Curve Shortening Flow of Closed Curves in the Plane and on Curved Surfaces

David Toharia

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By: David Toharia

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____________________________ Examiner

Dr. Pawel Gora

____________________________ Examiner

Dr. Galia Dafni

____________________________ Supervisor

Dr. Alina Stancu

Approved by ____

Dr. Lea Popovic, Graduate Program Director

__________ 2025 ____

Dr. Pascale Sicotte, Dean of the Faculty of Arts and Science

ABSTRACT

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The curve shortening flow of smooth curves, also referred to as flow by curvature, has seen detailed study in the last four decades. In this paper, we go over the main existence theorems of the theory, with an focus on making the presentation clear and self-contained. Starting with convex curves in the plane, we examine the Gage-Hamilton theorem, which proves the flow of these curves exists and converges to a point. We inspect the original proof by Gage and Hamilton. We next look at Grayson's Theorem which proves the same for non-convex curves. Instead of Grayson's original proof, we study a proof by Bryan and Andrews. The aim is to highlight the core arguments and make the proofs more accessible. We seek to emphasize the intuition underlying the key ideas. Finally, we turn to the flow of curves on Riemannian surfaces. We present some of the necessary differential geometry material which is otherwise typically assumed. We compare results on the surface to their analogues in the plane and examine how the curvature affects the respective proofs.

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Chapter 1

Introduction

Among geometric flows, curve shortening flow, the flow of curves according to their curvature, is one of the simplest and most intuitive to conceive of. The equation of the flow presents itself as a heat equation, furthering its alluring simplicity. Nonetheless, curve shortening has yielded a rich theory, with conclusive answers to questions about the existence of solutions and the nature of limiting sets. There exist many natural generalizations and in this sense the study of curve shortening can serve also as a useful stepping stone to pass to more advanced geometric flows.

In curve shortening, curves are flowed at each point in the direction of the normal vector with a speed equal to the curvature.

Definition 1.0.1. *A family of curves $X(u,t)$ is said to be undergoing curve shortening flow if it satisfies:*

$$\frac{\partial X}{\partial t} = kN. \tag{1.1}$$

Since the quantity kN is in fact the second derivative with respect to arc length, one can re-express curve shortening in a form reminescint of the classical heat equation.

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2}. \tag{1.2}$$

For this reason, curve shortening is often referred to as a geometric heat equation. Indeed as we shall see, curves undergoing curve shortening share many qualitative features with solutions of the heat equation.

The study of curve shortening started in the context of material science with work by Mullins on annealing boundaries in metals. The main theoretical work started with a series of papers by Gage and Hamilton. The focus was on closed convex curves. Gage and Hamilton addressed the primary initial inquiries of any geometric flow: do solutions exist, for how long, what does the limit look like and what can be said about the convergence. These queries were then addressed in the case of non-convex closed curves by Grayson.

In this work, we shall review the main results concerning curve shortening, both in the plane and on an arbitrary Riemannian surface. The emphasis will be on making the results easy to understand and the presentation self-contained. We strive to achieve a balance between completeness and clarity. In particular, one aspect of the original development of the theory which is somewhat brushed over is the nature of curve shortening as a parabolic partial differential equation. Although it is intuitively clear, rigorously establishing this fact is not so immediate. As this characterization is fundamentally important in order to apply tools from the parabolic theory, we have chosen to spend a moment to examine this question.

Next, for the case of non-convex curves, we will examine the original proof by Gage and Hamilton. We will try to emphasize the intuition behind the arguments. For convex curves, rather than Grayson's original proof, we will look at a more modern proof based on an inequality by Andrews and Bryan. This latter proof is a bit more straightforward, although does not generalize as readily to extensions of curve-shortening. In either case, we seek to highlight the core ideas to make the proofs readily understandable.

In the case of curve shortening on surfaces, one needs to deal with the complicated differential machinery. Results are typically presented already assuming some familiarity with this material. We instead choose to present some of the required material e.g. definitions and results related to connections and the curvature tensor. I do believe there is value in this presentation, both to make curve shortening on surfaces more accessible and also because the computations specific to curve shorting serve to highlight aspects of the general theory. For instance the direct carry-over of point-wise results, which should be true on a heuristic level, is shown to be true through the computations showcasing the aptness of the definitions of connections and covariant derivatives. On the other hand, the necessity of the curvature tensor emerges naturally when seeking to compute the evolution of the (curves') curvature. In this way I found studying curve-shortening aided in understanding aspects of this general theory, hence the latter's inclusion.

As with any flow, the initial questions to be posed are whether solutions exist and, if so, are they unique. Is existence only short-term or can something be said of long-term existence? Can the limit sets (if any) be characterized? As we shall see, these questions all have answers in the case of the plane. The analogy with the heat equation is a good one: the curve shortening induces an "averaging out" of curves, making them tend to a circle before decaying to a point. Specifically, we describe the limit of the flow as a "round point", that is, the curves shrink to a point, but upon rescaling the curves converge to a circle. However, the convergence to a point occurs in finite time. This is in major contrast to solutions of the heat equation, which can take infinite time to decay to steady state.

The intuition behind these results can be envisioned by considering Figure 1.1. Wherever the curve is convex, the flow drives the curve inwards, leading to to the collapse of the curve to a point. If the curve has a concavity, then the flow will tend to "inflate" out this concavity until the curve becomes convex. Finally, regions a high curvature flow faster then less regions; this induces the averaging out of the curvature, causing the curve to approach a circle.

We shall start with the case of the plane, deriving evolution equations for several geometric quantities in Chapter 2. We shall also prove that embedded curves stay embedded under the flow.

Next, in Chapter 3 we shall follow focus on convex curves, proving the Theorem of Gage-Hamilton. Namely we will show that the flow exists until the area becomes zero, that the

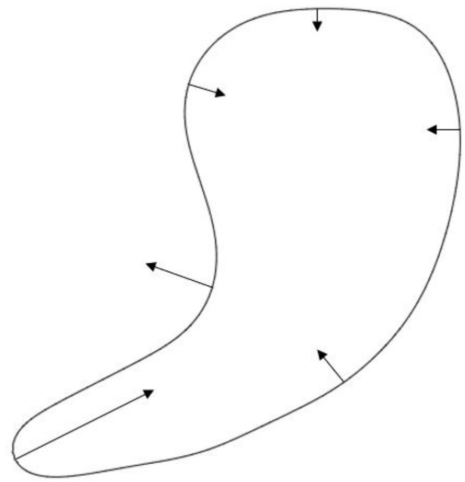

Figure 1.1: Direction of the flow along a non-convex curve

curves converge to a round point and that this convergence is smooth. In Chapter 4 we shall look at the case of non-convex curves, proving Grayson's theorem. The proof shall not be the original by Grayson, but instead one by Bryan and Andrews.

Finally, in Chapter 5 we shall turn to curve shortening on surfaces. We shall present some of the required differential geometry theory and derive the evolution equation for the curvature. We shall then examine the major results concerning convergence of shortening flows on surfaces.

Finally, we will end by listing a few applications of curve shortening flow both in mathematics and the applied sciences. These applications revolve around properties specific to curve-shortening and thus highlight the ongoing relevance of this particular flow.

Chapter 2

Curve Shortening in the Plane

2.1 Preliminaries

Recall that a curve in the plane is a mapping $X : [a, b] \to \mathbb{R}^2, u \mapsto X(u)$. Formally, the mapping is distinct from the **image of the curve** $\{X(u) : u \in [a, b]\}$, although we will often abuse notation and refer to the image by X. A curve is **closed** if $X(a) = X(b)$ and it is **smooth** if it is of class C^∞ as a function on $[a, b]$.

We will want to look at curves which are **regular**: the first derivative $X'(u)$ is never zero. In fact, we will eventually want to consider **embedded** curves. Formally a curve is an embedding if the mapping X is a diffeomorphism onto its image. One can think of curves which do not have any self-intersections nor any cusps. Note that embedded curves are automatically regular.

The derivative $X'(u)$ can be thought of as the velocity of the curve and its norm the speed. We will denote the speed by $v = \|X'(u)\|$. As long as a curve is regular, we can define the arc length parameter of the curve s, by $s = \int_a^u v\,du$. We can also define the **unit tangent vector** $T(u) = \frac{X'(u)}{v} = \frac{dX}{ds}$ and the **unit normal vector** N (taken to be orthogonal to T). If the curve is moreover C^2, we can define the **curvature** of the curve k as the scalar satisfying $\frac{1}{v}\frac{dT}{du} = \frac{dT}{ds} = kN$. The relationship between T, N and their derivatives are summed up by the Frenet equations.

Proposition 2.1.1. FRENET EQUATIONS, [1] *Let X be a regular C^2 curve. There exists a unit tangent vector field T, a unit normal vector field N along with a scalar function k, defined along X, satisfying*

$$\frac{dT}{ds} = kN, \qquad \frac{dN}{ds} = -kT. \tag{2.1}$$

We can now define the curve shortening flow for a family of curves in the plane.

Definition 2.1.1. *Given a smooth regular curve $X_0 : [a, b] \longrightarrow \mathbb{R}^2$, the family of curves $X : [a, b] \times [0, T) \longrightarrow \mathbb{R}^2$ is said to be undergoing curve shortening flow if it satisfies:*

$$\begin{cases} \dfrac{\partial X}{\partial t} = kN \\ X(u, 0) = X_0(u). \end{cases} \tag{2.2}$$

The curve $X_0(u)$ is referred to as the initial curve for the flow.

Note that the family of curves is parametrized by t. We think of this as a time parameter and describe the curves as evolving in time, with a normal speed, under the curve shortening flow. This is not to be confused with the speed v of a given curve, which should properly be referred to as the speed of the parametrization in u.

2.2 Basic Computations

Before examining the limit behavior of the flow, we want to establish how various geometric quantities evolve under the flow. In particular, we will want to know how the length and area evolve. Before getting to these, we will first have to examine the arc length of the curve and its tangent and normal vectors. Most of these results follow from direct computations and can also be found in [1] and [7].

2.2.1 Arc length and velocity

As in Definition 2.1.1, at each time t we have a curve $X(\cdot, t)$ parametrized by the parameter u, taking values in $[a, b]$. This u is independent of the time variable t. However for computing geometric quantities of the curves, it is useful to consider the arc length parameter s. The issue is that s will not be independent of t; in particular, the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ will not commute.

Note the relationship between the differential operators in s and u:

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}. \tag{2.3}$$

The following lemma establishes how the parametrization speed v evolves with the flow:

Lemma 2.2.1. Evolution of the curve speed *The parametrization speed, $v(u,t)$ evolves under the curve shortening flow according to:*

$$\frac{\partial v}{\partial t} = -k^2 v. \tag{2.4}$$

Proof. We have:

$$\frac{\partial}{\partial t}\left(v^2\right) = \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial u}\cdot\frac{\partial X}{\partial u}\right) = 2\frac{\partial^2 X}{\partial u \partial t}\cdot\frac{\partial X}{\partial u}. \tag{2.5}$$

We have freely commuted the derivatives with respect to u and t. We now use the assumption of curve shortening flow (2.2) and the Frenet equations (Proposition 2.1.1).

$$\frac{\partial}{\partial t}\left(v^2\right) = 2\frac{\partial}{\partial u}(kN)\cdot vT = 2v\left(\frac{\partial k}{\partial u}N\cdot T + k\frac{\partial N}{\partial u}\cdot T\right) = -2v^2k^2. \tag{2.6}$$

On the other hand

$$\frac{\partial}{\partial t}\left(v^2\right) = 2v\frac{\partial v}{\partial t} \tag{2.7}$$

and the result follows. □

We will now want to establish how to interchange the differential operators of t and s.

Proposition 2.2.1. *For a curve undergoing curve shortening flow, the commutation of the derivatives in time t and arc length s is given by:*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + k^2\frac{\partial}{\partial s}. \tag{2.8}$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial u}\right) &= \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial t}\frac{\partial}{\partial u} && (2.9)\\ &= \frac{k^2}{v}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial u}\frac{\partial}{\partial t} && (2.10)\\ &= k^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}. && (2.11) \end{aligned}$$

□

2.2.2 Tangent and Normal vectors, Curvature

Before computing the variations of the length and area, we will need to know how the tangent and normal vectors evolve. The computations are straightforward and follow from applying Proposition 2.2.1.

Proposition 2.2.2.

$$\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s}N, \tag{2.12}$$

$$\frac{\partial N}{\partial t} = -\frac{\partial k}{\partial s}T. \tag{2.13}$$

Proof.

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial^2 X}{\partial t \partial s} \\ &= \frac{\partial^2 X}{\partial s \partial t} + k^2 \frac{\partial X}{\partial s} \\ &= \frac{\partial (kN)}{\partial s} + k^2 T \\ &= \frac{\partial k}{\partial s} N + k \frac{\partial N}{\partial s} + k^2 T.\end{aligned}$$

Using the Frenet equation $\frac{\partial N}{\partial s} = -kT$ gives the result. To show the claim for N, note that $N \cdot T = 0$. Taking the derivative:

$$\frac{\partial N}{\partial t} \cdot T + N \cdot \frac{\partial T}{\partial t} = 0.$$

Using the already proven equation (2.12), this implies $\frac{\partial N}{\partial t} \cdot T = -\frac{\partial k}{\partial s}$. As a unit norm vector, the derivative of N must be lie orthogonal to N, this is in the direction of T. Hence, $\left\| \frac{\partial N}{\partial t} \right\| = \left| \frac{\partial k}{\partial s} \right|$ and equation (2.13) is proven. □

As a first immediate consequence we can express the time evolution of the curvature.

Proposition 2.2.3. (The Evolution of the Curvature) *The curvature, k, evolves under curve shortening flow according to the following differential equation:*

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3. \tag{2.14}$$

Proof. We can compute the time derivative of N using the Frenet equation $\frac{\partial T}{\partial s} = kN$:

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{k}\frac{\partial T}{\partial s}\right) \\ &= -\frac{1}{k^2}\frac{\partial k}{\partial t}\frac{\partial T}{\partial s} + \frac{1}{k}\frac{\partial^2 T}{\partial t \partial s} \\ &= -\frac{1}{k}\frac{\partial k}{\partial t}N + \frac{1}{k}\frac{\partial^2 T}{\partial s \partial t} + k\frac{\partial T}{\partial s} \\ &= -\frac{1}{k}\frac{\partial k}{\partial t}N + \frac{1}{k}\frac{\partial^2 T}{\partial s \partial t} + k^2 N.\end{aligned}$$

Using equation (2.12),

$$\frac{\partial^2 T}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial k}{\partial s}N\right) = \frac{\partial^2 k}{\partial s^2}N + \frac{\partial k}{\partial s}\left(-\frac{\partial k}{\partial s}T\right). \tag{2.15}$$

On the other hand, from equation (2.13), $\frac{\partial N}{\partial t} = -\frac{\partial k}{\partial s}T$. Gathering all the terms in the direction of N, we find

$$0 = -\frac{1}{k}\frac{\partial k}{\partial t} + \frac{1}{k}\frac{\partial^2 k}{\partial s^2} + k^2. \tag{2.16}$$

Multiplying through by k and rearranging gives the result. □

As we shall see in the next section, equation (2.14) will prove very useful in establishing convergence of the flow. We can immediately remark that the equation is a parabolic PDE, albeit a non-linear one. The classical methods for studying parabolic equations will thus be heavily used in analyzing the problem. Before getting there, we finish this section by computing the evolution of the length and area.

2.2.3 Length and Area

We can immediately calculate the evolution of the length.

Proposition 2.2.4. *Under the curve shortening flow, the length L of a closed embedded curve flows according to:*

$$\frac{\partial L}{\partial t} = -\int_\gamma k^2\, ds. \tag{2.17}$$

Proof. Using Proposition 2.2.1:

$$\begin{aligned}\frac{\partial L}{\partial t} &= \frac{\partial}{\partial t}\left(\int_a^b v\,du\right) \\ &= \int_a^b \frac{\partial v}{\partial t}\,du \\ &= \int_a^b -k^2 v\,du \\ &= -\int_\gamma k^2\,ds.\end{aligned}$$

□

We can now turn to the area. The computation is not too much longer. Remarkably, the derivative of the area is a constant. This immediately imposes an upper bound on the duration of a flow starting from a particular curve.

Proposition 2.2.5. *For a closed, embedded curve, the area A evolves according to:*

$$\frac{\partial A}{\partial t} = -2\pi. \tag{2.18}$$

Proof. Let $X(u,t) = (x(u,t), y(u,t))$. Note that the tangent vector is $T = \frac{1}{v}(x', y')$. It follows that the normal vector is $N = \frac{1}{v}(-y', x')$. By Green's theorem:

$$\begin{aligned}A(t) &= \frac{1}{2}\int_a^b x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\,du \\ &= -\frac{1}{2}\int_a^b vX\cdot N\,du.\end{aligned}$$

Thus:

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= -\frac{1}{2}\int_a^b \frac{\partial v}{\partial t} X \cdot N + v\frac{\partial X}{\partial t} \cdot N + vX \cdot \frac{\partial N}{\partial t}\, du \\
&= -\frac{1}{2}\int_a^b -k^2 vX \cdot N + kv - vX \cdot \frac{\partial k}{\partial s} T\, du.
\end{aligned}
$$

Note that $v\frac{\partial k}{\partial s} = \frac{\partial k}{\partial u}$. Using integration by parts

$$
\begin{aligned}
\int_a^b X \cdot \frac{\partial k}{\partial u} T\, du &= (kX \cdot T)|_a^b - \int_a^b \frac{\partial}{\partial u}(X \cdot T) k\, du \\
&= -\int_a^b kv + k^2 vN\, du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial A}{\partial t} &= -\int_a^b kv\, du \\
&= -\int_X k\, ds \\
&= -2\pi.
\end{aligned}
$$

□

To finish this section, we also include some lemmas which will be useful later, relating the area and length of convex curves to the **support function** h. Recall that the support function, defined by $h(s) = \langle \gamma(s), -N(s) \rangle$ is the distance from the origin to the support line orthogonal to $N(s)$. Convexity ensures that there is a unique support line in each direction, so h is well-defined.

Lemma 2.2.2. *For a closed, embedded convex curve parametrized by arc length:*

$$
A = \frac{1}{2}\int_0^L h(s)\, ds, \qquad L = \int_0^L hk\, ds. \tag{2.19}
$$

Proof. Let $\gamma(s) = \big(x(s),\, y(s)\big)$ be our curve; then we have $N(s) = \big(-y'(s),\, x'(s)\big)$. Hence:

$$\begin{aligned}\int_0^L h(s)\,ds &= \int_0^L \langle \gamma(s), -N(s)\rangle\,ds \\ &= \int_0^L x(s)y'(s) - y(s)x'(s)\,ds \\ &= 2A,\end{aligned} \tag{2.20}$$

where the last equality follows from Green's Theorem. For the second equality, note that if γ is parametrized by arc length, then:

$$\begin{aligned}\langle \gamma, kN\rangle &= \langle \gamma, \gamma''\rangle \\ &= \frac{d}{ds}\langle \gamma, \gamma'\rangle - 1.\end{aligned} \tag{2.21}$$

Hence:

$$\begin{aligned}\int_0^L hk\,ds &= \int_0^L \langle \gamma, -kN\rangle\,ds \\ &= \int_0^L 1\,ds - \langle \gamma, \gamma'\rangle\Big|_0^L \\ &= L.\end{aligned} \tag{2.22}$$

□

2.3 Short-term existence and Parabolic theory

The curve shortening equation is an example of a nonlinear parabolic partial differential equation. As a consequence, we can almost immediately conclude short-term existence of solutions from the general theory. We can also make use of the maximum principle for the further analysis of the equation. We will illustrate how one witnesses the parabolicity of the equation and give an overview of these results. One can consult [4] for general results of the theory pertaining to curve shortening.

We recall the definition of a **parabolic PDE**: denoting by $v = v(t, u)$ the quantity of interest, a parabolic PDE is one which can be written in the form

$$v_t = F(t, u, v, v_u, v_{uu}), \tag{2.23}$$

where F is elliptic. For our purposes, as we only have a single parameter u aside from time, ellipticity means that the derivative of F in the v_{uu} argument is positive. Note that this definition is for scalar equations. However, the curve shortening equation is a vector equation.

Heuristically, the flow equation takes on the appearance of a parabolic PDE, particularly when we express it using arc length:

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2} \tag{2.24}$$

(since $kN = \frac{dT}{ds}$). Of course, the arc length depends on the time t, so this expression merely masks the non-linearity of the equation. To apply general theorems, we would like to have a more rigorous way of realizing the flow as a parabolic PDE. One approach, detailed in [4] is to represent the evolving curves as graphs of functions. This can be done over the coordinate axis, or quite cleverly, using graphs over a fixed curve. One can take Γ to be a smooth curve near the initial curve. Then, for small enough t, one can express the flow as

$$X(s, t) = \Gamma(s) + d(s, t)N(s)\,, \tag{2.25}$$

with N being the unit normal of Γ. The function d will be smooth and for small enough t can be shown to satisfy a parabolic PDE.

Knowing that we have a parabolic equation, the first immediate result is short-term existence and uniqueness of solutions.

Theorem 2.3.1. (Short-term Existence, [9]) *If X_0 is a closed smooth embedded initial curve, then there exists a time ϵ such that a solution $X : [a, b] \times [0, \epsilon)$ satisfying the curve shortening equation (2.2). The curves $X(\cdot, t)$ are smooth and embedded.*

Note that unlike the (genuine) heat equation we cannot immediately make claims about the long-term existence of solutions. The non-linearity of the curve shortening flow means we cannot (yet) rule out the possibility that a solution develops kinks or crossings, that is, ceases to be embedded. Showing that this does not occur will be the focus of the next section.

We also recall the maximum principle. Roughly speaking, the maximum principle states that the maximum of a solution to a parabolic equation can only occur at the (parabolic) boundary. In the case of a periodic spatial variable, this simplifies to meaning the maximum occurs at time $t = 0$. We can state a version of the maximum principle for periodic functions.

Proposition 2.3.1. *([1]) Let $f : [0, 2\pi] \times [0, T) \to \mathbb{R}$ be a positive function, periodic in its first variable, which solves the equation*

$$\frac{\partial f}{\partial t} = a\frac{\partial^2 f}{\partial v^2} + b\frac{\partial f}{\partial v} + cf. \tag{2.26}$$

The functions a, b and c are bounded functions with $a \geq 0$ and $c \leq 0$. Then

$$f(v, t) \leq \max\{f(v, 0) : v \in [0, 2\pi]. \tag{2.27}$$

Similarly, one has a minimum principle if the functions a, b and c are bounded functions with $a \geq 0$ and $c \geq 0$.

2.4 Embedded curves remain embedded

Having shown short-term existence of solutions, we now wish to describe the long-term behavior of the flow. Since our parabolic equations are non-linear, there is a possibility that the solutions may develop kinks or self-intersections. However, as it turns out, this does not happen; provided that the initial curve is itself embedded and subject to a bound on the curvature, the solution will remain embedded under the flow.

Theorem 2.4.1. [1] *Let $X(u,t)$ be a family of curves undergoing curve shortening flow. If the initial curve $X_0(\cdot)$ is embedded and if the curvature is uniformly bounded: $|k(u,t)| < M$, for some $M > 0$, then the family of curves remains embedded. That is, for every $t \in [0,T)$, $X(\cdot,t)$ is an embedded curve.*

The proof will revolve around the function $f : [a,b]^2 \times [0,T)$, defined as the distance squared between points on the curve:

$$f(u_1,u_2,t) = \|X(u_1,t) - X(u_2,t)\|^2. \tag{2.28}$$

The proof, originally due to Gage and Hamilton, has a geometric interpretation. We can imagine the hypothetical first point at which a curve self-intersects. This self-intersection would have to occur tangentially and at points of the curve with distinct curvature (else the curve segments would be flowing together). However this would mean that the segment of curve with greater curvature had to move through the other segment, such that there must have already been an intersection immediately prior.

For the complete careful proof, we will have to consider multiple cases. To help handle one of the cases, we will need a lemma.

Lemma 2.4.1. *With f defined as above and k bounded by M, we have the following lower bound for f:*

$$f(u_1,u_2,t) \geq \left(\frac{2}{M}\sin\left(\frac{M}{2}|s_2 - s_1|\right)\right)^2 \tag{2.29}$$

See [7] for a proof of this lemma. We can now prove Theorem 2.4.1.

Proof. The curves $X(u,t)$ will be embedded provided that $f = 0$ implies $u_1 = u_2$. We shall thus analyze the possible points at which f can approach zero. Calculating the derivatives of f, using the curve shortening equation for the time derivative, we find

$$\frac{\partial f}{\partial t} = 2\langle X(u_1,t) - X(u_2,t), k(u_1,t)N(u_1,t) - k(u_2,t)N(u_2,t)\rangle \tag{2.30}$$

$$\frac{\partial f}{\partial s_i} = 2\langle X(u_i,t) - X(u_j,t), T(u_i,t)\rangle \tag{2.31}$$

$$\frac{\partial^2 f}{\partial s_i^2} = 2 + k(u_i,t)\langle X(u_i,t) - X(u_j,t), N(u_i,t)\rangle. \tag{2.32}$$

$$\frac{\partial^2 f}{\partial s_1 \partial s_2} = -2\langle T(u_1,t), T(u_2,t)\rangle \tag{2.33}$$

(Here j is the complement of i, either 1 or 2.) Thus, f satisfies the equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial s_1^2} + \frac{\partial^2 f}{\partial s_2^2} - 4. \tag{2.34}$$

At first glance it seems this equation is parabolic and we might hope to apply a minimum principle to bound f away from zero. However, the derivatives are with respect to the arc length s, which masks the non-linearity of the equation. A more careful analysis is required.

In fact, we know that $f = 0$ whenever $u_1 = u_2$. We will need to stay away from these points to use a minimum principle argument. We split into cases by defining the set

$$E = \left\{ (u_1, u_2, t) : |s_1 - s_2| < \frac{\pi}{M} \right\}. \tag{2.35}$$

We can deal with the set E using Lemma 2.4.1. There is a geometric intuition: the bound on the curvature implies that within a small enough arc length interval (defined in E), a curve cannot curl back on itself to self-intersect.

Specifically, let $(u_1, u_2, t) \in E$ and suppose $f(u_1, u_2, t) = 0$. Thus, the lower bound of f from Lemma 2.4.1 must also be zero, i.e. $\sin\left(\frac{M}{2}|s_2 - s_1|\right) = 0$. Using the bound on the interval from E, we have $\frac{M}{2}|s_2 - s_1| < \frac{\pi}{2}$. The only possible root for the sine is therefore at zero. Hence $s_1 = s_2$. The curve cannot intersect itself at distinct points in E. This deals with the first case.

We now consider the complement of E. On this set we will show that a minimum principle holds: the minimum value of f must be at the boundary. So let us first examine the boundary

$$\partial E^c = \left\{(u_1, u_2, t) : |s_1 - s_2| = \frac{\pi}{M}\right\} \cup \left\{(u_1, u_2, 0) : |s_1 - s_2| \geq \frac{\pi}{M}\right\} \tag{2.36}$$

In the first component, when $|s_1 - s_2| = \frac{\pi}{M}$, Lemma 2.4.1 implies that $f \geq (2/M)^2$. The second component, when $t = 0$, corresponds to the initial curve. By assumption, this curve does not self-intersect. So, as we are away from the points $u_1 = u_2$, f has a strictly positive minimum on this component. Thus f has a strictly positive minimum on all of ∂E^c. Call it m.

We argue as in the maximum principle to show that the minimum of f is not in the interior of E^c. Let $g = f + \epsilon t$ for some $\epsilon \geq 0$. The function g satisfies the equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} - 4 + \epsilon. \tag{2.37}$$

Suppose g attains a value $m - \delta$ in the interior of E^c. By continuity, there is a first time t' such that this value is attained at a point (u_1', u_2', t'). The largest possible value for g at $t = 0$ is m. So at $t = t'$, g must be non-increasing in time. Moreover, if we fix the time at t', we are left to consider a single compact curve. This implies that (u_1', u_2', t') is a minimum on this curve. Thus, at this point

$$\frac{\partial g}{\partial t} \leq 0, \quad \frac{\partial g}{\partial u_1} = \frac{\partial g}{\partial u_1} = 0 \quad \text{and} \quad |\mathbf{H}_g| \geq 0, \tag{2.38}$$

where $\mathbf{H}_g$ is the Hessian of g with respect to u_1 and u_2. In the standard proof of the maximum principle, we have but to consider the trace of $\mathbf{H}_g$ to draw the contradiction. However, as our equation has a -4 term, we must work a little harder. We first calculate the derivatives of g with respect to s_i:

$$\frac{\partial g}{\partial s_i} = \frac{\partial g}{\partial u_i}\frac{du_i}{ds_i} \tag{2.39}$$

$$\frac{\partial^2 g}{\partial s_i^2} = \frac{\partial^2 g}{\partial u_i^2}\left(\frac{du_i}{ds_i}\right)^2 + \frac{\partial g}{\partial u_i}\frac{d^2 u_i}{ds_i^2} \tag{2.40}$$

$$\frac{\partial^2 g}{\partial s_1 \partial s_2} = \frac{\partial^2 g}{\partial u_1 \partial u_2}\frac{du_1}{d2_1}\frac{du_2}{ds_2}. \tag{2.41}$$

Thus at the point (u_1', u_2', t'), we have

$$\frac{\partial g}{\partial s_i} = 0 \quad \text{for } i = 1, 2 \tag{2.42}$$

$$\frac{\partial^2 g}{\partial s_1^2}\frac{\partial^2 g}{\partial s_2^2} - \left(\frac{\partial^2 g}{\partial s_1 \partial s_2}\right)^2 = \left(\frac{\partial^2 g}{\partial u_1^2}\frac{\partial^2 g}{\partial u_2^2} - \left(\frac{\partial^2 g}{\partial u_1 \partial u_2}\right)^2\right)\left(\frac{du_1}{ds_1}\right)^2\left(\frac{du_2}{ds_2}\right)^2 \geq 0. \tag{2.43}$$

Comparing with equation (2.31), the first equality tells us that the tangents at u_1 and u_2 are parallel. Thus, looking at equation (2.33), we find

$$\frac{\partial^2 g}{\partial s_1 \partial s_2} = \pm 2. \tag{2.44}$$

In this way, along with the AGM inequality, we obtain the result:

$$\frac{\partial^2 g}{\partial s_1^2} + \frac{\partial^2 g}{\partial s_2^2} \geq 2\sqrt{\frac{\partial^2 g}{\partial s_1^2}\frac{\partial^2 g}{\partial s_2^2}} \geq 2\left|\frac{\partial^2 g}{\partial s_1 \partial s_2}\right| = 4. \tag{2.45}$$

The contradiction now follows by examining equation (2.37), evaluating at the point (u_1', u_2', t'). The left-hand side is non-positive, while the right-hand side is bounded below by ϵ.

Hence, the function g cannot attain the value $m - \delta$. So $g \geq m$ on the interior of E^c Taking $\epsilon \to 0$, the result carries over to f. In particular $f > 0$ on E^c. This completes the proof, as we have shown that $f = 0$ iff $u_1 = u_2$. □

Chapter 3

Convex Curves in the Plane

We now restrict ourselves to the flow of strictly convex curves. We say that curve, of at least class C^2, is strictly convex if the curvature is never 0. We consider the case where the initial curve is sufficiently regular, strictly convex, which, as it turns out, guarantees that the flow remains convex and we will show that the limit of the flow is always a circular point.

3.1 The curvature of convex curves remains bounded

Theorem 2.4.1 requires, as an assumption, a bound on the curvature. The next step is thus to examine the curvature function. It can be shown that for convex curves, as long as the area is non-zero the curvature is indeed bounded and that, moreover, the flow can always be extended.

The approach in establishing these facts is to reduce our problem to a scalar equation for the curvature. This is possible in the case of convex curves, as we can use the turning angle as a reparameterization for our curves.

We have already computed the equation satisfied by k in Proposition 2.2.3. We have:

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3. \tag{3.1}$$

We would like to be able to show that this equation is equivalent to the curve shortening equation. We know, from the fundamental theorem of plane curves, that given a prescribed curvature function it is possible to reconstruct a unit speed curve (unique up to rigid motions) having that curvature, [14]. One might hope this could be used to reconstruct curves at each fixed time t.

However, the issue is that for solutions of curve shortening, the domain of the arc length parameter s depends on t. Because of this dependence, the problem of solving (3.1) is not well-defined. To resolve this, we have to re-express the equation using independent parameters.

For strictly convex curves this can be done using the **turning angle**. We recall that for smooth curves, the turning angle is the smooth function θ defined along the curve, which satisfies

$$T = (\cos\theta, \sin\theta). \tag{3.2}$$

The curvature is given in terms of the turning angle by $k = \frac{d\theta}{ds}$. For strictly convex curves, every point of the curve is uniquely associated to an angle. We thus have a well-defined reparameterization of our curve, defined on $[0, 2\pi]$.

Through the chain rule, equation (3.1) becomes

$$\frac{\partial k}{\partial t} = k^2 \frac{\partial^2 k}{\partial \theta^2} + k^3, \tag{3.3}$$

when using the turning angle parametrization. The solutions of (3.3) have their domain on $[0, 2\pi] \times [0, T)$ so the PDE is well-defined.

If a solution $k(\theta, t)$ is obtained from equation (3.3) we would like to recover the family of flowing curve. At each time t, we can make use of a version of the fundamental theorem for plane curves, specific to the turning angle parametrization.

Proposition 3.1.1. (Characterization of Strictly Convex Curves, [1]) *Given a smooth function $k : [0, 2\pi] \to \mathbb{R}$ which is positive and 2π-periodic, there exists a regular, closed, smooth and strictly convex curve $X : [0, 2\pi] \to \mathbb{R}^2$ which is unique up to rigid motions and for which k is the curvature if and only if k satisfies*

$$\int_0^{2\pi} \frac{(\cos\theta, \sin\theta)}{k(\theta)}\, d\theta = 0. \tag{3.4}$$

Moreover, given such a k, up to a rigid motion the curve can be expressed as

$$X(\theta) = \int_0^{\theta} \frac{(\cos\alpha, \sin\alpha)}{k(\alpha)}\, d\alpha. \tag{3.5}$$

The consequence of Proposition 3.1.1 is that solutions $X(\theta, t)$ to curve shortening can be reconstructed from solutions $k(\theta, t)$ of equation (3.3).

Proposition 3.1.2. *([1]) For a strictly convex initial curve X_0, the existence of smooth solutions $X : [0, 2\pi] \times [0, T) \to \mathbb{R}^2$ to the curve shortening equation:*

$$\begin{cases} \dfrac{\partial X}{\partial t} = kN \\ X(\theta, 0) = X_0(\theta). \end{cases} \tag{3.6}$$

is equivalent to the existence of smooth solutions $k : [0, 2\pi] \times [0, T) \to \mathbb{R}$ to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial k}{\partial t} = k^2 \dfrac{\partial^2 k}{\partial \theta^2} + k^3 \\ k(\theta, 0) = k_0(u). \end{cases} \tag{3.7}$$

where k_0 is positive, smooth, and satisfies

$$\int_0^{2\pi} \frac{(\cos\theta, \sin\theta)}{k_0(\theta)}\, d\theta = 0. \tag{3.8}$$

The proof of Proposition 3.1.2 is simply a computation to show that solutions $k(\theta, t)$ satisfy the condition of Proposition 3.1.1.

With this equivalence theorem in hand, we can now turn our attention to the scalar PDE (3.3). Applying techniques from the theory of parabolic PDEs is now much more direct.

For instance, applying the minimum principle, we have that $k_{min}(t)$ is non-decreasing. We thus immediately establish that strictly convex curves remain strictly convex under the flow.

A more difficult result is to show that the curvature is uniformly bounded.

Theorem 3.1.1. ([7]) *If $X(\cdot, t) : [a, b] \times [0, T)$ is a solution to the curve shortening flow and the areas of the curves stay bounded away from 0, then the curvature k is uniformly bounded on $[a, b] \times [0, T)$.*

The proof is somewhat long and involves establishing a succession of estimates. First, using that the area is bounded below, it can be shown that the median curvature $k^* = \sup\{b \mid k(\theta, t) > b \text{ in some interval of length } \theta\}$ satisfies

$$k^* \leq \frac{L(t)}{A(t)}$$

for curve length L and area A. This bound, along with Wirtinger's inequality can be used to show that

$$H(t) = \int_0^{2\pi} \ln(k) d\theta$$

is bounded in time. This in turn can be used to establish a uniform bound on k.

3.2 The flow extends until the area is zero

We have established in Theorem 3.1.1 that the curvature is uniformly bounded for the duration of the flow. The hypothesis in Theorem 2.4.1 is therefore satisfied, so we know that solutions of the flow remain embedded. What we would like to know next is how long the solutions will exist. As it turns out, the flow can be extended as long as the hypothesis of Theorem 3.1.1 remains satisfied; that is, until the area reaches zero.

Theorem 3.2.1. *[1] For strictly convex curves, the curve shortening flow can be extended until the area bounded by the curves becomes zero.*

To prove this result, we consider a solution $k(\theta, t)$ for the Cauchy problem in equation (3.7). We know that the solution exists for a short time interval $[0, \epsilon)$. If, at the final time, the limiting function $k(\theta, \epsilon)$ is still smooth, then it can be used as a new initial condition to extend the solution.

To produce the smooth limit, we will use the Arzelà-Ascoli Theorem. Applying this theorem requires us to show that the curvature $k(\theta, t)$ and its derivatives each form a family of equicontinuous functions. The technique used is a standard argument involving the maximum principle. We will start with the first derivative.

Lemma 3.2.1. *If the curvature functional $k(\theta, t)$ is uniformly bounded on $[0, 2\pi] \times [0, T)$, then so is $\frac{\partial k}{\partial \theta}(\theta, t)$.*

Proof. We should want to compute the time derivative of $\frac{\partial k}{\partial \theta}$ and apply the maximum principle. Using 3.3 we find

$$\frac{\partial}{\partial t}\left(\frac{\partial k}{\partial \theta}\right) = k^2 \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial k}{\partial \theta}\right) + 2k \frac{\partial k}{\partial \theta} \frac{\partial}{\partial \theta}\left(\frac{\partial k}{\partial \theta}\right) + 3k^2 \frac{\partial k}{\partial \theta}. \tag{3.9}$$

We cannot immediately apply the maximum principle. The final term means that $\frac{\partial k}{\partial \theta}$ can grow exponentially. So, instead, we consider the function $e^{\alpha t} \frac{\partial k}{\partial \theta}$. Calculating the time evolution for this function we obtain

$$\frac{\partial}{\partial t}\left(e^{\alpha t}\frac{\partial k}{\partial \theta}\right) = k^2\frac{\partial^2}{\partial \theta^2}\left(e^{\alpha t}\frac{\partial k}{\partial \theta}\right) + 2k\frac{\partial k}{\partial \theta}\frac{\partial}{\partial \theta}\left(e^{\alpha t}\frac{\partial k}{\partial \theta}\right) + (\alpha + 3k^2)e^{\alpha t}\frac{\partial k}{\partial \theta}. \tag{3.10}$$

Knowing that k is uniformly bounded, we can choose a large enough negative α to ensure that $\alpha + 3k^2$ is negative. We can thus apply the maximum principle and deduce that $e^{\alpha t}\frac{\partial k}{\partial \theta}$ has its maximum at time $t = 0$. Hence, for every $(\theta, t) \in [0, 2\pi] \times [0, T)$,

$$\frac{\partial k}{\partial \theta}(\theta, t) \leq e^{-\alpha T} \max\left\{\frac{\partial k}{\partial \theta}(\theta, 0) : \theta \in [0, 2\pi]\right\}. \tag{3.11}$$

We thus have our uniform bound. □

We next need to show that the second derivative is also bounded. This is a somewhat more involved computation. To ease notation, we use prime notation to denote derivatives with respect to θ.

Lemma 3.2.2. *If the functions k and k' are uniformly bounded on $[0, 2\pi] \times [0, T)$, then so is the function k''.*

For the full details of the computations see [1]. Briefly, the idea is to first examine the function $\int_0^{2\pi}(k'')^4\, d\theta$. Using integration by parts along with equation (3.3) and the fact that k and k' are bounded, this function is found to be bounded. With this bound, we can use another integration by parts computation with equation (3.3) to find that $\int_0^{2\pi}(k''')^2$ is bounded. We can now use the Sobolev inequality

$$\max\left|f^2\right| \leq C\int \left|f'\right|^2 + |f|^2 \tag{3.12}$$

to obtain the bound on k''.

Having found bounds on the first two derivatives, it is now much easier to chain these results into bounds for the third and higher derivatives.

Lemma 3.2.3. *For $n \geq 3$, if the functions $k, k', ..., k^{(n-1)}$ are uniformly bounded, then so is the function $k^{(n)}$.*

Proof. Using induction and equation (3.3) we find that $k^{(n)}$ satisfies

$$\frac{\partial}{\partial t}k^{(n)} = k^2 k^{(n+2)} + 2nkk'k^{(n+1)} + p(k, k', ..., k^{(n-1)})k^{(n)} + q(k, k', ..., k^{(n-1)}) \tag{3.13}$$

where p and q are polynomials of the lower derivatives. By assumption, the lower derivatives are bounded, so the polynomials themselves can also be bounded. We can now repeat the same argument used for the first derivative, in Lemma 3.2.1. Since equation (3.13) implies that $k^{(n)}$ grows at most exponentially, we consider the function $e^{\alpha t}k^{(n)}$. Choosing an appropriate value of α allows us to use the maximum principle and obtain the bound for $k^{(n)}$. □

We can now notice that the bound on $\frac{\partial k}{\partial \theta}$ implies the equicontinuity of the family of functions $\{k(\theta, t) : t \in [0, T)\}$. Indeed, by the mean value theorem,

$$|k(\theta_1, t) - k(\theta_2, t)| \leq \left|\frac{\partial k}{\partial \theta}(c, t)\right| |\theta_1 - \theta_2| \leq M|\theta_1 - \theta_2|. \tag{3.14}$$

The uniform bound, M, does not depend on t, so the functions are in fact uniformly Lipschitz.

Likewise, the bound on $k^{(n)}$ implies the equicontinuity of the family of functions $\{k^{(n-1)}(\theta, t) : t \in [0, T)\}$. We can now prove Theorem 3.2.1.

Proof. Assuming that the area is not zero, Theorem 3.1.1 and Lemmas 3.2.1, 3.2.2 and 3.2.3 show that k and all its derivatives are bounded. As per the above discussion, this implies that for each n $\{k^{(n)}(\theta, t) : t \in [0, T)\}$ is an equicontinuous family of functions. Thus, by Arzelá-Ascoli Theorem, each family has a continuous limit $k^{(n)}(\theta, T)$ as $t \to T$. The convergence is uniform for each n. By the properties of uniform convergence, the derivative relations are maintained; i.e. $k^{(n+1)}(\theta, T)$ is the derivative of $k^{(n)}(\theta, T)$.

We thus have a C^∞ limit function $k(\theta, T)$. This function can be used as a new initial condition in the Cauchy problem (3.7) to extend the solution to some time $T + \epsilon$. We can continue to extend the solution in this way until the area becomes zero. □

We can draw a further notable fact from Theorem 3.2.1. We found in Proposition 2.2.5 that $\frac{\partial A}{\partial t} = -2\pi$. Hence, for any closed curve the curve shortening flow exists for only finite time! Phrased otherwise, the solutions converge in finite time. This is another qualitative contrast with the standard heat equation, for which solutions require infinite time to converge. In the next section, we shall characterize the limit sets.

3.3 Gage's Theorem: Characterizing the limit set

Having established the existence of the solution, the next step is to characterize the limit set for convex curves. This is the object of Gage's Theorem:

Theorem 3.3.1. (Gage's Theorem, [6]) *Let $X(u,t)$ be a family of C^2 strictly convex curves undergoing curve shortening flow. If the area vanishes in finite time; $\lim_{t\to T} A(t) = 0$, then the isoperimetric ratio tends to 4π:*

$$\lim_{t\to T} \frac{L(t)^2}{A(t)} = 4\pi. \tag{3.15}$$

Moreover, when the curves are rescaled such that they bound regions of constant area, these regions will converge in the Hausdorff sense to a circle.

Intuitively, the Hausdorff distance between two sets in $\mathbb{R}^2$ is the smallest amount of "inflation" required for each set to completely engulf the other. Formally:

Definition 3.3.1. *Let A and B be (non-empty) sets in $\mathbb{R}^2$. The Hausdorff distance between A and B is defined as:*

$$d_H(A,B) = \inf\{r > 0 : A \subseteq B_r \text{ and } B \subseteq A_r\}, \tag{3.16}$$

where

$$A_r = \{x \in \mathbb{R}^2 : d(x, A) \leq r\}. \tag{3.17}$$

It can be shown that the Hausdorff distance is a pseudometric on the space of subsets of $\mathbb{R}^2$ (and is in fact a true metric when considering only compact subsets). Convergence in the Hausdorff sense is then defined like convergence in metric spaces:

Definition 3.3.2. *A sequence of subsets $A_n \subset \mathbb{R}^2$ is said to converge in the Hausdorff sense to $A \subset \mathbb{R}^2$ if*

$$\lim_{n\to\infty} d_H(A_n, A) = 0. \tag{3.18}$$

3.3.1 Proving Gage's Theorem

The key step to proving Gage's Theorem is to show the existence of a non-negative functional F, defined on closed, convex and C^1 curves γ, which satisfies the inequality:

$$\pi \frac{L}{A} \leq (1 - F(\gamma)) \int_\gamma k^2 \, ds. \tag{3.19}$$

The functional must satisfy two additional properties:

- If $F(\gamma_i)$ converges to zero for a sequence of curves and the sequence of renormalized curves is uniformly bounded, then the regions bounded by the renormalized curves must converge to the unit circle in the Hausdorff sense.
- $F(\gamma) = 0$ if and only if γ is a circle.

The existence of this functional is the content of Gage's Isoperimetric Inequality. Assuming this result for now, the proof of Theorem 3.3.1 will follow using the properties of F. First, a lemma:

Lemma 3.3.1. *[6] For a family of curves $X(u,t)$ undergoing curve shortening flow, if $\lim_{t\to T} A(t) = 0$, then*

$$\liminf_{t\to T} L(t) \left(\int_{X(\cdot,t)} k^2 ds - \pi \frac{L(t)}{A(t)} \right) \leq 0. \tag{3.20}$$

Proof. We first compute the derivative of the isoperimetric ratio, using Propositions 2.2.4 and 2.2.5:

$$\frac{\partial}{\partial t}\left(\frac{L(t)^2}{A(t)}\right) = -2\frac{L(t)}{A(t)}\left(\int_{X(\cdot,t)} k^2 ds - \pi\frac{L(t)}{A(t)}\right). \tag{3.21}$$

Now, look in a neighborhood $(T-\delta, T]$ of T. For contradiction, assume that there is $\epsilon > 0$ such that, for every $t \in (T-\delta, T]$, we have

$$L\left(\int_X k^2 ds - \pi\frac{L}{A}\right) \geq \epsilon. \tag{3.22}$$

Then

$$\frac{\partial}{\partial t}\left(\frac{L(t)^2}{A(t)}\right) \leq -2\frac{\epsilon}{A(t)} \qquad \forall t \in (T-\delta, T]. \tag{3.23}$$

Integrating over $(T-\delta, t]$, using Proposition 2.2.5 yields:

$$\frac{L(t)^2}{A(t)} - \frac{L(T-\delta)^2}{A(T-\delta)} \leq \frac{\epsilon}{\pi}\Big(\ln(A(t)) - \ln(A(T-\delta))\Big). \tag{3.24}$$

Rearranging, this becomes

$$\frac{L(t)^2}{A(t)} \leq \frac{L(T-\delta)^2}{A(T-\delta)} + \frac{\epsilon}{\pi}\Big(\ln(A(t)) - \ln(A(T-\delta))\Big). \tag{3.25}$$

From the isoperimetric inequality, the left-hand side is bounded below for all times t by 4π. However, as t approaches T, the term $\ln(A(t))$ makes the right-hand side approach negative infinity. This is a contradiction.

Hence, for every δ and every ϵ there must be some $t \in (T-\delta, T]$ such that

$$L\left(\int_X k^2 ds - \pi\frac{L}{A}\right) < \epsilon, \tag{3.26}$$

which is the desired conclusion. □

We can now prove Gage's Theorem, Theorem 3.3.1.

Proof. The total curvature for a convex simple curve is 2π. Using Cauchy-Schwarz, we obtain

$$4\pi^2 = \left(\int k\, ds\right)^2 \leq \left(\int k^2\, ds\right)\left(\int 1\, ds\right) = L \int k^2\, ds. \tag{3.27}$$

Now bringing in our functional F, inequality (3.19) can be rearranged as

$$F(X)\left(\int k^2\, ds\right) \leq \int k^2\, ds - \pi\frac{L}{A}. \tag{3.28}$$

Multiplying both sides of (3.28) by L and using (3.27) gives the following inequality, in which the t-dependence is made explicit:

$$4\pi^2 F(X(\cdot,t)) \leq L(t)\left(\int_{X(\cdot,t)} k(t)^2\, ds - \pi\frac{L(t)}{A(t)}\right). \tag{3.29}$$

We can now apply Lemma 3.3.1 which tells us that the lim inf of the right-hand side is zero. Since the functional F is non-negative, the left-hand side also has a lim inf of zero. We thus have a sequence of times $\{t_i\}_i$ such that

$$\lim_{i\to\infty} F(X(\cdot,t_i)) = 0. \tag{3.30}$$

To draw the conclusion from the properties of the functional F, we must first show that the renormalized curves are uniformly bounded.

We tentatively define the renormalized curves to have constant area π by setting:

$$Y(\cdot,t) = \sqrt{\frac{\pi}{A(t)}}X(\cdot,t). \tag{3.31}$$

The issue in doing this rescaling is in the choice of origin, which must be chosen to lie inside all the curves $X(\cdot,t)$. However, for convex curves, the flow leads to a contraction of the curves. The family is thus nested: if $t < s$, then $X(\cdot,s) \subseteq X(\cdot,t)$. Hence, the intersection $\bigcap_{t\in[0,T]} X(\cdot,t)$ is non-empty and an origin can be chosen from this set.

Choosing the origin in this way causes no issues, since all the relevant properties of the family $X(\cdot, t)$ are translation invariant: curvature, normal vectors, length, area and, indeed, the flow itself.

To show boundedness of the family $Y(\cdot, t)$, we will need Bonnesen's inequality which states that for a simple closed curve, we have:

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A} (r_{out} - r_{in})^2 . \tag{3.32}$$

Here, r_{out} and r_{in} are the radii of the circumscribed and inscribed circles respectively. Noting that the radii are scaled by $\sqrt{\frac{\pi}{A(t)}}$ when going from X to Y we have:

$$\begin{aligned} \pi\Big(r_{out,\, Y(\cdot,t)} - r_{in,\, Y(\cdot,t)}\Big)^2 &= \frac{\pi^2}{A(t)}\Big(r_{out,\, X(\cdot,t)} - r_{in,\, X(\cdot,t)}\Big)^2 \\ &\leq \frac{L(t)^2}{A(t)} - 4\pi. \end{aligned} \tag{3.33}$$

On the other hand, we computed the derivative of the isoperimetric ratio in Lemma 3.3.1 (see equation (3.21)). Combining with equation (3.29), we find that the isoperimetric ratio must be decreasing:

$$\frac{\partial}{\partial t}\left(\frac{L(t)^2}{A(t)}\right) \leq -\frac{2}{A} 4\pi^2 F(X(\cdot, t)) \leq 0. \tag{3.34}$$

Hence,

$$\pi\Big(r_{out,\, Y(\cdot,t)} - r_{in,\, Y(\cdot,t)}\Big)^2 \leq \frac{L(0)^2}{A(0)} - 4\pi. \tag{3.35}$$

Moreover, since the curves $Y(\cdot, t)$ have constant area π, the inscribed circles for these curves must in turn have area at most π. Hence, the inner radii are always at most 1: $r_{in,\, Y(\cdot,t)} \leq 1$.

We thus deduce that there is a uniform upper bound R for the outer radii of the curves Y: $r_{out,\, Y(\cdot,t)} \leq R$, for every $t \in [0, T]$.

The origin was chosen such that it would lie inside the curves $Y(\cdot, t)$. It will thus also lie inside the circumscribed circles. The center of the circumscribed circle can therefore be at most distance R from the origin. Hence, the curve $Y(\cdot, t)$ must be contained in a circle of radius $2R$ and this holds for every time t.

In particular, the sequence $Y(\cdot, t_i)$ is uniformly bounded for the sequence t_i determined above. By the properties of the functional F, the sequence of regions bounded by $Y(\cdot, t_i)$ converges in the Hausdorff to the unit circle. It then follows from the continuity of L and A that the isoperimetric ratio along this subsequence of times, $\frac{L(t_i)^2}{A(t_i)}$, converges to 4π. In fact, since we showed the ratio is non-increasing, this convergent subsequence implies convergence for the entire sequence in time:

$$\lim_{t \to T} \frac{L(t)^2}{A(t)} = 4\pi. \tag{3.36}$$

Note that the isoperimetric ratio is invariant under scaling, so the above result holds equally for $X(\cdot, t)$ and $Y(\cdot, t)$. Finally, using equation 3.33 and taking $t \to T$, we deduce that

$$r_{out, Y(\cdot,t)} - r_{in, Y(\cdot,t)} \xrightarrow{t \to T} 0. \tag{3.37}$$

That is, the inscribed and circumscribed circles $Y(\cdot, t)$ are collapsing into each other; it follows that the family $Y(\cdot, t)$ is itself converging to a circle as $t \to T$. □

This concludes the proof of Gage's Theorem, under the assumption of the existence of the functional F. This is quite a strong assumption as F and its properties do a good deal of heavy lifting for the proof. The next section will deal with proving the existence of this functional.

3.3.2 Existence of Gage's Functional

Proposition 3.3.1. (Gage's Functional [6]) *There exists a non-negative functional F defined for all convex, closed, C^1 curves γ satisfying*

(i) $\pi \frac{L}{A} \leq (1 - F(\gamma)) \int_\gamma k^2 \, ds$.

(ii) *If $F(\gamma_i)$ converges to zero for a sequence of curves and the sequence of renormalized curves are uniformly bounded, then the regions bounded by the renormalized curves* *must converge to the unit circle in the Hausdorff sense.*

(iii) $F(\gamma) = 0$ *if and only if* γ *is a circle.*

Proof. First, the result will be proven for curves which are symmetric with respect to the origin. Then a symmetrization argument will be used to extend the result to all convex curves.

1. γ *is symmetric with respect to the origin*

For symmetric curves, the desired functional will be shown to take the form

$$E(\gamma) = 1 + \frac{\pi r_{in} r_{out}}{A} - \frac{2\pi(r_{out} + r_{in})}{L}. \tag{3.38}$$

We will show that E satisfies properties (ii) and (iii) of the proposition, along with the intermediate inequality

$$AL - \pi \int_0^L h(s)^2 \, ds \geq AL\, E(\gamma), \tag{3.39}$$

where h is the support function of γ. At the end we will show this intermediate inequality directly implies property (i) in the proposition.

To better understand where E comes from, we start by examining an equivalent formulation of Bonnesen's inequality. For simple closed curves, we have

$$rL - A - \pi r^2 \geq 0 \tag{3.40}$$

whenever $r \in [r_{in}, r_{out}]$. Viewed as a function of r, equation (3.40) is concave down. The graph will lie above the straight line joining the points at $r = r_{in}$ and $r = r_{out}$:

$$rL - A - \pi r^2 \geq (r - r_{in})\frac{r_{out}L - A - \pi r_{out}^2}{(r_{out} - r_{in})} + (r - r_{out})\frac{r_{in}L - A - \pi r_{in}^2}{(r_{out} - r_{in})} \geq 0 \tag{3.41}$$

for $r \in [r_{in}, r_{out}]$. Expanding and simplifying yields:

$$rL - A - \pi r^2 \geq rL - A - \pi r(r_{out} + r_{in}) + \pi r_{out} r_{in} \geq 0. \tag{3.42}$$

We will now use the assumption of symmetry; doing so by bringing in the support function h of γ. We recall that the support function, defined by $h(s) = \langle \gamma(s), -N(s) \rangle$ is the distance from the origin to the support line orthogonal to $N(s)$. Here, we are using the arc length parametrization for γ.

For a curve symmetric about the origin, the *width in the direction* N (the distance between the two parallel support lines orthogonal to N which sandwich γ) is twice the support function. Moreover, since γ is convex, the width in any direction is bounded by the diameters of the inscribed and circumscribed circles. This implies

$$r_{in} \leq h(s) \leq r_{out}. \tag{3.43}$$

Therefore, at every point of γ, the support function satisfies inequality (3.42):

$$h(s)L - A - \pi \left(h(s)\right)^2 \geq h(s)L - A - \pi\, h(s)(r_{out} + r_{in}) + \pi r_{out} r_{in} \geq 0. \tag{3.44}$$

To obtain the desired inequality (3.39), we need to integrate with respect to arc length. We use Lemma 2.2.2 and obtain:

$$AL - \pi \int_0^L h(s)^2\, ds \geq AL + \pi r_{in} r_{out} L - 2\pi A(r_{out} + r_{in}). \tag{3.45}$$

The functional $E(\gamma)$ is defined as in equation (3.38) such that the right side of equation (3.45) is $AL\, E(\gamma)$; replacing E and rearranging, the equation becomes

$$1 - E(\gamma) \geq \frac{\pi}{AL} \int_0^L h(s)^2\, ds. \tag{3.46}$$

This is equivalent to equation (3.39), which is what we desired to show.

We can now show that E is zero exactly when γ is a circle. From the definition of E (equation (3.38)) it is clear that if γ is a circle, that is $r_{in} = r_{out} = r$, then $E(\gamma) = 0$.

Suppose now that $E(\gamma) = 0$. The right-hand side of equation (3.45) is then zero. Since this was the result of integrating the right-hand side of equation (3.44) and the latter is non-negative, it follows that this latter is always zero. Recall that equation (3.44) is simply the result of evaluating the line segment in (3.41) at $r = h$. Since this line is non-negative, there are two possibilities. Either the segment is identically zero, or one of the end-points is zero, with $r = h$ being that endpoint.

In the first possibility, this implies that r_{in} and r_{out} are the roots of the quadratic in Bonnesen's inequality, equation (3.40). This is only possible if γ is a circle (see [16])

In the second possibility, h is a root of equation (3.40) and is equal to either r_{in} or r_{out}. Thus, h must be a constant, which from Lemma 2.2.2 we find to be $2A/L$. Replacing the first instance of h in the right expression of equation (3.44), we have

$$A - \pi\, h(r_{out} + r_{in}) + \pi r_{out} r_{in} = 0. \tag{3.47}$$

Swapping $R - in$ or r_{out} for h (whichever it equals) we find $A = \pi\, h^2$. Replacing with the constant value of h, we find that $A = 4\pi A^2/L^2$. Rearranging, this is the equality case of the isoperimetric inequality. Hence, γ must be a circle.

Finally, we can show that E satisfies the property (ii). Let γ_i be a sequence of closed convex curves symmetric with respect to the origin. Let α_i be the renormalized curves, defined by

$$\alpha_i = \sqrt{\frac{\pi}{A_i}}\gamma_i. \tag{3.48}$$

We suppose that the curves α_i are uniformly bounded, so there is some R such that all the

α_i are contained in the ball of radius R, and also we suppose that

$$\lim_{i\to\infty} E(\gamma_i) = 0. \tag{3.49}$$

From equation (3.38), we see that E is invariant under scaling. So, we also have

$$\lim_{i\to\infty} E(\alpha_i) = 0. \tag{3.50}$$

We now apply the Blaschke selection theorem, which states that every sequence of uniformly bounded convex sets has a subsequence converging in the Hausdorff sense to a convex limit. This yields a subsequence α_{i_j} which converges to some convex curve α. The functional E is defined in terms of L, A, r_{in} and r_{out}, all continuous quantities, so E is also continuous. It follows that

$$E(\alpha) = \lim_{j\to\infty} E(\alpha_{i_j}) = 0. \tag{3.51}$$

We have already established that $E(\alpha) = 0$ implies that α must be a circle. The same is also true for any other convergent subsequence of α_i. Hence, the whole sequence must in fact converge to a circle.

2. *General case*

We will prove the result in general for any convex closed γ. The strategy will be to symmetrize γ; we wish to find a line segment cutting γ into two regions of equal area. Moreover, we require that the tangents of γ at the endpoints of the segment be parallel.

To see that this is possible, note that for every point $\gamma(s)$ along the curve, there is a unique point $\gamma(\tilde{s})$ so that the segment between these points divides the area in two equal parts. Thus, we can define the function

$$f(s) = \langle T(s) \times T(\tilde{s}), \hat{z} \rangle, \tag{3.52}$$

where $T(s)$ is the tangent vector at $\gamma(s)$, (which we view as lying in $\mathbb{R}^3$ for the cross product

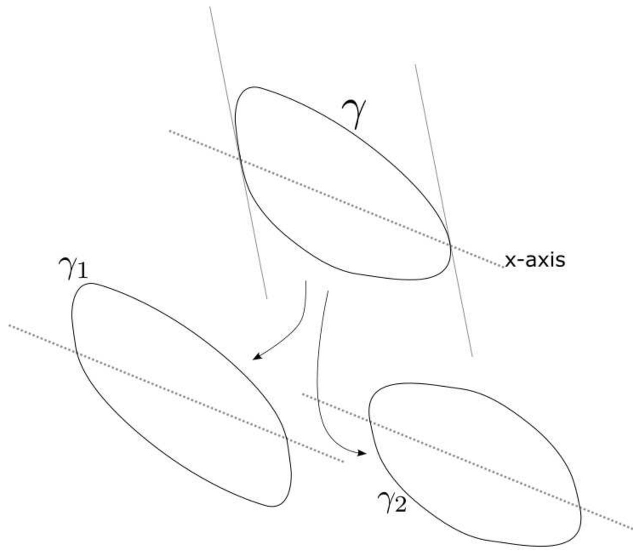

Figure 3.1: The curve γ is cut at parallel tangents into halves having equal areas. The curve halves γ_1 and γ_2 are reflected to form symmetric curves.

computation) and $\hat{z}$ is the unit vector normal to the xy plane. Note that since γ is C^1, f will be continuous. Moreover, $f(s) = -f(\tilde{s})$. An application of the intermediate value property gives a value s^* for which $f(s^*) = 0$. Hence, at this point, $T(s^*)$ and $T(\tilde{s^*})$ must be parallel by properties of the cross product.

With our line segment in hand, choose coordinates so that the segment lies on the x-axis, with the center of the segment at the origin. Denote the halves of γ lying above and below the x-axis by γ_1 and γ_2, respectively. Consider next the reflections of each half through the origin, denote these $-\gamma_1$ and $-\gamma_2$. We thus have two closed convex curves, $\gamma_1 \cup (-\gamma_1)$ and $\gamma_2 \cup (-\gamma_2)$ which are symmetric about the origin (see Figure 3.1).

Here we see why we required the tangents to parallel at the endpoint of our segment; this ensures that the curves $\gamma_1 \cup (-\gamma_1)$ and $\gamma_2 \cup (-\gamma_2)$ are C^1. Also, of course, the regions enclosed by both the curves will be equal to A, the area enclosed by γ. As for the length, denote by L_1 and L_2 the lengths of γ_1 and γ_2. In this way $L_1 + L_2 = L$, the length of γ.

Since we have symmetry, we know that the functional E (equation (3.38)) satisfies the properties of the proposition when applied to $\gamma_1 \cup (-\gamma_1)$ and $\gamma_2 \cup (-\gamma_2)$. In particular, equation (3.39) holds:

$$2AL_n - \pi \int_{\gamma_n \cup (-\gamma_n)} h(s)^2 \, ds \geq 2AL_n \, E(\gamma_n \cup (-\gamma_n)), \tag{3.53}$$

for $n = 1, 2$.

Adding the inequalities for each curve, we obtain

$$AL - \pi \int_\gamma h(s)^2 \, ds \geq AL \left(\frac{L_1}{L} \, E(\gamma_1 \cup (-\gamma_1)) + \frac{L_2}{L} \, E(\gamma_2 \cup (-\gamma_2)) \right). \tag{3.54}$$

(We have divided by two, since by symmetry the integrals over γ_i and their reflection are the same.)

At this stage, we might hope to simply set $F(\gamma)$ as the quantity in parenthesis on the right-hand side. The functional F would then satisfy the intermediate inequality (3.39). However, the issue is that F may not be well-defined; the quantity in parenthesis may depend on the choice of line segment used to cut γ. To get around this, define $F(\gamma)$ as a supremum:

$$F(\gamma) = \sup \left(\frac{L_1}{L} \, E(\gamma_1 \cup (-\gamma_1)) + \frac{L_2}{L} \, E(\gamma_2 \cup (-\gamma_2)) \right), \tag{3.55}$$

where the supremum is taken over all choices of line segment described as above. Note that, in general, we therefore cannot obtain an explicit formula for F.

We thus have that F satisfies inequality (3.39). We can rearrange this as:

$$1 - F(\gamma) \geq \frac{\pi}{AL} \int_\gamma h(s)^2 \, ds. \tag{3.56}$$

We can now show that this immediately implies the property (i) of the proposition.

Using Lemma 2.2.2 and applying Cauchy-Schwarz:

$$L^2 = \left(\int_0^L hk \, ds \right)^2 \leq \int_0^L h^2 \, ds \int_0^L k^2 \, ds. \tag{3.57}$$

By multiplying equation (3.56) with $\int k^2$, and using equation (3.57), we obtain:

$$\big(1 - F(\gamma)\big) \int_\gamma k^2 \, ds \geq \pi \frac{L}{A}. \tag{3.58}$$

This is property (i).

We would now like to show that property (ii) holds for F. We consider a sequence of curves γ_i and their renormalizations $\alpha_i = \sqrt{\frac{\pi}{A_i}}\gamma_i$. We are supposing that the curves α_i are uniformly bounded; so there is some R such that the ball of radius R contains all the curves α_i.

We can first note that since E is invariant under scaling, so is F. So $F(\gamma_i) = F(\alpha_i)$.

For each γ_i, we can consider a pair of symmetrized curves $\big(\gamma_1 \cup (-\gamma_1)\big)_i$ and $\big(\gamma_2 \cup (-\gamma_2)\big)_i$ produced as above. The choice of these curves will not matter (in the case where there is more than one way to cut γ). These in turn give rise to symmetrized curves for the renormalizations: $\big(\alpha_1 \cup (-\alpha_1)\big)_i$ and $\big(\alpha_2 \cup (-\alpha_2)\big)_i$.

From the definition of F (equation (3.55)), and since E is non-negative, we have

$$F(\alpha_i) \geq \frac{\frac{1}{2} L\Big(\big(\alpha_n \cup (-\alpha_n)\big)_i\Big)}{L(\alpha_i)} E\Big(\big(\alpha_n \cup (-\alpha_n)\big)_i\Big), \tag{3.59}$$

for $n = 1, 2$.

Knowing that the areas of the renormalized curves are π the isoperimetric inequality gives that the lengths of the curves $\big(\alpha_n \cup (-\alpha_n)\big)_i$ $(n = 1, 2)$ are bounded below by 2π.

Also, since α_i are convex and contained in the ball of radius R, we obtain an upper bound on the length: $L(\alpha_i) \leq 2\pi R$. Hence,

$$F(\alpha_i) \geq \frac{1}{2R} E\Big(\big(\alpha_n \cup (-\alpha_n)\big)_i\Big), \tag{3.60}$$

for $n = 1, 2$.

If we assume, as in the hypothesis of property (ii), that $F(\alpha_i) \to 0$, then we also have

$E\Big(\big(\alpha_n \cup (-\alpha_n)\big)_i\Big) \to 0$ for $n = 1, 2$. We have already shown that E satisfies property (ii) when applied to symmetric curves; so we have that the sequences $\big(\alpha_1 \cup (-\alpha_1)\big)_i$ and $\big(\alpha_2 \cup (-\alpha_2)\big)_i$ converge to the unit circle as $i \to \infty$. Equivalently, the inner and outer radii of these symmetrized curves are converging to each other (and to 1).

However, we also have that

$$r_{in}(\alpha_i) \geq \min\Big(r_{in}\big(\alpha_1 \cup (-\alpha_1)\big)_i\,,\, r_{in}\big(\alpha_2 \cup (-\alpha_2)\big)_i\Big) \tag{3.61}$$

and

$$r_{out}(\alpha_i) \leq \max\Big(r_{out}\big(\alpha_1 \cup (-\alpha_1)\big)_i\,,\, r_{out}\big(\alpha_2 \cup (-\alpha_2)\big)_i\Big). \tag{3.62}$$

Thus, the inner and outer radii of the sequence α_i are converging to each other (and to 1). We conclude that the α_i are indeed converging to the unit circle. This proves property (ii).

It only remains to show property (iii). First, if γ is a circle, then regardless of how we cut γ, the symmetrized curves $\gamma_1 \cup (-\gamma_1)$ and $\gamma_2 \cup (-\gamma_2)$ will also be circles. Thus, E applied to the symmetrized curves will always be zero. So, in turn, $F(\gamma) = 0$.

Conversely, suppose $F(\gamma) = 0$. It follows that $E(\gamma_n \cup (-\gamma_n)) = 0$ $(n = 1, 2)$, for any choice of symmetrized curves. Thus, the curves $\gamma_1 \cup (-\gamma_1)$ and $\gamma_2 \cup (-\gamma_2)$ must be circles, since E was shown to satisfy property (iii) for symmetric curves. This means the inner and outer radii for the symmetrized curves are equal. Applying inequalities (3.61) and (3.62) to γ, we can conclude that γ also has equal inner and outer radii and must then also be a circle. □

3.3.3 C^∞ convergence of the curvature

Having proven Gage's theorem, we have established the continuous convergence of curves to a point. In fact, it is possible to prove that the convergence is of class C^∞. First, using geometric arguments, it is possible to prove that the rescaled curvature convergence uniformly to 1, giving a C^2 type convergence.

Theorem 3.3.2. *[7] The rescaled curvature* $k(\theta, t)\sqrt{2T - 2t}$ *converges uniformly to 1 as* $t \to T$.

See [7] for the proof.

Even further it is possible to prove a C^∞ convergence, that is, all the derivatives of k converge to 0 uniformly.

Theorem 3.3.3. *[7] All derivative of* k *converge uniformly to 0 as* $t \to T$.

For the original proof, see [7]. For more details on the explicit calculations, see also [1]. Roughly, the proof uses similar techniques as employed in Section 3.2. However, these methods must applied to the rescaled curvature. The computations end up being far more tedious.

Chapter 4

Non-Convex Curves

4.1 Grayson's Theorem: Non-convex curves become convex

Theorem 4.1.1. (Grayson's Theorem, [9]) *If $X_0 : [a,b] \longrightarrow \mathbb{R}^2$ is a closed, smooth and simple curve, then there exists a family of curves $X(\cdot,t)$ undergoing curve shortening flow, initialized by X_0. The flow exists for all $t \in [0,T]$ and the curves converge to a point as $t \to T$. The normalized curves converge to a circle in C^∞ norm.*

We already know the result holds for convex curves by Gage's theorem (Theorem. 3.3.1). What is needed to be proven is that all curves eventually become convex.

The strategy will be to first examine length-normalized curves. The most important tool of the proof, an inequality by Andrews and Bryan, will give a lower bound for the euclidean distance between points on the curve: $d \geq f(\ell, \tau - b)$; given in terms of the arc length distance along the curve, ℓ and a carefully chosen function: $f(x,\tau) = 2e^{\tau} \arctan\left(e^{-\tau} \sin\left(x/2\right)\right)$.

The importance of this function f is revealed by defining an auxiliary function $g = \inf\left(e^{\tau} : d \geq f(\ell, -\tau)\right)$. As inputs, g has two distinct points of the curve. However, in the limit as the points become the same, g will equal an expression in terms of the curvature. This will be used to show that the renormalized flow exists for all time and that the curvature converges uniformly to 1.

4.1.1 Andrews and Bryan Inequality

The first step is to renormalize the curves to produce curves of constant length. However, as it will turn out, we will want to not only rescale the curves, but also to rescale the time variable. For a family of curves $X(\cdot, t)$, $t \in [0, T]$ with length $L(t) = L(X(\cdot, t))$, define as a new time variable

$$\tau(t) = \int_0^t \left(\frac{2\pi}{L(s)}\right)^2 ds. \tag{4.1}$$

The range of τ is $[0, \tau(T))$; τ is monotone and thus invertible on this interval. We can now define the rescaled curves:

$$Y(\cdot, \tau) = \frac{2\pi}{L(t)} X(\cdot, t), \tag{4.2}$$

where $t = t(\tau)$. We now have curves Y with constant length 2π. We will denote the curvature function of these curves by $\kappa = \kappa(\cdot, \tau)$ (keeping in mind that the curvature of the curves X is k).

If we assume that the original family X satisfies the curve shortening equation, then the family Y will also satisfy a similar flow equation.

Lemma 4.1.1. *The length-normalized family of curves $Y(\cdot, \tau)$ satisfies*

$$\frac{\partial Y}{\partial \tau} = \kappa N + \overline{\kappa^2} Y, \tag{4.3}$$

where $\overline{\kappa^2}$ is the average squared curvature:

$$\overline{\kappa^2} = \frac{1}{2\pi} \int_Y \kappa^2 \, ds. \tag{4.4}$$

Proof. □

We can now state the key inequality:

Theorem 4.1.2. (Andrews and Bryan Inequality, [1])

If $Y(\cdot, \tau) : [a, b] \times [0, \tau(T)) \to \mathbb{R}^2$ is a family of curves satisfying the normalized flow, then there exists a constant $b \in \mathbb{R}$ such that for every u_1, $u_2 \in [a.b]$ and $\tau \in [0, \tau(T))$ we have

$$d(u_1, u_2, \tau) \geq f(\ell(u_1, u_2, \tau), \tau - b) \tag{4.5}$$

with

$$f(x, \tau) = 2e^{\tau} \arctan\left(e^{-\tau} \sin\left(\frac{x}{2}\right)\right). \tag{4.6}$$

Here, d denotes the euclidean distance between points on the curve:

$$d(u_1, u_2, \tau) = \|Y(u_1, \tau) - Y(u_2, \tau)\| \tag{4.7}$$

and ℓ denotes the length of the curve segment between the two points:

$$\ell(u_1, u_2, \tau) = \int_{u_1}^{u_2} ds. \tag{4.8}$$

The proof of this theorem is somewhat long and involved. However, one of the first steps is proving a lemma which helps elucidate the role of the function f. The result of this lemma will also be needed to prove Grayson's theorem, Theorem 4.1.1.

We define an auxiliary function $g(u_1, u_2, \tau)$ for $u_1 \neq u_2$ by:

$$g(u_1, u_2, \tau) = \inf\{e^s : d(u_1, u_2, \tau) \geq f(\ell(u_1, u_2, \tau), -s)\}. \tag{4.9}$$

Equivalently, we have that

$$-\ln(g(u_1, u_2, \tau)) = \sup\{s : d(u_1, u_2, \tau) \geq f(\ell(u_1, u_2, \tau), s)\}. \tag{4.10}$$

We then have the following result:

Lemma 4.1.2. *The function g can be extended continuously to a function $g : [a,b]^2 \times [0, \tau(T)) \to \mathbb{R}$ by defining on the diagonal*

$$g(u,u,\tau) = \sqrt{\frac{\max\{\kappa^2 - 1, 0\}}{2}}. \tag{4.11}$$

As a consequence, at each fixed time τ, g attains a finite maximum on $[a,b]^2$.

4.1.2 Proving Grayson's Theorem

We will need a few lemmas.

Lemma 4.1.3. *Let $f : [0,\infty) \to [0.\infty)$ be an increasing C^1 function with $f(0) = 0$. Let $h : \gamma \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function defined on a curve γ. Define the sets*

$$A_h(r) = \lambda(\{x \in \gamma : |h(x)| \geq r\}) \tag{4.12}$$

using the Lebesgue measure λ. Then

$$\int_\gamma f(|h(x)|)\, dx = \int_0^\infty f'(r) A_h(r)\, dr. \tag{4.13}$$

Lemma 4.1.4. *(Gagliardo-Nirenberg Type Inequality) If $h : \gamma \subset \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function defined on a closed regular curve γ, then*

$$(\max_\gamma |h|)^4 \leq 3 \int_\gamma (h')^2\, ds \int_\gamma h^2\, ds. \tag{4.14}$$

We next need an analogous lemma for the length-normalized flow.

Lemma 4.1.5. *If $Y(\cdot,\tau) : [a,b] \times [0,\tau(T)) \to \mathbb{R}^2$ is a family of curves satisfying the normal-*

ized flow, then the following equations hold:

$$\frac{\partial}{\partial \tau}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial \tau} + (\kappa^2 - \overline{k^2})\frac{\partial}{\partial s} \tag{4.15}$$

$$\frac{\partial \kappa}{\partial \tau} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3 - \overline{\kappa^2}\kappa. \tag{4.16}$$

We will also need a version of the Wirtinger's Inequality

Lemma 4.1.6. *If $f : [a,b] \to \mathbb{R}$ is a C^1 function with $f(a) = f(b) = 0$, then*

$$\int_a^b (f(x))^2\, dx \le \frac{(b-a)^2}{\pi^2} \int_a^b (f'(x))^2\, dx. \tag{4.17}$$

With these lemmas, we can now prove Grayson's theorem, Theorem 4.1.1.

Proof. We again consider the function $g(u_1, u_2, \tau)$ as defined in equation (4.9) and extend it continously by Lemma 4.1.2. From its definition, the function g satisfies

$$-\ln(g(u_1, u_2, \tau)) = \sup\{s : d(u_1, u_2, \tau) \ge f(\ell(u_1, u_2, \tau), s)\}. \tag{4.18}$$

The function f is continuous in its second argument, moreover, by Lemma (4.1.2) g is bounded, so the sup is attained and we have equality in this case:

$$d(u_1, u_2, \tau) = f(\ell(u_1, u_2, \tau), -\ln(g(u_1, u_2, \tau))). \tag{4.19}$$

Using now the Andrews and Bryan inequality (Theorem (4.1.2)), we know there exists a constant $b \in \mathbb{R}$ such that

$$f(\ell(u_1, u_2, \tau), -\ln(g(u_1, u_2, \tau))) \ge f(\ell(u_1, u_2, \tau), \tau - b))) \tag{4.20}$$

for every point (u_1, u_2, τ).

Since f is increasing in its second argument, we have

$$-\ln(g(u_1, u_2, \tau)) \geq \tau - b. \tag{4.21}$$

Equivalently:

$$g(u_1, u_2, \tau) \leq e^{b-\tau}. \tag{4.22}$$

Taking now the limit as $u_1 \to u_2$, and using Lemma 4.1.2, we have

$$\sqrt{\frac{\max\{\kappa(u,\tau)^2 - 1, 0\}}{2}} = g(u, u, \tau) \leq e^{b-\tau}. \tag{4.23}$$

Since this is true for every u, by solving for κ and taking sup, we find:

$$\sup_u \kappa(u, \tau)^2 \leq 1 + 2e^{2(b-\tau)}. \tag{4.24}$$

We can use this result to show that the final time of the length renormalized flow, $\tau(T)$, is in fact infinite. We recall that the renormalized curves were defined as

$$Y(\cdot, \tau) = \frac{2\pi}{L(t)} X(\cdot, t). \tag{4.25}$$

We can write $X(\cdot, t) = \frac{L(t)}{2\pi} Y(\cdot, \tau)$. We seek to make the τ dependence on the left hand side explicit, so we compose through the inverse function $t = t(\tau)$. Let $\lambda(\tau)$ denote the scalar function of τ given by $\lambda(\tau) = \frac{L(t(\tau))}{2\pi}$.

Note that from the way τ is defined, equation (4.1), $\frac{d\tau}{dt} = \left(\frac{2\pi}{L(t)}\right)^2$. Hence, $\frac{dt}{d\tau} = \left(\frac{L(t)}{2\pi}\right)^2 = \lambda(\tau)^2$. We thus have $X(\cdot, t) = \lambda(\tau) Y(\cdot, \tau)$. We can find an expression for λ since we know that X satisfies the curve shortening flow equation:

$$\frac{\partial X}{\partial t} = \frac{d\lambda}{dt} Y + \lambda \frac{\partial Y}{\partial t} \tag{4.26}$$
$$= \frac{d\lambda}{dt}\frac{d\tau}{dt} Y + \lambda \frac{\partial Y}{\partial \tau}\frac{d\tau}{dt} \tag{4.27}$$
$$= \lambda'(\tau)\left(\frac{2\pi}{L(t)}\right)^2 Y + \lambda\left(\kappa N + \overline{\kappa^2} Y\right)\left(\frac{2\pi}{L(t)}\right)^2 \qquad \text{using Lemma 4.1.1} \tag{4.28}$$
$$= kN + \left(\frac{2\pi}{L(t)}\right)^2 \left(\lambda'(\tau) + \lambda\overline{\kappa^2}\right) Y. \tag{4.29}$$

We have used that $\kappa(\cdot,\tau) = \lambda(\tau)k(\cdot,t)$, following from X and Y being rescalings of each other, along with $\lambda(\tau) = \frac{L(t(\tau))}{2\pi}$. As X satisfies the curve shortening flow, we have $\lambda'(\tau) + \lambda\overline{\kappa^2} = 0$. Using also the initial condition $\lambda(0) = \frac{L(X_0)}{2\pi}$, we obtain

$$\lambda(\tau) = \frac{L(X_0)}{2\pi} \exp\left(-\int_0^\tau \overline{\kappa^2}(s)\, ds\right). \tag{4.30}$$

Using this expression for λ, the relation $\kappa(\cdot,\tau) = \lambda(\tau)k(\cdot,t)$ and equation 4.24, we compute:

$$k(\cdot,t)^2 = \frac{1}{\lambda(\tau)^2}\kappa(\cdot,\tau)^2 \tag{4.31}$$
$$\leq \frac{4\pi^2}{L(X_0)^2} \exp\left(2\int_0^\tau \overline{\kappa^2}(s)\, ds\right)\left(1 + 2e^{2(b-\tau)}\right) \tag{4.32}$$
$$\leq \frac{4\pi^2}{L(X_0)^2} \exp\left(2\int_0^{\tau(T)} \overline{\kappa^2}(s)\, ds\right)\left(1 + 2e^{2(b)}\right). \tag{4.33}$$

If $\tau(T)$ is assumed to be finite, the above inequality would imply that the curvature k is bounded uniformly in time t. We know this does not happen; as $t \to T$, the area goes to 0 and the curvature cannot remain bounded. Thus, we must have $\tau(T) = \infty$; the length renormalized flow exists for all time τ.

We now seek to show that κ converges uniformly to 1; equivalently, that $\kappa - 1$ converges uniformly to 0.

To do so, apply Lemma 4.1.4 to $h = \kappa - 1$:

$$(\max_Y |\kappa - 1|)^4 \leq 3 \int_Y (\kappa')^2 \, ds \int_Y (\kappa - 1)^2 \, ds. \tag{4.34}$$

Note that the maximum is taken over points of the curve Y and s is the arc length parameter for Y. We will show that the left hand side of the inequality converges to 0 as $\tau \to \infty$.

The time decay is obtained from the second integral using equation (4.24):

$$\int_Y (\kappa - 1)^2 \, ds = \int_0^{2\pi} \kappa^2 - 2\kappa + 1 \, ds \tag{4.35}$$

$$= \int_0^{2\pi} \kappa^2 - 1 \, ds \tag{4.36}$$

$$\leq 4\pi e^{2(b-\tau)}, \tag{4.37}$$

where we have used that $\int_0^{2\pi} \kappa \, ds = 2\pi$.

We thus only need to show that $\int_Y (\kappa')^2 \, ds$ is bounded for all τ.

We do so using the result of the following differential inequality: if f satisfies

$$\frac{df}{d\tau} \leq -\alpha f + C \tag{4.38}$$

for strictly positive constants α and C, then f will satisfy

$$f(\tau) \leq \frac{C}{\alpha} + (f(0) - \frac{C}{\alpha})e^{-\alpha\tau}. \tag{4.39}$$

We take as our function $f(\tau) = (\kappa')^2 + B\kappa^2$ for a constant B. Using an integration by parts computation, one can show that f will satisfy the differential inequality for all τ sufficiently large, after fixing the constant B to be large enough.

Thus, $\int_Y (\kappa')^2 \, ds \leq \int_Y (\kappa')^2 + B\kappa^2 \, ds$ is bounded, as required.

Hence,

$$\lim_{\tau \to \infty} \max_{Y} |\kappa - 1| = 0. \tag{4.40}$$

That is, the curvature κ converges uniformly to 1. This implies that after some time τ, the length renormalized curves Y become convex. This means in turn that after some time t the original curves X become convex. Gage's Theorem takes over from this point and yields the desired convergence for the family X. □

Chapter 5

Curve Shortening on Surfaces

The natural generalization of curve shortening on the plane is the case of evolving curves on surfaces. The first difficulty when working on an arbitrary surface is to wade through the differential machinery associated to an abstract non-flat structure.

5.1 Differential Machinery and Computations

Let (M, g) be a Riemannian surface. We recall that the **Riemannian metric** g amounts to a smoothly-varying choice of inner-product on each tangent space T_pM of M. We consider closed smooth curves $X(u)$ on M, $X : [a, b] \to M$. A family of curves $X(u, t)$, is said to be under-going curve shortening flow if

$$\frac{\partial X}{\partial t} = kN. \tag{5.1}$$

The equation looks the same as in the case of the plane, however each of the terms may now require some elaboration.

The term $\frac{\partial X}{\partial t}$ denotes a vector field along the curves $X(u, t)$. Specifically, at each point $p = X(u, t)$, the tangent vector $\frac{\partial X}{\partial t}|_p \in T_pM$ is the vector that acts on smooth functions

$f : M \to \mathbb{R}$ according to:

$$\left.\frac{\partial X}{\partial t}\right|_p (f) = \frac{\partial (f \circ X)}{\partial t}(u,t). \tag{5.2}$$

Note that $f \circ X$ is a function from $\mathbb{R}$ to $\mathbb{R}$ so the derivative is taken in the usual sense. For ease of notation, we will sometimes denote $\frac{\partial X}{\partial t}$ by simply $\frac{\partial}{\partial t}$ depending on whether we wish to emphasize the geometric vector nature or the differential operator nature of the object.

Defining the curvature k and the normal vector N for the curve $X(\cdot, t)$ is done in a similar way as in the plane. Along the curve we have a well-defined notion of tangent vector: the vector $\frac{\partial X}{\partial u}$, which is defined in the same way as above. For any smooth $f : M \to \mathbb{R}$, $\frac{\partial X}{\partial u}|_p$ is the vector which acts according to

$$\left.\frac{\partial X}{\partial u}\right|_p (f) = \frac{\partial (f \circ X)}{\partial u}(u,t). \tag{5.3}$$

Again for the ease of notation, we will sometimes denote $\frac{\partial X}{\partial u}$ by simply $\frac{\partial}{\partial u}$.

Since M possesses a Riemannian structure, we can define the unit tangent vector T. The velocity of the curve (with respect to the parametrization in u) is given by $v = \left\| \frac{\partial X}{\partial u} \right\|_g$. The unit tangent vector is then given by $T = \frac{1}{v}\frac{\partial X}{\partial u}$, and T is the tangent vector of the curve when the curve is reparametrized by the arc length parameter s. For this reason, we can also denote T by $\frac{\partial}{\partial s}$.

On every tangent space along the curves, we can now choose the normal vector N, to be the inward-facing unit vector orthogonal to T, where inward is with respect to the domain on M bounded by the curve. Thus, T and N form a frame of basis vectors, tangent to the surface, along the curve. To define k, the curvature of the curve, we now need to recall the notion of a connection.

We recall that a **connection**, ∇, generalizes the notion of directional derivative of vector fields in $\mathbb{R}^n$ to the manifold setting. A connection takes as input two vector fields and outputs a new vector field, and satisfies the following properties:

a. Linearity over real functions on M in the "direction" vector:

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y; \tag{5.4}$$

b. Linearity over $\mathbb{R}$ in the argument vector:

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2; \tag{5.5}$$

c. A product rule, for $f : M \to \mathbb{R}$

$$\nabla(fX) = X(f)Y + f \nabla_X Y. \tag{5.6}$$

Note that, as in the first term, the meaning of applying the connection to a scalar function is to take the derivative with respect to the direction vector: $\nabla_X f = X(f)$.

In the Riemannian setting, there is a special connection, called the **Levi-Civita connection**, which satisfies two additional properties. First, this connection is said to be "compatible with the metric". This amounts to obeying a product rule with the metric:

$$\nabla_X \langle Y, Z \rangle_g = \langle \nabla_X Y, Z \rangle_g + \langle Y, \nabla_X Z \rangle_g. \tag{5.7}$$

Secondly, the Levi-Civita connection is said to be "torsion-free", meaning

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{5.8}$$

where the latter is the **Lie bracket** of two vector fields:

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \text{ for all } f \in C^\infty(M),$$

smooth functions on M.

One more comment about the notation: we will denote the covariant derivatives along the vector fields $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial s}$ by simply ∇_t, ∇_u and ∇_s (instead of $\nabla_{\frac{\partial}{\partial t}}$, etc.).

With the connection in hand, we can now properly define the curvature k of a curve and even establish the Frenet equations, in much the same way as in the plane.

We can first note that any vector field of constant norm must be orthogonal to its derivative. Indeed taking for instance the unit tangent vector, which satisfies $\langle T, T\rangle_g = 1$, we have

$$\begin{aligned}\nabla_T \langle T, T\rangle_g &= \langle \nabla_T T, T\rangle_g + \langle T, \nabla_T T\rangle_g \\ &= 2\langle \nabla_T T, T\rangle_g \\ &= 0.\end{aligned}$$

Hence, $\nabla_T T$ is orthogonal to T at every point; $\nabla_T T$ must thus point in the direction of N. This implies the existence of a smooth function k such that $\nabla_T T = kN$.

Note that by the same argument $\nabla_T N$ must point in the direction of T. Also since $\langle T, N\rangle_g = 0$, we have

$$\nabla_T \langle T, N\rangle_g = \langle \nabla_T T, N\rangle_g + \langle T, \nabla_T N\rangle_g = 0. \tag{5.9}$$

This implies $\nabla_T N = -kT$. We have thus obtained the Frenet equations:

$$\begin{cases} \nabla_T T = kN \\ \nabla_T N = -kT. \end{cases} \tag{5.10}$$

We also note the following relation between covariant derivatives:

$$\nabla_s = \frac{1}{v}\nabla_u. \tag{5.11}$$

With the differential machinery in hand, we can now turn our attention back to the curve shortening problem. The initial computations are quite similar to those in the plane. For instance, we can prove results analogous to Lemma 2.2.1 and Proposition 2.2.1.

Proposition 5.1.1. *Along any smooth curve undergoing the curve shortening evolution on a surface, the following evolution equations hold:*

$$\frac{\partial v}{\partial t} = -k^2 v, \tag{5.12}$$

$$\left[\frac{\partial}{\partial t}, T\right] = k^2 T. \tag{5.13}$$

Proof. Firstly, since the parameters u and t are independent, we have that $\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right] = 0$. Indeed for any $f : M \to \mathbb{R}$, we have

$$\begin{aligned} \frac{\partial}{\partial u}\frac{\partial}{\partial t}(f) &= \frac{\partial}{\partial u}\left(\frac{\partial(f \circ X)}{\partial t}\right) = \frac{\partial(f \circ X)}{\partial u \partial t} = \frac{\partial(f \circ X)}{\partial t \partial u} \\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial u}(f) \end{aligned} \tag{5.14}$$

Using the torsion-free property of the connection this implies that

$$\nabla_u \frac{\partial}{\partial t} = \nabla_t \frac{\partial}{\partial u}. \tag{5.15}$$

Multiplying through by $\frac{1}{v}$ we have:

$$\frac{1}{v}\nabla_t \frac{\partial}{\partial u} = \frac{1}{v}\nabla_u \frac{\partial}{\partial t} = \nabla_T \frac{\partial X}{\partial t} = \nabla_T k N. \tag{5.16}$$

Knowing this, we have:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2v}\frac{\partial v^2}{\partial t} = \frac{1}{2v}\nabla_t \left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial u} \right\rangle_g = \frac{1}{v}\left\langle \nabla_t \frac{\partial X}{\partial u}, \frac{\partial X}{\partial u} \right\rangle_g \\ &= \left\langle \nabla_T k N, \frac{\partial X}{\partial u} \right\rangle_g \\ &= \left\langle \nabla_T k N, v T \right\rangle_g . \end{aligned} \tag{5.17}$$

Using the chain rule for connections and the Frenet equations:

$$\begin{aligned} \nabla_T kN &= T(k)N + k\nabla_T N \\ &= T(k)N - k^2 T. \end{aligned} \tag{5.18}$$

Hence, from the orthogonality of T and N, we obtain $\frac{\partial v}{\partial t} = -k^2 v$. To prove the second part, we again use the properties of the connection:

$$\begin{aligned} \left[\frac{\partial}{\partial t}, T\right] &= \nabla_t T - \nabla_T \frac{\partial}{\partial t} = \nabla_t \frac{1}{v}\frac{\partial}{\partial u} - \frac{1}{v}\nabla_u \frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\frac{\partial}{\partial u} + \frac{1}{v}\nabla_t \frac{\partial}{\partial u} - \frac{1}{v}\nabla_u \frac{\partial}{\partial t} \\ &= -\frac{1}{v^2}\frac{\partial v}{\partial t}\frac{\partial}{\partial u} \\ &= k^2 T. \end{aligned} \tag{5.19}$$

□

It is not too surprising that Lemma 2.2.1 and Proposition 2.2.1 have carried over directly from the plane. Indeed, these results concern only vectors as applied to functions at a point. The above proof shows more of the niceness of the definition of covariant derivatives. We have not yet seen the effect of the curvature of the surface.

This will come up when we turn to examine the evolution of the curves' curvature k. Indeed, k is defined implicitly through a covariant derivative: $\nabla_T T = kN$. To extract $\frac{\partial k}{\partial t}$, we should apply a covariant derivative with respect to $\frac{\partial}{\partial t}$:

$$\nabla_t \nabla_T T = \nabla_t (kN). \tag{5.20}$$

To compute this, we will need to commute the covariant derivatives. Here is where we finally find a deviation from the planar case. Indeed, from an intrinsic point of view, it is exactly the failure of covariant derivatives to commute that detects the curvature of a surface. This measure of the failure of commutation is captured by the curvature tensor.

Definition 5.1.1. *The curvature tensor is a mapping on vector fields yielding a new vector field, defined by:*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5.21}$$

A basic result about the curvature tensor shows its use in commuting covariant derivatives along families of curves:

Proposition 5.1.2. *Given a family of curves X parametrized by t, for any smooth vector field V along X:*

$$\nabla_u \nabla_t V - \nabla_t \nabla_u V = R\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right) V. \tag{5.22}$$

See [12] for a proof.

Another important concept which the curvature tensor captures is the notion of sectional curvature, or in the context of a surface, the **Gaussian curvature**. Given a basis of orthonormal vectors X, Y the sectional curvature of the plane spanned by these vectors is defined by:

$$K(X,Y) = \langle R(X,Y)X, Y \rangle_g. \tag{5.23}$$

It is a standard result to show that K does not depend on the choice of basis and that, in the case of an embedded surface, K agrees with the Gauss curvature obtained from the second fundamental form (see [5]). With the last proposition in mind, we can describe the evolution of the curvature k:

Proposition 5.1.3. *For a family of curves evolving under the curve shortening flow on a surface, the curvature of the curves satisfies the evolution equation:*

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3 + kK, \tag{5.24}$$

where K is the Gauss curvature of the surface at the corresponding point along the curve.

Before proving the result, we establish a lemma. Compare to Proposition 2.2.2 on the plane.

Lemma 5.1.1. *The tangent and normal vectors evolve according to:*

$$\nabla_t T = \frac{\partial k}{\partial s} N \qquad \nabla_t N = -\frac{\partial k}{\partial s} T. \tag{5.25}$$

Proof. We compute, using the torsion free property and Proposition 5.1.1:

$$\begin{aligned}
\nabla_t T &= \nabla_T \frac{\partial}{\partial t} + k^2 T && (5.26)\\
&= \nabla_T k N + k^2 T && (5.27)\\
&= \frac{\partial k}{\partial s} N + k \nabla_T N + k^2 T && (5.28)\\
&= \frac{\partial k}{\partial s} N. && (5.29)
\end{aligned}$$

In the last line, we have also used a Frenet equation and we recall that the quantity $\frac{\partial k}{\partial s}$ refers to the tangent vector T being applied to k.

Commuting $\nabla_t N$ follows by similar arguments used to calculate the second Frenet equation. Since N is unit length, its derivative must be orthogonal to it and hence must be colinear with T. Applying ∇_t to the equation $\langle N, T \rangle$ yields the result. □

We can now prove Proposition 5.1.3.

Proof. We pick up from equation (5.20). On the left-hand side, using Lemma 5.1.1, we compute:

$$\begin{aligned}
\nabla_t (kN) &= \frac{\partial k}{\partial t} N + k \nabla_t N && (5.30)\\
&= \frac{\partial k}{\partial t} N - k \frac{\partial k}{\partial t} T. && (5.31)
\end{aligned}$$

On the right-hand side, we first re-express the derivatives in terms of the orthogonal basis T, N, using that $\frac{\partial X}{\partial t} = kN$:

$$\nabla_t \nabla_T T = k \nabla_N \nabla_T T. \tag{5.32}$$

Next, we use the curvature tensor to express the interchange of these derivatives:

$$\nabla_N \nabla_T T = \nabla_T \nabla_N T + \nabla_{[N,T]} T - R(T, N)T. \tag{5.33}$$

Consider the first term:

$$\begin{aligned}
\nabla_T \nabla_N T &= \nabla_T \frac{1}{k} \nabla_t T && (5.34)\\
&= \nabla_T \frac{1}{k} \frac{\partial k}{\partial s} N && (5.35)\\
&= \frac{-1}{k^2} \left(\frac{\partial k}{\partial s} \right)^2 N + \frac{1}{k} \frac{\partial^2 k}{\partial s^2} N - \frac{\partial k}{\partial s} T. && (5.36)
\end{aligned}$$

Now, consider the second term. First, we calculate the commutator:

$$\begin{aligned}
[N, T] = NT - TN &= \frac{1}{k} \frac{\partial}{\partial t} T - T \frac{1}{k} \frac{\partial}{\partial t} && (5.37)\\
&= \frac{1}{k} \frac{\partial}{\partial t} T + \frac{1}{k^2} \frac{\partial k}{\partial s} \frac{\partial}{\partial t} - \frac{1}{k} T \frac{\partial}{\partial t} && (5.38)\\
&= \frac{1}{k} \left[\frac{\partial}{\partial t}, T \right] + \frac{1}{k^2} \frac{\partial k}{\partial s} \frac{\partial}{\partial t} && (5.39)\\
&= kT + \frac{1}{k^2} \frac{\partial k}{\partial s} \frac{\partial}{\partial t}. && (5.40)
\end{aligned}$$

Thus, using properties of the covariant derivative followed by Lemma 5.1.1 and a Frenet equation:

$$\begin{aligned}
\nabla_{[N,T]} T &= k \nabla_T N + \frac{1}{k^2} \frac{\partial k}{\partial s} \nabla_t T && (5.41)\\
&= k^2 N + \frac{1}{k^2} \left(\frac{\partial k}{\partial s} \right)^2 N. && (5.42)
\end{aligned}$$

Putting all the terms together, we get:

$$k \left(\frac{-1}{k^2} \left(\frac{\partial k}{\partial s} \right)^2 N + \frac{1}{k} \frac{\partial^2 k}{\partial s^2} N - \frac{\partial k}{\partial s} T + k^2 N + \frac{1}{k^2} \left(\frac{\partial k}{\partial s} \right)^2 N - R(T, N)T \right)$$

$$= \frac{\partial k}{\partial t} N - k \frac{\partial k}{\partial t} T, \tag{5.43}$$

and, thus,

$$\frac{\partial^2 k}{\partial s^2} N - k \frac{\partial k}{\partial s} T + k^3 N - k R(T, N) T = \frac{\partial k}{\partial t} N - k \frac{\partial k}{\partial t} T. \tag{5.44}$$

Applying the metric to the above vector field and N, gives:

$$\frac{\partial^2 k}{\partial s^2} + k^3 - k \langle R(T, N) T, N \rangle_g = \frac{\partial k}{\partial t}, \tag{5.45}$$

concluding the proof. □

5.2 Existence and Characterization of Solutions

After establishing these basic facts, the next steps are to show existence of solutions to the flow and to describe the limit sets. The nature of the flow's limit depends on whether the flow exists for all time or only for finite time. In this finite case, the flow will converge to a point. If the flow exists for all time, the curvature will converge uniformly to zero, from which we can conclude that a subsequence of the evolving curves converge to a geodesic.

Proposition 5.2.1. (Existence of solutions, [4]) *Given a smooth closed curve X_0 on M, there exists a maximal time T such that that the curve shortening flow exists and is unique on $[0, T)$. Moreover, if T is finite then the curvature k has no upper bound in the limit $t \to T$.*

We outline the main ideas of the proof. The approach is to translate the curve shortening equation on M into a parabolic PDE for real functions, so that we may use results we already know. The key claim is that the initial curve X_0 can be extended to an immersion $\sigma : [a, b] \times [-1, 1] \to M$ in such a way that $X_0 = \sigma|_{[a,b]\times\{0\}}$. Then, curves which are near X_0 can be expressed through the immersion; as $\sigma(u, f(u))$ for an appropriate smooth real function f.

In particular, there is an f such that for small enough values of t a solution to the curve shortening flow $X(u,t)$ can be expressed through σ: $X(u,t) = \sigma(u, f(u,t))$.

One can then use the pull-back of σ to express the metric g with smooth real functions on $[a,b] \times [-1,1]$. These can be used to compute expressions for T, N and eventually k. Using these expressions and starting from the equation of curve shortening flow, one can then derive a PDE for the function f. The existence of solutions for the latter is equivalent to the existence of solutions to the flow. The PDE in question is parabolic (though nonlinear) so methods similar to those used for establishing the flow in the plane can be employed. Likewise, the same arguments as those in the plane guarantee that the flow remains embedded as it evolves.

Having shown existence, the next step is to analyze the limiting behavior of the flow. There are two possibilities, depending on whether the extinction time T is finite or not. If T is finite, we find that the flow converges to a point, just as in the plane.

Theorem 5.2.1. ([4]) *Let $X(u,t)$ be a solution to the curve shortening flow on M and let $[0,T)$ be its maximal interval of existence. If T is finite, the curves $X(\cdot,t)$ converge to a point as $t \to T$.*

We sketch the ideas of the proof. The argument is by contradiction. Using as a starting point Grayson's original proof in the plane, one can show that if the limit is not a point, then the flow nonetheless converges in the Hausdorff sense to a limit curve X^*. An argument with the Sturm oscillation theorem shows that along curves of the flow, the number of inflection points cannot increase, thus there are only a finite number of inflection points. In turn, this means that X^* can only have finite many singularities. Away from these singularities, the flow converges smoothly. One then investigates the flow of the curve near each of the singularities. Around each singularity one can find a small enough disc on which to set isothermal coordinates. In these coordinates, an isoperimetric type inequality can be established, describing the flow of arcs in the neighborhood of the singularity.

In the isothermal coordinates, techniques from the plane can be supplemented with this inequality. Specifically, a blow-up argument shows that in fact no singularities are formed in the limit. Thus the limit curve is smooth. This contradicts its definition as the limit curve, since the smoothness allows the curve to be used as an initial flow, extending the flow to a further time.

We can then look at the case where T is infinite; we find the curvature will converge uniformly to zero.

Theorem 5.2.2. ([4]) *Let $X(u,t)$ be a solution to the curve shortening flow on M and let $[0,T)$ be its maximal interval of existence. If T is infinite, then the curvature k converges uniformly to zero as $t \to \infty$. A subsequence of the flow then converges to an embedded closed geodesic of M.*

The proof of this result is a little more straight-forward. First one establishes that in the limit $t \to \infty$, the integrals $\int k^2\,ds$ and $\int k_s^2\,ds$ converge to zero. These follow from a number of integral estimates, using the key insight that for the extinction time to be infinite, the length must be bounded below. These results imply the curvature itself converges uniformly to zero from the one-dimensional Sobolev inequality $|k(s)|^2 \leq c(\|k\|_{L^2}^2 + \|k_s\|_{L^2}^2)$, for some constant c, and all $s \in [0, L(t)]$.

Next one can show that k and all its derivatives vanish. This is done by similar arguments as those in Section 3.2 applied to equation (5.24). The Arzela-Ascoli Theorem then yields convergence of a subsequence to a closed geodesic. Knowing that the flow keeps curves embedded, one can argue that this limiting geodesic must also be embedded.

Chapter 6

Closing Remarks

As we have covered in this work, many of the basic questions about a given flow are quite satisfactorily answered for curve shortening in the plane. For closed curves the flow exists and its limiting behavior is understood. Turning to riemannian surfaces, although not as thoroughly, it is still possible to answer questions about the flow's existence. Besides these existence results, curve shortening flow posses many further avenues of exploration. There are many quite interesting properties of the flow, along with a number of potential applications, both in pure math and beyond. We shall spend this final chapter by surveying a few of these properties and the applications they open.

An early major application of curve shortening flow was its potential use in providing an alternate proof of the 3-geodesics Theorem of Lusternik and Schnirelmann.

Theorem 6.0.1. Lusternik and Schnirelmann *A 2-sphere with smooth Riemannian metric has at least 3 simple closed geodesics.*

The approach, outlined by Grayson in [10], uses a result akin to Theorem 5.2.2 to produce 3 distinct geodesics through curve shortening. Proofs of this same result had already been explored using other types of flows to produce the geodesics. The advantage of curve shortening, as Grayson explains, is that it is a continuous flow and it is easy to guarantee that curve remain embedded (by the analogue of Theorem 2.4.1, which holds on surfaces).

next, there are a few application of curve shortening that rely on the attractive property of smoothening curves. Much like the heat equation, curve shortening produces solutions that are smooth even if the initial data is not as regular. More precisely, as long as the curvature k is uniformly bounded, all derivatives of k will exist at positive times. For a proof of this fact, consult [2]. The argument uses similar techniques as employed in Section 3.2 to obtain bounds on the derivatives of k, using the evolution equation of k (equation 2.14) and the maximum principle.

This smoothing property is used for applications in the context of image processing. As detailed in a book by Cao ([3]), for the purpose of preparing an image for further analysis, a method of smoothing curves is needed to remove noise and make shape detection easier. The smoothing process should have certain properties; to name a few, it should be intrinsic (independent of parametrization), local and invariant under isometries. Under this list of desired properties, curve shortening emerges as a natural candidate.

More recently, curve shortening has been considered for use in drone robotics, specifically for the task of path-planning [11]. In this application, curve shortening is valuable not only for its smoothing property but also for being the gradient of the length functional. That is to say, curve shortening is the flow that minimizes lengths most efficiently. The idea is that a drone will first calculate a rough initial trajectory for its flight path. This path can then be smoothened and length optimized by applying a form of curve shortening.

Another instance where curve shortening crops up is in the study of reaction-diffusion equations. This is maybe not too surprising, as curve shortening is itself a geometric heat equation. In a paper by Rubinstein *et al* [15], reaction-diffusion equation are studied in a limiting case where the diffusion term dominate over the reaction term. The behavior of wave fronts are examined in various cases, according to the nature of the reaction function V. In the case when $V = 0$, that is there is no reaction and diffusion is asymptotically small, it turns out that wave fronts travel according to curve shortening.

Beyond these applications and its intrinsic interest, the study of curve shortening is also valuable as a primer to the study of more advanced geometric flows. For instance, curve shortening is a special case of the more general mean curvature flow. The latter has many properties in common with curve shortening, but also many additional challenges due to the higher dimensionality. Under active research, a particular question of interest is extending mean curvature flow through singularities.

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