Étale Cohomology and the Galois Representation attached to a Modular Form

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In this thesis, we introduce the theory of étale cohomology, assuming basics in algebraic geometry and sheaf theory on topological spaces. In particular, we define étale morphisms and develop the theory of sheaves on the étale site alongside some of the most important examples.

In the last two chapters, we collect, mostly without proofs, some of the most important results on étale cohomology and apply them to outline Deligne's construction of the Galois representation attached to a modular form, illustrating the usefulness of that cohomology theory in number theory.

Contents

Introduction

Consider a continuous map $f\colon X \to Y$ of topological spaces and let $\mathcal{F}$ be a sheaf on Y. The map f induces, via the adjunction map, a homomorphism of global sections

$$\Gamma(Y, \mathcal{F}) \xrightarrow{\text{can}} \Gamma(Y, f_* f^* \mathcal{F}) \cong \Gamma(X, f^* \mathcal{F}).$$

By the general theory of derived functors, the right derived functors of $\Gamma(Y, -)$ form a universal δ-functor, and this homomorphism induces homomorphisms between the sheaf cohomology groups $H^i(Y, \mathcal{F}) \xrightarrow{f^*} H^i(X, f^* \mathcal{F})$ for all natural numbers $i \geq 0$.

If we compare this to singular cohomology from algebraic topology, we note that there we get maps

$$H^i(Y, \mathbb{Z}) \xrightarrow{f^*} H^i(X, \mathbb{Z})$$

for all $i \geq 0$, where the functor on the right-hand side does not depend on f, i.e., it is intrinsically defined by X.

We can also archive this in sheaf cohomology by considering the constant sheaf $\mathcal{F} \coloneqq \underline{\mathbb{Z}}_Y$: for any map f we have a canonical isomorphism $f^* \underline{\mathbb{Z}}_Y \cong \underline{\mathbb{Z}}_X$, and hence the maps constructed above can be written as $H^i(Y, \underline{\mathbb{Z}}_Y) \xrightarrow{f^*} H^i(X, \underline{\mathbb{Z}}_X)$, with the codomain now independent of f. It is a classical result in algebraic topology that for paracompact locally contractible spaces (e.g. a compact real manifold), these two approaches give naturally isomorphic functors.

But when it comes to algebraic geometry and, say, an (irreducible) variety defined over an algebraically closed field (other than $\mathbb{C}$, perhaps), both of these approaches are flawed in some sense: The analytic topology and the Zariski topology on the variety are so different from one another that singular cohomology generally tends to give undesirable results, and while sheaf cohomology is regularly used to study the geometry of varieties, the coefficients used are not usually constant for a simple reason: a constant sheaf on an irreducible space is flasque, and therefore all of its higher cohomology groups vanish.

As a result of this, when using sheaf cohomology in algebraic geometry, we rarely, if ever make use of functoriality in the first variable of $H^i(X, \mathcal{F})$ and so cannot apply many of the same tricks applied in topology.

This motivates the need for a different cohomology theory in algebraic geometry, more reminiscent of the situation in topology. There exists more than one such theory, but in this thesis, we focus only on étale cohomology (and a closely related variant of it, the l-adic cohomology groups). We will see, for example, that for algebraic curves (even when defined over a field of characteristic $p > 0$), these cohomology groups are very similar to the singular cohomology groups of curves over $\mathbb{C}$ (interpreted as complex manifolds).

We begin by briefly introducing the purely geometric concept of étale morphisms, and then use them to define the étale site. Our investigation of the cohomology theory will be a mixture of results on general sites, and results more specific to the étale site. One such result that is very specific to the étale site is a categorical equivalence between a certain subcategory of étale sheaves and representations of the étale fundamental group, which is a scheme-theoretic analog of the topological fundamental group.

Étale cohomology also plays an important role in number theory: Because the étale fundamental group (which we will not construct in this paper) computes in many cases groups also interesting in number theory, we can use étale sheaves and the general theory surrounding them to construct and study representations of these interesting groups.

To illustrate this, we will outline Deligne's construction of the Galois representation attached to a modular form in the last chapter of this thesis.

Chapter 1

Étale Morphisms

The first goal for this paper is to develop the theory of étale cohomology, which we will do, roughly speaking, in two steps: First, we will introduce étale morphisms and their geometric properties, and then we develop the theory of sheaves on a general site.

Either one of these two parts is going to be a pretty dry read without the other in mind, and we begin with the geometric aspects of étale morphisms solely because they could, in a different paper, also be introduced without the étale site in mind, while it seems hard to imagine an overview of sheaf theory on general sites without the étale site as one of the first examples.

1.1 Flat morphisms

We are ultimately going to define étale morphisms as a morphism of schemes that is flat and unramified, so we first introduce both of these notions independently.

Flat morphisms of schemes are a direct generalization of the concept of a flat algebra over a ring, in the sense that an algebra $A \to B$ is flat if and only if the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat. We therefore begin with the following reminder to commutative algebra, as found, for example, in [Mat80, Chapter 2].

Reminder. *Let A be a ring and M be an A-module. Then the functor $-\otimes_A M$ is right exact, as it admits the right adjoint $\operatorname{Hom}_A(M, -)$. We say that M is flat if the functor $-\otimes M$ is also exact on the left.*

Reminder. *Let $f\colon A \to B$ be a homomorphism of commutative rings. Then, the following are equivalent:*

i) B is flat over A, i.e. it is flat when interpreted as an A-module.

ii) The extension of scalars functor $-\otimes_A B$ is exact.

iii) For every prime $\mathfrak{p} \in \operatorname{Spec} B$, the local homomorphism of rings $A_\mathfrak{q} \to B_\mathfrak{p}$, where $\mathfrak{q} := f^{-1}(\mathfrak{p})$, is flat.

This third condition fits the scheme setting very well, and so we make the following definition:

Definition 1.1.1. *Let $f: X \to S$ be a morphism of schemes and let $x \in X$. We say that f is flat in x iff $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,f(x)}$-algebra. We say that it is flat iff it is flat in every point $x \in X$.*

Remark. *It is immediate that any open immersion is flat. Closed immersions, on the other hand, are typically not flat.*

Assuming a solid basis in commutative algebra, it is easy to prove the following two permanence properties for flatness.

Proposition 1.1.2. *a) A composition of two flat morphisms is again flat.*

b) Any base change of a flat morphism is flat.

Proof. Both statements are easily reduced to the affine case, where the statements are standard results from commutative algebra (cf. [Mat80, Chapter 2, Section3]). □

Next, we introduce faithful flatness, a concept which looks interesting enough algebraically to be talked about in many introductions to commutative algebra (usually using a different definition, cf. Lemma 1.1.4), but turns out to have a rather geometric aspect to it, as should be apparent from our definition.

Definition 1.1.3. *Let $f: X \to S$ be a morphism. We say that f is faithfully flat iff it is flat and surjective.*

Remark. *Since being surjective and being flat are stable under base change and composition, and are local on the target, the same is true for faithful flatness.*

For the reader's convenience, we briefly recall the affine preliminaries used in the following results.

Lemma 1.1.4. *Let $f: A \to B$ be a ring homomorphism. Then the following conditions are equivalent:*

i) $f^*: \operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat.*

ii) Any sequence of A-modules $M' \to M \to M''$ is exact if and only the induced sequence of B-modules $M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$ is exact.

iii) B is a flat A-algebra and for any A-module M we have $M \cong 0$ if and only if $M \otimes_A B \cong 0$.

iv) B is a flat A-algebra and for any maximal ideal $\mathfrak{m} \subseteq A$ we have $\mathfrak{m}B \neq B$.

Proof. See [Mat80, Chapter 2, Section 4]. □

Corollary 1.1.5. *A local homomorphism $f: A \to B$ of local rings is flat if and only if it is faithfully flat.*

Proof. This follows from the last condition in Lemma 1.1.4. □

Having introduced our two main definitions for this section, we now turn our attention to studying two geometric properties of flat morphisms, the first of which being that a flat morphism is, under a mild finiteness condition, an open morphism. This allows us to factor such morphisms into a faithfully flat morphism followed by an open immersion, and will help us to generalize the concept of an open cover of a topological space once we start developing the theory of sheaves on the étale site.

Lemma 1.1.6. *Let $f\colon X \to S$ be a flat morphism of schemes and $x \in X$. Then we have $f(\operatorname{Spec} \mathcal{O}_{X,x}) = \mathcal{O}_{S,f(x)}$, where we consider the spectra of the local rings as subsets of X and S respectively.*

Proof. We always have $f(\operatorname{Spec} \mathcal{O}_{X,x}) \subseteq \mathcal{O}_{S,f(x)}$, and since, by Corollary 1.1.5, we know that $\mathcal{O}_{X,x}$ is a faithfully flat $\mathcal{O}_{S,f(x)}$-algebra, the surjectivity statement from Lemma 1.1.4 implies the other inclusion. $\square$

Corollary 1.1.7. *A morphism of schemes that is locally of finite presentation and flat is universally open.*

Proof. Since being locally of finite presentation and being flat are both stable under base change, it suffices to show that such a morphism is open. Since the morphism is still locally of finite presentation and flat after we compose with an open immersion, we only have to show that its image is open.

This is a slightly technical but completely general statement using only the property established in Lemma 1.1.6, see for example [GW20, Corollary 10.72.]. $\square$

The second property will turn out to be as essential for sheaf theory on a site as it looks useless without it in mind. Together with Corollary 1.1.7, it makes flatness so important in fact, that all commonly used sites in algebraic geometry use this concept in their definition. Its usefulness will only become apparent in the proof of Corollary 2.2.10, so that this last part of the section could safely be skipped until needing it there.

We begin with the following general definition, which lends itself well to contexts also outside of algebraic geometry when working with sites.

Definition 1.1.8. *In any category $\mathcal{C}$, a morphism $f\colon X \to S$ is called an effective epimorphism iff the fiber product $X \times_S X$ exists and the diagram*

$$X \times_S X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} S$$

is exact, i.e. $X \to S$ is the coequalizer of the two projections.

With a bit of creativity, this diagram could hint (contravariantly) at the relation to the sheaf condition also familiar from classical sheaf theory.

To prepare our main result, we first cover the affine case. Note that exactness at A and B is precisely saying that f is an effective epimorphism in the category $\mathbf{Ring}^{op}$, commonly identified with the category of affine schemes.

Lemma 1.1.9. *Let $f\colon A \to B$ be a faithfully flat ring homomorphism. Then the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{d^1} B^{\otimes 2} \xrightarrow{d^2} \dots \xrightarrow{d^{r-1}} B^{\otimes r} \xrightarrow{d^r} \dots$$

of A-modules is exact, where

- *$B^{\otimes r}$ is the r-fold tensor product of B with itself over A,*
- *$d^r \coloneqq \sum_{i=1}^{r}(-1)^{i+1}e_i$ and*
- *$e_i(b_1 \otimes \dots \otimes b_r) \coloneqq b_1 \otimes \dots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \dots \otimes b_r \in B^{r+1}$.*

Proof. It is trivial to check that the sequence is a chain complex (this does not use the faithfully flat condition). First, we assume that f admits a section g, i.e. a ring homomorphism such that $gf = \mathrm{id}_A$. We want to show that the identity of the complex is null-homotopic. Let $r \geq 1$ and define

$$k_r\colon B^{\otimes r} \to B^{\otimes r-1}$$

by $k_r(b_1 \otimes \dots \otimes b_r) \coloneqq g(b_1)(b_2 \otimes \dots \otimes b_r)$. We compute

$$\begin{aligned}
&(d^{r-1}k_r + k_{r+1}d^r)(b_1 \otimes \dots \otimes b_r)\\
=&g(b_1)\sum_{i=2}^{r}(-1)^i e_i(b_2 \otimes \dots \otimes b_r) + \left(b_1 \otimes \dots \otimes b_r + \sum_{i=2}^{r}(-1)^{i+1}g(b_1)\cdot e_i(b_2 \otimes \dots \otimes b_r)\right)\\
=&b_1 \otimes \dots \otimes b_r,
\end{aligned}$$

concluding this step of the proof, since the only position we have not yet checked is at $r = 0$, where the desired equation is trivially satisfied by the assumption that g is a section.

In general, for any A-algebra A', the analogously defined sequence for the base change morphism $f'\colon A' \to B' \coloneqq A' \otimes_A B$ is canonically isomorphic to the sequence obtained by the extending scalars to A' via the natural isomorphism $B'^{\otimes r} \cong (B^{\otimes r}) \otimes_A A'$.

If we apply this to $A' = B$, the base change morphism $B \to B \otimes_A B$ admits a section induced by the multiplication map, implying that our sequence is exact after we extend scalars to B. But by assumption, B is a faithfully flat A-algebra, so Lemma 1.1.4 yields the desired result. □

Having covered the basic local information we need, the global case follows from a rather standard gluing argument. Note that, compared to the affine case, we have to introduce a finiteness condition here for it to work.

Theorem 1.1.10. *A faithfully flat morphism $f\colon X \to S$ that is of finite presentation is a universally effective epimorphism in the category of schemes.*

Proof. Since being of finite presentation and being faithfully flat are stable under base change, the universality is clear if we can show that such morphisms are effective epimorphisms.

We have to show that the diagram

$$X \times_S X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} S$$

is exact, i.e. that there exists for every morphism $z\colon X \to Z$ with $z\pi_1 = z\pi_2$ a unique factorization $z'\colon S \to Z$ satisfying $z'f = z$. If X, S and Z are all affine, we are in the situation of Lemma 1.1.9, so we have a unique factorization.

As a next step, we assume that X and S are still affine, but Z could be an arbitrary scheme. We begin by showing that, if one exists, a factorization is unique. This only uses the faithful flatness: Let z_1' and z_2' be two morphisms with the property $z_1'f = z_2'f = z$. Since f is surjective, the two maps on the underlying topological space agree. We may work locally and choose for a point $s \in S$ an affine open neighborhood $z_1'(s) = z_2'(s) \in V \subseteq Z$ and an affine open neighborhood $s \in U \subseteq {z'}_1^{-1}(V) = {z'}_2^{-1}(V)$. The two restricted morphisms $U \to V \subseteq S$ agree, since they agree after composing with the faithfully flat (affine) morphism $f\colon f^{-1}(U) \to U$, and since $\Gamma(U, \mathcal{O}_S) \to \Gamma(f^{-1}(U), \mathcal{O}_X)$ is injective (cf. Lemma 1.1.9).

To show the existence of such a factorization, we only have to define it locally on S, because of the uniqueness just proved. Let $s \in S$ and $x \in X$ with $f(x) = s$. Let $z(x) \in V \subseteq Z$ be an affine open neighborhood. Then $f(z^{-1}(V)) \subseteq S$ is open (cf. Corollary 1.1.7), and so we can choose an affine open $s \in U \subseteq f(z^{-1}(V))$.

We want to show that $f^{-1}(U) \subseteq z^{-1}(V)$, so let $x_1 \in f^{-1}(U)$ and choose $x_2 \in z^{-1}(V)$ with $f(x_1) = f(x_2)$. Let $x' \in X \times_S X$ with $\pi_1(x') = x_1$ and $\pi_2(x') = x_2$ (to see existence of such a point, consider a common field extension of $\kappa(x_1)$ and $\kappa(x_2)$ over $\kappa(f(x_1))$) and compute

$$z(x_1) = z(\pi_1(x')) = z(\pi_2(x')) = z(x_2) \in V.$$

Finally, this implies that $z|_{f^{-1}(U)}$ factors over the affine open $V \subseteq Z$, and our problem is reduced to the already treated case where Z was also affine.

Lastly, for the general case with X, S and Z all arbitrary schemes we can still immediately reduce to the case where S is affine. Since we assumed f to be quasi-compact, X is a finite union of affine open subschemes $\bigcup_{i=1}^n X_i$. The canonical morphism $X^* := \coprod_{i=1}^n X_i \to S$ is clearly still faithfully flat and X^* is affine. We get the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S,Z) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X,Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathrm{Hom}(X \times_S X, Z) \arrow[d] \\
\mathrm{Hom}(S,Z) \arrow[r] & \mathrm{Hom}(X^*,Z) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(X^* \times_S X^*, Z)
\end{tikzcd},$$

where the second row is exact by our previous step, so an easy diagram chase concludes the proof. □

1.2 Unramified morphisms

Recall that we want to define étale morphisms as flat and unramified morphisms, so we now introduce the second of these two properties. Again, depending on one's background, this could be seen as a direct generalization of an already familiar concept, this time from algebraic number theory.

While flatness is perhaps a bit hard to build an intuition for, as it is a purely algebraic condition, unramified morphisms are a different story, even outside of their use case in number theory, as we will see in an example later. We begin with a definition that is rather close to the one studied in number theory, and then we develop some useful equivalent characterizations.

Definition 1.2.1. *Let X and S be schemes and let $f\colon X \to S$ be a morphism locally of finite presentation. We say that f is unramified at a point $x \in X$ iff $f_p^{\#}(\mathfrak{m}_{f(x)})\,\mathcal{O}_{X,x} = \mathfrak{m}_x$ (in other words: the image of $\mathfrak{m}_{f(x)}$ generates the ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$) and the residue field extension $\kappa(x)/\kappa(f(x))$ is finite separable.*

We say the morphism f is unramified iff it is unramified at every point $x \in X$.

Remark. *It is easy to see from this definition that any immersion is unramified.*

Example. *We want to explore the relation between Definition 1.2.1 and the perhaps more common concept from number theory. Let $A \subseteq B$ be a finite extension of DVRs and let π and π' be uniformizers of A and B, respectively. This situation naturally arises in number theory when trying to determine the splitting behavior of a prime.*

The induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is trivially unramified as long as the extension of the fields of fractions is separable, as for example in the case where they are number fields. Unramifiedness of this morphism hence solely depends on the maximal ideal of $(\pi') \subseteq B$. As B is a DVR we have $\pi B = (\pi')^e \subseteq B$ for a unique integer $e \geq 1$, called the ramification index.

Therefore, we see that the morphism is unramified if and only if the ramification index is $e = 1$, again under the assumption that the residue field extension is separable, which is usually the case in algebraic number theory because finite fields are perfect.

We also want to illustrate the property in a more geometric situation. To do that, it will be useful to first have a few characterizations of unramified morphisms on hand, which we will state after preluding with a technical lemma.

Lemma 1.2.2. *Let $f\colon X \to S$ be a morphism of schemes. Let $s \in S$ and $x \in X_s$. Then $\mathcal{O}_{X_s,x} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$.*

Proof. The question is local in nature, so we may assume that both X and S are affine, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$. Write $s =: \mathfrak{p} \subseteq A$. Then $\mathfrak{q} =: x \in X_s$ is a prime ideal in $\Gamma(X_s, \mathcal{O}_{X_s}) \cong B \otimes_A \kappa(s) \cong (B/\mathfrak{p}B)_{\mathfrak{p}}$, and is therefore of the form $(\mathfrak{q}'/\mathfrak{p}B)_{\mathfrak{p}} \subseteq (B/\mathfrak{p}B)_{\mathfrak{p}}$ for a uniquely determined prime ideal $\mathfrak{q}' \subseteq B$.

We have a canonical isomorphism

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} \kappa(s) \cong B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} \cong (B/\mathfrak{p}B)_{\mathfrak{q}} \cong ((B/\mathfrak{p}B)_{\mathfrak{p}})_{\mathfrak{q}} \cong (B \otimes_A \kappa(s))_{\mathfrak{q}} \cong (B \otimes_A \kappa(s))_{\mathfrak{q}'},$$

where in the second to last step we use the fact that the image of an element $a \in A \setminus \mathfrak{p}$ is already in $B \setminus \mathfrak{q}$, because $\mathfrak{p}$ is by assumption the preimage of $\mathfrak{q}$. $\square$

Proposition 1.2.3. *Let $f\colon X \to S$ be a morphism locally of finite presentation. Then the following are equivalent:*

i) The morphism f is unramified.

ii) For every point $s \in S$, the fiber $X_s \to \operatorname{Spec} \kappa(s)$ is unramified.

iii) For every geometric point $\operatorname{Spec} K \cong \bar{s} \to S$, the geometric fiber $X_{\bar{s}}$ is unramified.

iv) For every point $s \in S$, the fiber X_s is isomorphic (as $\operatorname{Spec} \kappa(s)$-scheme) to a disjoint union $\coprod_{i \in I} \operatorname{Spec} k_i$, where each k_i is a finite separable field extension of $\kappa(s)$.

Proof.

- $i) \iff ii)$: Being locally of finite presentation is stable under base change, so the fiber is still locally of finite presentation. Let $x \in X_s$. From Lemma 1.2.2 we get $\mathcal{O}_{X_s,x} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \kappa(s) \cong \mathcal{O}_{X,x} / \mathfrak{m}_s \mathcal{O}_{X,x}$. This is a finite separable field extension of $\kappa(s)$ if and only if f is unramified at x.

- $ii) \implies iv)$: This may be checked on an open cover, so let $\operatorname{Spec} A \cong U \subseteq X_s$ be an affine subscheme. Then A is a finitely generated $\kappa(s)$-algebra, hence Noetherian. As it is unramified by assumption, it is immediate that all the localizations of A at prime ideals are finite separable field extensions of $\kappa(s)$, so $\dim A = 0$ and A is artinian. Like any artinian ring, A is therefore isomorphic to the finite product of its localisations, which, as we have seen, are finite separable field extensions of $\kappa(s)$, completing the argument.

- $iv) \implies ii)$: This is obvious.

- $iv) \implies iii)$: Let $s \in S$ be the image of $\bar{s}$. Then $\bar{s}$ factors over the canonical morphism $\operatorname{Spec} \kappa(s) \to S$, and $X_{\bar{s}} \cong X_s \times_{\operatorname{Spec} \kappa(s)} \bar{s} \cong \coprod_{i \in I} \operatorname{Spec}(k_i \otimes_{\kappa(s)} K)$ for each k_i a finite separable field extension by assumption. Keeping in mind that K is separably closed, it is straightforward to check that $k_i \otimes_{\kappa(s)} K \cong K^{[k_i \colon \kappa(s)]}$. This implies that the geometric fiber is a disjoint union of copies of $\bar{s}$, and therefore unramified.

- $iii) \implies ii)$: We may again assume that $X_s \cong \operatorname{Spec} A$ and therefore also $X_{\bar{s}} \cong \operatorname{Spec}(A \otimes_{\kappa(s)} K)$ are affine. We already know that the explicit description of an unramified morphism over a field from $iv)$ applies, so $A \otimes_{\kappa(s)} K \cong K^n$ as a K-algebra, for some $n \geq 0$. But this implies that A must have been an n-dimensional $\kappa(s)$-algebra, and in particular finite and therefore an artinian ring. Then X_s is discrete and we may further assume that A is a local artinian ring.

  We have to show that A is a finite separable field extension, using that $A \otimes_{\kappa(s)} K \cong K^n$ for some $n \geq 1$ (the case $n = 0$ appears only when the fiber is empty). Since A has a unique prime ideal $\mathfrak{m}$, every morphism $A \to K$ factors uniquely through $A/\mathfrak{m}$ and using the universal property of the tensor product, we obtain

  $$\operatorname{Hom}_{\kappa(s)}(A/\mathfrak{m}, K) \cong \operatorname{Hom}_{\kappa(s)}(A, K) \cong \operatorname{Hom}_K(A \otimes_{\kappa(s)} K, K) \cong \operatorname{Hom}_K(K^n, K),$$

  so we get $[A/\mathfrak{m} \colon \kappa(s)]_s = n$ for the separable degree of $A/\mathfrak{m}$. This yields a chain of inequalities

  $$n = [A/\mathfrak{m} \colon \kappa(s)]_s \leq [A/\mathfrak{m} \colon \kappa(s)] \leq [A \colon \kappa(s)].$$

  But as discussed above, the dimension $[A \colon \kappa(s)]$ of A over $\kappa(s)$ is n as well, so all the inequalities are actually equalities, implying that $A \cong A/\mathfrak{m}$ is a field and a separable extension of $\kappa(s)$.

$\square$

Before we continue providing even more useful characterizations, we now give our example that is more geometric in nature.

Example. *Let k be an algebraically closed field and consider the morphism of affine schemes corresponding to*

$$k[T] \to k[X,Y]/(XY)$$
$$T \mapsto X + Y.$$

Intuitively, it maps the union of the two axes in $\mathbb{A}_k^2$ to the affine line by assigning to a point the respective non-zero coordinate. The origin is mapped to the origin.

We want to show that this morphism is ramified at the origin, and unramified outside of it. The automorphism of $k[X,Y]$ given by $X \mapsto X - Y$ and $Y \mapsto Y$ shows that $k[X,Y]/(XY)$ is isomorphic, as a $k[T]$-algebra, to the algebra $k[T][Y]/(T-Y)(Y)$. From this representation, it is easy to see, using the Chinese Remainder Theorem, that the fiber at any point outside the origin is isomorphic to $\operatorname{Spec}(k \times k) \cong \operatorname{Spec} k \coprod \operatorname{Spec} k$.

At the origin the fiber is $\operatorname{Spec} k[Y]/(Y^2)$, and so Proposition 1.2.3 implies that our morphism is unramified when restricted to the open subscheme without the origin, but not globally.

The next characterization of unramified morphisms will make it very easy to work with them formally, as it relates this property to a property of the relative cotangent sheaf.

Proposition 1.2.4. *Let $f\colon X \to S$ be a morphism locally of finite presentation and let $x \in X$. Then the following are equivalent:*

i) The morphism f is unramified at x.

ii) The cotangent sheaf $\Omega^1_{X/S}$ vanishes at x.

iii) There is an open neighborhood $x \in U \subseteq X$ such that $U \subseteq X \to X \times_S X$ is an open immersion.

Proof.

- *i)* $\implies$ *ii)* : We write $s := f(x)$. In general, we have $\Omega^1_{X/S,x} \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}$. Our assumptions imply that the diagram

$$\begin{tikzcd} \mathcal{O}_{S,s} \arrow[r] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\ \kappa(s) \arrow[r] & \kappa(x) \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \kappa(s) \end{tikzcd}$$

  is co-Cartesian and that $\Omega^1_{X/S,x}$ is a finitely generated $\mathcal{O}_{X,x}$-module. We find $\Omega^1_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \cong \Omega^1_{\kappa(x)/\kappa(s)} \cong 0$, as $\kappa(x)/\kappa(s)$ is a finite separable extension. Now Nakayama's lemma implies $\Omega^1_{X/S,x} \cong 0$.

- *ii)* $\implies$ *iii)* : The question is local, so we may assume that both schemes are affine, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$. Then the module $\Omega^1_{B/A}$ is constructed as the quotient $I/I^2 \subseteq (B \otimes_A B)/I \cong B$, where I is the kernel of the multiplication map. Since B is a finitely generated A-algebra, the ideal I is finitely generated, so Nakayama's lemma implies that $I_x \cong 0$, as $I_x/I_x^2 \cong 0$ by assumption and $I_x \subseteq \mathfrak{p}(B \otimes_A B)_{\mathfrak{p}}$, where $\mathfrak{p}$ denotes the prime ideal corresponding to $x \in V(I) \cong \operatorname{Spec} B$. But since I is finitely generated, it has closed support and therefore vanishes on an open subset of X, implying that the immersion is open there.

- *iii)* $\implies$ *ii)* : We continue to work in the setting from the previous implication. But if $I_x/I_x^2 \neq 0$, then we definitely have $I_x \neq 0$ and therefore, there cannot be an open subset on which the restriction would be an open immersion.

- *ii)* $\implies$ *i)* : Since $\Omega^1_{X/S}$ is locally of finite type, there is an affine open neighborhood $x \in U \cong \operatorname{Spec} A$ such that $\Omega^1_{X/S}|_U \cong 0$, so we want to show that $U \to S$ is unramified. Because the cotangent sheaf behaves well under base change, we may apply Proposition 1.2.3 to assume that $S \cong \operatorname{Spec} K$ for K a separably closed field.

  Let $\mathfrak{m} \subseteq A$ be a prime ideal. Since A is finitely generated over K, the field extension $\kappa(\mathfrak{m})/K$ is finitely generated. The relative cotangent sequence of differentials immediately implies that $\Omega^1_{\kappa(\mathfrak{m})/K} \cong 0$, so $\kappa(\mathfrak{m})/K$ has transcendence degree zero [Har77, II, Theorem 8.6A.], is finite and then it also has to be separable. Therefore $\dim A = 0$ and A is artinian, meaning we may assume it is local. We have an isomorphism $K \to A \to A/\mathfrak{m}$, and by [Har77, II, Proposition 8.7.] we have $\mathfrak{m}/\mathfrak{m}^2 \cong \Omega^1_{A/K} \otimes_B K \cong 0$, so Nakayama's lemma shows that $\mathfrak{m} \cong 0$ and $A \cong A/\mathfrak{m}$, finishing the proof.

$\square$

Corollary 1.2.5. *Let $f\colon X \to S$ be a morphism locally of finite presentation. Then the following are equivalent:*

i) The morphism f is unramified.

ii) We have $\Omega^1_{X/S} \cong 0$.

iii) $X \to X \times_S X$ is an open immersion.

Proof. Follows immediately from Proposition 1.2.4. $\square$

With this in mind, it is now very easy to prove the usual permanence properties for unramified morphisms.

Corollary 1.2.6. *a) A composition of two unramified morphisms is again unramified.*

b) Any base change of an unramified morphism is unramified.

Proof. a) This follows immediately from Corollary 1.2.5 and the relative cotangent sequence of differentials.

b) This follows from Corollary 1.2.5 and the stability of the cotangent sheaf under base change.

$\square$

Furthermore, we can use the condition on the relative cotangent sheaf to show that the property of being unramified is a generic property, since the same is true for a sheaf of finite type being zero. More precisely, we have:

Corollary 1.2.7. *Let $f\colon X \to S$ be a morphism locally of finite presentation. Then the set*

$$\{x \in X \colon f \text{ is unramified at } x\} \subseteq X$$

is open.

Proof. By Proposition 1.2.4, this set is nothing but the complement of the support of $\Omega^1_{X/S}$, and since we assumed that f is locally of finite presentation (we only need locally of finite type), this sheaf is of finite type, and hence has closed support. □

1.3 Étale Morphisms

We have been working towards a definition for étale morphisms and built a solid basis for flat and unramified morphisms. This allows us to essentially just combine the results from the previous two sections to already get a lot of basic facts for this section.

After having hinted at it multiple times now, we officially introduce our main definition for this chapter for later reference and state the basic permanence properties it satisfies.

Definition 1.3.1. *A morphism of schemes $X \to Y$ is called étale iff it is flat and unramified.*

Corollary 1.3.2. *a) A composition of two étale morphisms is again étale.*

b) Any base change of an étale morphism is étale.

Proof. Follows from Proposition 1.1.2 and Corollary 1.2.6 □

One interesting way in which the two properties of being flat and being unramified interact with one another is the following useful proposition.

Proposition 1.3.3. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "s"'] & & Y \arrow[dl, "t"] \\ & S & \end{tikzcd}$$

be a commutative triangle of schemes, and assume that s is étale and that t is unramified. Then f is étale.

Proof. We work with the commutative diagram

$$\begin{tikzcd} & X \arrow[r, "s"] & S \\ X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[u, "\pi_X"] \arrow[r, "\pi_Y"] \arrow[d, "f \times 1"] & Y \arrow[u, "t"'] \\ Y \arrow[r, "\Delta"'] & Y \times_S Y & \end{tikzcd},$$

where Γ_f is the graph morphism of f and Δ is the diagonal morphism.

It is elementary to check that both squares are Cartesian, so we can apply Corollary 1.3.2 to see that Γ_f (cf. Corollary 1.2.5 and note that open immersions are étale) and π_Y are both étale, so their composition f is as well. □

With this, we conclude our first overview of the geometry of étale morphisms, and so we are ready to turn our attention towards the étale site, though we will still be adding more specialized purely geometric results as we need them.

Chapter 2

The Étale Site and Étale Cohomology

Having developed a solid understanding of étale morphisms, we now explain our reason for doing so: The étale site and étale cohomology.

As we explained earlier, a big part of the general theory could be developed, at least in concept, completely independently from the étale site, or any site commonly used in algebraic geometry for that matter. But having introduced the concept of étale morphisms in the first chapter, we will introduce it along with some key examples and distinctions from the Zariski site, the site used for talking about sheaves on a topological space.

2.1 Sites, Presheaves and Sheaves: The General Theory

In this section, we introduce the abstract groundwork for generalizing the concept of sheaf theory and sheaf cohomology on a topological space to that on a general site, which is a much more flexible setting that is applicable also in other fields of mathematics.

The main idea is this: If X is a topological space, then a presheaf (of sets) on X is simply a functor

$$\mathcal{F}\colon Open(X)^{op} \to \mathbf{Set}.$$

For it to be called a sheaf, this functor has to satisfy the so-called sheaf condition: For any open $U \subseteq X$ and any open cover $\{U_i \subseteq U\}_{i\in I}$ of U, the sequence

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{j,k\in I}\mathcal{F}(U_j\cap U_k)$$

is exact, i.e. $\mathcal{F}(U)$ is the equalizer of the two possible restrictions.

The point is that this definition does not make too much use of the fact that X is a topological space. A presheaf can be defined for any category, not just on $Open(X)$, and for the sheaf condition, we just need a concept equivalent to an "open" cover. This leads us the following two definitions, first generalizing topological spaces, and then the concept of a sheaf:

Definition 2.1.1. *A site is a category $\mathcal{C}$ equipped with a designated class of so-called covering families (or just covers) $\{\varphi_i\colon U_i \to U\}_{i\in I}$ of morphisms in $\mathcal{C}$, satisfying the following axioms:*

- *If $\varphi\colon V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi\}$ is a covering family (consisting of one element).*

- *If $\{U_i \to U\}_{i\in I}$ is a cover and for each $i \in I$ we have a cover $\{V_{ij} \to U_i\}_{j\in J}$, then the family $\coprod_{i\in I}\{V_{ij} \to U_i \to U\}_{j\in J}$ is also a cover.*

- *If $\{U_i \to U\}_{i\in I}$ is a cover and $V \to U$ is any morphism in $\mathcal{C}$, then $\{U_i \times_U V \to V\}_{i\in I}$ is again a covering family (we implicitly assume that all relevant fiber products exist).*

Definition 2.1.2. *Let $\mathcal{C}$ be a site. A presheaf for the site $\mathcal{C}$ with values in $\mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$ is a functor $F\colon C^{op} \to \mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$. Note that this definition has nothing to do with the covers.*

A sheaf for the site $\mathcal{C}$ with values in $\mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$ is a presheaf such that for any covering family $\{U_i \to U\}_{i\in I}$, the sequence

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{j,k\in I}\mathcal{F}(U_j \times_U U_k)$$

is exact.

Example. *The X be a topological space. We define the site associated to X as the site having the underlying category $Open(X)$, the open subsets of X ordered by inclusion and that has as covers precisely the families $\{U_i \subseteq U\}_{i\in I}$ where $\bigcup_{i\in I} U_i = U$.*

Remark. *The classical definition of a sheaf on a space X clearly coincides with our Definition 2.1.2 of a sheaf for the site associated with X.*

Apart from the fact that classical sheaf theory is a special case of the theory of sheaves on a site, we also have our main site of interest, the étale site.

Definition 2.1.3. *Let X be a locally Noetherian scheme. We define two sites associated to X:*

- *The Zariski site X_{Zar}: The underlying category is the full subcategory of $\mathbf{Sch}/X$ consisting of the open immersions, and a family $\{\iota_i\colon U_i \to U\}_{i\in I}$ is a cover iff $\bigcup_{i\in I}\iota_i(U_i) = U$.*

- *The étale site $X_{\acute{e}t}$: The underlying category is the full subcategory of $\mathbf{Sch}/X$ consisting of étale morphisms that are of finite type, and a family $\{f_i\colon U_i \to U\}_{i\in I}$ is a cover iff $\bigcup_{i\in I} f_i(U_i) = U$.*

To illustrate what we need the étale site for, we first have to develop some theory, in particular on the category of sheaves and exact sequences.

A big part of this is done in a way that reminds us of the one for sheaves on topological spaces if we imagine the objects of our site to be the opens of a topological space.

Definition 2.1.4. *Let $\mathcal{C}$ be a site. We define the category $\mathcal{P}(\mathcal{C})_\mathcal{A}$ of presheaves with values in $\mathcal{A} = \mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$ to be the functor category $Fun(\mathcal{C}^{op}, \mathcal{A})$, i.e. a morphism of presheaves is just a natural transformation. We denote by $\mathcal{S}(\mathcal{C})_\mathcal{A} \subseteq \mathcal{P}(\mathcal{C})_\mathcal{A}$ the full subcategory of sheaves.*

If $\mathcal{A} = \mathbf{Mod}_R$, we simply write $\mathcal{S}(\mathcal{C})_R \coloneqq \mathcal{S}(\mathcal{C})_{\mathbf{Mod}_R}$ (resp. $\mathcal{P}(\mathcal{C})_R \coloneqq \mathcal{P}(\mathcal{C})_{\mathbf{Mod}_R}$). If $R = \mathbb{Z}$, we write $\mathcal{S}(\mathcal{C}) \coloneqq \mathcal{S}(\mathcal{C})_{\mathbf{Mod}_\mathbb{Z}}$ (resp. $\mathcal{P}(\mathcal{C}) \coloneqq \mathcal{P}(\mathcal{C})_{\mathbf{Mod}_\mathbb{Z}}$) and call it the category of abelian sheaves (resp. presheaves).

One small technical difficulty arises with this definition from set theory: If our site is not essentially small, then there is no guarantee that the category of presheaves or sheaves is locally small. In fact, the category of presheaves of sets will never be locally small in that case:

Lemma 2.1.5. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is essentially small if and only if the functor category $Fun(\mathcal{C}, \mathbf{Set})$ and $\mathcal{C}$ itself are both locally small.*

Proof. See [FS95]. □

There are multiple ways of circumventing this issue. We could work with universes or, as some references on the topic do, just ignore the problem entirely. However, we are going to only work with essentially small sites, and so the following lemma guarantees us that our sheaf and presheaf categories are always locally small.

Lemma 2.1.6. *Let $\mathcal{C}$ be an essentially small category and $\mathcal{D}$ be a locally small category. Then $Fun(\mathcal{C}, \mathcal{D})$ is locally small.*

Proof. We may assume that $\mathcal{C}$ is small. For two functors $F, G \colon \mathcal{C} \to \mathcal{D}$, their natural transformations belong to the set

$$\prod_{A \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(FA, GA).$$

□

This restriction also explains why our definition of the étale site (cf. Definition 2.1.3) differs slightly from the one used for example in [Tam94]: We require our schemes to be étale and of finite type, instead of just requiring them to be étale (and hence also locally of finite type), because we have to ensure that our site is essentially small.

We also only define the étale site for schemes that are locally Noetherian, because this condition ensures that all open immersions lie within our site (they are only locally of finite type in general), which will turn out useful for applications, as open immersions are one of the most basic examples of étale morphisms and having access to them essentially allows us to perform all tricks familiar from the Zariski site also on the étale site.

With this out of the way, we are now ready to collect our first real categorical properties of the category of presheaves, all of which are basic properties of functor categories in general.

Proposition 2.1.7. *Let $\mathcal{C}$ be an essentially small site and let $\mathcal{A} = \mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$. We have the following properties:*

- *$\mathcal{P}(\mathcal{C})_{\mathcal{A}}$ is locally small.*
- *$\mathcal{P}(\mathcal{C})_{\mathcal{A}}$ has all small limits and colimits, and they are computed pointwise.*
- *If $\mathcal{A} = \mathbf{Mod}_R$, then $\mathcal{P}(\mathcal{C})_{\mathcal{A}}$ is an abelian category satisfying AB5 and AB4*.*

Proof. The first statement follows from Lemma 2.1.6, the second statement is a standard result on functor categories and the last statement follows from the second one. □

Of course, we usually do not care too much about presheaves, and restrict our attention only to those satisfying the sheaf property. While this subcategory could initially seem pretty hard to study formally, one of the main results already familiar from the site $Open(X)$ associated to a topological space X comes in handy here: The existence of a sheafification functor left adjoint to the inclusion.

Theorem 2.1.8. *Let $\mathcal{C}$ be an essentially small site and let $\mathcal{A} = \mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$. Then the inclusion $\mathcal{S}(\mathcal{C})_{\mathcal{A}} \hookrightarrow \mathcal{P}(\mathcal{C})_{\mathcal{A}}$ admits an exact left adjoint $(-)^{\#}\colon \mathcal{P}(\mathcal{C})_{\mathcal{A}} \to \mathcal{S}(\mathcal{C})_{\mathcal{A}}$, called sheafification.*

Proof. See [Art62, Chapter II, Theorem 1.1.]. □

With access to this, all of the categorical properties we need and are familiar with from working on topological spaces follow from a very formal argument, making heavy use of the adjunction.

Corollary 2.1.9. *Let $\mathcal{C}$ be an essentially small site and let $\mathcal{A} = \mathbf{Set}/\mathbf{Ring}/\mathbf{Mod}_R$. We have the following properties:*

- *$\mathcal{S}(\mathcal{C})_{\mathcal{A}}$ has all small limits, and they are computed in $\mathcal{P}(\mathcal{C})_{\mathcal{A}}$, i.e. pointwise.*
- *$\mathcal{S}(\mathcal{C})_{\mathcal{A}}$ has all small colimits, and they are computed by taking the colimit in $\mathcal{P}(\mathcal{C})_{\mathcal{A}}$, and then sheafifying.*
- *If $\mathcal{A} = \mathbf{Mod}_R$, then $\mathcal{S}(\mathcal{C})_{\mathcal{A}}$ is an abelian category satisfying `AB5` and `AB3*`.*

Proof. For the first statement, all we have to check is that the pointwise limit of sheaves still satisfies the sheaf property from Definition 2.1.2. But this property is spelled out in terms of a certain limit, and limits are exchangeable.

The second statement is a direct consequence of the fact that sheafification, like any left adjoint functor, is cocontinuous.

For the last statement, in order to show that the category is abelian, all that remains to show is that image and coimage are canonically isomorphic. But this canonical morphism is just the sheafification of the respective morphism between the presheaf-coimage and -image, and this is an isomorphism by Proposition 2.1.7. The fact that it satisfies `AB3*` is clear from the first statement and for `AB5`, we only have to check the left exactness of filtered colimits (since any colimit is always right exact). This follows from the fact that $\mathcal{P}(\mathcal{C})_{\mathcal{A}}$ satisfies `AB5` (cf. Proposition 2.1.7), and the way colimits are computed in $\mathcal{S}(\mathcal{C})_{\mathcal{A}}$ (because sheafification is exact). □

We also want to generalize the concept of continuous maps between spaces: For a continuous map $f\colon X \to Y$ we obtain, via taking preimages, a functor $f^{-1}\colon Open(Y) \to Open(X)$ that preserves open covers and commutes with intersections.

It is straightforward to generalize this to our setup using just these two properties, but it should be noted that, to match our intuition coming from continuous maps, a morphism of sites is a functor between the underlying categories in the opposite direction, which may be a bit confusing in this purely abstract setting.

Definition 2.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be two sites. A morphism of sites from $\mathcal{C}$ to $\mathcal{D}$ is a functor $f\colon \mathcal{D} \to \mathcal{C}$ of the underlying categories such that:*

- *For every cover $\{U_i \to U\}_{i\in I}$ of $\mathcal{D}$, the induced family $\{f(U_i) \to f(U)\}_{i\in I}$ is a cover of $\mathcal{C}$.*
- *For every cover $\{U_i \to U\}_{i\in I}$ of $\mathcal{D}$ and every morphism $V \to U$ is $\mathcal{D}$ the natural map*

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is an isomorphism for all $i \in I$.

Pretty much by construction, our first example, of course, is the morphism of sites associated to a continuous map between topological spaces.

Example. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then taking preimages defines a morphism $f^{-1}\colon Open(Y) \to Open(X)$ of the associated sites.*

But also in the setting more relevant to us in this paper, we get an obvious morphism of the respective étale sites associated to a morphism of locally Noetherian schemes.

Example. *Let $f\colon X \to Y$ be a morphism of locally Noetherian schemes. Then the base change along f defines a morphism $X_{ét} \to Y_{ét}$, which we typically also denote by f. The first condition follows from the fact that surjectivity is stable under base change (applied to $\coprod_{i\in I} U_i \to U$), and the second condition follows from the general fact that limits commute with other limits.*

Of course, defining a morphism of sites simply as a functor with some mild permanence properties is way more flexible than just these two examples could ever hint at. A nice application of this flexibility can be seen later in Definition 2.2.19, after having introduced more background on the pushforward and pullback functors assigned to a morphism of sites.

From now on, we will restrict our attention only to sheaves with values in module categories, as this is our main case of interest and it lets us avoid notational clutter. Also note that this means that in the following, all our sheaf and presheaf categories are going to be abelian categories by Corollary 2.1.9 and Proposition 2.1.7.

To this end, fix a ring R and write for an arbitrary site $\mathcal{C}$ in the following $\mathcal{P}(\mathcal{C}) \coloneqq \mathcal{P}(\mathcal{C})_{\mathbf{Mod}_R}$ and $\mathcal{S}(\mathcal{C}) \coloneqq \mathcal{S}(\mathcal{C})_{\mathbf{Mod}_R}$.

With this in mind, we will now define the familiar pushforward and pullback functors associated to a morphism of sites in this setting. As before, we are more interested in the respective functors between the sheaf categories, but first, we will go over the presheaf case using a general result on functor categories, and only then turn our attention towards sheaves.

Proposition 2.1.11 (Kan Extension)**.** *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of sites and assume that $\mathcal{D}$ is essentially small. Then $f_p \coloneqq (-) \circ f\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ admits a left adjoint $f^p\colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ given by*

$$\mathcal{F} \mapsto \left(U \mapsto \operatorname*{colim}_{U \to fV} \mathcal{F}(V) \right)$$

with the obvious restriction maps, where the colimit runs over all arrows of the given form.

Proof. See [Lan98, Chapter X, Section 3] (and dualize). $\square$

Remark. *Proposition 2.1.11 has nothing to do with sites; the only reason we assume $\mathcal{C}$ and $\mathcal{D}$ to be sites is so that we can use the earlier introduced notation for presheaves.*

The fact that the pushforward and pullback functors are a pair of adjoints immediately implies that the pushforward functor is left exact and the pullback functor is right exact. Keeping our working example of a topological space in mind, there is, of course, no reason to expect the pushforward functor to also be exact on the right, but the pullback functor is exact, which would require a different argument.

However, in the present generality, this fails to be true. Luckily, we can salvage the situation by introducing a very mild condition on the domain, which will be satisfied in all the application relevant for us, and also explains the exactness of the pullback in the familiar case of topological spaces.

Corollary 2.1.12. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of essentially small sites and assume that $\mathcal{C}$ has finite limits. Then the colimit defining f^p is filtered and f^p is exact.*

Proof. Obvious. $\square$

Although the pullback of a presheaf is, in general, hard to compute explicitly, as should be obvious from its construction, in the special case where the presheaf is representable, we get an easy description of its pullback using the Yoneda lemma:

Proposition 2.1.13. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of essentially small sites, let $X \in \mathcal{D}$ be an abelian group object and consider the abelian presheaf $\operatorname{Hom}_{\mathcal{D}}(-, X) \in \mathcal{P}(\mathcal{D})$. Then $f^p \operatorname{Hom}_{\mathcal{D}}(-, X) \cong \operatorname{Hom}_{\mathcal{C}}(-, fX)$ (in particular, we find that fX is an abelian group object itself).*

Proof. This follows from the Yoneda lemma: For any presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ we have naturally

$$\begin{aligned} \operatorname{Hom}_{\mathcal{P}(\mathcal{C})}(f^p \operatorname{Hom}_{\mathcal{D}}(-, X), \mathcal{F}) &\cong \operatorname{Hom}_{\mathcal{P}(\mathcal{D})}(\operatorname{Hom}_{\mathcal{D}}(-, X), f_p\mathcal{F}) \cong \\ &\cong f_p\mathcal{F}(X) \cong \mathcal{F}(f(X)) \cong \operatorname{Hom}_{\mathcal{P}(\mathcal{C})}(\operatorname{Hom}_{\mathcal{C}}(-, fX), \mathcal{F}). \end{aligned}$$

$\square$

Now that we have covered the preliminaries for presheaves, we want to study how these two functors interact with the sheaf condition. Once more, the same phenomenon witnessed for continuous maps can also be observed here:

Proposition 2.1.14. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of essentially small sites. Then the pushforward functor $f_p\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ restricts to a functor $f_*\colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{D})$.*

Proof. Let $F \in \mathcal{S}(\mathcal{C})$. We have to show that $F(f(-))\colon \mathcal{D}^{op} \to \mathbf{Mod}_R$ is a sheaf. Let $\{U_i \to U\}_{i\in I}$ be a cover of $\mathcal{D}$. We get from Definition 2.1.10 the isomorphism

$$\begin{tikzcd} F(f(U)) \arrow[r] \arrow[d, equal] & \prod_{i\in I} F(f(U_i)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{j,k\in I} F(f(U_j \times_U U_k)) \arrow[d, "\cong"] \\ F(f(U)) \arrow[r] & \prod_{i\in I} F(f(U_i)) \arrow[r, shift left] \arrow[r, shift right] & \prod_{j,k\in I} F(f(U_j) \times_{f(U)} f(U_k)) \end{tikzcd}$$

and that the lower sequence is exact, as $\{f(U_i) \to f(U)\}_{i\in I}$ is a cover of $\mathcal{C}$. $\square$

Of course, the presheaf pullback also preserving sheaves fails to be true, as we know from topological spaces, but just as in that case, we can simply sheafify the result to get the left adjoint to the pushforward on sheaves.

Proposition 2.1.15. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of essentially small sites. Then the pushforward functor $f_*\colon \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{D})$ admits a left adjoint, given by $f^* := (f^p(-))^\#\colon \mathcal{S}(\mathcal{D}) \to \mathcal{S}(\mathcal{C})$, called the pullback functor.*

Proof. By combining the results from Theorem 2.1.8 and Proposition 2.1.11 we get a natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}(\mathcal{C})}(f^*(-), -) \cong \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(f^p(-), -) \cong \mathrm{Hom}_{\mathcal{P}(\mathcal{D})}(-, f_p(-)) \cong \mathrm{Hom}_{\mathcal{S}(\mathcal{D})}(-, f_*(-)),$$

where we used Proposition 2.1.14 in the last step. □

By combining two previous results, we immediately see that the pullback functor on sheaves is exact, as long as the one on presheaves is, so, for example, under the same mild conditions from Corollary 2.1.12.

Corollary 2.1.16. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a morphism of essentially small sites and assume that $\mathcal{C}$ has finite limits. Then the pullback functor f^* is exact.*

Proof. By Proposition 2.1.15, this functor is the composition of two functors, which are both exact according to Theorem 2.1.8 and Corollary 2.1.12. □

2.2 The Étale Site

After this short general review of sites, we are now ready to specialize our results to the étale site and combine it with our results from the first chapter. We start by noting that all of our results for general sites also hold for the étale and Zariski sites, as both of them have all finite limits.

All sheaves still have, unless explicitly stated otherwise, values in $\mathbf{Mod}_R$ for some ring R.

Lemma 2.2.1. *Let X be a locally Noetherian scheme. Then both the Zariski site X_{Zar} and the étale site $X_{\acute{e}t}$ have all finite limits.*

Proof. This is obvious for X_{Zar}. The étale site $X_{\acute{e}t}$ has a terminal object and fiber products by Corollary 1.3.2. Therefore, it has finite products and in any category with finite products and fiber products, we may construct the equalizer of two morphisms $f, g\colon X \to Y$ as the pullback

$$\begin{tikzcd} eq(f,g) \arrow[r] \arrow[d] & X \arrow[d, "(f;g)"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}.$$

It is well known that any category with finite products and equalizers has all finite limits. □

Since having finite limits was the sole condition in Corollary 2.1.16 for the pullback functor to be exact, we get this familiar result also for the étale site.

Corollary 2.2.2. *Let X be a locally Noetherian scheme and let $\mathcal{C}$ be an arbitrary essentially small site. Let f be a morphism from either the Zariski or the étale site of X to $\mathcal{C}$. Then the pullback functor on sheaves f^* and on presheaves f^p are both exact.*

Proof. Follows from Lemma 2.2.1, Corollary 2.1.16 and Corollary 2.1.12. $\square$

One important concept we have for sheaves on topological spaces are stalks, and yet they were completely absent from the general theory we developed in the first section of this chapter. This is because, in this generality, there is no good theory of stalks on a site in the following sense:

A stalk of a sheaf is its pullback to a "point", and what we would want are enough of these "points" to detect exactness, i.e. a sequence of sheaves should be exact if and only if its pullbacks to all "points" are exact.

This is handy because, at least in topology, a sheaf on a point is easy to describe: there is only one non-empty open and so the category of abelian sheaves on a point is equivalent, via the global sections functor, to the category of abelian groups, where exactness is a reasonably well understood concept, and way more hands-on than a sequence of sheaves on a more complicated space.

This leads us to the following idea of what a "point" of a site $\mathcal{C}$ should be in our general setting: a site $\mathcal{P}$ with a morphism $\mathcal{P} \to \mathcal{C}$ such that the global sections functor on $\mathcal{S}(\mathcal{P})$ (exists and) is an equivalence.

While there is no hope in general to have enough of these points to detect exactness, it is well-known that we do for the Zariski site of a scheme and, in fact, we also do for the étale site. Note that, by Proposition 1.2.3, the étale site of a geometric point of a scheme defines a "point" in the sense of this discussion of stalks, and so we make the following definition:

Definition 2.2.3. *Let X be a locally Noetherian scheme, $x \in X$ and $\bar{x}\colon \bar{x} \to \operatorname{Spec} \kappa(x) \to X$ a geometric point of X, i.e. a morphism from $\bar{x} = \operatorname{Spec} K$ to X where K is a separably closed field. Let $\mathcal{F}$ be a presheaf on $X_{ét}$. The étale stalk of $\mathcal{F}$ at $\bar{x}$ is defined to be $\mathcal{F}_{\bar{x}} \coloneqq (\bar{x}^p \mathcal{F})(\bar{x})$.*

A scheme has, of course, way more geometric points than its underlying topological space. Luckily, the stalk at a geometric point depends, up to (non-canonical) isomorphism, only on the underlying topological point. This allows us to do the computation of stalks for only a much smaller collection of points.

Lemma 2.2.4. *We continue the notation from Lemma 2.2.4. The étale stalk of $\mathcal{F}$ at x is independent, up to isomorphism, of the choice of separably closed field K.*

Proof. It given by a particular colimit described in Proposition 2.1.11, namely over the category that has as objects morphisms from $\bar{x}$ to $V_{\bar{x}}$ for some étale morphism $V \to X$. But this category is obviously equivalent, by the universal property of the fiber product, to the category of commutative triangles

$$\begin{tikzcd} & V \arrow[dr, "ét"] & \\ \bar{x} \arrow[ur] \arrow[rr] & & X \end{tikzcd},$$

and as long as K is separably closed, this category does not depend on the exact choice of K. $\square$

We have defined the étale stalk to be the global sections of the presheaf pullback to a geometric point. The next lemma shows that we could just as well have defined it to be the global sections of the sheaf pullback, since the result is canonically isomorphic.

Lemma 2.2.5. *Let K be a separably closed field, $\bar{x} \coloneqq \operatorname{Spec} K$ and let $\mathcal{F}$ be a presheaf on $\bar{x}_{\acute{e}t}$. Then we canonically have $\mathcal{F}(\bar{x}) \cong \mathcal{F}^{\#}(\bar{x})$.*

Proof. By Proposition 1.2.3, any étale morphism $U \to \bar{x}$ is a disjoint union of copies of $\bar{x}$. Keeping this in mind, it is easy to see from the universal property of sheafification that $\mathcal{F}^{\#}(U) \cong \mathcal{F}^{\pi_0(U)}(U)$, where $\pi_0(U)$ is the (finite) number of copies of $\bar{x}$ that U consists of.

The statement we are trying to prove is the $U = \bar{x}$ special case of that general statement. $\square$

We are also used to a presheaf and its sheafification having the same stalk, and this is also true here via a quite formal argument.

Proposition 2.2.6. *Let X be a locally Noetherian scheme, let $\mathcal{F} \in \mathcal{P}(X_{\acute{e}t})$ be a presheaf and let $\bar{x}\colon \bar{x} \to X$ be a geometric point. Then we have a natural isomorphism $\mathcal{F}_{\bar{x}} \cong (\mathcal{F}^{\#})_{\bar{x}}$.*

Proof. If $f\colon X \to Y$ denotes any morphism of essentially small sites and $\mathcal{F}$ is a presheaf on Y, then we canonically have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{S}(X)}((f^p\mathcal{F})^{\#}, -) \cong \operatorname{Hom}_{\mathcal{P}(X)}(f^p\mathcal{F}, -) \cong \operatorname{Hom}_{\mathcal{P}(Y)}(\mathcal{F}, f_p(-)) \cong \\ \cong \operatorname{Hom}_{\mathcal{S}(Y)}(\mathcal{F}^{\#}, f_*(-)) \cong \operatorname{Hom}_{\mathcal{S}(Y)}(f^*(\mathcal{F}^{\#}), -),\end{aligned}$$

i.e. $(f^p\mathcal{F})^{\#} \cong f^*(\mathcal{F}^{\#})$. Combining this with Lemma 2.2.5 in our specific situation, we get

$$\mathcal{F}_{\bar{x}} = \bar{x}^p\mathcal{F}(\bar{x}) \cong (\bar{x}^p\mathcal{F})^{\#}(\bar{x}) \cong \bar{x}^*(\mathcal{F}^{\#})(\bar{x}) \cong (\bar{x}^p(\mathcal{F}^{\#}))^{\#}(\bar{x}) \cong \bar{x}^p(\mathcal{F}^{\#})(\bar{x}) = (\mathcal{F}^{\#})_{\bar{x}}.$$

$\square$

After these basic preliminaries for étale stalks, we now come to the main point of interest: their ability to detect exact sequences.

Theorem 2.2.7. *Let X be a locally Noetherian scheme and let $\{\bar{x}\colon \bar{x} \to X\}_{x\in X}$ be a choice of geometric points, one for each point of the underlying topological space of X. Then a sequence*

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

of sheaves on $X_{\acute{e}t}$ is exact if and only if the sequence

$$\mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}}$$

is exact for each $x \in X$.

Proof. The only if is implied by Corollary 2.2.2 anyway, and for the converse see [Mil80, Chapter II, Theorem 2.15]. $\square$

Since we have skipped over some of the tools required to effectively compute stalks, we also want a more elementary description of exactness, and especially for a morphism of étale sheaves to be an epimorphism.

Theorem 2.2.8. *Let X be a locally Noetherian scheme and consider the sequence*

$$0 \to \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \to 0$$

of étale sheaves on X. Then we have:

- *The sequence is exact in $\mathcal{S}(X_{ét})$ at $\mathcal{F}'$ and $\mathcal{F}$ if and only if it is exact in $\mathcal{P}(X_{ét})$.*
- *The sequence is exact at $\mathcal{F}''$ if and only if for each section $s \in \mathcal{F}''(U \to X)$ there exists an étale cover $\{U_i \to U\}_{i \in I}$ and elements $s_i \in \mathcal{F}(U_i)$ such that $g(s_i) = s_{|U_i}$ for all $i \in I$.*

Proof. The first statement is an immediate consequence of Theorem 2.1.8.

The second statement also relies on the sheafification, because the cokernel of g is just the sheafification of its presheaf cokernel by Corollary 2.1.9. Since we have not constructed this functor here, we have to refer to [Mil80, Chapter II, Theorem 2.15]. □

Recall that in our section on flat morphisms, we have proven a rather technical condition concerning the relation between faithfully flat morphisms and effective epimorphisms (cf. Theorem 1.1.10). While we lacked the relevant context to explain this theorem's usefulness back then, we can now come back and fix that. As it turns out, on the étale site, all presheaves represented by a scheme (not necessarily in the étale site) are actually sheaves, which will be our first major source for examples of étale sheaves.

Note first that when we fix a scheme A, then the statement of the theorem yields an exact sequence

$$\mathrm{Hom}(S, A) \xrightarrow{f} \mathrm{Hom}(X, A) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathrm{Hom}(X \times_S X, A),$$

which is to say, assuming X and S are étale schemes over some base, $\mathrm{Hom}(-, A)$ satisfies the sheaf condition for the étale cover $\{X \to S\}$.

To see that this is essentially sufficient for representable presheaves to satisfy the sheaf condition for all covers, we just need the following preliminary result:

Proposition 2.2.9. *Let X be a locally Noetherian scheme and $\mathcal{F} \in \mathcal{P}(X_{ét})$. Then $\mathcal{F}$ is a sheaf if and only if it satisfies the sheaf condition for covers of the form*

- *$\{U_i \to U\}_{i \in I}$, where all the U_i are open immersions,*
- *$\{V \to U\}$, where $V \to U$ is a single surjective morphism of affine schemes.*

Proof. The "only if" is clear anyway, and the converse can be found in [Tam94, Chapter II, Lemma 3.1.1]. □

Corollary 2.2.10. *Let X be a locally Noetherian scheme and let $A \to X$ be an abelian group scheme over X. Then the abelian presheaf $\mathrm{Hom}_X(-, A) \in \mathcal{P}(X_{ét})$ is a sheaf.*

Proof. By Proposition 2.2.9 we only have to check the sheaf condition for specific kinds of covers. For the Zariski covers (i.e. the open immersions), the sheaf condition is the well-known gluing of schemes. For the surjective single morphism, this is implied by Theorem 1.1.10. □

Note that such a presheaf is not necessarily representable in the sense of Proposition 2.1.13, as A was not assumed to be étale and of finite type.

However, if it is, then Proposition 2.1.13 applies and we get a nice way of computing the pullbacks of these sheaves, in particular their stalks.

Corollary 2.2.11. *Let $f\colon X \to Y$ be a morphism of locally Noetherian schemes and let $A \to Y$ be an abelian group scheme over Y that is étale and of finite type. Then $f^* \operatorname{Hom}_{Y_{\acute{e}t}}(-, A) \cong \operatorname{Hom}_{X_{\acute{e}t}}(-, A \times_Y X)$.*

Proof. We only need to sheafify the according isomorphism from Proposition 2.1.13, but by Corollary 2.2.10 the representable presheaf there is already a sheaf, and hence isomorphic to its sheafification. □

Now that we have a reasonably good understanding of representable étale sheaves, we are finally in a good spot to introduce our first and also most important examples of étale sheaves.

Example. *Let X be a locally Noetherian scheme. The abelian étale presheaves*

- $\mathbb{G}_a\colon U \mapsto \mathcal{O}_U(U)$,
- $\mathbb{G}_m\colon U \mapsto \mathcal{O}_U(U)^*$ *and*
- $\mu_n\colon U \mapsto \mu_n(\mathcal{O}_U(U))$

are sheaves, as they are represented by

- $\mathbb{A}^1_X$,
- $X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$ *and*
- $X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[T]/(T^n - 1)$

respectively, where μ_n denotes the group of n-th roots of unity. Also note that the scheme representing μ_n is represented by a finite étale scheme, so we can apply Corollary 2.2.11 to easily compute its stalks: if $\bar{x}\colon \bar{x} := \operatorname{Spec} K \to X$ is a geometric point, then

$$\mu_{n,\bar{x}} \cong \operatorname{Hom}_{\bar{x}}(\bar{x}, \operatorname{Spec} K[T]/(T^n - 1)) \cong \operatorname{Hom}_K(K[T]/(T^n - 1), K) \cong \mu_n(K) = \{a \in K\colon a^n = 1\}.$$

Now that we have both some hands-on examples of étale sheaves and an initial understanding of exact sequences, we can introduce one of the most important short exact sequences in étale cohomology, and also highlight a key difference to the Zariski site.

Proposition 2.2.12. *Let $n \in \mathbb{N}$ and let X be a locally Noetherian scheme such that n is invertible in X. Then the sequence of abelian étale sheaves*

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 0$$

is exact. The analogous statement for the Zariski site is false in general, even for $X = \mathbb{A}^1_{\mathbb{C}}$ and $n \neq 1$.

Proof. The sequence is exact at the first two positions almost by definition, even for the Zariski site.

The morphism $\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$ is not surjective on the Zariski site for $\mathbb{A}^1_{\mathbb{C}}$, as can easily be seen at the (Zariski) stalk of the origin.

To see how the étale site fixes this issue, we want to use Theorem 2.2.8. Let $U \to X$ be a finite type étale morphism and let $s \in \mathcal{O}_U(U)^*$.

Let $\operatorname{Spec} A_i = U_i' \subseteq U$ be an open affine cover of U and let $U_i \to U_i'$ be the affine morphism corresponding to $A_i \to A_i[X_i]/(X_i^n - s_{|U_i'})$. We want to show that $\{U_i \to U\}_{i \in I}$ is an étale cover and the $X_i \in \mathcal{O}_{U_i}(U_i)^*$ satisfy the condition from Theorem 2.2.8.

All claims are obvious by construction if we can show that $U_i \to U_i'$ is étale. It is certainly flat, as $A_i[X_i]/(X_i^n - s_{|U_i'})$ is a free A_i-module of rank n.

To show that it is unramified, we want to use Corollary 1.2.5. A standard computation shows that the module of relative Kähler differentials Ω^1 of this extension is isomorphic to

$$\frac{A[X_i]/(X_i^n - s_{|U_i'})}{(X_i^n - s_{|U_i'}) + (nX_i^{n-1})} = \frac{A[X_i]/(X_i^n - s_{|U_i'})}{(X_i^n - s_{|U_i'} - \frac{X_i}{n} nX_i^{n-1})} = \frac{A[X_i]/(X_i^n - s_{|U_i'})}{(s_{|U_i'})} \cong 0,$$

so we are done. $\square$

Next, we want to study how étale sheaves behave under restriction to open and closed subschemes. First, we define a category that, in some sense, splits up the datum of a given sheaf into a datum of sheaves on certain subschemes.

Definition 2.2.13. *Let X be a locally Noetherian scheme and let $i\colon Z \to X$ be a closed immersion. Denote by $j\colon U \subseteq X$ the complement of $i(Z)$. We define the category $\mathcal{T}(X)$ of triples $(\mathcal{F}_1), \mathcal{F}_2, \phi)$, where $\mathcal{F}_1 \in \mathcal{S}(Z_{ét}), \mathcal{F}_2 \in \mathcal{S}(U_{ét})$ and $\phi\colon \mathcal{F}_1 \to i^*j_*\mathcal{F}_2$, that has as morphisms pairs $(f_1\colon \mathcal{F}_1 \to \mathcal{F}_1', f_2\colon \mathcal{F}_2 \to \mathcal{F}_2')$ such that*

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "\phi"] \arrow[d, "f_1"] & i^*j_*\mathcal{F}_2 \arrow[d, "i^*j_*f_2"] \\ \mathcal{F}_1' \arrow[r, "\phi'"] & i^*j_*\mathcal{F}_2' \end{tikzcd}$$

commutes.

The reason for introducing this category is that it is equivalent, via a rather naturally defined functor, to the category of sheaves on X.

Theorem 2.2.14. *We continue the notation from Definition 2.2.13. The functor*

$$\begin{aligned} \mathcal{S}(X_{ét}) &\to \mathcal{T}(X) \\ \mathcal{F} &\mapsto (i^*\mathcal{F}, j^*\mathcal{F}, \phi), \end{aligned}$$

where $\phi\colon i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}$ is the pullback along i of the canonical morphism $\mathcal{F} \to j_*j^*\mathcal{F}$, is an equivalence of categories. A sequence

$$(\mathcal{F}_1'', \mathcal{F}_2'', \phi'') \to (\mathcal{F}_1, \mathcal{F}_2, \phi) \to (\mathcal{F}_1', \mathcal{F}_2', \phi')$$

is exact in $\mathcal{T}(X)$ if and only if the sequences

$$\mathcal{F}_1'' \to \mathcal{F}_1 \to \mathcal{F}_1' \quad \text{and} \quad \mathcal{F}_2'' \to \mathcal{F}_2 \to \mathcal{F}_2'$$

are both exact in $\mathcal{S}(X_{ét})$.

Proof. See [Mil80, Chapter II, Theorem 3.10]. □

This interpretation of the category of sheaves allows us to define additional comparison functors to or from the category of étale sheaves on a subscheme, in particular the extension by zero functor in the special case of an open immersion.

Definition 2.2.15. *Using the equivalence from Theorem 2.2.14, it is possible to define six functors*

$$\mathcal{S}(Z_{ét}) \;\underset{\overset{\xleftarrow{i^!}}{}}{\overset{\overset{\xleftarrow{i^*}}{}}{\xrightarrow{i_*}}}\; \mathcal{S}(X_{ét}) \;\underset{\overset{\xleftarrow{j_*}}{}}{\overset{\overset{\xleftarrow{j_!}}{}}{\xrightarrow{j^*}}}\; \mathcal{S}(U_{ét})$$

by

$$\begin{array}{ccccccc} \mathcal{F}_1 & \leftarrow\!\shortmid & (\mathcal{F}_1, \mathcal{F}_2, \phi) & & (0, \mathcal{F}_2, 0) & \leftarrow\!\shortmid & \mathcal{F}_2 \\ \mathcal{F}_1 & \mapsto & (\mathcal{F}_1, 0, 0) & & (\mathcal{F}_1, \mathcal{F}_2, \phi) & \mapsto & \mathcal{F}_2 \\ \ker \phi & \leftarrow\!\shortmid & (\mathcal{F}_1, \mathcal{F}_2, \phi) & & (i^* j_* \mathcal{F}_2, \mathcal{F}_2, id) & \leftarrow\!\shortmid & \mathcal{F}_2 \end{array}$$

Apart from the extension by zero functor, we will not really make use of this, and so we note, more for the sake of illustrating its usefulness outside of this paper, the following short exact sequence of étale sheaves.

Corollary 2.2.16. *Let X be a locally Noetherian scheme, let $i\colon Z \to X$ be a closed immersion and denote by $j\colon U \subseteq X$ the open immersion of $U := X \setminus i(Z)$. Then for each sheaf $\mathcal{F} \in \mathcal{S}(X_{ét})$, we functorially have a short exact sequence*

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

Proof. Under the equivalence from Theorem 2.2.14 this sequence corresponds to the sequence

$$0 \to (0, \mathcal{F}_2, 0) \to (\mathcal{F}_1, \mathcal{F}_2, \phi) \to (\mathcal{F}_1, 0, 0) \to 0.$$

□

This is the sequence that yields the long exact sequence of a subspace in (relative) singular cohomology (cf. [Har77, Chapter III, Exercise 2.3]), and as soon as we introduce the concept of étale cohomology, we would theoretically be able to define an analogous long exact sequence also for the étale cohomology groups. But we will not make use of this.

We are now slowly moving towards introducing cohomology, but before we do this, we first collect some statements on the functors between sites we have that will come in useful later.

The next lemma is as basic as it is essential to working with universal δ-functors, and is mainly stated here for later reference, since it is also regularly used in ordinary sheaf cohomology.

Lemma 2.2.17. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories. Assume F admits an exact left adjoint $G\colon \mathcal{B} \to \mathcal{A}$. Then F preserves injectives, i.e. for each injective object $I \in \mathcal{A}$, the object $FI \in \mathcal{B}$ is also injective.*

Proof. There is an isomorphism of functors

$$\mathrm{Hom}_{\mathcal{B}}(-, FI) \cong \mathrm{Hom}_{\mathcal{A}}(G(-), I),$$

and by assumption, the right-hand side is a composition of exact functors, and hence itself an exact functor. □

This lemma interacts nicely with Definition 2.2.15, as we have a lot of exact adjoints there.

Proposition 2.2.18. *a) Every functor in Definition 2.2.15 is left adjoint to the functor directly below it.*

b) The functors i^*, i_*, j^* and $j_!$ are exact; $i^!$ and j_* are left exact.

c) The compositions $i^*j_!, i^!j_!, i^!j_*$ and j^*i_* are all zero.

d) The functors i_*, j_*, j^* and $i^!$ preserve injective objects.

Proof. The statements a)-c) are clear by construction, d) follows immediately from the fact that each functor has an exact left adjoint and Lemma 2.2.17. □

In Definition 2.2.15 we defined, for an open immersion, the extension by zero functor, which is also familiar from ordinary sheaf cohomology. There, an open immersion is the only kind of morphism where we could reasonably expect an extension by zero, but on the étale site, we have more objects in our underlying category.

We want to define the extension by zero also for them. Here, the more general setup from Definition 2.1.10 yields a nice explanation of the, admittedly elementary, fact that the presheaf pullback functor along an étale morphism (resp. an open immersion in the Zariski case) preserves sheaves and is usually referred to as the restriction: It has an interpretation as a pushforward functor along a second morphism of sites that we can only define when the morphism of schemes is itself étale (resp. an open immersion).

The extension by zero functor will then turn out to simply be its exact left adjoint, which exists by the general theory developed in Proposition 2.1.15.

Definition 2.2.19. *Let X be a locally Noetherian scheme and let $f\colon U \to X$ be an étale morphism of finite type. In addition to the usual associated morphism of sites $f\colon U_{\acute{e}t} \to X_{\acute{e}t}$, there is also the morphism of sites*

$$\tilde{f}\colon X_{\acute{e}t} \to U_{\acute{e}t}$$
$$(V \to U \xrightarrow{f} X) \hookleftarrow (V \to U).$$

We denote the induced pushforward functor on sheaves by $(-)_{|U} := \tilde{f}_*$. Concretely, if $\mathcal{F}$ is a sheaf on $X_{\acute{e}t}$, then $\mathcal{F}_{|U}(V \to U) = \mathcal{F}(V \to X)$.

It is not hard to see that $\tilde{f}_p \cong f^p$, since the category over which the colimit defining $f^p\mathcal{F}(V \to U)$ is taken has a final object for each $V \to U \in U_{ét}$, where $\mathcal{F}$ denotes an arbitrary presheaf over $X_{ét}$.

This alternate description of f^p shows that in the situation at hand, f^p preserves sheaves and hence there is no need to sheafify in the definition of f^*. We also obtain a functor $f_! := \tilde{f}^*$ left adjoint to f^*, called extension by zero. This functor is exact by Corollary 2.2.2 and by uniqueness of adjoints, it agrees with extension by zero functor from Definition 2.2.15 in the special case where f is an open immersion.

Remark. *Of course, there also exists the exact left adjoint $\tilde{f}^p$ of f^p, but since in the following we will only need it once, namely to show that $\mathcal{P}(X_{ét})$ has enough injectives, we will not give a separate name to it.*

In principle, we would now be ready to introduce cohomology, but as is the case for topological spaces, there are some considerable computational benefits to working on Noetherian schemes (or spaces), which will also be useful when computing certain cohomology groups, and so we briefly state the two main improvements we get on Noetherian schemes. Also note that we are always assuming our schemes to be locally Noetherian anyway, so we are now only really assuming quasi-compactness in addition to our usual standing assumption.

Proposition 2.2.20. *Let X be a Noetherian scheme. Denote by $X_{ét(f)}$ the site that has the same underlying category as $X_{ét}$, but has as covers only finite surjective families of morphisms (instead of arbitrary surjective families). Then we have $\mathcal{P}(X_{ét(f)}) = \mathcal{P}(X_{ét})$ and $\mathcal{S}(X_{ét(f)}) = \mathcal{S}(X_{ét})$.*

Proof. It is clear by definition $\mathcal{P}(X_{ét(f)}) = \mathcal{P}(X_{ét})$, as the presheaf categories do not depend on the choice of covers. We also have $\mathcal{S}(X_{ét(f)}) \supseteq \mathcal{S}(X_{ét})$, as a presheaf that satisfies the sheaf condition for all surjective families certainl satisfies it for finite surjective families.

For the reverse inclusion, let $\mathcal{F} \in \mathcal{S}(X_{ét(f)})$ and let $\mathcal{U} := \{U_i \to U\}_{i\in I}$ be an arbitrary cover for $X_{ét}$. If we assume that we have a section $s \in \mathcal{F}(U)$ such that $s_{|U_i} = 0$ for all $i \in I$, then $s = 0$ because there exists a finite subcover of $\mathcal{U}$ (because the images of étale morphisms are open and U is quasi-compact, as X was assumed to be Noetherian). Similarly, if we are given sections $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_{i|U_i\times_U U_j} = s_{j|U_i\times_U U_j}$, then for each finite subcover of $\mathcal{U}$ there exists a gluing s, and by the already proven uniqueness, they all agree. But now we have $s_{|U_i} = s_i$, since every morphism $U_i \to U$ is part of some finite subcover. $\square$

Corollary 2.2.21. *Let X be a Noetherian scheme. Then any small filtered colimit of sheaves, computed in $\mathcal{P}(X_{ét})$, is already a sheaf; so, in particular, it is already the colimit in $\mathcal{S}(X_{ét})$ with no need for sheafification.*

Proof. Let $\mathcal{F}\colon I \to \mathcal{S}(X_{ét})$ be a small filtered diagram. We want to show that $\operatorname{colim}_{i\in I} \mathcal{F}_i$, computed in $\mathcal{P}(X_{ét})$, satisfies the sheaf condition. By Proposition 2.2.20 it suffices to check finite covers $\{U_j \to U\}_{j\in J}$, but since filtered colimits preserve finite products, the sequence

$$0 \to \operatorname*{colim}_{i\in I} \mathcal{F}_i(U) \to \prod_{j\in J} \operatorname*{colim}_{i\in I} \mathcal{F}_i(U_j) \to \prod_{i,k\in J} \operatorname*{colim}_{i\in I} \mathcal{F}_i(U_j \times_U U_k)$$

is isomorphic to the sequence

$$0 \to \operatorname*{colim}_{i\in I} \mathcal{F}_i(U) \to \operatorname*{colim}_{i\in I} \prod_{j\in J} \mathcal{F}_i(U_j) \to \operatorname*{colim}_{i\in I} \prod_{i,k\in J} \mathcal{F}_i(U_j \times_U U_k),$$

which is exact by the exactness of filtered colimits. □

2.3 Cohomology

Now we finally come to cohomology. Having already generalized the concept of a sheaf, most results in this section follow from the abstract general machinery also used to develop the cohomology theory of sheaves on topological spaces, i.e. the theory of derived functors.

The one thing we need to derive additive functors (on the right) is, of course, enough injective objects in our abelian category. As it turns out, the category of étale sheaves admits a generator, defined in an analogous way to the usual generator on the Zariski site.

Proposition 2.3.1. *Let X be a locally Noetherian scheme. Then $\mathcal{P}(X_{ét})$ and $\mathcal{S}(X_{ét})$ have enough injectives.*

Proof. We prove the statement only for $\mathcal{S}(X_{ét})$, as this proof can easily be adapted to $\mathcal{P}(X_{ét})$ (cf. the Remark following Definition 2.2.19).

It is well known (cf. [Gro57, Theorem 1.10.1]) that it suffices to find a set of generators for $\mathcal{S}(X_{ét})$, as we already know that it satisfies `AB5` and `AB3`* from Corollary 2.1.9.

Remember that we write $\mathcal{S}(\mathcal{C}) \coloneqq \mathcal{S}(\mathcal{C})_{\mathbf{Mod}_R}$. Let $f\colon U \to X \in X_{ét}$ and denote by $\underline{R}_U$ the constant presheaf on $U_{ét}$ associated to R. Then the sheaf $f_!(\underline{R}_U^{\#})$ has the property

$$\begin{aligned}\operatorname{Hom}_{\mathcal{S}(X_{ét})}(f_!(\underline{R}_U^{\#}), \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{S}(U_{ét})}(\underline{R}_U^{\#}, \mathcal{F}_{|U}) \cong \operatorname{Hom}_{\mathcal{P}(U_{ét})}(\underline{R}_U, \mathcal{F}_{|U}) \cong \\ \cong \operatorname{Hom}_R(R, \mathcal{F}(U)) \cong \mathcal{F}(U),\end{aligned}$$

where this isomorphism is natural in $\mathcal{F}$.

Since the isomorphism classes of $X_{ét}$ form a (small) set, we get by choosing a representative $f\colon U \to X$ for each of these classes a set of generators $\{f_!(\underline{R}_U^{\#})\}_f$, as when $\mathcal{G} \subsetneq \mathcal{F}$ is a proper subsheaf, then there exists an f such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ is not surjective. □

With this, we can define all the familiar derived functors on the category of étale sheaves on a locally Noetherian scheme X.

Definition 2.3.2. *Let X be a locally Noetherian scheme. We have several left exact functors:*

- *The inclusion $\underline{H}^0_{ét}\colon \mathcal{S}(X_{ét}) \subseteq \mathcal{P}(X_{ét})$, which is left exact as it has the sheafification as a left adjoint.*
- *For any object $U \in X_{ét}$ we have the section functor $H^0_{ét}(X, U; -) \coloneqq \Gamma(U; -)\colon \mathcal{S}(X_{ét}) \to$ $\mathbf{Mod}_R$, which is left exact as the composition of the left exact functor $\underline{H}^0$ with the exact section functor on presheaves.*
- *As a special case of that, we get the global section functor $H^0_{ét}(X; -) \coloneqq H^0_{ét}(X, X; -) =$ $\Gamma(X; -)\colon \mathcal{S}(X_{ét}) \to \mathbf{Mod}_R$ associated to the identity $X \xrightarrow{id} X \in X_{ét}$.*

Using Proposition 2.3.1, we can define the following associated derived functors:

- *The cohomology presheaf* $\underline{H}^\bullet_{ét}(X, -) \coloneqq R^\bullet \underline{H}^0_{ét}(X, -)\colon \mathcal{S}(X_{ét}) \to \mathcal{P}(X_{ét})$.
- $H^\bullet_{ét}(X, U; -) \coloneqq R^\bullet H^0_{ét}(X, U; -)\colon \mathcal{S}(X_{ét}) \to \mathbf{Mod}_R$.
- *The étale cohomology groups* $H^\bullet_{ét}(X, -) \coloneqq R^\bullet H^0_{ét}(X, -)\colon \mathcal{S}(X_{ét}) \to \mathbf{Mod}_R$.

Usually, the only functors out of these that are actually used are the étale cohomology groups, and for a good reason: the three universal δ-functors we defined are all related and, in fact, we can (more or less) use the derived functors of the global sections to accurately describe the other two, making use of the restriction functor.

Lemma 2.3.3. *Let X be a locally Noetherian scheme and let $j\colon U \to X$ be an étale morphism of finite type. Then we have canonical isomorphisms of δ-functors $H^\bullet_{ét}(X, U; -) \cong H^\bullet_{ét}(U; (-)_{|U}) \cong \underline{H}^\bullet_{ét}(X, -)(U)$.*

Proof. We obviously have $H^0_{ét}(X, U; -) \cong H^0_{ét}(U; (-)_{|U}) \cong \underline{H}^0_{ét}(X, -)(U)$, so we only need to show that $H^\bullet_{ét}(U; (-)_{|U})$ and $\underline{H}^\bullet_{ét}(X, -)(U)$ are universal δ-functors.

$H^\bullet_{ét}(U; (-)_{|U})$ is a δ-functor because $(-)_{|U}$ is exact by Definition 2.2.19, and to show universality we have to show that the $H^i_{ét}(U; (-)_{|U})$ are effaceable for $i \geq 1$ (cf. [Gro57, Proposition 2.2.1]). But this is clear as $(-)_{|U}$ has the exact left adjoint $j_!$ from Definition 2.2.19, and hence preserves injective objects.

For $\underline{H}^\bullet_{ét}(X, -)(U)$ it suffices to note that the evaluation functor $\mathrm{ev}_U\colon \mathcal{P}(X_{ét}) \to \mathbf{Mod}_R$ is exact, as effacability is obvious. □

Of course, we also want a relative version of the cohomology groups, also known as the higher direct images. Again, to define them, we need nothing but the abstract theory of derived functors.

Definition 2.3.4. *Let $f\colon X \to S$ be a morphism of locally Noetherian schemes. The derived functors $R^i f_*\colon \mathcal{S}(X_{ét}) \to \mathcal{S}(S_{ét})$ are called the higher direct images of f. Note that f_* is left exact by Proposition 2.1.15.*

Just like in the topological setting, the higher direct images are nothing but the sheafification of the pushforward of the cohomology presheaf on the domain, a result which is interesting not just for direct computations, but will also be the main step in showing that our generalization of flasque resolutions may be used to compute higher direct images.

Proposition 2.3.5. *Let $f\colon X \to S$ be a morphism of locally Noetherian schemes. Then we have a canonical isomorphism of δ-functors $R^\bullet f_* \cong (f_p \underline{H}^\bullet_{ét}(X, -))^\#$.*

Proof. We know from Proposition 2.1.14 that $f_* \cong (f_p \underline{H}^0_{ét}(X, -))^\#$, so again we only need to show that $(f_p \underline{H}^\bullet_{ét}(X, -))^\#$ is a universal δ-functor.

The functor f_p is exact by definition, and sheafification is exact by Theorem 2.1.8, so the right-hand side is a δ-functor, and it is universal again by the effacability criterion [Gro57, Proposition 2.2.1]. □

As useful as injective objects are to set up the whole derived functor machinery, they are not too useful for explicit calculations. Indeed, we showed the existence of enough injectives in the category of étale sheaves in Proposition 2.3.1 via an abstract theorem from Grothendieck's Tohoku paper. As a result, we have not seen a single example of an injective sheaf so far, and they are generally very hard to describe explicitly.

As a result of this, we need to fall back on acyclic resolutions if we want to compute the étale cohomology of a sheaf. Therefore, our next goal will be to get our hands on a reasonably large class of objects that are acyclic for all the derived functors we commonly use.

The basic lemma that will allow us to do so is the following:

Lemma 2.3.6. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories and assume that $\mathcal{A}$ has enough injectives. Let T be a class of objects in $\mathcal{A}$ that is stable under isomorphism and such that:*

a) for every object $M \in \mathcal{A}$ exists a monomorphism $M \hookrightarrow X$ with $X \in T$,

b) whenever a direct sum $M \oplus N$ of two objects $M, N \in \mathcal{A}$ lies in T, then we have $M \in T$ and

c) if $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathcal{A}$ with $M', M \in T$, then $M'' \in T$ and $0 \to FM' \to FM \to FM'' \to 0$ is exact.

Then all injectives are in T and all objects of T are F-acyclic, allowing us to compute the derived functors of F using T-resolutions.

Proof. Let $I \in \mathcal{A}$ be injective. Then by a), there exists a short exact sequence $0 \to I \to X \to X/I \to 0$ with $X \in T$. Because I is injective, this sequence splits and we have $X \cong I \oplus X/I$, so because of b) we also have $I \in T$.

Now let $X \in T$ and choose an injective resolution $0 \to X \to I^\bullet$. We define inductively objects Z^i such that

$$0 \to X \to I^0 \to Z^1 \to 0,$$
$$0 \to Z^i \to I^i \to Z^{i+1} \to 0$$

are exact. By induction and c), all the Z^i are in T. Again by c), all the sequences

$$0 \to FZ^i \to FI^i \to FZ^{i+1} \to 0$$

are exact and so we have $R^iF(X) \cong 0$ for $i \geq 1$. □

Since, in contrast to presheaves on a topological space, we do not have a unique restriction map between two objects on the étale site, the approach of defining flasque presheaves as the ones who have all of their restriction maps being surjective does not work anymore, and so we have to come up with something else.

Unfortunately, our solution is much less hands-on than in the topological setting, which might be the reason why a different name is commonly used to refer to them.

Definition 2.3.7. *Let X be a locally Noetherian scheme. We say that a sheaf $\mathcal{F} \in \mathcal{S}(X_{ét})$ is flabby iff it is $\underline{H}^0_{ét}(X, -)$-acyclic.*

Being acyclic for this functor easily implies the conditions from Lemma 2.3.6 also for all other functors we care about.

Corollary 2.3.8. *Let X be a locally Noetherian scheme. Then any flabby sheaf is*

- $\underline{H}^0_{ét}(X, -)$*-acyclic,*
- $H^0_{ét}(X, -)$*-acyclic,*
- $H^0_{ét}(X, U; -)$*-acyclic for all* $U \in X_{ét}$*,*
- f_**-acyclic for any morphism* $f\colon X \to S$ *to a locally noetherian scheme.*

Proof. The first point is just the definition of being flabby. For the remaining points, we only have to check that the class of flabby sheaves satisfies the conditions of Lemma 2.3.6. The first two conditions are independent of the functor considered, and hence are always satisfied. The third condition is implied by the long exact sequence in cohomology and Lemma 2.3.3 for the second and third point we are trying to prove, and by the long exact sequence and Proposition 2.3.5 for the last point. □

Lemma 2.3.9. *Let X be a locally Noetherian scheme and let $\mathcal{I} \in \mathcal{S}(X_{ét})$ be an injective sheaf. Then $\mathcal{I}$ is also injective as a presheaf.*

Proof. The inclusion $\mathcal{S}(X_{ét}) \subseteq \mathcal{P}(X_{ét})$ has an exact left adjoint, the sheafification functor from Theorem 2.1.8, and hence preserves injectives by Lemma 2.2.17. □

Another common computational tool for sheaf cohomology are spectral sequences, in particular the Leray spectral sequence. As it turns out, it can be seen as a special case of the spectral sequence of composed functors, another general result found in Grothendieck's Tohoku paper.

To apply it here, we have to show that in every case our first functor (out of the two composed ones) sends injective objects to acyclics. In all cases, this is implied by the adjunctions we have combined with Lemma 2.2.17.

Lemma 2.3.10. *Let $f\colon X \to S$ be a morphism of locally Noetherian schemes. Then f_* preserves injectives.*

Proof. Follows from Lemma 2.2.17 because f_* admits the exact left adjoint f^* by Proposition 2.1.15 and Corollary 2.2.2. □

As a result, we obtain, just like in the topological setting, the classical Leray spectral sequence, and also the spectral sequence relating the higher direct images of two morphisms of schemes.

Theorem 2.3.11 (Leray Spectral Sequence I)**.** *Let $f\colon X \to S$ be a morphism of locally Noetherian schemes. Then for each sheaf $\mathcal{F} \in \mathcal{S}(X_{ét})$ we have spectral sequences*

- $H^p_{ét}(S, R^q f_* \mathcal{F}) \implies H^{p+q}_{ét}(X, \mathcal{F})$ *and*
- $\underline{H}^p_{ét}(S, R^q f_* \mathcal{F}) \implies \underline{H}^{p+q}_{ét}(X, \mathcal{F})$.

Proof. We prove only the existence of the first spectral sequence as the second spectral sequence is constructed analogously.

We have $H^0_{\acute{e}t}(S, f_*(-)) \cong H^0_{\acute{e}t}(X, -)$ by definition of the functor f_*. Since f_* preserves injectives by Lemma 2.3.10, this spectral sequence is a special case of the Grothendieck spectral sequence of composed functors (cf. [Gro57, Theorem 2.4.1]). □

Theorem 2.3.12 (Leray Spectral Sequence II). *Let $X \xrightarrow{f} Y \xrightarrow{g}$ be two morphisms of locally Noetherian schemes. Then, for each sheaf $\mathcal{F} \in \mathcal{S}(X_{\acute{e}t})$, we have a spectral sequence*

$$R^p g_*(R^q f_* \mathcal{F}) \implies R^{p+q}(gf)_* \mathcal{F}.$$

Proof. This is another special case of the Grothendieck spectral sequence of composed functors for the same reasons as in Theorem 2.3.11. □

Lemma 2.3.13. *Let X be a Noetherian scheme. Then a filtered colimit of flabby sheaves is again flabby.*

Proof. We can detect flabby sheaves using Čech cohomology, which we have omitted in this paper. Therefore, we refer to [Mil80, Chapter III, Remark 3.6]. □

Theorem 2.3.14. *Let X be a Noetherian scheme and let $\mathcal{F} \colon I \to \mathcal{S}(X_{\acute{e}t})$ be a small filtered diagram. Then there is a canonical isomorphism of δ-functors $\operatorname{colim}_{i \in I} H^\bullet_{\acute{e}t}(X, \mathcal{F}_i) \cong H^\bullet_{\acute{e}t}(X, \operatorname{colim}_{i\in I} \mathcal{F}_i)$.*

Proof. We know that $\mathcal{S}(X_{\acute{e}t})$ satisfies `AB3`* and has enough injectives, so $Fun(I, \mathcal{S}(X_{\acute{e}t}))$ has enough injectives by a general argument, see for example [Wei94, Exercise 2.3.7]. Both sides are δ-functors as filtered colimits are exact, and in degree zero, the statement is implied by Corollary 2.2.21. Hence, it suffices to show that both sides are effacable.

By Lemma 2.3.13 we can just show that an injective diagram $I \to \mathcal{S}(X_{\acute{e}t})$ is necessarily composed out of injective sheaves.

Consider, for $i \in I$, the functor

$$Fun(I, \mathcal{S}(X_{\acute{e}t})) \xrightarrow{ev_i} Fun(\{*\}, \mathcal{S}(X_{\acute{e}t})) \cong \mathcal{S}(X_{\acute{e}t}),$$

induced by the functor $* \mapsto i$. By Proposition 2.1.11, this has a left adjoint given by

$$\mathcal{F} \mapsto \left(j \mapsto \bigoplus_{\mathrm{Hom}_I(j,i)} \mathcal{F} \right),$$

which is obviously exact by Proposition 2.1.7. Now Lemma 2.2.17 implies that ev_i preserves injective objects, and so we are done. □

2.4 The Étale Cohomology of Curves

Now that we have developed all of the relevant theory, let us come to an example. We ultimately want to compute the étale cohomology groups of a smooth projective and connected curve over an algebraically closed field. As it turns out, the result is very similar to the one we get when computing the singular cohomology of such a curve over $\mathbb{C}$, interpreted as a Riemann surface, at least for finite coefficients.

But before that, we want to briefly illustrate the case of what happens on an arbitrary field.

Theorem 2.4.1. *Let k be a field and choose a separable closure $\bar{k}$. Then, for each abelian sheaf $\mathcal{F}$ the étale stalk at $\bar{x} := \operatorname{Spec} \bar{k}$ has a canonical structure of a discrete $\operatorname{Gal}(\bar{k}/k)$-module, and this induces an equivalence of categories.*

Under this equivalence, the global sections functor corresponds to the (global) invariants functor, and so the theory of étale cohomology on a field is equivalent to its Galois cohomology.

Proof. By Definition 2.2.3 we have $\mathcal{F}_{\bar{x}} \cong \operatorname{colim}_{\bar{x} \to X} \mathcal{F}(X) \cong \operatorname{colim}_{k \subseteq k' \subseteq \bar{k}} \mathcal{F}(\operatorname{Spec} k')$, where the second colimit is taken over all finite subextensions that are Galois over k, as this subcategory is cofinal. There is an obvious $G := \operatorname{Gal}(\bar{k}/k)$-action on that last colimit, and it is discrete by construction as a colimit over obviously discrete G-modules.

We write implicitly $\mathcal{F}(A)$ for $\mathcal{F}(\operatorname{Spec} A)$ and define a functor in the opposite direction: For a discrete G-module M, define $\mathcal{F}_M$ as the functor

$$\prod_i k_i \mapsto \prod_i M^{\operatorname{Gal}(\bar{k}/k_i)}$$

with the obvious restriction maps. It is easy to see that composing them in one way is isomorphic to the identity on discrete G-modules, and so we only compute explicitly the other possible composition, as this computation depends on the sheaf condition.

We have to check that $\mathcal{F}(k') \to \mathcal{F}_{\bar{x}}^{\operatorname{Gal}(\bar{k}/k')} \cong \operatorname{colim}_{k' \subseteq k_0 \subseteq \bar{k}} \mathcal{F}(\operatorname{Spec} k_0)^{\operatorname{Gal}(\bar{k}/k')}$ is an isomorphism, i.e. we have to show that for each finite Galois extension $k' \subseteq k_0$ we have $\mathcal{F}(k') \cong \mathcal{F}(\operatorname{Spec} k_0)^{\operatorname{Gal}(\bar{k}/k')}$. This is the sheaf condition of $\mathcal{F}$ on the one-element cover $\{\operatorname{Spec} k_0 \to \operatorname{Spec} k'\}$:

$$0 \to \mathcal{F}(k') \to \mathcal{F}(k_0) \xrightarrow{\Delta - \prod_\sigma \sigma} \prod_{\operatorname{Gal}(k_0/k')} \mathcal{F}(k_0)$$

is exact, where Δ denotes the diagonal map, so $\mathcal{F}(k')$ is exactly the subgroup of elements x such that $\sigma x = x$ for all $\sigma \in \operatorname{Gal}(k_0/k')$.

The statement on global sections is clear. □

Now let us come to the case of a normal projective (and irreducible) curve X over an algebraically closed field k. Denote by $\eta \colon \operatorname{Spec} \kappa(X) \to X$ the generic point, and by $|X|$ the set of closed points, in one-to-one correspondence with $X(k)$.

Our first objective will be to compute the cohomology groups of $\mathbb{G}_m$.

Lemma 2.4.2. *We have*

$$H^i_{\acute{e}t}(X, \mathbb{G}_m) \cong \begin{cases} k^* & \text{if } i = 0 \\ Pic(X) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

Proof. The sequence

$$0 \to \mathbb{G}_{m,X} \to \eta_* \mathbb{G}_{m,\eta} \to \bigoplus_{x \in |X|} x_* \underline{\mathbb{Z}}_x \to 0$$

is exact. The only interesting part is surjectivity anyway, and this we can show using Theorem 2.2.8 and only Zariski opens.

For the long exact sequence in étale cohomology, we see that it suffices to show that $H^i_{ét}(X, \eta_* \mathbb{G}_{m,\eta}) \cong H^i_{ét}(X, \bigoplus_{x \in |X|} x_* \underline{\mathbb{Z}}_x) \cong 0$ for $i \geq 1$, as the we would have $H^1_{ét}(X, \mathbb{G}_m)$ as the cokernel of the degree map

$$\kappa(X)^* \to \bigoplus_{x \in |X|} \mathbb{Z},$$

a common definition of the Picard group, at least for sufficiently smooth schemes. Note that we are implicitly using Corollary 2.2.21 to compute the global sections of the direct sum.

Étale cohomology commutes with the direct sum by Theorem 2.3.14, and for each $x \in |X|$ we have

$$H^i_{ét}(X, x_* \underline{\mathbb{Z}}_x) \cong H^i_{ét}(x, \underline{\mathbb{Z}}_x) \cong H^i(k, \mathbb{Z}) \cong 0$$

as x is a closed immersion, and hence x_* is exact by Proposition 2.2.18 and preserves injectives by Lemma 2.3.10. The last two isomorphisms follow from Theorem 2.4.1 and the fact that k is algebraically closed.

For $\eta_* \mathbb{G}_{m,\eta}$, we want to use the Leray Spectral Sequence I, so we begin by computing the higher direct images of η. By Proposition 2.3.5, they are described as the sheafification of the presheaf

$$U \mapsto H^i_{ét}(U \times_X \operatorname{Spec} \kappa(X), \mathbb{G}_m),$$

which vanishes for $i \geq 1$ because $\kappa(X)$ has transcendence degree 1 over k, and Tsen's theorem from Galois cohomology (cf. [NSW08, Corollary 6.5.5]). Therefore, we have, as before,

$$H^i_{ét}(X, \eta_* \mathbb{G}_m) \cong H^i_{ét}(\operatorname{Spec} \kappa(X), \mathbb{G}_m) \cong H^i(\kappa(X), \kappa(X)^*) \cong 0$$

for $i \geq 1$, again because of Tsen's theorem. This finishes our proof. □

With this, the computation of the cohomology groups we are interested in is a breeze:

Theorem 2.4.3. *Let $n \in \mathbb{N}$ such that $\operatorname{char} k \nmid n$. Then*

$$H^i_{ét}(X, \mu_n) \cong \begin{cases} \mu_n(k) & \text{if } i = 0 \\ Pic^0(X)[n] & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3. \end{cases}$$

Note that we have (non-canonical) isomorphisms $\mu_n(k) \approx \mathbb{Z}/n\mathbb{Z}$ and $Pic(X)[n] \approx (\mathbb{Z}/n\mathbb{Z})^{2g}$, where g is the genus of X. This mirrors the result we get for a surface of genus g in singular cohomology as $\mu_n \approx \underline{\mathbb{Z}/n\mathbb{Z}}_X$ on X, again non-canonically.

Proof. By assumption, n is invertible in X, so by Proposition 2.2.12 we get a short exact sequence of étale sheaves

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 0.$$

Lemma 2.4.2 and the long exact sequence imply our result, at least after noting that there is an exact sequence

$$0 \to Pic^0(X) \to Pic(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

and applying the 5-lemma.

For the fact that $Pic^0(X) \xrightarrow{n} Pic^0(X)$ is surjective and that $\ker(Pic^0(X) \xrightarrow{n} Pic^0(X)) \approx (\mathbb{Z}/n\mathbb{Z})^{2g}$ we refer to [BG06, Section 8.7].

The isomorphism $\mu_n \approx \underline{\mathbb{Z}/n\mathbb{Z}}_X$ is given by choosing a primitive n-th root of unity, as can be seen by considering stalks. $\square$

Chapter 3

Main Theorems

In this section we collect, mainly without proofs, some of the main theorems we will be using later on. Most of them do not hold for all sheaves, so we begin by discussing restrictions on the sheaves we will be working with.

3.1 Finiteness Conditions

The first and most basic restriction we introduce is that of being torsion. Note that, by definition, an abelian group A is torsion if and only if each element in it is annihilated by some positive integer n, or, phrased in a more categorical way,

$$\operatorname*{colim}_n A[n] \cong \bigcup_n A[n] = A,$$

where $A[n]$ denotes the kernel of multiplication by n, which is, by definition, the set of all elements killed by n.

This condition generalized immediately to the category of sheaves, but, due to the rather explicit description we have for colimits of sheaves, there is also a second equivalent condition, which in practice is easier to check.

Lemma 3.1.1. *Let X be a locally Noetherian scheme and let $\mathcal{F}$ be an abelian sheaf on $X_{\acute{e}t}$. Then the following conditions are equivalent:*

i) The canonical monomorphism $\operatorname*{colim}_n \mathcal{F}[n] \to \mathcal{F}$ is surjective, hence an isomorphism. Here, the colimit is taken over the natural numbers, partially ordered by divisibility and $\mathcal{F}[n]$ denotes the kernel of multiplication by n.

ii) $\mathcal{F}$ is isomorphic to the sheafification of a presheaf that takes values in torsion abelian groups.

Proof. $i) \implies ii)$: The subsheaves $\mathcal{F}[n]$ take values in torsion abelian groups by construction, and since that presheaf colimit is computed pointwise, the colimit $\operatorname*{colim}_n \mathcal{F}[n]$ computed in presheaves also takes values in torsion abelian groups. The sheaf colimit is just the sheafification of this presheaf.

ii) $\implies$ *i)* : Let $\mathcal{F}'$ be an abelian presheaf with values in torsion abelian groups such that $(\mathcal{F}')^\# \cong \mathcal{F}$. Then

$$\mathcal{F} \cong (\mathcal{F}')^\# \cong (\operatorname*{colim}_n \mathcal{F}'[n])^\# \cong \operatorname*{colim}_n (\mathcal{F}'[n]^\#) \cong \operatorname*{colim}_n (\mathcal{F}'^\#[n]) \cong \operatorname*{colim}_n \mathcal{F}[n].$$

□

Definition 3.1.2. *Let X be a locally Noetherian scheme and let $\mathcal{F}$ be an abelian sheaf on $X_{ét}$. We say that $\mathcal{F}$ is a torsion sheaf iff it satisfies one (and hence both) of the equivalent conditions of Lemma 3.1.1.*

Note that by Corollary 2.2.21, a sheaf on a Noetherian site is torsion if and only if it takes values in torsion abelian groups.

As the condition is stated via a canonical morphism being an isomorphism, it can be checked on stalks.

Proposition 3.1.3. *Let X be a locally Noetherian scheme and let $\mathcal{F}$ be a sheaf on $X_{ét}$. Then $\mathcal{F}$ is torsion if and only if all of its stalks are torsion groups.*

Proof. We have to check when the canonical morphism $\operatorname*{colim}_n \mathcal{F}[n] \to \mathcal{F}$ is an isomorphism. This can be checked on stalks by Theorem 2.2.7, and since the pullback functor preserves colimits (it has a right adjoint) as well as kernels (cf. Corollary 2.2.2), the induced map on stalks is

$$\operatorname*{colim}_n (\mathcal{F}_{\bar{x}})[n] \to \mathcal{F}_{\bar{x}},$$

which is an isomorphism if and only if the stalk at $\bar{x}$ is a torsion group. □

On a Noetherian scheme, we have the following nice coherence result for the cohomology groups:

Corollary 3.1.4. *Let X be a Noetherian scheme and let $\mathcal{F}$ be a torsion sheaf on $X_{ét}$. Then the étale cohomology groups $H^q_{ét}(X, \mathcal{F})$ are torsion groups for all $q \geq 0$.*

Proof. By Theorem 2.3.14 we have $H^q_{ét}(X, \mathcal{F}) \cong \operatorname*{colim}_n H^q_{ét}(X, {}_n\mathcal{F})$, so we may assume ${}_n\mathcal{F} = \mathcal{F}$. But then multiplication by n equals the zero map, and the same is therefore true on the cohomology groups, concluding the proof. □

Being torsion is the main technical condition that gets most of the important theorems to work, but to relate sheaves to continuous representations of certain profinite groups, something we will obviously need in our last chapter, we need a way stronger and more specialized condition.

The result will ultimately come from the general theory of finite étale coverings and the étale fundamental group, so our goal is to classify the abelian sheaves represented by finite étale group schemes. We begin with the following definition.

Definition 3.1.5. *Let X be a locally Noetherian scheme and let A be an abelian group. The sheafification of the constant presheaf $U \mapsto A$ is called the constant sheaf associated to A and denoted by $\underline{A}_X$. An arbitrary sheaf $\mathcal{F}$ on $X_{ét}$ is called locally constant iff there exists a cover $\{U_i \to X\}_{i \in I}$ such that $\mathcal{F}_{|U_i}$ is isomorphic to some constant sheaf for each $i \in I$.*

This property is, of course, preserved by taking the pullback.

Lemma 3.1.6. *Let $f\colon X \to Y$ be a morphism of locally Noetherian sheaves. Let A be an abelian group and let $\mathcal{F}$ be a locally constant sheaf on $Y_{\acute{e}t}$. Then $f^*\underline{A}_Y$ is canonically isomorphic to $\underline{A}_X$ and $f^*\mathcal{F}$ is again locally constant.*

Proof. The first statement is implied by the Yoneda lemma:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}(X_{\acute{e}t})}(f^*\underline{A}_Y, -) \cong \mathrm{Hom}_{\mathcal{S}(Y_{\acute{e}t})}(\underline{A}_Y, f_*(-)) \cong \mathrm{Hom}_{\mathcal{P}(Y_{\acute{e}t})}(\Delta(A), f_*(-)) \cong \\ \cong \mathrm{Hom}_{\mathbf{Ab}}(A, \Gamma(Y, f_*(-))) \cong \mathrm{Hom}_{\mathbf{Ab}}(A, \Gamma(X, -)) \cong \mathrm{Hom}_{\mathcal{S}(X_{\acute{e}t})}(\underline{A}_X, -),\end{aligned}$$

where Δ denotes the constant functor $U \mapsto A$.

The second statement follows from the first: if $\{U_i \to Y\}_{i\in I}$ is a cover on $Y_{\acute{e}t}$ with the property that $\mathcal{F}_{|U_i}$ is constant for each $i \in I$, then $\{U_i \times_Y X \to X\}_{i\in I}$ is a cover of $X_{\acute{e}t}$ such that $(f^*\mathcal{F})_{|U_i\times_Y X} \cong f^*(\mathcal{F}_{|U_i})$ is constant for each $i \in I$. □

The second definition we need to classify the sheaves represented by finite étale group schemes is obviously necessary (cf. Corollary 2.2.11): It has to have finite stalks.

Definition 3.1.7. *Let X be a locally Noetherian scheme and let $\mathcal{F}$ be an abelian étale sheaf. We say that $\mathcal{F}$ is finite iff all of its étale stalks are finite groups.*

The combination of these two properties turns out to be just the thing we need, via a descent argument.

Proposition 3.1.8. *Let X be a locally Noetherian scheme and let $\mathcal{F}$ be an abelian étale sheaf. Then $\mathcal{F}$ is locally constant and finite if and only if it is represented by a group scheme over X that is finite and étale.*

Proof. See [Mil80, Chapter V, Proposition 1.1]. □

Definition 3.1.9. *Let X be a locally Noetherian scheme. An abelian étale sheaf on X is called lisse or smooth iff it satisfies one and hence both equivalent properties from Proposition 3.1.8.*

To relate this to representations of profinite groups, we need the étale fundamental group. We only collect the main results that we will need here, see [Len08] for a nice exposition on it.

Theorem 3.1.10. *Let X be a connected scheme equipped with a geometric point $\bar{x}$. Then the automorphism group $\pi_1^{\acute{e}t}(X, \bar{x})$ of the functor*

$$\mathrm{Hom}_X(\bar{x}, -)\colon \mathbf{F\acute{E}t}/X \to \mathbf{set},$$

*where $\mathbf{F\acute{E}t}/X$ denotes the category of finite étale maps (usually called étale coverings, as they are automatically surjective) over X, is a profinite group and the canonical factorization of that functor over the category $\pi_1^{\acute{e}t}(X, \bar{x})$-**set** of finite sets with a $\pi_1^{\acute{e}t}(X, \bar{x})$-action is an equivalence of categories.*

If $\bar{x}'$ denotes a second geometric point of X, then there is an isomorphism $\pi_1^{\acute{e}t}(X, \bar{x}) \to \pi_1^{\acute{e}t}(X, \bar{x}')$, unique up to inner automorphism.

If Y is a second connected scheme and $f\colon X \to Y$ is a morphism of schemes, then the isomorphism of functors $\mathrm{Hom}_Y(f(\bar{x}), -) \cong \mathrm{Hom}_X(\bar{x}, X \times_Y -)$ induces a continuous homomorphism of profinite groups $\pi_1^{\acute{e}t}(X, \bar{x}) \to \pi_1^{\acute{e}t}(Y, f(\bar{x}))$, turning $\pi_1^{\acute{e}t}(-, -)$ into functor from pointed connected schemes to profinite groups.

Proof. See [Len08]. $\square$

The equivalence of categories extends to an equivalence of the categories of abelian group objects, and so we find:

Theorem 3.1.11. *Let X be a connected locally Noetherian scheme with a geometric point $\bar{x}$. Then we have an equivalence of categories*

$$\{\mathcal{F} \in \mathcal{S}(X_{\acute{e}t})\colon \mathcal{F} \textit{ lisse}\} \simeq \{\textit{Commutative group objects in } \mathbf{F\acute{E}t}\} \simeq$$
$$\simeq \{\textit{Commutative group objects in } \pi_1^{\acute{e}t}(X,\bar{x})\} \simeq \pi_1^{\acute{e}t}(X,\bar{x})\text{-}\mathbf{mod},$$

where the last category denotes the category of finite $\pi_1^{\acute{e}t}(X,\bar{x})$-modules. The first equivalence is given by the Yoneda embedding, and the second one is the one induced by Theorem 3.1.10.

Note that, by construction and Corollary 2.2.11, the underlying abelian group of the representation associated to a lisse sheaf $\mathcal{F}$ is nothing but its étale stalk $\mathcal{F}_{\bar{x}}$.

This, of course, also extends to an equivalence of the full subcategories of objects killed by an integer n to obtain

$$\{\mathcal{F} \in \mathcal{S}(X_{\acute{e}t})_{\mathbb{Z}/n\mathbb{Z}}\colon \mathcal{F} \textit{ lisse}\} \simeq \pi_1^{\acute{e}t}(X,\bar{x})\text{-}\mathbf{mod}_{\mathbb{Z}/n\mathbb{Z}},$$

where the second category denotes the category of finite $\mathbb{Z}/n\mathbb{Z}$-modules with a $\pi_1^{\acute{e}t}(X,\bar{x})$-action.

Proof. Obvious by what we have done so far. $\square$

Now that we have the connection to representations, we just have to quickly introduce one more condition, the usefulness of which will only really become apparent when we introduce l-adic sheaves later in this chapter.

Definition 3.1.12. *Let X be a Noetherian scheme and let $\mathcal{F}$ be an abelian étale sheaf on X. We say that $\mathcal{F}$ is constructible iff there is a finite jointly surjective family of locally closed subschemes $j\colon Y \to X$ such that $\mathcal{F}_{|Y} := j^*\mathcal{F}$ is locally constant and finite for each of the subschemes.*

Note that any lisse sheaf is trivially constructible, and any constructible sheaf is automatically torsion by Proposition 3.1.3.

Proposition 3.1.13. *Let X be a Noetherian scheme. As they are in particular finite, constructible sheaves are torsion sheaves. A torsion sheaf is constructible if and only if it is a Noetherian object (i.e. satisfies the ascending chain condition for subsheaves) in the (abelian) category of all torsion sheaves.*

Proof. See [FK88, Chapter I, Proposition 4.8]. $\square$

3.2 The Proper Base Change Theorem

One of the most important results, both for explicit computations of higher direct images and also to define, for example, cohomology with compact support, is the proper base change theorem, which relates the higher direct images of a proper map to that of its base change along some second morphism of schemes.

The comparison morphism can be defined for an arbitrary commutative square, a fact that we will also be using later on.

Theorem 3.2.1 (Proper Base Change Theorem)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & S \end{tikzcd}$$

be a commutative square of locally Noetherian schemes. Then there exists a canonical morphism of δ*-functors* $f^*(R^\bullet\pi_*(-)) \to R^\bullet\pi'_*(f'^*(-))$*, which, if the square is Cartesian and* π *is proper, is even an isomorphism on the subcategory of abelian torsion sheaves.*

Proof. We only construct the base change morphism, for the proof that it is an isomorphism under the stated hypothesis, see [SGAIV73].

As $f^*(R^\bullet\pi_*(-))$ is a universal δ-functor (obviously, $R^\bullet\pi_*(-)$ is one, and f^* is exact), it suffices to construct a morphism in degree zero, and this we do via a sequence of adjunction formulas:

Giving a morphism

$$f^*\pi_* \to \pi'_*f'^*$$

is equivalent to giving a morphism

$$\pi_* \to f_*\pi'_*f'^* \cong \pi_*f'_*f'^*,$$

and there we have the canonical morphism $\pi_*(\mathrm{id} \xrightarrow{\text{can}} f'_*f'^*)$, finishing the construction. □

The most common form in which we use Theorem 3.2.1 is to compute stalks of higher direct images:

Corollary 3.2.2. *Let* $f\colon X \to S$ *be a proper morphism of locally Noetherian schemes and let* $\bar{x}$ *be a geometric point of* S*. Then there is a canonical isomorphism of* δ*-functors* $R^\bullet f_*(-)_{\bar{x}} \cong H^\bullet_{\acute{e}t}(X_{\bar{x}}, (-)_{|X_{\bar{x}}})$*.*

Proof. Immediate from Theorem 3.2.1. □

Now we want to see how we use this to define cohomology, or higher direct images, with compact (proper) support. First, we need the following definition:

Definition 3.2.3. *Let* $\pi\colon X \to S$ *be a morphism of schemes. We say that* π *is compactifiable iff there exists a scheme* $\overline{X}$ *as well as an open immersion* $j\colon X \hookrightarrow \overline{X}$ *and a proper morphism* $\bar{\pi}\colon \overline{X} \to S$ *such that the resulting triangle*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & \overline{X} \arrow[dl, "\bar{\pi}"] \\ & S & \end{tikzcd}$$

commutes.

Since this condition could be very hard to check from scratch, we note that such a compactification exists under quite general circumstances.

Theorem 3.2.4 (Nagata)**.** *Let $f\colon X \to S$ be a separated morphism of finite type to a quasi-compact and quasi-separated scheme S. Then f is compactifiable.*

Proof. See [Sta25, Tag 0F3T]. □

The reason this preliminary definition is simple: Even though the usual definition of direct images with compact support from algebraic topology technically generalizes by using the left exact functor that assignes to a sheaf its subsheaf of sections that have proper support, but its derived functors give the wrong result in general, see [FK88, Chapter 8] for a nice example of this fact.

However, for a compactifiable topological space, there is a separate definition that works, and as it turns out, this definition also generalizes nicely to our setting and gives the "right" result.

Since it depends, a priori, on the chosen compactification, we first show that this is in fact not the case.

Proposition 3.2.5. *Let $\pi\colon X \to S$ be a compactifiable morphism of locally Noetherian schemes and let $\overline{X}_1$ and $\overline{X}_2$ be two compactifications. Then, for each $q \geq 0$, there is a natural isomorphism $H^q_{ét}(\overline{X}_1, (j_1)_!\mathcal{F}) \cong H^q_{ét}(\overline{X}_2, (j_2)_!\mathcal{F})$, where $\mathcal{F}$ is a torsion sheaf.*

Proof. Note that the canonical map $X \to \overline{X}_1 \times_S \overline{X}_2$ is a quasi-compact immersion, so there is a closed immersion such that X is isomorphic to an open subset of it, and so, by replacing $\overline{X}_1$ by that closed immersion, we may assume that there is a morphism $f\colon \overline{X}_1 \to \overline{X}_2$ such that the obvious diagram commutes.

Such a morphism is automatically proper, so we want to apply the proper base change theorem (note that $(j_1)_!$, as an exact left adjoint, preserves colimits and kernels, and hence torsion sheaves) and Definition 2.2.15 to show that $f_*(j_1)_! \cong (j_2)_!$ and $R^i f_*(j_!(-)) \cong 0$ for $i \geq 1$, as the result then follows from Theorem 2.3.12. It is easy to show that the sheaves $R^i f_*(j_1)_!(\mathcal{F})$ vanish on $\overline{X}_2 \setminus X$, so we focus on the inclusion $X \hookrightarrow \overline{X}_2$.

We work in the diagram

$$\begin{tikzcd} X \arrow[r, hook, "j_1'"] \arrow[dr, equal] & f^{-1}(X) \arrow[r, hook, "i"] \arrow[d, "f'"] & \overline{X}_1 \arrow[d, "f"] \\ & X \arrow[r, hook, "j_2"] & \overline{X}_2 \end{tikzcd}.$$

Then f' is separated as the base change of the separated morphism f, and so we find that j_1' is proper by cancellation. This implies that $f^{-1}(X)$ splits (over X) into a disjoint union of X and some other (proper) X-scheme $Y \xrightarrow{y} X$.

Now we have $\mathcal{S}((X \coprod Y)_{ét}) \simeq \mathcal{S}(X_{ét}) \times \mathcal{S}(Y_{ét})$ and via a standard argument for universal δ-functors $R^\bullet f'_*(-,-) \cong R^\bullet \operatorname{id}_*(-) \oplus R^\bullet y_*(-)$.

Under this equivalence, the functor $(j_1')_!$ sends a sheaf $\mathcal{F}$ to $(\mathcal{F}, 0)$, so Theorem 3.2.1 applied to the Cartesian square in our diagram and Definition 2.2.15 yield the isomorphisms mentioned above. □

Definition 3.2.6. *Let $\pi\colon X \to S$ be a compactifiable morphism of locally Noetherian schemes. We define for a torsion sheaf $\mathcal{F}$ the i-th higher direct image with compact support $R^i\pi_!\mathcal{F} := R^i\pi_*(j_!\mathcal{F})$, where $j\colon X \hookrightarrow X'$ denotes an open S-immersion into a proper S-scheme as in Definition 3.2.3. This is well-defined by Proposition 3.2.5.*

There is also a version of the proper base change theorem for these higher direct images with compact support. It holds for any compactifiable morphism, and, in particular, for a proper morphism we just get back Theorem 3.2.1.

Theorem 3.2.7 (Proper Base Change Theorem II)**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & S \end{tikzcd}$$

be a Cartesian square of locally Noetherian schemes and assume that π is compactifiable. Then π' is compactifiable and there is a canonical isomorphism of δ-functors $f^*(R^\bullet\pi_!(-)) \cong R^\bullet\pi'_!(f'(-))$ (for abelian torsion sheaves).

Proof. Both properness and open immersions are preserved by base change, so we get the diagram

$$\begin{tikzcd} X' \arrow[r, hook, "j'"] \arrow[d, "f'"'] & \overline{X'} \arrow[r, "\bar{\pi}'"] \arrow[d, "\bar{f}"] & S' \arrow[d, "f"] \\ X \arrow[r, hook, "j"'] & \overline{X} \arrow[r, "\bar{\pi}"'] & S \end{tikzcd},$$

where both lines are compactifications of π and π' respectively, and all squares are Cartesian.

It is elementary to check that $j_!\bar{f}^* \cong f'^*j'_!$, by checking that they have isomorphic right adjoints. By combining this with the old version of the proper base change theorem Theorem 3.2.1, we get

$$f^*(R^\bullet\bar{\pi}_!(-)) = f^*(R^\bullet\bar{\pi}_*(j_!(-))) \cong R^\bullet\bar{\pi}'_*(\bar{f}^*(j_!(-))) \cong R^\bullet\bar{\pi}'_*(j'_!(f'^*(-))) = R^\bullet\bar{\pi}'_!(f'^*(-)).$$

□

We also record, without proof, the trace mapping for later use in Deligne's construction.

Theorem 3.2.8. *For each finite morphism $\pi\colon X \to S$ of locally Noetherian schemes there exists, for all étale sheaves $\mathcal{F}$ on X, a canonical morphism $Tr_\pi\colon \pi_*\pi^*\mathcal{F} \to \mathcal{F}$ that is natural in $\mathcal{F}$ and, assuming that it is of constant rank n, has the property that the composition*

$$\mathcal{F} \xrightarrow{can} \pi_*\pi^*\mathcal{F} \xrightarrow{Tr_\pi} \mathcal{F}$$

is just multiplication by n.

This assignment is stable under base change in the following sense: Let

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & S \end{tikzcd}$$

be a Cartesian square of locally Noetherian schemes and let $\mathcal{F}$ be a torsion sheaf. Then the diagram

$$\begin{tikzcd} \pi'_*\pi'^*f^*\mathcal{F} \arrow[r, "\cong"] \arrow[dr, "\mathrm{Tr}_{\pi'}"'] & \pi'_*f'^*\pi^*\mathcal{F} \arrow[r, "\cong"] & f^*\pi_*\pi^*\mathcal{F} \arrow[dl, "f^*\mathrm{Tr}_\pi"] \\ & f^*\mathcal{F} & \end{tikzcd}$$

commutes.

Proof. See [SGAIV73, Exposé XVII, Théorème 6.2.3]. □

3.3 Definition of l-adic cohomology

Fix for the whole section a prime number $l \in \mathbb{N}$.

Our ultimate goal is to construct a representation on an l-adic number field, but so far we only really have a good cohomology theory for torsion sheaves, and they tend to give cohomology groups which are also torsion, cf. Corollary 3.1.4. This means that such a cohomology group will certainly not admit a vector space structure over any characteristic zero field.

The trick we employ to fix this issue is to go to a limit of such groups, in a similar way to how we construct the l-adic integers $\mathbb{Z}_l$.

Definition 3.3.1. *Let X be a Noetherian scheme and let l be a prime number. An l-adic sheaf $\mathcal{F}$ on $X_{\acute{e}t}$ is a projective system*

$$\mathcal{F}_1 \longleftarrow \mathcal{F}_2 \longleftarrow \mathcal{F}_3 \longleftarrow \dots \longleftarrow \mathcal{F}_n \longleftarrow \mathcal{F}_{n+1} \longleftarrow \dots$$

of constructible abelian étale sheaves on X such that for each $n \geq 1$ the given morphism induces an isomorphism $\mathcal{F}_n \xleftarrow{\cong} \mathcal{F}_{n+1}/(l^n\mathcal{F}_{n+1})$. Note that this implies that for each $n \geq 1$, the sheaf $\mathcal{F}_n \cong \mathcal{F}_{n+1}/(l^n\mathcal{F}_{n+1})$ is killed by l^n, so all the $\mathcal{F}_n$ are torsion sheaves.

We say that an l-adic sheaf $\mathcal{F}$ is constant or lisse iff each of the $\mathcal{F}_n$ is (since each of the components is finite by definition, being lisse is the same as being locally constant).

A morphism of two l-adic sheaves $\mathcal{F}$ and $\mathcal{G}$ is a morphism of the two diagrams $\{\mathcal{F}_n\}_{n\geq 1}$ and $\{\mathcal{G}_n\}_{n\geq 1}$, i.e. a system of morphisms $\{\varphi_n\colon \mathcal{F}_n \to \mathcal{G}_n\}_{n\geq 1}$ that makes all the obvious squares commute.

This gives us a well-defined and abelian category of l-adic sheaves on X.

The reason for this definition is that, although the limit of such systems exists by Corollary 2.1.9, simply taking its cohomology does not yield desirable results for us. Instead, we define the cohomology of such a system in a slightly different way.

Definition 3.3.2. *Let X be a Noetherian scheme, let l be a prime number and let $\mathcal{F}$ be an l-adic sheaf. We define the i-th étale cohomology groups of $\mathcal{F}$ to be $H^i_{ét}(X, \mathcal{F}) := \lim_n H^i_{ét}(X, \mathcal{F}_n)$. This makes $H^i_{ét}(X, \mathcal{F})$ into a functor via the obvious action on morphisms.*

As the group $H^i_{ét}(X, \mathcal{F}_n)$ is killed by l^n, it is a $\mathbb{Z}/l^n\mathbb{Z}$-module, and the cohomology group $H^i_{ét}(X, \mathcal{F}) := \lim_n H^i_{ét}(X, \mathcal{F}_n)$ has a canonical $\mathbb{Z}_l := \lim_n \mathbb{Z}/l^n\mathbb{Z}$-module structure.

In the same way, we can define the stalk of $\mathcal{F}$ at some geometric point $\bar{x}$ as the $\mathbb{Z}_l$-module $\mathcal{F}_{\bar{x}} := \lim_n \mathcal{F}_{n,\bar{x}}$.

There is also a way to make sense out of l-adic higher direct images. The problem is that while the pullback functor is exact and hence preserves the defining property of an l-adic sheaf, the same is not true for even the usual direct image functor, and so we have to restrict ourselves to a small class of sheaves: constructible l-adic sheaves.

Theorem 3.3.3. *Let $f \colon X \to S$ be a morphism of locally Noetherian schemes. Then, for all constructible l-adic sheaves $\mathcal{F}$ the naive (i.e. pointwise) higher direct images $R^i f_* \mathcal{F}$ are constructible l-adic sheaves.*

If f is compactifiable, then the same is true for the sheaves $R^i f_! \mathcal{F}$.

Proof. See [SGAV77]. □

To get from a finitely generated $\mathbb{Z}_l$-module to a finite-dimensional $\mathbb{Q}_l$-vector space, we can simply extend scalars to $\mathbb{Q}_l$. Categorically, this has the effect of killing precisely those modules that are torsion, i.e. that are killed by some power of l.

This property is obviously stable under extensions, kernels and cokernels, and so it forms a thick Serre subcategory, and the category of finite-dimensional $\mathbb{Q}_l$-vector spaces can be defined as the Serre quotient category of finitely generated $\mathbb{Z}_l$-modules modulo its torsion objects. This interpretation is useful, as it extends nicely to more general abelian categories.

Definition 3.3.4. *Let X be a Noetherian scheme. We define the category of $\mathbb{Q}_l$-sheaves as the quotient of l-adic (or from now on: $\mathbb{Z}_l$-) sheaves modulo its thick Serre subcategory of $\mathbb{Z}_l$-sheaves killed by a power of l.*

This property is obviously preserved by any additive functor, and so we get by the universal property of the Serre quotient functors $R^i f_*$, $R^i f_!$, $H^i_{ét}(X, -)$ etc. defined also on $\mathbb{Q}_l$-sheaves.

Morally speaking, if a statement is true for (torsion) étale sheaves, then an equivalent statement is also true for $\mathbb{Z}_l$- and $\mathbb{Q}_l$-sheaves, by assembling it via the limit and localization.

Of particular relevance for us: the proper base change theorem and the connection between representations of the fundamental group and lisse sheaves. Because the correct statement is a bit less obvious for the second case compared to the first one, we state it separately.

Theorem 3.3.5. *Let X be a connected locally Noetherian scheme, equipped with a geometric point $\bar{x}$. Then we have equivalences of categories*

$$\{\textit{lisse } \mathbb{Z}_l\textit{-sheaves}\} \simeq \left\{ \begin{matrix} \textit{continuous } \pi_1^{ét}(X, \bar{x})\textit{-representations} \\ \textit{of finitely generated } \mathbb{Z}_l\textit{-modules} \end{matrix} \right\}$$

and, via going to the respective quotient categories,

$$\{\textit{lisse } \mathbb{Q}_l\textit{-sheaves}\} \simeq \left\{\begin{matrix} \textit{continuous } \pi_1^{\acute{e}t}(X, \bar{x})\textit{-representations} \\ \textit{of finite-dimensional } \mathbb{Q}_l\textit{-vector spaces} \end{matrix}\right\}.$$

The underlying $\mathbb{Z}_l$-module (resp. $\mathbb{Q}_l$-vector space) of the representation associated to a lisse $\mathbb{Z}_l$-sheaf (resp. $\mathbb{Q}_l$-vector space) is again given by the stalk at $\bar{x}$ as defined in Definition 3.3.2 (resp. the extension of scalars along $\mathbb{Q}_l$ thereof).

Proof. This follows, for the most part, from assembling the statement component-wise from Theorem 3.1.11 and a limit argument, and then studying the respective Serre quotients for the second equivalence, see [SGAV77, Exposé VI, 1.2.4] and [SGAV77, Exposé VI, 1.4.2] for details. □

Chapter 4

Deligne's Construction of the Galois Representation attached to a Modular Form

We want to explain how all of this theory we have developed allows us to construct certain Galois representations with good properties.

The goal is to show that there exists, for each normalized eigenform $f \in S_k(\Gamma_1(N), \chi)$ of level $N \geq 5$, weight $k \geq 2$, and where $\chi\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is some character, a 2-dimensional, l-adic Galois representation

$$\rho_{f,l}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}_2(K_{f,l})$$

over some finite extension $K_{f,l}/\mathbb{Q}_l$, such that $\rho_{f,l}$ is unramified away from N and l, and that $\rho_{f,l}(\varphi_p)$ satisfies the polynomial equation $1 - a_p(f)X + p^{k-1}\chi(p)X^2 = 0$ for any prime $p \nmid Nl$, where $\varphi_p \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes any relative Frobenius element for p.

To do this, we follow the same general strategy that Eichler and Shimura followed to show the weight $k = 2$ case. Fix for the whole chapter the level $N \geq 5$, the weight $k \geq 2$ and let $l \nmid N$ be a prime number.

4.1 The General Strategy

We will construct a universal representation (independent of the eigenform f), using Theorem 3.3.5, that will turn out to have a module structure over the $\mathbb{Q}_l$-algebra $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l$, making it into a free module of rank 2 over that ring. Here, $\mathbb{T}_N \subseteq \operatorname{End}_{\mathbb{C}}(S_k(N))$ denotes the Hecke algebra over $\mathbb{Z}$.

After showing that this module structure is compatible with the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, any choice of basis will give us a representation

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}_2(\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l),$$

and, as in the weight $k = 2$ case there is, for each normalized eigenform $f \in S_k(\Gamma_1(N), \chi)$, a finite extension $K_{f,l}/\mathbb{Q}_l$ and a surjection $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l \twoheadrightarrow K_{f,l}$ such that $T_n \mapsto a_n(f)$ and $\langle n \rangle \mapsto \chi(n)$. For a detailed construction of this step, see [DS16, Chapter 9, Section 5].

The problem is hence reduced to understanding the Hecke and diamond operators in $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l$. As in the weight $k = 2$ case, the universal elliptic curve is heavily involved in our construction, and so we begin with the following two definitions.

Definition 4.1.1. *Let S be a scheme. An elliptic curve over S is a proper and smooth S-scheme $f\colon E \to S$ with a distinguished section $0 \in E(S)$ such that all geometric fibers are elliptic curves (in the usual sense). It is canonically isomorphic to its own relative Picard scheme $Pic^0_{E/S}$, and hence has the structure of a commutative S-group scheme whose unit is the chosen section 0.*

For a natural number $n \geq 1$, we denote by E_n the kernel of the multiplication by n endomorphism of E/S. It is a finite and flat S-scheme.

Definition 4.1.2. *Let $f\colon E \to S$ be an elliptic curve and assume that f is a morphism of $\mathbb{Z}[\frac{1}{n}]$-schemes. A level n structure on E/S is an S-isomorphism*

$$\alpha_n \colon \underline{\mathbb{Z}/n\mathbb{Z}}^2_S \xrightarrow{\cong} E_n.$$

As it turns out, there is a universal elliptic curve with level n structure, in the following sense:

Theorem 4.1.3. *Let $n \geq 3$. Then the functor*

$$\begin{aligned} \mathbf{Sch}/\mathbb{Z}[\frac{1}{n}] &\to \mathbf{Set} \\ S &\mapsto \{E/S \text{ elliptic curve with level } n \text{ structure } \alpha_n\}/\cong \end{aligned}$$

is representable by an affine smooth curve $M_n/\mathbb{Z}[\frac{1}{n}]$, which admits a smooth compactification $\overline{M}_n/\mathbb{Z}[\frac{1}{n}]$ such that $\overline{M}_n \setminus M_n$ is finite and étale over $\mathbb{Z}[\frac{1}{n}]$.

Proof. See [Del71]. $\square$

With this in mind, we are ready to define our universal representation. Note that for a compactifiable morphism f of locally Noetherian schemes

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & \overline{X} \arrow[dl, "\bar{\pi}"] \\ & S & \end{tikzcd},$$

there is a canonical morphism of δ-functors $R^\bullet\pi_! \to R^\bullet\pi_*$, constructed on the level of derived categories as $R\pi_! = (R\bar{\pi}_*)j_! = R\bar{\pi}_*Rj_! \to R\bar{\pi}_*Rj_* = R\pi_*$. Therefore, for each $i \geq 0$, we get a functor given by the image of this natural morphism $R^i\tilde{f} := \operatorname{im}(R^if_! \to R^if_*)$.

Denote by a the structure morphism of the modular curve $M_N/\mathbb{Z}[\frac{1}{N}]$ and by $f\colon E \to M_N$ the universal elliptic curve over it. Consider the l-adic sheaf $\mathcal{F}_{k,l} := Sym^{k-2}R^1f_*\mathbb{Z}_l$ on M_N, where $\mathbb{Z}_l$ denotes the lisse l-adic sheaf

$$\underline{\mathbb{Z}/l\mathbb{Z}}_E \leftarrow \underline{\mathbb{Z}/l^2\mathbb{Z}}_E \leftarrow \underline{\mathbb{Z}/l^3\mathbb{Z}}_E \leftarrow \dots$$

on E.

It follows from general theory (cf. [Del71]) that the sheaves $\mathcal{F}_{k,l}$, $R^1a_*(\mathcal{F}_{k,l})$ and $R^1a_!(\mathcal{F}_{k,l})$ are all lisse on $\operatorname{Spec}\mathbb{Z}[\frac{1}{Nl}]$. Therefore, $R^1\tilde{a}(\mathcal{F}_{k,l})$ is also lisse away from l and hence gives, by Theorem 3.3.5, a continuous $\pi_1^{ét}(\operatorname{Spec}\mathbb{Z}[\frac{1}{Nl}],\overline{\mathbb{Q}})$-representation on the finite-dimensional $\mathbb{Q}_l$-vector space $W_l \coloneqq R^1\tilde{a}(\mathcal{F}_{k,l})_{\overline{\mathbb{Q}}} \otimes \mathbb{Q}_l$.

As explained in [Len08], the group $\pi_1^{ét}(\operatorname{Spec}\mathbb{Q},\overline{\mathbb{Q}})$ is canonically isomorphic to $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the morphism $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}[\frac{1}{Nl}]$ induces a surjection $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \pi_1^{ét}(\operatorname{Spec}\mathbb{Z}[\frac{1}{Nl}],\overline{\mathbb{Q}})$, identifying the latter group with the Galois group $\operatorname{Gal}(M/\mathbb{Q})$, where M is the maximal extension of $\mathbb{Q}$ unramified away from N and l.

This shows that W_l is actually a Galois representation, and it is unramified away from N and l by construction. Next, we want to show that there is a natural $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l$-module action on it.

4.2 The Action of the Hecke Algebra

Now that we have a Galois representation, we need to understand how the Hecke algebra acts on it. First, note that we have the following alternative description for the underlying vector space of W_l.

Proposition 4.2.1. *We have a canonical isomorphism of finite-dimensional $\mathbb{Q}_l$-vector spaces*

$$W_l \cong \tilde{H}^1_{ét}(M_N \otimes \overline{\mathbb{Q}}, \mathcal{F}_{k,l}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong \tilde{H}^1(M_N^{an}, \operatorname{Sym}^{k-2} R^1f_*\underline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}_l,$$

where $\tilde{H}^1_{ét}(M_N \otimes \overline{\mathbb{Q}}, -)$ denotes the functor $R^1\tilde{f}$ associated to the morphism $f\colon M_N \otimes \overline{\mathbb{Q}} \to \operatorname{Spec}\overline{\mathbb{Q}}$ just like above and $\tilde{H}^1(M_N^{an}, -)$ denotes the image of $H^1_c(M_N^{an}, -) \to H^1(M_N^{an}, -)$ in singular cohomology.

Proof. The first isomorphism follows from the Proper Base Change Theorem II 3.2.7, and the second isomorphism follows from a standard comparison result between singular cohomology and étale cohomology with constructible coefficients, see for example [FK88, Chapter I, Section 11]. □

We know that the Hecke algebra $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}$ acts on $\tilde{H}^1(M_N^{\mathrm{an}}, \operatorname{Sym}^{k-2} R^1f_*\underline{\mathbb{Q}})$, and Deligne showed that we have an isomorphism of $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{C}$-modules

$$\tilde{H}^1(M_N^{\mathrm{an}}, \operatorname{Sym}^{k-2} R^1f_*\underline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \cong S_k(N) \oplus \overline{S_k(N)},$$

the so-called Shimura isomorphism.

This shows (using the q-expansion) that $\tilde{H}^1(M_N^{\mathrm{an}}, \operatorname{Sym}^{k-2} R^1f_*\underline{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ is a free $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{C}$-module of rank 2, and hence $\tilde{H}^1(M_N^{\mathrm{an}}, \operatorname{Sym}^{k-2} R^1f_*\underline{\mathbb{Q}})$ is a free $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}$-module of rank 2.

By extending scalars to $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l$, we obtain the promised module structure on the vector space W_l. Our last goal for this section is to show that the action of $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}_l$ and $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ commute.

It obviously suffices to check this for Hecke operators T_p, where p is a prime, and the diamond operators. Categorically speaking, the condition of commuting with the Galois action is saying that we have an endomorphism in the category of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations. Using the correspondence Theorem 3.3.5, we therefore have to show that our endomorphisms on the stalk W_l lift to endomorphisms of the $\mathbb{Q}_l$-sheaf $R^1\tilde{a}(\mathcal{F}_{k,l})$.

Unfortunately, this construction requires some heavy arithmetic geometry as well as an explicit description of the Shimura isomorphism, which is why we will only outline the strategy here. We begin with the following definition for the diamond operators. In the following, let $p \nmid N$ denote a prime.

Definition 4.2.2. *We denote by $I_p\colon M_N \to M_N$ the automorphism defined by the automorphism of functors $(E, \alpha_N) \mapsto (E, \alpha_N \cdot p)$. It acts on $R^1\tilde{a}(\mathcal{F}_{k,l})$ via*

$$I_p^*\colon R^1\tilde{a}(\mathcal{F}_{k,l}) = R^1\tilde{a}(\mathrm{Sym}^{k-2} R^1 f_*\mathbb{Z}_l) \xrightarrow{I_p^*} R^1\tilde{a}(I_p^*(\mathrm{Sym}^{k-2} R^1 f_*\mathbb{Z}_l)) \cong R^1\tilde{a}(\mathrm{Sym}^{k-2} R^1 f_*\mathbb{Z}_l),$$

where the last isomorphism is induced by the Proper Base Change Theorem II 3.2.7.

The construction for the Hecke operators T_p involves another moduli problem:

Proposition 4.2.3. *Let $p \nmid N$ be a prime number. Then the functor*

$$\mathbf{Sch}/\mathbb{Z}[\frac{1}{N}] \to \mathbf{Set}$$

associating to a scheme S the set of isomorphism classes of commutative diagrams of S-schemes

$$\begin{tikzcd} & \underline{\mathbb{Z}/N\mathbb{Z}}^2_S \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ E_N \arrow[rr] \arrow[d] & & F_N \arrow[d] \\ E \arrow[rr, "\varphi"] & & F \end{tikzcd},$$

where $\varphi\colon E \to F$ is a p-isogeny of relative elliptic curves and α, α' are isomorphisms, is represented by a $\mathbb{Z}[\frac{1}{N}]$-scheme $M_{N,p}$. The morphisms $q_1\colon M_{N,p} \to M_N$ and $q_2\colon M_{N,p} \to M_N$ corresponding to the morphisms of functors assigning to each diagram as above the elliptic curve $E \to S$ with level N structure α and $F \to S$ with level N structure α' respectively are both finite.

Proof. See [Del71]. □

While we could (like Deligne in [Del71]) use the scheme $M_{N,p}$ to construct the desired lift of the operator T_p, it turns out that its normalization also works, and is better suited for computations. Therefore (and since we do not prove it here anyway) we define the lift of T_p directly with the normalization, but not before collecting some useful facts on it.

Consider the two commutative diagrams

$$\begin{tikzcd} & \underline{\mathbb{Z}/N\mathbb{Z}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\alpha^{(p)}"] & \\ E_N \arrow[rr] \arrow[d] & & E_N^{(p)} \arrow[d] \\ E \arrow[rr, "F"] & & E^{(p)} \end{tikzcd} \qquad \begin{tikzcd} & \underline{\mathbb{Z}/N\mathbb{Z}^2} \arrow[dl, "\frac{\alpha^{(p)}}{p}"'] \arrow[dr, "\alpha"] & \\ E_N^{(p)} \arrow[rr] \arrow[d] & & E_N \arrow[d] \\ E^{(p)} \arrow[rr, "V"] & & E \end{tikzcd}$$

of $M_N \otimes \mathbb{F}_p$-schemes, where E is the universal elliptic curve pulled back, F is the relative Frobenius and V is its dual (induced by the contravariant functor $Pic^0_{-/M_N \otimes \mathbb{F}_p}$), called the "Verschiebung".

They induce by Proposition 4.2.3 two morphisms Φ_1 and Φ_2 from $M_N \otimes \mathbb{F}_p$ to $M_{N,p}$, which together induce a morphism

$$\Phi\colon M_N \otimes \mathbb{F}_p \coprod M_N \otimes \mathbb{F}_p \to M_{N,p} \otimes \mathbb{F}_p.$$

For the two morphisms $q_1, q_2\colon M_{N,p} \otimes \mathbb{F}_p \to M_N \otimes \mathbb{F}_p$ from Proposition 4.2.3 we find

$$q_1 \circ \Phi = \mathrm{id} \coprod (I_p^{-1} \circ F)$$
$$q_2 \circ \Phi = F \coprod \mathrm{id},$$

where F denotes the absolute Frobenius of $M_N \otimes \mathbb{F}_p$.

Proposition 4.2.4. *i) Denote by $M'_{N,p} \to M_{N,p}$ the normalization of $M_{N,p}$. The morphisms $q'_1, q'_2\colon M'_{N,p} \to M_N$ induced by composition with q_1, q_2 are still finite and flat.*

ii) The morphism Φ can be factored through a surjective morphism

$$\Phi'\colon M_N \otimes \mathbb{F}_p \coprod M_N \otimes \mathbb{F}_p \to M'_{N,p} \otimes \mathbb{F}_p.$$

Proof. See [Del71]. □

With this out of the way, we now come to the construction of T_p as an endomorphism of $R^1\tilde{a}\mathcal{F}_{k,l}$.

Consider the commutative diagram

$$\begin{tikzcd} & E_1 \arrow[rr, "\varphi"] \arrow[dl] \arrow[dr, "u"] & & E_2 \arrow[dl, "v"'] \arrow[dr] & \\ E \arrow[dr, "f"'] & & M'_{N,p} \arrow[dl, "q'_1"] \arrow[dr, "q'_2"'] & & E \arrow[dl, "f"] \\ & M_N & & M_N & \end{tikzcd},$$

where f is the universal elliptic curve over M_N and u, v, φ are all induced by the universal diagram over $M_{N,p}$ and the base change along its normalization. Note that both the left and the right square are Cartesian.

Theorem 4.2.5. *Denote for each compactifiable $\mathbb{Z}[\frac{1}{N}]$-scheme X the higher direct image (resp. the higher direct image with compact support and its image under the canonical map) by $R^\bullet a_*(X,-)$ (resp. $R^\bullet a_!(X,-)$ and $R^\bullet \tilde{a}(X,-)$) along the structure morphism of $X/\mathbb{Z}[\frac{1}{N}]$. Note that, despite the notation, there is no reason to assume that the functors $R^\bullet \tilde{a}(X,-)$ form a δ-functor.*

Then the Hecke operator $T_p\colon W_l \to W_l$ is modeled by the endomorphism

$$\begin{aligned} R^1\tilde{a}(M_N, \operatorname{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l)) &\xrightarrow{q_1'^*} R^1\tilde{a}(M_{N,p}, q_1'^* \operatorname{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l)) \cong \\ \cong R^1\tilde{a}(M'_{N,p}, \operatorname{Sym}^{k-2}(R^1 v_*\mathbb{Z}_l)) &\cong R^1\tilde{a}(M'_{N,p}, \operatorname{Sym}^{k-2}(\mathrm{id}^* R^1 v_*\mathbb{Z}_l)) \xrightarrow{\varphi^*} \\ \xrightarrow{\varphi^*} R^1\tilde{a}(M'_{N,p}, \operatorname{Sym}^{k-2}(R^1 u_*\varphi^*\mathbb{Z}_l)) &\cong R^1\tilde{a}(M'_{N,p}, \operatorname{Sym}^{k-2}(R^1 u_*\mathbb{Z}_l)) \cong \\ \cong R^1\tilde{a}(M_{N,p}, q_2'^* \operatorname{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l)) &\xrightarrow{q'_{2*}} R^1\tilde{a}(M_N, \operatorname{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l)), \end{aligned}$$

meaning that this endomorphism induces T_p on W_l. Here, φ^* is the map induced by the base change morphism from Theorem 3.2.1 and q'_{2*} denotes the trace mapping induced by the finite flat map q'_2 (cf. Theorem 3.2.8).

Similarly, the endomorphism I_p^* from Definition 4.2.2 is identified with the diamond operator $\langle p\rangle\colon W_l \to W_l$.

Proof. See [Del71]. □

4.3 Generalizing the Eichler-Shimura Relation

Now that we have our 2-dimensional Galois representation over $\mathbb{T}_N \otimes_{\mathbb{Q}} \mathbb{Q}_l$, we want to prove the promised polynomial relation for the Frobenius. As in the weight $k = 2$ case, this is done by investigating how the construction behaves under reduction modulo p.

Our main aim is to prove the following theorem.

Theorem 4.3.1 (Deligne)**.** *As before, let $M/\mathbb{Q}$ be the maximal extension unramified away from N and l, let φ_p be a relative Frobenius element for a prime $p \nmid Nl$ in* Gal$(M/\mathbb{Q})$ *and denote by $F\colon W_l \to W_l$ the endomorphism φ_p^{-1}. There is a canonically defined l-adic scalar product on W_l, and we denote by V the transpose of F via that scalar product.*

Then we have

$$T_p = F + I_p^* V \qquad , \qquad FV = VF = p^{k-1}$$

and hence

$$1 - T_p X + I_p^* p^{k-1} X^2 = (1 - FX)(1 - I_p^* V X).$$

In particular, φ_p is a zero of that last polynomial.

Fix a prime $p \nmid Nl$ and choose an algebraic closure $\overline{\mathbb{F}}_p/\mathbb{F}_p$. We want to understand the restriction $\tilde{T}_p := T_{p|\operatorname{Spec}\mathbb{F}_p}$ of the Hecke operator. First, note that the restriction

$$R^1\tilde{a}(M_N, \operatorname{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l))_{|\operatorname{Spec}\mathbb{F}_p}$$

is again locally constant by Lemma 3.1.6, and hence corresponds to some continuous l-adic $\pi_1^{ét}(\operatorname{Spec}\mathbb{F}_p, \overline{\mathbb{F}}_p)$-representation.

Lemma 4.3.2. *Let $f\colon X \to Y$ be a morphism of connected locally Noetherian schemes, let $\bar{x} \to X$ be a geometric point and let $\mathcal{F}$ be a lisse $\mathbb{Q}_l$-sheaf. Then the $\pi_1^{\acute{e}t}(X, \bar{x})$-representation corresponding under Theorem 3.3.5 to $f^*\mathcal{F}$ is the representation*

$$\pi_1^{\acute{e}t}(X, \bar{x}) \xrightarrow{f_*} \pi_1^{\acute{e}t}(Y, f(\bar{x})) \to \mathrm{Aut}_{\mathbb{Q}_l}(\mathcal{F}_{f(\bar{x})}),$$

where the latter representation is the one corresponding to $\mathcal{F}$ under Theorem 3.3.5.

Proof. By construction, it suffices to prove the corresponding result for lisse $\mathbb{Z}/l^n\mathbb{Z}$-sheaves.

In that case, there is, by Proposition 3.1.8, a finite étale scheme G/Y such that $\mathcal{F} \cong \mathrm{Hom}_Y(-, G)$. Therefore, we have $f^*\mathcal{F} \cong \mathrm{Hom}_X(-, G \times_Y X)$ (cf. Corollary 2.2.11) and the statement is clear by the construction of $f_*\colon \pi_1^{\acute{e}t}(X, \bar{x}) \to \pi_1^{\acute{e}t}(Y, f(\bar{x}))$ in [Len08]. $\square$

There is an isomorphism between $\pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{Nl}], \overline{\mathbb{Q}}) \cong \mathrm{Gal}(M/\mathbb{Q})$ and $\pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{Nl}], \overline{\mathbb{F}}_p)$, unique up to inner automorphism. It is clear by construction that $\pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{F}_p, \overline{\mathbb{F}}_p) \to \pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{Nl}], \overline{\mathbb{F}}_p)$ sends the Frobenius in $\pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{F}_p, \overline{\mathbb{F}}_p) \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ to an absolute Frobenius element in $\mathrm{Gal}(M/\mathbb{Q})$.

Therefore, to prove the sum decomposition of T_p in Theorem 4.3.1, we can consider its restriction $\tilde{T}_p$ to $\mathrm{Spec}\,\mathbb{F}_p$. Note that by Theorem 3.2.7 and Theorem 3.2.8, the endomorphism $\tilde{T}_p$ is constructed in an analogous way to Theorem 4.2.5, with the whole diagram base changed to $\mathbb{F}_p$. We begin with the following rather general lemma.

Lemma 4.3.3. *Let S be a Noetherian scheme, and assume we have four compactifiable S-schemes X, Y, Z_1 and Z_2. Let $\mathcal{F}$ be a $\mathbb{Z}_l$-sheaf on X and $\mathcal{G}$ a $\mathbb{Z}_l$-sheaf on Y. Assume we have a commutative diagram of S-schemes and two morphisms of sheaves*

$$y_1^*\mathcal{G} \xrightarrow{z_1} x_1^*\mathcal{F} \qquad\qquad y_2^*\mathcal{G} \xrightarrow{z_2} x_2^*\mathcal{F}$$

$$\begin{tikzcd}
& Z_1 \arrow[r, "f"] \arrow[dl, "x_1"'] \arrow[drr, "y_1"'] & Z_2 \arrow[dll, "x_2"] \arrow[dr, "y_2"] & \\
X & & & Y
\end{tikzcd}$$

such that $f^*z_2 = z_1$, the morphisms to Y are proper and the ones to X are finite flat. Lastly, assume that for each geometric point $\bar{x} \to Z_2$, the multiplicity of $\bar{x}$ in the fiber $x_2^{-1}(x_2(\bar{x}))$ is equal to the sum of multiplicities of points $\bar{y} \in f^{-1}(\bar{x})$ in their respective fibers $x_1^{-1}(x_1(\bar{y}))$.

Then, the diagram

$$\begin{tikzcd}
R^i\tilde{a}(Y, \mathcal{G}) \arrow[r, "y_1^*"] \arrow[d, equal] & R^i\tilde{a}(Z_1, y_1^*\mathcal{G}) \arrow[r, "z_1"] & R^i\tilde{a}(Z_1, x_1^*\mathcal{F}) \arrow[r, "x_{1*}"] & R^i\tilde{a}(X, \mathcal{F}) \arrow[d, equal] \\
R^i\tilde{a}(Y, \mathcal{G}) \arrow[r, "y_2^*"] & R^i\tilde{a}(Z_2, y_2^*\mathcal{G}) \arrow[r, "z_2"] \arrow[u, "f^*"] & R^i\tilde{a}(Z_2, x_2^*\mathcal{F}) \arrow[r, "x_{2*}"] \arrow[u, "f^*"] & R^i\tilde{a}(X, \mathcal{F})
\end{tikzcd}$$

commutes, where x_{1*} and x_{2*} denote the respective trace maps.

Proof. Commutativity of the first two squares is obvious, and the commutativity of the last one relies on the construction of the trace map, so we refer to [Del71]. $\square$

It is here where the model for T_p using the normalization of M_N comes in handy, as it allows us to combine the last lemma with Proposition 4.2.4 to obtain a decomposition of T_p into a sum of certain endomorphisms.

Corollary 4.3.4. *The restriction $\tilde{T}_p$ is given by the sum of endomorphisms defined, as in Theorem 4.2.5, by the two diagrams (implicitly over $\mathbb{F}_p$)*

$$\begin{tikzcd}
 & E \arrow[rr, "F_{E/M_N}"] \arrow[dl, equal] \arrow[dr] & & E^{(p)} \arrow[dl] \arrow[dr] & \\
E \arrow[dr, "f"'] & & M_N \arrow[dl, equal] \arrow[dr, "F"'] & & E \arrow[dl, "f"] \\
 & M_N & & M_N &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (I_p^{-1})^* E^{(p)} \arrow[rr, "V"] \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr, equal] & \\
E \arrow[dr, "f"'] & & M_N \arrow[dl, "I_p^{-1} \circ F"] \arrow[dr, equal] & & E \arrow[dl, "f"] \\
 & M_N & & M_N &
\end{tikzcd}.$$

Proof. This is an immediate application of Lemma 4.3.3 to the diagram

$$\begin{tikzcd}
 & M_n \otimes \mathbb{F}_p \coprod M_n \otimes \mathbb{F}_p \arrow[r, "\Phi'"] \arrow[dl] \arrow[drr] & M'_{n,p} \otimes \mathbb{F}_p \arrow[dll, "q'_1"] \arrow[dr, "q'_2"] & \\
M_n \otimes \mathbb{F}_p & & & M_n \otimes \mathbb{F}_p
\end{tikzcd}$$

after noting that by Proposition 4.2.4 we have

$$q'_1 \circ \Phi' = \mathrm{id} \coprod (I_p^{-1} \circ F)$$
$$q'_2 \circ \Phi' = F \coprod \mathrm{id},$$

so we only have to check the conditions for Lemma 4.3.3. The pullback $\Phi^*(\varphi^*)$ (cf. Theorem 4.2.5) is computed using the fact that the base change morphism is stable under composition (cf. [SGAIV73, Expose XII, Proposition 4.4]) and the condition on the multiplicities of geometric points is in [Del71]. $\square$

This turns out to be just the decomposition we were looking for in Theorem 4.3.1:

Proposition 4.3.5. *The endomorphism F, defined as in Theorem 4.2.5 by the diagram*

$$\begin{tikzcd}
 & E \arrow[rr, "F_{E/M_N}"] \arrow[dl, equal] \arrow[dr] & & E^{(p)} \arrow[dl] \arrow[dr] & \\
E \arrow[dr, "f"'] & & M_N \arrow[dl, equal] \arrow[dr, "F"'] & & E \arrow[dl, "f"] \\
 & M_N & & M_N &
\end{tikzcd},$$

is the inverse of the Frobenius $\varphi_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

The diagram

$$\begin{tikzcd}
 & (I_p^{-1})^*E^{(p)} \arrow[rr, "V"] \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr, equal] & \\
E \arrow[dr, "f"'] & & M_N \arrow[dl, "I_p^{-1}\circ F"] \arrow[dr, equal] & & E \arrow[dl, "f"] \\
 & M_N & & M_N &
\end{tikzcd}$$

defines the composition $I_p^* \circ V$, where V is the transpose of F with respect to the l-adic inner product mentioned in Theorem 4.3.1 and I_p* denotes the automorphism defined in Definition 4.2.2 (restricted to $\operatorname{Spec}\mathbb{F}_p$*).*

Furthermore, we have $FV = VF = p^{k-1}$.

Proof. As in Proposition 4.2.1, we may view $R^1\tilde{a}(M_N, \mathrm{Sym}^{k-2}(R^1 f_*\mathbb{Z}_l))_{|\operatorname{Spec}\mathbb{F}_p}$ as the cohomology group $\tilde{H}^1_{\acute{e}t}(M_N \otimes \overline{\mathbb{F}}_p, \mathcal{F}_{k,l}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, and therefore we can refer to [SGAV77, Exposé XV, Section 2, $n°3$] for the computation of F.

For the second endomorphism, the l-adic inner product and the relation $FV = VF = p^{k-1}$ we refer to [Del71] directly. □

This concludes the proof of Theorem 4.3.1. Combining all of our results, we have proved:

Theorem 4.3.6. *Let $k \geq 2$, let $N \geq 5$ and let $f \in S_k(\Gamma_1(N), \chi)$ be a normalized eigenform, where $\chi\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ is some character. Then there is a 2-dimensional l-adic Galois representation*

$$\rho_{f,l}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_2(K_{f,l}),$$

where $K_{f,l}$ is an appropriate l-adic number field, such that $\rho_{f,l}$ is unramified away from Nl, and such that for any prime $p \nmid Nl$, any relative Frobenius $\varphi_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfies the equation

$$1 - a_p(f)\rho_{f,l}(\varphi_p) + p^{k-1}\chi(p)\rho_{f,l}(\varphi_p)^2 = 0.$$

Bibliography

[Art62] Michael Artin. *Grothendieck Topologies*. Harvard Unversity Lecture Notes. 1962.

[BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine Geometry*. Cambridge University Press, 2006.

[Del71] Pierre Deligne. “Formes modulaires et reprśentations l-adique”. In: *Séminaire N. Bourbaki* (1971).

[DS16] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*. Springer, 2016.

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Etale Cohonology and the Weil Conjecture*. Springer, 1988.

[FS95] Peter Freyd and Ross Street. “On the Size of Categories”. In: *Theory and Applications of Categories* 1.9 (1995), pp. 174–178.

[Gro57] Alexander Grothendieck. “Sur quelques points d’algèbre homologique”. In: *Tôhoku Mathematical Journal* (1957).

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. Springer, 2020.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Lan98] Saunder Mac Lane. *Categories for the Working Mathematician*. Springer, 1998.

[Len08] H. W. Lenstra. *Galois theory for schemes*. `https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`. Accessed 01-08-2025. 2008.

[Mat80] Hideyuki Matsumura. *Commutative Algebra*. Springer, 1980.

[Mil80] James S. Milne. *Étale Cohomology*. Princeton Universitiy Press, 1980.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*. Springer, 2008.

[SGAIV73] M. Artin, A. Grothendieck, and J. L. Verdier. *Théorie des topos et cohomologie étale des schémas (SGA IV) Tome 3*. Springer, 1973.

[SGAV77] A. Grothendieck. *Cohomologie l-adique et Fonctions L (SGA V)*. Springer, 1977.

[Sta25] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2025.

[Tam94] Günter Tamme. *Introduction to Étale Cohomology*. Springer, 1994.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.