Stability of Localized Solutions to Lattice Dynamical Systems

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Abstract

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This thesis focuses on the stability of spatial localized solutions of lattice dynamical systems (LDSs). In particular, I focus on the stability of spatially localized single- and multi-pulse solutions to lattice dynamical systems. By linearizing the nonlinear system around steady-state solutions and applying exponential dichotomy theory, an isomorphism between the localized solutions and the stable front and back solutions is constructed. Then, the proof constructs Evans functions about the localized solutions from Evans functions of the front and back solutions. It gives that eigenvalues of the front and back solutions lead to nearby eigenvalues for the localized solution, which can in turn be used to verify instability of the solution. Furthermore, Rouché's theorem is used to prove that the number of roots in the single-pulse solution is the sum of front and back solutions', and the multi-pulse solution's is the sum of each single-pulse solution's. This work is widely applicable to a range of scalar LDSs, particularly the well-studied discrete Nagumo equation. Finally, this thesis concludes with a discussion of possible avenues for future work.

Acknowledgments

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Chapter 1

Introduction

Lattice dynamical systems are mathematical frameworks used to describe the evolution of dynamic systems over discrete, individual spatial locations. These systems are characterized by state variables that evolve according to coupled ordinary differential equations, often leading to complex behaviors, such as the emergence of localized patterns or coherent structures. Such localized patterns, also referred to as spatially localized structures, are solutions where nontrivial features arise within a larger homogeneous background.

Localized patterns take the form of a spatially homogeneous background with a nontrivial or activated state existing within a compact region of space [4]. These patterns are not merely theoretical ideas, they have real-world implications across various scientific disciplines to help us predict, control, and better understand how complex systems behave. For instance, these patterns have been observed in ecosystems [14], where vegetation clusters form distinct regions; in semiconductors [17], where charge distributions stabilize in specific areas; and in social systems such as crime hotspots [11], where criminal activity concentrations occur in compact regions. Localized patterns often emerge as a result of the interplay between local growth and spreading processes, as described by reaction-diffusion equations. In such systems, front and back solutions, which are monotonic spatially-extended solutions connecting two homogeneous states, play a critical role not only in understanding the existence of localized behavior but also in determining the stability of localized structures [15, 13]. In this thesis, the connection between the stability of front and back solutions to a localized solution is made rigorous by appealing to the theory of the Evans function.

Historically, lattice dynamical systems have been studied to understand the effect of spatial inhomogeneity coming from the discrete jumps between spatial locations. Early work by Keener [9] focused on the propagation or its failure of waves in coupled systems of excitable cells, establishing foundational principles that continue to influence contemporary research. More recently, Bramburger and Sandstede studied the existence and bifurcation structure of localized patterns in one [6] and two [5] discrete spatial dimensions. The spatial evolution of localized patterns is a particularly challenging problem due to their dependence on multiple interconnected components. Recent work on the spatially-continuous Swift-Hohenberg equation has demonstrated that the stability of localized structures can be understood in terms of the stability of front and back solutions [13]. Here we outline a similar framework by using front and back solutions to better understand the stability properties of localized solutions to lattice dynamical systems.

In this work, the Evans function is used to analyze the stability of N-pulse solutions that have exactly $N \geq 1$ significant non-zero, disconnected, and bounded region(s). A closely related strategy has been used to demonstrate the existence of snaking localized solutions in continuous space [15, 13], thus forming the basis for the work herein. The main objective is to demonstrate the stability of localized solutions by utilizing the framework of the Evans function. The results herein rely on an initial assumption that there exist front and back solutions, which can roughly be glued together to create localized/pulse solutions [12, 6, 3]. Since these front and back solutions represent heteroclinic orbits to a related spatial dynamical system, they decay exponentially in space toward their limiting states and thus provide the existence of exponential dichotomies to study their stability. An isomorphism is then established between the exponential dichotomies of localized solutions and front/back solutions as they decay to the homogeneous background state. This allows for a construction of the Evans function for these asymptotic solutions. The main results are: eigenvalues of the front and back solutions lead to eigenvalues of the localized solution, and their eigenvalues become exponentially close together as the region of localization grows. In addition, Rouché's theorem [10, Theorem on p. 111] is used to determine that the number of eigenvalues in a single-pulse solution is the sum of the eigenvalues in front and back solutions' and the number of eigenvalues in a multi-pulse solution is the sum of the eigenvalues in each single pulse solution. Thus, a clear approach to analyzing the stability of localized solutions is proven for lattice dynamical systems.

This thesis is organized as follows. In Chapter 2, we present the general framework for lattice dynamical systems that we study. We further provide a brief overview of the work on the existence of single- and multi-pulse solutions, while also discussing a specific system where out hypotheses can be verified. In Chapter 3, as it has been proven that the pulse solutions can indeed be created by gluing the front and back solutions together, we will demonstrate the stability for single- and multi-pulse solutions by using the Evans function, respectively. Chapter 4 summarizes and discusses potential future research avenues.

Chapter 2

Background

In this chapter, we provide foundational concepts and key results related to lattice dynamical systems. These elements form the theoretical groundwork for the developments in Chapter 3.

2.1 Lattice Dynamical Systems

Lattice dynamical systems are infinite systems of coupled ordinary differential equations, for which variables are arranged according to a regular structure as lattice points, with some rules that depend on their neighbours and even long-distance connections. The general form of systems on the one-dimensional integer lattice $\mathbb{Z}$ can be described by

$$\dot{u}_i = f_i(\{u_j\}_{j\in\mathbb{Z}}) \tag{2.1}$$

for all $i \in \mathbb{Z}$, where $\dot{u}_i$ is the time derivative du/dt, u_i are time-dependent functions, and f_i are functions that typically only depend on finitely many states u_j, termed the *neighbours*.

One of the most typical and widely studied examples in lattice dynamical systems is the reaction-diffusion class of systems. Many researchers have studied various properties of thier solutions, such as traveling wave solutions, pattern formation, and synchronization [16, 18, 19, 1]. Given that coupled map lattices often exhibit symmetry and reversibility, we will focus on lattice systems of Nagumo-type [7]. These models are derived from the Nagumo partial differential equation (PDE) taking the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u,\mu), \tag{2.2}$$

where $\partial^2 u/\partial x^2$ is the second spatial derivative, and f is a bistable nonlinearity. Some common examples in the study of pulses and pattern formation are $f(u,\mu) = u(1-u)(u-\mu)$ and $f(u,\mu) = \mu u + 2u^3 - u^5$ with $\mu \in [0,1]$.

To move from the PDE framework to a lattice dynamical system one can use a finite-difference approximation. First, we consider evenly-spaced points arranged in a line, spaced apart by length Δx, and introduce the notation $U_n := u(n\Delta x, t)$. To approximate $\partial^2 u/\partial x^2$ at grid points, we will use neighboring values. By using a Taylor series expansion, we can expand $U_{n+1} := u((n+1)\Delta x, t)$ and $U_{n-1} := u((n-1)\Delta x, t)$ around U_n

$$U_{n+1} = \overbrace{u(n\Delta x, t)}^{U_n} + \Delta x \frac{\partial u}{\partial x}(n\Delta x, t) + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}(n\Delta x, t) + O\left((\Delta x)^3\right),$$
$$U_{n-1} = u(n\Delta x, t) - \Delta x \frac{\partial u}{\partial x}(n\Delta x, t) + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}(n\Delta x, t) + O\left((\Delta x)^3\right).$$

Adding these two expansions together and solving for $\partial^2 u/\partial x^2$ gives

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{n+1} - 2U_n + U_{n-1}}{(\Delta x)^2}$$

which is the central difference approximation for the second derivative. Then, we define a scaled diffusion coefficient $d = 1/(\Delta x)^2$, representing the coupling strength between neighboring lattice sites. Substituting this into (2.2), we obtain the discrete analogue of the Nagumo PDE

$$\dot{U}_n = d(U_{n+1} + U_{n-1} - 2U_n) + f(U_n, \mu) \tag{2.3}$$

where $\dot{U}_n$ is the time derivative of a state U_n which only depends on time, $d(U_{n+1} + U_{n-1} - 2U_n)$ is the discrete diffusion operation, and $f(U_n, \mu)$ is the nonlinear reaction with a bifurcation parameter μ. Lattice systems of the form (2.3) will be the focus of investigation in this thesis.

A primary goal in discretizing space is to study the effect of breaking the continuous spatial symmetry that is exhibited by the Nagumo PDE. Our main research object will be steady-state solutions which are solutions to the lattice dynamical system (2.3) that do not change in time. In steady-state, the system is in perfect balance and thus does not exhibit any temporal evolution, meaning $\dot{U}_n = 0$ for all n. Steady states are some of the most widely-studied potential limit states of dynamical systems, and as we show in Section 2.3 below, are the only possible limit states for the LDSs that we study herein due to a gradient-flow structure in a Hilbert space.

2.2 Existence of Localized Solutions

Solutions of (2.3) with $\dot{U}_n = 0$ are steady states, and we can recast the steady-state equation as a first-order discrete dynamical system by setting $(u_n, v_n) = (U_{n-1}, U_n)$ to get

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = 2v_n - u_n - \dfrac{1}{d} f(v_n, \mu) \end{cases} \tag{2.4}$$

where μ is a bifurcation parameter that can be used to determine the existence, stability, and structure of localized solutions. Inspired by (2.4), in this section we consider planar discrete-time dynamical systems for full generality to other LDSs. Since this thesis does not focus on bifurcations, we consider the bifurcation parameter μ to be fixed and the evolution of $\mathbf{u}_n$ governed by a discrete-time dynamical system of the form that is not dependent on μ,

$$\mathbf{u}_{n+1} = F(\mathbf{u}_n), \tag{2.5}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth mapping function, ensuring a unique relationship between $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$ at every step. We assume that F is a diffeomorphism and therefore backwards-in-space orbits are also uniquely determined by the inverse mapping

$$\mathbf{u}_{n-1} = F^{-1}(\mathbf{u}_n).$$

Localized solutions manifest themselves as an orbits to (2.5), asymptotically connecting to a fixed point as $n \to \pm\infty$ and are bounded away from this fixed point for finitely many iterations. Specifically, these fixed points exhibit different steady states at $\mathbf{u} = \mathbf{0}$ and $\mathbf{u} = \mathbf{u}^*$, typically stemming from nonlinear interactions such as a bistable function f.

Hypothesis 2.1. The system possesses a symmetry operator $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ ($\mathcal{R} \circ \mathcal{R} = id$) such that:

$$F^{-1}(\mathbf{u}_n) = \mathcal{R} \circ F \circ \mathcal{R}(\mathbf{u}_n)$$

If we apply $\mathcal{R}$ to a trajectory $\{\mathbf{u}_n\}_{n\in\mathbb{Z}}$ and set $\mathbf{v}_n = \mathcal{R}\mathbf{u}_n$, it will result in a solution $\{\mathbf{v}_n\}_{n\in\mathbb{Z}} = \{\mathcal{R}\mathbf{u}_{-n}\}_{n\in\mathbb{Z}}$ which follows the same path in reverse order. If $\mathbf{u}_n$ is a bounded sequence to (2.5), then so it $\mathcal{R}\mathbf{u}_{-n}$ too.

Hypothesis 2.2. $\mathbf{0}$ and $\mathbf{u}^*$ are hyperbolic fixed points for F of (2.5).

Equilibrium points $\mathbf{0}$ and $\mathbf{u}^*$ are hyperbolic, which means the linearization is characterized by the absence of any eigenvalue on the unit circle around an equilibrium. Since the fixed points $\mathbf{0}$ and $\mathbf{u^*}$ are hyperbolic, the reverser $\mathcal{R}$ guarantees that they are saddles, and the stable manifold theorem guarantees that one-dimensional stable manifolds

$$W^{\mathrm{s}}(\mathbf{0}) := \left\{\{\mathbf{u}_n\}_{n\in\mathbb{Z}} : \lim_{n\to\infty} \mathbf{u}_n = \mathbf{0}\right\} \qquad \text{and} \qquad W^{\mathrm{s}}(\mathbf{u}^*) := \left\{\{\mathbf{u}_n\}_{n\in\mathbb{Z}} : \lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}^*\right\}$$

and one-dimensional unstable manifolds

$$W^{\mathrm{u}}(\mathbf{0}) := \left\{\{\mathbf{u}_n\}_{n\in\mathbb{Z}} : \lim_{n\to-\infty} \mathbf{u}_n = \mathbf{0}\right\} \qquad \text{and} \qquad W^{\mathrm{u}}(\mathbf{u}^*) := \left\{\{\mathbf{u}_n\}_{n\in\mathbb{Z}} : \lim_{n\to-\infty} \mathbf{u}_n = \mathbf{u}^*\right\}$$

exist for the equilibria $\mathbf{0}$ and $\mathbf{u}^*$. We also notice the reversibility of F and manifolds for equilibria, with symmetry operator $\mathcal{R}$ it holds

$$W^{\mathrm{s}}(\mathbf{0}) = \mathcal{R}W^{\mathrm{u}}(\mathbf{0}) \qquad \text{and} \qquad W^{\mathrm{s}}(\mathbf{u}^*) = \mathcal{R}W^{\mathrm{u}}(\mathbf{u}^*).$$

Definition 2.1. A **heteroclinic orbit** is a solution that connects two distinct steady-state solutions $\mathbf{0}$ and $\mathbf{u}^*$ such that $\lim_{n\to-\infty} \mathbf{u}_n = \mathbf{0}$ ($\lim_{n\to-\infty} \mathbf{u}_n = \mathbf{u}^*$) and $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}^*$ ($\lim_{n\to\infty} \mathbf{u}_n = \mathbf{0}$). A **homoclinic orbit** is a trajectory returns to the same fixed point such that $\lim_{n\to\pm\infty} \mathbf{u}_n = \mathbf{0}$ ($\lim_{n\to\pm\infty} \mathbf{u}_n = \mathbf{u}^*$).

In spatial dynamics formulation (2.4), if a localized solution satisfies $(u_n, v_n) \to (0,0)$ as $n \to \pm\infty$, and its trajectory starts at the trivial equilibrium $(0,0)$, spends arbitrarily many iterations near the excited equilibrium (u^*, u^*), and then returns to the trivial equilibrium $(0,0)$, it is referred to as a homoclinic solution. Furthermore, a single-pulse solution corresponds to its trajectory that actives one such excursion away from $(0,0)$ to (u^*, u^*) and finally return to $(0,0)$, while a multi-pulse solution corresponds to its trajectory that actives multiple excursions before finally returning $(0,0)$.

Hypothesis 2.3. $W^{\mathrm{u}}(\mathbf{0})$ and $W^{\mathrm{s}}(\mathbf{u}^*)$ intersect transversely, and the reverser $\mathcal{R}$ gives that $W^{\mathrm{s}}(\mathbf{0})$ and $W^{\mathrm{u}}(\mathbf{u}^*)$ intersect transversely too.

Since the manifolds in Hypothesis 2.3 are assumed to intersect transversely, it implies that $W^{\mathrm{u}}(\mathbf{0}) \cap W^{\mathrm{s}}(\mathbf{u}^*) \neq \emptyset$ and $W^{\mathrm{s}}(\mathbf{0}) \cap W^{\mathrm{u}}(\mathbf{u}^*) \neq \emptyset$ give front and back solutions of (2.5), respectively, that are robust under small perturbations. As the consequence, they are heteroclinic orbits from $\mathbf{0}$ to $\mathbf{u}^*$ and $\mathbf{u}^*$ to $\mathbf{0}$, respectively. In addition, Hypothesis 2.1 implies that if there exists a front solution $\{U_n^f\}_{n\in\mathbb{Z}}$ of (2.3) such that

$$\lim_{n\to-\infty} U_n^f = 0 \qquad \text{and} \qquad \lim_{n\to\infty} U_n^f = u^*,$$

then there also exists a corresponding a back solution $\{U_n^b\}_{n\in\mathbb{Z}} = \{\mathcal{R}U_{-n}^f\}_{n\in\mathbb{Z}}$ such that

$$\lim_{n\to-\infty} U_n^b = u^* \qquad \text{and} \qquad \lim_{n\to\infty} U_n^b = 0.$$

It allowes for spatial matching to form a localized solution (homoclinic orbit) through gluing front and back solutions. The localized solution $\{\mathbf{u}_n\}_{n\in\mathbb{Z}}$ may therefore be symmetrical, satisfying $\{\mathbf{u}_n\}_{n\in\mathbb{Z}} = \mathcal{R}\{\mathbf{u}_n\}_{n\in\mathbb{Z}}$. This symmetry could be either inversion such that $\mathbf{u}_{-n} = \mathcal{R}\mathbf{u}_n$ or bipartite such that $\mathbf{u}_{-n+1} = \mathcal{R}\mathbf{u}_n$ for all $n \in \mathbb{Z}$. More specifically, symmetric solutions with an axis of symmetry at $n = 0$ are on-site solutions and at $n = 1/2$ are off-site solutions. This study of heteroclinic orbits in lattice dynamical systems is essential for understanding localized structures.

Theorem 2.1. *([6, Theorem 2.4]) Suppose Hypotheses 2.1, 2.2, and 2.3 are satisfied. Then, there exists $M \gg 1$ such that for all $N \geq M$ there exists a homoclinic orbit of (2.5) that spends N iterations in a neighbourhood of u^*.*

These homoclinic solutions in Theorem 2.1 are called single-pulse solutions to the LDS. The reason is that there is a single compact activated component to the pattern whose length is determined by N.

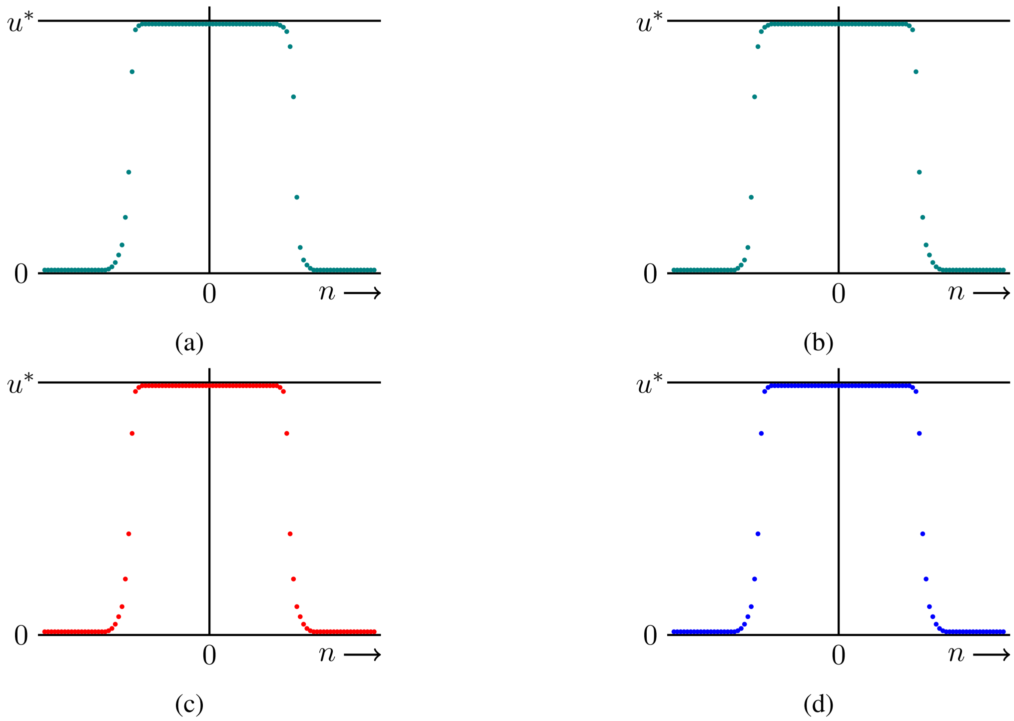

Figure 2.1: A visual depiction of the results of Theorem 2.1. (a) and (b) Asymmetric solutions that are mapped into each other by the reverser $\mathcal{R}$; (c) On-site and (d) off-site symmetric solutions that are mapped into themselves by the reverser $\mathcal{R}$. Roughly, on-site solutions have an odd number of points that are activated on the plateau, while off-site have an even number.

In the proof of Theorem 2.1, the local coordinate transformations are applied to express the system in a simplified standard form, thereby providing a clear analysis of the interactions between stable and unstable manifolds. First of all, a smooth coordinate transformation to local coordinates about u^* that is guaranteed by the stable manifold theorem, which maps u_n to a new set of variables $v_n = (v_n^{\mathrm{s}}, v_n^{\mathrm{u}})$, representing the stable and unstable components of the solution, respectively. Then, reversibility conditions implies that for every solution v_n^{s} in the forward time direction, there exists a solution v_n^{u} in the opposite direction. To further characterize solutions, there exists a unique solution v_n that remains close to u^* for all n, for trajectories that remain in a neighborhood of u^*. Therefore, the transformed system enables the identification of trajectories that transition between fixed points, laying the groundwork for constructing homoclinic orbits.

Then, symmetric homoclinic orbits are constructed with reverser $\mathcal{R}$ and other corresponding symmetric matching conditions, and they are further divided into on- or off-site solutions. Moreover, asymmetric homoclinic orbits can be constructed without reverser $\mathcal{R}$. They often bifurcate from symmetric ones via pitchfork bifurcations and come in pairs, mapped into each other by $\mathcal{R}$. With these matching conditions, both symmetric and asymmetric homoclinic orbits have been proven to exist.

Theorem 2.2. *([3, Theorem 2.4]) Suppose Hypotheses 2.1, 2.2, and 2.3 are satisfied. For each* $k \geq 2$*, there exists* $M_k \gg 1$ *so that for each set of integers* $N_1, \ldots, N_{2k-1} \geq M_k$ *there is a homoclinic orbit of* (2.5) *that spends* N_1 *iterates in a neighbourhood of* u^**, followed by* N_2 *iterates in a neighbourhood of* 0*, followed by* N_3 *iterates in a neighbourhood of* u^**, and so on* (see Figure 2.2).

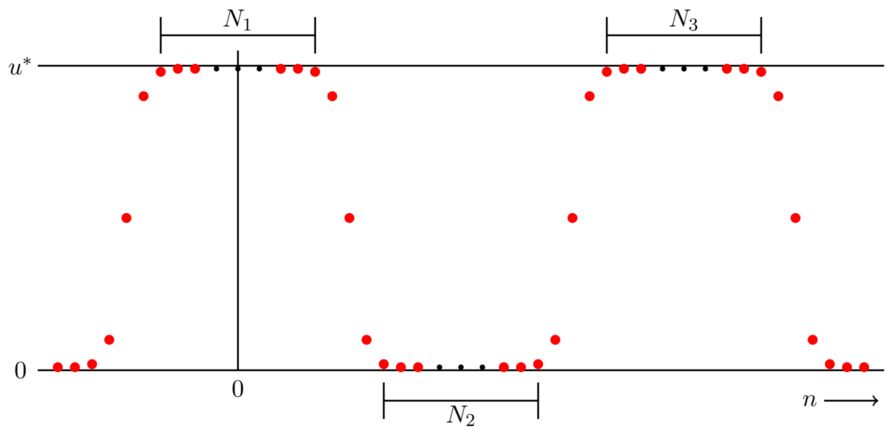

Figure 2.2: A visual depiction of the results of theorem 2.2. The k-pulse solution is an oscillating homoclinic orbit. In a compact region, it activates the plateau near u^* a total of k times and rapidly transitions to near zero $k - 1$ times, and it ends its oscillations and returns to zero outside the region.

These homoclinic solutions in Theorem 2.2 are called multi-pulse solutions to the LDS becauese they have $k \geq 2$ compact activated components that are separated by regions where the pattern is near zero (as given by the N_j with j even).

The proof of Theorem 2.2 as an extension of Theorem 2.1 which is limited to the single-pulse solution only, a smooth coordinate transformation of the local coordinates about 0 has also been analyzed. By the analogous steps as the local coordinates about u^*, it establishes that trajectories originating from u^* eventually return to a neighbourhood of 0 and vice versa. Similarly, symmetric 2-pulse solution can be constructed by gluing of orbit segments satisfying specific symmetry conditions, while asymmetric 2-pulse solutions fail to satisfy the reverser $\mathcal{R}$ to have more flexible combinations. In this way one can construct single- and multi-pulse solutions to the LDSs as homoclinic and multi-homoclinic orbits of (2.5), respectively.

Theorem 2.1 deals with the existence of single-pulse solutions, while theorem 2.2 will deal with the existence of multi-pulse solutions. In Chapter 3, we will analyze the stability of both single- and multi-pulse solutions, beginning with the former and then bootstrapping the proof to the latter.

2.3 Application to the Cubic-Quintic Model

The results of the previous subsection can be applied to specific LDSs to explicitly confirm the hypotheses. In particular, consider the real cubic–quintic discrete Ginzburg–Landau equation studied in [6]

$$\dot{U}_n = d(U_{n+1} + U_{n-1} - 2U_n) - \mu U_n + 2U_n^3 - U_n^5 \tag{2.6}$$

which is significant for understanding nonlinear dynamics in optical systems. It is also the normal form for a subcritical pitchfork bifurcation which is known to exhibit bistability.

For $\mu \in [0,1]$, there are five steady-state solutions 0 and $\pm\sqrt{1 \pm \sqrt{1-\mu}}$. Recasting the steady-state equation of (2.6) by setting $(u_n, v_n) = (U_{n-1}, U_n)$ to get

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = 2v_n - u_n - \dfrac{1}{d}(-\mu v_n + 2v_n^3 - v_n^5) \end{cases} \tag{2.7}$$

and then the eigenvalues at the fixed points (u^*, u^*) to the mapping (2.7) satisfy

$$\lambda^2 - \lambda\left(2 + \frac{1}{d}\left(\mu - 6(u^*)^2 + 5(u^*)^4\right)\right) + 1 = 0.$$

Setting $a = \left(\mu - 6(u^*)^2 + 5(u^*)^4\right)/d$ and using the quadratic formula, we solve

$$\lambda = \frac{2 + a \pm \sqrt{(2+a)^2 - 4}}{2}.$$

When $u^* = 0, \pm\sqrt{1 \pm \sqrt{1-\mu}}$,

$$a_0 = \frac{\mu}{d}, \qquad a_{\pm\sqrt{1+\sqrt{1-\mu}}} = \frac{4 - 4\mu + 4\sqrt{1-\mu}}{d}, \qquad a_{\pm\sqrt{1-\sqrt{1-\mu}}} = \frac{4 - 4\mu - 4\sqrt{1-\mu}}{d}.$$

For $\mu \in (0,1)$ and positive d, a_0 and $a_{\pm\sqrt{1+\sqrt{1-\mu}}}$ are greater than 0, so λ_0 and $\lambda_{\pm\sqrt{1+\sqrt{1-\mu}}}$ are non-zero, and their norms are not equal to 1, implying that fixed points 0 and $\pm\sqrt{1+\sqrt{1-\mu}}$ are hyperbolic. On the other side, $a_{\pm\sqrt{1-\sqrt{1-\mu}}}$ are less than 0, and the norms of $\lambda_{\pm\sqrt{1-\sqrt{1-\mu}}}$ are

$$\begin{aligned}\left|\lambda_{\pm\sqrt{1-\sqrt{1-\mu}}}\right| &= \left|\frac{2+a_{\pm\sqrt{1-\sqrt{1-\mu}}} \pm \sqrt{(2+a_{\pm\sqrt{1-\sqrt{1-\mu}}})^2-4}}{2}\right| \\ &= \frac{1}{4}\left|2+a_{\pm\sqrt{1-\sqrt{1-\mu}}} \pm i\sqrt{4-\left(2+a_{\pm\sqrt{1-\sqrt{1-\mu}}}\right)^2}\right| \\ &= \frac{1}{4}\left(\left(2+a_{\pm\sqrt{1-\sqrt{1-\mu}}}\right)^2+4-\left(2+a_{\pm\sqrt{1-\sqrt{1-\mu}}}\right)^2\right)\end{aligned}$$

which is exactly equal to 1, so fixed points $\pm\sqrt{1-\sqrt{1-\mu}}$ are non-hyperbolic. Therefore, this lays the foundation for our study of heteroclinic orbits from 0 to $u^* = \sqrt{1+\sqrt{1-\mu}}$, or other heteroclinic orbits, as well as localized solutions.

Moreover, to understand long-term behavior of the system (2.6), system (2.6) can be shown as a gradient flow on the Hilbert space ℓ^2 definded by

$$\ell^2 = \left\{ \{x_n\}_{x\in\mathbb{Z}} \ : \ \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}.$$

Indeed, (2.7) is the negative gradient of the potential

$$E(U) = \sum_{n\in\mathbb{Z}} \left(\frac{d}{2}(U_{n+1}-U_n)^2 + \frac{d}{2}(U_{n-1}-U_n)^2 + \frac{1}{2}\mu U_n^2 - \frac{1}{2}U_n^4 + \frac{1}{6}U_n^5 \right), \tag{2.8}$$

$E(U) : \ell^2 \to \mathbb{R}$, evolving in the direction of steepest descent. Now, we can replace $\dot{U}_n$ by a negative derivative of the potential with respect to U_n, that is $\dot{U}_n = -\frac{\partial E(U)}{\partial U_n}$. Then, we take a quadratic form of the energy associated with a difference between neighboring points to obtain (2.8). The potential decreases over time, eventually the system will converge to one of these steady-state solutions. In other words, no matter where the system starts, it always tends towards an equilibrium as $t \to \infty$. Since (2.6) is a gradient flow, this motivates only looking at steady-state solutions since these are the only possible limit states for the differential equation.

The verification of Hypothesis 2.3 for heteroclinic orbits of (2.7) connecting $(0,0)$ and (u^*, u^*) was proven in [6], thus allowing for the application of Theorems 2.1 and 2.2 to provide the existence of localized single- and multi-pulse solutions. Single-pulse solutions to (2.7) are shown in Figure 2.1. Furthermore, these results were extended to prove the existence of a heteroclinic connection between $(0,0)$ and cyclic/periodic solutions of (2.7) to apply Theorem 2.1 to demonstrate the existence of localized solutions with oscillator plateaus. Different from flat plateaus constructed by heteroclinic connections between $\mathbf{0}$ and $\mathbf{u}^*$, the oscillation plateaus where the solution profile exhibits periodic oscillations around non-zero steady states over a finite spatial region before decaying toward to 0 have been studied. Similarly, these oscillation plateaus can also be glued together by the front and back solutions, and their stabilities can be analyzed through front and back solutions. In particular, F in the map (2.5) can be replaced with a second iterate map F^2. This means that 2-cycle becomes a pair of the fixed points $\{\mathbf{u}_1^*, \mathbf{u}_2^*\}$ such that $\mathbf{u}_1^* = F(\mathbf{u}_2^*)$ and $\mathbf{u}_2^* = F(\mathbf{u}_1^*)$. Therefore, the heteroclinic connection between $\mathbf{0}$ and 2-cycle can be transformed into a standard heteroclinic connection defined in Section 2.2, and all the previous theory can apply to this second iterate map. These steady-state solutions enriches the diversity of spatial localized solutions in LDSs.

Chapter 3

Stability Analysis

The previous section reviewed results on the existence of localized patterns to LDSs. The natural follow-up investigation is to understand their local stability properties. In this chapter, we work to understand the stability of localized solutions to LDSs using knowledge of the stability of front and back solutions. In particular, in Section 3.1 we analyze the stability of single-pulse solutions and then in Section 3.2 we extend this analysis to multi-pulse solutions.

3.1 Single-Pulse Solutions

In this section, we begin by the main result concerning the stability of localized single-pulse solutions, and then it is followed by a detailed proof, drawing upon the framework of Evans function via exponential dichotomies.

3.1.1 Main Result for Single-Pulse Solutions

We begin by assuming $U_n^*(\mu)$ is a localized steady-state single-pulse solution of (2.3), which we recall is guaranteed to exist by Theorem 2.1 in Section 2.2. Then, we set $U_n = U_n^*(\mu) + V_n e^{\lambda t}$ where V_n is time-independent. If λ has a positive real part, then the solution is unstable. Plugging into (2.3), we get

$$\begin{aligned}\dot{U}_n^*(\mu) + \lambda V_n e^{\lambda t} &= d\big((U_{n+1}^*(\mu) + V_{n+1}e^{\lambda t}) + (U_{n-1}^*(\mu) + V_{n-1}e^{\lambda t}) - 2(U_n^*(\mu) + V_n e^{\lambda t})\big)\\ &\quad + f(U_n^*(\mu) + V_n e^{\lambda t}, \mu)\end{aligned}$$

and linearizing about $V_n = 0$ for all n gives

$$\begin{aligned}\dot{U}_n^*(\mu) + \lambda V_n e^{\lambda t} = d\big((U_{n+1}^*(\mu) + V_{n+1}e^{\lambda t}) + (U_{n-1}^*(\mu) + V_{n-1}e^{\lambda t}) - 2(U_n^*(\mu) + V_n e^{\lambda t})\big) \\ + f(U_n^*(\mu), \mu) + f'(U_n^*(\mu), \mu)V_n e^{\lambda t}.\end{aligned} \tag{3.1}$$

Since $U_n^*(\mu)$ is a localized steady-state solution of (2.3), it follows that

$$\dot{U}_n^*(\mu) = d(U_{n+1}^*(\mu) + U_{n-1}^*(\mu) - 2U_n^*(\mu)) + f(U_n^*(\mu), \mu)$$

and so (3.1) can be simplified to

$$\lambda V_n = d(V_{n+1} + V_{n-1} - 2V_n) + f'(U_n^*(\mu), \mu)V_n.$$

The above linear LDS can be re-cast as a first-order discrete dynamical system by setting $(\bar{u}_n, \bar{v}_n) = (V_{n-1}, V_n)$ to get

$$\begin{cases} \bar{u}_{n+1} = \bar{v}_n \\ \bar{v}_{n+1} = 2\bar{v}_n - \bar{u}_n + \dfrac{f'(U_n^*(\mu), \mu)}{d}\bar{v}_n + \dfrac{\lambda}{d}\bar{v}_n. \end{cases} \tag{3.2}$$

Furthermore, we can rewrite (3.2) as

$$\overline{V}_{n+1} = A_n(\lambda)\overline{V}_n, \quad n \in \mathbb{Z}, \quad A_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 2 - (f'(U_n^*(\mu), \mu) - \lambda)/d \end{pmatrix}, \quad \overline{V}_n = \begin{pmatrix} \bar{u}_n \\ \bar{v}_n \end{pmatrix}. \tag{3.3}$$

We now recall the definition of an exponential dichotomy to better understand (3.3).

Definition 3.1. ([2, Definition 2.1]) A linear discrete dynamical system $V_{n+1} = A_n V_n$ has an exponential dichotomy on the integer interval

$$I = \{n \in \mathbb{Z} : n_- \leq n \leq n_+\}, \qquad n_\pm \in \mathbb{Z} \cup \{\pm\infty\}$$

if there exists some constants $C, \alpha > 0$ such that

$$P_n \Phi(n, m) = \Phi(n, m) P_m, \qquad n, m \in I$$

and

$$
\begin{aligned}
|\Phi(n,m)P_m| &\le Ce^{-\alpha(n-m)}, \qquad n \ge m \text{ in } I \\
|\Phi(n,m)(I-P_m)| &\le Ce^{-\alpha(m-n)}, \qquad n \le m \text{ in } I
\end{aligned}
$$

where $\Phi(n,m)$ is a solution operator of $V_{n+1} = A_n V_n$, defined as

$$
\Phi(n,m) = \begin{cases} A_{n-1}A_{n-2}\dots A_{m+1}A_m & n > m \\ I & n = m \\ A_n^{-1}A_{n+1}^{-1}\dots A_{m-2}^{-1}A_{m-1}^{-1} & n < m \end{cases} \tag{3.4}
$$

and P_m ($P_m^2 = P_m$) is a projector in $\mathbb{R}^2$.

Define $\Phi^{\mathrm{s}}(n,m;\lambda) := \Phi(n,m)P_m$, $\Phi^{\mathrm{u}}(n,m;\lambda) := \Phi(n,m)(I-P_m)$, and the associated projections $P^{\mathrm{s,u}}(n;\lambda) := \Phi^{\mathrm{s,u}}(n,n;\lambda)$. The range of $P^{\mathrm{s,u}}(n;\lambda)$ is denoted as $\mathrm{Ran}\, P^{\mathrm{s,u}}(n;\lambda)$, representing stable and unstable eigenspaces such that $\mathbb{R}^2 = \mathrm{Ran}\, P^{\mathrm{s}}(n;\lambda) \oplus \mathrm{Ran}\, P^{\mathrm{u}}(n;\lambda)$ at every n. Therefore, V_n can be uniquely decomposed as $V_n = V_n^{\mathrm{s}} + V_n^{\mathrm{u}}$ where $V_n^{\mathrm{s}} = P_n V_n \in \mathrm{Ran}\, P^{\mathrm{s}}(n;\lambda)$ and $V_n^{\mathrm{u}} = (I - P_n)V_n \in \mathrm{Ran}\, P^{\mathrm{u}}(n;\lambda)$, and then we obtain

$$
V_n = \Phi(n,m)V_m = \Phi(n,m)P_mV_m + \Phi(n,m)(I-P_m)V_m = \Phi(n,m)V_m^{\mathrm{s}} + \Phi(n,m)V_m^{\mathrm{u}} = V_n^{\mathrm{s}} + V_n^{\mathrm{u}}.
$$

Directly following results from the definition of exponential dichotomy,

$$
\begin{aligned}
\|V_n^{\mathrm{s}}\| = \|\Phi(n,m)P_mV_m^{\mathrm{s}}\| &\le Ce^{-\alpha(n-m)}\|V_m^{\mathrm{s}}\|, \qquad n \ge m \text{ in } I \\
\|V_n^{\mathrm{u}}\| = \|\Phi(n,m)(I-P_m)V_m^{\mathrm{u}}\| &\le Ce^{-\alpha(m-n)}\|V_m^{\mathrm{u}}\|, \qquad n \le m \text{ in } I,
\end{aligned}
$$

evolving that the stable (unstable) component of V_n exponentially decays as $n \to \infty$ ($n \to -\infty$), while the unstable (stable) component of V_n exponentially grows as $n \to \infty$ ($n \to -\infty$).

Following the definition of exponential dichotomy, we now introduce a hypothesis that will be used in the stability analysis.

Hypothesis 3.1. For some fixed μ, we have that $f'(0,\mu)$ and $f'(u^*(\mu),\mu)$ are negative.

Based on this hypothesis, we can establish the following lemma to identify the key condition for the system (3.3) to have exponential dichotomy.

Lemma 3.1. If $\lambda \notin [f'(0,\mu) - 4d, f'(0,\mu)]$, then the system

$$\begin{cases} \bar{u}_{n+1} = \bar{v}_n \\ \bar{v}_{n+1} = 2\bar{v}_n - \bar{u}_n - \dfrac{f'(0,\mu)}{d}\bar{v}_n + \dfrac{\lambda}{d}\bar{v}_n \end{cases} \tag{3.5}$$

has an exponential dichotomy on $\mathbb{Z}$. We denote these dichotomies by $\Phi_0^{\mathrm{s,u}}(n,m;\lambda)$ and denote the associated projections by $P_0^{\mathrm{s,u}}(n;\lambda)$.

Similarly, if $\lambda \notin [f'(u^*,\mu) - 4d, f'(u^*,\mu)]$, then the system

$$\begin{cases} \bar{u}_{n+1} = \bar{v}_n \\ \bar{v}_{n+1} = 2\bar{v}_n - \bar{u}_n - \dfrac{f'(u^*(\mu),\mu)}{d}\bar{v}_n + \dfrac{\lambda}{d}\bar{v}_n \end{cases} \tag{3.6}$$

has an exponential dichotomy on $\mathbb{Z}$. We denote these dichotomies by $\Phi_{u^*}^{\mathrm{s,u}}(n,m;\lambda)$ and denote the associated projections by $P_{u^*}^{\mathrm{s,u}}(n;\lambda)$.

Before proceeding with the proof of Lemma 3.1, we comment that the values of λ in these intervals constitute the essential spectrum. Precisely, the essential spectrum is part of the spectrum of the linearization about a bounded solution to an LDS for which there is no exponential dichotomy. Coupling Hypothesis 3.1 with the assumption that $d > 0$ gives that the essential spectrum is contained in the left half of the complex plane, meaning that the essential spectrum cannot lead to instability of the front, back, or localized solutions. Thus, we focus here exclusively on the point spectrum, which is defined below as roots of the Evans function away from the essential spectrum.

Proof. The system (3.5) can be rewritten in the form $\overline{V}_{n+1} = A^0(\lambda)\overline{V}_n$ where

$$A^0(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 2 - (f'(0,\mu) - \lambda)/d \end{pmatrix} \tag{3.7}$$

is a constant matrix, so the existence of exponential dichotomies is equivalent to $A^0(\lambda)$ not having eigenvalues on unit circle in the complex plane. The characteristic polynomial of $A^0(\lambda)$ is

$$\tilde{\lambda}^2 - \left(2 - \frac{f'(0,\mu) - \lambda}{2}\right)\tilde{\lambda} + 1 = 0$$

and we use the quadratic formula to obtain eigenvalues $\tilde{\lambda}$ of $A^0(\lambda)$

$$\tilde{\lambda} = \frac{2 - \left(f'(0,\mu) - \lambda\right)/d \pm \sqrt{\left(2 - \left(f'(0,\mu) - \lambda\right)/d\right)^2 - 4}}{2}$$

To ensure $|\tilde{\lambda}| \neq 1$, $\left(2 - \left(f'(0,\mu) - \lambda\right)/d\right)^2 - 4$ cannot be negative, leading

$$2 - \big(f'(0,\mu) - \lambda\big)/d > 2 \qquad \text{or} \qquad 2 - \big(f'(0,\mu) - \lambda\big)/d < -2$$

Therefore, $\lambda > f'(0,\mu)$ or $\lambda < f'(0,\mu) - 4d$ confirms that $A^0(\lambda)$ is hyperbolic, so the system (3.5) has an exponential dichotomy on $\mathbb{Z}$. The proof for the system (3.6) is the same. □

Hypothesis 3.2. There exists front and back solutions to (2.3), denoted $\{U_n^f(\mu)\}_{n\in\mathbb{Z}}$ and $\{U_n^b(\mu)\}_{n\in\mathbb{Z}}$, and constants $C > 0$, $\eta > 1$ such that

$$\begin{aligned}
\left|U_n^f(\mu) - u^*(\mu)\right| &\leq C\eta^{-n}, \qquad && n \geq 0\\
\left|U_n^b(\mu) - 0\right| &\leq C\eta^{-n}, \qquad && n \geq 0\\
\left|U_n^f(\mu) - 0\right| &\leq C\eta^{n}, \qquad && n \leq 0\\
\left|U_n^b(\mu) - u^*(\mu)\right| &\leq C\eta^{n}, \qquad && n \leq 0.
\end{aligned}$$

The back solution can be understood as the "inverse" of the front solution, which represents a transition in the opposite direction. If we assume reversibility, as was done for the existence results in Hypothesis 2.1, the front and back solutions are symmetrically-related under the reverser, and so the existence of one implies the existence of the other. Sometimes, the back solution may not be a perfect reversal of the front solution, and they are not always symmetric. However, only symmetric front and back are considered in Hypothesis 3.2, the reason will be explained in Hypothesis 3.3.

Lemma 3.2. The system

$$\begin{cases} \bar{u}_{n+1} = \bar{v}_n \\ \bar{v}_{n+1} = 2\bar{v}_n - \bar{u}_n - \dfrac{f'(U^j(\mu),\mu)}{d}\bar{v}_n + \dfrac{\lambda}{d}\bar{v}_n \end{cases} \tag{3.8}$$

with $j = f, b$ has two exponential dichotomies on $\mathbb{Z}^+$ and $\mathbb{Z}^-$. We denote these dichotomies by $\Phi_{U^j}^{+,\mathrm{s/u}}(n,m;\lambda)$ and $\Phi_{U^j}^{-,\mathrm{s/u}}(n,m;\lambda)$, and denote the associated projections by $P_{U^j}^{+,\mathrm{s/u}}(n;\lambda)$ and $P_{U^j}^{-,\mathrm{s/u}}(n;\lambda)$.

Proof. Consider two difference equations $\overline{V}_{n+1} = A^{u^*}(\lambda)\overline{V}_n$ which is the system (3.6) and $\overline{V}_{n+1} = A^+_{f,n}(\lambda)\overline{V}_n$ which is the system (3.8) with front solutions on $\mathbb{Z}^+$, and Hypothesis 3.2 implies that

$$A^{u^*}(\lambda) - A^+_{f,n}(\lambda) \to 0 \quad \text{as} \quad n \to \infty.$$

Directly following from [2, Proposition 2.5], the result shows that the system $\overline{V}_{n+1} = A^+_{f,n}(\lambda)\overline{V}_n$ has an exponential dichotomy on $\mathbb{Z}^+$. Similarly, we can use two difference equations $\overline{V}_{n+1} = A^0(\lambda)\overline{V}_n$ which is the system (3.5) and $\overline{V}_{n+1} = A^-_{f,n}(\lambda)\overline{V}_n$ which is the system (3.8) with front solutions on $\mathbb{Z}^-$ to show that the system $\overline{V}_{n+1} = A^-_{f,n}(\lambda)\overline{V}_n$ has an exponential dichotomy on $\mathbb{Z}^-$. □

Hypothesis 3.3. The localized solutions $\{U_n^*(\mu)\}_{n\in\mathbb{Z}}$ are parametried by constants $N \geq N_*$ for some $N_* \geq 1$, so that there exists constants $C > 0$, $\eta > 1$ such that

$$U_n^*(\mu) = \begin{cases} U^f_{n+N}(\mu) + w^{f,-}_{n+N}(N), & n \leq -N \\ U^f_{n+N}(\mu) + w^{f,+}_{n+N}(N), & -N \leq n \leq 0 \\ U^b_{n+N}(\mu) + w^{b,+}_{n-N}(N), & 0 \leq n \leq N \\ U^b_{n+N}(\mu) + w^{b,-}_{n-N}(N), & N \leq 0 \end{cases} \tag{3.9}$$

and

$$|w_n^{f,-}(N)| \leq C\eta^{-N+n}, \qquad n \leq 0 \tag{3.10}$$

$$|w_n^{f,+}(N)| \leq C\eta^{-N}, \qquad 0 \leq n \leq N \tag{3.11}$$

$$|w_n^{b,-}(N)| \leq C\eta^{-N}, \qquad N \leq n \leq 0 \tag{3.12}$$

$$|w_n^{b,+}(N)| \leq C\eta^{-N-n}, \qquad 0 \leq n \tag{3.13}$$

Since $U_n^*(\mu)$ is unique when $n = -N, 0, N$, we have

$$\begin{aligned} w_0^{f,-}(N) - w_0^{f,+}(N) = 0, \\ \left(U_N^f(\mu) + w_N^{f,+}(N)\right) - \left(U_N^b(\mu) + w_N^{b,+}(N)\right) = 0, \\ w_0^{b,+}(N) - w_0^{b,-}(N) = 0. \end{aligned}$$

Hypothesis 3.3 says that the single-pulse solution of interest here is constructed by "gluing" a front and a back together, with small corrections w, and inequalities (3.10)−(3.13) imply that the localized solution hardly deviates from the fronts or the backs for N large. Different from symmetric single-pulse solution, asymmetric single-pulse solution can be seen as gluing of front and back solutions which do not reflect each other, and small perturbations could potentially destabilize the pulse. They may remain stable if they are constructed by gluing both stable front and back solutions. On the other side, the study of the stability of symmetric pulses is more common.

In the following theorem we show that roots of the Evans functions for the front and back solutions, denoted $D_f(\lambda)$ and $D_b(\lambda)$ below, lead to nearby roots of the Evans function for the localized solution, defined as $D_N(\lambda)$ below.

Theorem 3.1. *Suppose Hypotheses 3.1, 3.2, and 3.3 are satisfied, and define*

$$\Omega = \mathbb{C} \setminus \{[f'(0,\mu) - 4d, f'(0,\mu)] \cup [f'(u^*,\mu) - 4d, f'(u^*,\mu)]\}$$

and Evans functions of front and back solutions

$$\begin{aligned} D_f(\lambda) &:= \det \left(\operatorname{Ran} P_{U^f}^{-,\mathrm{u}}(0;\lambda) \quad \operatorname{Ran} P_{U^f}^{+,\mathrm{s}}(0;\lambda) \right), \\ D_b(\lambda) &:= \det \left(\operatorname{Ran} P_{U^b}^{-,\mathrm{u}}(0;\lambda) \quad \operatorname{Ran} P_{U^b}^{+,\mathrm{s}}(0;\lambda) \right), \end{aligned}$$

where $P_{U^f}^{\pm,\mathrm{s/u}}(0;\lambda)$ and $P_{U^b}^{\pm,\mathrm{s/u}}(0;\lambda)$ are projections to the systems $\overline{V}_{n+1} = A_{f,n}^{\pm}(\lambda)\overline{V}_n$ and $\overline{V}_{n+1} = A_{b,n}^{\pm}(\lambda)\overline{V}_n$, respectively. Fix $\lambda_* \in \Omega$ and suppose that for $m_f, m_b \geq 0$ and for some $\delta > 0$, we have

$$D_f(\lambda) = (\lambda - \lambda_*)^{m_f} + \mathrm{O}\left(|\lambda - \lambda_*|^{m_f+1}\right), \tag{3.14}$$

$$D_b(\lambda) = (\lambda - \lambda_*)^{m_b} + \mathrm{O}\left(|\lambda - \lambda_*|^{m_b+1}\right) \tag{3.15}$$

for $\lambda \in B_\delta(\lambda_*)$. Then, we can define an analytic function $D_N(\lambda)$ such that

$$D_N(\lambda) = \left(D_f(\lambda) + \mathrm{O}(e^{-\alpha N})\right)\left(D_b(\lambda) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N}), \tag{3.16}$$

and there exists a δ_* sufficiently small, with $\delta > \delta_* > 0$, and an $N_* > 0$ sufficiently large, such that the following hold uniformly in $N \geq N_*$:

(i) *$D_N(\lambda)$ has precisely $m_f + m_b$ roots, counted with multiplicity, in $B_{\delta_*}(\lambda_*)$. These values of λ are $\mathrm{O}(e^{-\alpha N})$ close to λ_*, with $\alpha > 0$.*

(ii) *The system (3.2) has a bounded, nontrivial solution at $\lambda \in B_{\delta_*}(\lambda_*)$ if and only if $D_N(\lambda) = 0$.*

The proof is in the following subsection since it is long and requires multiple steps.

3.1.2 Proof of Theorem 3.1

Using (3.9), we can solve the eigenvalue problem (3.2) in pieces by solving the equations

$$\overline{V}^{f,-}_{n+1} = A^{f,-}_n(\lambda)\overline{V}^{f,-}_n, \qquad n \leq 0 \tag{3.17}$$

$$\overline{V}^{f,+}_{n+1} = A^{f,+}_n(\lambda)\overline{V}^{f,+}_n, \qquad 0 \leq n \leq N \tag{3.18}$$

$$\overline{V}^{b,-}_{n+1} = A^{b,-}_n(\lambda)\overline{V}^{b,-}_n, \qquad -N \leq n \leq 0 \tag{3.19}$$

$$\overline{V}^{b,+}_{n+1} = A^{b,+}_n(\lambda)\overline{V}^{b,+}_n, \qquad 0 \leq n \tag{3.20}$$

and then constructing the solution as

$$\overline{V}_n = \begin{cases} \overline{V}^{f,-}_{n+N}, & n \leq -N \\ \overline{V}^{f,+}_{n+N}, & -N \leq n \leq 0 \\ \overline{V}^{b,-}_{n-N}, & 0 \leq n \leq N \\ \overline{V}^{b,+}_{n-N}, & N \leq n \end{cases}$$

Thus, we have a bounded solution to (3.2) if and only if we have solutions to (3.17)–(3.20) along with the matching conditions

$$\overline{V}^{f,+}_N - \overline{V}^{b,-}_{-N} = 0, \tag{3.21}$$

$$\overline{V}_0^{f,+} - \overline{V}_0^{f,-} = 0, \tag{3.22}$$

$$\overline{V}_0^{b,+} - \overline{V}_0^{b,-} = 0. \tag{3.23}$$

From Lemma 3.8, we have exponential dichotomies for each equation (3.17)−(3.20), which can be chosen so that they depend analytically on $\lambda \in B_\delta(\lambda_*)$ for $\delta > 0$ small. That is, we have

$$\left.\begin{aligned} \left|\Phi_{U^*}^{f,-,\mathrm{s}}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi_{U^*}^{f,-,\mathrm{u}}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad n, m \le 0, \tag{3.24}$$

$$\left.\begin{aligned} \left|\Phi_{U^*}^{f,+,\mathrm{s}}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi_{U^*}^{f,+,\mathrm{u}}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad 0 \le n, m \le N, \tag{3.25}$$

$$\left.\begin{aligned} \left|\Phi_{U^*}^{b,-,\mathrm{s}}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi_{U^*}^{b,-,\mathrm{u}}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad -N \le n, m \le 0, \tag{3.26}$$

$$\left.\begin{aligned} \left|\Phi_{U^*}^{b,+,\mathrm{s}}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi_{U^*}^{b,+,\mathrm{u}}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad 0 \le n, m. \tag{3.27}$$

We denote the associated projections $P_{U^*}^{i,\pm,j}(n;\lambda) := \Phi_{U^*}^{i,\pm,j}(n,n;\lambda)$ where $i = f, b$ for front/back solutions and $j = \mathrm{s}, \mathrm{u}$ for stable/unstable components. Moreover, as a consequence of Hypothesis 3.1, we have that

$$\left|P_{U^*}^{f,+,\mathrm{u}}(N;\lambda) - P_{u^*}^{\mathrm{u}}(0;\lambda)\right| \le Ce^{-\alpha N}, \tag{3.28}$$

$$\left|P_{U^*}^{b,-,\mathrm{s}}(-N;\lambda) - P_{u^*}^{\mathrm{s}}(0;\lambda)\right| \le Ce^{-\alpha N}. \tag{3.29}$$

We now let $a := (a^{f,+}, a^{b,-}) \in \widetilde{V}_a$, $b := (b^{f,-}, b^{f,+}, b^{b,-}, b^{b,+}) \in \widetilde{V}_b$ and $\lambda \in \widetilde{V}_\lambda$, where the spaces $\widetilde{V}_a$, $\widetilde{V}_b$, $\widetilde{V}_\lambda$ are defined as follows:

$$\begin{aligned} \widetilde{V}_a &:= \mathrm{Ran}\, P_{u^*}^{\mathrm{u}}(0;\lambda_*) \oplus \mathrm{Ran}\, P_{u^*}^{\mathrm{s}}(0;\lambda_*) \\ \widetilde{V}_b &:= \left(\mathrm{Ran}\, P_{U^*}^{f,-,\mathrm{u}}(0;\lambda_*) \oplus \mathrm{Ran}\, P_{U^*}^{f,+,\mathrm{s}}(0;\lambda_*)\right) \oplus \left(\mathrm{Ran}\, P_{U^*}^{b,-,\mathrm{u}}(0;\lambda_*) \oplus \mathrm{Ran}\, P_{U^*}^{b,+,\mathrm{s}}(0;\lambda_*)\right) \\ \widetilde{V}_\lambda &:= B_\delta(\lambda_*) \subset \Omega \end{aligned}$$

where $\widetilde{V}_a$ and $\widetilde{V}_b$ are endowed with the maximum norm over components such that $\|\widetilde{V}_a\|_\infty = \max\{\|a^{f,+}\|, \|a^{b,-}\|\}$ and $\|\widetilde{V}_b\|_\infty = \max\{\|b^{f,-}\|, \|b^{f,+}\|, \|b^{b,-}\|, \|b^{b,+}\|)$.

For δ sufficiently small and $N > N_*$ sufficiently large, we can then write solutions to the eigenvalue problem (3.2) for the localized structure U_n^* in fixed-point form as

$$\overline{V}_n = \begin{cases} \overline{V}_n^{f,-} = \Phi_{U^*}^{f,-,\mathrm{u}}(n,0;\lambda)b^{f,-} & n \leq 0 \\ \overline{V}_n^{f,+} = \Phi_{U^*}^{f,+,\mathrm{s}}(n,0;\lambda)b^{b,+} + \Phi_{U^*}^{f,+,\mathrm{u}}(n,N;\lambda)a^{f,+} & 0 \leq n \leq N \\ \overline{V}_n^{b,-} = \Phi_{U^*}^{b,-,\mathrm{s}}(n,-N;\lambda)a^{b,-} + \Phi_{U^*}^{b,-,\mathrm{s}}(n,0;\lambda)b^{b,-} & -N \leq n \leq 0 \\ \overline{V}_n^{b,+} = \Phi_{U^*}^{b,+,\mathrm{s}}(n,0;\lambda)b^{b,+} & 0 \leq n. \end{cases} \tag{3.30}$$

Lemma 3.3. *There exists an N_* such that for all $N > N_*$, the following holds uniformly in N. There exists an operator $F : \widetilde{V}_\lambda \times \widetilde{V}_b \to \widetilde{V}_a$ such that $\overline{V}_n$ as given by (3.30) with $a = F(\lambda)b$ solves (3.16) for any b and λ. F is analytic in λ and linear in b, and satisfies*

$$|F(\lambda)b| \leq Ce^{-\alpha N}|b|. \tag{3.31}$$

Proof. To begin, substituting from (3.30) into (3.21) gives us:

$$\begin{aligned} 0 &= \Phi_{U^*}^{f,+,\mathrm{s}}(N,0;\lambda)b^{b,+} + \Phi_{U^*}^{f,+,\mathrm{u}}(N,N;\lambda)a^{f,+} - \Phi_{U^*}^{b,-,\mathrm{s}}(-N,-N;\lambda)a^{b,-} - \Phi_{U^*}^{b,-,\mathrm{s}}(-N,0;\lambda)b^{b,-} \\ &= \left(P_{U^*}^{f,+,\mathrm{u}}(N;\lambda) - P_{u^*}^{\mathrm{u}}(0;\lambda)\right)a^{f,+} + a^{f,+} + \left(P_{u^*}^{\mathrm{s}}(0;\lambda) - P_{U^*}^{b,-,\mathrm{s}}(N;\lambda)\right)a^{b,-} - a^{b,-} \\ &\quad + \Phi_{U^*}^{f,+,\mathrm{s}}(N,0;\lambda)b^{b,+} - \Phi_{U^*}^{b,-,\mathrm{s}}(-N,0;\lambda)b^{b,-}. \end{aligned} \tag{3.32}$$

Define

$$\begin{aligned} G(\lambda)(a,b) = &\left(P_{U^*}^{f,+,\mathrm{u}}(N;\lambda) - P_{u^*}^{\mathrm{u}}(0;\lambda)\right)a^{f,+} + \left(P_{u^*}^{\mathrm{s}}(0;\lambda) - P_{U^*}^{b,-,\mathrm{s}}(N;\lambda)\right)a^{b,-} \\ &+ \Phi_{U^*}^{f,+,\mathrm{s}}(N,0;\lambda)b^{b,+} - \Phi_{U^*}^{b,-,\mathrm{s}}(-N,0;\lambda)b^{b,-}. \end{aligned}$$

$G(\lambda)(a,b)$ is analytic in λ because all projections and solutions operators involved are analytic in lambda. Moreover, we can see that G is linear in a and b. From (3.28) and (3.29) and definition of the dichotomies,

$$|G(\lambda)(a,b)| \leq Ce^{-\alpha N}(|a| + |b|). \tag{3.33}$$

Define the map K as

$$K : \widetilde{V}_a \to \mathbb{C}^2$$
$$(a^{f,+}, a^{b,-}) \mapsto a^{f,+} - a^{b,-}$$

Since $\widetilde{V}_a = \operatorname{Ran} P^{\mathrm{u}}_{u^*}(0;\lambda_*) \oplus \operatorname{Ran} P^{\mathrm{s}}_{u^*}(0;\lambda_*) = \mathbb{C}^2$, K is a linear isomorphism. Equation (3.32) can be written as

$$\begin{aligned}
&\Phi^{b,-,\mathrm{s}}_{U^*}(-N,0;\lambda)b^{b,-} + \Phi^{f,+,\mathrm{s}}_{U^*}(N,0;\lambda)b^{b,+} \\
&\qquad\qquad = (a^{f,+} - a^{b,-}) + \left(P^{f,+,\mathrm{u}}_{U^*}(N;\lambda) - P^{\mathrm{u}}_{u^*}(0;\lambda)\right) a^{f,+} + \left(P^{\mathrm{s}}_{u^*}(0;\lambda) - P^{b,-,\mathrm{s}}_{U^*}(N;\lambda)\right) a^{b,-} \\
&G(\lambda)(0,b) = K(a) + G(\lambda)(a,0) = (K + G(\lambda)J)(a)
\end{aligned}$$

where $J(a) := (a,0)$. By (3.31), for $N \geq N_*$ with N_* sufficiently large, $(K + G(\lambda)J)$ is invertible so that we have the solution operator

$$a = -(K + G(\lambda)J)^{-1}G(\lambda)(0,b) =: F(\lambda)b.$$

Finally, from (3.33) we have (3.31) for the above defined $F(\lambda)b$. □

Following from Lemma 3.3 and substituting from (3.30) into (3.22) and (3.23) gives us

$$\begin{aligned}
0 &= \Phi^{f,+,\mathrm{s}}_{U^*}(0,0;\lambda)b^{b,+} + \Phi^{f,+,\mathrm{u}}_{U^*}(0,N;\lambda)a^{f,+} - \Phi^{f,-,\mathrm{u}}_{U^*}(0,0;\lambda)b^{f,-} \\
&= P^{f,+,\mathrm{s}}_{U^*}(0;\lambda)b^{b,+} - P^{f,-,\mathrm{u}}_{U^*}(0;\lambda)b^{f,-} + \Phi^{f,+,\mathrm{u}}_{U^*}(0,N;\lambda)(F(\lambda)b)^{f,+}
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
0 &= \Phi^{b,+,\mathrm{s}}_{U^*}(0,0;\lambda)b^{b,+} - \Phi^{b,-,\mathrm{s}}_{U^*}(0,-N;\lambda)a^{b,-} - \Phi^{b,-,\mathrm{s}}_{U^*}(0,0;\lambda)b^{b,-} \\
&= P^{b,+,\mathrm{s}}_{U^*}(0;\lambda)b^{b,+} - P^{b,-,\mathrm{s}}_{U^*}(0;\lambda)b^{b,-} - \Phi^{b,-,\mathrm{s}}_{U^*}(0,-N;\lambda)(F(\lambda)b)^{b,-}
\end{aligned} \tag{3.35}$$

$(F(\lambda)b)^{f,+}$ and $(F(\lambda)b)^{b,-}$ are the components of $F(\lambda)b$ in $\operatorname{Ran} P^{\mathrm{u}}_{u^*}(0;\lambda_*)$ and $\operatorname{Ran} P^{\mathrm{s}}_{u^*}(0;\lambda_*)$, respectively. According to the exponential dichotomies (3.25)−(3.26) and Lemma 3.3, we obtain

$$\Phi^{f,+,\mathrm{u}}_{U^*}(0,N;\lambda)(F(\lambda)b)^{f,+} = \mathrm{O}(e^{-2\alpha N}|b|), \tag{3.36}$$

$$\Phi^{b,-,\mathrm{s}}_{U^*}(0,-N;\lambda)(F(\lambda)b)^{b,-} = \mathrm{O}(e^{-2\alpha N}|b|) \tag{3.37}$$

uniformly in λ near λ_*. Rewrite (3.34) and (3.35) and combine them in matrix form,

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix} &= \left[\begin{pmatrix}-P^{f,-,\mathrm{u}}_{U^*}(0;\lambda) & P^{f,+,\mathrm{s}}_{U^*}(0;\lambda) & 0 & 0\\ 0 & 0 & -P^{b,-,\mathrm{u}}_{U^*}(0;\lambda) & P^{b,+,\mathrm{s}}_{U^*}(0;\lambda)\end{pmatrix}\right.\\ &\qquad\left. + \begin{pmatrix}\Phi^{f,+,\mathrm{u}}_{U^*}(0,N;\lambda) & 0\\ 0 & -\Phi^{b,-,\mathrm{s}}_{U^*}(0,-N;\lambda)\end{pmatrix} F(\lambda)\right] b\\ &=: [P_N(\lambda)+R_N(\lambda)]b.\end{aligned}$$

From (3.36) and (3.37), we can observe that $R_N(\lambda) = \mathrm{O}(e^{-2\alpha N})$ uniformly in λ near λ_*. Define $\hat{P}^{f,-,\mathrm{u}}_{U^*}(0;\lambda)$ as $P^{f,-,\mathrm{u}}_{U^*}(0;\lambda)$ restricted to $\operatorname{Ran} P^{f,-,\mathrm{u}}_{U^*}(0;\lambda_*)$

$$\hat{P}^{f,-,\mathrm{u}}_{U^*}(0;\lambda) := P^{f,-,\mathrm{u}}_{U^*}(0;\lambda)\Big|_{\operatorname{Ran} P^{f,-,\mathrm{u}}_{U^*}(0;\lambda_*)}$$

and similarly for the other projections $P^{f,+,\mathrm{s}}_{U^*}(0;\lambda)$, $P^{b,-,\mathrm{u}}_{U^*}(0;\lambda)$, and $P^{b,+,\mathrm{s}}_{U^*}(0;\lambda)$, then we obtain

$$\hat{P}_N(\lambda) = \begin{pmatrix}-\hat{P}^{f,-,\mathrm{u}}_{U^*}(0;\lambda) & \hat{P}^{f,+,\mathrm{s}}_{U^*}(0;\lambda) & 0 & 0\\ 0 & 0 & -\hat{P}^{b,-,\mathrm{u}}_{U^*}(0;\lambda) & \hat{P}^{b,+,\mathrm{s}}_{U^*}(0;\lambda)\end{pmatrix}.$$

Define $\hat{R}_N(\lambda)$ as $R_N(\lambda)$ restricted to $\tilde{V}_b$

$$\hat{R}_N(\lambda) = R_N(\lambda)\Big|_{V_b}.$$

Now, we can define

$$D_N(\lambda) := \det\left(\hat{P}_N(\lambda) + \hat{R}_N(\lambda)\right).$$

Since $\hat{R}_N(\lambda)$ is a small deviation exponentially approching 0 as $N \to \infty$, there exists a C which does not depend on λ or N such that for $\lambda \in B_\delta$ with δ sufficiently small and $N \geq N_*$, with N_* sufficiently large,

$$\tilde{R}_N(\lambda) \leq C|\hat{R}_N(\lambda)| = Ce^{-2\alpha N}.$$

Therefore, we can rewrite $D_N(\lambda)$ as

$$\det\left(\hat{P}_N(\lambda)\right) + \tilde{R}_N(\lambda) = 0.$$

Since $\hat{P}_N(\lambda)$ is block diagonal, its determinate is just the multiplication of determinates of two blocks,

$$\det\left(\hat{P}_N(\lambda)\right) = \det\left(-\hat{P}^{f,-,\mathrm{u}}_{U^*}(0;\lambda) \quad \hat{P}^{f,+,\mathrm{s}}_{U^*}(0;\lambda)\right) \cdot \det\left(-\hat{P}^{b,-,\mathrm{u}}_{U^*}(0;\lambda) \quad \hat{P}^{b,+,\mathrm{s}}_{U^*}(0;\lambda)\right)$$

where $\hat{P}^{f,-,\mathrm{u}}_{U^*}(0;\lambda)$, $\hat{P}^{f,+,\mathrm{s}}_{U^*}(0;\lambda)$, $\hat{P}^{b,-,\mathrm{u}}_{U^*}(0;\lambda)$, and $\hat{P}^{b,+,\mathrm{s}}_{U^*}(0;\lambda)$ restricted in λ near λ_*, and

$$\mathrm{Ran}\, P^{f,-,\mathrm{u}}_{U^*}(0;\lambda)\Big|_{\mathrm{Ran}\, P^{f,-,\mathrm{u}}_{U^*}(0;\lambda_*)} \to \mathrm{Ran}\, P^{f,-,\mathrm{u}}_{U^*}(0;\lambda)$$

is an isomorphism with uniform bound in λ near λ_*, and similarly for $\mathrm{Ran}\, P^{f,+,\mathrm{s}}_{U^*}(0;\lambda)$, $\mathrm{Ran}\, P^{b,-,\mathrm{u}}_{U^*}(0;\lambda)$, and $\mathrm{Ran}\, P^{b,+,\mathrm{s}}_{U^*}(0;\lambda)$. So then we have a nontrivial solution if and only if

$$\det\left(\mathrm{Ran}\, P^{f,-,\mathrm{u}}_{U^*}(0;\lambda) \quad \mathrm{Ran}\, P^{f,+,\mathrm{s}}_{U^*}(0;\lambda)\right) \cdot \det\left(\mathrm{Ran}\, P^{b,-,\mathrm{u}}_{U^*}(0;\lambda) \quad \mathrm{Ran}\, P^{b,+,\mathrm{s}}_{U^*}(0;\lambda)\right) + \hat{R}_N(\lambda) = 0.$$

Now since we also have that $|P^{f,-,\mathrm{u}}_{U^*}(0;\lambda) - P^{-,\mathrm{u}}_{U^f}(0;\lambda)| \leq Ce^{-\alpha N}$, and analogously for projections associated with the other exponential dichotomies, we have

$$\begin{aligned} 0 = &\left(\det\left(\mathrm{Ran}\, P^{-,\mathrm{u}}_{U^f}(0;\lambda) \quad \mathrm{Ran}\, P^{+,\mathrm{s}}_{U^f}(0;\lambda)\right) + \mathrm{O}(e^{-\alpha N})\right) \cdot \\ &\left(\det\left(\mathrm{Ran}\, P^{-,\mathrm{s}}_{U^b}(0;\lambda) \quad \mathrm{Ran}\, P^{b,+,\mathrm{s}}_{U^b}(0;\lambda)\right) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N}) \end{aligned}$$

which can then be written

$$D_N(\lambda) = \left(D_f(\lambda) + \mathrm{O}(e^{-\alpha N})\right)\left(D_b(\lambda) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N}) = 0. \tag{3.38}$$

Since, by assumption, (3.14) and (3.15) hold for $\lambda \in B_\delta(\lambda_*)$.

To prove the next lemma, we evoke Rouché's theorem, which is stated as follows:

Theorem 3.2. *([10, Theorem on p. 111]) Let $K \subset \mathbb{C}$ a closed, bounded region with ∂K a simple closed contour, and suppose $f, g : K \to \mathbb{C}$ are analytic with*

$$|g(z)| < |f(z)|, \qquad z \in \partial K.$$

Then, f and g have the same number of zeros in K, counting multiplicities.

Lemma 3.4. *Suppose that (3.14) and (3.15) hold for $\lambda \in B_{\delta_0}(\lambda_*)$ and $D_N(\lambda)$ is as defined in (3.16) for $\lambda \in B_{\delta_1}(\lambda_*)$. Then there exists a δ_* satisfying $\min\{\delta_0, \delta_1\} > \delta_* > 0$ such that for $N_* > 0$ sufficiently large, $D_N(\lambda)$ has exactly $m_f + m_b$ zeroes in $B_{\delta_*}(\lambda_*)$, uniformly in $N \geq N_*$, which are $\mathrm{O}(e^{-\alpha N/h(m_f,m_b)})$ close to λ_*, with $h(m_f, m_b) = 2\max\{m_f, m_b\} + \epsilon$ and $\epsilon > 0$ arbitrarily small.*

Proof. We let $z = \lambda - \lambda_*$ and expand (3.16) as

$$\begin{aligned}
D_N(\lambda) &= D_f(\lambda)D_b(\lambda) + D_f(\lambda)\,\mathrm{O}(e^{-\alpha N}) + D_b(\lambda)\,\mathrm{O}(e^{-\alpha N}) + \mathrm{O}(e^{-2\alpha N}) + \mathrm{O}(e^{-2\alpha N}) \\
&= z^{m_f+m_b} + z^{m_f}\,\mathrm{O}(z^{m_b+1}) + z^{m_b}\,\mathrm{O}(z^{m_f+1}) + \mathrm{O}(z^{m_f+m_b+2}) + z^{m_f}\,\mathrm{O}(e^{-\alpha N}) \\
&\quad + \mathrm{O}(e^{-\alpha N}z^{m_f}) + z^{m_b}\,\mathrm{O}(e^{-\alpha N}) + \mathrm{O}(e^{-\alpha N}z^{m_b}) + \mathrm{O}(e^{-2\alpha N}) \\
&= z^{m_f+m_b} + \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(z^{m_f+m_b+2}) + \mathrm{O}(e^{-\alpha N}z^{m_f}) + \mathrm{O}(e^{-\alpha N}z^{m_f+1}) \\
&\quad + \mathrm{O}(e^{-\alpha N}z^{m_b}) + \mathrm{O}(e^{-\alpha N}z^{m_b+1}) + \mathrm{O}(e^{-2\alpha N}) \\
&= z^{m_f+m_b} + \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N}z^{m_f}) + \mathrm{O}(e^{-\alpha N}z^{m_b}) + \mathrm{O}(e^{-2\alpha N})
\end{aligned}$$

Then, we define

$$\begin{aligned}
f(z) &= z^{m_f+m_b} \\
g(z) &= \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N}(z^{m_f} + z^{m_b})) + \mathrm{O}(e^{-2\alpha N})
\end{aligned}$$

with an appropriate contour $K = B_r(0)$ for some $r > 0$, ensuring that the error terms in $g(z)$ are small enough. We wish to find the smallest r such that f and g have the same number of zeros in K near λ_*. We require

$$\mathrm{O}(r^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N}(r^{m_f} + r^{m_b})) + \mathrm{O}(e^{-2\alpha N}) < r^{m_f+m_b}. \tag{3.39}$$

By rewiriting (3.39), we have

$$\mathrm{O}(e^{-2\alpha N}) < r^{m_f+m_b}\left(1 - \mathrm{O}(r) - \mathrm{O}(e^{-\alpha N}(r^{-m_b} + r^{-m_f}))\right). \tag{3.40}$$

For N_* sufficiently large, there exists a constant C such that we can write (3.40) as

$$Ce^{-2\alpha N} < r^{m_f+m_b}\left(1 - Cr - Ce^{-\alpha N}(r^{-m_b} + r^{-m_f})\right). \tag{3.41}$$

The choice of r gives the distance over which we can apply Rouché's theorem, and we choose $r = C^{-1}e^{-\alpha N/h(m_f,m_b)}$,

$$C^2 e^{-2\alpha N} < e^{-\alpha N\frac{m_f+m_b}{h(m_f,m_b)}}\left(1 - e^{-\alpha N\frac{1}{h(m_f,m_b)}} - e^{-2\alpha N\frac{m_f}{h(m_f,m_b)}} - e^{-2\alpha N\frac{m_b}{h(m_f,m_b)}}\right)$$

where $h(m_f, m_b)$ is a function to be determined. we require $\frac{m_f+m_b}{h(m_f,m_b)} \leq 1$ as well as $\frac{mf}{h(m_f,m_b)} < \frac{1}{2}$ and $\frac{m_b}{h(m_f,m_b)} < \frac{1}{2}$, where we require strict inequalities to ensure the second factor in (3.41) can be made arbitrarily close to 1 without requiring dependence of the constant on m_f or m_b. Thus, we need $h(m_f, m_b) > 2\max\{m_f, m_b\}$, and so we conclude that $r = \mathrm{O}(e^{-\alpha N/h(m_f,m_b)})$, where $h(m_f, m_b) = 2\max\{m_f, m_b\} + \epsilon$, where $\epsilon > 0$. Therefore, $D_N(\lambda)$ has exactly $m_f + m_b$ zeros in a small neighborhood of λ_*, exponentially closing to λ_* for large N. □

3.2 Multi-Pulse Solutions

In Section 3.1, the stability of a single-pulse localized solution was analyzed and compared to that of front and back solutions. We can take this further by noting that a 2-pulse solution is the combination of two single-pulse solutions The stable single-pulse solution can be constructed using the symmetry of front and back solutions. Therefore, a symmetric 2-pulse solution is stable in each component if it is constructed from two stable single-pulse solutions that are reflections of each other. Although many asymmetric 2-pulse solutions are often unstable, especially when glued of unstable single-pulse solutions. However, if an asymmetric 2-pulse solution is the gluing of two stable single-pulse solutions, and when the single-pulse solution is such that its localized plateau remains near u^* for a longer stretch than the other single-pulse, it can still remain stable. The key is that both one-pluse solutions must be stable, so the 2-pulse solution glued of them can also be stable under appropriate conditions. This intuition extends inductively to N-pulse solutions that are comprised of N single-pulses glued together.

3.2.1 Main Result for Multi-Pulse Solutions

We begin by assuming $U_n^*(\mu)$ is a localized steady-state multi-pulse solution of (2.3), guaranteed to exist by Theorem 2.2 in Section 2.2. Then, we make the connection between multi-pulses and the single-pulses of the previous subsections precise with the following assumption.

Hypothesis 3.4. The localized solutions $\{U_n^*(\mu)\}_{n\in\mathbb{Z}}$ are parametried by constants $N_1,\ldots,N_k, M_1,\ldots,M_{2k} \geq N_*$ for some $N_* \geq 1$, so that there exists constants $C > 0$, $\eta > 1$ such that

$$U_n^{*,i}(\mu) = \begin{cases} U^f_{n+N_i}(\mu) + w^{f,-}_{n+N_i}(N_i), & n \in [-N_i - M_{2i-1} : -N_i] \\ U^f_{n+N_i}(\mu) + w^{f,+}_{n+N_i}(N_i), & n \in [-N_i : 0] \\ U^b_{n-N_i}(\mu) + w^{b,-}_{n-N_i}(N_i), & n \in [0 : N_i] \\ U^b_{n-N_i}(\mu) + w^{b,+}_{n-N_i}(N_i), & n \in [N_i : N_i + M_{2i}] \end{cases} \tag{3.42}$$

and by taking the "gluing" point of the front and back of the first pulse $U_n^{*,1}(\mu)$ as $n = 0$, the localized solution is $U_n^*(\mu) = U^{*,i}_{n-S_i}(\mu)$ if $n \geq S_i - N_i - M_{2i-1}$, $M_1 = M_{2k} = \infty$, $S_i = \sum_{j=2}^{2i-1} N_{\lfloor j/2+1 \rfloor} + M_j$ for all $i \in [1 : k]$, and

$$|w_n^{f,-}(N_i)| \leq C\eta^{-N+n}, \qquad n \in [-M_{2i-1} : 0] \tag{3.43}$$

$$|w_n^{f,+}(N_i)| \leq C\eta^{-N}, \qquad n \in [0 : N_i] \tag{3.44}$$

$$|w_n^{b,-}(N_i)| \leq C\eta^{-N}, \qquad n \in [-N_i : 0] \tag{3.45}$$

$$|w_n^{b,+}(N_i)| \leq C\eta^{-N-n}, \qquad n \in [0 : M_{2i}] \tag{3.46}$$

where $N = \min\{N_1,\ldots,N_k, M_1,\ldots,M_{2k}\}$. Since $U_n^*(\mu)$ is unique when $n = S_i - N_i - M_{2i-1}$, we have

$$w_0^{f,-}(N_i) - w_0^{f,+}(N_i) = 0, \tag{3.47}$$

$$\left(U^f_{N_i}(\mu) + w^{f,+}_{N_i}(N_i)\right) - \left(U^b_{-N_i}(\mu) + w^{b,-}_{-N_i}(N_i)\right) = 0, \tag{3.48}$$

$$w^{b,-}_0(N_i) - w^{b,+}_0(N_i) = 0, \tag{3.49}$$

$$\left(U^b_{M_{2i}}(\mu) + w^{b,+}_{M_{2i}}(N_i)\right) - \left(U^f_{-M_{2i+1}}(\mu) + w^{f,-}_{-M_{2i+1}}(N_{i+1})\right) = 0. \tag{3.50}$$

The "gluing" method in Hypothesis 3.4 is very similar to that in Hypothesis 3.2. Equations (3.47) and (3.49) ensures the uniqueness of front and back solutions, equation (3.49) glues together front and back solutions near u^*, and additional equation (3.50) glues together single-pulse solutions at their tails near 0.

Theorem 3.3. *Suppose Hypotheses 3.1, 3.2, and 3.4 are satisfied, and define*

$$\Omega = \mathbb{C} \setminus \{[f'(0,\mu) - 4d, f'(0,\mu)] \cup [f'(u^*,\mu) - 4d, f'(u^*,\mu)]\} \tag{3.51}$$

and Evans functions of front and back solutions

$$D_f(\lambda) := \det\left(\operatorname{Ran} P^{-,\mathrm{u}}_{U^f}(0;\lambda) \quad \operatorname{Ran} P^{+,\mathrm{s}}_{U^f}(0;\lambda)\right), \tag{3.52}$$

$$D_b(\lambda) := \det\left(\operatorname{Ran} P^{-,\mathrm{u}}_{U^b}(0;\lambda) \quad \operatorname{Ran} P^{+,\mathrm{s}}_{U^b}(0;\lambda)\right), \tag{3.53}$$

where $P^{\pm,\mathrm{s/u}}_{U^f}(0;\lambda)$ and $P^{\pm,\mathrm{s/u}}_{U^b}(0;\lambda)$ are projections to the systems $\overline{V}_{n+1} = A^{\pm}_{f,n}(\lambda)\overline{V}_n$ and $\overline{V}_{n+1} = A^{\pm}_{b,n}(\lambda)\overline{V}_n$, respectively. Fix $\lambda_* \in \Omega$ and suppose that for $m^f, m^b \geq 0$ and for some $\delta > 0$, we have

$$D_f(\lambda) = (\lambda - \lambda_*)^{m^f} + \mathrm{O}\left(|\lambda - \lambda_*|^{m^f+1}\right), \tag{3.54}$$

$$D_b(\lambda) = (\lambda - \lambda_*)^{m^b} + \mathrm{O}\left(|\lambda - \lambda_*|^{m^b+1}\right) \tag{3.55}$$

for $\lambda \in B_\delta(\lambda_*)$. Then, we can define an analytic function $D_N(\lambda)$ such that

$$D_N(\lambda) = \prod_{i=1}^{k} \left(D_f(\lambda) + \mathrm{O}(e^{-\alpha N})\right)\left(D_b(\lambda) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N}) \tag{3.56}$$

and there exists a δ_* sufficiently small, with $\delta > \delta_* > 0$, and an $N_* > 0$ sufficiently large, such that the following hold uniformly in $N \geq N_*$ where $N = \min\{N_1, \dots, N_k, M_1, \dots, M_{2k}\}$:

(i) *$D_N(\lambda)$ has precisely $k(m^f + m^b)$ roots, counted with multiplicity, in $B_{\delta_*}(\lambda_*)$. These values of λ are $\mathrm{O}(e^{-\alpha N})$ close to λ_*, with $\alpha > 1$.*

(ii) *The system* (3.2) *has a bounded, nontrivial solution at* $\lambda \in B_{\delta_*}(\lambda_*)$ *if and only if* $D_N(\lambda) = 0$.

The proof is in the following subsection by using the similar strategies as in Subsection 3.1.2.

3.2.2 Proof of Theorem 3.3

To begin, using (3.42) we can solve the eigenvalue problem (3.2) in pieces by solving the equations of each pulse

$$\overline{V}_{n+1}^{f,-,i} = A_n^{f,-,i}(\lambda)\overline{V}_n^{f,-,i}, \qquad n \in [-M_{2i-1} : 0] \tag{3.57}$$

$$\overline{V}_{n+1}^{f,+,i} = A_n^{f,+,i}(\lambda)\overline{V}_n^{f,+,i}, \qquad n \in [0 : N_i] \tag{3.58}$$

$$\overline{V}_{n+1}^{b,-,i} = A_n^{b,-,i}(\lambda)\overline{V}_n^{b,-,i}, \qquad n \in [-N_i : 0] \tag{3.59}$$

$$\overline{V}_{n+1}^{b,+,i} = A_n^{b,+,i}(\lambda)\overline{V}_n^{b,+,i}, \qquad n \in [0 : M_{2i}] \tag{3.60}$$

and then constructing the solution as

$$\overline{V}_n^i = \begin{cases} \overline{V}_{n+N_i}^{f,-,i}, & n \in [-N_i - M_{2i-1} : -N_i] \\ \overline{V}_{n+N_i}^{f,+,i}, & n \in [-N_i : 0] \\ \overline{V}_{n-N_i}^{b,-,i}, & n \in [0 : N_i] \\ \overline{V}_{n-N_i}^{b,+,i}, & n \in [N_i : N_i + M_{2i}]. \end{cases}$$

Thus, we have a bounded solution to (3.2) if and only if we have solutions to (3.57)−(3.60) along with the matching conditions

$$\overline{V}_0^{f,+,i} - \overline{V}_0^{f,-,i} = 0, \tag{3.61}$$

$$\overline{V}_{N_i}^{f,+,i} - \overline{V}_{-N_i}^{b,-,i} = 0, \tag{3.62}$$

$$\overline{V}_0^{b,+,i} - \overline{V}_0^{b,-,i} = 0, \tag{3.63}$$

$$\overline{V}_{M_{2i}}^{b,-,i} - \overline{V}_{-M_{2i+1}}^{f,+,i+1} = 0. \tag{3.64}$$

From Lemma 3.8, we have exponential dichotomies for each equation (3.57)−(3.60), which can be chosen so that they depend analytically on $\lambda \in B_\delta(\lambda_*)$ for $\delta > 0$ small. That is, we have

$$\left.\begin{aligned} \left|\Phi^{f,-,\mathrm{s}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi^{f,-,\mathrm{u}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad n,m \in [-M_{2i-1}:0], \tag{3.65}$$

$$\left.\begin{aligned} \left|\Phi^{f,+,\mathrm{s}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi^{f,+,\mathrm{u}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad n,m \in [0:N_i], \tag{3.66}$$

$$\left.\begin{aligned} \left|\Phi^{b,-,\mathrm{s}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi^{b,-,\mathrm{u}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad n,m \in [-N_i:0], \tag{3.67}$$

$$\left.\begin{aligned} \left|\Phi^{b,+,\mathrm{s}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(n-m)}, \quad n \ge m \\ \left|\Phi^{b,+,\mathrm{u}}_{U^*,i}(n,m;\lambda)\right| &\le Ce^{-\alpha(m-n)}, \quad n \le m \end{aligned}\right\} \quad n,m \in [0:M_{2i}]. \tag{3.68}$$

We denote the associated projections $P^{j,\pm,l}_{U^*,i}(n;\lambda) := \Phi^{j,\pm,l}_{U^*,i}(n,n;\lambda)$ where $j = f, b$ for front/back solutions and $l = \mathrm{s}, \mathrm{u}$ for stable/unstable components in individual pulses. Moreover, as a consequence of Hypothesis 3.1, we have that

$$\left|P^{f,-,\mathrm{s}}_{U^*,i}(-M_{2i-1};\lambda) - P^{\mathrm{s}}_0(0;\lambda)\right| \le Ce^{-\alpha N}, \tag{3.69}$$

$$\left|P^{f,+,\mathrm{u}}_{U^*,i}(N_i;\lambda) - P^{\mathrm{u}}_{u^*}(0;\lambda)\right| \le Ce^{-\alpha N}, \tag{3.70}$$

$$\left|P^{b,-,\mathrm{s}}_{U^*,i}(-N_i;\lambda) - P^{\mathrm{s}}_{u^*}(0;\lambda)\right| \le Ce^{-\alpha N}, \tag{3.71}$$

$$\left|P^{b,+,\mathrm{u}}_{U^*,i}(M_{2i};\lambda) - P^{\mathrm{u}}_0(0;\lambda)\right| \le Ce^{-\alpha N}. \tag{3.72}$$

We now let $a_i^{u^*} := (a_i^{f,+,u^*}, a_i^{b,-,u^*}) \in \widetilde{V}_a^{u^*}$, $a_i^0 := (a_i^{b,+,0}, a_i^{f,-,0}) \in \widetilde{V}_a^0$, $b_i := (b_i^{f,-}, b_i^{f,+}, b_i^{b,-}, b_i^{b,+}) \in \widetilde{V}_b$, and $\lambda \in \widetilde{V}_\lambda$, where the spaces $\widetilde{V}_a^{u^*}$, $\widetilde{V}_a^0$, $\widetilde{V}_b$, $\widetilde{V}_\lambda$ are defined as follows:

$$\begin{aligned} \widetilde{V}_a^{u^*} &:= \bigoplus_{i=1}^{k} \mathrm{Ran}\, P^{\mathrm{u}}_{u^*}(0;\lambda_*) \oplus \mathrm{Ran}\, P^{\mathrm{s}}_{u^*}(0;\lambda_*) \\ \widetilde{V}_a^0 &:= \bigoplus_{i=1}^{k-1} \mathrm{Ran}\, P^{\mathrm{u}}_0(0;\lambda_*) \oplus \mathrm{Ran}\, P^{\mathrm{s}}_0(0;\lambda_*) \\ \widetilde{V}_b &:= \bigoplus_{i=1}^{k} \Big(\mathrm{Ran}\, P^{f,-,\mathrm{u}}_{U^*,i}(0;\lambda_*) \oplus \mathrm{Ran}\, P^{f,+,\mathrm{s}}_{U^*,i}(0;\lambda_*)\Big) \oplus \Big(\mathrm{Ran}\, P^{b,-,\mathrm{u}}_{U^*,i}(0;\lambda_*) \oplus \mathrm{Ran}\, P^{b,+,\mathrm{s}}_{U^*,i}(0;\lambda_*)\Big) \end{aligned}$$

$$\widetilde{V}_\lambda := B_\delta(\lambda_*) \subset \mathbb{C}$$

where $\widetilde{V}_a^{u^*}$, $\widetilde{V}_a^0$ and $\widetilde{V}_b$ are endowed with the maximum norm over components such that $\|\widetilde{V}_a^{u^*}\|_\infty = \max_{i\in[1:k]}\{\|a_i^{f,+,u^*}\|, \|a_i^{b,-,u^*}\|\}$, $\|\widetilde{V}_a^0\|_\infty = \max_{i\in[1:k-1]}\{\|a_i^{b,+,0}\|\|a_i^{f,-,0}\|\}$ and $\|\widetilde{V}_b\|_\infty = \max_{i\in[1:k]}\{\|b_i^{f,-}\|, \|b_i^{f,+}\|, \|b_i^{b,-}\|, \|b_i^{b,+}\|\}$.

For δ sufficiently small and $N_1, \dots, N_k, M_1, \dots, M_{2k} > N_*$ sufficiently large, we can then write solutions to the eigenvalue problem (3.2) for the localized structure U_n^* as

$$\overline{V}_n^{f,-,i} = \Phi_{U^*,i}^{f,-,\mathrm{s}}(n, -M_{2i-1}; \lambda) a_{i-1}^{f,-,0} + \Phi_{U^*,i}^{f,-,\mathrm{u}}(n, 0; \lambda) b_i^{f,-} \qquad n \in [-M_{2i-1} : 0] \tag{3.73}$$

$$\overline{V}_n^{f,+,i} = \Phi_{U^*,i}^{f,+,\mathrm{s}}(n, 0; \lambda) b_i^{f,+} + \Phi_{U^*,i}^{f,+,\mathrm{u}}(n, N_i; \lambda) a_i^{f,+,u^*} \qquad n \in [0 : N_i] \tag{3.74}$$

$$\overline{V}_n^{b,-,i} = \Phi_{U^*,i}^{b,-,\mathrm{s}}(n, -N_i; \lambda) a_i^{b,-,u^*} + \Phi_{U^*,i}^{b,-,\mathrm{u}}(n, 0; \lambda) b_i^{b,-} \qquad n \in [-N_i : 0] \tag{3.75}$$

$$\overline{V}_n^{b,+,i} = \Phi_{U^*,i}^{b,+,\mathrm{s}}(n, 0; \lambda) b_i^{b,+} + \Phi_{U^*,i}^{b,+,\mathrm{u}}(n, M_{2i}; \lambda) a_i^{b,+,0} \qquad n \in [0 : M_{2i}] \tag{3.76}$$

with $a_0^{f,-,0} = a_k^{b,+,0} = 0$.

In the next lemma, we will solve matching conditions (3.62) and (3.64).

Lemma 3.5. *There exists an N_* such that for all $N > N_*$, the following holds uniformly in N. There exists operators $F_1 : \widetilde{V}_\lambda \times \widetilde{V}_b \to \widetilde{V}_a^{u^*}$ and $F_2 : \widetilde{V}_\lambda \times \widetilde{V}_b \to \widetilde{V}_a^0$ such that $\overline{V}_n$ as given by (3.73)–(3.76) with $a^{u^*} = F_1(\lambda)b$ and $a^0 = F_2(\lambda)b$ solves (3.56) for any b and λ. F_1 and F_2 are analytic in λ and linear in b, and satisfies*

$$|F_1(\lambda)b| \le Ce^{-\alpha N}|b|, \tag{3.77}$$

$$|F_2(\lambda)b| \le Ce^{-\alpha N}|b|. \tag{3.78}$$

Proof. For all $i \in [1 : k]$, substituting from (3.73)–(3.76) into euqations (3.62) and (3.64) gives us

$$\begin{aligned} 0 &= \Phi_{U^*,i}^{f,+,\mathrm{s}}(N_i, 0; \lambda) b_i^{b,+} + \Phi_{U^*,i}^{f,+,\mathrm{u}}(N_i, N_i; \lambda) a_i^{f,+,u^*} \\ &\qquad - \Phi_{U^*,i}^{b,-,\mathrm{s}}(-N_i, -N_i; \lambda) a_i^{b,-,u^*} - \Phi_{U^*,i}^{b,-,\mathrm{s}}(-N_i, 0; \lambda) b_i^{b,-} \\ &= a_i^{f,+,u^*} - a_i^{b,-,u^*} + \Phi_{U^*,i}^{f,+,\mathrm{s}}(N_i, 0; \lambda) b_i^{b,+} - \Phi_{U^*,i}^{b,-,\mathrm{s}}(-N_i, 0; \lambda) b_i^{b,-} \\ &\qquad + \left(P_{U^*,i}^{f,+,\mathrm{u}}(N_i; \lambda) - P_{u^*}^{\mathrm{u}}(0; \lambda)\right) a_i^{f,+,u^*} + \left(P_{u^*}^{\mathrm{s}}(0; \lambda) - P_{U^*,i}^{b,-,\mathrm{s}}(-N_i; \lambda)\right) a_i^{b,-,u^*} \end{aligned} \tag{3.79}$$

and

$$
\begin{aligned}
0 &= \Phi^{b,+,\mathrm{s}}_{U^*,i}(M_{2i},0;\lambda)b^{b,+}_i + \Phi^{f,+,\mathrm{u}}_{U^*,i}(M_{2i},M_{2i};\lambda)a^{b,+,0}_i \\
&\quad - \Phi^{f,+,\mathrm{u}}_{U^*,i+1}(-M_{2i+1},-M_{2i+1};\lambda)a^{f,-,0}_i - \Phi^{f,-,\mathrm{u}}_{U^*,i+1}(M_{2i+1},0;\lambda)b^{f,-}_{i+1} \\
&= a^{b,+,0}_i - a^{f,-,0}_i + \Phi^{b,+,\mathrm{s}}_{U^*,i}(M_{2i},0;\lambda)b^{b,+}_i - \Phi^{f,-,\mathrm{u}}_{U^*,i+1}(-M_{2i+1},0;\lambda)b^{f,-}_{i+1} \\
&\quad + \left(P^{b,+,\mathrm{u}}_{U^*,i}(M_{2i};\lambda) - P^{\mathrm{u}}_0(0;\lambda)\right)a^{b,+,0}_i + \left(P^{\mathrm{s}}_0(0;\lambda) - P^{f,-,\mathrm{s}}_{U^*,i+1}(-M_{2i+1};\lambda)\right)a^{f,-,0}_i
\end{aligned}
\tag{3.80}
$$

Define

$$
\begin{aligned}
G_1(\lambda)(a^{u^*}_i,b_i) &:= \left(P^{f,+,\mathrm{u}}_{U^*,i}(N_i;\lambda) - P^{\mathrm{u}}_{u^*}(0;\lambda)\right)a^{f,+,u^*}_i + \left(P^{\mathrm{s}}_{u^*}(0;\lambda) - P^{f,+,\mathrm{s}}_{U^*,i}(-N_i;\lambda)\right)a^{b,-,u^*}_i \\
&\quad - \Phi^{f,+,\mathrm{s}}_{U^*,i}(N_i,0;\lambda)b^{b,+}_i + \Phi^{b,-,\mathrm{s}}_{U^*,i}(-N_i,0;\lambda)b^{b,-}_i
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(\lambda)(a^0_i,b_i) &:= \left(P^{b,+,\mathrm{u}}_{U^*,i}(M_{2i};\lambda) - P^{\mathrm{u}}_0(0;\lambda)\right)a^{b,+,0}_i + \left(P^{\mathrm{s}}_0(0;\lambda) - P^{f,-,\mathrm{s}}_{U^*,i}(-M_{2i+1};\lambda)\right)a^{f,-,0}_i \\
&\quad - \Phi^{b,+,\mathrm{s}}_{U^*,i}(M_{2i},0;\lambda)b^{b,+}_i + \Phi^{f,-,\mathrm{u}}_{U^*,i+1}(-M_{2i+1},0;\lambda)b^{f,-}_{i+1}.
\end{aligned}
$$

Now, $G_1(\lambda)(a^{u^*},b)$ and $G_2(\lambda)(a^0,b)$ are analytic in λ and linear in a^{u^*} and b, and a^0 and b, respectively. We also notice that $G_2(\lambda)(a^0,b)$ is different from $G_1(\lambda)(a^{u^*},b)$, the elements in a^0 or b are not multiples, so b^f_1 and b^b_k are not assigned with some element in a^0. Moreover, from (3.69) and (3.72) and definition of the dichotomies,

$$
|G_1(\lambda)(a^{u^*},b)| \le Ce^{-\alpha N}(|a^{u^*}| + |b|), \tag{3.81}
$$

$$
|G_2(\lambda)(a^0,b)| \le Ce^{-\alpha N}(|a^0| + |b|). \tag{3.82}
$$

Define the map K_1 and K_2 as

$$
\begin{aligned}
K_1 : \widetilde{V}^{u^*}_a &\to \bigoplus_{i=1}^{k} \mathbb{C}^2 \\
(a^{f,+,u^*}_i, a^{b,-,u^*}_i) &\mapsto a^{f,+,u^*}_i - a^{b,-,u^*}_i
\end{aligned}
$$

and

$$K_2 : \widetilde{V}_a^0 \to \bigoplus_{i=1}^{k-1} \mathbb{C}^2$$
$$(a_i^{b,+,0}, a_i^{f,-,0}) \mapsto a_i^{b,+,0} - a_i^{f,-,0}$$

Since $\widetilde{V}_a^{u^*} = \bigoplus_{i=1}^{k} \operatorname{Ran} P_{u^*}^{\mathrm{u}}(0;\lambda_*) \oplus \operatorname{Ran} P_{u^*}^{\mathrm{s}}(0;\lambda_*) = \bigoplus_{i=1}^{k} \mathbb{C}^2$, and $\widetilde{V}_a^0 = \bigoplus_{i=1}^{k} \operatorname{Ran} P_{u^*}^{\mathrm{u}}(0;\lambda_*) \oplus \operatorname{Ran} P_{u^*}^{\mathrm{s}}(0;\lambda_*) = \bigoplus_{i=1}^{k-1} \mathbb{C}^2$, K_1 and K_2 are linear isomorphisms. Equations (3.79) and (3.80) can be written as

$$G_{1/2}(\lambda)(0,b) = K_{1/2}(a^{u^*/0}) + G_{1/2}(\lambda)(a^{u^*/0},0) = (K_{1/2} + G_{1/2}(\lambda)J_{1/2})(a^{u^*/0}) \tag{3.83}$$

where $J_{1/2}(a^{u^*/0}) := (a^{u^*/0},0)$. By (3.77) and (3.78), for $N \geq N_*$ with N_* sufficiently large, $(K_{1/2} + G_{1/2}(\lambda)J_{1/2})$ is invertible so that we have the solution operator

$$a^{u^*/0} = -(K_{1/2} + G_{1/2}(\lambda)J_{1/2})^{-1} G_{1/2}(\lambda)(0,b) =: F_{1/2}(\lambda)b. \tag{3.84}$$

Finally, from (3.81)–(3.82) and (3.83), we have (3.77) and (3.78). □

Using operators F_1 and F_2 from Lemma 3.5 and substituting from (3.73)–(3.76) into equations (3.61) and (3.63) gives us

$$\begin{aligned} 0 &= \Phi_{U^*,i}^{f,+,\mathrm{s}}(0,0;\lambda)b_i^{f,+} + \Phi_{U^*,i}^{f,+,\mathrm{u}}(0,N_i;\lambda)a_i^{f,+,u^*} - \Phi_{U^*,i}^{f,-,\mathrm{s}}(0,M_{2i-1};\lambda)a_{i-1}^{f,-,0} - \Phi_{U^*,i}^{f,-,\mathrm{u}}(0,0;\lambda)b_i^{f,-} \\ &= P_{U^*,i}^{f,+,\mathrm{s}}(0;\lambda)b_i^{f,+} - P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda)b_i^{f,-} \\ &\qquad + \Phi_{U^*,i}^{f,+,\mathrm{u}}(0,N_i;\lambda)(F_1(\lambda)b)_i^{f,+,u^*} - \Phi_{U^*,i}^{f,-,\mathrm{s}}(0,-M_{2i-1};\lambda)(F_2(\lambda)b)_{i-1}^{f,-,0} \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} 0 &= \Phi_{U^*,i}^{b,+,\mathrm{s}}(0,0;\lambda)b_i^{b,+} + \Phi_{U^*,i}^{b,+,\mathrm{u}}(0,M_{2i};\lambda)a_i^{b,+,0} - \Phi_{U^*,i}^{b,-,\mathrm{s}}(0,-N_i;\lambda)a_i^{b,-,u^*} - \Phi_{U^*,i}^{b,-,\mathrm{u}}(0,0;\lambda)b_i^{b,-} \\ &= P_{U^*,i}^{b,+,\mathrm{s}}(0;\lambda)b_i^{b,+} - P_{U^*,i}^{b,-,\mathrm{u}}(0;\lambda)b_i^{b,-} \\ &\qquad + \Phi_{U^*,i}^{b,+,\mathrm{u}}(0,M_{2i};\lambda)(F_2(\lambda)b)_i^{b,+,0} - \Phi_{U^*,i}^{b,-,\mathrm{s}}(0,-N_i;\lambda)(F_1(\lambda)b)_i^{b,-,u^*}. \end{aligned} \tag{3.86}$$

$(F_1(\lambda)b)_i^{f,+,u^*}$ and $(F_1(\lambda)b)_i^{b,-,u^*}$ are the components of $F_1(\lambda)b$ in $\operatorname{Ran} P_{u^*}^{\mathrm{u}}(0;\lambda_*)$ and $\operatorname{Ran} P_{u^*}^{\mathrm{s}}(0;\lambda_*)$; $(F_2(\lambda)b)_{i-1}^{f,-,0}$ and $(F_2(\lambda)b)_i^{b,+,0}$ are the components of $F_2(\lambda)b$ in $\operatorname{Ran} P_0^{\mathrm{s}}(0;\lambda_*)$ and $\operatorname{Ran} P_0^{\mathrm{u}}(0;\lambda_*)$, respectively. According to the exponential dichotomies (3.65)–(3.68) and lemma 3.5, we obtain

$$\Phi_{U^*,i}^{f,-,\mathrm{s}}(0,M_{2i-1};\lambda)(F_2(\lambda)b)_{i-1}^{f,-,0} = \mathrm{O}(e^{-2\alpha N}|b|), \tag{3.87}$$

$$\Phi_{U^*,i}^{f,+,\mathrm{u}}(0,N_i;\lambda)(F_1(\lambda)b)_i^{f,+,u^*} = \mathrm{O}(e^{-2\alpha N}|b|), \tag{3.88}$$

$$\Phi_{U^*,i}^{b,-,\mathrm{s}}(0,-N_i;\lambda)(F_1(\lambda)b)_i^{b,-,u^*} = \mathrm{O}(e^{-2\alpha N}|b|), \tag{3.89}$$

$$\Phi_{U^*,i}^{b,+,\mathrm{u}}(0,M_{2i};\lambda)(F_2(\lambda)b)_i^{b,+,0} = \mathrm{O}(e^{-2\alpha N}|b|) \tag{3.90}$$

uniformly in λ near λ_*. Temporarily rewrite solution operators and projectors so that the following equations do not exceed the lateral length:

$$\begin{aligned}
P_{f,i}^{-,\mathrm{u}} &:= P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda) & P_{f,i}^{+,\mathrm{s}} &:= P_{U^*,i}^{f,+,\mathrm{s}}(0;\lambda)\\
P_{b,i}^{-,\mathrm{u}} &:= P_{U^*,i}^{b,-,\mathrm{u}}(0;\lambda) & P_{b,i}^{+,\mathrm{s}} &:= P_{U^*,i}^{b,+,\mathrm{s}}(0;\lambda)\\
\Phi_{f,i}^{+,\mathrm{u}} &:= \Phi_{U^*,i}^{f,+,\mathrm{u}}(0,N_i;\lambda) & \Phi_{b,i}^{-,\mathrm{s}} &:= \Phi_{U^*,i}^{b,-,\mathrm{s}}(0,-N_i;\lambda)\\
\Phi_{f,i}^{-,\mathrm{s}} &:= \Phi_{U^*,i}^{f,-,\mathrm{s}}(0,-M_{2i-1};\lambda) & \Phi_{b,i}^{+,\mathrm{u}} &:= \Phi_{U^*,i}^{b,+,\mathrm{u}}(0,M_{2i};\lambda)
\end{aligned}$$

Rewrite (3.85) and (3.86) and combine them in matrix form,

$$\begin{pmatrix}0\\0\\\vdots\\0\\0\end{pmatrix} = \left[\begin{pmatrix}
-P_{f,1}^{-,\mathrm{u}} & P_{f,1}^{+,\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -P_{b,1}^{-,\mathrm{u}} & P_{b,1}^{+,\mathrm{s}} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -P_{f,k}^{-,\mathrm{u}} & P_{f,k}^{+,\mathrm{s}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -P_{b,k}^{-,\mathrm{u}} & P_{b,k}^{+,\mathrm{s}}
\end{pmatrix}\right.$$

$$\left. + \begin{pmatrix}
\Phi_{f,1}^{+,\mathrm{u}} & 0 & 0 & 0 & 0\\
0 & -\Phi_{b,1}^{-,\mathrm{s}} & 0 & 0 & 0\\
0 & 0 & \ddots & 0 & 0\\
0 & 0 & 0 & \Phi_{f,k}^{+,\mathrm{u}} & 0\\
0 & 0 & 0 & 0 & -\Phi_{b,k}^{-,\mathrm{s}}
\end{pmatrix} F_1(\lambda) + \begin{pmatrix}
-\Phi_{f,1}^{-,\mathrm{s}} & 0 & 0 & 0 & 0\\
0 & \Phi_{b,1}^{+,\mathrm{u}} & 0 & 0 & 0\\
0 & 0 & \ddots & 0 & 0\\
0 & 0 & 0 & -\Phi_{f,k}^{-,\mathrm{s}} & 0\\
0 & 0 & 0 & 0 & \Phi_{b,k}^{+,\mathrm{u}}
\end{pmatrix}\begin{pmatrix}0\\ \\F_2(\lambda)\\ \\0\end{pmatrix}\right] b$$

$$=: [P_N(\lambda) + R_N^1(\lambda) + R_N^2(\lambda)]b$$

From (3.87)−(3.90), we can observe that

$$R_N^1(\lambda) = \mathrm{O}(e^{-2\alpha N}) \qquad \text{and} \qquad R_N^2(\lambda) = \mathrm{O}(e^{-2\alpha N})$$

uniformly in λ near λ_*. If $b \neq 0$ and $[P_N(\lambda) + R_N^1(\lambda) + R_N^2(\lambda)]b = 0$, we can find a bounded nontrivial solution for the eigenvalue problem (3.57)−(3.62) uniformly in λ near λ_*. Define $\hat{P}_{f,i}^{-,\mathrm{u}}$ as $P_{f,i}^{-,\mathrm{u}}$ restricted to $\mathrm{Ran}\, P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda^*)$

$$\hat{P}_{f,i}^{-,\mathrm{u}} := P_{f,i}^{-,\mathrm{u}}\Big|_{\mathrm{Ran}\, P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda^*)}$$

and similarly for the other projections $P_{U^*,i}^{f,+,\mathrm{s}}$, $P_{U^*,i}^{b,-,\mathrm{u}}$, and $P_{U^*,i}^{b,+,\mathrm{s}}$.

$$\hat{P}_N(\lambda) = \begin{pmatrix} -\hat{P}_{f,1}^{-,\mathrm{u}} & \hat{P}_{f,1}^{+,\mathrm{s}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{P}_{b,1}^{-,\mathrm{u}} & \hat{P}_{b,1}^{+,\mathrm{s}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\hat{P}_{f,k}^{-,\mathrm{u}} & \hat{P}_{f,k}^{+,\mathrm{s}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{P}_{b,k}^{-,\mathrm{u}} & \hat{P}_{b,k}^{+,\mathrm{s}} \end{pmatrix}$$

Define $\hat{R}_N^1(\lambda)$ as $R_N^1(\lambda)$ restricted to $\widetilde{V}_b$

$$\hat{R}_N^1(\lambda) = R_N^1(\lambda)\Big|_{\widetilde{V}_b}$$

and similarly for $R_N^2(\lambda)$. Define

$$D_N(\lambda) := \det\left(\hat{P}(\lambda) + \hat{R}_N^1(\lambda) + \hat{R}_N^2(\lambda)\right)$$

and there exists a bounded nontrivial solution at $\lambda \in B_{\delta_*}(\lambda_*)$ if and only if $D_N(\lambda) = 0$. When N is large enough, $\hat{R}_N^1(\lambda)$ and $\hat{R}_N^2(\lambda)$ are small deviations exponentially approching 0 for $\hat{P}_N(\lambda)$, we can rewrite

$D_N(\lambda)$ as

$$\det\left(\hat{P}(\lambda)\right) + \tilde{R}(\lambda) = 0$$

where

$$\tilde{R}_N(\lambda) = \mathrm{O}\left(\left|\hat{R}_N^1(\lambda) + \hat{R}_N^2(\lambda)\right|\right) = \mathrm{O}(e^{-2\alpha N}). \tag{3.91}$$

So, there exists a C which does not depend on λ or N such that for $\lambda \in B_\delta(\lambda_*)$ with δ sufficiently small and $N \geq N_*$, with N_* sufficiently large,

$$\tilde{R}_N(\lambda) \leq Ce^{-2\alpha N}.$$

Since $\hat{P}_N(\lambda)$ is block diagonal, its determinate is just the multiplication of determinates of these blocks,

$$\det\left(\hat{P}(\lambda)\right) = \prod_{i=1}^{k} \det\begin{pmatrix} -\hat{P}_{f,i}^{-,\mathrm{u}} & \hat{P}_{f,i}^{+,\mathrm{s}} \end{pmatrix} \cdot \det\begin{pmatrix} -\hat{P}_{b,i}^{-,\mathrm{u}} & \hat{P}_{b,i}^{+,\mathrm{s}} \end{pmatrix}.$$

Each $\hat{P}$ in $\hat{P}(\lambda)$ is previously defined, and

$$\mathrm{Ran}\, P_{f,i}^{-,\mathrm{u}}\Big|_{\mathrm{Ran}\, P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda_*)} \to \mathrm{Ran}\, P_{f,i}^{-,\mathrm{u}}$$

is an isomorphism with uniform bound in λ near λ_*, and similarly for $\mathrm{Ran}\, P_{U^*,i}^{f,+,\mathrm{s}}$, $\mathrm{Ran}\, P_{U^*,i}^{b,-,\mathrm{u}}$, and $\mathrm{Ran}\, P_{U^*,i}^{b,+,\mathrm{s}}$. So, we have a bounded nontrivial solution if and only if

$$D_N(\lambda) = \prod_{i=1}^{k} \det\begin{pmatrix} -\,\mathrm{Ran}\, P_{f,i}^{-,\mathrm{u}} & \mathrm{Ran}\, P_{f,i}^{+,\mathrm{s}} \end{pmatrix} \cdot \det\begin{pmatrix} -\,\mathrm{Ran}\, P_{b,i}^{-,\mathrm{u}} & \mathrm{Ran}\, P_{b,i}^{+,\mathrm{s}} \end{pmatrix} + \tilde{R}_N(\lambda) = 0.$$

Since corrections (3.43)−(3.46) have been added to both front and back of each pulse (3.42), we also have that

$$|P_{U^*,i}^{f,-,\mathrm{u}}(0;\lambda) - P_{U^f,i}^{-,\mathrm{u}}(0;\lambda)| \leq Ce^{-\alpha N},$$
$$|P_{U^*,i}^{f,+,\mathrm{s}}(0;\lambda) - P_{U^f,i}^{-,\mathrm{s}}(0;\lambda)| \leq Ce^{-\alpha N},$$

$$|P^{b,-,\mathrm{u}}_{U^*,i}(0;\lambda) - P^{-,\mathrm{u}}_{U^b,i}(0;\lambda)| \le Ce^{-\alpha N},$$
$$|P^{b,+,\mathrm{s}}_{U^*,i}(0;\lambda) - P^{-,\mathrm{s}}_{U^b,i}(0;\lambda)| \le Ce^{-\alpha N}.$$

By performing operations similar to $\tilde{R}_N$, we obtain

$$0 = \prod_{i=1}^{k} \left(\det\left(\mathrm{Ran}\, P^{-,\mathrm{u}}_{U^f,i}(0;\lambda) \quad \mathrm{Ran}\, P^{+,\mathrm{s}}_{U^f,i}(0;\lambda)\right) + \mathrm{O}(e^{-\alpha N})\right) \cdot \left(\det\left(\mathrm{Ran}\, P^{-,\mathrm{u}}_{U^b,i}(0;\lambda) \quad \mathrm{Ran}\, P^{+,\mathrm{s}}_{U^b,i}(0;\lambda)\right) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N})$$

which can then be written

$$D_N(\lambda) = \prod_{i=1}^{k} \left(D_f(\lambda) + \mathrm{O}(e^{-\alpha N})\right)\left(D_b(\lambda) + \mathrm{O}(e^{-\alpha N})\right) + \mathrm{O}(e^{-2\alpha N}) = 0.$$

Since, by assumption, (3.52) and (3.53) hold for $\lambda \in B_\delta(\lambda_*)$.

Lemma 3.6. *Suppose that* (3.54) *and* (3.55) *hold for* $\lambda \in B_{\delta_0}(\lambda_*)$ *and* $D_N(\lambda)$ *is as defined in* (3.56) *for* $\lambda \in B_{\delta_1}(\lambda_*)$. *Then there exists a* δ_* *satisfying* $\min\{\delta_0, \delta_1\} > \delta_* > 0$ *such that for* $N_* > 0$ *sufficiently large,* $D_N(\lambda)$ *has exactly* $k(m_f + m_b)$ *zeroes in* $B_{\delta_*}(\lambda_*)$*, uniformly in* $N \ge N_*$*, which are* $\mathrm{O}(e^{-\alpha N/(k\cdot h(m_f,m_b))})$ *close to* λ_**, with* $h(m_f, m_b) = 4\max\{m_f, m_b\} + \epsilon$ *and* $\epsilon > 0$ *arbitrarily small.*

Recall Theorem 3.2, it will be used again to prove Lemma 3.6.

Proof. To begin, we let $z = \lambda - \lambda_*$,

$$\begin{aligned}
D_N(\lambda) &= \prod_{i=1}^{k} \left(z^{m_f+m_b} + \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N} z^{m_f}) + \mathrm{O}(e^{-\alpha N} z^{m_b})\right) + \mathrm{O}(e^{-2\alpha N}) \\
&= \sum_{i=0}^{k} \binom{k}{i} z^{(k-i)(m_f+m_b)} \left(\mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N} z^{m_f}) + \mathrm{O}(e^{-\alpha N} z^{m_b})\right)^i + \mathrm{O}(e^{-2\alpha N}) \\
&= z^{k(m_f+m_b)} + \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N} z^{m_f}) + \mathrm{O}(e^{-\alpha N} z^{m_b}) + h.o.t + \mathrm{O}(e^{-2\alpha N}).
\end{aligned}$$

Then, we define

$$f(z) = z^{k(m_f+m_b)}$$
$$g(z) = \mathrm{O}(z^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N}(z^{m_f+m_b} + z^{m_f} + z^{m_b})) + \mathrm{O}(e^{-2\alpha N})$$

with an appropriate contour $K = B_r(0)$ for some $r > 0$, ensuring that the error terms in $g(z)$ are small enough. We wish to find the smallest r such that f and g have the same number of zeros in K near λ_*. We require

$$\mathrm{O}(r^{m_f+m_b+1}) + \mathrm{O}(e^{-\alpha N}(r^{m_f} + r^{m_b})) + \mathrm{O}(e^{-2\alpha N}) < r^{k(m_f+m_b)}. \tag{3.92}$$

By rewiriting (3.92), we have

$$\mathrm{O}(e^{-2\alpha N}) < r^{k(m_f+m_b)} \left(1 - \mathrm{O}(r^{1+(k-1)(m_f+m_b)}) - \mathrm{O}(e^{-\alpha N}(r^{-km_b-(k-1)m_f} + r^{-km_f-(k-1)m_b}))\right). \tag{3.93}$$

For N_* sufficiently large, there exists a constant C such that we can write (3.93) as

$$Ce^{-2\alpha N} < r^{k(m_f+m_b)} \left(1 - Cr^{1+(k-1)(m_f+m_b)} - Ce^{-\alpha N}(r^{-km_b-(k-1)m_f} + r^{-km_f-(k-1)m_b})\right). \tag{3.94}$$

The choice of r gives the distance over which we can apply Rouché's theorem, and we chooce $r = C^{-1}e^{-\alpha N/(k\cdot h(m_f,m_b))}$,

$$C^2 e^{-2\alpha N} < e^{-\alpha N \frac{m_f+m_b}{h(m_f,m_b)}} \left(1 - e^{-\alpha N \frac{1+(k-1)(m_f+m_b)}{k\cdot h(m_f,m_b)}} - e^{-2\alpha N \frac{k(m_f+m_b)-m_f}{k\cdot h(m_f,m_b)}} - e^{-2\alpha N \frac{k(m_f+m_b)-m_b}{k\cdot h(m_f,m_b)}}\right)$$

where $h(m_f, m_b)$ is a function to be determined. We require $\frac{m_f+m_b}{h(m_f,m_b)} \leq 1$ as well as $\frac{k(m_f+m_b)-m_f}{k\cdot h(m_f,m_b)} < \frac{1}{2}$ and $\frac{k(m_f+m_b)-m_b}{k\cdot h(m_f,m_b)} < \frac{1}{2}$, where we require strict inequalities to ensure the second factor in (3.94) can be made arbitrarily close to 1 without requiring dependence of the constant on m_f or m_b. We can simplify above three inequalities to clearly obtain the necessary conditions for $h(m_f, m_b)$,

$$m_f + m_b \leq h(m_f, m_b),\ 2m_b + 2m_f\left(1 - \frac{1}{k}\right) < h(m_f, m_b),\ \text{and } 2m_f + 2m_b\left(1 - \frac{1}{k}\right) < h(m_f, m_b).$$

Thus, we chooce $h(m_f, m_b) > 4\max\{m_f, m_b\}$ to hold these inequalities for all $k \geq 2$, and so we conclude that $r = \mathrm{O}(e^{-\alpha N/(k\cdot h(m_f,m_b))})$, where $h(m_f, m_b) = 4\max\{m_f, m_b\} + \epsilon$, where $\epsilon > 0$. Therefore, $D_N(\lambda)$ has exactly $k(m_f + m_b)$ zeros in a small neighborhood of λ_*, exponentially closing to λ_* for large N. $\square$

Chapter 4

Conclusion

This thesis analyzes the stability of spatially localized single- and multi-pulse solutions of the discrete Nagumo equation. On the basis of the existence of spatially localized solutions [6, 3], linearizing the nonlinear system around the steady-state solutions and applying exponential dichotomy theory, an isomorphism between localized solutions and front and back solutions is constructed. Then, deriving the spectral condition through the Evans function, stable front and back solutions are sufficient to obtain a stable localized solution. For single-pulse solutions, the stability requires symmetry between front and back solutions and ensures exponential decay of perturbations. Extending the analysis to multi-pulse solutions, it indicates that if each constituent pulse is individually stable and symmetrically bonded in space, the stability can be attained. If these multi-pulse solutions are asymmetrically bonded in space, the stability can also be maintained under appropriate conditions. This method is not limited to analyzing discrete Nagumo equation, but can also be widely applied to other lattice equations, partial differential equations, or approximate solutions of certain nonlinear systems with similar structures.

In this thesis, we consider only the stability of localized steady-state solutions. However, in many nonlinear systems, asymmetric solutions are more common than symmetric solutions, especially in equations involving bifurcation phenomena or symmetry breaking. For systems, such as the Nagumo equation or other nonlinear reaction-diffusion systems, as the bifurcation parameter μ changes, the solution may undergo bifurcation. The originally symmetric solution may become unstable, while the asymmetric solution may become a new stable solution. The stability of asymmetric pulse solutions and their appropriate conditions are also valuable and worth studying. Especially in practical applications, the study of the stability of asymmetric solutions can be more attractive.

In the future research, the stability of spatial localized solutions can be extended to the more complex situation of time-periodic localized solutions [1], often called breathers. These solutions possess both spatiality and temporal periodicity, representing a subtle balance between temporal and spatial oscillations. In Hodgkin-Huxley model, time-periodic localized solutions can effectively simulate information transmission and signal propagation between neurons, and better explain phenomena such as pulse coupling and synchronicities of neurons [8]. The existing research on localized synchrony patterns in the bistable Ginzburg−Landau system [1] could be applied to study this model as my future potential research. In various other application fields, time-periodic localized solutions also play a crucial role. In the future research, we will conduct multi-scale analysis, such as bifurcation phenomena and numerical simulation methods, to reveal the appropriate conditions and stability for the formation of local periodic solutions in various nonlinear systems, providing new perspectives and methods for solving practical problems.

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