w} = 0.77$. The p-value for the null hypothesis $H_0 : \lambda_{1w} = \lambda_{2w}$ is 0.0043, indicating a significant difference between the spillovers from White and Black peers.

These analysis suggests that the observed greater peer effect from black friends, compared to white friends, may be attributed to selection bias in the formation of friendships. Friendships are not formed randomly but are influenced by preferences. White students may have more opportunities to connect with other white peers, while interactions with black peers are more likely to occur in the context of basketball clubs. Ignoring the endogenous nature of network connections in estimating peer effects could lead to an overestimation of the influence of black students.

Following this, I analyze the impact of joining sports clubs on white students' friendship connection. Table 3.8 presents the effects of the three most popular sports—basketball, baseball, and football—on the number of black and white friends reported by white students. The dependent variables are each white student's white friends number or black friends number; the factors I considered here include the relative size of the corresponding racial group in a school and a dummy variable for club participation, and the interaction term of club participation and relative group size. There might exists some unobservable variables impact both club participation and friendship connections, including personalities and social abilities. I use student's general health condition and energy level as IV on the variable of basketball participation.

The empirical results indicate that while participation in basketball clubs doesn't significantly affect the number of white friends among white students, it does increase their number of black friends. This points to the endogeneity of network structures in such settings. Ignoring the friendships formed through club participation can lead to a biased estimation of peer effects. Specifically, the influence of club members on each other's behavior might be overvalued. Moreover, these findings support the idea that cultural activities, like being part of a club, encourage students to form new friendships across racial lines, helping to reduce racial segregation.

My analysis reveals distinct preferences for club participation among white and black students, with baseball being the top choice for white students, and basketball preferred by black students. This preference hierarchy leads to a noticeable racial disparity in club memberships, particularly in basketball. To delve deeper into how the popularity of sports affects student behavior across different racial groups, I focus the analysis on basketball and baseball clubs.

I observe a clear pattern of coordination within racial groups, evidenced by the positive correlation between the proportion of a racial group in a school and its participation rate in specific sports: an increase in the white student population at a school is associated with higher participation in basketball clubs among white students; similarly, a higher presence of black students in a school leads to increased enrollment in baseball clubs by black students. This pattern does not replicate for white students in baseball or black students in basketball, indicating a unique interplay between sports participation and prevailing social norms.

The decision to participate in a sports club often transcends simple interest in the sport itself, serving instead a social function by offering avenues for new friendships. This underlines the significant role of the social aspect of these clubs in forming friendship networks and potentially easing racial divisions.

3.4.3 Estimates on Social Interaction Term

Analysis on White students in Basketball Participation

This section delves into the factors influencing the decision of white students to join basketball clubs. My analysis utilizes two metrics of group size: the total student population at each school and the specific population of white students. The outcomes are detailed in the regression results presented in Table 3.9, with a robustness check utilizing the white student population as an explanatory variable.

Racial proportion: The racial makeup of a school significantly impacts student choices. The Average Marginal Effects (AME), detailed in Table 3.11, suggest that an increase in the white student proportion by 1% raises the likelihood of their participation in basketball clubs by 0.03%. This finding implies that the influence of societal norms on individual behavior diminishes in schools with a higher concentration of white students, who thereby exert more significant influence within their social circles. However, it's crucial to note that AME may not capture non-linear relationships between population proportion and participation rate effectively, especially in the context of large schools which dominate the binary choice analysis. Additionally, the predominance of students from larger schools in the binary choice analysis means that the AME may not accurately reflect the dynamics in smaller schools. To address this, I further disaggregate the analysis by separating students from smaller and larger schools to more precisely estimate the effects of social interactions.

Participation Rate: My findings indicate that social interaction plays a crucial role in club participation decisions. Specifically, a 1% increase in white students' participation rate correlates with a 0.42% increase in the likelihood of a white student joining the basketball club. In contrast, high participation rates among other racial groups can deter white students' participation, particularly if the black student population exceeds 3% in the school. This complex interplay highlights the significant role of cultural activities in fostering communal ties and understanding the multifaceted nature of students' decision-making processes.

School Size: The size of the school emerges as a critical factor, with larger schools showing a negative impact on the likelihood of students joining basketball clubs. This relationship is quantified in the AME for school size, which indicates increasing the school size by a factor of e reduces club participation probability by 1.73%. These findings suggest that beyond facility constraints, larger school environments may dilute personal interactions and weaken social cohesion, affecting students' extracurricular engagement.

Intersection of School Size and Club Participation: I introduce an interaction term between school size and club participation. This term reveals that in larger schools, the influence of a club on student choices becomes significantly more pronounced. This insight suggests that as schools grow in size, the role of clubs in shaping social networks and providing a sense of belonging gains importance, potentially counteracting the impersonal nature of larger school environments. This may be due to greater cohesion among club members in these settings.

Black Students in Basketball Participation:

The analysis of black students' decisions to join basketball clubs is detailed in Table **??**. Unlike what might be expected, the data reveals that the rate of participation among black students in basketball clubs is not influenced by the proportion of black students within the school. This suggests that having a larger number of black students in a school does not necessarily lead to increased participation in basketball among black students. This finding points to a lack of coordinated group behavior or peer influence among black students regarding participation in this sport.

Participation in Baseball Club

The analysis extended to the participation decisions of white and black students in baseball clubs, with results documented in Table 3.12 for white students and Table **??** for black students. Baseball's popularity traditionally leans more towards white students.

An interesting discovery from the analysis is that the participation of black students in baseball clubs shows a positive correlation with the proportion of black students within the school, suggesting a significant coordination effect. A higher presence of black students seems to foster an environment where forming same-race friendships within the club is more likely, thus increasing the perceived value of joining the club for black students.

Conversely, an interesting pattern emerges where the involvement of white students in baseball clubs appears to encourage greater participation among black students, marking a departure from the basketball participation trends. In basketball, increased popularity of the sport among black students has been associated with a reduced participation rate among white students. This contrast underscores the complex interplay of majority and minority group dynamics within societal structures. While majority groups might predominantly seek connections within their racial or ethnic circles, minority groups navigate the dual objectives of strengthening intra-group bonds and integrating more extensively with the broader community, as reflected in their diverse participation patterns in school clubs.

Small Schools v.s. Large Schools.

My analysis further reveals that white students exhibit more variability in their club participation rates when they are in the majority within smaller schools. Notably, in smaller schools where white students form the majority, their participation rates significantly exceed the general social norm, although this average participation rate can vary considerably from one school to another.

To explore the differences between small and large schools, I divided students into two categories based on school size and conducted separate logit regressions for each group. The outcomes of this analysis are detailed in Table 3.16.

In small schools, the relationship between students' decisions to join the basketball club and their racial proportion exhibits a quadratic increase, indicating a nonlinear growth in participation as the proportion of their racial group increases. Conversely, in large schools, this relationship is more linear, with the participation decision showing a direct proportionality to their racial makeup. Moreover, the influence of racial proportion on club participation diminishes as school size increases, suggesting that social norms wield less sway in a small micro-social environment.

The social interaction effects appear more pronounced in larger schools than in smaller ones. This phenomenon could be attributed to a higher level of clustering among club members in larger schools, where club participants are more inclined to form friendships with fellow members compared to non-members. In contrast, small schools show minimal differences in friendship patterns between club members and non-members, indicating lower levels of matching bias and homophily. This differentiation points to the complex dynamics of social interaction and club participation across schools of varying sizes, highlighting the influence of institutional scale on the formation of social networks within school clubs.

3.4.4 Dynamics in Friendship Connections

Stylized Facts 1. *Club members tend to make more friends within their club. A stronger clustering effect is observed among club members in larger schools.*

A comparison of friendship networks between two waves, one year apart, shows that club members are more likely to form new connections with other members. Despite the formation of new friendships and the dissolution of old ones, the total number and composition of friendships remain stable across the sample. Club members tend to build more connections with other members, with the proportion of within-club friends exceeding the club participation rate at the school. In contrast, non-members' friendships are more randomly distributed across the school.

To investigate the dynamics of student friendships over time, I analyzed the transition from wave 1 (1984-1985) to wave 2 (1986), focusing on those who participated in home interviews during both years. Students were asked to list up to ten friends, with a maximum of five male and five female friends. By integrating this subset of individuals with data from school questionnaires collected in the first year, I crafted a dataset of $1,557$ students, laying the groundwork for the analysis.[4]

A comparative study was conducted to explore the changes in the number of friendships reported

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[4] The analysis is grounded in a subset of this data, in which students were asked to nominate no more than 10 friends. However, most of the interviews solicited information on no more than micro-social environment friends. Despite the smaller sample size, this subset offers rich insights into the dynamics of friendship connections over time.

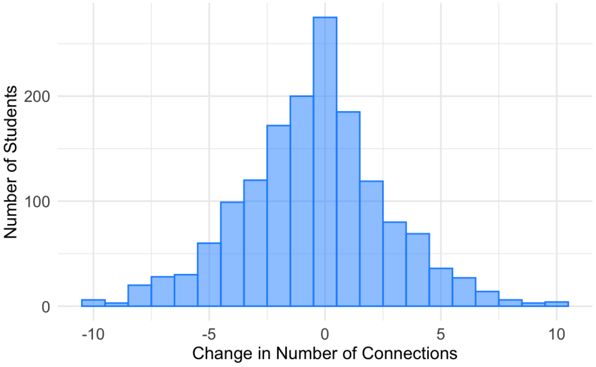

Figure 3.2: Distribution of Students by Change of Friendship Connections

Note: The mean of this difference is -0.47, indicating that, on average, students have 0.47 fewer friends in the second year than in the first. The first quantile, median, and third quantile of the difference are -2, 0, and 1 connection, respectively.

at the two time points. The variation in friendship counts is depicted in 3.2, showing a distribution that approximates a normal curve, primarily clustered around zero. This pattern suggests a general stability in the number of friendships over the studied period.

From the second wave's responses, I documented $8,687$ friendship connections, averaging 5.6 friends per student. Among these, $4,063$ friends were identifiable from the first wave's in-school questionnaires, offering concrete insights into the continuity and evolution of these friendships. A detailed analysis of the students' friendship networks, especially focusing on newly established connections within the last year, allowed us to identify recent friendships and critically assess potential biases in friendship selection reflected in the empirical findings.

Of the $4,063$ traceable connections, $2,289$ represented new friendships formed during the second wave. This fluidity—where students dissolve old ties and create new ones—reveals that, on average, 56% of the connections are newly formed. This suggests that while individual friendships are dynamic, the overall network size maintains equilibrium.

This exploration into the dynamics of friendship formation within schools, especially in relation to basketball club participation, unveils key insights into how these cultural activities influence social networks.

There's a notable distinction in friendship network structures between small and large schools. In larger schools, there's a marked trend among club members to form friendships within their

Table 3.5: Matching Bias in Friendship Networks (Across School Sizes)

Type of schools	Type of students	New friends-Part.	Total Friends-Part.	Sample size
Small (52.1% participation)	Non-participants Club participants	52.80% 62.14%	51.85% 70..48%	67 73
Medium (40.6% participation)	Non-participants Club participants	35.05% 62.38%	33.21% 67.91%	101 69
Large (14.4% participation)	Non-participants Club participants	15.84% 30.24%	14.78% 32.12%	1066 180

Note: This analysis includes 1, 557 students from 12 schools, categorized by school size: small (six schools with fewer than 100 students), medium (four schools with 100 to 200 students each), and large (two schools with student populations of 776 and 1, 743, respectively). The variables "New friends-Part." and "Total Friends-Part." in the table represent, respectively, the percentage of new connections that are club participants and the overall percentage of a student's friends who are club participants in wave 2, differentiated by various student groups.

group, indicating a strong matching bias. This contrasts with small schools, where the networks of club participants and non-participants are more integrated, suggesting less segregation based on club involvement.

The detailed breakdown in table 3.5 shows how friendship compositions vary with school size and club participation. In smaller schools, where the club participation rate averages 52.1%, club members show a slight preference for forming new connections within their group, with 62.1% of new friendships being intra-club for participants. For non-participants, 52.8% of new friendships are still within the club circle, albeit to a lesser extent.

The trend intensifies in medium-sized schools, where 62.4% of new friendships by club members are within the club, against 35.1% for non-participants. This pattern suggests an escalating inclination towards forming connections within the club as school size grows.

Large schools exhibit the most pronounced matching bias, with club members making 30.2% of their new friendships within the club, starkly contrasting the 15.8% rate among non-participants. This disparity highlights a significant preference for intra-club socialization in larger settings, solidifying the club's role as a key social junction.

These trends align with my regression analyses regarding the interplay between social interaction effects and school size. It highlights the nuanced impact of school size on the nature of social

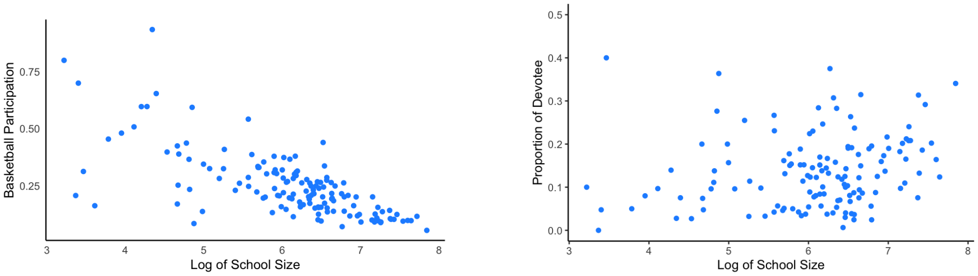

Figure 3.3: Club Participation and Devotee Proportion by School Size

interactions within clubs. While the likelihood of joining a basketball club decreases with increasing school size, the social cohesion and intensity of interactions within clubs grow in larger schools, promoting the development of robust, club-centric social networks.

3.4.5 Participation Rate and Ratio of Devotee by School Size

Stylized Facts 2. *In this analysis, students are classified as either Connectors or Devotees. As school size increases, the overall participation rate decreases, but the proportion of Devotees among club members rises.*

A club member is defined as a Devotee if basketball is the only club they participate in; otherwise, they are categorized as a Connector. Connectors tend to join clubs to expand their social networks, while Devotees are primarily motivated by a passion for the sport.

The data show that Connectors have more friendships than both Devotees and non-members, while Devotees' friendships are more diverse across racial groups.

School size. The relationship between school size and club participation is visually represented in left-hand graph in Figure 3.3, highlighting the clear negative correlation. This trend is largely due to the capacity constraints faced by the clubs. As school size increases, these constraints become more significant, reducing students' opportunities to engage in club sports.

Devotee and Connectors:

In this analysis, I classify students into two types: Connectors and Devotees. A club member is identified as a Devotee if basketball is the only club they participate in; otherwise, they are labeled a Connector. Connectors typically join clubs to expand their social networks, while Devotees are motivated by their passion for the sport.

Connectors measure success by the quantity and quality of interpersonal connections they develop. These agents value the social structure of the club and are motivated by the opportunity to engage with more people. In contrast, Devotees are agents who join the club due to their intrinsic love for the sport the club focuses on. They derive their primary satisfaction from the level of skill and talent present within the club environment. Their decisions within the club are motivated by opportunities to improve their own abilities and to engage in high-level competitions or practices.

The data reveals that Connectors tend to have more social connections than Devotees—for instance, white Connectors have an average of 7.8 connections, compared to 6.7 for white Devotees. However, Connectors are more affected by school size; the proportion of Connectors in a club decreases as school size increases, regardless of racial group. This trend is illustrated in right-hand graph in Figure 3.3. This decline is largely driven by the rising costs of participation due to capacity constraints in larger schools. While Devotees benefit from the competitive, high-talent environment that larger schools often provide, Connectors do not gain the same advantages in such settings.

On the other hand, White Devotees form more connections across racial groups than both Connectors and Non-Members, as illustrated in Table 3.6 and Figure 3.4. From a social planning perspective, increasing the capacity for extracurricular activities to include more Devotees could enhance social cohesion. Additionally, as more Connectors join the clubs, they tend to build more connections. Although their level of homophily may not change, the increase in connections can still contribute to greater social integration. Furthermore, White Connectors serve as a bridge between White students outside the club and White Devotees, creating indirect links between White students and club members from other racial groups.

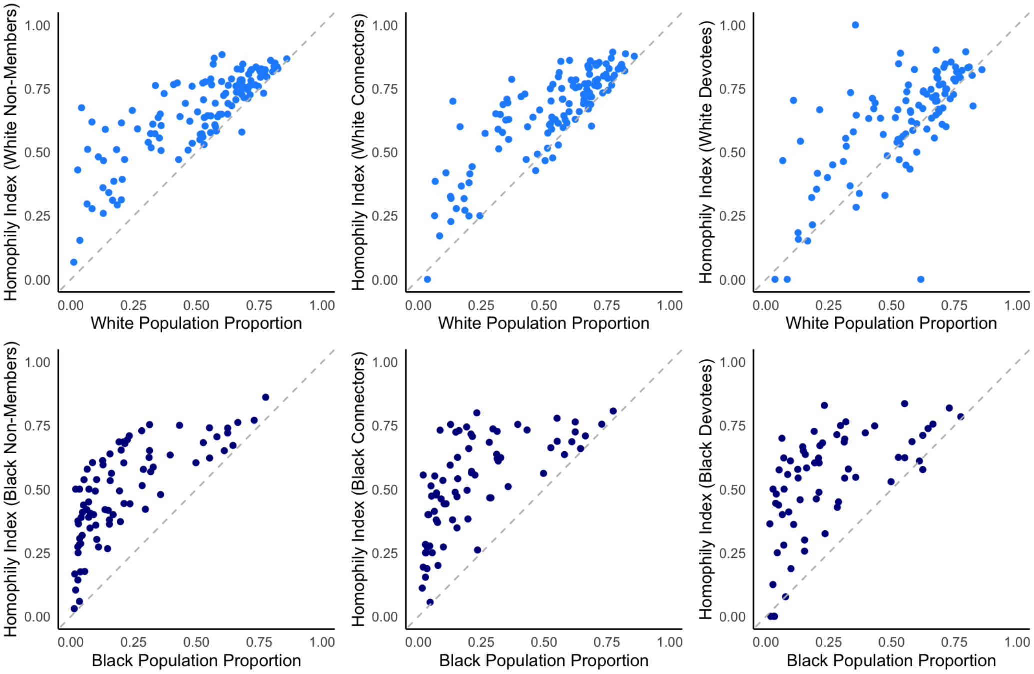

Figure 3.4: Homophily Index for Different Types of Students

Note: Students are categorized into three groups: Connectors, Devotees, and Non-Members. The first row of the graph represents White students, while the second row corresponds to Black students. The homophily index is defined as the proportion of connections within the same racial group, and the graph displays the average homophily across schools.
If connections were formed randomly, the homophily index would align with the racial proportion, represented by the 45-degree line on the graph. A point above this line indicates that students in that school are more likely to form connections within their own racial group, reflecting inbreeding homophily. The further the point is above the 45-degree line, the greater the level of racial segregation.

Table 3.6: Homophily Indices by Student Type (%)

	Non-Members	Connector	Devotee
White	73.0	71.7	64.1
Black	62.6	59.9	58.9
Hispanic	58.6	36.4	45.6
Asian	45.8	46.5	47.1

Notes: Among white students, 17.0% are club members, of which 10.0% are devotees. Among black students, 27.8% are club members, of which 19.6% are devotees. Among Hispanic students, 14.5% are club members, of which 17.0% are devotees. Among Asian students, 15.1% are club members, of which 14.9% are devotees.

3.5 Conclusion

This study delves into sports club participation among US adolescents, shedding light on the the role of cultural activities in shaping social interactions. A central finding from the peer-effects analysis is the overvaluation of peer effects due to the endogenous nature of networks structures. Social clubs emerge as pivotal platforms for social engagement, serving as essential venues for establishing new connections.

While instrumental variables (IV) help mitigate the direct spillover effects of an individual's activities on their friends, they fall short of addressing the underlying endogeneity within the network structure itself. My analysis reveals that compared with white peers, black peers exert more spillovers on white students' decisions to participate in basketball, with the use of IV potentially magnifying this discrepancy.

Further examination of basketball participation's impact on friendship connections among white students shows that while it does not significantly affect the number of white friends, it notably increases the number of black friends. This outcome underscores the endogenous selection bias inherent in evaluating peer effects and confirms the complex interplay between club participation and friendship formation.

Building on this, the study leverages average school-wide club participation rates as an exogenous factor to investigate how social interactions influence club membership decisions. A significant discovery from this analysis is that white students show a higher propensity to engage in basketball clubs as their representation increases within the school. Conversely, black students are more inclined to participate in baseball clubs when they form the majority. This trend suggests a form of coordination among students, with each group gravitating towards sports that are traditionally favored by the other racial group.

A comparison between smaller and larger schools reveals contrasting participation dynamics and network structures. While cultural activities and club participation rates are lower in larger schools, these institutions foster stronger connections among club members compared to non-members. This finding is consistent with regression results, which suggest that the influence of social interactions on club participation decisions is more pronounced in larger schools.

Additionally, previous studies have highlighted lower levels of racial homophily in smaller schools. My research extends this understanding by suggesting that the high rates of participation in cultural activities within small schools may contribute to this phenomenon. Recognizing the coordinated nature of students' participation, policymakers can implement more nuanced and context-specific interventions. For instance, providing targeted subsidies for basketball programs in predominantly white neighborhoods and for baseball in predominantly black neighborhoods could encourage diverse participation and promote inclusivity. This tailored approach addresses the unique cultural preferences of different communities and fosters social cohesion through strategic support for sporting and cultural activities.

Table 3.7: Peer Effects on Club Participation Across Racial Groups

	Club Participation					
	Split Fixed Effects			Pooled Fixed Effects		
	Baseline	IV	IV	Baseline	IV	IV
White γ_1	2.63^{***}	3.05^{***}	2.93^{***}	2.43^{***}	1.42^{***}	1.37^{**}
	(0.15)	(0.44)	(0.68)	(0.08)	(0.29)	(0.42)
White γ_2	2.72^{***}	4.54^{***}	4.58^{***}	2.44^{***}	3.70^{***}	3.63^{***}
	(0.36)	(0.78)	(0.76)	(0.39)	(0.77)	(0.74)
White γ_3	2.25^{***}	3.02^{***}	2.91^{***}	2.03^{***}	1.54^{***}	1.38^{***}
	(0.14)	(0.49)	(0.56)	(0.12)	(0.36)	(0.42)
Black γ_1	2.59^{***}	1.54	1.74	2.14^{***}	−1.25	−1.35
	(0.47)	(1.04)	(1.23)	(0.48)	(0.97)	(1.00)
Black γ_2	2.02^{***}	2.39^{***}	2.68^{***}	1.59^{***}	0.57^{*}	0.45
	(0.20)	(0.36)	(0.55)	(0.12)	(0.29)	(0.34)
Black γ_3	1.60^{***}	1.77^{***}	1.96^{***}	1.17^{***}	0.01	−0.13
	(0.18)	(0.42)	(0.55)	(0.17)	(0.36)	(0.40)
Other γ_1	1.42^{***}	0.57	0.49	1.10^{***}	-1.52^{***}	-1.63^{***}
	(0.15)	(0.50)	(0.70)	(0.11)	(0.32)	(0.48)
Other γ_2	2.99^{***}	4.96^{***}	5.03^{***}	2.62^{***}	3.98^{***}	3.90^{***}
	(0.21)	(0.32)	(0.40)	(0.16)	(0.29)	(0.39)
Other γ_3	1.69^{***}	0.73	0.81	1.37^{***}	-1.31^{***}	-1.55^{**}
	(0.11)	(0.51)	(0.71)	(0.10)	(0.35)	(0.47)
Black	0.71^{***}	0.66^{***}	0.61^{***}	0.75^{***}	0.84^{***}	0.85^{***}
White	-0.43^{***}	-0.55^{***}	-0.53^{***}	-0.49^{***}	-0.65^{***}	-0.66^{***}
Gender	0.45^{***}	0.45^{***}	0.44^{***}	0.49^{***}	0.50^{***}	0.50^{***}
Age	-0.08^{***}	-0.08^{***}	-0.08^{***}	-0.07^{***}	-0.11^{***}	-0.12^{***}
Health	0.12^{***}	0.11^{***}	0.11^{***}	0.11^{***}	0.13^{***}	0.13^{***}
Energy	0.20^{***}	0.20^{***}	0.20^{***}	0.20^{***}	0.21^{***}	0.21^{***}
Connector	1.23^{***}	1.16^{***}	1.16^{***}	1.23^{***}	1.25^{***}	1.26^{***}
Ctrl. Var.	✓	✓	✓	✓	✓	✓
Clustered SE			✓			✓
Log Likelihood	−24334.66	−25217.60	−25217.67	−24563.24	−25356.50	−25359.29
Deviance	48669.32	50435.21	50435.34	49126.47	50713.01	50718.57
Observations	63230	63259	63230	63259	63259	63259

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Notes: *Split Fixed Effects Method* refers to assigning separate school-level fixed effects for students with no friends and those with at least one friend within each school. *Pooled Fixed Effects Method* uses the same school-level fixed effect for all students, regardless of their friendship status.

Table 3.8: Impact of Sports Activities on the Racial Composition of White Students' Friendships

	Number of White Friends			Number of Black Friends				
	White	White	White IV	Black	Black	Black IV	Black IV	Black IV
OwnRacialProp	3.44***	3.18***	2.69***	0.84***	0.74***	0.86***	0.76***	0.76***
	(0.08)	(0.09)	(0.13)	(0.02)	(0.02)	(0.03)	(0.05)	(0.05)
Basketball	0.27***	−0.70***	6.94***	0.04***	−0.00	0.09	0.06	0.05
	(0.04)	(0.15)	(0.54)	(0.01)	(0.01)	(0.05)	(0.06)	(0.05)
Basketball×RacialProp		1.55***			0.72***		0.77	0.89*
		(0.23)			(0.06)		(0.56)	(0.43)
Baseball	0.44***	0.43**	−1.00***	0.00	0.00	−0.01	−0.01	−0.01
	(0.04)	(0.14)	(0.13)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)
Baseball×RacialProp		0.02			0.02		0.04	
		(0.22)			(0.05)		(0.10)	
Football	0.28***	0.06	−0.52***	0.01*	0.01	0.01	−0.00	0.01
	(0.05)	(0.17)	(0.09)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Football×RacialProp		0.34			0.10		0.09	
		(0.27)			(0.07)		(0.10)	
Log(SchSize)	0.20***	0.20***	0.63***	0.01	0.00	0.01	0.01	0.01
	(0.02)	(0.02)	(0.05)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)
Gender	−0.51***	−0.51***	−0.78***	−0.01**	−0.01**	−0.01**	−0.01**	−0.01**
	(0.03)	(0.03)	(0.05)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Age	−0.03**	−0.03**	0.18***	−0.01***	−0.01***	−0.00*	−0.00*	−0.00*
	(0.01)	(0.01)	(0.02)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Years	0.13***	0.13***	0.08***	0.00**	0.00**	0.00*	0.00**	0.00**
	(0.01)	(0.01)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
Avg.Grade	−0.38***	−0.38***	−0.12***	0.01***	0.01***	0.01***	0.01***	0.01***
	(0.02)	(0.02)	(0.03)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
R^2	0.09	0.09	−0.73	0.05	0.06	0.05	0.05	0.05
Adj. R^2	0.09	0.09	−0.73	0.05	0.06	0.05	0.05	0.05
Num. obs.	33690	33690	33690	33690	33690	33690	33690	33690

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Note: 'RacialProp' refers to the proportion of a specific racial group within a school. The first three columns assess the impact of club participation on the number of white friends among white students, facilitating comparison with the impact on the number of black friends depicted in the last five columns. The Variance Inflation Factor (VIF) shows elevated levels in the regression of the 2^{nd} column and reaches extremely high levels in the 3^{rd} column. Especially, a negative R^2 value in the Instrumental Variable (IV) analysis concerning white friendships indicates that basketball participation does not contribute to an increase in the number of white friends. The first three columns are included for comparative analysis regarding the number of black friends.

Table 3.9: White Students’ Choices in Basketball Club

	White Students’ Participation in Basketball Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-0.57^{*}	-0.12	-0.53	0.32	0.25	0.88^{*}
	(0.28)	(0.31)	(0.32)	(0.35)	(0.35)	(0.41)
White Participation $_{-i}$	4.29^{***}	4.15^{***}	4.28^{***}	3.79^{***}	2.25^{**}	1.17
	(0.20)	(0.22)	(0.26)	(0.24)	(0.85)	(1.04)
White Proportion	0.50^{***}	0.31^{**}	0.43^{***}	0.29^{*}	0.30^{**}	0.29^{*}
	(0.11)	(0.11)	(0.11)	(0.13)	(0.12)	(0.13)
log(School Size)	-0.14^{***}	-0.12^{***}	-0.09^{**}	-0.15^{***}	-0.18^{***}	-0.25^{***}
	(0.03)	(0.03)	(0.03)	(0.04)	(0.04)	(0.05)
Gender	0.50^{***}	0.25^{***}	0.25^{***}	0.24^{***}	0.25^{***}	0.24^{***}
	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Age	-0.15^{***}	-0.15^{***}	-0.14^{***}	-0.15^{***}	-0.14^{***}	-0.15^{***}
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Years	-0.02	-0.01	-0.02	-0.02	-0.01	-0.02
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Health	0.20^{***}	0.14^{***}	0.14^{***}	0.14^{***}	0.14^{***}	0.14^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Energy	0.27^{***}	0.15^{***}	0.15^{***}	0.15^{***}	0.15^{***}	0.15^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Avg.Grade	-0.24^{***}	-0.26^{***}	-0.26^{***}	-0.26^{***}	-0.25^{***}	-0.25^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
AttentionDiff	0.14^{***}	0.10^{**}	0.10^{**}	0.10^{**}	0.10^{**}	0.10^{**}
	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Accepted	0.19^{***}	0.15^{***}	0.15^{***}	0.15^{***}	0.15^{***}	0.14^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Other Ptcp.× Black Pres.	-0.54^{***}	-0.51^{***}		-0.34	-0.54^{***}	-0.44^{*}
	(0.13)	(0.14)		(0.19)	(0.14)	(0.19)
Other Race Participation			0.12			
			(0.30)			
White Ptcp.$_{-i}$ ×log(Size)					0.32^{*}	0.43^{**}
					(0.14)	(0.16)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	−13947.60	−11852.37	−11858.50	−11095.64	−11849.80	−11092.35
Deviance	27895.20	23704.74	23717.01	22191.28	23699.61	22184.71
Observations	35590	35590	35590	33227	35590	33227
Schools	139	139	139	123	139	123

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.10: Black Students' Choices in Basketball Club

	Black Students' Participation in Basketball Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-0.65	0.25	-0.16	0.62	2.12^{*}	2.00
	(0.54)	(0.60)	(0.64)	(0.71)	(0.88)	(1.06)
Black Participation $_{-i}$	3.09^{***}	2.98^{***}	2.64^{***}	2.85^{***}	-3.41	-1.56
	(0.35)	(0.38)	(0.42)	(0.43)	(2.25)	(2.57)
Black Proportion	-0.65^{***}	-0.35^{*}	-0.55^{**}	-0.39	-0.31	-0.38
	(0.16)	(0.17)	(0.21)	(0.21)	(0.17)	(0.21)
log(School Size)	-0.10	-0.08	-0.03	-0.12	-0.35^{**}	-0.32^{*}
	(0.06)	(0.06)	(0.07)	(0.07)	(0.11)	(0.13)
Gender	0.97^{***}	0.79^{***}	0.79^{***}	0.85^{***}	0.79^{***}	0.85^{***}
	(0.06)	(0.07)	(0.07)	(0.08)	(0.07)	(0.08)
Age	-0.09^{***}	-0.12^{***}	-0.12^{***}	-0.13^{***}	-0.13^{***}	-0.13^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Years	-0.06^{*}	-0.03	-0.03	-0.03	-0.02	-0.03
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Health	0.12^{***}	0.07	0.07	0.08^{*}	0.07^{*}	0.08^{*}
	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)
Energy	0.22^{***}	0.18^{***}	0.18^{***}	0.16^{***}	0.18^{***}	0.16^{***}
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Avg.grade	-0.08^{*}	-0.10^{*}	-0.10^{*}	-0.10^{*}	-0.10^{*}	-0.10^{*}
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
AttentionDiff	0.12^{*}	0.14^{*}	0.14^{*}	0.13^{*}	0.14^{*}	0.13^{*}
	(0.06)	(0.06)	(0.06)	(0.06)	(0.06)	(0.06)
Accepted	0.07^{*}	0.05	0.05	0.03	0.05	0.03
	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Other Ptcp.× White Pres.	0.31	0.05		0.23	0.17	0.25
	(0.37)	(0.41)		(0.54)	(0.41)	(0.53)
Other Race Participation			1.13			
			(0.68)			
Black Ptcp.$_{-i}$ ×log(Size)					0.98^{**}	0.67
					(0.34)	(0.38)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	-4126.26	-3419.31	-3417.95	-3037.95	-3415.22	-3036.45
Deviance	8252.51	6838.62	6835.89	6075.90	6830.43	6072.90
Observations	7774	7774	7774	6878	7774	6878
Schools	101	101	101	88	101	88

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.11: Impact of Key Factors on Club Participation: AME Analysis

	Average Marginal Effects (AME) of Factors on Club Participation Choice					
	Basketball Participation (10^{-2})		Baseball Participation (10^{-2})		Football Participation (10^{-2})	
	White Stds.	Black Stds.	White Stds.	Black Stds.	White Stds.	Black Stds.
Same-Race Ptcp.$_{-i}$(%)	0.42^{***}	0.44^{***}	0.60^{***}	0.32^{***}	0.61^{***}	0.21^{**}
	(0.03)	(0.06)	(0.04)	(0.09)	(0.04)	(0.06)
Race Proportion(%)	0.03^{*}	-0.06	0.00	0.12^{***}	0.01	-0.03
	(0.01)	(0.03)	(0.01)	(0.03)	(0.01)	(0.02)
Log(SchoolSize)	-1.73^{***}	-1.97	0.74	0.98	0.78	-0.76
	(0.40)	(1.09)	(0.42)	(0.86)	(0.31)	(0.69)
Other-Race Ptcp.(%)	-0.02^{*}	0.03	0.05^{***}	0.28^{***}	0.03^{**}	0.03
	(0.01)	(0.07)	(0.01)	(0.07)	(0.01)	(0.07)
Age	-1.58^{***}	-1.99^{***}	-0.24	-0.10	-0.46^{***}	-0.53^{*}
	(0.15)	(0.37)	(0.15)	(0.30)	(0.12)	(0.26)
Gender	2.60^{***}	12.83^{***}	-0.43	0.72	27.97^{***}	32.95^{***}
	(0.42)	(1.10)	(0.45)	(1.03)	(0.67)	(1.09)
Avg.Grade	-2.71^{***}	-1.56^{*}	-1.13^{***}	-0.26	0.59^{**}	-0.01
	(0.26)	(0.66)	(0.27)	(0.56)	(0.20)	(0.49)
Accepted	1.54^{***}	0.42	0.97^{***}	0.62	1.25^{***}	-0.04
	(0.23)	(0.57)	(0.25)	(0.49)	(0.19)	(0.43)
Black/White Pres.	-1.07^{*}	0.61	1.97^{***}	5.22^{***}	1.00^{**}	0.38
	(0.47)	(1.29)	(0.54)	(1.23)	(0.38)	(0.98)

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Note: This table presents the key factors influencing students' decisions to participate in basketball, baseball, and football, as determined by Average Marginal Effect (AME) analysis. The AME reveals the average impact of a one-unit change in each variable on the likelihood of club participation, accounting for any non-linear relationships. The values are expressed in terms of exponential 10^{-2}, reflecting the percentage change in participation rates per unit change in the factors.

Table 3.12: White Students' Choices in Baseball Club

	White Students' Participation in Baseball Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-2.31^{***}	-2.70^{***}	-2.60^{***}	-2.83^{***}	-2.40^{***}	-2.78^{***}
	(0.24)	(0.26)	(0.25)	(0.28)	(0.33)	(0.38)
White Participation $_{-i}$	5.34^{***}	4.72^{***}	4.15^{***}	4.83^{***}	3.20^{**}	4.54^{***}
	(0.23)	(0.24)	(0.31)	(0.27)	(1.15)	(1.36)
White Proportion	0.24^{*}	0.09	-0.15	0.03	0.09	0.04
	(0.10)	(0.10)	(0.10)	(0.12)	(0.10)	(0.12)
log(School Size)	0.01	0.07^{*}	0.06^{*}	0.06	0.02	0.05
	(0.03)	(0.03)	(0.03)	(0.03)	(0.05)	(0.06)
Gender	0.41^{***}	-0.05	-0.05	-0.03	-0.05	-0.03
	(0.03)	(0.03)	(0.03)	(0.04)	(0.03)	(0.04)
Age	-0.09^{***}	-0.03^{**}	-0.02^{*}	-0.02	-0.03^{**}	-0.02
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Years	-0.03^{**}	-0.04^{**}	-0.04^{***}	-0.05^{***}	-0.04^{**}	-0.05^{***}
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Health	0.11^{***}	0.05^{*}	0.05^{**}	0.06^{**}	0.05^{*}	0.06^{**}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Energy	0.27^{***}	0.19^{***}	0.19^{***}	0.18^{***}	0.19^{***}	0.19^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Avg.grade	-0.07^{***}	-0.09^{***}	-0.08^{***}	-0.09^{***}	-0.09^{***}	-0.09^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
AttentionDiff	0.04	-0.03	-0.03	-0.03	-0.03	-0.03
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Accepted	0.13^{***}	0.08^{***}	0.08^{***}	0.08^{***}	0.08^{***}	0.08^{***}
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Other Ptcp.× Black Pres.	0.21	0.96^{***}		0.95^{***}	0.94^{***}	0.94^{***}
	(0.18)	(0.20)		(0.26)	(0.20)	(0.26)
Other Race Participation			1.05^{**}			
			(0.33)			
White Ptcp.$_{-i}$ ×log(Size)					0.26	0.05
					(0.19)	(0.22)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	-16005.35	-13534.65	-13541.13	-12590.72	-13533.74	-12590.70
Observations	35590	35590	35590	33227	35590	33227
Schools	139	139	139	123	139	123

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.13: Black Students' Choices in Baseball Club

	Black Students' Participation in Baseball Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-2.72^{***}	-4.00^{***}	-4.27^{***}	-3.85^{***}	-3.75^{***}	-3.94^{***}
	(0.52)	(0.56)	(0.62)	(0.64)	(0.86)	(0.96)
Black Participation $_{-i}$	3.83^{***}	3.76^{***}	4.13^{***}	2.99^{***}	1.84	3.73
	(0.56)	(0.59)	(0.58)	(0.69)	(5.02)	(5.61)
Black Proportion	0.48^{*}	1.19^{***}	0.88^{***}	1.06^{***}	1.18^{***}	1.07^{***}
	(0.20)	(0.21)	(0.18)	(0.26)	(0.21)	(0.26)
log(School Size)	-0.01	0.12	0.17^{*}	0.09	0.08	0.11
	(0.06)	(0.07)	(0.08)	(0.08)	(0.12)	(0.14)
Gender	0.58^{***}	0.11	0.11	0.07	0.11	0.07
	(0.07)	(0.09)	(0.09)	(0.09)	(0.09)	(0.09)
Age	-0.05^{*}	-0.02	-0.03	-0.01	-0.02	-0.01
	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Years	-0.03	0.01	0.01	0.02	0.01	0.02
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Health	0.09^{*}	0.02	0.02	-0.00	0.02	-0.00
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Energy	0.14^{**}	0.08	0.08	0.07	0.08	0.07
	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
Avg.Grade	-0.02	-0.00	0.01	-0.02	-0.00	-0.02
	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
AttentionDiff	0.07	0.03	0.03	-0.01	0.03	-0.01
	(0.07)	(0.07)	(0.07)	(0.08)	(0.07)	(0.08)
Accepted	0.07	0.05	0.04	0.06	0.05	0.06
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Other Ptcp.× White Pres.	2.30^{***}	2.60^{***}		3.08^{***}	2.60^{***}	3.09^{***}
	(0.59)	(0.62)		(0.71)	(0.62)	(0.72)
Other Race Participation			2.63^{**}			
			(0.86)			
Black Ptcp.$_{-i}$ ×log(Size)					0.31	-0.12
					(0.82)	(0.91)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	-3039.08	-2645.67	-2649.99	-2363.51	-2645.60	-2363.50
Deviance	6078.16	5291.34	5299.97	4727.03	5291.20	4727.01
Observations	7774	7774	7774	6878	7774	6878
Schools	101	101	101	88	101	88

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.14: White Students' Choices in Football Club

	White Students' Participation in Football Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-5.44^{***}	-6.14^{***}	-5.95^{***}	-6.66^{***}	-5.94^{***}	-6.30^{***}
	(0.28)	(0.31)	(0.31)	(0.35)	(0.33)	(0.36)
White Participation $_{-i}$	7.26^{***}	7.14^{***}	6.24^{***}	8.21^{***}	7.44^{***}	8.78^{***}
	(0.38)	(0.41)	(0.69)	(0.47)	(0.44)	(0.50)
White Proportion	0.18	−0.00	−0.22	0.14	−0.03	0.06
	(0.13)	(0.14)	(0.13)	(0.15)	(0.14)	(0.16)
log(School Size)	−0.03	0.08^{*}	0.06	0.11^{**}	0.06	0.08^{*}
	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Gender	3.66^{***}	3.67^{***}	3.66^{***}	3.75^{***}	3.67^{***}	3.75^{***}
	(0.09)	(0.09)	(0.09)	(0.09)	(0.09)	(0.10)
Age	-0.10^{***}	-0.08^{***}	-0.07^{***}	-0.06^{***}	-0.08^{***}	-0.06^{***}
	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Years	−0.02	−0.01	−0.01	−0.03	−0.01	−0.03
	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Health	0.17^{***}	0.09^{***}	0.10^{***}	0.09^{**}	0.09^{***}	0.09^{**}
	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Energy	0.13^{***}	0.02	0.02	0.01	0.02	0.01
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Avg.Grade	0.07^{**}	0.12^{***}	0.12^{***}	0.08^{**}	0.12^{***}	0.08^{**}
	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
AttentionDiff	0.15^{***}	0.10^{*}	0.10^{*}	0.11^{**}	0.10^{*}	0.11^{**}
	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Accepted	0.22^{***}	0.17^{***}	0.17^{***}	0.17^{***}	0.17^{***}	0.17^{***}
	(0.02)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Other Ptcp.× Black Pres.	0.36	0.88^{***}		0.78^{**}	0.82^{***}	0.67^{*}
	(0.22)	(0.24)		(0.30)	(0.24)	(0.30)
Other Race Participation			1.13^{*}			
			(0.53)			
White Ptcp.$_{-i}$ ×log(Size)					−0.11	-0.20^{**}
					(0.06)	(0.06)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	−9557.61	−8225.54	−8230.06	−7704.03	−8223.80	−7699.21
Deviance	19115.22	16451.08	16460.12	15408.06	16447.60	15398.41
Observations	35590	35590	35590	33227	35590	33227
Schools	139	139	139	123	139	123

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.15: Black Students' Choices in FootballClub

	Black Students' Participation in Football Club					
	(1)	(2)	(3)	(4)	(5)	(6)
(Intercept)	-4.20^{***}	-4.16^{***}	-5.64^{***}	-3.88^{***}	-3.94^{***}	-3.54^{***}
	(0.60)	(0.66)	(0.72)	(0.74)	(0.67)	(0.76)
Black Participation $_{-i}$	3.61^{***}	3.61^{***}	1.80^{*}	2.69^{***}	3.62^{***}	2.48^{**}
	(0.59)	(0.63)	(0.74)	(0.76)	(0.63)	(0.77)
Black Proportion	−0.33	−0.05	−0.35	−0.24	−0.06	−0.28
	(0.20)	(0.22)	(0.21)	(0.29)	(0.23)	(0.29)
log(School Size)	−0.12	−0.10	0.07	−0.10	−0.11	−0.12
	(0.07)	(0.07)	(0.08)	(0.08)	(0.07)	(0.08)
Gender	3.77^{***}	3.87^{***}	3.88^{***}	3.82^{***}	3.86^{***}	3.81^{***}
	(0.14)	(0.15)	(0.15)	(0.16)	(0.15)	(0.16)
Age	−0.04	−0.05	−0.05	-0.06^{*}	−0.05	−0.06
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Years	-0.08^{*}	-0.07^{*}	-0.07^{*}	−0.05	−0.07	−0.05
	(0.03)	(0.04)	(0.04)	(0.04)	(0.04)	(0.04)
Health	0.20^{***}	0.13^{**}	0.14^{**}	0.15^{**}	0.13^{**}	0.15^{**}
	(0.04)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
Energy	0.17^{**}	0.12^{*}	0.13^{*}	0.12^{*}	0.12^{*}	0.13^{*}
	(0.05)	(0.06)	(0.06)	(0.06)	(0.06)	(0.06)
Avg.Grade	−0.05	0.00	−0.00	0.00	0.00	0.00
	(0.05)	(0.05)	(0.05)	(0.06)	(0.05)	(0.06)
AttentionDiff	0.18^{*}	0.22^{**}	0.22^{**}	0.20^{*}	0.22^{**}	0.20^{*}
	(0.07)	(0.08)	(0.08)	(0.08)	(0.08)	(0.08)
Accepted	0.05	0.02	0.02	−0.01	0.02	−0.01
	(0.04)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
Other Ptcp.× White Pres.	0.63	−0.26		0.43	0.02	0.81
	(0.72)	(0.79)		(0.96)	(0.80)	(0.98)
Other Race Participation			5.32^{***}			
			(1.25)			
Black Ptcp.$_{-i}$ ×log(Size)					-0.20^{*}	-0.25^{*}
					(0.10)	(0.12)
Individual Ctrl Var.		✓	✓	✓	✓	✓
School Ctrl Var.				✓		✓
Log Likelihood	−2329.92	−1978.66	−1969.54	−1777.04	−1976.65	−1774.67
Deviance	4659.83	3957.31	3939.09	3554.09	3953.30	3549.35
Observations	7774	7774	7774	6878	7774	6878
Schools	101	101	101	88	101	88

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Table 3.16: Compare Students' Participation Between Small and Large Schools

	White Student's Participation in Basketball Club			
	Small Schools		Large Schools	
	(1)	(2)	(3)	(4)
(Intercept)	-0.95	-0.27	0.84	1.21^{*}
	(0.52)	(0.57)	(0.56)	(0.60)
White Participation $_{-i}$	3.91^{***}	3.98^{***}	4.50^{***}	4.46^{***}
	(0.32)	(0.32)	(0.33)	(0.33)
White Proportion	-0.28	-2.58^{***}	0.50^{***}	-0.66
	(0.21)	(0.78)	(0.14)	(0.72)
White Proportion^2		2.27^{**}		1.12
		(0.74)		(0.68)
log(School Size)	-0.01	-0.01	-0.26^{***}	-0.28^{***}
	(0.06)	(0.06)	(0.06)	(0.06)
Gender	0.40^{***}	0.40^{***}	0.17^{***}	0.17^{***}
	(0.06)	(0.06)	(0.05)	(0.05)
Age	-0.11^{***}	-0.13^{***}	-0.14^{***}	-0.14^{***}
	(0.02)	(0.02)	(0.02)	(0.02)
Years	0.07^{***}	0.07^{***}	-0.08^{***}	-0.08^{***}
	(0.02)	(0.02)	(0.02)	(0.02)
Health	0.10^{**}	0.10^{**}	0.17^{***}	0.17^{***}
	(0.04)	(0.04)	(0.03)	(0.03)
Energy	0.18^{***}	0.18^{***}	0.13^{***}	0.13^{***}
	(0.04)	(0.04)	(0.03)	(0.03)
Avg. Grade	-0.30^{***}	-0.30^{***}	-0.21^{***}	-0.21^{***}
	(0.04)	(0.04)	(0.03)	(0.03)
Accepted	0.15^{***}	0.15^{***}	0.14^{***}	0.14^{***}
	(0.03)	(0.03)	(0.03)	(0.03)
Black Presence × Other Race Ptcp.	-0.86^{***}	-0.71^{**}	-0.73^{***}	-0.64^{**}
	(0.24)	(0.24)	(0.20)	(0.21)
Log Likelihood	-4073.33	-4068.70	-7802.67	-7801.35
Deviance	8146.67	8137.40	15605.34	15602.70
Num. obs.	9933	9933	25910	25910

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Chapter 4

Benefits from Social Engagement: A Model and Empirical Analysis

4.1 Introduction

In Chapter 3, an enhanced Linear-in-Means (LIM) model was used to estimate peer effects on students' decisions regarding basketball participation. The analysis revealed that the influence of White students on their Black peers was stronger than the reverse. Additionally, it was found that homophily exists among club members, who tend to build new connections with each other. Traditional peer effects analysis often relies on the assumption that network structures are exogenous; however, social networks and individual behaviors are interdependent, evolving together. When individuals decide on their actions, they form expectations about the types of friendships they will establish within a given social environment. This dynamic introduces selection bias in friendship formation, as individuals are more inclined to connect with others who share similar traits.

To better understand the motivations behind these behaviors, this chapter develops a model that examines how social norms influence students' decision-making processes. The model emphasizes the trade-off between the effort required to participate in a club and the benefits gained from forming friendships within the club. Based on prevailing social norms, students form expectations about their payoffs from social engagement when deciding whether to participate.

Using this framework, the equilibrium participation rate is estimated, representing the point at which the additional payoff from joining the club equals the participation cost. This equilibrium is influenced by the size of the racial group, the participation decisions of other students, and the benefits derived from social engagement.

The empirical findings support the model, confirming that social engagement significantly influences students' club participation. Fitting the model to the data reveals that social engagement is more valuable for students in larger schools. The estimated parameters also highlight differing patterns of behavior between White and Black students. While White students tend to develop more friendships after joining the club and receive larger net benefits, Black students do not increase their total number of friendships as much. Instead, they form more connections within the club. When the Black racial group is large enough to facilitate interactions outside the club, Black students are less likely to participate, a behavior contrasting with that of White students.

Among White students, club members establish more connections than non-members (7.7 versus 7.2 friends per student, and 5.4 versus 4.5 friends identified in the data). Additionally, racial homophily is lower among White club members compared to other White students (73.0% versus 70.9%), indicating that club participation helps to alleviate racial segregation.

The analysis presented in this paper provides insights into the social dynamics within schools, demonstrating how club participation fosters cross-group interactions and promotes social cohesion. By estimating the benefits derived from friendship connections, this model enhances our understanding of the role that cultural and sports clubs play in shaping students' social networks.

4.2 Model

In this model, student's decision to join is influenced by personal talent in the club's activity and the prevailing social norms regarding participation, denoted as h. The analysis examines the balance students face between the benefits of joining and the costs, exploring the conditions under which they choose to participate and how these decisions affect overall participation rates across different racial groups.

The primary benefit of joining a club lies in enhanced social connections. Drawing on findings from previous studies on homophily and social networks, I propose that students derive greater payoffs by forming friendships with peers who share similar interests or racial backgrounds. The expected benefits are higher when the likelihood of forming such friendships increases.

The social norm is common knowledge among all students, shaping their expectations and choices. a student's expected payoff is intricately connected to their personal choices, and the prevailing cultural norm h which is endogenously determined by each student's participation decision.

4.2.1 Students' Social Payoff and Optimization Problem

If students choose to join the club, then he need to distribute their efforts between two spheres: within the club and outside the club. The total effort, denoted by E, is the sum of effort spent on forming friendship with club members (x_m) and with other students in the school (x_s), such that $x_m + x_s = E$.

The primary goal for students is to maximize their social payoff, which is modeled as:

$$V_J(x_m, x_s) = v_m x_m^\alpha + v_s x_s^\alpha$$

subject to the constraint

$$x_m + x_s = E.$$

The variable v_m and v_s represents the benefits per friend within and outside the club, respectively. The function x^α describes the number of new friendships as a function of effort x, where α in the interval $(0, 1)$ signifies diminishing returns on increased effort. The number of friends for a club member is defined as $n_m = x_m^\alpha$ for within-club connections, and $n_s = x_s^\alpha$ for external connections.

The optimal allocation of effort between the two channels is determined by setting the marginal utilities derived from each type of effort equal to each other: $\frac{\partial}{\partial x_m}\left(v_m x_m^\alpha\right) = \frac{\partial}{\partial x_s}\left(v_s x_s^\alpha\right)$. Simplifying the equation, we arrive at the relationship:

$$\frac{x_m}{x_s} = \left(\frac{v_m}{v_s}\right)^{\frac{1}{1-\alpha}}.$$

From this, the efforts v_m and v_s are calculated as following:

$$\begin{aligned} x_m &= \frac{v_m^{\frac{1}{1-\alpha}}}{v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}} E \\ x_s &= E - x_m \end{aligned}$$

Under the optimal effort allocations, the social payoff from joining the club can be expressed as:

$$V_J^* = \left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}}\right)^{1-\alpha} E^{\alpha}.$$

The net benefit of club participation, $\Delta V(h)$, represents the additional payoff from social engagement. It is calculated as the difference in payoffs from joining versus not joining:

$$\Delta V(v_m, v_s, \alpha, E) = E^{\alpha}\left[\left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}}\right)^{1-\alpha} - v_s\right]$$

where the payoff of not joining the club is given by $v_s E^{\alpha}$. In this scenario, students spend all their effort, E, on making friends outside the club and receive a per-connection benefit of v_s for each friend. Given that $\frac{1}{1-\alpha}$ is greater than 1, by the property of function[1] $f(x) = (a^x + b^x)^{1/x}$, the additional payoff ΔV is guaranteed to be positive, reinforcing the utility of club participation.

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[1] Property of $f(x) = ((a^x + b^x)^{1/x}$, where $x > 1$:

- If $x \longrightarrow 1$, $f(x)$ approaches $a + b$
- If $x \longrightarrow \infty$, $f(x)$ approaches $max\,\{a, b\}$
- In general, for any $x > 1$, $f(x) \in (max\,\{a, b\}\,,\; a + b)$.

4.2.2 Cost of Participation

This model assumes that each student's cost of joining a basketball club is determined by a fixed baseline cost reduced by their individual talent. Talent captures unobserved heterogeneity in students' ease of participation, including factors such as motivation, physical ability, or prior experience.

Assumption 4.1: Distribution of Participation Cost *Each student i faces a participation cost given by:*

$$c_i = a - \theta_i,$$

where $a > 0$ is a constant and θ_i represents individual talent. A higher value of θ_i implies a lower participation cost. When $\theta_i > a$, the cost becomes negative, indicating that participation provides a net intrinsic benefit to the student.

The talent parameter θ_i is independently and identically distributed across students. I consider two benchmark distributions for θ_i:

Example 1 (Exponential Talent Distribution)

$$\theta_i \sim Exp(\lambda), \qquad \text{with support } [0, \infty).$$

Example 2 (Normal Talent Distribution)

$$\theta_i \sim \mathcal{N}(\mu, \sigma^2), \qquad \text{with support } (-\infty, \infty).$$

This formulation interprets the cost of participation as decreasing in the student's natural talent. The parameter a reflects the highest possible cost that a student with no talent would face, while θ_i introduces randomness in student ability.

Because c_i can be negative, some students experience net utility gains from participating in the club, independent of peer influences or prevailing social norms. These students are intrinsically motivated to participate, regardless of their peers' choices or the prevailing social norms. In the context of evolutionary game theory, such individuals resemble 'mutants'—agents whose optimal strategy deviates from the social equilibrium due to their inherent characteristics.

In each period, a proportion of new students joins the school, and they decide whether to participate in clubs based on their individual costs. By the Law of Large Numbers, as the school population increases, the average cost for these new students converges in probability to the expected value. In smaller schools, however, the costs for these new entrants are more likely to deviate from the broader population distribution, leading to greater variation and instability in social norms. In contrast, larger schools have a higher number of new students, resulting in cost distributions that align more closely with the overall population, making shifts in established social norms less likely.

4.2.3 Equilibrium Analysis

In the model, the participation rate threshold h^* that determines the decision to join or not is defined by equalizing the additional payoff from joining and the corresponding cost ($\Delta V = \bar{C}$):

$$E^{\alpha}\left[\left(v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}\right)^{1-\alpha} - v_s\right] = a - F^{-1}(1-h) \tag{4.1}$$

where $F^{-1}(1-h)$ represents the inverse cumulative density function (CDF) of talent θ. The left-hand side (LHS) of (4.1) quantifies the additional payoff from joining the club, while the right-hand side (RHS), denoted as $\overline{C}$, represents the cutoff cost level when a proportion h of the population joins the club. This cutoff cost is the highest expense at which only agents with costs lower than or equal to this threshold will choose to participate in the club.

The variable h represents the participation rate, indicating that students with higher talent—and correspondingly lower learning costs— decide to participate. The cumulative distribution function (CDF), denoted as $1 - h = F(x)$, indicates that a proportion $1 - h$ of students have natural talent less than or equal to x. This establishes a cost threshold at $a - x = a - F^{-1}(1 - h)$. Consequently, the fraction h, comprising students with higher talent, faces costs not exceeding this threshold.

Therefore, in a sufficiently large population that accurately reflects the actual talent distribution, a rational and well-informed student should join the club if and only if their talent exceeds the threshold talent level, determined by $F^{-1}(1 - h^*)$. The value h^* is derived from the equilibrium condition specified in the equation (4.1). Consequently, the cost threshold, which corresponds to the participation rate h, is defined as:

$$\bar{C}(h) = a - F^{-1}(1 - h)$$

Closed-Form under Exponential Talent Distribution When individual talent follows an exponential distribution with rate parameter $\lambda > 0$, we have:

$$F(\theta_i) = 1 - e^{-\lambda\theta_i}, \qquad \text{for } \theta_i \in [0, \infty).$$

Thus, the inverse CDF[2] is:

$$F^{-1}(1 - h) = -\frac{ln(h)}{\lambda},$$

and the equilibrium condition becomes:

$$E^{\alpha}\left[\left(v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}\right)^{1-\alpha} - v_s\right] = a + \frac{ln(h^*)}{\lambda}$$

where $v_m(w,h^*)$ is the benefit under the equilibrium state h^*.

Our model specifically addresses the payoff dynamics for students from racial group g, considering the relative group size w within the interval $(0, 1)$. We analyze the participation rate for group g, denoted by h; and use h_b to represent the average participation rate of other racial groups, considered as an exogenous factor. The per-connection benefits $v_m\,(w, h)$ and $v_s(w)$ are influenced by both the relative size and the participation rate of group g. This framework builds on previous research on homophily, which highlights type-sensitive preferences as a key factor.

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[2] The inverse cumulative density function is defined as $F^{-1}((1-h), \lambda) = -\frac{ln(h)}{\lambda}$, which behaves as an increasing and concave function. Specifically, as h approaches 0 from the positive side, the RHS tends to negative infinitive ($\lim RHS_{h\to 0^+} = -\infty$), and as h approaches 1 from the negative side, the RHS converges to a ($\lim RHS_{h\to 1^-} = a$).

Assumption 4.2 (Type-Sensitive Preference). *Students prefer interacting with peers who share similar traits. The expected value of friendships formed through a particular channel increases with the likelihood of encountering peers with the same traits.*

1. *The benefits per connection* $v_m(w, h)$ *and* $v_s(w)$*, monotonically increase with the relative size* w *of a given racial group, as expressed by* $\frac{\partial v_m}{\partial w} > 0$ *and* $\frac{\partial v_s}{\partial w} > 0$.
2. *For club members, the benefit per connection,* $v_m(w, h)$*, also increases with their racial group's participation rate* h*, as expressed by* $\frac{\partial v_m}{\partial h} > 0$.

Random Matching Example. In the random matching model, students are assumed to choose their friends randomly. This simplification allows us to focus on how the racial composition of the overall population influences the per-connection benefits, which are solely impacted by the racial proportions within the population. Specifically, the benefits are calculated as follows:

$$\begin{cases} v_m = c + \frac{hw_i}{hw+h_b(1-w)} v_{homo} \\ v_s = w v_{homo} \end{cases}$$

where v_{homo} represents the baseline benefit from homophilous race friendship, h_b is the participation rate for other racial groups, and c is a constant that benefits of club membership.

Other Cases: Biased Matching. We further explore the dynamics of club participation under conditions where matching is not random but biased, meaning students from the same racial group g are more likely to encounter each other. This scenario better reflects realistic social interactions and can lead to different outcomes in terms of club benefits:

1. **Example 1: Power Bias.** The benefit within the club is expressed as a power function of the racial proportion: $v_m = c + \left(\frac{hw_i}{hw+h_b(1-w)}\right)^b v_{homo}$, and outside the club as $v_s = w^b v_{homo}$ where $b \in (0, 1)$ quantifies the degree of bias in matching.
2. **Example 2: Linear-Threshold Bias.** Within the club, the benefit increases linearly up to a threshold: $v_m = c + d\frac{hw}{hw+h_b(1-w)} v_{homo}$, where $d > 1$, applicable when $d\frac{hw}{hw+h_b(1-w)} \leq 1$;

$v_m = c + v_{homo}$ otherwise. Outside the club, the biased matching benefit is: $v_s = dwv_{homo}$ when $dw \leq 1$.

These examples demonstrate how variations in the bias of matching processes influence the degree of racial homophily in social connections, which in turn affects the net benefits derived from club participation. The first example illustrates a gradual and continuous increase in both the probability of matching and the associated benefits as the racial proportion, w, increases. In the second example, the benefit within the club increases linearly until the proportion of same-race meetings maxes out at one, at which point the benefit curve exhibits a distinct kink, indicating that further increases in racial proportion yield no additional benefits.

Lemma 4.1 (Monotonic Growth of Joining Payoff Advantage). *An individual's net benefit* $\Delta V(w, h)$ *increases monotonically with the participation rate* h *among students form the same group, as indicated by* $\frac{\partial \Delta V}{\partial h} > 0$. *This suggests that the incentive to join the club strengthens as more students participate.*

In particular, we demonstrate in the Appendix that $\frac{\partial \Delta V}{\partial h} = x_m^{\alpha} \frac{\partial v_m}{\partial h}$, and from Assumption 4.2, we derive that $\frac{\partial \Delta V}{\partial h} > 0$. The essence of Lemma 4.1 captures the internal dynamics of the club: as the participation rate h among students of racial group g increases, so does the likelihood of a club member from this group matching with another member of the same racial group. This dynamic fosters a clear, positive trend in the net benefit $\Delta V(h)$, which aligns individual decisions to join with the growing communal participation. Proof of Lemma 4.1 is provided in Appendix D on page 149.

Lemma 4.2 (Advantage of Racial Majority in Club Benefits). *For any fixed participation rate* h*, as the racial proportion increases, students receive a greater net benefit,* $\Delta V(w, h)$*, if the within-club benefit* v_m *increases faster than that of an outside-club friendship connection* v_s*. This relationship is formally expressed as:* $\frac{\partial \Delta V}{\partial w} > 0$ *if* $\frac{\partial v_m}{\partial w} > \frac{\partial v_s}{\partial w}$*, regardless of the specific values of* w *and* h*.*

Lemma 4.2': (Influence of the Ratio of Club Benefits on Net Benefits). *For any fixed participation rate* h*, as their group's racial proportion increases, students receive a greater net benefit,*

$\Delta V(w, h)$, if the ratio between within-club benefit v_m and outside-club benefit v_s increases. This relationship is formally expressed as: $\frac{\partial \Delta V}{\partial w} > 0$ if $\frac{\partial}{\partial w}\left(\frac{v_m}{v_s}\right) > 0$.

As the racial group becomes larger within the society, students from this group are more likely to encounter each other in both matching scenarios. Increased same-race matching within the club, as compared to outside, enhances the benefits derived from club participation. This lemma validates that increased racial homophily within the club leads to a higher net benefit, underlining the advantage of a racial majority in club benefits. The detailed proof of this relationship is provided in the Appendix on page 150.

Using the example of unbiased matching, when the participation rate h is less than h_b and the relative size w of a racial group is large, we find that the condition $\frac{\partial v_m}{\partial w} > \frac{\partial v_s}{\partial w}$ is satisfied. The proof is provided in the Appendix. This outcome is supported by empirical data, which demonstrate that when the relative size of a racial group, such as white students, is sufficiently large, there is a positive correlation between their participation rate and their proportion within the population.

It is also essential to recognize that the dynamics of racial homophily affecting club benefits are not limited to unbiased matching scenarios but also extend to any form of biased matching. In such environments, the per-connection benefits, both within (v_m) and outside (v_m) the club, tend to increase more significantly as racial proportion w enlarges. This increase in benefits is influenced by the degree of bias present in interactions both within and outside the club. Generally, if the matching bias favoring same-race interactions is stronger within the club than outside, the conditions outlined in Lemma 4.2 (Lemma 4.2') are less stringent and more likely to be met. This scenario suggests that increased club internal homophily can amplify the advantages conferred by a racial majority, reinforcing the pattern of higher participation rates among larger racial groups.

Assumption 4.3 (Cost Dominance at Full Participation). *The maximum potential cost, a, exceeds the highest possible net payoff for joining the club, especially as the proportion of club members h approaches 1. This can be formally expressed as: $a > \Delta V(h)$, when $h \rightarrow 1$.*

This assumption ensures that even if all students in society opt to join the club, there will be some, particularly those with the least talent and consequently higher relative costs, for whom the payoff from joining does not exceed the payoff from not joining. This assumption prevents a corner solution where all students uniformly decide to join the club, maintaining diversity in strategic behavior.

Proposition 4.1 (Existence of Equilibrium). *The existence of equilibrium state is guaranteed under Assumption 4.3.*

Solving (4.1) for h, we can determine the proportion h^* in the equilibrium state. As $h \to 0$, the cost approaches negative infinite for the most talent students int he society, where $\Delta V > Cost$. Conversely, as $h \to 1$, it follows from Assumption 4.3 that $\Delta V < Cost$, ensuring the presence of at least one solution h^* where $\Delta V = Cost$ within the interval $h \in (0, 1)$. Assuming private information about talent, a student will choose to join the club if and only if his talent is high enough such that his cost is lower than the cutoff point $c \leq c^* = a - F^{-1}(1 - h^*)$. By Assumption 4.3, at least one solution for h^* exists, guaranteeing an equilibrium state.

Proposition 4.2 (Equilibrium Uniqueness under Single Crossing Condition). Let $\Delta V(w, h)$ denote the average marginal benefit from club participation, where $w \in (0, 1)$ is fixed, and h is the endogenous participation rate. Let $\bar{C}(h)$ denote the cutoff cost of participation for students.

Define the function as:

$$F(h) \equiv \Delta V(w, h) - \bar{C}(h).$$

Under the single crossing condition, if $F(h)$ is strictly decreasing in h on the interval $h \in (0, 1)$, then the equilibrium participation rate $h^* \in (0, 1)$, defined by the fixed-point condition $\Delta V(h^*) = \bar{C}(h^*)$, exists and is unique.

Corollary 4.2.1 (Equilibrium Uniqueness under Exponential Talent Distribution). *Suppose talent $\theta_i \sim Exp(\lambda)$, and cost is given by $c_i = a - \theta_i$. The uniqueness of the equilibrium participation rate $h^* \in (0, 1)$ depend on the average talent level, given by $\mathbb{E}[\theta_i] = 1/\lambda$.*

A unique equilibrium exists when $1/\lambda$ is sufficiently large. In contrast, multiple equilibria may arise when the average talent level is low.

To analyze equilibrium uniqueness, define the function: $F \equiv \Delta V - \bar{C}$, and consider the partial derivative $\frac{\partial F}{\partial h} = \frac{\partial \Delta V}{\partial h}\frac{\partial v_m}{\partial h} - \frac{\partial Cost}{\partial h}$. then we can get the equation

$$\frac{\partial F}{\partial h} = x_m^{\alpha}\frac{\partial v_m}{\partial h} - \frac{1}{\lambda}\frac{1}{h}$$

where x_m^{α} represents the number of friends a student builds within the club. If $\frac{\partial F}{\partial h} < 0$ throughout the interval $h \in (0, 1)$, then there is only one intersection between the curves ΔV and $Cost$.

The specific condition, which determines the uniqueness or multiplicity of the equilibrium states, depends on the functional forms of v_m. In the scenario, for instance, that students meet others randomly, the per-connection benefit is defined as:

$$v_m = c + \frac{hw}{hw + h_b(1-w)} v_{homo}$$

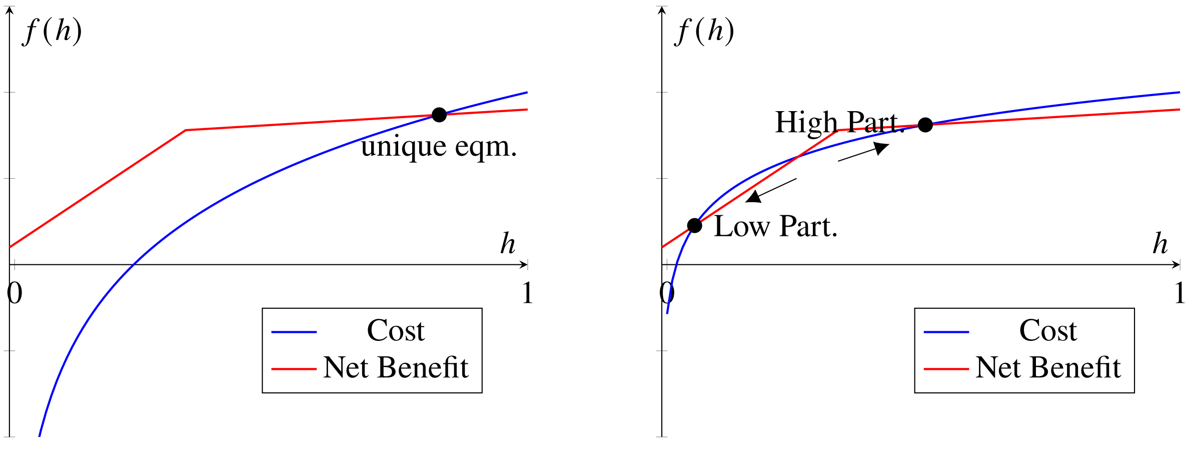
The derivative is expressed as:

$$\frac{\partial v_m}{\partial h} = \frac{h_b w(1-w)}{(hw + h_b(1-w))^2} v_{homo} < \frac{1}{4h}$$

Therefore, the condition $\frac{\partial F}{\partial h} < 0$ is satisfied if $\frac{1}{\lambda} > \frac{1}{4} x_m^{\alpha} v_{homo}$.

When $\frac{1}{\lambda}$ is high, implying a small rate parameter λ in the exponential distribution, talent is more evenly distributed across different levels. Consequently, the cost function increases gradually across all $h \in (0, 1)$. However, if $\frac{1}{\lambda}$ is small, the exponential distribution is highly skewed towards less talented students, and the cost increases steeply from negative infinity as h increase from 0 but becomes flat when h is high.

Assuming that Black students, on average, have higher basketball talent than White students, the exponential talent distribution for Black students is flatter, corresponding to a higher average talent level ($1/\lambda$). This results in a more gradual increase in the cost curve as the participation



a. Unique Equilibrium under High $1/\lambda$ b. Two Stable Equilibria Under Low $1/\lambda$

Figure 4.1: Equilibria Under Different λ Condition

Note: The left panel illustrates the case with $1/\lambda = 7$, representing a higher average talent level. The right panel shows the case with $1/\lambda = 2.8$ corresponding to a lower average talent level.

rate rises. Figure 4.1(a) illustrates this scenario, where the higher $1/\lambda$ yields a unique equilibrium due to the flatter cost curve. In contrast, 4.1(b) presents the case of a lower $1/\lambda$, where the cost curve increases steeply at low participation rates and then levels off as h approaches one, reflecting the scarcity of high-talent individuals in the population. Multiple equilibria may arise under such conditions, where the cost curve intersects the benefit curve more than once.

In scenarios where the majority of students in a racial group are less talented, the cost curve exhibits a higher degree of concavity compared to cases with a more uniformly distributed talent. This might result in multiple equilibria. The low participation equilibrium characterizes situations where only the rare talented students choose to join at low cost, while the high participation equilibrium describes scenarios where students can receive a relatively high benefit from club participation when the participation rate in their own racial group is sufficient to cover the high learning cost.

In states of multiple equilibria, the net benefit curve and cost curve intersect more than once. Because the net benefit curve is above cost curve near $h = 0$ and below it near $h = 1$, there exists an odd number of intersections between the two curves, as shown in Figure 4.1. This figure depicts a cost curve that increases dramatically at the beginning and then becomes relatively flat, with three intersections; however, only the first and third intersections represent stable equilibria. If the participation rate h h deviates from these stable points and falls between the two equilibria, it will eventually trend towards either the low or high participation equilibrium.

If h deviates between the first stable (low participation) and the unstable equilibrium, students at this level experience costs that exceed the net benefits ($\Delta V(h) < Cost(h)$). In this situation, students whose talent and associated costs make participation less rewarding than the net benefits at h h will opt out or choose not to join, leading to a decrease in h. This adjustment continues until h stabilizes at the low-participation equilibrium, where the net benefits equal the costs.

Conversely, if h deviates between the unstable and the third stable (high participation) equilibrium, students will perceive that the benefits of joining outweigh the costs ($\Delta V(h) > Cost(h)$). As a result, more students will decide to join, causing h to increase. This adjustment will continue until the participation rate reaches the high-participation equilibrium, where the costs and benefits are balanced.

This self-correcting mechanism ensures that deviations in the participation rate h lead it to converge to the nearest stable equilibrium, whether that is a state of lower or higher participation.

Lemma 4.3 (Equilibrium Stability via Deviation Conditions). *We defined the function* $F \equiv \Delta V - Cost$. *In a stable equilibrium state* h^*, *the partial derivative* $\frac{\partial F}{\partial h}$ *must be negative in the vicinity of the equilibrium.*

This condition ensures that any slight deviation from the equilibrium participation rate motivates students in the following period to adjust back towards the equilibrium state, thereby confirming its stability. To satisfy this condition, the net benefit curve ΔV must cross the cost curve from above.

Proposition 4.3 (Stability of the Equilibrium States).

- *If a unique equilibrium state* h^* *exists, it is stable.*
- *If multiple equilibrium states* $\{h_1^*, h_2^*, \cdots h_I^*\}$, *where* $I = 3, 5, 7, ...$ *(an odd number), are*

present, then the minimum h^*_{min} and maximum h^*_{max} participation rates are stable.

Proposition 4.4 (Net Benefit and Participation Equilibrium). *In a stable equilibrium state, as a racial group's proportion increases, leading to a greater net benefit ($\frac{\partial \Delta V}{\partial w} > 0$), then the stable participation rate h^* of the racial group will correspondingly increase.*

Proof. Define $F \equiv \Delta V - Cost = 0$ as previously discussed, where ΔV is a function of group size w and the participation rate h, and $Cost = a + \frac{ln(h)}{\lambda}$ is a function that depends solely on h. The rate of change of the participation rate h with respect to racial proportion w is given by the derivative:

$$\frac{dh}{dw} = -\frac{\frac{\partial F}{\partial w}}{\frac{\partial F}{\partial h}} = -\frac{\frac{\partial \Delta V}{\partial w}}{\frac{\partial F}{\partial h}}$$

From Lemma 4.3, we know that in any stable equilibrium state, the partial derivative $\frac{\partial F}{\partial h}$ is negative. Therefore, the sign of $\frac{dh}{d_w}$ is determined by the sign of $\frac{\partial \Delta V}{\partial w}$. This relationship can be formally expressed as: if $\frac{\partial \Delta V}{\partial w} > 0$, then $\frac{dh}{d_w} > 0$.

This proposition reinforces the critical insight that an increase in the racial proportion, which enhances net benefits (ΔV), directly contributes to an increase in participation rates within that racial group. This substantiates the significant influence of racial dynamics on social engagement patterns. Additionally, Lemma 4.2 and its extended version, Lemma 4.2', establish the conditions under which $\frac{\partial \Delta V}{\partial w}$ is positive, further articulating how changes in racial proportion affect club benefits.

Proposition 4.4' (Impact of Racial Composition on Participation Equilibrium) *In a stable equilibrium state, as a racial group's proportion increases, if the benefit per connection within club v_m increases faster than that of outside club v_s, then the participation rate h^* of the racial group correspondingly rises. This relationship is captured by two key conditions near the equilibrium point h^*:*

1. $\frac{\partial v_m}{\partial w} > \frac{\partial v_s}{\partial w}$ *implies* $\frac{dh^*}{dw} > 0$*;*

2. $\frac{\partial}{\partial w}\left(\frac{v_m}{v_s}\right) > 0$ *implies* $\frac{dh^*}{dw} > 0$.

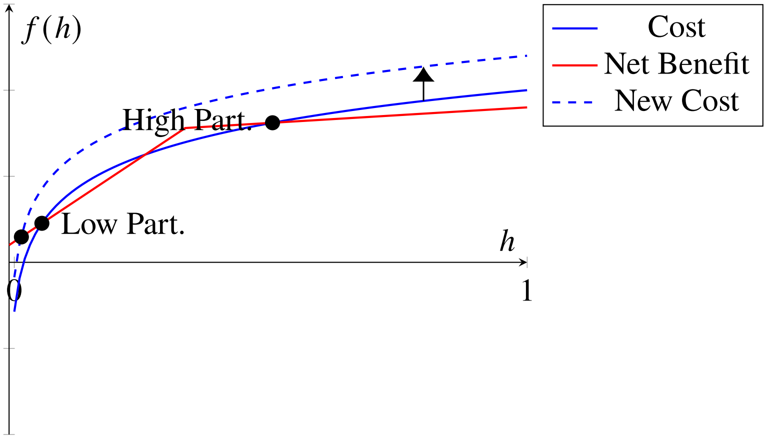


Figure 4.2: Increased Cost

Note: When costs uniformly increase across all agents, the constant in the cost distribution rises. Consequently, the cutoff curve shifts upwards, reflecting the elevated cost levels.

4.2.4 Uniform Cost Increase and Its Implications

The participation rate within a school's club is influenced by the school size, which potentially imposes a capacity limit on club membership. The empirical data from this study indicates a clear negative correlation between the participation rate and school size. In larger schools, where competition to join clubs is more intense due to these capacity constraints, students are often required to invest more time in developing their skills. We hypothesize that this results in a uniform increase in costs across all students. Consequently, while the talent distribution remains unchanged, the cost curve uniformly shifts upward. An increase in the constant maximum potential cost a is depicted in Figure 4.2.

As the cost cutoff curve $\bar{C}(h)$ shifts upward due to increased competition and capacity limits, the higher participation equilibria may no longer be sustainable. However, the lowest stable equilibrium persists regardless of how significantly costs escalate. Moreover, if multiple equilibria exist, when the cost increase is substantial enough, the lowest stable equilibrium becomes the only one remaining, effectively resulting in a unique stable equilibrium.

Next, we examine the impact of these cost increases on a specific stable equilibrium. Generally, as costs rise uniformly across all students, those whose original costs were marginally below the net benefit may no longer find club participation economically feasible under the new cost structure. Consequently, only students with higher talent, who can absorb the increased costs, continue to participate. Therefore, in a specific stable equilibrium state, the stable equilibrium h^* decreases as costs systematically increase.

Proposition 4.5 (Equilibrium Response to Cost Increases). *When costs increase uniformly across a school:*

- *The lowest stable equilibrium remains.*
- *The stable participation rate h^* for any racial group decreases within the same equilibrium.*

Proof. Consider the cost function $a + \frac{ln(h)}{\lambda}$. Define $F \equiv \Delta V - Cost = 0$. The rate of change of the participation rate h with respect to the cost parameter a is given by:

$$\frac{dh}{da} = -\frac{\frac{\partial F}{\partial a}}{\frac{\partial F}{\partial h}} = -\frac{-\frac{\partial Cost}{\partial a}}{\frac{\partial F}{\partial h}} = \frac{1}{\frac{\partial F}{\partial h}}$$

Given that $\partial F/\partial h$ is negative in any stable equilibrium, as established in Lemma 4, it follows that dh/da is negative. This indicates that the stable participation rate h^* decreases as the generalized cost parameter a increases.

In addition, in cases with multiple stable equilibria, this shift in the cost curve may cause initial conditions that previously converged to a higher equilibrium to now converge to the lower equilibrium.

4.2.5 Connectors and Devotees:

The analysis in this section is conducted under the assumption that talent follows an *exponential distribution*. This assumption allows for analytical tractability and enables a clear characterization of threshold behavior.

The exponential distribution was therefore chosen because it allows for a transparent and interpretable structure in which type differentiation can be rigorously derived.

Building on our previous analysis, this section differentiates between two club participant types: Connectors and Devotees. This classification allows us to delve deeper into the motivations and behavioral patterns that differentiate members within the club setting.

In this section, we differentiate between Connectors and Devotees to better understand the underlying mechanisms that drive club participation. Connectors are individuals who join clubs primarily to expand their social networks, appreciating the club's social structure for its broad networking opportunities. Conversely, Devotees are motivated by a passion for the club's activities, such as sports or other specific interests, focusing on skill enhancement and personal achievement within the club environment.

For Connectors, the utility function is defined as:

$$U_c = \Delta V(v_m, v_s, \alpha, E)$$

where ΔV represents the net benefits derived from social connections.

For Devotees, their utility is augmented by their intrinsic motivation to improve their skills, formalized as:

$$U_d = \Delta V(v_m, v_s, \alpha, E) + \gamma E \left(t_j | \, j \text{ in club}\right)$$

where $\gamma \in [0, 1)$ represents a synergy factor, quantifying the additional utility derived from the average talent level within the club; and $E(t_j)$ denotes the average talent of all club members.

The assumption that γ falls within the range $[0, 1)$ ensures that students value the average talent of the club members appropriately (less than their own talent level), preventing an overvaluation that could disrupt the model's stability. This assumption is critical, especially if agents can continuously adjust their behavior based on average talents within the club.

Devotees, compared to Connectors, receive additional utility from their deeper engagement with the club's core activities, which influences their decision to remain active members even under varying cost conditions. This distinction plays a crucial role in understanding how different motivations impact participation thresholds and rates under dynamic club environments.

I assume that student talent follows an exponential distribution with parameter λ. For club members categorized as either Connectors (c) or Devotees (d), the minimum talent threshold is denoted as t_m, where $m \in \{c, d\}$. Thus, the participation rate among type-m students is defined as $h_m = P(T_m \geq t_m) = e^{-\lambda t_m}$.

When cost changes are uniform across all students, the participation rates for Connectors and Devotees shift at the same rate *if and only if* synergy parameter $\gamma = 0$. Under such scenario, if there is a change in cost, Δc, the minimum talent threshold for both types of club members adjust accordingly, such that $t'_m = t_m + \Delta c$. Consequently, given $\gamma = 0$, the new participation rate h'_m relative to the original rate h_m adheres to the rate $h'_m / h_m = e^{-\lambda \Delta c}$, applicable to both Connectors and Devotees.

In scenarios where the systemic cost parameter, a, is identical for both Connectors and Devotees, theoretical participation rates should align across student types. If a differs, although distinct participation rates, $h_c \neq h_d$, emerge, the rate of change remains consistent, $h'_c / h_c = h'_d / h_d$, provided that $\gamma = 0$.

In contexts where synergy parameters $\gamma > 0$ and there is a uniform adjustment in costs across students, the participation rate for Devotees will change at a slower rate compared to that for Connectors. For example, with a systematic increase in costs, the requisite minimum talent, t_m, escalates for both groups, thereby reducing the participation rates h_c and h_m. Unlike the scenario where $\gamma = 0$, Devotees potentially benefit from enhanced synergy in clubs with higher-skilled members, resulting in a less pronounced decline in their participation rate compared to Connectors.

Proposition 4.6: (Differential Impact of Cost on Connectors and Devotees) If a synergy benefit exists for Devotees ($\gamma > 0$), their participation rate's responsiveness to a uniform cost alteration is slower compared to Connectors:

1. The participation rate for both Connectors and Devotees decreases, but the rate of decline is faster for Connectors. This is represented by:$\frac{h'_c}{h_c} < \frac{h'_d}{h_d} < 1$;
2. When costs decrease: The participation rate for both types increases, but Connectors' rate of

increase is faster than that of Devotees. This relationship is captured by: $1 < \frac{h'_d}{h_d} < \frac{h'_c}{h_c}$.

These adjustments in participation rates consistently show either an increase or decrease, depending on the direction of the cost changes. Despite the increased benefits from synergy, the adverse effects of higher costs on Devotees cannot be completely mitigated. A formal proof of this proposition is provided in Appendix D on page 152.

4.3 Empirical Analysis and Model Validation

4.3.1 Methodology and Estimation Results

In this section, I fit the model to the data and estimate the key parameters to understand how social dynamics influence club participation. The first step involves estimating the discount factor α for both White and Black students, based on the observed number of friendships. The results show that the factor for White students is less than one (0.787) , indicating they invest less effort in forming friendships, with diminishing marginal utility from each additional connection. On the other hand, the number of friendships among Black club members and non-members is relatively similar, resulting in an estimated parameter for Black students close to one. This helps explain the distinct participation behaviors observed: White students' participation rates increase with racial size, while those of Black students decrease.

With the estimated discount factor $\hat{\alpha}$, I proceed to determine the effort level $\hat{E}$ for each student. Using the theoretical model, I then conduct a counterfactual analysis to explore how each student's friendship structure would change if they were a club member versus a non-member. From this, I derive the net benefits $\hat{\Delta V}$. Lastly, a reduced-form solution is employed to estimate the benefits of cross-school ties v_s within each school, showing how these benefits change with racial composition and school size.

Estimation of the discount factor α

At the outset, students are classified into two groups: club members and non-members. The model posits that both types of students exert equivalent levels of effort in building social connections. However, empirical data reveal variability in the number of friends, ranging from 0 to 10, which contradicts the initial assumption of uniform effort across student types. To address this discrepancy, it is assumed that the effort in forming social connections for both groups follows an identical distribution.

In the model, the number of friends for a club member is defined as $n_m = x_m^\alpha$ for within-club connections, and $n_s = x_s^\alpha$ for external connections, where $x_m + x_s = E$. The effort level for a club member can thus be quantified by $\hat{E}_J = n_m^{1/\alpha} + n_s^{1/\alpha}$, and for a non-member by $\hat{E}_{NJ} = n_{NJ}^{1/\alpha}$, where n_{NJ} representing the number of friends for non-members. Determination of the optimal α requires minimizing the discrepancies between the distributions of $\hat{E}_J$ and $\hat{E_{NJ}}$.

I employ the Kolmogorov-Smirnov Statistic (K-S test) to measure the maximum divergence between the empirical cumulative distribution functions (ECDFs) of the two groups. Given that the number of friends is concentrated on specific discrete values, the data contain exact duplicates, leading to ties in the empirical cumulative distribution functions (ECDFs). As a result, the classical computation of the K-S test is not applicable. To address this issue, a jittering method is introduced by adding a small, random Gaussian noise $N(\mu = 0, \sigma = 0.01)$ to each estimated $\hat{E}_i$, thereby eliminating ties.

To enhance the reliability of our findings, given the slight variations introduced by jittering, we adopt a bootstrap method. This involves:

1. Estimate Effort: Calculate effort $\hat{E}_J$ for club members and $\hat{E_{NJ}}$ for non-members.
2. Bootstrap Analysis: Conduct $10,000$ bootstrap iterations, each involving a resampled dataset comprising a total of N students, with varying counts of each student type.
3. Jittering: In each iteration, add Gaussian noise $N(0, 0.01)$ on each $\hat{E}_i$.
4. Kolmogorov-Smirnov Test: Apply the K-S test in each iteration to identify the α that minimizes differences in the ECDFs of the two student groups, denoted as $\hat{\alpha}$.
5. Statistical Analysis of α: Analyze the $10,000$ $\hat{\alpha}$ values to determine their statistical properties and identify the optimal $\hat{\alpha}$.

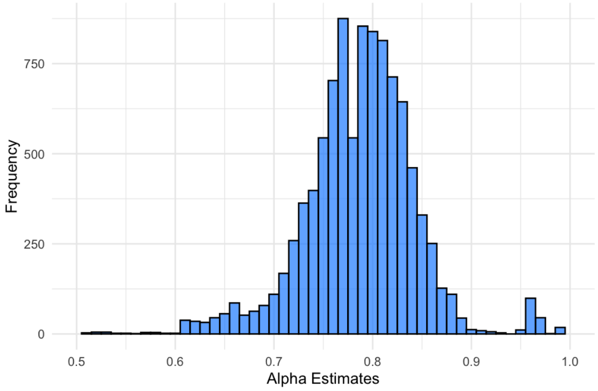


Figure 4.3: Bootstrap Distribution of Alpha Estimates

Figure 4.3 illustrates the distribution of the $10,000$ $\hat{\alpha}$ values, showing a mean of 0.779, a mode of 0.794, and a median of 0.787, with a 95% confidence Interval ranging from 0.637 to 0.884. Subsequent analyses will use the median value of $\hat{\alpha} = 0.78733$ is utilized. Figure 4.4 depicts the density distributions of social efforts for both types of students.

Impacted by the value of α

In this model, the within-club friends and external friends for club members are denoted as $n_m = x_m^\alpha$ and $n_s = x_s^\alpha$ respectively, while for non-club members, the number of friends are represented as $n_{NJ} = E^\alpha$. In the specific case where $\alpha = 1$, the relation $x_m^\alpha + x_s^\alpha = E^\alpha$ implies $n_m + n_s = n_{NJ}$, indicating that students do not form additional friendship connections through club participation. While they may form new friendships with other club members after joining, their total number of social connections remains unchanged. Conversely, when $\alpha < 1$, the inequality $x_m^\alpha + x_s^\alpha > E^\alpha$ suggests that students can establish more friendship connections as a result of club participation. Therefore, compared to those whose total number of friendships remains constant, students in this scenario gain greater benefits from joining the club. Furthermore, this disparity in benefits becomes more pronounced as α decreases.

Figure 4.4: Comparison of Density Distributions for Social Effort ($\alpha = 0.787$)

Note: In both distributions, random noise is added. For the non-members' distribution, the original estimated consists of a limited number of discrete values, leading to a multimodal distribution even after jittering. It is important to note that although the plot appears smoothed, the actual distribution features more pronounced peaks.

The value of α also influences the ratio benefits ratio $r_j = \frac{v_m}{v_s}$. In situations where α approaches 1, the ratio $\frac{v_m}{v_s} = \left(\frac{x_m}{x_s}\right)^{1-\alpha}$ tends to 1, irrespective of the values of x_m and x_s. Thus, the benefits ratio r_j hovers around 1 for any school j. In such cases, club participation neither increases friendships nor enhances the value of friendship connections. The benefits from social connections are similar for all students, implying that a student's decision to participate is primarily influenced by their intrinsic qualities rather than by external factors like the relative size of their racial group.

Empirical analysis reveals that $\alpha = 0.787$ for white students, suggesting that white club members establish more friendship connections than non-members (7.7 versus 7.2 friends per student). In contrast, $\alpha = 0.96$ for black students, resulting in a nearly equal number of social connections among club members and non-members (6.4 versus 6.2 friends per student). This observation explains why the participation rate among white students correlates with their relative group size, whereas this trend is less pronounced among black students.

Estimation of Net Benefit ΔV

The model given in Section 4 provides the equation of net benefit, $\Delta V = v_m n_m + v_s n_s - v_s n_{NJ}$, where v_m and v_s represent per-connection values; n_m, n_s, and n_{NJ} denote a student's number of within-club connections, external connections, and the total connections if the student decides not to join the club, respectively.

Empirical data reveal a broad variability in the friendship structure of club members. Some members primarily form friendship connections within the club, while others build more external connections. To accurately estimate how social connections impact students' participation decisions, it is assumed that each club member assigns a distinct value to the ratio $r = \frac{v_m}{v_s}$. A higher value of r implies that a club member tends to make more friends within the club, whereas a lower value suggests a preference for external friendships.

The net benefit ΔV can thus be represented as:

$$\Delta V = v_s \left(r \cdot n_m + n_s - n_{NJ} \right)$$

For club member i in school j, the ratio r_{ij} can be calculated by:

$$r_{ij} = \frac{v_{m_{ij}}}{v_{s_{ij}}} = \left(\frac{x_{m_{ij}}}{x_{s_{ij}}} \right)^{1-\alpha} = \left(\frac{n_{m_{ij}}}{n_{s_{ij}}} \right)^{\frac{1-\alpha}{\alpha}}$$

Since there is no direct data on a non-member's network structure after they join the club, both per-connection values v_m and v_s are context-dependent and vary by school. Therefore, the school-level average rate r_j is used to evaluate non-members' preferences, assuming it remains consistent across all non-members within school j. Defining the average within-club and external connections for club members of a particular race in school j as $\bar{n}_{m_j}$ and $\bar{n}_{s_j}$, respectively, the school-level average rate r_j is estimated by:

$$\hat{r_j} = \left(\frac{\bar{n}_{m_j}}{\bar{n}_{s_j}}\right)^{\frac{1-\alpha}{\alpha}}$$

In the data, students are classified as either club members, who have two types of connections, n_m and n_s, or non-members with total connections n_{NJ}. Estimating the connections if a student were of the opposite type involves using their calculated effort levels.

For instance, if student i is a club member, their effort level $\hat{E_{ij}}$.is estimated, and the number of friends they would have if they were a non-member can be calculated as:

$$\hat{n}_{NJ_{ij}} = \hat{E_{ij}}^{\alpha}$$

Conversely, for a non-member, determining the optimal allocation of their effort $\hat{E}_i$ on two types of friends if they were to join a club is required. The optimal allocation is derived from:

$$\begin{aligned} \hat{x}_{s_{ij}} &= \frac{\hat{E_{ij}}}{\hat{r}_j^{\frac{1}{1-\alpha}} + 1} \\ \hat{x}_{m_{ij}} &= \hat{E_{ij}} - \hat{x}_{s_{ij}} \end{aligned}$$

Thus, the number of friends for a num-member i in school j is driven as :

$$\begin{aligned} \hat{n_{s_{ij}}}(\hat{E}_{ij}, \hat{r_j}) &= \left(\hat{x_{s_{ij}}}(\hat{E}_{ij}, \hat{r_j})\right)^{\hat{\alpha}} \\ \hat{n_{m_{ij}}}(\hat{E}_{ij}, \hat{r_j}) &= \left(\hat{x_{m_{ij}}}(\hat{E}_{ij}, \hat{r_j})\right)^{\hat{\alpha}} \end{aligned}$$

Estimation of Net Benefit

The net benefit can finally be represented as:

$$\Delta\hat{V}_{ij} = v_{s_j}\Delta\hat{n}_{ij}$$

where $\Delta\hat{n}_{ij}$ denotes the relative discrepancy in the number of friends, which is given by (4.2). The parameter v_{s_j}, representing the value of an external friendship in school j, is the only parameter not estimated in the preceding steps.

$$\Delta\hat{n}_{ij} = \begin{cases} \hat{r_{ij}} n_{m_{ij}} + n_{s_{ij}} - \hat{n_{NJ_{ij}}} & \text{if } i \text{ is a club member} \\ \\ \hat{r_j} \hat{n_{m_{ij}}} + \hat{n_{s_{ij}}} - n_{NJ_{ij}} & \text{if } i \text{ is a non-member} \end{cases} \tag{4.2}$$

Figure 4.5 presents the density distributions for both the increase in number of friendships and the relative discrepancy $\Delta\hat{n}_{ij}$, with data grouped by student type. Compared to non-members, club members exhibit a larger relative friendship discrepancy. This is because some members have significantly more within-club connections, indicating a higher value placed on the ratio $r = \frac{v_m}{v_s}$. Consequently, under this scenario, the relative discrepancy is magnified.

4.3.2 Reduced form Estimation

In the subsequent analysis, a logistic regression model is adopted to verify the existence of the impact from benefits $\Delta\hat{n}_{ij}$ on students' participation decision. This model enables the estimation of the value of social connections in the decision-making process and examines how the benefit per connection v_j varies across different schools.

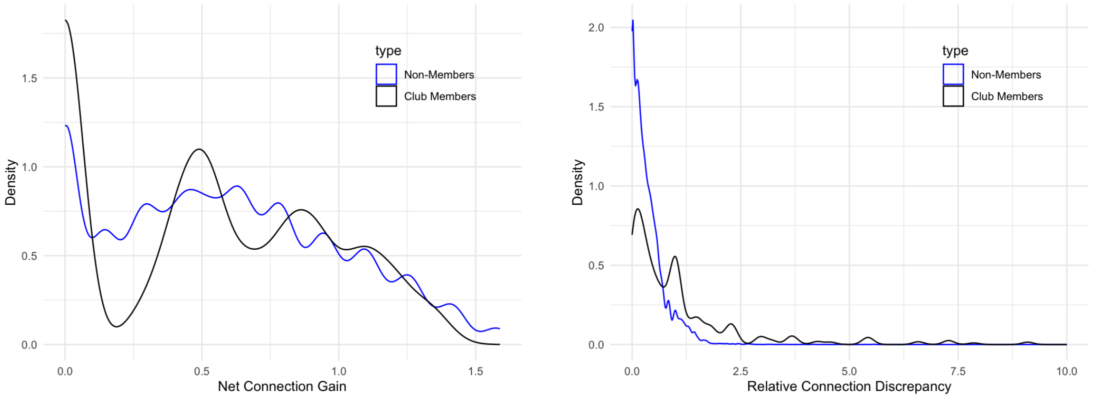


Figure 4.5: Density Distributions of Friendship Gains and Relative Discrepancies by Student Type

Note: The first distribution (left) shows the Net Friendship Gain, representing the increase in the number of connections when a student opts to join a club, calculated as $n_m + n_s - n_{NJ}$. The second distribution (right) illustrates the Relative Connection Discrepancy, defined as $r \cdot n_m + n_s - n_{NJ}$, which measures the relative difference in friendship connections based on the type of student.

The logistic model primarily focuses on assessing the influence of social connections on participation. Formally, the model is expressed as:

$$y_{ij} = v_j^s \Delta \hat{n}_{ij} + \beta X + \epsilon_j + \epsilon_{ij} \text{ with } i = 1, ..., n_j \text{ and } j = 1, ..., J. \tag{4.3}$$

where y_{ij} represents the participation of student i in school j, $\Delta \hat{n}_{ij}$ denotes the potential social engagement if the student opts to participate, v_j^s is the benefit per across-school friendship (v_s) in school j, βX_{ij} captures the student's personal characteristics, ϵ_j represents the fixed effects for school j, and $\epsilon_{ij}$is the individual error term.

The primary aim of this regression model is to determine whether benefits influence students' decisions to participate in clubs. Additionally, it allows for the estimation of the per-connection benefit v_s in each school within the dataset, enabling further analysis of how these values v_j^svary with school size and the proportion of White students across different schools. The logistic regression is conducted, and the results are presented in Table **??**.

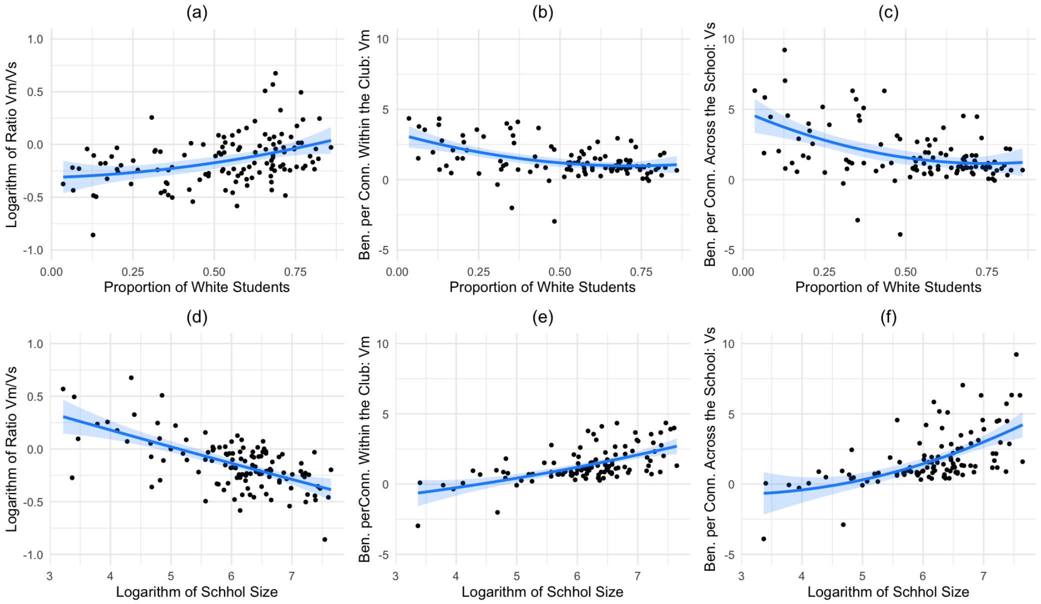


Figure 4.6: Relationships Among v_m/v_s, v_m, v_s, Racial Proportion, and School Size

Empirical Confirmation of Model Predictions: Benefits from Social Engagement and Participation Rates

In the theoretical framework established in Section 4.2, Lemma 4.2' asserts that if the ratio of within-club to outside-club benefits, $\frac{v_m}{v_s}$, increases as the racial proportion rises, then the net benefits ΔV will also increase with the racial proportion. Proposition 4.4' further further explains that this relationship leads to a corresponding rise in the stable participation rate h^* for the racial group in question.

Empirical evidence aligns with this theoretical assertion, showing a clear trend among White students where the ratio of per-connection benefits, $\frac{v_m}{v_s}$, increases as the racial proportion grows. This relationship is illustrated in Figure 4.6 (a), which depicts a positive correlation between the benefit ratio and racial proportion, thereby reinforcing the model's predictions.

The subgraphs in Figure 4.6 illustrate how the per-connection values v_m and v_s, as well as the ratio v_m/v_s, are influenced by racial proportion and school size, he corresponding correlation

Table 4.1: Correlation Matrix for Variables Shown in Figure 8

Correlation Matrix	Ratio v_m/v_s	v_m	v_s
Relative Racial Size w	0.369	−0.427	−0.446
School Size	−0.518	0.565	0.586

values are presented in Table 4.1. The estimates shown here are derived from regressions without school-level fixed effects but account for various school characteristics, including location, region, and type (private, public, or Catholic).

In the analysis, students were separated into members and non-members. A total of 24 schools were removed from the dataset: 10 schools were excluded because they had fewer than three students of at least one type (member or non-member); one school was excluded for yielding a negative value when estimating the ratio v_m/v_s; and 13 schools were excluded due to missing information on school characteristics. Ultimately, the final dataset consisted of 110 schools.

Figure 4.7 presents the coefficients v_j^s in regression (4.3), standard errors, and p-values for each group. Schools are categorized by size into four groups: 26 Small (<300 students), 30 Medium-Small (300-550 students), 29 Medium-Large (550–800 students), and 25 Large (>800 students).

Generally, as school size increases, the value of both types of friendships rises, but the value of across-school friendships (v_s) grows at a faster rate than that of within-club friendships (v_m). Conversely, as the proportion of White students in the population increases, the per-connection value for both types of friendships declines, with across-school friendships (v_s) experiencing a sharper decrease. These results, however, contradict the assumptions in the previous model, where per-connection benefits were expected to increase as the racial proportion rises in a school.

In the data, we observe that the number of friendships increases with racial group size, indicating

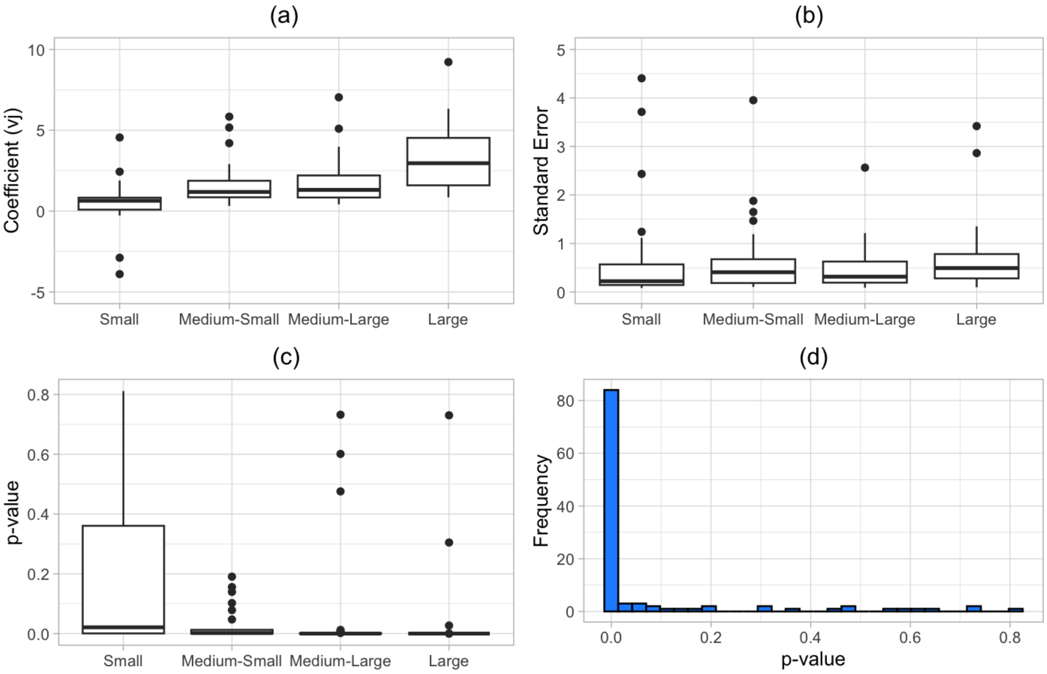


Figure 4.7: Coefficients, Standard Errors, and p-Values Grouped by School Size

Note: Schools are grouped by size in each graph.
(a) Coefficient Distribution ($v_j{}^s$) by School Size
(b) Standard Error Distribution by School Size
(c) p-Value Distribution by School Size
(d) Histogram of p-Values
In total, there are 68 schools with p-values < 0.001, 12 schools with p-values < 0.01, 9 schools with p-values < 0.05, and 21 schools with insignificant results.

that students tend to have more friends in this scenario. One explanation is that when the proportion of White students rises, the increased homogeneity may allow students to build friendship connections with less effort (x_m and x_s). This leads to two outcomes: an increase in the total number of friends and a decrease in the absolute value of per-connection benefits (v_m and v_s). Additionally, the value of across-school friendships (v_s) may decrease more sharply, as students find it easier to make same-race friends without needing to engage as much outside their existing circles.

4.4 Conclusion

In this chapter, I develop a benefit-driven model to provide a mechanism that explains the interplay between social norms and students' decisions, as well as its impact on social networks. This model offers a framework to understand how students' social interactions and participation choices shape their broader social environment.

Previous research has suggested that smaller schools exhibit lower levels of racial homophily. This study contributes to this discussion by demonstrating that high rates of participation in cultural activities may play a crucial role in this phenomenon. This study provides insights that can inform policy interventions aimed at promoting inclusivity and social cohesion. By recognizing the coordinated behavior in students' participation, social planners can design more targeted programs that cater to the specific cultural preferences of different communities. For example, implementing subsidies for basketball programs in predominantly White neighborhoods and for baseball in predominantly Black neighborhoods could foster inclusivity and encourage diverse participation, thereby enhancing social cohesion.

While this study provides valuable insights, it also has certain limitations. The analysis relies on cross-sectional data, which may not fully capture the temporal evolution of social networks and participation decisions. Additionally, the study focuses on specific sports, which may limit the generalizability of the findings to other types of cultural activities. Future research could expand on this work by using longitudinal data to observe changes over time, or by exploring how other extracurricular activities influence peer effects and social interactions across different demographic groups.

Appendix to Chapter 4

Proof of Lemma 4.1 (Monotonic Growth of Joining Payoff Advantage).

We aim to demonstrate that the partial derivative of the net benefit ΔV with respect to the participation rate h, $\frac{\partial \Delta V}{\partial h}$, is positive. This derivative can be expressed through the chain rule as follows:

$$\frac{\partial \Delta V}{\partial h} = \frac{\partial \Delta V}{\partial v_m} \frac{\partial v_m}{\partial h}$$

Given the functional form of the net benefit ΔV:

$$\Delta V = E^{\alpha} \left[\left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}} \right)^{1-\alpha} - v_s \right]$$

we can get

$$\begin{aligned} \frac{\partial \Delta V}{\partial h} &= E^{\alpha} \left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}} \right)^{-\alpha} v_m^{\frac{\alpha}{1-\alpha}} \frac{\partial v_m}{\partial h} \\ &= \left(\frac{v_m^{\frac{1}{1-\alpha}}}{v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}} E \right)^{\alpha} \frac{\partial v_m}{\partial h} \end{aligned}$$

Using the relationship between effort allocation within the club x_m and the per-connection benefit v_m and v_s, we have:

$$x_m = \frac{v_m^{\frac{1}{1-\alpha}}}{v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}} E$$

Substituting x_m into the derivative, we simplify $\frac{\partial \Delta V}{\partial h}$:

$$\frac{\partial \Delta V}{\partial h} = x_m^{\alpha} \frac{\partial v_m}{\partial h}$$

where x_m^{α} is the number of connections a student builds within the club. Given that $\frac{\partial v_m}{\partial h} > 0$ by Assumption 4.2, we conclude that

$$\frac{\partial \Delta V}{\partial h} > 0$$

This shows that the net benefit ΔV increases as the participation rate h increases.

Proof of Lemma 4.2 (Advantage of Racial Majority in Club Benefits).

$$\Delta V(v_m, v_s, \alpha, E) = E^{\alpha} \left[\left(v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}} \right)^{1-\alpha} - v_s \right]$$

$$\begin{aligned}
\frac{\partial \Delta V}{\partial w} &= E^{\alpha} \left[(1-\alpha) \left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}} \right)^{-\alpha} \left(\frac{1}{1-\alpha} v_m^{\frac{\alpha}{1-\alpha}} \frac{\partial v_m}{\partial w} + \frac{1}{1-\alpha} v_s^{\frac{\alpha}{1-\alpha}} \frac{\partial v_s}{\partial w} \right) - \frac{\partial v_s}{\partial w} \right] \\
&= E^{\alpha} \left[\frac{v_m^{\frac{\alpha}{1-\alpha}} \frac{\partial v_m}{\partial w} + v_s^{\frac{\alpha}{1-\alpha}} \frac{\partial v_s}{\partial w}}{\left(v_m^{\frac{1}{1-\alpha}} + v_s^{\frac{1}{1-\alpha}} \right)^{\alpha}} - \frac{\partial v_s}{\partial w} \right] \\
&= \left(E \frac{v_m^{\frac{1}{1-\alpha}}}{v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}} \right)^{\alpha} \frac{\partial v_m}{\partial h} + \left(E \frac{v_s^{\frac{1}{1-\alpha}}}{v_s^{\frac{1}{1-\alpha}} + v_m^{\frac{1}{1-\alpha}}} \right)^{\alpha} \frac{\partial v_s}{\partial h} - E^{\alpha} \frac{\partial v_s}{\partial w} \\
&= x_m^{\alpha} \frac{\partial v_m}{\partial h} + x_s^{\alpha} \frac{\partial v_s}{\partial h} - E^{\alpha} \frac{\partial v_s}{\partial w} \\
&= x_m^{\alpha} \frac{\partial v_m}{\partial w} + \left(x_s^{\alpha} - E^{\alpha} \right) \frac{\partial v_s}{\partial w}
\end{aligned}$$

Given the assumption $\frac{\partial v_s}{\partial w_i} \geq 0$ (as stated in Assumption 4.2), and the relationship $x_m + x_s = E$, which implies $x_m^\alpha + x_s^\alpha \geq E^\alpha$, we conclude:

$$\frac{\partial \Delta V}{\partial w_i} \geq x_m^\alpha \left(\frac{\partial v_m}{\partial w_i} - \frac{\partial v_s}{\partial w_i} \right)$$

This expression demonstrates that the net benefit increases as the within-club benefit v_m increases faster than the outside-club benefit v_s.

Proof of Lemma 4.2'

Starting with the formula for the partial derivative of net benefit with respect to w, $\frac{\partial \Delta V}{\partial w} = x_m^\alpha \frac{\partial v_m}{\partial w} + \left(x_s^\alpha - E^\alpha \right) \frac{\partial v_s}{\partial w}$, we normalize by dividing through by x_s^α, leading to:

$$\begin{aligned}
\frac{\frac{\partial \Delta V}{\partial w}}{x_s^\alpha} &= \left(\frac{x_m}{x_s} \right)^\alpha \frac{\partial v_m}{\partial w} + \left(1 - \left(1 + \frac{x_m}{x_s} \right)^\alpha \right) \frac{\partial v_s}{\partial w} \\
&\geq \left(\frac{x_m}{x_s} \right)^\alpha \frac{\partial v_m}{\partial w} - \frac{x_m}{x_s} \frac{\partial v_s}{\partial w} \\
&= \left(\frac{x_m}{x_s} \right)^\alpha \frac{\partial v_m}{\partial w} - \left(\frac{x_m}{x_s} \right)^\alpha \left(\frac{x_m}{x_s} \right)^{1-\alpha} \frac{\partial v_s}{\partial w} \\
&= \left(\frac{x_m}{x_s} \right)^\alpha \left(\frac{\partial v_m}{\partial w} - \frac{v_m}{v_s} \frac{\partial v_s}{\partial w} \right)
\end{aligned}$$

where the inequality is derived from the property that $(x+1)^\alpha - 1 < x$ for any $x > 0$ and $\alpha \in (0, 1)$.

Additionally, the partial derivative of the ratio $\frac{v_m}{v_s}$ with respect to w is given by: $\frac{\partial}{\partial w} \left(\frac{v_m}{v_s} \right) = \frac{1}{v_s} \left(\frac{\partial v_m}{\partial w} - \frac{v_m}{v_s} \frac{\partial v_s}{\partial w} \right)$, implying that $\frac{\partial}{\partial w} \left(\frac{v_m}{v_s} \right) > 0$ is equivalent to $\frac{\partial v_m}{\partial w} - \frac{v_m}{v_s} \frac{\partial v_s}{\partial w} > 0$. Thus, we can conclude:

$$\frac{\partial \Delta V}{\partial w} > 0 \text{ if } \frac{\partial}{\partial w} \left(\frac{v_m}{v_s} \right) > 0$$

Proof of $\frac{\partial v_m}{\partial w} > \frac{\partial v_s}{\partial w}$ in unbiased matching:

This proof examines the conditions under which $\frac{\partial v_m}{\partial w} > \frac{\partial v_s}{\partial w}$. We begin by detailing the formulas for these derivatives in the context of random matching:

$$\frac{\partial v_m}{\partial w} = \frac{hh_b}{[hw + h_b(1-w)]^2} v_{homo}$$

$$\frac{\partial v_s}{\partial w} = v_{homo}$$

To understand how $\frac{\partial v_m}{\partial w}$ compares to $\frac{\partial v_s}{\partial w}$, consider the expression $[hw_i + h_b(1-w)]^2$ which transitions: from h_b^2 as $w \longrightarrow 0$ to h^2 as $w \longrightarrow 1$. This squared term in the denominator indicates that the value of $\frac{\partial v_m}{\partial w_i}$ depends critically on the magnitudes of h and h_b relative to w:

1. Case $h < h_b$: The derivative $\frac{\partial v_m}{\partial w_i}$ starts lower than $\frac{\partial v_s}{\partial w_i}$ when w is small but surpasses it as w increases, reflecting a transition as the racial group's representation in the population grows.
2. Case $h = h_b$: Here, $\frac{\partial v_m}{\partial w_i} = \frac{\partial v_s}{\partial w_i}$ consistently across all w.
3. Case $h > h_b$: In this scenario, $\frac{\partial v_m}{\partial w_i}$ exceeds $\frac{\partial v_s}{\partial w_i}$ for small w, suggesting that a greater proportion of the same race within the club significantly enhances the net benefits right from a lower threshold of racial proportion.

Proof of Proposition 4.6 (Differential Impact of Cost on Connectors and Devotees)

Assume that a proportion α of club members are Connectors and $1-\alpha$ are Devotees. Given the initial participation rates for Connectors and Devotees as h_c and h_d respectively, the average talent value among club members is expressed as $\bar{t} = \alpha \bar{t_c} + (1-\alpha)\bar{t_d} = \frac{1-log(h_c^{\alpha} h_d^{1-\alpha})}{\lambda}$.[3] We use $\tilde{h} = h_c^{\alpha} h_d^{1-\alpha}$

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[3] By the property of exponential distributions, the expected talent, given survival beyond the threshold t, is $E(T|T \geq t) = \frac{1}{\lambda} + t = \frac{1-log(h_m)}{\lambda}$, where $h_m = P(T|T \geq t)$ represents the participation rate for type-m members. Therefore, the general average talent $\bar{t}$ is $\alpha\frac{1-log(h_c)}{\lambda} + (1-\alpha)\frac{1-log(h_d)}{\lambda}$, which can be simplified as $\frac{1-log(h_c^{\alpha} h_d^{1-\alpha})}{\lambda}$.

denote the geometric mean of the participation rates.

Case: Increase in Costs ($\Delta c > 0$)

Consider a scenario where the cost increases, denoted $\Delta c > 0$. The new participation rate for Devotees, h'_d is calculated by:

$$h'_d = e^{-\lambda(t_d + \Delta c - \gamma \Delta t)} \tag{4.4}$$

where $\Delta t = \bar{t}' - \bar{t} = \frac{1}{\lambda} \log\left(\frac{\tilde{h}}{\tilde{h}'}\right)$ quantifies the change in average talent due to the cost adjustment. Therefore, the ratio of the participation rate $\frac{h'_d}{h_d} = e^{-\lambda(\Delta c - \gamma \Delta t)}$.

1. If $\tilde{h}' < \tilde{h}$, the term $\gamma \Delta t = \frac{\gamma}{\lambda} \log\left(\frac{\tilde{h}}{\tilde{h}'}\right)$ is positive, indicating that: $\frac{h'_d}{h_d} > \frac{h'_c}{h_c} = e^{-\lambda \Delta c}$.

   This implies that the decrease in participation rate for Devotees is less severe than that for Connectors due to the synergy benefits derived from the club's talent composition.

2. If $\tilde{h}' > \tilde{h}$, it leads to a contradiction.

   As demonstrated in the statement above, we can get $\frac{h'_d}{h_d} < \frac{h'_c}{h_c} < 1$. However, this condition suggests that if both groups' participation rates decrease, the average rate $\tilde{h}$ must also decrease, conflicting with the assumption that $\tilde{h}' > \tilde{h}$.

We must next demonstrate that, under the condition where $\Delta c > 0$, the inequality $\frac{h'_d}{h_d} < 1$ necessarily holds.

We prove by contradiction. Assume $\Delta c > 0$ leads to an increase in participation rate for Devotees, $h'_d \geq h_d$. According to formula 4.4, this increase suggests that the synergy benefits outweigh the detrimental effects of increased cost, $\gamma \Delta t \geq \Delta c$. From this, we derive: $\frac{\tilde{h}'}{\tilde{h}} \leq e^{-\frac{\lambda \Delta c}{\gamma}}$

However, the new to old average participation rate ratio contradicts this, given by:

$$\begin{aligned}\frac{\tilde{h}'}{\tilde{h}} &= \frac{h_c'^{\alpha} h_d'^{1-\alpha}}{h_c^{\alpha} h_d^{1-\alpha}} \\ &> \frac{h_c'}{h_c} = e^{-\lambda \Delta c}\end{aligned}$$

Therefore, the participation rate among Devotees must decrease, ($h_d' < h_d$), when the cost increases.

In conclusion, when costs increase, the ratio of the new to old participation rates satisfies $\frac{h_c'}{h_c} < \frac{h_d'}{h_d} < 1$. Similarly, when costs decrease, we can establish that $1 < \frac{h_d'}{h_d} < \frac{h_c'}{h_c}$, showing an increase in participation rates, with Devotees experiencing a less pronounced increase compared to Connectors.

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