Fay's identities, Goldman bracket and Integrable Systems

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Abstract

Fay's identities, Goldman bracket and Integrable Systems

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The results presented in this thesis contribute to the understanding of the interplay between the geometry of Riemann surfaces and their moduli space and the theory of integrable systems. First, in Chapter 2, we present a new degeneration of Fay's trisecant identity leading to algebro-geometric solutions of the Schwarzian Kadomtsev-Petviashvili equation in terms of Riemann theta functions. Then, in Chapter 3, we introduce a new set of log-canonical coordinates on the $SL(2, \mathbb{C})$ character variety of compact Riemann surfaces; these coordinates are constructed by combining shear type coordinates with length-twist type coordinates.

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Contribution of Authors

Chapter 1 is an original contribution from the author.
Chapter 2 is an original contribution from Christian Klein and the author, published in Proceedings of the Royal Society A, Volume 479, Issue 2273, (May 2023).
Chapter 3 is an original contribution from Marco Bertola, Dmitry Korotkin and the author. A preprint of a preliminary version is pre-published on arXiv.
Chapter 4 is an original contribution from the author.

Contents

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Glossary

$\mathcal{A}$	Abel map. 2
$\mathbb{B}$	Riemann matrix of b-periods. 2
β_γ	logarithmic variable conjugate to ℓ_γ. 6
$\mathcal{C}$	closed oriented Riemann surface of genus g. 2
$\mathcal{C}^{(i)}$	open subsurface obtained by cutting along a contour γ_j. 8
D_a	directional derivative: $\mathrm{D}_a = \sum_{i=1}^{g} v_{i0}\, \partial_{z_i}$. 3
ℓ_γ	logarithmic "length" coordinate, $\ell_\gamma = \ln \lambda_\gamma$. 6
Γ	embedded ciliated graph on $\mathcal{C}$. 6
γ	simple closed contour on $\mathcal{C}$. 7
Γ_{trin}	trinion graph of a complete trinion decomposition. 8
$G(\Gamma_{trin})$	generating function relating θ_{FN} and θ_{LC}. 11
$\widehat{\mathcal{C}}$	open subsurface of genus $\widehat{g}$ obtained by cutting along a single contour γ. 7
$\mathcal{H}_g$	Siegel upper half space. 2
$Jac(\mathcal{C})$	Jacobian variety of C. 2
$\mathrm{Li}_2(x)$	Euler dilogarithm. 11

$\mathcal{M}_g$ — moduli space of compact Riemann surfaces of genus g. 5

M_γ — monodromy matrix around γ. 5

Ω — Goldman symplectic form on $\mathcal{V}_g$. 5

$\Omega^1(\mathcal{C})$ — space of holomorphic 1-forms on $\mathcal{C}$. 15

$\boldsymbol{\omega}$ — g-dimensional vector of normalized holomorphic 1-forms on $\mathcal{C}$. 2

ω_{WP} — Weil–Petersson symplectic form on $\mathcal{P}_g$ and $\mathcal{M}_g$. 5

$\mathcal{T}_g$ — Teichmüller space. 9

$\pi_1(\mathcal{C})$ — fundamental group of $\mathcal{C}$. 5

$\Sigma^{(i)}$ — triangulation fat-graph on $\mathcal{C}^{(i)}$. 8

$\widehat{\Sigma}$ — triangulation fat-graph on $\widehat{\mathcal{C}}$. 7

$\widetilde{\Sigma}$ — triangulation fat-graph on $\widetilde{\mathcal{C}}$. 7

τ_γ — Fenchel–Nielsen twist coordinate. 6

$\widetilde{\mathcal{C}}$ — open subsurface of genus $\widetilde{g}$ obtained by cutting along a single contour γ. 7

$\Theta(\mathbf{z}, \mathbb{B})$ — Riemann theta function with zero characteristic. 3

θ_{FN} — symplectic potential in Fenchel–Nielsen coordinates. 10

θ_{LC} — symplectic potential in log-canonical coordinates. 10

Θ^* — theta function with an odd, non-singular characteristic. 3

$\mathcal{V}_g$ — $SL(2,\mathbb{C})$ character variety of $\mathcal{C}$. 5

z_e — shear coordinate associated with an edge e. 6

$\zeta(e)$ — logarithm of the shear coordinate associated with an edge e. 8

Chapter 1

Introduction

The notion of integrability in mathematical physics first appeared at the end of the nineteenth century with the notion of *Liouville integrability* in Hamiltonian mechanics. These results showed that a Hamiltonian system of dimension $2N$ can be solved by quadratures, provided that there are N conserved quantities [44], [4]. Such ideas found far reaching generalizations and enrichment during the 1960s with the discovery of completely integrable systems with infinite number of degrees of freedom. These systems typically admit special solutions called *solitons* [48]. The most famous example is the Korteweg–de Vries (KdV) equation [40]:

$$\phi_t + \phi_{xxx} - 6\phi\phi_x = 0 \,, \qquad (x \in \mathbb{R}, \quad t \geq 0) \,.$$

This equation admits a soliton moving to the right with constant velocity c:

$$\phi(x,t) = -\frac{1}{2\cosh^2(x - ct - a)} \,,$$

where a is an arbitrary constant. The 1970s witnessed the emergence of powerful new tools for the study of integrable systems, including *Inverse Scattering Transform* [1] and the notion of *Lax pairs* [43]. These ideas opened unexpected bridges with algebraic geometry, where Riemann surfaces and their moduli spaces play a key role. In this context, Novikov proposed a conjecture characterizing Jacobian varieties in terms of solutions to the Kadomtsev–Petviashvili (KP) hierarchy. This conjecture was later proved by T. Shiota, who used Fay's identities [59].

Chapter 2 is dedicated to the construction of algebro-geometric solutions to integrable nonlinear partial differential equations (PDEs) in terms of multidimensional Riemann theta functions using degenerations of Fay's trisecant identity. To describe these results in more detail, let us introduce some notation.

Let $\mathcal{C}$ be a compact Riemann surface of genus $g \in \mathbb{N}$, equipped with a canonical basis of homology cycles

$$a_1, \ldots, a_g \ , \qquad b_1, \ldots, b_g \ ,$$

satisfying the standard intersection conditions:

$$a_i \circ b_j = \delta_{ij} \ , \qquad a_i \circ a_j = 0 \ , \qquad b_i \circ b_j = 0 \ , \qquad i, j = 1, \ldots, g \ .$$

Let $\boldsymbol{\omega}$ be the g-dimensional vector of holomorphic 1-forms, normalized by

$$\int_{a_i} \omega_j = \delta_{ij} \ , \qquad i, j = 1, \ldots, g \ .$$

The matrix of b-periods is defined by

$$\mathbb{B}_{ij} = \int_{b_i} \omega_j \ , \qquad i, j = 1, \ldots, g \ .$$

This matrix $\mathbb{B} \in \mathcal{H}_g$ is a Riemann matrix, that is, a symmetric complex matrix with positive-definite imaginary part. The Abel map

$$\mathcal{A} : P \mapsto \int_{P_0}^{P} \boldsymbol{\omega}$$

is an holomorphic map from the Riemann surface $\mathcal{C}$ into its *Jacobian variety* $Jac(\mathcal{C}) = \mathbb{C}^g / \Lambda$, where the period lattice Λ is given by

$$\Lambda = \{ \mathbf{m} + \mathbb{B}\mathbf{n} : \mathbf{m}, \mathbf{n} \in \mathbb{Z}^g \} \, .$$

Given the above data one can construct algebro-geometric solutions to integrable partial differential equations in terms of multidimensional *Riemann theta functions*

$$\Theta_{\mathbf{p},\mathbf{q}}(\mathbf{z}, \mathbb{B}) = \sum_{\mathbf{N} \in \mathbb{Z}^g} \exp \left\{ \mathrm{i}\pi \left\langle \mathbb{B} \left(\mathbf{N} + \mathbf{p} \right), \mathbf{N} + \mathbf{p} \right\rangle + 2\pi i \left\langle \mathbf{z} + \mathbf{q}, \mathbf{N} + \mathbf{p} \right\rangle \right\} \ ,$$

where $\mathbf{z} \in \mathbb{C}^g$, and $\mathbf{p}$, $\mathbf{q} \in \mathbb{R}^g$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product:

$$\langle \mathbf{N}, \mathbf{z} \rangle = \sum_{i=1}^{g} N_i z_i \ .$$

Riemann theta functions can be viewed as holomorphic sections of a line bundle over the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ of the compact Riemann surface $\mathcal{C}$.

Quasi-periodic solutions to integrable nonlinear PDEs expressed in terms of Riemann theta functions were first discovered during the 1970s, via the *Baker-Akhiezer function* - a function on $\mathcal{C}$ with an essential singularity - originally introduced by Clebsh and Gordan. A more 'direct' method for obtaining such solutions was developed by D. Mumford et al. in the 1980s [49]. Their approach is based on *Fay's trisecant identity* [18], a remarkable identity for Riemann theta functions:

$$\Theta^*_{ad}\Theta^*_{cb}\Theta_{ac}(\mathbf{z})\Theta_{bd}(\mathbf{z}) + \Theta^*_{ca}\Theta^*_{db}\Theta_{bc}(\mathbf{z})\Theta_{ad}(\mathbf{z}) = \Theta^*_{cd}\Theta^*_{ab}\Theta(\mathbf{z})\Theta_{a+b,c+d}(\mathbf{z}) \,, \tag{1.1}$$

where we use the notation

$$\Theta_{ab}(\mathbf{z}) = \Theta\left(\mathbf{z} + \int_a^b \boldsymbol{\omega}, \mathbb{B}\right), \qquad \Theta^*_{ab} = \Theta^*\left(\int_a^b \boldsymbol{\omega}, \mathbb{B}\right), \tag{1.2}$$

with $\Theta(\mathbf{z}, \mathbb{B})$ denoting a Riemann theta function with zero characteristic, $\Theta^*(\mathbf{z}, \mathbb{B})$ a Riemann theta function with an odd, non-singular characteristic (whose precise definition is recalled in the beginning of Chapter 2), and where $a, b, c, d \in \mathcal{C}$ are arbitrary points on $\mathcal{C}$.

To study degenerations of identity (1.1), one considers limits in which two or more of the points a, b, c, d coalesce. This requires Taylor expansions of the Abel map. The expansion of the Abel map at a point $P \in \mathcal{C}$ near a point $a \in \mathcal{C}$ is written in the form

$$\mathcal{A}_i(P) = \sum_{j=0}^{\infty} v_{ij} \frac{\tau_a^{j+1}}{(j+1)!} \,, \quad i = 1, \dots, g \,, \tag{1.3}$$

where τ_a is a local parameter in the vicinity of a that also contains P. From this expansion, one defines directional derivatives that act on a function $f(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^g$, as follows:

$$\begin{gathered} \mathrm{D}_a = \sum_{i=1}^{g} v_{i0}\partial_{z_i} \,, \qquad \mathrm{D}'_a = \sum_{i=1}^{g} v_{i1}\partial_{z_i} \,, \qquad \mathrm{D}''_a = \sum_{i=1}^{g} \frac{v_{i2}}{6}\partial_{z_i} \,, \\ \mathrm{D}^{(n)}_a = \sum_{i=1}^{g} \frac{v_{in}}{(n+1)!}\partial_{z_i} \,, \quad n \in \mathbb{N} \,. \end{gathered} \tag{1.4}$$

By substituting these expansions into (1.1), one can obtain various degenerate forms of Fay's trisecant identity, involving some combination of directional derivatives of Riemann theta functions. These degenerate identities can be used to directly solve some integrable nonlinear PDE's by identification.

In [49] this framework has been applied to obtain solutions of the Toda lattice, Sine-Gordon and Kadomtsev-Petviasvili equations. Further reductions, involving special classes of

Riemann surfaces (e.g. hyperelliptic or trigonal), yield solutions to the Korteweg-de-Vries and Boussinesq equations, respectively. Extensions of this approach have also produced solutions to the Ernst equation and the Camassa–Holm equation [38] [36].

More recently, C. Kalla derived a new degeneration of Fay's identity in [35], which we prove again in Chapter 2, Section 2.3. Using this identity, known solutions of Davey-Stewartson and vector nonlinear Schrödinger equations have been re-derived.

The first result of this thesis is the derivation of another higher-order degeneration of Fay's trisecant identity

$$\begin{aligned} 0 &= 2(\mathrm{D}_a U)^2 \mathrm{D}''_a \mathrm{D}_a U - 2\mathrm{D}_a U \mathrm{D}''_a U \mathrm{D}_a^2 U - (\mathrm{D}_a U)^2 \mathrm{D}_a^4 U + 4\mathrm{D}_a U \mathrm{D}_a^3 U \mathrm{D}_a^2 U \\ &\quad - 3(\mathrm{D}_a^2 U)^3 + 3(\mathrm{D}'_a U)^2 \mathrm{D}_a^2 U - 3(\mathrm{D}_a U)^2 (\mathrm{D}'_a)^2 \mathrm{D}_a U \ , \end{aligned} \tag{1.5}$$

where $a, b \in \mathcal{C}$ are two points on the Riemann surface lying inside a chosen fundamental polygon, the directional derivatives D_a, D_b, D'_a, D''_a are defined by (1.4) and the function U is given by

$$U = \mathrm{D}_b \ln \left[\Theta(\mathrm{z})\Theta^*_{ba}\right] \ ,$$

with D_b acting on both theta functions.

The second main result is the construction of a new class of algebro-geometric solutions to the Schwarzian Kadomtsev-Petviashvili (SKP) equation using identity (1.5). The SKP equation

$$\Big(\frac{\phi_t}{\phi_x} - \frac{1}{2}\{\phi; x\}\Big)_x - \frac{3}{2}\Big(\frac{\phi_y}{\phi_x}\Big)_y - \frac{3}{4}\Big(\frac{\phi_y^2}{\phi_x^2}\Big)_x = 0 \ , \tag{1.6}$$

where $\{\phi; x\}$ denotes the Schwarzian derivative in x defined by

$$\{\phi; x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2 \ ,$$

was first introduced in the context of *Painlevé analysis* of the KP equation in [62], where it appeared as a singularity manifold equation. The same work established its connections to the KP, modified KdV, and Harry Dym equations via Miura and Bäcklund transformations. Its complete integrability was later proved in [8].

Using identity (1.5), one can verify that the function

$$\phi(x, y, t) = \mathrm{D}_b \ln \left[\Theta^*_{ab}\Theta(x\mathbf{v}_0(a) + y\mathbf{v}_1(a) + t\mathbf{v}_2(a) + \mathbf{d})\right]$$

where $(x, y, t) \in \mathbb{R}^3$, $\mathbf{d} \in \mathbb{C}^g$, and $\mathbf{v}_j = (v_{1j}, ..., v_{gj}) \in \mathbb{C}^g$ for $j = 0, 1, 2$, with v_{ij} the coefficients of the Taylor expansion (2.4) near a of the Abel map, solves the SKP equation (2.3).

Similarly to the KP equation, reductions to the Schwarzian KdV and Boussinesq equations, which require elimination of one variable, are obtained by restricting to special classes of Riemann surfaces, hyperelliptic and trigonal, respectively.

The SKP equation itself belongs to an infinite integrable hierarchy, as shown in [52, 53]. These equations and their discrete analogues have attracted recent attention [13, 41], notably through their connections to inversive geometry and dimer models.

Moreover, Fay's trisecant identity is a special case of a general identity involving an arbitrary number of points [18]. Dubrovin [16] has derived a fully degenerate version of this general identity and proven that it encodes the entire KdV hierarchy using Lax pairs formalism. Similar conjectures are presented at the end of Chapter 2 for the Schwarzian KP hierarchy.

In Chapter 3, we study the symplectic structure of the $SL(2,\mathbb{C})$ character varieties of compact Riemann surfaces and the moduli space of compact Riemann sufraces $\mathcal{M}_g$. More specifically, we introduce new explicit sets of log-canonical coordinates for those spaces.

The $SL(2,\mathbb{C})$ character variety $\mathcal{V}_g$ of a compact Riemann surface $\mathcal{C}$ of genus g with fundamental group $\pi_1(\mathcal{C})$ is the quotient space

$$\mathcal{V}_g \simeq \mathrm{Hom}\left(\pi_1(\mathcal{C}), SL(2,\mathbb{C})\right) / \sim\ ,$$

where the equivalence relation $\sim$ is up to conjugation in $SL(2,\mathbb{C})$. We denote by $M_\gamma \in SL(2,\mathbb{C})$ the *monodromy matrix* corresponding to a contour $\gamma \in \pi_1(\mathcal{C})$. Traces of those matrices provide local coordinates on $\mathcal{V}_g$. W. Goldman in [25] introduced the following Poisson bracket on $\mathcal{V}_g$

$$\{\mathrm{tr}M_\gamma, \mathrm{tr}M_{\tilde{\gamma}}\}_G = \sum_{p\in\gamma\cap\tilde{\gamma}} \nu(p)\left(\mathrm{tr}M_{\gamma\circ_p\tilde{\gamma}} - \frac{1}{2}\mathrm{tr}M_\gamma\,\mathrm{tr}M_{\tilde{\gamma}}\right)\ ,$$

where $\gamma, \tilde{\gamma} \in \pi_1(\mathcal{C})$ and $\nu(p)$ is the contribution of the point p to the intersection index of γ and $\tilde{\gamma}$. On $\mathcal{V}_g$ the Goldman bracket $\{\cdot,\cdot\}_G$ is non-degenerate and we denote by Ω the corresponding symplectic form. For the Teichmüller space of compact Riemann surfaces $\mathcal{T}_g$ (the relationship between $\mathcal{V}_g$ and $\mathcal{T}_g$ is recalled page 8), the form Ω coincides with the Weil-Petersson symplectic form ω_{WP} [66].

A well-known set of Darboux coordinates for the Weil-Petersson symplectic form ω_{WP} in $\mathcal{T}_g$ is given by the *Fenchel-Nielsen twist-length coordinates*. Fenchel-Nielsen coordinates are associated to an arbitrary trinion decomposition of $\mathcal{C}$. Such a decomposition is obtained by cutting $\mathcal{C}$ along $m = 3g-3$ closed, non-intersecting geodesics. This produces $2g-2$ trinions, that is, spheres with three holes $\mathcal{T}^{(i)}$. Let ℓ_{γ_j} be the hyperbolic length of γ_j. The original surface $\mathcal{C}$ is recovered by regluing the trinions along their boundaries. In this process, there is an S^1 freedom of rotation. This rotational freedom is described by a *twist* τ_{γ_j} corresponding to an oriented arc length from a chosen origin. The $6g-6$ real parameters $\{\ell_{\gamma_j}, \tau_{\gamma_j}\}_{j=1}^{3g-3} \in \mathbb{R}^+ \times \mathbb{R}$ are the Fenchel-Nielsen coordinates on $\mathcal{M}_g$ [20], [31]. By analytic continuation to the complex domain, these give Darboux coordinates on $\mathcal{V}_g$, called complex Fenchel–Nielsen coordinates. S. Wolpert proved the following expression for the Weil-Petersson symplectic form ω_{WP} on $\mathcal{M}_g$ in Fenchel-Nielsen coordinates [66]

$$\omega_{WP} = \sum_{j=1}^{3g-3} \mathrm{d}\tau_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \, . \tag{1.7}$$

Let $\mathcal{C}$ have n punctures, denote by $\mathcal{T}_{g,n}$ the corresponding Teichmüller space. There exists a set of log-canonical coordinates on $\mathcal{T}_{g,n}$ due to Thurston called shear coordinates, later generalized by Fock and Goncharov to higher Teichmüller theory [21]. These coordinates are defined for any ideal triangulation of $\mathcal{C}$ with vertices at the punctures. Namely, pick two adjacent triangles $T_1 = [x_1, x_2, x_3]$ and $T_2 = [x_1, x_3, x_4]$ with a common edge $e = [x_1, x_3]$. In a fundamental domain of $\pi_1(\mathcal{C})$ in the upper half-plane $\mathbb{H}$ the points $x_1, \dots, x_4$ lie on $\partial\mathbb{H}$ in counterclockwise order. Then the shear coordinate z_e on the edge e is defined as [45]

$$z_e = \ln \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)} \, .$$

These coordinates are log-canonical for the Weil-Petersson symplectic form on $\mathcal{T}_{g,n}$, and Darboux coordinates can be obtained as linear combinations with constant coefficients.

In the unpunctured case, the explicit parameterization of $\mathcal{V}_g$ using shear coordinates remained unclear. Results in this direction were obtained by F. Bonahon and I. Kim [11], who discussed the relationship between Fock-Goncharov coordinates and Goldman coordinates for the space of convex projective structures on a closed surface with negative Euler characteristic, using trinion decompositions. More recently, M. Shapiro and L. Chekhov also used Fock-Goncharov coordinates to parameterize $\mathcal{V}_2$ [12] and discussed some examples of higher genera.

New systems of log-canonical coordinates on $\mathcal{V}_g$ are constructed as mixtures of complex shear type coordinates $\zeta_{e_j} = \ln z_{e_j}$ and complex twist-length type coordinates $\{\beta_\gamma, \ell_\gamma\}$ (the complex toric variable β_γ is closely related to the complex Fenchel-Nielsen twist τ_γ).

Using the framework of [7] we equip $\mathcal{C}$ with an embedded ciliated graph Γ with $SL(2,\mathbb{C})$ jump matrices on its edges. A canonical symplectic form is associated with this graph and jump matrices assignment; this canonical symplectic form turns out to coincide with an extension of the Goldman symplectic form Ω.

The simplest illustration of the general idea starts with a single contour γ dividing $\mathcal{C}$ into two surfaces with boundaries $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$, of genera $\widetilde{g}$ and $\widehat{g}$, respectively. A pair of complex twist-length type coordinates $\{\beta_\gamma, \ell_\gamma\}$ is associated with γ. Collapsing the boundaries $\widetilde{\gamma}$ and $\widehat{\gamma}$ to points $\widetilde{v}$ and $\widehat{v}$, we equip $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ with triangulation fat-graphs $\widetilde{\Sigma}$ and $\widehat{\Sigma}$, each with a single vertex at $\widetilde{v}$ and $\widehat{v}$. The complex shear type coordinates $\widetilde{z}_{e_i}$ and $\widehat{z}_{e_i}$ appear in the off-diagonal entries of the jump matrices $\widetilde{S}_{e_i}$ and $\widehat{S}_{e_i}$ on the edges $\tilde{e}_i$ and $\widehat{e}_i$ of $\widetilde{\Sigma}$ and $\widehat{\Sigma}$, respectively.

By amalgamation of the graphs $\widetilde{\Sigma}$ and $\widehat{\Sigma}$, together with some additional structure, we obtain the embedded ciliated graph Γ in $\mathcal{C}$ shown in Figure 1.1.

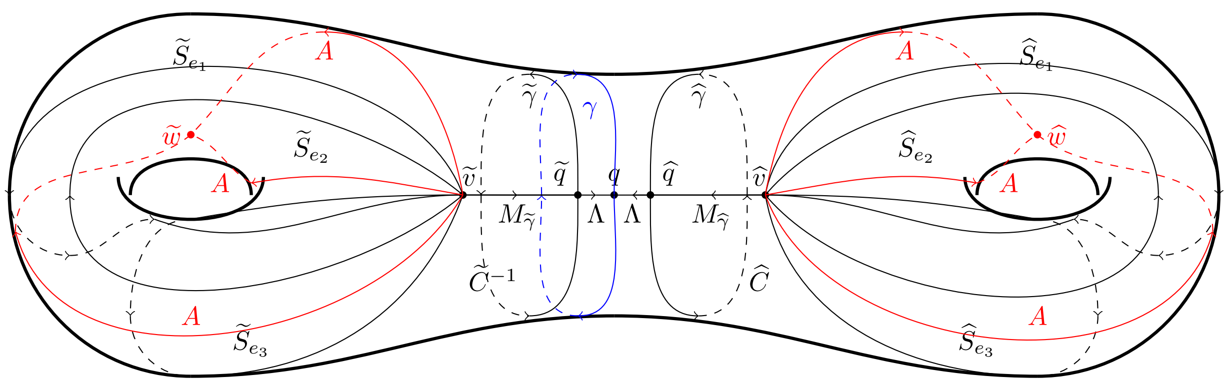

Figure 1.1: The graph Γ drawn on $\mathcal{C}$ for $g = 2$.

Denote by $\{\widetilde{\zeta}_{e_i} = \ln \widetilde{z}_{e_i}\}_{i=1}^{6\widetilde{g}-3}$ the logarithms of the shear type coordinates on $\mathcal{V}_{\widetilde{g},1}$ and by $\{\widehat{\zeta}_{e_j} = \ln \widehat{z}_{e_j}\}_{j=1}^{6\widehat{g}-3}$ the logarithms of the shear type coordinates on $\mathcal{V}_{\widehat{g},1}$. These coordinates satisfy the following linear constraint

$$2\sum_{i=1}^{6\widetilde{g}-3} \widetilde{\zeta}_{e_i} = 2\sum_{j=1}^{6\widehat{g}-3} \widehat{\zeta}_{e_j} = \ell_\gamma \, . \tag{1.8}$$

Resolving the constraint we define the following set of independent coordinates:

$$\left\{ \{\widetilde{\zeta}\}_{i=2}^{6\widetilde{g}-3}, \, \{\widehat{\zeta}_{e_i}\}_{i=2}^{6\widehat{g}-3}, \ell_\gamma, \beta_\gamma \right\} \, . \tag{1.9}$$

Our first result is that the Goldman symplectic form Ω on $\mathcal{V}_g$ can be written in log-canonical coordinates (1.9) as follows

$$\Omega = \Omega_0 + \widetilde{\Omega}_1 + \widehat{\Omega}_1 + \widetilde{\Omega}_2 + \widehat{\Omega}_2 \ ,$$

where

$$\Omega_0 = 2\mathrm{d}\beta_\gamma \wedge \mathrm{d}\ell_\gamma \ ,$$

$$\widetilde{\Omega}_1 = \sum_{\substack{i,j=2 \\ i<j}}^{6\widetilde{g}-3} (\widetilde{c}_{ij} - \widetilde{c}_{1j} + \widetilde{c}_{1i})\, \mathrm{d}\widetilde{\zeta}_{e_i} \wedge \mathrm{d}\widetilde{\zeta}_{e_j} \ , \qquad \widehat{\Omega}_1 = \sum_{\substack{k,l=2 \\ k<l}}^{6\widehat{g}-3} (\widehat{c}_{kl} - \widehat{c}_{1l} + \widehat{c}_{1k})\, \mathrm{d}\widehat{\zeta}_{e_k} \wedge \mathrm{d}\widehat{\zeta}_{e_l} \ ,$$

$$\widetilde{\Omega}_2 = \frac{1}{2} \sum_{m=2}^{6\widetilde{g}-3} \widetilde{c}_{1m} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widetilde{\zeta}_{e_m} \ , \qquad \widehat{\Omega}_2 = \frac{1}{2} \sum_{n=2}^{6\widehat{g}-3} \widehat{c}_{1n} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widehat{\zeta}_{e_n} \ .$$

The integer coefficients $\widetilde{c}_{ij}$, $\widehat{c}_{kl}$ lie in the interval between -4 and 4. They are determined by the order of the edges of the one-valent graphs $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ at their respective vertices $\widetilde{v}$ and $\widehat{v}$.

This result is then generalized to an arbitrary system of closed contours $\{\gamma_1, \gamma_2, ..., \gamma_m\}$, $1 \le m \le 3g-3$, splitting $\mathcal{C}$ into n Riemann surfaces $\mathcal{C}^{(i)}$ with $k^{(i)}$ boundary components for $i = 1, ..., n$. There are now m pairs of coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{m}$. Each surface $\mathcal{C}^{(i)}$ has a triangulation graph $\Sigma^{(i)}$ drawn on it and we associate complex shear type coordinates $\zeta(e)$ to the $6g^{(i)} - 6 + 3k^{(i)}$ edges of $\Sigma^{(i)}$. In this situation, one can write the Goldman symplectic form on $\mathcal{V}_g$ as

$$\Omega = \sum_{i=1}^{n} \Omega^{(i)} + \sum_{j=1}^{m} \mathrm{d}\beta_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ , \tag{1.10}$$

where

$$\Omega^{(i)} = \sum_{v \in V(\Sigma^{(i)})} \sum_{\substack{e,\widetilde{e} \perp v \\ e<\widetilde{e}}} \mathrm{d}\zeta_e \wedge \mathrm{d}\zeta_{\widetilde{e}} \ . \tag{1.11}$$

When m reaches its maximal value, that is, $m = 3g-3$, the surface $\mathcal{C}$ decomposes into $2g-2$ trinions, $\mathcal{C}$ decomposes into $2g-2$ trinions $\mathcal{C}^{(i)}$. Then, all shear type coordinates can be expressed in terms of the complex boundary "lengths" of $\mathcal{C}^{(i)}$, due to the constraints (1.8). This yields $3g-3$ log-canonical coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$ for the Goldman symplectic form Ω on $\mathcal{V}_g$.

Define the *trinion graph* Γ_{trin} whose vertices correspond to individual trinions, and whose edges describe how the boundary components are glued together. Denote by ℓ_e the "complex oriented length" of the trinion's boundary component corresponding to the edge e. Then, the form Ω takes the expression

$$\Omega = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e,\widetilde{e} \perp v \\ e < \widetilde{e}}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\widetilde{e}} + \sum_{j=1}^{3g-3} \mathrm{d}\beta_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ . \tag{1.12}$$

Here, for each vertex $v \in V(\Gamma_{trin})$, the inner sum is over pairs of the three edges incident to v, according to the inherited ordering of the triangulation edges of the ciliated graph Γ. The expression (1.12) is independent of the position of the cilium.

When restricted to $\mathbb{R}$, the real character variety $\mathcal{V}_g^{\mathbb{R}}$ has several connected components [27]. In Section 3.10, we prove that the representations constructed by our framework lie in the Fuchsian component. When projected to $PSL(2,\mathbb{R})$ each conjugacy class of such representations is in one to one correspondence with a point of the Teichmüller space $\mathcal{T}_g$. Therefore, our sets of log-canonical coordinates are well fined on $\mathcal{T}_g$ and the form Ω coincides with the Weil-Petersson symplectic form ω_{WP}, given by equation (1.7). By results of S. Wolpert [66], the form ω_{WP} is invariant under the action of the mapping class group and therefore descends to an open dense subset of $\mathcal{M}_g$.

On $\mathcal{M}_g$, the log-canonical coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$ can be related to the real Fenchel-Nielsen coordinates $\{\tau_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$. The real length coordinates ℓ_{γ_j} are the same in both systems, while each real twist coordinate τ_{γ_j} differs from β_{γ_j} by a function that depends only on the lengths.

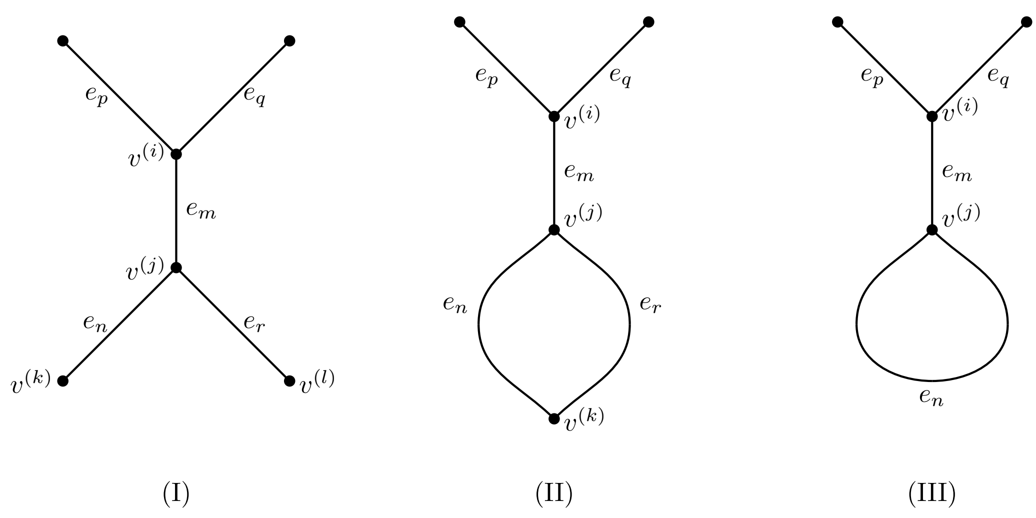

Figure 1.2: Three possible configurations of two vertices $v^{(i)}$ and $v^{(j)}$ of Γ_{trin}

Consider a *standard edge* $e_m = [v^{(i)}, v^{(j)}]$ connecting two vertices $v^{(i)}$ and $v^{(j)}$ of Γ_{trin}. Let e_p and e_q be the two other edges incident to $v^{(i)}$, with $e_q < e_p < e_m$. Similarly, let e_n and e_r be the two other edges incident to $v^{(j)}$, with $e_n < e_m < e_k$ (see Figure 1.2, configurations (I) and (II)).

The relation between the Fenchel-Nielsen twist τ_{e_m}, the toric variable β_{e_m}, and the eigenvalues

$$\lambda_{e_m} = e^{\pm \ell_{e_k}}, \qquad \lambda_{e_p} = e^{\pm \ell_{e_p}}, \qquad \lambda_{e_q} = e^{\pm \ell_{e_q}}, \qquad \lambda_{e_n} = e^{\pm \ell_{e_n}}, \qquad \lambda_{e_k} = e^{\pm \ell_{e_k}}$$

follows from the geometric definition in $\mathbb{H}$ of the Fenchel-Nielsen twist associated to the gluing of the trinions $\mathcal{T}^{(i)}$ and $\mathcal{T}^{(j)}$ corresponding to $v^{(i)}$ and $v^{(j)}$, respectively. One obtains:

$$\tau_{e_m} = \beta_{e_m} + \ln \Delta(\lambda_{e_q}, \lambda_{e_p}; \lambda_{e_m}) + \ln \Delta(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) \ , \tag{1.13}$$

where the function $\Delta(x, y; z)$ is defined by

$$\Delta(x, y; z) = \left[\frac{(1 + xyz)(y + xz)(x + yz)}{y^2(z^2 - 1)^2(z + yx)} \right]^{1/2} .$$

For a *loop edge* $e_n = [v^{(j)}, v^{(j)}]$, cf. Figure 1.2, configuration (III), equation (1.13) becomes

$$\tau_{e_n} = \beta_{e_n} + \ln \Delta^2(\lambda_{e_n}, \lambda_{e_m}; \lambda_{e_n}) \ . \tag{1.14}$$

Consider a symplectic potential (or Liouville form) θ_{FN} for ω_{WP} in Fenchel-Nielsen coordinates such that $\mathrm{d}\theta_{FN} = \omega_{WP}$. A possible choice (see equation (1.7)) is:

$$\theta_{FN} = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{e \perp v} \tau_e \mathrm{d}\ell_e \ .$$

Similarly, for the same trinion graph, a symplectic potential in log-canonical coordinates (cf. equation (1.12)) can be written as

$$\theta_{LC} = \theta_{LC}^{(1)} + \theta_{LC}^{(2)} \ ,$$

$$\theta_{LC}^{(1)} = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e, \widetilde{e} \perp v \\ e < \widetilde{e}}} \beta_e \mathrm{d}\ell_e \ , \qquad \theta_{LC}^{(2)} = -\frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e, \widetilde{e} \perp v \\ e < \widetilde{e}}} \ell_{\widetilde{e}} \mathrm{d}\ell_e \ .$$

The difference $\theta_{FN} - \theta_{FG}$ is an exact 1-form that depends only on lengths. It can be described in terms of a *generating function* $G(\Gamma_{trin})$, such that:

$$\theta_{FN} - \theta_{LC} = \sum_{v \in V(\Gamma_{trin})} \sum_{e \perp v} \frac{\partial G(\Gamma_{trin})}{\partial \ell_e} \mathrm{d}\ell_e \ . \tag{1.15}$$

Substituting equation (1.13), (1.14), into (1.15) and integrating gives the following expression for $G(\Gamma_{trin})$ ($\theta_{LC}^{(2)}$ is specifically chosen to cancel a maximum of logarithmic terms):

$$G(\Gamma_{trin}) = \frac{1}{2} \sum_{e_m = [v^{(j)}, v^{(i)}]} \left[g_0^{(j)}(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) \right] + \frac{1}{2} \sum_{e_n = [v^{(j)}, v^{(j)}]} g_1^{(j)}(\lambda_{e_m}; \lambda_{e_n}) \ , \tag{1.16}$$

where standard edges contributions are given by

$$\begin{aligned} g_0^{(j)}(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) &= -\frac{1}{2}\Big[\Phi(\lambda_{e_n}, \lambda_{e_r}, \lambda_{e_m}) + \mathrm{Li}_2(1 - \lambda_{e_m}^2) + \ln(\lambda_{e_m}^2)\ln(\lambda_{e_m}^2 - 1)\Big] \ , \\ g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) &= -\frac{1}{2}\Big[\Phi(\lambda_{e_p}, \lambda_{e_q}, \lambda_{e_m}) + \mathrm{Li}_2(1 - \lambda_{e_m}^2) + \ln(\lambda_{e_m}^2)\ln(\lambda_{e_m}^2 - 1)\Big] \ , \end{aligned}$$

and loop edges contributions by

$$g_1^{(j)}(\lambda_{e_m}; \lambda_{e_n}) = -\Phi(\lambda_{e_n}, \lambda_{e_m}, \lambda_{e_n}) - \mathrm{Li}_2(1 - \lambda_{e_n}^2) - \ln(\lambda_{e_n}^2)\ln(\lambda_{e_n}^2 - 1) - \ln\lambda_{e_m}\ln\lambda_{e_n} + \frac{\ln^2\lambda_{e_n}}{2} \ .$$

Here the auxiliary function Φ is defined as

$$\Phi(x, y, z) = \mathrm{Li}_2(-xyz) + \mathrm{Li}_2\Big(-\frac{xz}{y}\Big) + \mathrm{Li}_2\Big(-\frac{yz}{x}\Big) - \mathrm{Li}_2\Big(-\frac{z}{xy}\Big) \ ,$$

where $\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\mathrm{d}t$ denotes the Euler dilogarithm.

Finally, at the end of Section 3, we outline a method for constructing new systems of log-canonical coordinates on an arbitrary $SL(N, \mathbb{C})$ character variety of $\pi_1(\mathcal{C})$. This construction combines the Fock-Goncharov coordinates associated with the graphs and the eigenvalues plus toric variables associated with M_γ. Explicit computations for $g = 2$ and $g = 3$ are provided in an appendix.

Chapter 2

Higer order degenerations of Fay's identities and applications to integrable equations

Joint work with C. Klein

Abstract

Higher order degenerated versions of Fay's trisecant identity are presented. It is shown that they lead to solutions for Schwarzian Kadomtsev-Petviashvili equations.

2.1 Introduction

Solutions to integrable partial differential equations (PDEs) in terms of multi-dimensional theta functions on compact Riemann surfaces appeared in the 1970s in the search for quasi-periodic solutions, see for instance [14, 5] for a historic account. These solutions were constructed via the *Baker-Akhiezer* function, a function with an essential singularity on the Riemann surface first introduced by Clebsch and Gordan. Mumford and coworkers introduced in [49] a complementary approach based on Fay's celebrated *trisecant identity* for theta functions [18],

$$\Theta_{ad}^{*}\Theta_{cb}^{*}\Theta_{ac}(\mathbf{z})\Theta_{bd}(\mathbf{z})+\Theta_{ca}^{*}\Theta_{db}^{*}\Theta_{bc}(\mathbf{z})\Theta_{ad}(\mathbf{z})=\Theta_{cd}^{*}\Theta_{ab}^{*}\Theta(\mathbf{z})\Theta_{a+b,c+d}(\mathbf{z})\ ,$$

where we have introduced the notation

$$\Theta_{ab}^{*}=\Theta^{*}\left(\int_{a}^{b}\boldsymbol{\omega}\right)\ ,\qquad \Theta_{ab}(\mathbf{z})=\Theta\left(\mathbf{z}+\int_{a}^{b}\boldsymbol{\omega}\right)\ . \tag{2.1}$$

Here $\Theta(\mathbf{z})$, $\mathbf{z}\in\mathbb{C}^{g}$, is the g dimensional Riemann theta function with zero characteristic, $\Theta^{*}(\mathbf{z})$ is a theta function with an odd non-singular characteristic; see the definitions (2.6), (2.7), $\Theta^{*}=\Theta^{*}(0)=0$, and a,b,c,d are points on a Riemann surface $\mathcal{C}$ with genus g. The Abel map $\int_{a}^{b}\boldsymbol{\omega}$ between two points a and b on $\mathcal{C}$ is defined at the beginning of Section 2.2.1. Note that the name trisecant identity refers to secants on the so-called Kummer variety, see [60] for a comprehensive review.

Since Fay's identity (2.9) holds for arbitrary points a, b, c, d on the Riemann surface $\mathcal{C}$, it is possible to consider the identity in the limit that two or more points coincide[1]. This leads to identities between derivatives of theta functions making it possible to identify solutions to certain PDEs from degenerated identies. In [49] this was done for the Sine-Gordon equation and the Kadomtsev-Petviasvili (KP) equation. On special Riemann surfaces (hyperelliptic, trigonal) the latter solutions lead to algebro-geometric solutions for the Korteweg-de Vries (KdV) [49] and the Boussinesq equation [5]. In [38] previously known solutions to the Ernst

[1]Note that there are generalizations of Fay's identity to more than 4 points and degenerations thereof, see for instance [19, 15, 6] and references therein.

equation [39] were reconstructed via Fay's identity, see also [17], in [36] known solutions to the Camassa-Holm equation [23] were obtained with Mumford's approach. In [35] Kalla presented a new degenerated identity allowing to identify known solutions to the nonlinear Schrödinger [32, 57] and Davey-Stewartson equations [46] and to construct solutions to vector nonlinear Schrödinger equations in terms of theta functions. For a recent review on completely integrable dispersive PDEs, we refer to [5]. In this chapter we generalize Kalla's approach to higher order in the local parameter near the point a. We obtain with the above notation

Main theorem Part I

Let a, b be two points on a compact Riemann surface $\mathcal{C}$, lying inside of a chosen fundamental polygon. Let the derivatives D_a, D_b, D'_a, D''_a be defined as in (2.5) and let $U = \mathrm{D}_b \ln(\Theta\Theta^*_{ba})$ (D_b acts on both theta functions). Then U satisfies

$$\begin{aligned} 0 &= 2(\mathrm{D}_aU)^2\mathrm{D}''_a\mathrm{D}_aU - 2\mathrm{D}_aU\mathrm{D}''_aU\mathrm{D}^2_aU - (\mathrm{D}_aU)^2\mathrm{D}^4_aU + 4\mathrm{D}_aU\mathrm{D}^3_aU\mathrm{D}^2_aU \\ &\quad - 3(\mathrm{D}^2_aU)^3 + 3(\mathrm{D}'_aU)^2\mathrm{D}^2_aU - 3(\mathrm{D}_aU)^2(\mathrm{D}'_a)^2\mathrm{D}U\ . \end{aligned} \tag{2.2}$$

This identity has similarities to the classical identity (2.14) by Fay in the sense that it involves the derivatives D''_a, D'_a and D_a of $\Theta(\mathbf{z})$, but appears to be new. In contrast to the potential in (2.14), the function U also depends on a point b on the Riemann surface $\mathcal{C}$ which is distinct from a, but otherwise arbitrary.

We also prove

Main theorem Part II

The function $\phi(x,y,t) = \mathrm{D}_b \ln\left[\Theta^*_{ab}\Theta(x\mathbf{v}_0(a) + y\mathbf{v}_1(a) + t\mathbf{v}_2(a) + \mathbf{d})\right]$, $(x,y,t) \in \mathbb{R}^3$, solves the Schwarzian KP equation:

$$\left(\frac{\phi_t}{\phi_x} - \frac{1}{2}\{\phi;x\}\right)_x - \frac{3}{2}\left(\frac{\phi_y}{\phi_x}\right)_y - \frac{3}{4}\left(\frac{\phi_y^2}{\phi_x^2}\right)_x = 0\ , \tag{2.3}$$

where $\{\phi;x\}$ denotes the Schwarzian derivative along x; $\{\phi;x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$, where the indices denote partial derivatives with respect to the respective variable, and where $\mathbf{v}_j$, $j = 0,1,2$ has the components v_{ij}, $i = 1,\dots,g$ defined in (2.4).

The solution ϕ in terms of multi-dimensional theta functions for the Schwarzian KP equation seems to be new. The Schwarzian KP equation (2.3) appeared first in the Painlevé analysis of the KP equation in [62] as a singularity manifold equation. Its integrability was established in [8]. As in the case of the KP equation, a reduction to a Schwarzian KdV and Boussinesq equation is possible. It should be noted that these equations and their discrete analogues have gained some intention in the recent literature [13] [41].

This chapter is organized as follows: in section 2.2 we collect some basic definitions of quantities defined on a compact Riemann surface and known facts on Fay's identities. In section 2.3 we rederive identities (2.12) and (2.16) from identity (2.10) and prove the first part of the main theorem. In section 2.4 this is applied to integrable PDEs. We add some concluding remarks in section 2.5.

2.2 Preliminaries

In this section, we will collect some basic definitions and known facts on Fay's identities and applications.

2.2.1 Basic definitions

In this chapter we always consider a compact Riemann surface $\mathcal{C}$ of genus $g \in \mathbb{N}$ equipped with a canonical basis of cycles $a_1, \dots, a_g, b_1, \dots, b_g$ satisfying the intersection conditions

$$a_i \circ b_j = \delta_{ij} \ , \qquad a_i \circ a_j = 0 \ , \qquad b_i \circ b_j = 0 \ , \qquad i, j = 1, \dots, g \ .$$

The g-dimensional vector space over $\mathbb{C}$ of holomorphic 1-forms is denoted by $\Omega^1(\mathcal{C})$. An element $\boldsymbol{\omega} \in \Omega^1(\mathcal{C})$ is normalized by $\int_{a_i} \omega_j = \delta_{ij}$, $i, j = 1, \dots, g$. The matrix of b-periods $\mathbb{B}_{ij} = \int_{b_i} \omega_j$, $i, j = 1, \dots, g$, is a Riemann matrix, i.e., it is symmetric and has a positive definite imaginary part. The Abel map $\mathcal{A} : P \mapsto \int_{P_0}^{P} \boldsymbol{\omega}$ is an holomorphic map from the Riemann surface $\mathcal{C}$ into the *Jacobian* $Jac(\mathcal{C}) = \mathbb{C}^g/\Lambda$ where Λ is the lattice formed by the periods of the holomorphic 1-forms,

$$\Lambda = \{\mathbf{m} + \mathbb{B}\mathbf{n} : \mathbf{m}, \mathbf{n} \in \mathbb{Z}^g\} \ .$$

The expansion of the Abel map at a point $P \in \mathcal{C}$ near a point $a \in \mathcal{C}$ is written in the form,

$$\mathcal{A}_i(P) = \sum_{j=0}^{\infty} v_{ij} \frac{\tau_a^{j+1}}{(j+1)!} \ , \qquad i = 1, \dots, g \ , \tag{2.4}$$

where τ_a is a local parameter in the vicinity of a containing also P. We define the directional derivatives acting on a function $f(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^g$ as

$$\begin{aligned} \mathrm{D}_a &= \sum_{i=1}^{g} v_{i0} \partial_{z_i} \ , \qquad \mathrm{D}'_a = \sum_{i=1}^{g} v_{i1} \partial_{z_i} \ , \qquad \mathrm{D}''_a = \sum_{i=1}^{g} \frac{v_{i2}}{6} \partial_{z_i} \ , \\ \mathrm{D}_a^{(n)} &= \sum_{i=1}^{g} \frac{v_{in}}{(n+1)!} \partial_{z_i} \ , \qquad n \in \mathbb{N} \ . \end{aligned} \tag{2.5}$$

In other words, $\mathrm{D}_a f(\mathbf{z}) = \sum_{i=1}^{g} v_{i0}\partial_{z_i} f(\mathbf{z})$ and analogously for the other derivatives.

Multi-dimensional theta functions are the building blocks of meromorphic functions on Riemann surfaces. The theta function with characteristic $[\mathbf{p},\mathbf{q}]$ is defined as an infinite series,

$$\Theta_{\mathbf{pq}}(\mathbf{z},\mathbb{B}) = \sum_{\mathbf{N}\in\mathbb{Z}^g} \exp\left\{\mathrm{i}\pi\left\langle \mathbb{B}\left(\mathbf{N}+\mathbf{p}\right),\mathbf{N}+\mathbf{p}\right\rangle + 2\pi i\left\langle \mathbf{z}+\mathbf{q},\mathbf{N}+\mathbf{p}\right\rangle\right\}\ , \tag{2.6}$$

with $\mathbf{z}\in\mathbb{C}^g$ and $\mathbf{p}$, $\mathbf{q}\in\mathbb{R}^g$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product $\langle\mathbf{N},\mathbf{z}\rangle = \sum_{i=1}^{g} N_i z_i$. The properties of the Riemann matrix ensure that the series converges absolutely and that the theta function is an entire function on $\mathbb{C}^g$. A characteristic is called *singular* if the corresponding theta function vanishes identically. Half-integer characteristics with $2\mathbf{p},2\mathbf{q}\in\mathbb{Z}^g$ are called *even* if $4\langle\mathbf{p},\mathbf{q}\rangle = 0 \bmod 2$ and *odd* otherwise. Theta functions with odd (even) characteristic are odd (even) functions of the argument $\mathbf{z}$. The theta function with characteristic is related to the Riemann theta function Θ, the theta function with zero characteristic $\Theta = \Theta_{00}$, via

$$\Theta_{\mathbf{pq}}(\mathbf{z},\mathbb{B}) = \Theta(\mathbf{z}+\mathbb{B}\mathbf{p}+\mathbf{q})\exp\left\{\mathrm{i}\pi\left\langle \mathbb{B}\mathbf{p},\mathbf{p}\right\rangle + 2\pi\mathrm{i}\left\langle \mathbf{p},\mathbf{z}+\mathbf{q}\right\rangle\right\}\ . \tag{2.7}$$

A theta function with a chosen nonsingular half-integer characteristic $*$ is denoted by Θ^*.

2.2.2 Fay's identities

Theta functions on Jacobians satisfy Fay's celebrated trisecant identity [18]. We remind the notation $\Theta^*_{ab} = \Theta^*\left(\int_a^b \boldsymbol{\omega}\right)$ for theta functions with nonsingular half-integer characteristic, and $\Theta_{ab}(\mathbf{z}) = \Theta\left(\mathbf{z}+\int_a^b \boldsymbol{\omega}\right)$ for theta functions with zero characteristic. Fay's trisecant identity can be seen as a generalization of the classical relation between cross ratio functions for four arbitrary points a, b, c, d in the euclidean plane [56]. According to this point of view, let us define the function on $\mathcal{C}$

$$\lambda_{abcd} = \frac{\Theta^*_{ab}\Theta^*_{cd}}{\Theta^*_{ad}\Theta^*_{cb}}\ , \tag{2.8}$$

where a, b, c, d are four points on $\mathcal{C}$, all lying inside a chosen fundemental polygon. The function defined by (2.8) vanishes for $a=b$ and $c=d$ and has poles for $a=d$ and $b=c$. Note that this function is independent of the choice of nonsingular half-integer characteristic. This can be seen by rewriting λ_{abcd} in terms of prime forms (which are characteristic independent) as

$$\lambda_{abcd} = \frac{E(a,b)E(c,d)}{E(a,d)E(c,b)}\ ,$$

where the prime form $E(x,y)$ is defined by

$$E(x,y) = \frac{\Theta^*_{xy}}{\sqrt{h_*(x)}\sqrt{h_*(y)}} ,$$

with $h_*(x)$ a spinor satisfying $h_*(x) = \sum_{i=1}^{g} \frac{\partial \Theta^*}{\partial z_j} \omega_j(x)$.

Theorem 2.2.1 (Fay [18])**.** *Let a, b, c, d be four points on the Riemann surface $\mathcal{C}$, all lying inside a chosen fundamental polygon. Then with the above definitions the following identity holds*

$$\lambda_{cabd}\,\Theta_{bc}(\mathbf{z})\Theta_{ad}(\mathbf{z}) + \lambda_{cbad}\,\Theta_{ac}(\mathbf{z})\,\Theta_{bd}(\mathbf{z}) = \Theta(\mathbf{z})\;\Theta_{a+b,c+d}(\mathbf{z}) , \qquad \forall \mathbf{z} \in \mathbb{C}^g . \tag{2.9}$$

The integration paths in (2.9) have to be chosen in a way not to intersect the boundary of the fundamental polygon.

Degenerate versions of Fay's identity lead to identities for derivatives of theta functions. In the limit $d \to b$, one finds for (2.9)

Corollary 2.2.2 (Fay [18])**.** *Let a, b, c be three points on the Riemann surface $\mathcal{C}$, all lying inside a chosen fundamental polygon. Then the following identity holds*

$$\mathrm{D}_b \ln \frac{\Theta_{ac}(\mathbf{z})}{\Theta(\mathbf{z})} = p_1(a,b,c) + p_2(a,b,c)\frac{\Theta_{bc}(\mathbf{z})\Theta_{ab}(\mathbf{z})}{\Theta_{ac}(\mathbf{z})\Theta(\mathbf{z})} , \tag{2.10}$$

where

$$p_1(a,b,c) = \mathrm{D}_b \ln \frac{\Theta^*_{ab}}{\Theta^*_{cb}} , \qquad p_2(a,b,c) = \frac{\Theta^*_{ac}\mathrm{D}_b\Theta^*}{\Theta^*_{bc}\Theta^*_{ba}} . \tag{2.11}$$

In the limit $c \to a$, equation (2.10) yields

Corollary 2.2.3 (Fay [18])**.** *Let a, b be two points on the Riemann surface $\mathcal{C}$, both lying inside a chosen fundamental polygon. Then the following identity holds,*

$$\mathrm{D}_a\mathrm{D}_b \ln \Theta(\mathbf{z}) = q_1(a,b) + q_2(a,b)\frac{\Theta_{ba}(\mathbf{z})\Theta_{ab}(\mathbf{z})}{\Theta(\mathbf{z})^2} , \tag{2.12}$$

where

$$q_1(a,b) = \mathrm{D}_a\mathrm{D}_b \ln \Theta^*_{ab} , \qquad q_2(a,b) = \frac{\mathrm{D}_a\Theta^*\mathrm{D}_b\Theta^*}{(\Theta^*_{ab})^2} . \tag{2.13}$$

In the limit $b \to a$, equation (2.12) can be cast into the form

Corollary 2.2.4 (Fay [18])**.** *The following identity holds on the Riemann surface $\mathcal{C}$*

$$\mathrm{D}_a^4 \ln \Theta(\mathbf{z}) + 6(\mathrm{D}_a^2 \ln \Theta(\mathbf{z}))^2 + 3\mathrm{D}'_a\mathrm{D}'_a \ln \Theta(\mathbf{z}) - 2\mathrm{D}_a\mathrm{D}''_a \ln \Theta(\mathbf{z}) + c_1\mathrm{D}_a^2 \ln \Theta(\mathbf{z}) + c_2 = 0 , \tag{2.14}$$

where
$$c_1 = 2\frac{\mathrm{D}_a''\Theta^*}{\Theta^*} - 4\frac{\mathrm{D}_a^3\Theta^*}{\mathrm{D}_a\Theta^*} - 3\left(\frac{\mathrm{D}_a'\Theta^*}{\mathrm{D}_a\Theta^*}\right)^2 ; \tag{2.15}$$
the constant c_2 can be obtained by expanding q_1 and q_2 (2.13) in the considered limit to fourth order in the local parameter τ_a near a.

Kalla generalized the identity (2.12) to

Theorem 2.2.5 (Kalla [35])**.** *Let a, b be points on $\mathcal{C}$, then the following identity holds,*

$$\begin{aligned} 0 =& \mathrm{D}_a' \ln \frac{\Theta_{ab}(\mathbf{z})}{\Theta(\mathbf{z})} + \mathrm{D}_a^2 \ln \frac{\Theta_{ab}(\mathbf{z})}{\Theta(\mathbf{z})} + \left(\mathrm{D}_a \ln \frac{\Theta_{ab}(\mathbf{z})}{\Theta(\mathbf{z})} - K_1(a,b)\right)^2 \\ &+ 2\mathrm{D}_a^2 \ln \Theta(\mathbf{z}) + K_2(a,b) , \end{aligned} \tag{2.16}$$

where $K_1(a,b)$, $K_2(a,b)$ depend on the points a, b, but not on $\mathbf{z}$.

2.3 Proof of Main theorem Part I

In this section we will consider various identities following from Fay's identity (2.10) in the limit $c \to a$. First we identify the known relations (2.12) and (2.16), then we prove the first part of the main theorem (in this section the dependence in $\mathbf{z}$ in the arguments of the Riemann theta functions with zero characteristic is dropped in notation).
To this end we write (2.10) in the form

$$V^2 \mathrm{D}_b \frac{V(c)}{V} = \mathrm{D}_b\Theta^*\Theta^*_{ac}\Theta_{ab}\Theta_{bc} , \tag{2.17}$$

where we have put
$$V(c) = \Theta_{ac}\Theta^*_{cb} , \qquad V = V(a) = \Theta\Theta^*_{ab} . \tag{2.18}$$

2.3.1 Known degenerations of Fay's trisecant identity

For identity (2.17) we consider a Taylor expansion in the limit $c \to a$ in the local parameter τ_a in (2.4). In lowest order we get (2.12) in the form

$$\mathrm{D}_a\mathrm{D}_b \ln V = \frac{1}{V^2}\mathrm{D}_b\Theta^*\mathrm{D}_a\Theta^*\Theta_{ab}\Theta_{ba} . \tag{2.19}$$

In order τ_a^2, we get for (2.17)

$$\frac{V^2}{2}\mathrm{D}_b(\mathrm{D}_a^2 \ln V + \mathrm{D}_a' \ln V + (\mathrm{D}_a \ln V)^2) = \mathrm{D}_b\Theta^*\mathrm{D}_a\Theta^*\Theta_{ab}\Theta_{ba}\left(\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*} + \mathrm{D}_a \ln \Theta_{ba}\right) ,$$

which can be written with (2.19) in the form

$$\frac{1}{2}\mathrm{D}_b(\mathrm{D}_a^2\ln V+\mathrm{D}_a'\ln V+(\mathrm{D}_a\ln V)^2)=\mathrm{D}_a\mathrm{D}_b\ln V\left(\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*}+\mathrm{D}_a\ln\Theta_{ba}\right)\,. \tag{2.20}$$

Since this identity holds for all $\mathbf{z}\in\mathbb{C}$, it also holds for $\mathbf{z}$ replaced by $\mathbf{z}+\int_a^b$. This means $\Theta\mapsto\Theta_{ab}$, $\Theta_{ba}\mapsto\Theta$. As shown by Kalla [35], the difference between identity (2.20) and (2.20) after this shift of $\mathbf{z}$ reads

$$\begin{aligned}0=&\frac{1}{2}\mathrm{D}_b\left(\mathrm{D}_a'\ln\frac{\Theta_{ab}}{\Theta}+\frac{\mathrm{D}_a^2\Theta_{ab}}{\Theta_{ab}}-\frac{\mathrm{D}_a^2\Theta}{\Theta}+2\mathrm{D}_a\ln\Theta_{ba}^*\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\right)\\&+\mathrm{D}_a\mathrm{D}_b\ln\Theta\Theta_{ab}^*\mathrm{D}_a\ln\Theta_{ba}-\mathrm{D}_a\mathrm{D}_b\ln\Theta_{ab}\Theta_{ab}^*\mathrm{D}_a\ln\Theta-\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*}\mathrm{D}_a\mathrm{D}_b\ln\frac{\Theta_{ab}}{\Theta}\,.\end{aligned} \tag{2.21}$$

With (2.12), we get

$$\begin{aligned}0=&\frac{1}{2}\mathrm{D}_b\left(\mathrm{D}_a'\ln\frac{\Theta_{ab}}{\Theta}+\mathrm{D}_a^2\ln\frac{\Theta_{ab}}{\Theta}+\left(\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\right)^2+2\mathrm{D}_a^2\ln\Theta\Theta_{ba}^*\right)\\&+\left(\mathrm{D}_a\ln\Theta_{ba}^*-\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*}\right)\mathrm{D}_a\mathrm{D}_b\ln\frac{\Theta_{ab}}{\Theta}\,.\end{aligned} \tag{2.22}$$

Introducing the derivative $\nabla_b=\sum_{i=1}^g v_{i0}(b)\partial_{z_i}$ acting only on $\mathbf{z}$, we can write (2.22) in the form

$$\begin{aligned}0=&\nabla_b\left\{\frac{1}{2}\left(\mathrm{D}_a'\ln\frac{\Theta_{ab}}{\Theta}+\mathrm{D}_a^2\ln\frac{\Theta_{ab}}{\Theta}\right)+\left(\mathrm{D}_a\ln\Theta_{ba}^*-\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*}\right)\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\right.\\&\left.+\mathrm{D}_a^2\ln\Theta\Theta_{ab}^*+\frac{1}{2}\left(\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\right)^2\right\}\,.\end{aligned} \tag{2.23}$$

This implies that relation (2.16) holds with

$$\begin{aligned}K(a,b)=&\mathrm{D}_a'\ln\frac{\Theta_{ab}}{\Theta}+\mathrm{D}_a^2\ln\frac{\Theta_{ab}}{\Theta}+2\left(\mathrm{D}_a\ln\Theta_{ba}^*-\frac{\mathrm{D}_a'\Theta^*}{2\mathrm{D}_a\Theta^*}\right)\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\\&+2\mathrm{D}_a^2\ln\Theta\Theta_{ab}^*+\left(\mathrm{D}_a\ln\frac{\Theta_{ab}}{\Theta}\right)^2\,,\end{aligned} \tag{2.24}$$

where $K(a,b)$ just depends on a, b, but not on $\mathbf{z}$. It can be computed for instance by putting $\mathbf{z}=0$ on the right hand side of (2.24). This reproduces the proof of Theorem 2.2.5 from [35].

2.3.2 Third order terms

In third order of the local parameter τ_a we get for (2.10)

$$
\begin{aligned}
0 =& \mathrm{D}_b \left(\frac{1}{6}\mathrm{D}''_a \ln V + \frac{1}{2}\mathrm{D}_a\mathrm{D}'_a \ln V + \frac{1}{2}\mathrm{D}'_a \ln V \mathrm{D}_a \ln V + \frac{1}{6}\mathrm{D}^3_a \ln V \right. \\
&\left. + \frac{1}{2}\mathrm{D}^2_a \ln V \mathrm{D}_a \ln V + \frac{1}{6}(\mathrm{D}_a \ln V)^3 \right) \\
&- \mathrm{D}_a\mathrm{D}_b \ln V \left(\frac{\mathrm{D}''_a\Theta^*}{6\mathrm{D}_a\Theta^*} + \frac{\mathrm{D}^3_a\Theta^*}{6\mathrm{D}_a\Theta^*} + \frac{\mathrm{D}'_a\Theta^*}{2\mathrm{D}_a\Theta^*}\mathrm{D}_a \ln \Theta_{ba} + \frac{1}{2}\mathrm{D}'_a \ln \Theta_{ba} + \frac{\mathrm{D}^2_a\Theta_{ba}}{2\Theta_{ba}} \right) ,
\end{aligned}
\tag{2.25}
$$

where we have used (2.19). With (2.20), we can replace Θ_{ba} in (2.25) to obtain a relation only involving V. To this end we put

$$
F = \frac{1}{2}\mathrm{D}_b \left(\mathrm{D}'_a \ln V + \mathrm{D}^2_a \ln V + (\mathrm{D}_a \ln V)^2 \right) \tag{2.26}
$$

as well as

$$
C_1 = \frac{\mathrm{D}'_a\Theta^*}{2\mathrm{D}_a\Theta^*} \tag{2.27}
$$

which leads to (2.20) in the form

$$
\mathrm{D}_a \ln \Theta_{ba} = \frac{F}{\mathrm{D}_a\mathrm{D}_b \ln V} - C_1 \ . \tag{2.28}
$$

In addition we put

$$
\begin{aligned}
G =& \mathrm{D}_b \left(\frac{1}{6}\mathrm{D}''_a \ln V + \frac{1}{2}\mathrm{D}_a\mathrm{D}'_a \ln V + \frac{1}{2}\mathrm{D}'_a \ln V \mathrm{D}_a \ln V \right. \\
&\left. + \frac{1}{6}\mathrm{D}^3_a \ln V + \frac{1}{2}\mathrm{D}^2_a \ln V \mathrm{D}_a \ln V + \frac{1}{6}(\mathrm{D}_a \ln V)^3 \right)
\end{aligned}
\tag{2.29}
$$

and

$$
C_2 = \frac{\mathrm{D}''_a\Theta^*}{6\mathrm{D}_a\Theta^*} + \frac{\mathrm{D}^3_a\Theta^*}{6\mathrm{D}_a\Theta^*} \ , \tag{2.30}
$$

with which (2.25) takes the form

$$
G = \mathrm{D}_a\mathrm{D}_b \ln V \left(C_2 + C_1\mathrm{D}_a \ln \Theta_{ba} + \frac{1}{2}\mathrm{D}'_a \ln \Theta_{ba} + \frac{1}{2}\mathrm{D}^2_a \ln \Theta_{ba} + \frac{1}{2}(\mathrm{D}_a \ln \Theta_{ba})^2 \right) \ . \tag{2.31}
$$

Eliminating Θ_{ba} with (2.28) from (2.31) leads in a first step to

$$
\frac{G}{\mathrm{D}_a\mathrm{D}_b \ln V} - \frac{1}{2}\mathrm{D}_a \left(\frac{F}{\mathrm{D}_a\mathrm{D}_b \ln V} \right) - \frac{F^2}{2(\mathrm{D}_a\mathrm{D}_b \ln V)^2} = C_2 - \frac{C_1^2}{2} + \frac{1}{2}\mathrm{D}'_a \ln \Theta_{ba} \ . \tag{2.32}
$$

Differentiating with respect to D_a (note that the derivatives of the odd theta functions in c_1, c_2 vanish), we obtain with (2.28)

$$\mathrm{D}_a\left(\frac{G}{\mathrm{D}_a\mathrm{D}_b\ln V}-\frac{1}{2}\mathrm{D}_a\left(\frac{F}{\mathrm{D}_a\mathrm{D}_b\ln V}\right)-\frac{F^2}{2(\mathrm{D}_a\mathrm{D}_b\ln V)^2}\right)=\frac{1}{2}\mathrm{D}'_a\left(\frac{F}{\mathrm{D}_a\mathrm{D}_b\ln V}\right)\ . \tag{2.33}$$

We have with $U=\mathrm{D}_b\ln V$

$$\begin{aligned}&\frac{G}{\mathrm{D}_aD_b\ln V}-\frac{1}{2}\mathrm{D}_a\left(\frac{F}{\mathrm{D}_a\mathrm{D}_b\ln V}\right)-\frac{F^2}{2(\mathrm{D}_a\mathrm{D}_b\ln V)^2}\\&=\frac{\mathrm{D}''_aU}{6\mathrm{D}_aU}+\frac{\mathrm{D}'_a\mathrm{D}_aU}{4\mathrm{D}_aU}+\frac{1}{2}\mathrm{D}'_a\ln V-\frac{\mathrm{D}^3_aU}{12\mathrm{D}_aU}+\frac{(\mathrm{D}^2_aU)^2}{8(\mathrm{D}_aU)^2}-\frac{(\mathrm{D}'_aU)^2}{8(\mathrm{D}_aU)^2}\ ,\end{aligned} \tag{2.34}$$

thus we get for (2.33)

$$0=\frac{\mathrm{D}''_a\mathrm{D}_aU}{6\mathrm{D}_aU}-\frac{\mathrm{D}''_aU\mathrm{D}^2_aU}{(6\mathrm{D}_aU)^2}-\frac{\mathrm{D}^4_aU}{12\mathrm{D}_aU}+\frac{\mathrm{D}^3_aU\mathrm{D}^2_aU}{3(\mathrm{D}_aU)^2}-\frac{(\mathrm{D}^2_aU)^3}{4(\mathrm{D}_aU)^3}+\frac{(\mathrm{D}'_aU)^2\mathrm{D}^2_aU}{4(\mathrm{D}_aU)^3}-\frac{(\mathrm{D}'_a)^2\mathrm{D}_aU}{4\mathrm{D}_aU}\ . \tag{2.35}$$

identical to equation (2.2) which concludes the proof. Note that there are terms in (2.33) without a derivative D_a, but remarkably these terms all cancel leaving (2.34) a relation for terms all involving this derivative.

2.4 Applications to integrable PDEs

In this section we will apply relation (2.2) to integrable equations, prove the second part of the main theorem and consider various reductions on special Riemann surfaces as known for the KP case.

2.4.1 Main theorem Part II

To prove the second part of the main theorem, we define the function

$$\phi(x,y,t)=\mathrm{D}_b\ln\left[\Theta^*_{ab}\Theta(x\mathbf{v}_0(a)+y\mathbf{v}_1(a)+t\mathbf{v}_2(a)+\mathbf{d})\right]$$

and show that it solves the Schwarzian KP equation (2.3).

With our previous notations $\phi=\mathrm{D}_b\ln V=U$, moreover we can identify: $\mathrm{D}_a=\partial_x$,

$\mathrm{D}_a' = \partial_y$ and $\mathrm{D}_a'' = \partial_t$. Inserting the function ϕ into (2.3) we get:

$$\frac{\mathrm{D}_a''\mathrm{D}_a U}{\mathrm{D}_a U} - \frac{\mathrm{D}_a^2 U}{(\mathrm{D}_a U)^2}\mathrm{D}_a'' U - \frac{1}{2}\frac{\mathrm{D}_a^4 U}{\mathrm{D}_a U} + \frac{2}{(\mathrm{D}_a U)^2}\mathrm{D}_a^2 U \mathrm{D}_a^3 U$$
$$-\frac{3}{2(\mathrm{D}_a U)^3}(\mathrm{D}_a U)^3 + \frac{3}{2}\frac{\mathrm{D}_a^2 U}{(\mathrm{D}_a U)^3}(\mathrm{D}_a' U)^2 - \frac{3}{2}\frac{(\mathrm{D}_a')^2 U}{\mathrm{D}_a U} = 0 \ ,$$

which is equivalent to (2.2) thus proving this part of the theorem.
The Schwarzian KP equation can also be written in the form

$$\left(\frac{U_t}{U_x} - \frac{1}{2}\frac{U_{xxx}}{U_x} + \frac{3}{4}\frac{U_{xx}^2}{U_x^2}\right)_x + \frac{3}{2}\frac{U_{xx}}{U_x^3}U_y^2 - \frac{3}{2}\frac{U_{yy}}{U_x} = 0 \ . \tag{2.36}$$

Integrating with respect to x, we get

$$U_t - \frac{1}{2}U_{xxx} + \frac{3}{4}\frac{U_{xx}^2 - U_y^2}{U_x} + \frac{3}{2}U_x\partial_x^{-1}\left(\frac{U_{xy}U_y}{U_x^2} - \frac{U_{yy}}{U_x}\right) \ , \tag{2.37}$$

which can be written in the form

$$U_t - \frac{1}{2}U_{xxx} + \frac{3}{4}\frac{U_{xx}^2 - U_y^2}{U_x} - \frac{3}{2}U_x W_y \ , \tag{2.38}$$

where $W_x = U_y/U_x$. This is equation (13) in [8] after the change of time $t \mapsto 2t$.

2.4.2 Reductions on special Riemann surfaces

Let us restrict our attention to the special case of the Riemann surface $\mathcal{C}$ being hyperelliptic, i.e., given by the zero locus of the polynomial $P(\lambda,\mu) = \mu^2 - \prod_{j=1}^{N}(\lambda - \lambda_j)$ where $\lambda_j \in \mathbb{C}$, $j = 1,\ldots,N$, and $N = 2g+1$ or $N = 2g+2$, and denote by $\pi : \mathcal{C} \longrightarrow \mathbb{CP}^1$ the projection onto the Riemann sphere.
If a is a branch point of π then the hyperelliptic involution locally reads $\sigma : \tau_a \mapsto -\tau_a$. The pullback on $\Omega^1(\mathcal{C})$ of the hyperelliptic involution is given locally by: $\sigma^*\omega_j(a) = -\omega_j(a)$. Hence the Taylor expansion (2.4) of $\mathcal{A}_j$ around a must be even in τ_a, i.e $\mathbf{v}_1 = 0$ and thus $\mathrm{D}_{\mathrm{a}}' = 0$. Eliminating Θ_{ba} from (2.31) via (2.26), we get

Corollary 2.4.1. *Identity (2.2) reduces on hyperelliptic surfaces with a being a branch point to*

$$0 = \frac{1}{6}\mathrm{D}_a''\mathrm{D}_b \ln V - \frac{1}{12}\mathrm{D}_a^3\mathrm{D}_b \ln V - C_2\mathrm{D}_a\mathrm{D}_b \ln V + \frac{1}{8}\frac{(\mathrm{D}_a^2\mathrm{D}_b \ln V)^2}{\mathrm{D}_a\mathrm{D}_b \ln V} \ . \tag{2.39}$$

If we put again $U = \mathrm{D}_b \ln V$, $\mathrm{D}''_a = \partial_t$ and $\mathrm{D}_a = \partial_x$, we get for (2.39)

$$\frac{1}{6}U_t - \frac{1}{12}U_{xxx} - C_2 U_x + \frac{1}{8}\frac{U_{xx}^2}{U_x} = 0 \; . \tag{2.40}$$

This implies

Theorem 2.4.2. *The function* $\phi(x,t) = \mathrm{D}_b \ln \left[\Theta^*_{ab}\Theta(x\mathbf{v}_0(a) + t\mathbf{v}_2(a) + \mathbf{d})\right]$ *solves the Schwarzian KdV equation:*

$$\frac{1}{6}\frac{\phi_t}{\phi_x} - \frac{1}{12}\{\phi; x\} = C_2 \; . \tag{2.41}$$

As in Chapter 3.4 of [5] for KP, there is also a reduction to a Schwarzian Boussinesq equation. If the surface $\mathcal{C}$ is given by a trigonal curve, i.e., a curve on which a meromorphic function with a third order pole at a point $a \in \mathcal{C}$ and no other singularities exists. A simple example of such a curve is

$$\mu^4 = \prod_{i=1}^{4}(\lambda - E_i) \; . \tag{2.42}$$

In this case $\mathrm{D}''_a = 0$, and one can simplify (2.2).

Corollary 2.4.3. *Let* $\mathcal{C}$ *be a trigonal curve, with the point* a *being the single pole of third order of a meromorphic function. On such a surface, identity (2.2) reduces to*

$$0 = -\frac{\mathrm{D}_a^4 U}{12\mathrm{D}_a U} + \frac{\mathrm{D}_a^3 U \mathrm{D}_a^2 U}{3(\mathrm{D}_a U)^2} - \frac{(\mathrm{D}_a^2 U)^3}{4(\mathrm{D}_a U)^3} + \frac{(\mathrm{D}'_a U)^2 \mathrm{D}_a^2 U}{4(\mathrm{D}_a U)^3} - \frac{(\mathrm{D}'_a)^2 DU}{4\mathrm{D}_a U} \; . \tag{2.43}$$

In this setting,

Theorem 2.4.4. *The function* $\phi(x,t) = \mathrm{D}_b \ln \left[\Theta^*_{ab}\Theta(x\mathbf{v}_0(a) + t\mathbf{v}_1(a) + \mathbf{d})\right]$ *gives a solution to the Schwarzian Boussinesq equation [62]*

$$\Big(-\frac{1}{2}\{\phi; x\}\Big)_x - \frac{3}{2}\Big(\frac{\phi_t}{\phi_x}\Big)_t - \frac{3}{4}\Big(\frac{\phi_t^2}{\phi_x^2}\Big)_x = 0 \; . \tag{2.44}$$

2.5 Conclusion

In this chapter, we have studied degenerations of Fay's identities in higher order of the local parameter τ_a near the point a. The starting point was Fay's identity for 3 points on a Riemann surface in the form (2.17). The case of order τ_a^3 was studied in detail, leading to identity (2.2). As an application, we used this identity to find solutions under the form (2.4.1) to the Schwarzian KP equation (2.3). Reductions to special Riemann surfaces (hyperelliptic and trigonal curves) allowed us to further simplify (2.2), leading to solutions of the Schwarzian KdV and Schwarzian Boussinesq equations.

It would be natural to generalize this approach to higher orders of the parameter τ_a. A standard Taylor expansion of the quantity $V(c) = \Theta_{ac}\Theta^*_{cb}$ yields for the left hand side of equation (2.17)

$$\mathrm{D}_b \sum_{m=1}^{\infty} \frac{1}{m!V}\left(\tau \mathrm{D}_a + \frac{\tau^2}{2}\mathrm{D}'_a + \frac{\tau^3}{6}\mathrm{D}''_a + \ldots \frac{\tau^k}{k!}\mathrm{D}^{(k)}_a + \ldots\right)^m V. \tag{2.45}$$

Thus one gets in order τ^n

$$\mathrm{D}_b\left(\frac{\mathrm{D}^n_a V}{n!V} + \frac{n\mathrm{D}^{n-1}_a \mathrm{D}'_a V}{2(n-1)!V} + \ldots + \frac{\mathrm{D}^{(n)}_a V}{n!V}\right). \tag{2.46}$$

The same expansion can be obtained on the right hand side of (2.17) for Θ^*_{ac}, where all even order derivatives vanish for symmetry reasons, and for Θ_{bc} leading to derivatives of Θ_{ba}. As in section 3, the latter terms can be replaced via (2.28) by derivatives of Θ. As in (2.33), it will be necessary to differentiate with respect to D_a in general to eliminate all terms with Θ_{ba}. It is beyond the scope of this chapter to detail the resulting identities and to establish a potential relation to integrable PDEs and whether these are from a hierarchy of Schwarzian KP equations. An interesting question is also whether a similar approach can be applied to the degeneration of identity (2.12) in the limit $b \to a$ in higher orders of the local parameter near a, which would lead to a generalization of relation (2.14). This will be the subject of future research.

Another interesting aspect would be to relate the present work to the bilinear approach studied in [9], [42] and [52]. In the first two articles the authors describe the tau functions associated to the Schwarzian KP hierarchy. For the KP hierarchy let us recall that the tau function is a solution of Hirota's bilinear-bilocal equation [42]:

$$\int d\lambda \tau(\mathbf{s} - [\lambda])\tau(\mathbf{s}' + [\lambda]) \exp\left[\zeta(\mathbf{s} - \mathbf{s}', \lambda)\right] = 0\ . \tag{2.47}$$

Where the integration path is taken along a contour about infinity, $\mathbf{s} = (t_1, t_2, t_3 ...)$ is an infinite vector containing the time flows (with the convention $x = t_1, y = t_2, t = t_3$), $\mathbf{s} + [\lambda] = (t_1 + [\lambda]_1, t_2 + [\lambda]_2, ...)$, with $[\lambda]_i = \frac{1}{i}\lambda^{-i}$ and $\zeta(\mathbf{s}, \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n$.

Defining the polynomials p_n such that $\exp\left[\zeta(\mathbf{s}, \lambda)\right] = \sum_{n=0}^{\infty} p_n(\mathbf{s})\lambda^n$, and using Hirota's bilinear symbols:

$$\mathrm{D}^m_x(f.g) = (\partial_x - \partial_{x'})^m f(x, y, t, t_4, ...)g(x', y, t, t_4, ...)|_{x=x'}\ ,$$

one can show that equation (2.47) contains the following system of equations which generates the full KP hierarchy

$$\frac{1}{2}\mathrm{D}_1\mathrm{D}_n\tau.\tau = p_{n+1}(\tilde{\mathrm{D}})\tau.\tau\ . \tag{2.48}$$

With $\tilde{\mathrm{D}} = (\mathrm{D}_1, \frac{1}{2}\mathrm{D}_2, \frac{1}{3}\mathrm{D}_3, ...)$. It can be proven ([9], [59]) that this tau function satisfies the following addition formula for a, b, c and d some arbitrary complex numbers :

$$\begin{aligned}&(a-c)(d-b)\tau(\mathbf{x}+[a]+[c])\tau(\mathbf{x}+[d]+[b])+\\&+(d-a)(b-c)\tau(\mathbf{x}+[d]+[a])\tau(\mathbf{x}+[b]+[c])+\\&+(b-a)(d-c)\tau(\mathbf{x}+[b]+[a])\tau(\mathbf{x}+[d]+[c])=0\ .\end{aligned}$$

This addition formula is nothing more than Fay's trisecant identity when a tau function is written in terms of Riemann theta functions ([59]).

Let ϕ be a solution of the Schwarzian KP equation, then $\phi\tau$ is a tau function for the Schwarzian KP hierarchy, which means that

$$\frac{1}{2}\mathrm{D}_1\mathrm{D}_n\phi\tau.\phi\tau = p_{n+1}(\tilde{\mathrm{D}})\phi\tau.\phi\tau$$

generates the full Schwarzian KP hierarchy [42]. This observation, together with the well-known expression of the KP tau function in terms of Riemann theta functions motivates the following conjecture.

Conjecture 2.5.1. *There exists a quadratic form $\mathbf{Q}(\mathbf{t}) = \sum_{i,j} Q_{ij}t_it_j$, $Q_{ij} \in \mathbb{C}$, such that the function*

$$\tilde{\tau} = e^{\mathbf{Q}(\mathbf{t})}\Theta(\mathbb{V}\mathbf{t}+\mathbf{d})\mathrm{D}_b \ln\left[\Theta^*_{ab}\Theta(\mathbb{V}\mathbf{t}+\mathbf{d})\right]$$

is a tau function for the Schwarzian KP hierarchy with $\mathbb{V}$ a $g\times\infty$ matrix whose columns are the vectors associated to the different time flows.

The precise relationship between the higher order degenerations discussed above and such a tau function has to be explored in a future work.

2.6 Appendix: Links with other integrable systems

We collect in this appendix some facts on the Schwarzian PDEs appearing naturally in the context of higher order degenerations of Fay's identity. The Schwarzian KP, KdV and Boussinesq equations originally appear in a series of papers by Weiss [62], [63], in an extensive study of the Painlevé property of many integrable PDEs. These three Schwarzian equations were shown to be, in some sense, prototypical PDEs satisfying this property. They are also linked to some integrable systems that appear in physics through Bäcklund and Miura transforms. Here, we present these relationships.

First, observe that SKP, SKdV and Schwarzian Boussinesq equations are invariant under Möbius transformations. Namely, if a function ϕ solves the Schwarzian KP, Boussinesq or KdV equations then

$$\psi = \frac{A\phi + B}{C\phi + D}$$

with $AD - BC \neq 0$, A, B, C and D being some constants, is also a solution of these equations. Moreover, this transformation plays the role of an auto-Bäcklund transform for these three equations.

If $\phi(x, y, t)$ solves the Schwarzian KP equation

$$\left(\frac{\phi_t}{\phi_x} + \{\phi; x\}\right)_x + \left(\frac{\phi_y}{\phi_x}\right)_y + \frac{1}{2}\left(\frac{\phi_y^2}{\phi_x^2}\right)_x = 0 \ ,$$

then the Bäcklund transformed function

$$u(x, y, t) = 12\frac{\partial^2}{\partial x^2}\ln\phi + v$$

with

$$v = 3\frac{\phi_{xx}^2}{\phi_x^2} - 4\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_t}{\phi_x} - \frac{\phi_y^2}{\phi_x^2}$$

is a solution of the KP equation

$$u_{tx} + u_x^2 + uu_{xx} + u_{xxxx} + u_{yy} = 0 \ .$$

Similarly, the Schwarzian KdV equation is related to the classical KdV equation through Bäcklund transformations. Let $\phi(x, t)$ be a solution of

$$\frac{\phi_t}{\phi_x} - \{\phi; x\} = \lambda \ ,$$

then

$$u = 12\frac{\partial^2}{\partial x^2}\ln\phi + v$$

is a solution of

$$u_t + u_{xxx} + uu_x = 0 \ ,$$

where v is given by

$$v = 3\frac{\phi_{xx}^2}{\phi_x^2} - 4\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_t}{\phi_x} \ .$$

The very same Bäcklund transformations also relate solutions of the Schwarzian Boussinesq equation to solutions of the classical Boussinesq equation. Moreover, if we set $\lambda = 0$ then

$$w = 2\frac{\phi_x}{\phi} + \frac{\phi_{xx}}{\phi_x}$$

solves the modified KdV equation

$$w_t + \frac{\partial}{\partial x}\big(w_{xx} - \frac{w^3}{2}\big) = 0 \ .$$

Finally, let us consider the following Miura transform

$$x \to \phi \ , \qquad t \to t \ , \qquad \phi \to x \ ,$$

which implies

$$\{\phi; x\} = -\phi_x^2 \{x; \phi\} \ , \qquad \phi_x = \frac{1}{x_\phi} \ , \qquad x_t = -\frac{\phi_t}{\phi_x} \ .$$

Under this transformation, the Schwarzian KdV equation is rewritten as follows

$$x_\phi^2 x_t = \lambda x_\phi^2 + \{x; \phi\} \ ,$$

replacing the Schwarzian derivative $\{x; \phi\}$ by its explicit expression, we get

$$x_t = \lambda - \frac{1}{2}\Big(\frac{1}{x_\phi}\Big)_{\phi\phi} + \frac{3}{2}\Big(\frac{1}{x_\phi}\Big)^2_\phi \ .$$

Setting $v = x_\phi^{-1}$, the previous equation is equivalent to

$$v_t = v^3 v_{\phi\phi\phi}$$

i.e. the Harry-Dym equation.

Chapter 3

New systems of log-canonical coordinates on $SL(2,\mathbb{C})$ character varieties of compact Riemann surfaces

Joint work with M. Bertola and D. Korotkin

Abstract

We construct new sets of log-canonical coordinates on $SL(2,\mathbb{C})$ character varieties of compact Riemann surfaces. These coordinates are obtained by combining shear type coordinates with length-twist type coordinates. On the real component corresponding to the moduli space of compact Riemann surfaces $\mathcal{M}_g$, the generating function corresponding to the symplectomorphism between these new coordinates and Fenchel-Nielsen coordinates is explicitly computed.

3.1 Introduction

The $SL(2,\mathbb{C})$ character variety $\mathcal{V}_g$ of compact Riemann surfaces of genus g is equipped with the natural complex symplectic form, inverse to the Goldman Poisson bracket on $\mathcal{V}_g$ [26]. There exists a set of Darboux coordinates on $\mathcal{V}_g$ which is obtained by analytic continuation to the complex domain of the Fenchel-Nielsen coordinates on the Teichmüller space $\mathcal{T}_g$ [66].

Once one introduces n boundary components on the Riemann surface and considers the corresponding character variety, there exists a set of coordinates which are extensions of Thurston's shear coordinates and are log-canonical for the Goldman form on the symplectic leaves (the coordinates are called log-canonical if the coefficients of the 2-form written in terms of those coordinates are constant, the actual Darboux coordinates can then be obtained as linear combinations of those coordinates).

However, so far it remains unclear how to use the (complexified) shear coordinates efficiently on $\mathcal{V}_g$, although important steps in this direction have been made in [11]. In the case of $\mathcal{V}_2$, a set of log-canonical coordinates alternative to Fenchel-Nielsen were recently constructed in [12]; however, generalizing this construction to higher genus was still an open question.

In this chapter, we construct new systems of log-canonical coordinates on $\mathcal{V}_g$ for any g by combining the ideas behind the (complexified) shear coordinates and the twist-length coordinates. The technical framework is in [7], where for each graph on a Riemann surface and an assignment of $SL(2,\mathbb{C})$ matrices on edges, one associates a canonical two-form.

The main result is that we associate a new system of (local) log-canonical coordinates on $\mathcal{V}_g$ to any system of simple non-intersecting closed contours $\{\gamma_1,\dots,\gamma_m\}$, for any $m = 1,\dots,3g-3$. In the case of maximal $m = 3g-3$, the set of contours define a "trinion decomposition" of the Riemann surface into three-holed spheres (the *trinions* or *pairs of pants*). In this case, our coordinates are closely related (but not exactly coinciding) with the complexified Fenchel-Nielsen length-twist coordinates; their relationship is explained in section 3.5.

We start by presenting this construction for the simplest case $m = 1$, which corresponds to a single contour γ that separates $\mathcal{C}$ into two Riemann surfaces, $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ of genera $\widetilde{g}$ and $\widehat{g}$, respectively. The associated system of log-canonical coordinates on $\mathcal{V}_g$ consists of a pair of coordinates of length-twist type along γ, and also complex shear type coordinates on $SL(2,\mathbb{C})$ character varieties of Riemann surfaces $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ with the boundaries collapsed to a point.

More precisely, given a representation ρ in $\mathcal{V}_g$, let λ_γ and λ_γ^{-1} be the eigenvalues of the monodromy matrix $\rho(\gamma) = M_\gamma$, and denote $\ell_\gamma = \ln \lambda_\gamma$, with some choice of branch for the complex logarithm. Let us choose a point $\widetilde{v} \in \widetilde{\mathcal{C}}$ in a neighborhood of γ on the $\widetilde{\mathcal{C}}$ side, and similarly another point $\widehat{v} \in \widehat{\mathcal{C}}$ that is also close to γ but lying on the $\widehat{\mathcal{C}}$ side. Consider a fatgraph $\widetilde{\Sigma}$ on $\widetilde{\mathcal{C}}$ with only one vertex at $\widetilde{v}$, and a fatgraph $\widehat{\Sigma}$ on $\widehat{\mathcal{C}}$ with a single vertex at $\widehat{v}$. These graphs are chosen to triangulate $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ if their boundaries were collapsed to a point. The number of edges of $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ equals $\widetilde{n} = 6\widetilde{g} - 3$ and $\widehat{n} = 6\widehat{g} - 3$, respectively. The valences of the vertices $\widetilde{v}$ and $\widehat{v}$ are equal to $2\widetilde{n}$ and $2\widehat{n}$, respectively, since every edge of $\widetilde{\Sigma}$ (resp. $\widehat{\Sigma}$) comes to $\widetilde{v}$ (resp. $\widehat{v}$) twice.
To each edge, $\widetilde{e}_i$, of the graph $\widetilde{\Sigma}$ we assign a coordinate $\widetilde{\zeta}_{e_i} \in \mathbb{C}$ and $\widetilde{z}_{e_i} = \mathrm{e}^{\widetilde{\zeta}_{e_i}}$, which are complex coordinates of shear type parametrizing the character variety $\mathcal{V}_{\widetilde{g},1}$ of $\widetilde{\mathcal{C}}$. The complex "length" ℓ_γ of the separating loop γ is related to these coordinates by

$$2\sum_{i=1}^{\widetilde{n}} \widetilde{\zeta}_{e_i} = \ell_\gamma \ . \tag{3.1}$$

We shall continue calling $\widetilde{\zeta}_{e_i}$ shear or shear type coordinates in this chapter although they really coincide with logarithms of Thurston's shear coordinates only when the length of the boundary curve is zero. Similar coordinates $\widehat{\zeta}_{e_j}$'s and $\widehat{z}_{e_j}$'s are assigned to the edges $\widehat{e}_j$ of $\widehat{\Sigma}$ and they satisfy the similar relation

$$2\sum_{j=1}^{\widehat{n}} \widehat{\zeta}_{e_j} = \ell_\gamma \ . \tag{3.2}$$

Using (3.1), (3.2) we can eliminate $\widehat{\zeta}_1$ and $\widetilde{\zeta}_1$ so that the full set of independent complex local coordinates on $\mathcal{V}_g$ looks as follows:

$$\left\{ \{\widetilde{\zeta}_{e_i}\}_{i=2}^{\widetilde{n}},\ \{\widehat{\zeta}_{e_j}\}_{j=2}^{\widehat{n}}, \ell_\gamma, \beta_\gamma \right\} \ . \tag{3.3}$$

The total number of variables in the set (3.3) is given by $\widetilde{n} + \widehat{n} = 6g - 6$ which coincides with the dimension of $\mathcal{V}_g$. The coordinate β_γ is the complex toric variable along contour γ (similar but not coinciding with the complexification of the Fenchel-Nielsen twist). Then, by Theorem 3.7.1, the form Ω on $\mathcal{V}_g$ can be written as follows in terms of coordinates (3.3):

$$\Omega = \Omega_0 + \widetilde{\Omega}_1 + \widehat{\Omega}_1 + \widetilde{\Omega}_2 + \widehat{\Omega}_2 \ , \tag{3.4}$$

where

$$\Omega_0 = \mathrm{d}\beta_\gamma \wedge \mathrm{d}\ell_\gamma \ ,$$

$$\widetilde{\Omega}_1 = \sum_{\substack{i,j=2\\ i<j}}^{6\widetilde{g}-3} (\widetilde{c}_{ij} - \widetilde{c}_{1j} + \widetilde{c}_{1i})\, \mathrm{d}\widetilde{\zeta}_{e_i} \wedge \mathrm{d}\widetilde{\zeta}_{e_j} \ , \qquad \widehat{\Omega}_1 = \sum_{\substack{k,l=2\\ k<l}}^{6\widehat{g}-3} (\widehat{c}_{kl} - \widehat{c}_{1l} + \widehat{c}_{1k})\, \mathrm{d}\widehat{\zeta}_{e_k} \wedge \mathrm{d}\widehat{\zeta}_{e_l} \ ,$$

$$\widetilde{\Omega}_2 = \frac{1}{2}\sum_{m=2}^{6\widetilde{g}-3} \widetilde{c}_{1m} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widetilde{\zeta}_{e_m} \ , \qquad \widehat{\Omega}_2 = \frac{1}{2}\sum_{n=2}^{6\widehat{g}-3} \widehat{c}_{1n} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widehat{\zeta}_{e_n} \ .$$

The integer coefficients $\widetilde{c}_{ij}$, $\widehat{c}_{kl}$ lie in the interval between -4 and 4. They are determined by the order of the edges of the one-valent graphs $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ at their vertex.

This construction extends to an arbitrary system of simple non-intersecting contours $\gamma_1, \ldots, \gamma_m$, $1 \le m \le 3g-3$, which split $\mathcal{C}$ into n subsurfaces $\mathcal{C}^{(i)}$ with $k^{(i)}$ boundary components. Then we have m pairs of log-canonical coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^m$. Each subsurface $\mathcal{C}^{(i)}$ carries a triangulation graph $\Sigma^{(i)}$ with $6g^{(i)} - 6 + 3k^{(i)}$ edges, to which we assign shear coordinates ζ_e. In these coordinates, by Theorem 3.8.1, one can write the Goldman form on $\mathcal{V}_g$ as

$$\Omega = \sum_{i=1}^{n} \Omega^{(i)} + \sum_{j=1}^{m} \mathrm{d}\beta_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ , \tag{3.5}$$

where

$$\Omega^{(i)} = \sum_{v\in V(\Sigma^{(i)})} \sum_{\substack{e,\widetilde{e}\perp v\\ \tilde{e}<\widetilde{e}}} \mathrm{d}\zeta_e \wedge \mathrm{d}\zeta_{\widetilde{e}} \ .$$

When $m = 3g-3$, the surface $\mathcal{C}$ decomposes into $2g-2$ trinions. Since each $\mathcal{C}^{(i)}$ is a trinion, all shear type coordinates can be expressed in terms of the boundary lengths of $\mathcal{C}^{(i)}$, due to the constraints (3.1). This yields $3g-3$ log-canonical coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$ for the Goldman form Ω on $\mathcal{V}_g$.

Using the trinion graph Γ_{trin}, whose vertices correspond to trinions and edges to the gluings of their boundary components, one may rewrite Ω as (cf. Theorem 3.9.2)

$$\Omega = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e, \tilde{e} \perp v \\ e < \tilde{e}}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\tilde{e}} + \sum_{j=1}^{3g-3} \mathrm{d}\beta_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ . \tag{3.6}$$

For each vertex, the inner sum runs over pairs of incident edges, ordered according to the inherited orientation of the ciliated graph. This expression is independent of the choice of cilium.

When restricted to the real Fuchsian component which projects to the Teichmuller space $\mathcal{T}_g$, the form Ω coincides with the Weil-Petersson form ω_{WP}. In terms of Fenchel-Nielsen coordinates $\{\tau_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$, the form ω_{WP} can be written [66]

$$\omega_{WP} = \sum_{j=1}^{3g-3} \mathrm{d}\tau_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ . \tag{3.7}$$

The length coordinates ℓ_{γ_j} are the same in (3.7) and (3.6) while each Fenchel-Nielsen twist coordinate τ_{γ_j} differs from β_{γ_j} by adding a function that depends only on the lengths. The generating function of the symplectomorphism between log-canonical coordinates $\{\beta_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$ and Fenchel-Nielsen coordinates $\{\tau_{\gamma_j}, \ell_{\gamma_j}\}_{j=1}^{3g-3}$ can be expressed in terms of Euler dilogarithms.

This construction admits a straightforward generalization to the case of an arbitrary $SL(N, \mathbb{C})$ character variety of $\pi_1(\mathcal{C})$. Here, one needs to use the Fock-Goncharov coordinates associated to each fatgraph $\Gamma^{(i)}$ together with N eigenvalues and toric variables associated to each M_{γ_j}.

Some explicit computations of the Goldman symplectic form Ω for low genera are given in an appendix.

3.2 Main spaces

3.2.1 Character variety of a compact Riemann surface

Let $\mathcal{C}$ be an oriented compact Riemann surface of genus g. Then the standard generators $\{\alpha_k, \beta_k\}_{k=1}^{g}$ of the fundamental group $\pi_1(\mathcal{C})$ can be chosen such that

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = id \ , \tag{3.8}$$

where

$$[\alpha_i, \beta_i] = \alpha_i \beta_i^{-1} \alpha_i^{-1} \beta_i \ .$$

A point of the $SL(2,\mathbb{C})$ character variety $\mathcal{V}_g$ is represented by the equivalence class under conjugation of the set of monodromy matrices $\{M_{\alpha_1}, M_{\beta_1} ..., M_{\alpha_g}, M_{\beta_g}\} \subset SL(2,\mathbb{C})$ satisfying

$$\prod_{i=1}^{g} [M_{\alpha_i}, M_{\beta_i}] = \mathbb{I}_2 \ . \tag{3.9}$$

Traces of monodromy matrices can be used as coordinates on $\mathcal{V}_g$.
The Goldman Poisson bracket on $\mathcal{V}_g$ (the complexification of the Weil-Petersson bracket on $\mathcal{M}_g$) can be written as follows in terms of trace coordinates

$$\{\mathrm{tr}\, M_\gamma, \mathrm{tr}\, M_{\widetilde{\gamma}}\} = \sum_{p \in \gamma \cap \widetilde{\gamma}} \nu(p) \left(\mathrm{tr}\, M_{\gamma \circ_p \widetilde{\gamma}} - \frac{1}{2} \mathrm{tr}\, M_\gamma \, \mathrm{tr}\, M_{\widetilde{\gamma}} \right) \ , \tag{3.10}$$

where $\gamma, \widetilde{\gamma} \in \pi_1(\mathcal{C})$ and $\nu(p)$ is the contribution of the point p to the intersection index of γ and $\widetilde{\gamma}$.
The Goldman bracket on $\mathcal{V}_g$ is non-degenerate. The symplectic form inverting the Goldman's bracket on an arbitrary $SL(N,\mathbb{C})$ character variety of a compact Riemann surface was found in [3], but the corresponding log-canonical coordinates for them are unknown.
Below, we show how to construct systems of Darboux coordinates on $\mathcal{V}_g$ which can be naturally extended to higher $SL(N,\mathbb{C})$ character varieties using the machinery of Fock-Goncharov coordinates.

Let now $\mathcal{C}$ be a closed and oriented Riemann surface of genus g with $n \geq 1$ marked points $t_1, \ldots, t_n$. Then the standard generators $\{\gamma_j\}_{j=1}^n$, $\{\alpha_k, \beta_k\}_{k=1}^g$ of the fundamental group $\pi_1(\mathcal{C} \setminus \{t_j\})$ can be chosen such that:

$$\gamma_1 ... \gamma_n \prod_{i=1}^{g} [\alpha_i, \beta_i] = id \ .$$

Consider now a representation of $\pi_1(\mathcal{C} \setminus \{t_j\})$ in $SL(2,\mathbb{C})$. Then monodromy matrices corresponding to the standard generators of the fundamental group satisfy the same relation:

$$M_{\gamma_1} ... M_{\gamma_n} \prod_{i=1}^{g} [M_{\alpha_i}, M_{\beta_i}] = \mathbb{I}_2 \ . \tag{3.11}$$

We denote the eigenvalues of M_{γ_j} by λ_{γ_j} and $1/\lambda_{\gamma_j}$.
Define the character variety $\mathcal{V}_{g,n}$ as the space of monodromy representations modulo simulta-

neous conjugation by $SL(2,\mathbb{C})$. The coordinates on $\mathcal{V}_{g,n}$ can be again chosen to be the trace coordinates, i.e. $\mathrm{tr} M_\gamma$, for sufficiently many elements γ of the fundamental group.
The Goldman bracket on $\mathcal{V}_{g,n}$ is defined by the same formula (3.10). However, on $\mathcal{V}_{g,n}$ it is degenerate, with the Casimir elements being the eigenvalues of M_{γ_j}.

3.2.2 Extended character varieties

The character varieties $\mathcal{V}_{g,n}$ admit natural extensions $\mathcal{H}_{g,n}$ that possess nondegenerate Poisson structures. The idea of such an extension is to introduce canonical partners to the Casimirs $\ell_{\gamma_1}, ..., \ell_{\gamma_n}$ of the Goldman bracket on $\mathcal{V}_{g,n}$. Suppose that all monodromies M_{γ_k} are diagonalizable with

$$M_{\gamma_k} = C_k \Lambda_k C_k^{-1} \ .$$

Here we do not fix the order of the eigenvalues and eigenvectors of M_{γ_k}. The Casimirs ℓ_{γ_k} are defined by $\ell_{\gamma_k} = \pm \ln \lambda_{\gamma_k}$, where ln denotes a branch of the complex logarithm (on the real slice we will assume $\lambda_{\gamma_k} > 1$ and analytic continuation can be obtained by choosing the principal value).

Define the extended space $\mathcal{H}_{g,n}$ as follows

$$\mathcal{H}_{g,n} = \left\{ \{M_{\alpha_j}, M_{\beta_j}\}_{j=1}^g, \{C_k, \Lambda_k\}_{k=1}^n : \quad C_1\Lambda_1 C_1^{-1} \dots C_N \Lambda_N C_N^{-1} \prod_{i=1}^g [M_{\alpha_i}, M_{\beta_i}] = \mathbb{I}_2 \right\} / \sim \ . \tag{3.12}$$

The sets related by transformation $M_{\gamma_k} \to G M_{\gamma_k} G^{-1}$, $C_k \to G C_k$ for a fixed $G \in SL(2,\mathbb{C})$ (while Λ_k remain invariant) are assumed to be equivalent.

The space $\mathcal{H}_{g,n}$ for an arbitrary matrix dimension was studied in [33], [10] and [7]. It possesses a natural symplectic form [10, 7]. The canonical partners of the Casimirs are the coordinates β_{γ_k}, which are the logarithms of the so-called toric variables b_{γ_k}. The toric variables are the normalizing factors of the eigenvectors of M_{γ_k} encoded in the matrices C_k. In [7] this symplectic form was represented in a log-canonical form for an arbitrary $SL(N,\mathbb{C})$ group by combining Fock-Goncharov coordinates with the toric variables.

3.3 Graphs on surfaces and canonical symplectic form

Let Γ be a finite ciliated oriented graph embedded into a Riemann surface $\mathcal{C}$, with $V(\Gamma)$ and $E(\Gamma)$ denoting the sets of its vertices and edges respectively. The chosen cilium at each vertex induces the ordering of edges attached to this vertex. One writes $e < e'$ when the edge e precedes the edge e' in a counterclockwise order from the cilium.

We will use the following notation: $e = [v_1, v_2]$ denotes the oriented edge connecting v_1 to v_2 and $-e = [v_2, v_1]$ denotes the oriented edge connecting v_2 to v_1. If an edge e is incident to a vertex v we write $e \perp v$. We define the valency n_v of a given vertex v to be the total number of edges incident to v counted with multiplicity.

Definition 3.3.1. A pair (Γ, J) consisting of an oriented graph on $\mathcal{C}$ up to isotopy and a map $J : E(\Gamma) \to SL(2, \mathbb{C})$ is canonical if it satisfied the following conditions:

1. The map $J : E(\Gamma) \to SL(2, \mathbb{C})$ satisfies $J(-e) = J(e)^{-1}$. The matrix $J(e)$ is called the jump matrix associated to the edge e.

2. For each vertex $v \in V(\Gamma)$ of valence $n_v \geq 2$ with incident *incoming* edges $\{e_1, e_2, ..., e_{n_v}\}$ enumerated counterclockwise starting from an arbitrary chosen edge the local monodromy around v is trivial, i.e.

$$J(e_1)J(e_2)...J(e_{n_v}) = \mathbb{I}_2 \ . \tag{3.13}$$

To each pair (Γ, J) we associate the *canonical* two-form $\Omega(\Gamma)$ as in [7]:

$$\Omega(\Gamma) = \frac{1}{2} \operatorname{tr} \sum_{v \in V(\Gamma)} \sum_{l=1}^{n_v - 1} \left(J^{(v)}_{[1...l]}\right)^{-1} \mathrm{d}J^{(v)}_{[1...l]} \wedge \left(J^{(v)}_{l}\right)^{-1} \mathrm{d}J^{(v)}_{l} \ , \tag{3.14}$$

where $J^{(v)}_l = J(e_l)$, $e_l \perp v$, are the matrices defined by the map in Definition 3.3.1 corresponding to the edges $e_1, e_2, ..., e_{n_v}$ incident to v, and

$$J^{(v)}_{[1...l]} = J^{(v)}_1 J^{(v)}_2 ... J^{(v)}_l \ .$$

The form $\Omega(\Gamma)$ is invariant under the set of the following admissible moves [7]:

1. merging of two vertices v_1 and v_2 of valency $n_{v_1} + 1$ and $n_{v_2} + 1$ respectively, connected by a common edge e_0, to get a single vertex v of valency $n_{v_1} + n_{v_2}$,

2. zipping of two edges e_1 and e_2 with jump matrices J_1 and J_2 respectively, connecting the same two vertices v_1 and v_2, to get a single edge e with jump matrix $J_1 J_2$.

3.4 Standard graph and symplectic form on $\mathcal{V}_g$

Let $\mathcal{C}$ be a compact unpunctured Riemann surface of genus g. Choose a base point v_0 and a set of standard generators $\{\alpha_j, \beta_j\}_{j=1}^g$ of $\pi_1(\mathcal{C}, v_0)$ satisfying (3.8). Denote by M_{α_i}, M_{β_i} the corresponding monodromies which satisfy (3.11) for $n = 0$:

$$\prod_{j=1}^{g} [M_{\alpha_j}, M_{b_j}] = \mathbb{I}_2 \; . \tag{3.15}$$

Let us choose some oriented closed contours on $\mathcal{C}$ starting at v_0 representing the elements α_j, β_j; with a slight abuse of notations we denote these loops by the same letters. Consider the one-valent ciliated graph Γ_0 (which we call the *standard graph*, see Figure 3.1 (a)) with vertex at v_0 and $2g$ oriented edges $\{\alpha_j, \beta_j\}_{j=1}^{g}$; the valency of v_0 equals $4g$. Define the jump matrices on the edges of Γ_0 as follows:

$$J(\alpha_j) = M_{\beta_j} \; , \qquad J(\beta_j) = M_{\alpha_j} \; . \tag{3.16}$$

The cilium at v_0 is chosen between α_g and β_1 (see Fig 3.1 (a)). Notice that the relation (3.13) in the general definition of jump matrices on a given graph is written under assumption that all the edges e_j are incoming to the vertex. In our case, since every edge meets the vertex twice, while defining the canonical form (3.14) we need to inverse the orientation of $2g$ out of $4g$ half-edges incident to v_0 as shown in Fig 3.1 (b).

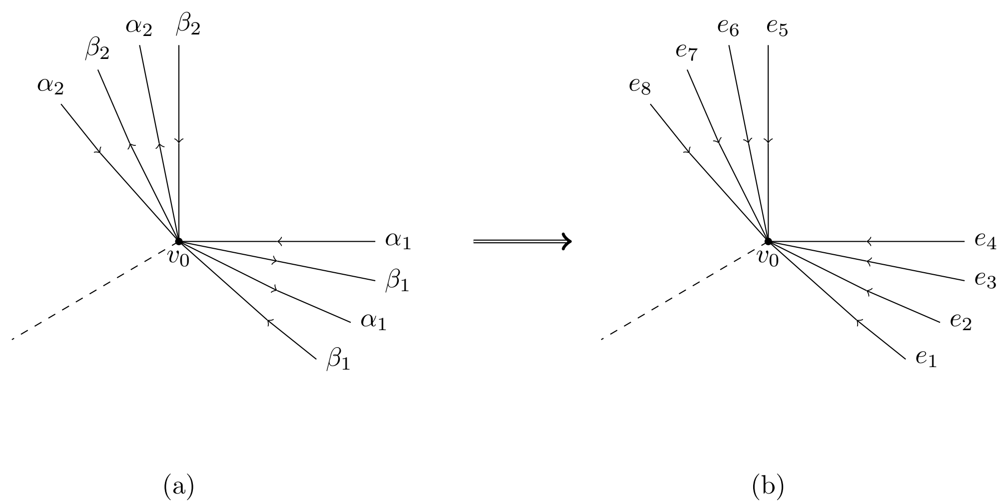

Figure 3.1: Orientation of the edges of the standard graph Γ_0 for $g = 2$.

Then we get:

$$J(e_1) = M_{\alpha_1} \; , \quad J(e_2) = M_{\beta_1}^{-1} \; , \quad J(e_3) = M_{\alpha_1}^{-1} \; , \quad J(e_4) = M_{\beta_1} \; , \ldots \; .$$

Under these conventions we get

$$J(e_1)J(e_2)...J(e_{4g}) = \prod_{i=1}^{g} M_{\alpha_i} M_{\beta_i}^{-1} M_{\alpha_i}^{-1} M_{\beta_i} = \mathbb{I}_2 \ .$$

Therefore, the standard graph Γ_0 with our choice of jump matrices satisfies the condition (3.13).

Applying the formula (3.14) to the pair (Γ_0, J) we get the symplectic form $\Omega(\Gamma_0)$ which can be written as follows:

$$\Omega(\Gamma_0) = \frac{1}{2} \sum_{j=1}^{2g-1} \operatorname{tr} \left(K_{[1...j]}^{-1} dK_{[1...j]} \wedge K_j^{-1} dK_j \right) \ , \tag{3.17}$$

where K_j is the jth matrix in the product (3.15); $K_{[1...j]}$ denotes the ordered product of matrices entering in the same product (3.15) up to j; for instance

$$K_{[1]} = M_{\alpha_1} \ , \quad K_{[12]} = M_{\alpha_1} M_{\beta_1}^{-1} \ , \quad K_{[123]} = M_{\alpha_1} M_{\beta_1}^{-1} M_{\alpha_1}^{-1} \ , \ldots \ .$$

Proposition 3.4.1. *The form $\Omega(\Gamma_0)$ given by (3.17) coincides with the Goldman symplectic form Ω on $\mathcal{V}_g$.*

Proof. The symplectic form Ω inverting the Goldman bracket on the $SL(N, \mathbb{C})$ character variety of $\pi_1(\mathcal{C})$ was computed by Alekseev and Malkin [3]. The form of [3] can be represented via (3.14) for a special graph G and the jump matrices [7]. In turns out that the Alekseev-Malkin graph can be transformed to the standard graph Γ_0 by a series of admissible moves preserving the symplectic form. Therefore, the form $\Omega(\Gamma_0)$ coincides with the Goldman symplectic form (for an arbitrary $SL(N, \mathbb{C})$ case).

□

3.5 Parametrization of $\mathcal{V}_g$ by plumbing two lower genus Riemann surfaces

Here, we are going to construct another oriented ciliated graph on $\mathcal{C}$ (with jump matrices on its edges) which can be transformed to Γ_0 by a series of admissible moves. This allows us to get an alternative representation for the form $\Omega(\Gamma_0)$ which will be used in the next section to construct new systems of log-canonical coordinates on $\mathcal{V}_g$.

3.5.1 Oriented graph on $\mathcal{C}$ by amalgamation of standard graphs on two Riemann surfaces of lower genera

Consider two topological Riemann surfaces, $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$, of genera $\widetilde{g}$ and $\widehat{g}$, respectively (such that $g = \widetilde{g} + \widehat{g}$) with one boundary component each. We denote by $\widetilde{\gamma}$ (resp. $\widehat{\gamma}$) a contour slightly shifted from the boundary to the interior of $\widetilde{\mathcal{C}}$ (resp. $\widehat{\mathcal{C}}$).

Pick a point $\widetilde{v}$ on $\widetilde{\mathcal{C}}$ near its boundary and a point $\widehat{v}$ on $\widehat{\mathcal{C}}$ also lying near its boundary. The fundamental group $\pi_1(\widetilde{\mathcal{C}}, \widetilde{v})$ is freely generated by the elements $\{\widetilde{\alpha}_i, \widetilde{\beta}_i\}_{i=1}^{\widetilde{g}}$ and we define the monodromy around the boundary of $\widetilde{\mathcal{C}}$ by

$$M_{\widetilde{\gamma}}^{-1} = \prod_{i=1}^{\widetilde{g}} \left[M_{\widetilde{\alpha}_i}, M_{\widetilde{\beta}_i} \right] \ . \tag{3.18}$$

We assume that $M_{\widetilde{\gamma}}$ is diagonalizable, $M_{\widetilde{\gamma}} = \widetilde{C}\widetilde{\Lambda}\widetilde{C}^{-1}$ where the matrix of eigenvectors $\widetilde{C}$ has a freedom of multiplication with $SL(2,\mathbb{C})$ diagonal matrix from the right.

Similarly $\pi_1(\widehat{\mathcal{C}}, \widehat{v})$ is freely generated by the elements $\{\widehat{\alpha}_j, \widehat{\beta}_j\}_{i=1}^{\widehat{g}}$ and we define the monodromy around the boundary of $\widehat{\mathcal{C}}$ by

$$M_{\widehat{\gamma}}^{-1} = \prod_{i=1}^{\widehat{g}} \left[M_{\widehat{\alpha}_i}, M_{\widehat{\beta}_i} \right] \ , \tag{3.19}$$

which is also assumed to be diagonalizable, $M_{\widehat{\gamma}} = \widehat{C}\widehat{\Lambda}\widehat{C}^{-1}$.

Define the following moduli space:

$$\widetilde{\mathcal{H}} = \left\{ \left\{ M_{\widetilde{\alpha}_j}, M_{\widetilde{\beta}_j} \right\}_{j=1}^{\widetilde{g}}, \ \ \{\widetilde{C}, \widetilde{\Lambda}\} : \quad \widetilde{C}\widetilde{\Lambda}\widetilde{C}^{-1} \prod_{i=1}^{\widetilde{g}} \left[M_{\widetilde{\alpha}_i}, M_{\widetilde{\beta}_i} \right] = \mathbb{I}_2 \right\} / \sim \ , \tag{3.20}$$

where $\widetilde{C}, \widetilde{\Lambda} \in SL(2,\mathbb{C})$ and $\widetilde{\Lambda} = \mathrm{diag}(\lambda_{\widetilde{\gamma}}, \lambda_{\widetilde{\gamma}}^{-1})$.

The equivalence relation $\sim$ is the same as in (3.12). The monodromies $\{M_{\widetilde{\alpha}_j}, M_{\widetilde{\beta}_j}\}$ can be chosen arbitrarily; they determine the matrix $\widetilde{\Lambda}$ uniquely. The matrix $\widetilde{C}$ is then defined up to the right multiplication with a diagonal matrix.

Similarly, consider the moduli space:

$$\widehat{\mathcal{H}} = \left\{ \left\{ M_{\widehat{\alpha}_j}, M_{\widehat{\beta}_j} \right\}_{j=1}^{\widehat{g}}, \{\widehat{C}, \widehat{\Lambda}\} : \quad \widehat{C}\widehat{\Lambda}\widehat{C}^{-1} \prod_{i=1}^{\widehat{g}} \left[M_{\widehat{\alpha}_i}, M_{\widehat{\beta}_i} \right] = \mathbb{I}_2 \right\} / \sim \ , \tag{3.21}$$

with $\widehat{C}, \widehat{\Lambda} \in SL(2,\mathbb{C})$ and $\widehat{\Lambda} = \mathrm{diag}(\lambda_{\widehat{\gamma}}, \lambda_{\widehat{\gamma}}^{-1})$.

The idea of the following proposition is to construct an element of $\mathcal{V}_g$ starting from a set of matrices representing a point of $\widetilde{\mathcal{H}}$ and a set of matrices representing a point of $\widehat{\mathcal{H}}$ that satisfy the additional constraint

$$\widehat{\Lambda} = \widetilde{\Lambda} \,. \tag{3.22}$$

We observe that for a diagonal matrix Λ, the inverse matrix is in the same adjoint orbit, namely;

$$\Lambda^{-1} = \mathfrak{b}^{-1}\Lambda\mathfrak{b} \,, \tag{3.23}$$

where

$$\mathfrak{b} = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \,.$$

Assuming (3.22) we can glue (topologically) the surfaces $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ along the boundary to get a topological surface $\mathcal{C}$ of genus $g = \widetilde{g} + \widehat{g}$. We denote by γ contour on $\mathcal{C}$ placed between $\widetilde{\gamma}$ with $\widehat{\gamma}$. The diagonal monodromy matrix representing γ is given by

$$\Lambda = \mathrm{diag}(\lambda_\gamma, \lambda_\gamma^{-1}) \,,$$

such that $\Lambda = \widehat{\Lambda} = \widetilde{\Lambda}$.

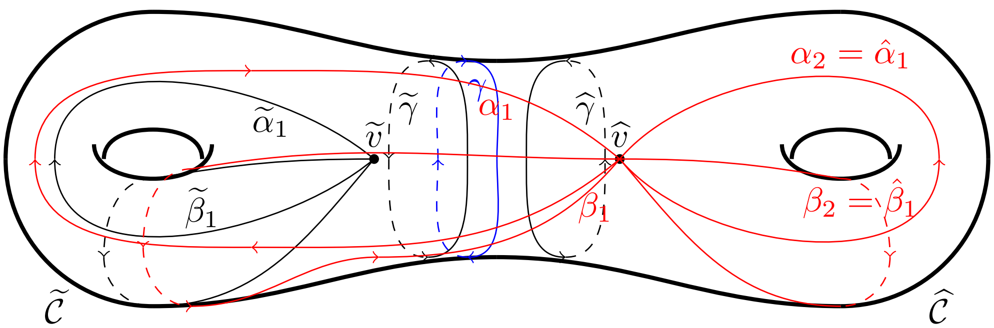

Figure 3.2: Generators of the fundamental groups $\pi_1(\widetilde{\mathcal{C}}, \widetilde{v})$ (black) and $\pi_2(\widehat{\mathcal{C}}, \widehat{v})$ and corresponding generators of $\pi_1(\mathcal{C}, \widehat{v})$ (red) for $g = 2$.

The generators $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of the fundamental group $\pi_1(\mathcal{C}, \widehat{v})$ are naturally inherited from the generators $\{\widetilde{\alpha}_j, \widehat{\alpha}_k\}$ and $\{\widetilde{\beta}_j, \widehat{\beta}_k\}$ of the fundamental groups of $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ as shown in Figure 3.2.

Let us now construct the $SL(2,\mathbb{C})$ monodromy representation of $\pi_1(\mathcal{C}, \widehat{v})$ starting from a point of the space (3.20) and a point of the space (3.21) satisfying the constraint (3.22). In other words, we are going to construct the map

$$F : \widetilde{\mathcal{H}} \times \widehat{\mathcal{H}}\Big|_{\widehat{\Lambda}=\widetilde{\Lambda}} \to \mathcal{V}_g \ . \tag{3.24}$$

Proposition 3.5.1. *Consider a set of matrices representing a point of $\widetilde{\mathcal{H}}$ and a set of matrices representing a point of $\widehat{\mathcal{H}}$ such that $\widehat{\Lambda} = \widetilde{\Lambda}$.*

Then the following set of $SL(2,\mathbb{C})$ matrices

$$M_{\alpha_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\alpha_i}}\widetilde{C}\mathfrak{b}\widehat{C}^{-1} \ , \qquad M_{\beta_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\beta_i}}\widetilde{C}\mathfrak{b}\widehat{C}^{-1}, \quad i = 1, ..., \widetilde{g} \ ,$$

$$M_{\alpha_{i+\widetilde{g}}} = M_{\widehat{\alpha_i}} \ , \qquad M_{\beta_{i+\widetilde{g}}} = M_{\widehat{\beta_i}} \ , \quad i = 1, ..., \widehat{g} \ . \tag{3.25}$$

defines a point of $\mathcal{V}_g$.

Proof. From the definitions (3.20) of $\widetilde{\mathcal{H}}$ and $\widehat{\mathcal{H}}$ one has

$$\begin{aligned} \widetilde{\Lambda} &= \widetilde{C}^{-1}\left(\prod_{i=1}^{\widetilde{g}}\left[M_{\widetilde{\alpha}_i}, M_{\widetilde{\beta}_i}\right]\right)^{-1}\widetilde{C} \ , \\ \widehat{\Lambda} &= \widehat{C}^{-1}\left(\prod_{i=1}^{\widehat{g}}\left[M_{\widehat{\alpha}_i}, M_{\widehat{\beta}_i}\right]\right)^{-1}\widehat{C} \ . \end{aligned} \tag{3.26}$$

Imposing $\widetilde{\Lambda} = \widehat{\Lambda}$ we get

$$\widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}\left(\prod_{i=1}^{\widetilde{g}}\left[M_{\widetilde{\alpha}_i}, M_{\widetilde{\beta}_i}\right]\right)\widetilde{C}\mathfrak{b}\widehat{C}^{-1}\left(\prod_{i=1}^{\widehat{g}}\left[M_{\widehat{\alpha}_i}, M_{\widehat{\beta}_i}\right]\right) = \mathbb{I}_2 \ . \tag{3.27}$$

Defining $M_{\alpha_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\alpha}_i}\widetilde{C}\mathfrak{b}\widehat{C}^{-1}$ and $M_{\beta_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\beta}_i}\widetilde{C}\mathfrak{b}\widehat{C}^{-1}$ for $i = 1, ..., \widetilde{g}$, equation (3.27) can be rewritten as:

$$\prod_{i=1}^{\widetilde{g}}[M_{\alpha_i}, M_{\beta_i}]\prod_{i=1}^{\widehat{g}}\left[M_{\widehat{\alpha}_i}, M_{\widehat{\beta}_i}\right] = \mathbb{I}_2 \ .$$

Now, defining $M_{\alpha_{i+\widetilde{g}}} = M_{\widehat{\alpha}_i}$ and $M_{\beta_{i+\widetilde{g}}} = M_{\widehat{\beta}_i}$ for $i = 1, ..., \widehat{g}$, this last identity can be rewritten:

$$\prod_{i=1}^{\widetilde{g}}[M_{\alpha_i}, M_{\beta_i}]\prod_{i=\widetilde{g}+1}^{\widetilde{g}+\widehat{g}}[M_{\alpha_i}, M_{\beta_i}] = \mathbb{I}_2 \ ,$$

which coincides with (3.9). □

Let us now use this plumbing formalism to construct the natural graph Γ_1 on $\mathcal{C}$ equipped with

jump matrices on its edges such that the canonical form $\Omega(\Gamma_1)$ coincides with the Goldman form Ω on $\mathcal{V}_g$.
The graph Γ_1 is constructed by amalgamation of three graphs:

- the graph $\widetilde{\Gamma}_1$ (which can be considered to be the standard graph on $\widetilde{\mathcal{C}}$ if the boundary component $\widetilde{\gamma}$ is shrunk to a regular point),
- the graph $\widehat{\Gamma}_1$ (which can be considered to be the standard graph on $\widehat{\mathcal{C}}$ if the boundary component $\widehat{\gamma}$ is shrunk to a regular point),
- and the five-vertex "plumbing" graph Γ_{pl}.

These graphs can be seen in Fig. 3.3. The five-valent graph Γ_{pl} has one-valent vertices at $\widetilde{v}$, $\widehat{v}$, and three four-valent vertices which we denote by $\widetilde{q}$, q and $\widehat{q}$; these are the points where the segment connecting $\widetilde{v}$ and $\widehat{v}$ crosses $\widetilde{\gamma}$, γ and $\widehat{\gamma}$, respectively.
The jump matrices on the edges of $\widetilde{\Gamma}$ are given by matrices $M_{\widetilde{\alpha}_j}, M_{\widetilde{\beta}_j}$; the jump matrices on the edges of $\widehat{\Gamma}$ are given by matrices $M_{\widehat{\alpha}_j}, M_{\widehat{\beta}_j}$ according to the rule (3.16).
Finally, the jump matrices on the edges of Γ_{pl} are given by

$$J([\widetilde{v},\widetilde{q}]) = J(\widetilde{e}_1) = M_{\widetilde{\gamma}} \;, \qquad J(\widetilde{e}_2) = J(\widehat{e}_2) = \Lambda \;, \qquad J([\widehat{v},\widetilde{q}]) = J(\widehat{e}_1) = M_{\widehat{\gamma}} \;,$$
$$J(\widetilde{\gamma}) = \widetilde{C} \;, \qquad J(\widehat{\gamma}) = \widehat{C} \;, \qquad J(\gamma) = \mathfrak{b} \;,$$

as shown in Fig. 3.4 (recall that $M_{\widetilde{\gamma}}$ and $M_{\widehat{\gamma}}$ are defined by (3.18) and (3.19) and that the orientation of γ is irrelevant because $-\mathfrak{b} = \mathfrak{b}^{-1} \equiv \mathfrak{b}$ are the same in $PSL(2,\mathbb{C})$).

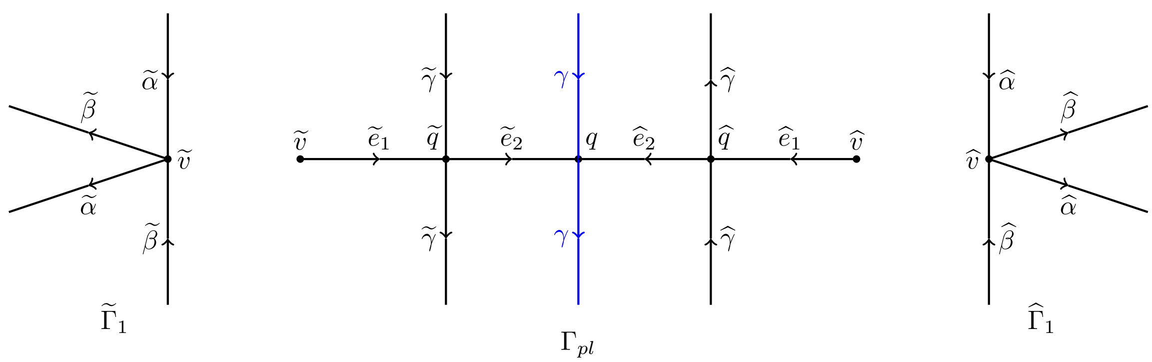

Figure 3.3: The graphs $\widetilde{\Gamma}_1$, Γ_{pl} and $\widehat{\Gamma}_1$ for $g = 2$.

One can verify that the local monodromies around all five vertices are trivial. Consider, for example, q, $\widetilde{q}$ and $\widetilde{v}$ (the computation at $\widehat{v}$ and $\widehat{q}$ is identical).

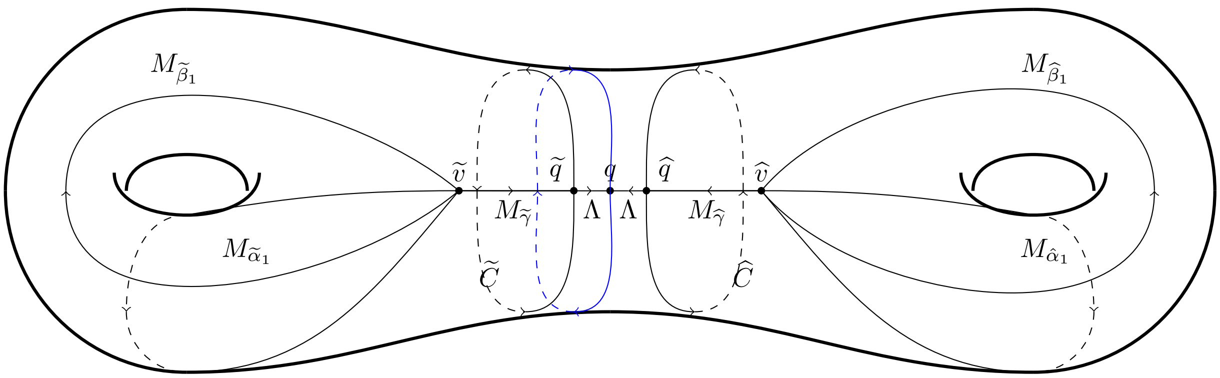

Figure 3.4: The amalgamated graph Γ_1 drawn on $\mathcal{C}$ for $g = 2$.

- For q: Let us choose the cilium position to be between the edge $[q, \widehat{q}]$ and the half edge corresponding to γ with the same orientation as $[q, \widehat{q}]$. The four edges incident to q in counterclockwise order have jump matrices: $J(e_1) = \mathfrak{b}^{-1}$, $J(e_2) = \Lambda$, $J(e_3) = \mathfrak{b}$ and $J(e_4) = \Lambda$. Then

$$J(e_1)J(e_2)J(e_3)J(e_4) = \mathfrak{b}^{-1}\Lambda\mathfrak{b}\Lambda = \Lambda^{-1}\Lambda = \mathbb{I}_2 \ .$$

- For $\widetilde{q}$: Let us choose the cilium position to be between the edge $\tilde{e} = [\widetilde{v}, \widetilde{q}]$ and the half edge corresponding to $\widetilde{\gamma}$ with the same orientation as $-\tilde{e}$. Then we have $J(e_1) = M_{\widetilde{\gamma}}$, $J(e_2) = \widetilde{C}$, $J(e_3) = \Lambda^{-1}$ and $J(e_4) = \widetilde{C}^{-1}$. Then

$$J(e_1)J(e_2)J(e_3)J(e_4) = M_{\widetilde{\gamma}}\widetilde{C}\Lambda^{-1}\widetilde{C}^{-1} = M_{\widetilde{\gamma}}M_{\widetilde{\gamma}}^{-1} = \mathbb{I}_2 \ .$$

- For $\widetilde{v}$: Let us choose the cilium position between the edge $\tilde{e}$ and the half edge corresponding to $\tilde{\beta}_1$ with the same orientation as $-\tilde{e}$. Then

$$J(e_1) = M_{\widetilde{\alpha}_1} \ , \qquad J(e_2) = M_{\widetilde{\beta}_1}^{-1} \ , \qquad J(e_3) = M_{\widetilde{\alpha}_1}^{-1} \ , \qquad J(e_4) = M_{\widetilde{\beta}_1} \ ,$$

$$J(e_5) = M_{\widetilde{\alpha}_2}, \qquad \dots, \qquad J(e_{4\widetilde{g}+1}) = J(\tilde{e}) = M_{\widetilde{\gamma}}^{-1} \ .$$

Therefore,

$$\prod_{i=1}^{4\widetilde{g}+1} J(e_i) = \prod_{i=1}^{\widetilde{g}} \left[M_{\widetilde{\alpha}_i}, M_{\widetilde{\beta}_i}\right] M_{\widetilde{\gamma}}^{-1} = \mathbb{I}_2 \ .$$

The graph Γ_1 can be transformed into the standard graph Γ_0 by a sequence of admissible moves in the sense of [7]:

1. Merge the vertices $\widetilde{v}$ with $\widetilde{q}$ and $\widehat{v}$ with $\widehat{q}$.

2. Merge the vertices q , $\widetilde{q}$ and $\widehat{q}$.

3. Zip together the three edges γ, $\widetilde{\gamma}$ and $\widehat{\gamma}$ whose jump matrices are respectively $\mathfrak{b}$, $\widetilde{C}^{-1}$ and $\widehat{C}$ to get a single edge γ_0 with jump matrix $J(\gamma_0) = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}$.

4. Zip the edge γ_0 with the edges $\{\widetilde{\alpha}_i, \widetilde{\beta}_i\}_{i=1}^{\widetilde{g}}$ to get edges $\{\alpha_i, \beta_i\}_{i=1}^{\widetilde{g}}$ with corresponding jump matrices:

$$J(\beta_i) = M_{\alpha_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\alpha}_i}\widetilde{C}\mathfrak{b}\widehat{C}^{-1} \quad J(\alpha_i) = M_{\beta_i} = \widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}M_{\widetilde{\beta}_i}\widetilde{C}\mathfrak{b}\widehat{C}^{-1}, \quad i = 1, ..., \widetilde{g}.$$

Therefore, the canonical symplectic form $\Omega(\Gamma_1)$ coincides with the canonical symplectic form $\Omega(\Gamma_0)$. In other words, one can use the graph Γ_1 to compute the Goldman symplectic form Ω on $\mathcal{V}_g$.

In the next section we construct another graph Γ_2 which can also be transformed to Γ_1 by a sequence of standard moves; the jump matrices on edges of Γ_2 will be expressed in terms of Fock-Goncharov coordinates, and, therefore, can be used to write Ω in the log-canonical form.

3.6 Triangulation of $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ and log-canonical coordinates

3.6.1 Graph Γ_2 from triangulations of $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$

For explicit computation of the Weil-Petersson form in terms of log-canonical coordinates we are going to construct one more graph on $\mathcal{C}$, which we denote by Γ_2, with jump matrices on its edges which make it equivalent to Γ_1 and Γ_0.

The graph Γ_2 and jump matrices on its edges are constructed via amalgamation of the following three graphs

1. Graph $\widetilde{\Gamma}_2$ constructed as follows. Fix an oriented triangulation (i.e. such that an orientation is assigned to each edge) $\widetilde{\Sigma}$ of $\widetilde{\mathcal{C}}$ with a single vertex $\widetilde{v}$ such that the boundary γ and contour $\widetilde{\gamma}$ fall into one of the triangles. The number of edges of $\widetilde{\Sigma}$ equals $6\widetilde{g} - 3$. The jump matrices on the edges $\widetilde{e}_j$ of $\widetilde{\Sigma}$ are chosen as follows

$$J(\widetilde{e}_j) = \widetilde{S}_{e_j} = \begin{pmatrix} 0 & 1/\widetilde{z}_{e_j} \\ -\widetilde{z}_{e_j} & 0 \end{pmatrix}, \quad \widetilde{z}_{e_j} \in \mathbb{C} .$$

Note that $\widetilde{S}_{e_j}^{-1} = -\widetilde{S}_{e_j} = \widetilde{S}_{-e_j}$. We also denote

$$\widetilde{\zeta}_{e_j} = \ln \widetilde{z}_{e_j} \ ,$$

for $j = 1, ..., 6\widetilde{g} - 3$.

Choose on each face $\widetilde{F}_k$ of $\widetilde{\Sigma}$ an extra internal vertex $\widetilde{w}_k$ connected to $\widetilde{v}$ by three internal edges oriented toward $\widetilde{w}_k$. To each internal edge (shown in red in Fig. 3.5 and Fig. 3.6) assign the constant jump matrix:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} .$$

2. Graph $\widehat{\Gamma}_2$ constructed as follows. Fix an oriented triangulation $\widehat{\Sigma}$ of $\widehat{\mathcal{C}}$ with a single vertex $\widehat{v}$ such that the boundary γ and contour $\widehat{\gamma}$ fall into one of the triangles. The number of edges of $\widehat{\Sigma}$ equals $6\widehat{g} - 3$. The jump matrices on edges $\widehat{e}_i$ of $\widehat{\Sigma}$ are chosen as follows

$$J(\widehat{e}_i) = \widehat{S}_{e_i} = \begin{pmatrix} 0 & 1/\widehat{z}_{e_i} \\ -\widehat{z}_{e_i} & 0 \end{pmatrix}, \qquad \widehat{z}_{e_i} \in \mathbb{C} \ ,$$

and denote $\widehat{\zeta}_{e_i} = \ln \widehat{z}_{e_i}$, for $i = 1, ..., 6\widehat{g} - 3$. Introduce on each face of $\widehat{\Sigma}$ a point $\widehat{w}_k$ together with three internal edges linking it to $\widehat{v}$. Assign the jump matrix A to each of these internal edges.

The variables $\widehat{\zeta}$ and $\widetilde{\zeta}$ are assumed to satisfy the constraint

$$\sum_{j=1}^{6\widetilde{g}-3} \widetilde{\zeta}_{e_j} = \sum_{i=1}^{6\widehat{g}-3} \widehat{\zeta}_{e_i} \tag{3.28}$$

or, equivalently,

$$\prod_{j=1}^{6\widetilde{g}-3} \widetilde{z}_{e_j} = \prod_{i=1}^{6\widehat{g}-3} \widehat{z}_{e_i} \ . \tag{3.29}$$

3. The "plumbing" 5-vertex graph part Γ_{pl} remains the same while the jump matrices on its edges are defined from the variables $\widetilde{\zeta}_{e_j}$ and $\widehat{\zeta}_{e_j}$ as follows.

The monodromy matrix $M_{\widetilde{\gamma}}$ (which is also the jump matrix $J([\widetilde{v}, \widetilde{q}])$ on Γ_{pl}) can be computed as follows and turns out to be lower-triangular

$$M_{\widetilde{\gamma}} = \left(\prod_{l=1}^{n_{\widetilde{v}}} A\widetilde{S}_{e_l} \right)^{-1} = \begin{pmatrix} \lambda_{\widetilde{\gamma}} & 0 \\ * & \lambda_{\widetilde{\gamma}}^{-1} \end{pmatrix} .$$

where

$$\lambda_{\widetilde{\gamma}} = \prod_{j=1}^{6\widetilde{g}-3} \widetilde{z}_{e_j}^2 \ . \tag{3.30}$$

Choose the special matrix $\widetilde{C}$ of eigenvectors of $M_{\widetilde{\gamma}}$ having the form

$$\widetilde{C} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \ .$$

Then the general $SL(2,\mathbb{C})$ matrix $\widetilde{C'}$ diagonalizing $M_{\widetilde{\gamma}}$ (such that $M_{\widetilde{\gamma}} = \widetilde{C'}\widetilde{\Lambda}\widetilde{C'}^{-1}$) has the form

$$\widetilde{C'} = \widetilde{C} \begin{pmatrix} b_{\widetilde{\gamma}}^{-1} & 0 \\ 0 & b_{\widetilde{\gamma}} \end{pmatrix} \ , \tag{3.31}$$

where the complex number $b_{\widetilde{\gamma}}$ is called the toric variable. The corresponding logarithmic coordinate is denoted $\beta_{\widetilde{\gamma}} = \ln b_{\widetilde{\gamma}}$.

Similarly, we define the matrix

$$M_{\widehat{\gamma}} = \left(\prod_{l=1}^{n_{\widehat{v}}} A\widehat{S}_{e_l} \right)^{-1} = \begin{pmatrix} \lambda_{\widehat{\gamma}} & 0 \\ * & \lambda_{\widehat{\gamma}}^{-1} \end{pmatrix} \ ,$$

with

$$\lambda_{\widehat{\gamma}} = \prod_{j=1}^{6\widehat{g}-3} \widehat{z}_{e_j}^2 \ , \tag{3.32}$$

the matrix $\widehat{C}$ of the form

$$\widehat{C} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \ ,$$

and the general matrix diagonalizing $M_{\widehat{\gamma}}$ of the form

$$\widehat{C'} = \widehat{C} \begin{pmatrix} b_{\widehat{\gamma}}^{-1} & 0 \\ 0 & b_{\widehat{\gamma}} \end{pmatrix} \ , \tag{3.33}$$

where $b_{\widehat{\gamma}}$ is another toric variable; we denote $\beta_{\widehat{\gamma}} = \ln b_{\widehat{\gamma}}$.

Note that the necessary condition

$$\lambda_{\widehat{\gamma}} = \lambda_{\widetilde{\gamma}}$$

is the corollary of the constraint (3.28).

In the sequel we shall drop the decorations and denote

$$\lambda_{\widetilde{\gamma}} = \lambda_{\gamma} = \lambda_{\widehat{\gamma}} \ .$$

Define also

$$\ell_{\gamma} = \ln \lambda_{\gamma} \ .$$

An example of graph Γ_2 for genus 2 surface $\mathcal{C}$ is shown in Fig. 3.5.

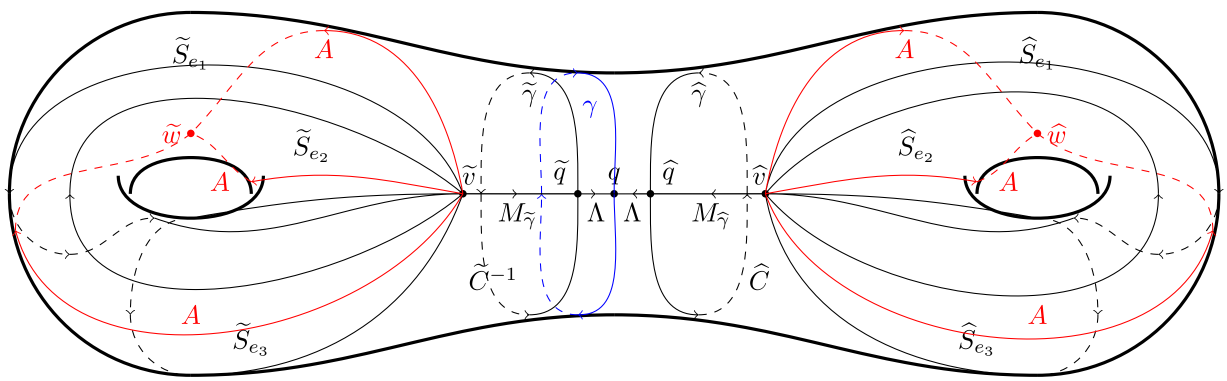

Figure 3.5: The graph Γ_2 drawn on $\mathcal{C}$ for $g = 2$.

Another representation of Γ_2 with explicitly shown triangulation is given schematically in Fig. 3.6. To get from Fig. 3.5 to Fig. 3.6 we cut the genus two surface $\mathcal{C}$ along the contour γ (shown in blue) positioned "in the middle" between $\widetilde{\gamma}$ and $\widehat{\gamma}$. Then the edge e connecting $\widetilde{q}$ and $\widehat{q}$ is cut in half, leaving the half-edges inside the small circles in both genus one "halfs" shown in Fig. 3.6.

The graph Γ_2 can be transformed into the graph Γ_1 by the following sequence of standard moves:

1. First, merge the internal vertices $\widetilde{w}_k$ with $\widetilde{v}$ and $\widehat{w}_k$ with $\widehat{v}$.

2. Second, zip together the edges with jump matrices A and the edges with jump matrices $\widetilde{S}_{e_i}$ (resp. $\widehat{S}_{e_i}$) to get edges with jump matrices $A\widetilde{S}_{e_i}$ or $A^{-1}\widetilde{S}_{e_i}$ (resp. $A\widehat{S}_{e_i}$ or $A^{-1}\widehat{S}_{e_i}$).

3. Third, zip together the edges of the triangulations $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ to get the set of edges $\{\widetilde{\alpha}_i, \widetilde{\beta}_i\}_{i=1}^{\widetilde{g}}$ and $\{\widehat{\alpha}_i, \widehat{\beta}_i\}_{i=1}^{\widehat{g}}$. The jump matrices

$$\{M_{\widetilde{\alpha}_j}\ , M_{\widetilde{\beta}_j}\ , M_{\widehat{\alpha}_j}\ , M_{\widehat{\beta}_j}\} \tag{3.34}$$

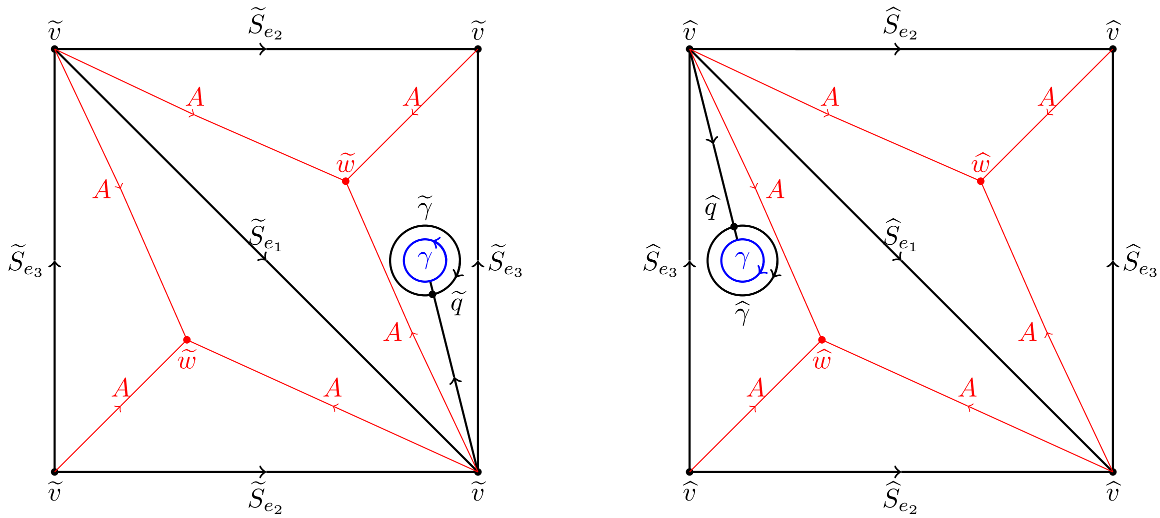

Figure 3.6: Triangulations of genus one components $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ for $g = 2$. The segments inside of the circles represent the two edges $[\widetilde{q}, q]$ and $[q, \widehat{q}]$ in Fig. 3.5.

on these new edges become then some products of $\widetilde{S}_{e_i}$ $\widehat{S}_{e_i}$, A and A^{-1}.

The Fuchsian monodromy matrices M_{α_j}, M_{β_j} can be obtained from matrices (3.34) using formulas (3.25) with $\widehat{C}$ and $\widetilde{C}$ replaced by the general diagonalizing matrices $\widetilde{C'}$ and $\widehat{C'}$ (3.31) and (3.33).
The matrix

$$\widetilde{C'}^{-1}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\widehat{C'} = \widetilde{C}^{-1}\begin{bmatrix}0 & -b_{\widetilde{\gamma}}\, b_{\widehat{\gamma}}\\ (b_{\widetilde{\gamma}}\, b_{\widehat{\gamma}})^{-1} & 0\end{bmatrix}\widehat{C} \tag{3.35}$$

depends only on the product of the toric variables; thus the Fuchsian monodromies depend only on the sum of their logarithms and we can define the new variable

$$\beta_\gamma = 2(\beta_{\widetilde{\gamma}} + \beta_{\widehat{\gamma}})\ . \tag{3.36}$$

Note that the variables $\widetilde{\zeta}_{e_j}$ and $\widehat{\zeta}_{e_j}$ are not independent but satisfy the constraint (3.28). The independent set of variables can be chosen as follows:

$$\left\{\{\widetilde{\zeta}_{e_i}\}_{i=2}^{6\widetilde{g}-3},\ \{\widehat{\zeta}_{e_i}\}_{i=2}^{6\widehat{g}-3}, \ell_\gamma, \beta_\gamma\right\}\ . \tag{3.37}$$

Then the remaining coordinates $\widetilde{\zeta}_1$ and $\widehat{\zeta}_1$ can be obtained from the logarithmic version of

(3.32):

$$\widetilde{\zeta}_{e_1} = \frac{1}{2}\ell_\gamma - \sum_{j=2}^{6\widetilde{g}-3} \widetilde{\zeta}_{e_j} \;, \qquad \widehat{\zeta}_{e_1} = \frac{1}{2}\ell_\gamma - \sum_{j=2}^{6\widehat{g}-3} \widehat{\zeta}_{e_j} \;. \tag{3.38}$$

3.7 Computation of Goldman symplectic form as $\Omega(\Gamma_2)$

We are in a position to state our main theorem.

Theorem 3.7.1. *The Goldman form Ω on $\mathcal{V}_g$ has constant coefficients in terms of variables (3.37) i.e. these variables are log-canonical for Ω. Namely, we have*

$$\Omega = \Omega_0 + \widetilde{\Omega}_1 + \widehat{\Omega}_1 + \widetilde{\Omega}_2 + \widehat{\Omega}_2 \;, \tag{3.39}$$

where

$$\Omega_0 = \mathrm{d}\beta_\gamma \wedge \mathrm{d}\ell_\gamma \;,$$

$$\widetilde{\Omega}_1 = \sum_{\substack{i,j=2\\ i<j}}^{6\widetilde{g}-3} (\widetilde{c}_{ij} - \widetilde{c}_{1j} + \widetilde{c}_{1i})\, \mathrm{d}\widetilde{\zeta}_{e_i} \wedge \mathrm{d}\widetilde{\zeta}_{e_j} \;, \qquad \widehat{\Omega}_1 = \sum_{\substack{k,l=2\\ k<l}}^{6\widehat{g}-3} (\widehat{c}_{kl} - \widehat{c}_{1l} + \widehat{c}_{1k})\, \mathrm{d}\widehat{\zeta}_{e_k} \wedge \mathrm{d}\widehat{\zeta}_{e_l} \;,$$

$$\widetilde{\Omega}_2 = \frac{1}{2}\sum_{m=2}^{6\widetilde{g}-3} \widetilde{c}_{1m} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widetilde{\zeta}_{e_m} \;, \qquad \widehat{\Omega}_2 = \frac{1}{2}\sum_{n=2}^{6\widehat{g}-3} \widehat{c}_{1n} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widehat{\zeta}_{e_n} \;,$$

The integer coefficients $\widetilde{c}_{ij}$, $\widehat{c}_{kl}$ lie in the interval between -4 and 4. They are determined by the order of the edges of the uni-valent graphs $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ at their vertices.

Proof. The symplectic form $\Omega(\Gamma_2)$ can be computed using the general formula (3.14) by summing the contributions of all four vertices $\widetilde{v}$, $\widehat{v}$, $\widetilde{q}$ and $\widehat{q}$ of the graph Γ_2.
The contributions coming from the vertices $\widetilde{q}$, $\widehat{q}$ and q are given by the following lemma:

Lemma 3.7.2. *The contribution of the vertex $\widetilde{q}$ to $\Omega(\Gamma_2)$ equals to $\Omega_{\widetilde{q}} = 2\mathrm{d}\beta_{\widetilde{\gamma}} \wedge \mathrm{d}\ell_\gamma$, the contribution of the vertex $\widehat{q}$ equals to $\Omega_{\widehat{q}} = 2\mathrm{d}\beta_{\widehat{\gamma}} \wedge \mathrm{d}\ell_\gamma$ and the contribution of the vertex q equals 0.*

Choose the cilium position to be between the edge $[\widetilde{v}, \widetilde{q}]$ and the half edge corresponding to $\widetilde{\gamma}$ with the same orientation as $[\widetilde{v}, \widetilde{q}]$. The four edges incident to $\widetilde{q}$ in counterclockwise order have jump matrices: $J(e_1) = J_1 = M_{\widetilde{\gamma}}$, $J(e_2) = J_2 = \widetilde{C'}$, $J(e_3) = J_3 = \Lambda^{-1}$ and $J(e_4) = J_4 = \widetilde{C'}^{-1}$.

According to equation (3.14) the contribution of $\widetilde{q}$ to $\Omega(\Gamma_2)$ reads:

$$\begin{aligned}\Omega_{\widetilde{q}} &= \frac{1}{2}\sum_{l=1}^{3}\mathrm{tr}\left(J_{[1...l]}^{-1}\mathrm{d}J_{[1...l]}\wedge J_l^{-1}\mathrm{d}J_l\right)\\ &= \frac{1}{2}\mathrm{tr}\left((J_1J_2)^{-1}\mathrm{d}(J_1J_2)\wedge J_2^{-1}\mathrm{d}J_2\right)+\frac{1}{2}\mathrm{tr}\left((J_1J_2J_3)^{-1}\mathrm{d}(J_1J_2J_3)\wedge J_3^{-1}\mathrm{d}J_3\right)\\ &= \frac{1}{2}\mathrm{tr}\left((M_{\widetilde{\gamma}}\widetilde{C'})^{-1}\mathrm{d}(M_{\widetilde{\gamma}}\widetilde{C'})\wedge \widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\right)+\frac{1}{2}\mathrm{tr}\left(\widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\wedge\Lambda\mathrm{d}(\Lambda^{-1})\right)\\ &= \frac{1}{2}\mathrm{tr}\left(M_{\widetilde{\gamma}}^{-1}\mathrm{d}M_{\widetilde{\gamma}}\wedge\widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\right)+\frac{1}{2}\mathrm{tr}\left(\Lambda^{-1}\mathrm{d}\Lambda\wedge\widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\right).\end{aligned}$$

Using the identity $M_{\widetilde{\gamma}}^{-1}\mathrm{d}M_{\widetilde{\gamma}}=\widetilde{C'}\Lambda^{-1}\widetilde{C'}^{-1}\left(\mathrm{d}\widetilde{C'}+\mathrm{d}\Lambda+\mathrm{d}(\widetilde{C'}^{-1})\right)$ we see that the first term is equal to the second one, so the contribution $\Omega_{\widetilde{q}}$ simplifies to:

$$\Omega_{\widetilde{q}}=\mathrm{tr}\left(\Lambda^{-1}\mathrm{d}\Lambda\wedge\widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\right)\ .$$

Using the definition (3.31) of $\widetilde{C'}$ we get:

$$\Omega_{\widetilde{q}}=\mathrm{tr}\left(\Lambda^{-1}\mathrm{d}\Lambda\wedge\widetilde{C'}^{-1}\mathrm{d}\widetilde{C'}\right)=-2\frac{\mathrm{d}\lambda_\gamma}{\lambda_\gamma}\wedge\frac{\mathrm{d}b_{\widetilde{\gamma}}}{b_{\widetilde{\gamma}}}=2\mathrm{d}\beta_{\widetilde{\gamma}}\wedge\mathrm{d}\ell_\gamma\ .$$

Similarly,

$$\Omega_{\widehat{q}}=-2\frac{\mathrm{d}\lambda_\gamma}{\lambda_\gamma}\wedge\frac{\mathrm{d}b_{\widehat{\gamma}}}{b_{\widehat{\gamma}}}=2\mathrm{d}\beta_{\widehat{\gamma}}\wedge\mathrm{d}\ell_\gamma\ .$$

The four edges incident to q have jump matrices $\mathfrak{b}^{-1}$, Λ, $\mathfrak{b}$ and Λ. Since the matrix $\mathfrak{b}$ is constant by immediate consequence of equation (3.14) the contribution of q to Ω is zero. □

To compute the contributions of the vertex $\widetilde{v}$ we can use the result of [7] which states that

$$\Omega_{\widetilde{v}}=\sum_{\substack{i,j=1\\ e_i<e_j}}^{2(6\widetilde{g}-3)}\mathrm{d}\widetilde{\zeta}_{e_i}\wedge\mathrm{d}\widetilde{\zeta}_{e_j}\ ,$$

where each edge comes to $\widetilde{v}$ twice. Eliminating repetitions of similar terms we get

$$\Omega_{\widetilde{v}}=\sum_{\substack{i,j=1\\ i<j}}^{6\widetilde{g}-3}\widetilde{c}_{ij}\mathrm{d}\widetilde{\zeta}_{e_i}\wedge\mathrm{d}\widetilde{\zeta}_{e_j}\ , \tag{3.40}$$

where $\widetilde{c}_{ij}$ are some (depending on the order of the edges at the vertex) integers from the range $\{-4,...,4\}$.

Similarly,

$$\Omega_{\widehat{v}} = \sum_{\substack{k,l=1 \\ k<l}}^{6\widehat{g}-3} \widehat{c}_{kl} \mathrm{d}\widehat{\zeta}_{e_k} \wedge \mathrm{d}\widehat{\zeta}_{e_l} \ , \tag{3.41}$$

where $\widehat{c}_{kl}$ are also some integers from the same set $\{-4, ..., 4\}$.
If one eliminates $\widetilde{\zeta}_1$ and $\widehat{\zeta}_1$ using the constraints (3.38) we get also

$$\Omega_{\widetilde{v}} = \sum_{\substack{i,j=2 \\ i<j}}^{6\widetilde{g}-3} (\widetilde{c}_{ij} - \widetilde{c}_{1j} + \widetilde{c}_{1i})\, \mathrm{d}\widetilde{\zeta}_{e_i} \wedge \mathrm{d}\widetilde{\zeta}_{e_j} + \frac{1}{2} \sum_{m=2}^{6\widetilde{g}-3} \widetilde{c}_{1m} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widetilde{\zeta}_{e_m}$$
$$\Omega_{\widehat{v}} = \sum_{\substack{k,l=2 \\ k<l}}^{6\widehat{g}-3} (\widehat{c}_{kl} - \widehat{c}_{1l} + \widehat{c}_{1k})\, \mathrm{d}\widehat{\zeta}_{e_k} \wedge \mathrm{d}\widehat{\zeta}_{e_l} + \frac{1}{2} \sum_{n=2}^{6\widehat{g}-3} \widehat{c}_{1n} \mathrm{d}\ell_\gamma \wedge \mathrm{d}\widehat{\zeta}_{e_n}$$

Summing up the contributions of all four vertices to $\Omega(\Gamma_2)$ we get (3.39).

3.8 Generalization to multiple cutting contours

A system of log-canonical coordinates generalizing (3.3) can be naturally associated to any system of m closed nonintersecting geodesics $\gamma_1, \dots \gamma_m$ for any $m = 1, \dots, 3g-3$. These contours split $\mathcal{C}$ into one or more Riemann surfaces with boundaries which we denote by $\mathcal{C}^{(1)}, \dots \mathcal{C}^{(n)}$. Denote the number of boundary components of $\mathcal{C}^{(i)}$ by $k^{(i)}$ and the genus by $g^{(i)}$; we have $\sum_{i=1}^n k^{(i)} = 2m$. The boundary components of each $\mathcal{C}^{(i)}$ are denoted $\gamma_j^{(i)}$, $j = 1, ..., k^{(i)}$, see Figure 3.7.
Then we can choose two points $v_k^\pm$ near the opposite sides of the curve γ_k. Consider now a fat-graph $\Sigma^{(i)}$ with $k^{(i)}$ vertices $v_k^\pm \in \mathcal{C}^{(i)}$ that provides triangulation of each $\mathcal{C}^{(i)}$ if the boundary components shrink to a point. The number of edges of $\Sigma^{(i)}$ equals $6g^{(i)} - 6 + 3k^{(i)}$, which coincides with the dimension of the moduli space of Riemann surfaces of genus $g^{(i)}$ with $k^{(i)}$ boundary components of variable length.
We associate complex shear coordinates $\zeta_{e_j}^{(i)}$, $j = 1, \dots, 6g^{(i)} - 6 + 3k^{(i)}$ to the edges of $\Sigma^{(i)}$; for each $e \in \Sigma^{(i)}$ we simply denote the coordinate by ζ_e. Adding internal vertices and edges we get a graph $\Gamma^{(i)}$ on each $\mathcal{C}^{(i)}$. The graph in $\mathcal{C}$ obtained by amalgamating all graphs $\Gamma^{(i)}$, in a manner similar to that of Section 3.6, is denoted by Γ.
There are m constraints that relate the oriented lengths ℓ_{γ_k} of γ_k to ζ_e's

$$\sum_{e \in E(\Sigma^{(i)}),\ e \perp v_k} \zeta_e = \ell_{\gamma_k} \ , \qquad k = 1, \dots m \ , \tag{3.42}$$

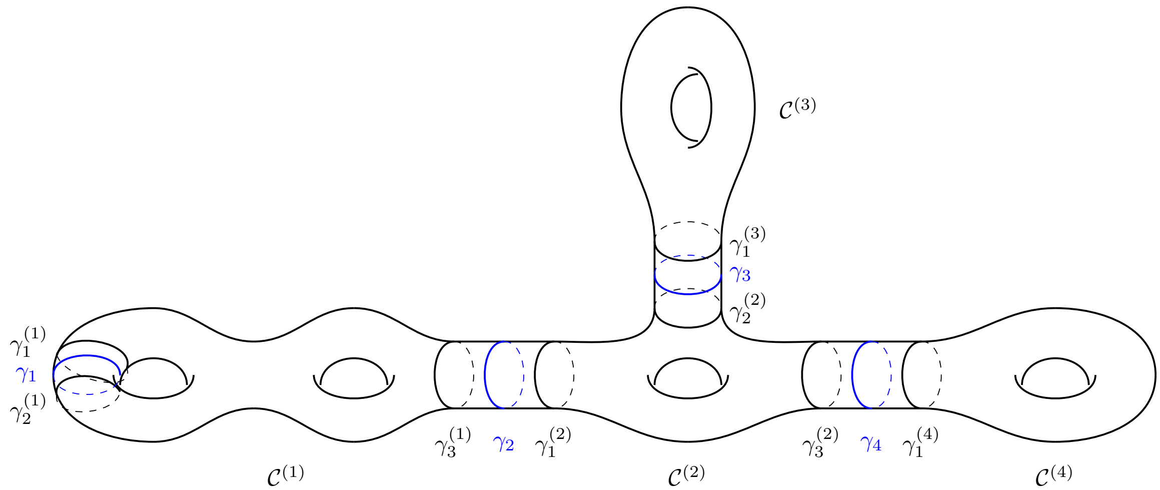

Figure 3.7: An example of multiple contours for $m = 4$ and $g = 5$.

where some terms on the left-hand side come with coefficient 1, and some other terms with coefficient 2 (depending on whether the edge "comes back" or not).
The system of constraints (3.42) is linear in terms of ζ_e. The system of log-canonical coordinates contains "complex lengths" ℓ_{γ_k} for $k = 1, \dots, m$ and the shear-type variables ζ_e (there are linear constraints (3.42) relating ℓ_{γ_k} to ζ_e's).

Theorem 3.8.1. *The Goldman form Ω can be represented as*

$$\Omega = \sum_{i=1}^{n} \Omega^{(i)} + \sum_{j=1}^{m} \mathrm{d}\beta_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} \ , \tag{3.43}$$

where

$$\Omega^{(i)} = \sum_{v \in V(\Sigma^{(i)})} \sum_{\substack{e, \widetilde{e} \perp v \\ e < \widetilde{e}}} \mathrm{d}\zeta_e \wedge \mathrm{d}\zeta_{\widetilde{e}} \ . \tag{3.44}$$

Proof. The proof is by direct computation of Ω as $\Omega(\Gamma)$, which is a straightforward generalization of the one given in the proof of Theorem 3.7.1. □

Remark 3.8.2. In (3.44) some terms repeat and some coordinates ζ_e should also be expressed by a linear combination of the remaining ones and ℓ_{γ_k} due to constraints (3.42).

3.9 Complete trinion decomposition

Let us discuss in detail the case of complete trinion decomposition. This decomposition is obtained by splitting $\mathcal{C}$ along a system S of $m = 3g-3$ closed non-intersecting geodesics $\gamma_1, ..., \gamma_{3g-3}$ to obtain $2g-2$ trinions -that is, spheres with three boundary components-denoted by $\mathcal{T}^{(j)}$.

Consider a single trinion $\mathcal{T}$. A triangulation on $\mathcal{T}$ is shown in Figure 3.8.

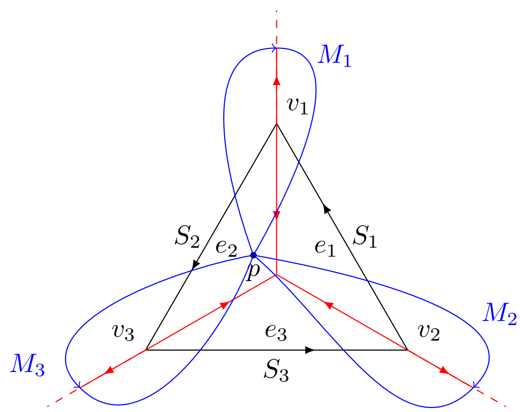

Figure 3.8: Triangulation and monodromies associated to a single trinion $\mathcal{T}$.

The triangulation in Figure 3.8 has edges ordered as follows:

$$\begin{aligned} e_1, e_2 \perp v_1, \quad e_2 < e_1 \ , \\ e_1, e_3 \perp v_2, \quad e_1 < e_3 \ , \\ e_2, e_3 \perp v_3, \quad e_3 < e_2 \ . \end{aligned} \tag{3.45}$$

The boundary components near the vertices v_1, v_2, v_3 are denoted by γ_1, γ_2, γ_3, respectively. The induced cyclic ordering of the boundary components, based on the edge ordering (3.45), is:

$$(\gamma_3, \gamma_2, \gamma_1) \ . \tag{3.46}$$

Taking the base point p inside the black triangle (Figure 3.8), the local $SL(2,\mathbb{C})$ monodromy matrices around v_j are denoted by M_j, with complex eigenvalues $-\lambda_j$ and $-1/\lambda_j$, $j = 1, 2, 3$. The shear-type coordinates z_{e_1}, z_{e_2} and z_{e_3} can be expressed in terms of λ_1, λ_2 and λ_3 as:

$$z_{e_1} = \sqrt{\frac{\lambda_3}{\lambda_2\lambda_1}} \ , \qquad z_{e_2} = \sqrt{\frac{\lambda_2}{\lambda_1\lambda_3}} \ , \qquad z_{e_3} = \sqrt{\frac{\lambda_1}{\lambda_3\lambda_2}} \ . \tag{3.47}$$

Then a representation $\rho : \pi_1(\mathcal{T}, p) \to SL(2,\mathbb{C})$ is explicitly given by the following three monodromies:

$$M_1 = \begin{bmatrix} -\lambda_1 & 0 \\ \frac{\lambda_1\lambda_2+\lambda_3}{\lambda_1\lambda_3} & -\frac{1}{\lambda_1} \end{bmatrix} , \qquad M_2 = \begin{bmatrix} \frac{\lambda_3}{\lambda_1} & \frac{-\lambda_2\lambda_3-\lambda_1}{\lambda_1\lambda_2} \\ \frac{\lambda_2\lambda_1+\lambda_3}{\lambda_1} & -\frac{\lambda_1\lambda_2^2+\lambda_3\lambda_2+\lambda_1}{\lambda_1\lambda_2} \end{bmatrix} , \qquad M_3 = \begin{bmatrix} -\frac{1}{\lambda_3} & \frac{-\lambda_2\lambda_3-\lambda_1}{\lambda_2} \\ 0 & -\lambda_3 \end{bmatrix} . \tag{3.48}$$

These monodromies satisfy the fundamental relation in homotopy:

$$M_1 M_2 M_3 = \mathbb{I}_2 , \qquad \mathrm{Tr}(M_j) = -\lambda_j - \frac{1}{\lambda_j} , \qquad j = 1,2,3 .$$

For an embedded trinion $\mathcal{T}^{(j)}$, the complex lengths of its boundary components must match the lengths of three geodesics in the system S. This matching imposes additional constraints. The coordinates $\{\ell_{\gamma_j}, \beta_{\gamma_j}\}_{j=1}^{3g-3}$ form a set of log-canonical coordinates on $\mathcal{V}_g$. The exact expression of the Goldman form Ω in those coordinates depends on the chosen trinion decomposition. Nevertheless, a general expression can be written using the "trinion graph" corresponding to a complete trinion decomposition.

Definition 3.9.1. The trinion graph Γ_{trin} of a complete trinion decomposition is the graph whose vertices are individual trinions. The edges of Γ_{trin} represent the boundary components where two trinions are glued (or a closed handle for a loop edge).

Figure 3.9, right, shows a vertex $v^{(j)}$ of the trinion graph. It corresponds to a trinion $\mathcal{T}^{(j)}$, which is glued along its boundaries to two other trinions $\mathcal{T}^{(j-1)}$ and $\mathcal{T}^{(j+1)}$.
Let the boundary components of $\mathcal{T}^{(j)}$ be $\gamma_1^{(j)}$, $\gamma_2^{(j)}$ and $\gamma_3^{(j)}$, ordered as (3.46). Figure 3.9, left, shows that $\gamma_1^{(j)}$ lies in a collar neighborhood of γ_m, $\gamma_2^{(j)}$ lies near γ_r and $\gamma_3^{(j)}$ lies near γ_n, where $\gamma_n, \gamma_r, \gamma_m \in S$ and $n, r, m \in \{1, 2, ..., 3g-3\}$. This induces the following ordering of the geodesics:

$$(\gamma_n, \gamma_r, \gamma_m) . \tag{3.49}$$

This is equivalent to the following ordering of the edges e_n, e_r and e_m incident to the vertex $v^{(j)}$ of Γ_{trin}:

$$(e_n, e_r, e_m) . \tag{3.50}$$

With the edge ordering (3.50) fixed, we formulate the following theorem.

Theorem 3.9.2. *Let Γ_{trin} be the trinion graph of a complete trinion decomposition. Then the Goldman form on $\mathcal{V}_g$ can be expressed as:*

$$\Omega = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e,\widetilde{e} \perp v \\ e < \widetilde{e}}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\widetilde{e}} + \sum_{e \in E(\Gamma_{trin})} \mathrm{d}\beta_e \wedge \mathrm{d}\ell_e , \tag{3.51}$$

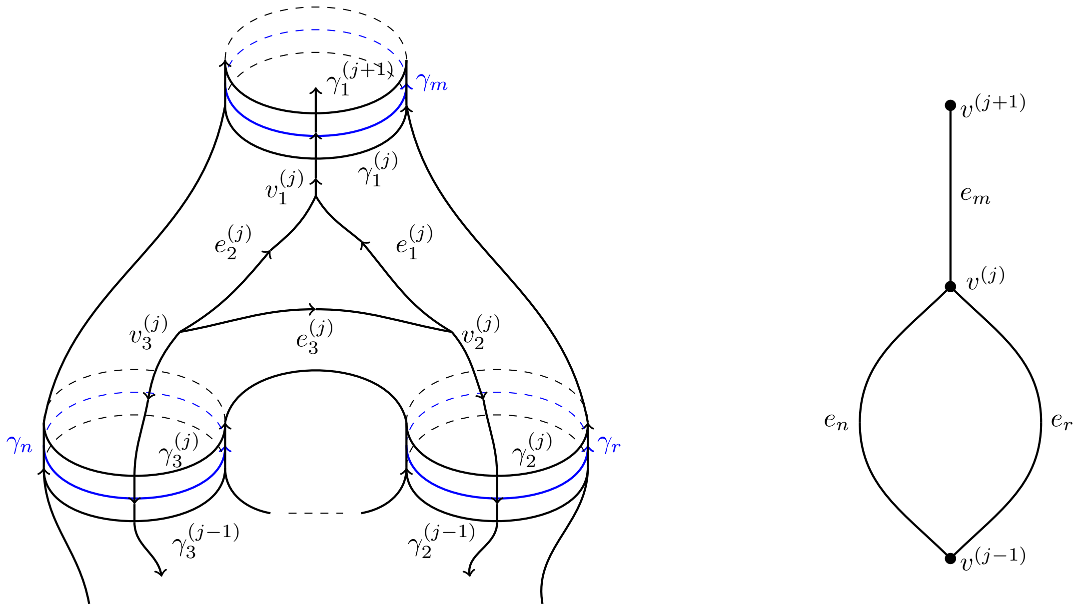

Figure 3.9: Gluing of a trinion $\mathcal{T}^{(j)}$ with two other trinions $\mathcal{T}^{(j-1)}$ and $\mathcal{T}^{(j+1)}$ -with graph $\Gamma^{(j)}$ drawn on $\mathcal{T}^{(j)}$- (left) and associated edges and vertices of the trinion graph (right).

where the $3g-3$ variables ℓ_e are the "complex oriented lengths" of the edges of the trinion graph, and β_e's are the conjugated variables.

Proof. Let $\{\lambda_i^{(j)}, b_i^{(j)}\}_{i=1,2,3}$ be the eigenvalues and toric variables associated with the $j-th$ trinion $\mathcal{T}^{(j)}$. Their logarithmic counterparts are $\{\ell_i^{(j)}, \beta_i^{(j)}\}_{i=1,2,3}$. These variables are pairwise duplicated due to trinion gluing. Applying (3.17) to the monodromies (3.48) yields the contribution of $\mathcal{T}^{(j)}$ to the symplectic form Ω:

$$\Omega_{\mathcal{T}^{(j)}} = \sum_{i=1}^{3} \left(\mathrm{d}\ell_{i+1}^{(j)} \wedge \mathrm{d}\ell_i^{(j)} + 2\mathrm{d}\beta_i^{(j)} \wedge \mathrm{d}\ell_i^{(j)} \right), \tag{3.52}$$

where the summation index i taken modulo 3. Importantly, the expression (3.52) is independent of the trinion configuration and is compatible with the ordering (3.46).

Let $v^{(j)}$ be a vertex of Γ_{trin} corresponding to $\mathcal{T}^{(j)}$. The three gluing constraints (for the configuration shown on Figure 3.9) imply:

$$\ell_3^{(j)} = \ell_3^{(j+1)} = \ell_{e_n}\,, \qquad \ell_2^{(j)} = \ell_2^{(j-1)} = \ell_{e_r}\,, \qquad \ell_1^{(j)} = \ell_1^{(j-1)} = \ell_{e_m}\,. \tag{3.53}$$

The corresponding toric variables are:

$$\beta_{e_n} = 2\left(\beta_3^{(j)} + \beta_3^{(j+1)}\right) \ , \qquad \beta_{e_r} = 2\left(\beta_2^{(j)} + \beta_2^{(j-1)}\right) \ , \qquad \beta_{e_m} = 2\left(\beta_1^{(j)} + \beta_1^{(j-1)}\right) \ .$$

These constraints correspond specifically to $\mathcal{T}^{(j)}$ glued to $\mathcal{T}^{(j-1)}$ and $\mathcal{T}^{(j+1)}$.
For a loop edge e_p incident to a vertex $v^{(q)}$ (corresponding to a "closed handle"), the constraints become:

$$l_1^{(q)} = l_2^{(q)} = l_{e_p} \ , \qquad \beta_{e_p} = 2\left(\beta_1^{(q)} + \beta_2^{(q)}\right) \ .$$

For a trinion $P^{(s)}$ glued to three distinct trinions $\mathcal{T}^{(s-2)}$, $\mathcal{T}^{(s-1)}$ and $\mathcal{T}^{(s+1)}$, one possible choice of constraints is:

$$\ell_3^{(s)} = \ell_1^{(s+1)} = \ell_{e_t} \ , \qquad \ell_2^{(s)} = \ell_3^{(s-1)} = \ell_{e_v} \ , \qquad \ell_1^{(s)} = \ell_3^{(s-2)} = \ell_{e_w} \ .$$

The corresponding toric variables are:

$$\beta_{e_t} = 2\left(\beta_3^{(s)} + \beta_1^{(s+1)}\right) \ , \qquad \beta_{e_v} = 2\left(\beta_2^{(s)} + \beta_3^{(s-1)}\right) \ , \qquad \beta_{e_w} = 2\left(\beta_1^{(s)} + \beta_3^{(s-2)}\right) \ .$$

In general, gluing of boundary components always falls into one of the three above cases. The particular choice of constraints does not affect the final form of Ω. Summing over all vertices v of the trinion graph and using the ordering (3.45), we obtain

$$\Omega = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e, \tilde{e} \perp v \\ e < \tilde{e}}} \mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\tilde{e}} + \sum_{e \in E(\Gamma_{trin})} \mathrm{d}\beta_e \wedge \mathrm{d}\ell_e \ .$$

The prefactor 1/2 in the first term corrects the duplication of variables due to gluing.

□

3.10 Application to $\mathcal{T}_g$ and $\mathcal{M}_g$

The $SL(2,\mathbb{R})$ character variety $\mathcal{V}_g^{\mathbb{R}}$ has several connected components classified by their Euler class (Euler numbers of the corresponding representations) [27]. In this section, we prove that the representations in $\mathcal{V}_g^{\mathbb{R}}$ constructed by plumbing of two or more lower genus Riemann surfaces with boundaries lie in the Fuchsian component of $\mathcal{V}_g^{\mathbb{R}}$. Projection to the Fuchsian component of the $PSL(2,\mathbb{R})$ character variety enables us to parametrize the Teichmüller space $\mathcal{T}_g$ using log-canonical coordinates. We explain in detail the argument for two Riemann surfaces, the generalization to multiple plumbing being straightforward. These results are already known in the existing literature; we only adapt them to our setting and give proofs

for the sake of self-completeness.
Given the setting of section 3.5, consider two Fuchsian representations $\widetilde{\rho} : \pi_1(\widetilde{\mathcal{C}}, \widetilde{v}) \to SL(2,\mathbb{R})$ and $\widehat{\rho} : \pi_1(\widehat{\mathcal{C}}, \widehat{v}) \to SL(2,\mathbb{R})$. Such Fuchsian representations are given by hyperbolic matrices

$$\left\{ \left\{ M_{\widetilde{\alpha}_j}, M_{\widetilde{\beta}_j} \right\}_{j=1}^{\widetilde{g}}, M_{\widetilde{\gamma}} \right\}, \qquad \left\{ \left\{ M_{\widehat{\alpha}_j}, M_{\widehat{\beta}_j} \right\}_{j=1}^{\widehat{g}}, M_{\widehat{\gamma}} \right\},$$

still satisfying relations (3.18) and (3.19), respectively. The representations $\widetilde{\rho}$ and $\widehat{\rho}$ act on $\mathbb{RP}^1$ by Möbius transformations, since $\mathbb{RP}^1 \cong S^1 = \mathbb{R}/\mathbb{Z}$ (by stereographic projection), one can lift these actions to $\mathbb{R}$. We denote by $R_{\widetilde{\rho}} : \mathbb{R} \to \mathbb{R}$ and by $R_{\widehat{\rho}} : \mathbb{R} \to \mathbb{R}$ the lifted actions of $\widetilde{\rho}$ and $\widehat{\rho}$, respectively.
Taking an element $M \in SL(2,\mathbb{R})$, its action on $\mathbb{RP}^1 \cong S^1$ can be lifted to an action $R_M : \mathbb{R} \to \mathbb{R}$. The Poincaré rotation number is then

$$\mathrm{rot}(R_M) = \lim_{n\to\infty} \frac{R_M^n(\theta_0) - \theta_0}{n} ,$$

for $\theta_0 \in \mathbb{R}$ not a fixed point. For hyperbolic elements this limit is an integer, independent of θ_0, and invariant under conjugation.
The Euler numbers $e(\widetilde{\rho})$ and $e(\widehat{\rho})$ are then defined to be [27]:

$$e(\widetilde{\rho}) = \mathrm{rot}\left(R_{M_{\widetilde{\gamma}}}\right) + \sum_{i=1}^{\widetilde{g}} \mathrm{rot}\left(\left[R_{M_{\widetilde{\alpha}_i}}, R_{M_{\widetilde{\beta}_i}}\right]\right), \qquad e(\widehat{\rho}) = \mathrm{rot}\left(R_{M_{\widehat{\gamma}}}\right) + \sum_{i=1}^{\widehat{g}} \mathrm{rot}\left(\left[R_{M_{\widehat{\alpha}_i}}, R_{M_{\widehat{\beta}_i}}\right]\right).$$

Lemma 3.10.1. *For the two Fuchsian representations $\widetilde{\rho} : \pi_1(\widetilde{\mathcal{C}}, \widetilde{v}) \to SL(2,\mathbb{R})$ and $\widehat{\rho} : \pi_1(\widehat{\mathcal{C}}, \widehat{v}) \to SL(2,\mathbb{R})$ we have:*

$$e(\widetilde{\rho}) = 2\widetilde{g} - 1 , \qquad e(\widehat{\rho}) = 2\widehat{g} - 1 .$$

Proof. Recall that $M_{\widetilde{\gamma}}$ and $M_{\widehat{\gamma}}$ are $SL(2,\mathbb{R})$ hyperbolic matrices diagonalizable over $\mathbb{R}$, with diagonal forms $\mathrm{diag}(\widetilde{\lambda}, 1/\widetilde{\lambda})$ and $\mathrm{diag}(\widehat{\lambda}, 1/\widehat{\lambda})$, respectively. For such matrices it is a classical fact that [27, 24]:

$$\mathrm{rot}\left(R_{M_{\widetilde{\gamma}}}\right) = \mathrm{rot}\left(R_{M_{\widehat{\gamma}}}\right) = \pm 1 .$$

We adopt the $+1$ convention, which corresponds to the orientation-preserving choice. If we close the surface $\widetilde{\mathcal{C}}$ by capping off the boundary, the resulting closed surface of genus $\widetilde{g}$ carries a Fuchsian representation with Euler number $2\widetilde{g} - 2$. Hence,

$$\sum_{i=1}^{\widetilde{g}} \mathrm{rot}\Big([R_{M_{\widetilde{\alpha}_i}}, R_{M_{\widetilde{\beta}_i}}]\Big) = 2\widetilde{g} - 2 .$$

The same argument applies for $\widehat{\mathcal{C}}$. Putting everything together it follows that

$$e(\widetilde{\rho}) = 2\widetilde{g} - 1 \ , \qquad e(\widehat{\rho}) = 2\widehat{g} - 1 \ .$$

□

Proposition 3.10.2. *Consider the representation $\rho : \pi_1(\mathcal{C}, \widehat{v}) \to SL(2, \mathbb{R})$ given by the set of matrices constructed in Proposition 3.5.1. We have $e(\rho) = 2g - 2$.*

Proof. For the representation ρ constructed in Proposition 3.5.1 we have

$$\begin{aligned}
e(\rho) &= \sum_{i=1}^{\widetilde{g}} \mathrm{rot}\left(R_{\widehat{C}\mathfrak{b}^{-1}\widetilde{C}^{-1}} \left[R_{M_{\widetilde{\alpha}_i}}, R_{M_{\widetilde{\beta}_i}} \right] R_{\widetilde{C}\mathfrak{b}\widehat{C}^{-1}} \right) + \sum_{i=1}^{\widehat{g}} \mathrm{rot}\left(\left[R_{M_{\widehat{\alpha}_i}}, R_{M_{\widehat{\beta}_i}} \right] \right) \\
&= \sum_{i=1}^{\widetilde{g}} \mathrm{rot}\left(\left[R_{M_{\widetilde{\alpha}_i}}, R_{M_{\widetilde{\beta}_i}} \right] \right) + \sum_{i=1}^{\widehat{g}} \mathrm{rot}\left(\left[R_{M_{\widehat{\alpha}_i}}, R_{M_{\widehat{\beta}_i}} \right] \right) \\
&= e(\widetilde{\rho}) + e(\widehat{\rho}) - \left(\mathrm{rot}\,(\Lambda) + \mathrm{rot}\left(\Lambda^{-1}\right) \right) \\
&= 2\widetilde{g} - 1 + 2\widehat{g} - 1 - 0 \\
&= 2g - 2 \ .
\end{aligned}$$

□

Therefore, the representation constructed in Proposition 3.5.1 lies in the Fuchsian component of $\mathcal{V}_g^{\mathbb{R}}$. Projecting to $PSL(2, \mathbb{R})$, we have the following corollary by Goldman's classification of connected components of $PSL(2, \mathbb{R})$ character varieties.

Corollary 3.10.3. *Let $\rho : \pi_1(\mathcal{C}) \to SL(2, \mathbb{R})$ be the representation constructed by Proposition 3.5.1, and let $\bar{\rho} : \pi_1(\mathcal{C}) \to PSL(2, \mathbb{R})$ be its projection. Then the conjugacy class of $\bar{\rho}$ defines a point of the Teichmüller space $\mathcal{T}_g$.*

Proof. By Proposition 3.10.2 the Euler number of the representation ρ is

$$e(\rho) = 2g - 2 \ .$$

The central subgroup $\{\pm \mathbb{I}_2\} \subset SL(2, \mathbb{R})$ acts trivially on $\mathbb{RP}^1$, therefore the Euler number of the induced $PSL(2, \mathbb{R})$ representation $\bar{\rho}$ coincides with $e(\rho)$. In particular $\bar{\rho}$ achieves the maximum possible value of the Euler number:

$$|e(\bar{\rho})| = 2g - 2 \ .$$

By Goldman's classification of connected components of $PSL(2,\mathbb{R})$ character varieties over compact Riemann surfaces [27, Corollary C], representations whose Euler number is maximal (in absolute value) are exactly the discrete, faithful representations. Therefore, $\bar{\rho}$ is discrete and faithful and its conjugacy class determines a point of the Teichmüller space $\mathcal{T}_g$.

□

The following lemma gives the other direction ; any point in $\mathcal{T}_g$ can be recovered from the data of a Fuchsian $SL(2,\mathbb{R})$ representation ρ.

Lemma 3.10.4. *Consider the following subsets of $\mathcal{V}_g^{\mathbb{R}}$:*

$$\mathcal{V}_g^+ = \{\, \rho^+ : \pi_1(\mathcal{C}) \to SL(2,\mathbb{R}) \mid \prod_{i=1}^{g}[M_{\alpha_i}, M_{\beta_i}] = +\mathbb{I}_2 \,\} /\!\!/ SL(2,\mathbb{R}) \ ,$$

where the representations $\rho^+ \in \mathcal{V}^+$ are assumed to be Fuchsian. Then the natural projection

$$\begin{aligned} \Pi^+ : \mathcal{V}_g^+ &\to \mathcal{T}_g \\ \rho^+ &\mapsto \bar{\rho} \end{aligned}$$

is surjective.

Proof. Every point of the Teichmüller space $\mathcal{T}_g$ corresponds to a Fuchsian representation

$$\bar{\rho} : \pi_1(\mathcal{C}) \to PSL(2,\mathbb{R}) \ ,$$

which can be realized as the side-pairing group of a fundamental $4g$-gon in the hyperbolic plane $\mathbb{H}$. Each side-pairing is an isometry of $\mathbb{H}$ and each such isometry has two preimages in $SL(2,\mathbb{R})$. W. Abikoff, in [2], proved that one can choose the lifts consistently by imposing a canonical sign convention (for instance, requiring the lower-left entry of each matrix to be positive). With this convention, the relation

$$\prod_{i=1}^{g}[M_{\alpha_i}, M_{\beta_i}] = +\mathbb{I}_2$$

holds in $SL(2,\mathbb{R})$. Thus every Fuchsian representation $\bar{\rho} : \pi_1(\mathcal{C}) \to PSL(2,\mathbb{R})$ has a canonical lift $\rho : \pi_1(\mathcal{C}) \to SL(2,\mathbb{R})$ lying in $\mathcal{V}_g^+$. Since every point of $\mathcal{T}_g$ arises from a Fuchsian representation, the projection

$$\Pi^+ : \mathcal{V}_g^+ \longrightarrow \mathcal{T}_g$$

is surjective. □

In other words, Corollary 3.10.3 and Lemma 3.10.4 prove that conjugacy classes of representations constructed by gluing of Fuchsian representations presented in Proposition 3.5.1 and points of the Teichmuller space $\mathcal{T}_g$ are in one to one correspondence.
We can deduce from the above results that expression (3.39) for the Goldman symplectic form Ω in Theorem 3.7.1 holds on $\mathcal{T}_g$. The form must therefore coincide with the Weil-Petersson symplectic form ω_{WP}. By the results of S. Wolpert, ω_{WP} is invariant under the action of the mapping class group and hence descends to the moduli space of Riemann surfaces $\mathcal{M}_g$.
The above reasoning can be generalized to an arbitrary number $1 \leq m \leq 3g-3$ of Fuchsian representations $\rho^{(i)} : \pi_1(\mathcal{C}^{(i)}) \to SL(2,\mathbb{R})$ (using the notation of Section 3.8). So, expression (3.43) for the Goldman form Ω also lifts to $\mathcal{T}_g$ (and $\mathcal{M}_g$). Essentially, Theorem 3.8.1 assigns a different set of log-canonical coordinates to any component of the Deligne-Mumford boundary of $\mathcal{M}_g$. The real slice for $m = 3g-3$ (complete trinion decomposition) is discussed in detail in the next section.

Remark 3.10.5. *Let $\mathcal{M}_{\widetilde{g},1}[\ell_{\widetilde{\gamma}}]$ (resp. $\mathcal{M}_{\widehat{g},1}[\ell_{\widehat{\gamma}}]$) be the moduli space of Riemann surfces of genus $\widetilde{g}$ (resp. $\widehat{g}$) with one geodesic boundary of length $\ell_{\widetilde{\gamma}}$ (resp. $\ell_{\widehat{\gamma}}$). Starting directly at the level of the moduli space $\mathcal{M}_g$ another idea of proof for Theorem 3.7.1 would be to parametrize each space $\mathcal{M}_{\widetilde{g},1}[\ell_{\widetilde{\gamma}}]$ and $\mathcal{M}_{\widehat{g},1}[\ell_{\widehat{\gamma}}]$ independently and then quotient $\mathcal{M}_{\widetilde{g},1}[\ell_{\widetilde{\gamma}}] \times \mathcal{M}_{\widehat{g},1}[\ell_{\widehat{\gamma}}]$ by the $\mathbb{R}$ action*

$$\beta_{\widetilde{\gamma}} \mapsto \beta_{\widetilde{\gamma}} + t\ , \qquad \beta_{\widehat{\gamma}} \mapsto \beta_{\widehat{\gamma}} + t\ , \qquad t \in \mathbb{R}\ .$$

The moment map $\mu : \mathcal{M}_{\widetilde{g},1}[\ell_{\widetilde{\gamma}}] \times \mathcal{M}_{\widehat{g},1}[\ell_{\widehat{\gamma}}] \to \mathbb{R}$ of this action is given by the Hamiltonian function

$$\mu(\ell_{\widetilde{\gamma}}, \ell_{\widehat{\gamma}}) = \ell_{\widetilde{\gamma}} - \ell_{\widehat{\gamma}}\ .$$

The moduli space $\mathcal{M}_g$ is symplectomorphic to $\mu^{-1}(0) /\!\!/ \mathbb{R}$, i.e:

$$\mathcal{M}_g \cong \mu^{-1}(0) /\!\!/ \mathbb{R}\ ,$$

and the Goldman symplectic form Ω can be written as in Theorem 3.7.1 on the reduced space $\mu^{-1}(0) /\!\!/ \mathbb{R}$.

3.11 Relationship to Fenchel-Nielsen coordinates

We continue to restrict our attention to the situation in which the log-canonical coordinates are defined on the Fuchsian component of the $SL(2,\mathbb{R})$ character variety $\mathcal{V}_g^{\mathbb{R}}$. Our goal is to establish a precise relationship to the real Fenchel-Nielsen coordinates in terms of algebraic expressions that can be simply extended to the complex Fenchel-Nielsen coordinates on the

$SL(2,\mathbb{C})$ character variety $\mathcal{V}_g$.

3.11.1 Dehn twist action on the variable β_γ

To simplify the discussion, we adopt for the rest of Chapter 3 a slightly different but equivalent convention for the toric actions on the spaces $\widetilde{\mathcal{H}}$ and $\widehat{\mathcal{H}}$. Instead of defining it on jump matrix $\widetilde{C}$ as $\widetilde{C} \mapsto \widetilde{C} b_{\widetilde{\gamma}}^{\boldsymbol{\sigma_3}}$ and $\widehat{C} \mapsto \widehat{C} b_{\widehat{\gamma}}^{\boldsymbol{\sigma_3}}$, we define a reduced toric action directly on the matrix $\mathfrak{b}$:

$$\mathfrak{b} \mapsto \mathfrak{T} = b_{\widetilde{\gamma}}^{-\boldsymbol{\sigma_3}} \mathfrak{b} b_{\widehat{\gamma}}^{\boldsymbol{\sigma_3}} = \begin{bmatrix} 0 & -\sqrt{b_\gamma} \\ \frac{1}{\sqrt{b_\gamma}} & 0 \end{bmatrix},$$

where $b_\gamma = (b_{\widetilde{\gamma}} b_{\widehat{\gamma}})^2$ is the toric variable on $\mathcal{V}_g^{\mathbb{R}}$. Thus the jump matrix on the curve γ is taken to be $\mathfrak{T}$. Figure 3.10 illustrates how a Dehn twist along γ deforms the graph Γ_2. Under such a twist, the log–canonical variable β_γ transforms as follows:

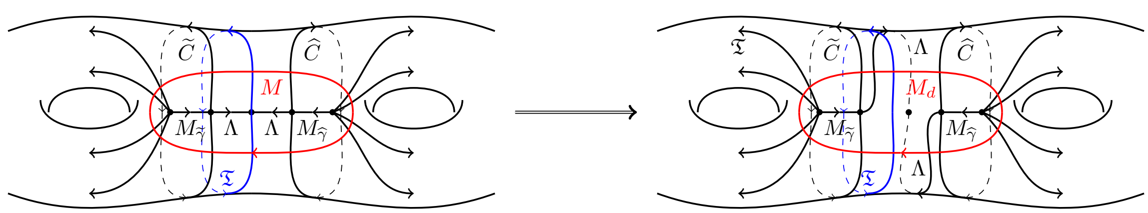

Figure 3.10: Action of a Dehn twist on the plumbing region.

Proposition 3.11.1. *A Dehn twist on the plumbing region acts on the log–canonical variable β_γ by:*

$$\beta_\gamma \mapsto \beta_\gamma + \ell_\gamma \ .$$

Proof. To see how the Dehn twist acts on the monodromies, consider the red loop corresponding to the trivial monodromy $M = \mathbb{I}_2$. Before the Dehn twist this monodromy (Figure 3.10 left) is given by

$$M = \widetilde{C}^{-1} M_{\widetilde{\gamma}^{-1}} \widetilde{C} \mathfrak{T} \widehat{C}^{-1} M_{\widehat{\gamma}^{-1}} \widehat{C} \mathfrak{T}^{-1} = \mathbb{I}_2 \ .$$

After the Dehn twist (Figure 3.10 right) it becomes

$$M_d = \widetilde{C}^{-1} M_{\widetilde{\gamma}^{-1}} \widetilde{C} \Lambda \mathfrak{T} \widehat{C}^{-1} M_{\widehat{\gamma}^{-1}} \widehat{C} \mathfrak{T}^{-1} \Lambda^{-1} = \mathbb{I}_2 \ .$$

Therefore, this twist only acts on $\mathfrak{T}$ by

$$\mathfrak{T} \mapsto \Lambda \mathfrak{T} \, .$$

Consequently, b_γ transforms as $b_\gamma \mapsto b_\gamma \lambda_\gamma$, which in logarithmic form gives $\beta_\gamma \mapsto \beta_\gamma + \ell_\gamma$. $\square$

For a closed geodesic γ, one can take the ratio of the real Fenchel-Nielsen twist parameter τ_γ over the length ℓ_γ. Then, the quantity (with some suitable choice of origin and orientation for τ_γ)

$$\theta_\gamma = 2\pi \, \frac{\tau_\gamma}{\ell_\gamma} \, ,$$

can be interpreted as an angle. In the present setting, the corresponding quantity is

$$2\pi \, \frac{\beta_\gamma}{\ell_\gamma} \, ,$$

where β_γ is the log-canonical coordinate conjugated to ℓ_γ. To describe the relation between β_γ and τ_γ, we consider next the real length-twist coordinates arising from the Fuchsian uniformization of a trinion.

3.11.2 Fuchsian uniformization of a trinion

Consider again a single trinion $\mathcal{T}$ with the triangulation shown in Figure 3.8. Take the Fuchsian representation $\rho : \pi_1(\mathcal{T}, p) \to SL(2, \mathbb{R})$ determined by the three hyperbolic monodromies whose explicit form is given by (3.48). The eigenvalues of M_j are labeled so that $\lambda_j > 1$. The edge variables z_{e_j}, $j = 1, 2, 3$, are now real and positive. They can still be expressed in terms of the eigenvalues of M_j via (3.47). The next lemma describes the fixed points of the Möbius transformations corresponding to the monodromies M_j in the upper half-plane $\mathbb{H}$.

Lemma 3.11.2. *For each monodromy M_j, let* $\mathrm{Fix}(M_j) = \{f_j^{(+)}, f_j^{(-)}\}$ *denote the pair of attracting and repelling fixed points of the associated Möbius transformation. Then*

$$\begin{aligned} \mathrm{Fix}(M_1) &= \left\{ \frac{(1-\lambda_1^2)\lambda_3}{\lambda_1\lambda_2 + \lambda_3}, 0 \right\} = \{f_1^{(+)}, 0\} \, , \\ \mathrm{Fix}(M_2) &= \left\{ \frac{\lambda_2\lambda_3 + \lambda_1}{\lambda_2\,(\lambda_1\lambda_2 + \lambda_3)}, 1 \right\} = \{f_2^{(+)}, 1\} \, , \\ \mathrm{Fix}(M_3) &= \left\{ \frac{\lambda_3(\lambda_2\lambda_3 + \lambda_1)}{\lambda_2\,(\lambda_3^2 - 1)}, \infty \right\} = \{f_3^{(+)}, \infty\} \, . \end{aligned} \qquad (3.54)$$

In addition, the fixed points satisfy

$$f_1^{(+)} < 0 < f_2^{(+)} < 1 < f_3^{(+)} .$$

Proof. The proof is obtained by direct inspection of the matrices (3.48) and recalling the running assumption that $\lambda_j > 1$. □

3.11.3 Gluing of two trinions and real Fenchel-Nielsen twist

Glue a trinion $\widetilde{\mathcal{T}}$ with monodromies $\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}$ (as in (3.48)) to another trinion $\widehat{\mathcal{T}}$ with monodromies $\widehat{M_1}, \widehat{M_2}, \widehat{M_3}$, defined in the same manner, along the axis of $\widetilde{M_3}$ (Figure 3.11). Let $\widetilde{C}_j$ (resp. $\widehat{C}_j$) denote the matrices diagonalizing $\widetilde{M_j}$ (resp. $\widehat{M_j}$).

Lemma 3.11.3. *Let $\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}$ be normalized monodromies as in (3.48) for a trinion $\widetilde{\mathcal{T}}$, and $\widehat{M_1}, \widehat{M_2}, \widehat{M_3}$ be similarly normalized monodromies for another trinion $\widehat{\mathcal{T}}$, with $\widetilde{\Lambda}_3 = \widehat{\Lambda}_3 = \Lambda$. Then the following $SL(2,\mathbb{R})$ matrices*

$$M_1 = \widetilde{M_1} , \qquad M_2 = \widetilde{M_2} , \qquad M_3 = \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{C}_3^{-1} \widehat{M_1} \widehat{C}_3 \mathfrak{T} \widetilde{C}_3^{-1} , \qquad M_4 = \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{C}_3^{-1} \widehat{M_2} \widehat{C}_3 \mathfrak{T} \widetilde{C}_3^{-1} ,$$

form a Fuchsian representation of $\pi_1(\mathcal{S}^2[\boldsymbol{\ell}], p)$, where $\mathcal{S}^2[\boldsymbol{\ell}]$ is a four-holed sphere with boundary lengths $\boldsymbol{\ell} = (\widetilde{\ell}_1, \widetilde{\ell}_2, \widehat{\ell}_1, \widehat{\ell}_2)$.

Proof. Diagonalize $\widetilde{M_3}$ (resp. $\widehat{M_3}$) by an upper-triangular matrix $\widetilde{C}_3$ (resp. $\widehat{C}_3$) so that $\widetilde{C}_3 \widetilde{\Lambda}_3 \widetilde{C}_3^{-1} = \widetilde{M_3}$ and $\widehat{C}_3 \widehat{\Lambda}_3 \widehat{C}_3^{-1} = \widehat{M_3}$. Impose $\widetilde{\Lambda}_3 = \widehat{\Lambda}_3 = \Lambda$. With M_1, M_2, M_3, M_4 defined above, one checks

$$\begin{aligned} M_1 M_2 M_3 M_4 &= \widetilde{M_1} \widetilde{M_2} \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{C}_3^{-1} \widehat{M_1} \widehat{M_2} \widehat{C}_3 \mathfrak{T} \widetilde{C}_3^{-1} \\ &= \widetilde{M_1} \widetilde{M_2} \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{C}_3^{-1} \widehat{M_3}^{-1} \widehat{C}_3 \mathfrak{T} \widetilde{C}_3^{-1} \\ &= \widetilde{M_1} \widetilde{M_2} \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{\Lambda}_3^{-1} \mathfrak{T} \widetilde{C}_3^{-1} \\ &= \widetilde{M_1} \widetilde{M_2} \widetilde{C}_3 \widehat{\Lambda}_3 \widetilde{C}_3^{-1} \\ &= \widetilde{M_1} \widetilde{M_2} \widetilde{C}_3 \widetilde{\Lambda}_3 \widetilde{C}_3^{-1} \\ &= \widetilde{M_1} \widetilde{M_2} \widetilde{M_3} = \mathbb{I}_2 . \end{aligned} \tag{3.55}$$

Hence, the set of generators $\{M_1, M_2, M_3, M_4\}$ defines a representation

$$\rho : \pi_1\big(\mathcal{S}^2[\boldsymbol{\ell}], p\big) \to SL(2,\mathbb{R}) ,$$

where $\mathcal{S}^2[\boldsymbol{\ell}]$ is a four-holed sphere with boundary lengths $\boldsymbol{\ell} = (\widetilde{\ell}_1, \widetilde{\ell}_2, \widehat{\ell}_1, \widehat{\ell}_2)$. Since all the matrices M_1, M_2, M_3, M_4 are hyperbolic and their axis in $\mathbb{H}$ are mutually disjoint, the subgroup of $SL(2,\mathbb{R})$ they generate is discrete and acts freely on $\mathbb{H}$ [31]. Therefore, the representation of $\pi_1(\mathcal{S}^2[\boldsymbol{\ell}], p)$ defined by these matrices is Fuchsian. $\square$

Remark 3.11.4. *Lemma 3.11.3 is essentially Klein's combination theorem [64] presented in a slightly different manner.*

Denote by $\mathrm{Ax}(\widetilde{M}_3)$ the axis of $\widetilde{M}_3$. Let γ be the separating contour on $\mathcal{S}^2[\boldsymbol{\ell}]$ given by $\gamma = \mathrm{Ax}(\widetilde{M}_3)/\langle\widetilde{M}_3\rangle$. It is a classical fact (see, e.g., [31, 64]) that the real Fenchel–Nielsen twist parameter τ_γ can be expressed as the hyperbolic distance along the axis of $\widetilde{M}_3$ between two distinguished points (see Fig. 3.11):

- the intersection point $\widetilde{p}_{FN}$ of the axis of $\widetilde{M}_3$ with the unique geodesic orthogonal to both the axis of $\widetilde{M}_3$ and the axis of M_2,
- the intersection point $\widetilde{p}'_{FN}$ of the axis of $\widetilde{M}_3$ with the unique geodesic orthogonal to both the axis of $\widetilde{M}_3$ and the axis of M_4.

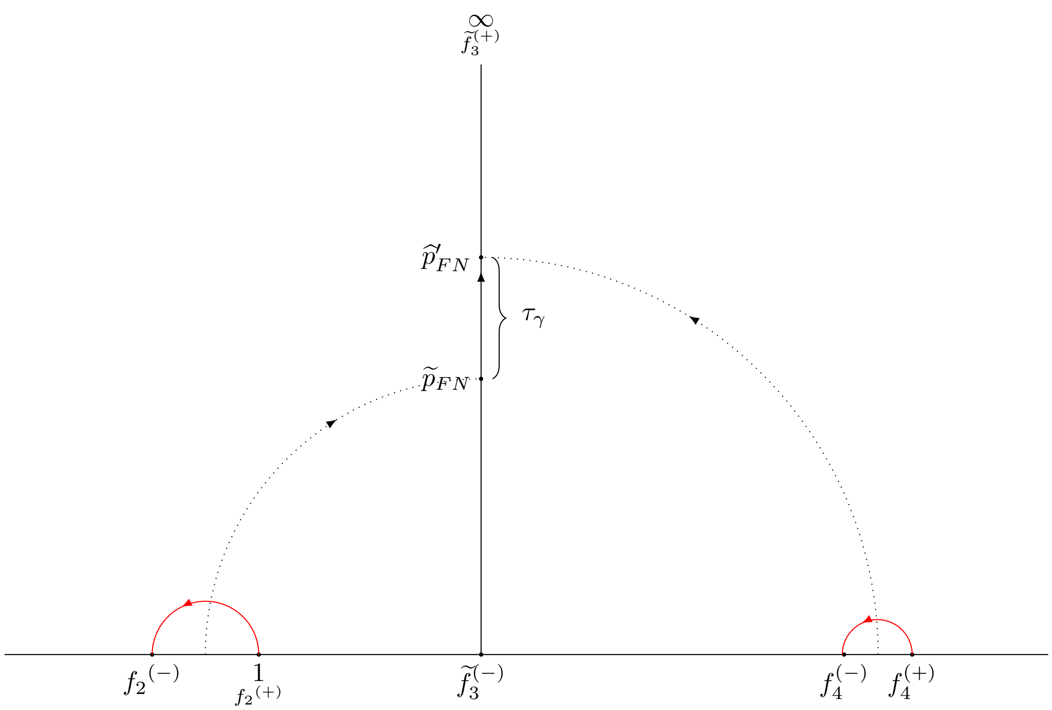

Figure 3.11: Axis of M_2, M_4 on $\mathbb{H}$ and Fenchel-Nielsen twist τ_γ.

Proposition 3.11.5. *For two trinions $\widetilde{\mathcal{T}}$ and $\widehat{\mathcal{T}}$ glued along the axis of $\widetilde{M_3}$, the real Fenchel–Nielsen twist τ_γ and the log-toric variable β_γ are related by*

$$\tau_\gamma = \beta_\gamma + \ln \Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \lambda) + \ln \Delta(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \lambda) \ , \tag{3.56}$$

where the rational function $\Delta(x, y; z)$ is given by:

$$\Delta(x, y; z) = \left[\frac{(1 + xyz)(y + xz)(x + yz)}{y^2(z^2 - 1)^2(z + xy)} \right]^{1/2} \ .$$

Proof. Consider for $\widetilde{\mathcal{T}}$ the unique geodesic on $\mathbb{H}$ orthogonal to both the axes of $\widetilde{M_3}$ and $\widetilde{M_2}$. Its intersection with the axis of $\widetilde{M_3}$ is (see Fig. 3.11)

$$\widetilde{p}_{FN} = \widetilde{f}_3^{(+)} + i\sqrt{\left(\widetilde{f}_3^{(+)} - \widetilde{f}_2^{(+)}\right)\left(\widetilde{f}_3^{(+)} - \widetilde{f}_2^{(-)}\right)} = \widetilde{f}_3^{(+)} + i\Delta(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \widetilde{\lambda}_3) \ .$$

For $\widehat{\mathcal{T}}$, before gluing, the geodesic mutually orthogonal to the axis of $\widehat{M_2}$ and the axis of $\widehat{M_3}$ intersects this same axis at:

$$\widehat{p}_{FN} = \widehat{f}_3^{(+)} + i\sqrt{\left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(+)}\right)\left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(-)}\right)} = \widehat{f}_3^{(+)} + i\Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \widehat{\lambda}_3) \ .$$

Upon gluing $\widetilde{\mathcal{T}}$ and $\widehat{\mathcal{T}}$ along the axis of $\widetilde{M_3}$, the generator $\widehat{M_2}$ is conjugated:

$$\widehat{M_2} \to M_4 = \widetilde{C}_3 \mathfrak{T}^{-1} \widehat{C}_3^{-1} \widehat{M_2} \widehat{C}_3 \mathfrak{T} \widetilde{C}_3^{-1} \ .$$

This conjugation maps the fixed points as

$$\widehat{f}_3^{(+)} \mapsto \widetilde{f}_3^{(+)} \ , \qquad \widehat{f}_2^{(+)} \mapsto f_4^{(+)} = \widetilde{f}_3^{(+)} + b_\gamma \left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(+)}\right)^{-1} \ ,$$
$$\widehat{f}_2^{(-)} \mapsto f_4^{(-)} = \widetilde{f}_3^{(+)} + b_\gamma \left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(-)}\right)^{-1} \ .$$

Hence,

$$\widehat{p}_{FN} \mapsto \widehat{p}'_{FN} = \widetilde{f}_3^{(+)} + \frac{ib_\gamma}{\sqrt{\left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(+)}\right)\left(\widehat{f}_3^{(+)} - \widehat{f}_2^{(-)}\right)}} = \widetilde{f}_3^{(+)} + ib_\gamma \Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \widehat{\lambda}_3)^{-1} \ .$$

By definition, the Fenchel–Nielsen twist is the following hyperbolic distance along the axis of

$\widetilde{M}_3$:

$$\tau_\gamma = \ell(\widetilde{p}_{FN}, \widehat{p}'_{FN}) = \ln\left[b_\gamma \Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \widehat{\lambda}_3)\Delta(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \widetilde{\lambda}_3)\right] = \beta_\gamma + \ln\Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \widehat{\lambda}_3) + \ln\Delta(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \widetilde{\lambda}_3)\ .$$

Imposing the constraint $\widetilde{\lambda}_3 = \lambda$ and $\widehat{\lambda}_3 = \lambda$ we get formula (3.56). □

Remark 3.11.6. *There is some arbitrariness in the choice of geodesics. We could also have taken the geodesic mutually orthogonal to the axis of $\widetilde{M}_1$ and $\widetilde{M}_3$ and the geodesic mutually orthogonal to the axis of M_3 and $\widetilde{M}_3$. The twist magnitude defined in this way would just differ from the one defined above by half a Dehn twist.*

3.11.4 Closing an handle : the non-separating case

In the previous subsection, we have only considered the case of gluing two *different* trinions. Let us now discuss the case where we *close a handle*, i.e., we transform a trinion $\mathcal{T}$ into a one-holed torus $\mathcal{T}^1[\ell_2]$ with boundary length $\ell_2 = \ln\lambda_2$, by identifying the geodesic boundaries corresponding to M_1 and M_3.
For monodromies (3.48), if we want to keep the diagonalizing matrices C_1 and C_3 lower and upper triangular respectively, with constant coefficients on the diagonal (this ensures that the coordinates $\{\beta_\gamma, \ell_\gamma, \ell_2\}$ are log-canonical for Ω in $\mathcal{M}_{1,1}[\ell_2]$, by (3.14)), we impose the constraint $\Lambda_1 = \Lambda_3^{-1} = \Lambda$. This condition is satisfied without permuting the eigenvalues. The corresponding b_γ-jump matrix $\mathfrak{B}$ must then be diagonal:

$$\mathfrak{B} = \begin{pmatrix} \sqrt{b_\gamma} & 0 \\ 0 & 1/\sqrt{b_\gamma} \end{pmatrix}\ .$$

The following lemma gives a Fuchsian representation of the one-holed torus $\mathcal{T}_{1,1}[\ell_2]$ under such a convention.

Lemma 3.11.7. *Let $\{M_1, M_2, M_3\}$ be monodromies as in (3.48), such that $\Lambda_1 = \Lambda_3^{-1} = \Lambda$. Define*

$$A = M_3\ , \qquad B = C_3\mathfrak{B}^{-1}C_1^{-1},$$

where C_1 (resp. C_3) diagonalizes M_1 (resp. M_3) and $\mathfrak{B}$ is the diagonal b-jump matrix. Then the pair $\{A, B\}$ defines a Fuchsian representation

$$\rho : \pi_1(\mathcal{T}^1[\ell_2], p) \to SL(2,\mathbb{R})\ ,$$

where $\mathcal{T}^1[\ell_2]$ is a one-holed torus with boundary length $\ell_2 = \ln\lambda_2$.

Proof. Set $A = M_3$ and $B = C_3\mathfrak{B}^{-1}C_1^{-1}$ as above, with the constraint $\Lambda_1 = \Lambda_3^{-1} = \Lambda$. Then one computes the commutator:

$$\begin{aligned} AB^{-1}A^{-1}B &= M_3C_1\mathfrak{B}^{-1}C_3^{-1}M_3^{-1}C_3\mathfrak{B}C_1^{-1} \\ &= M_3C_1\mathfrak{B}^{-1}\Lambda_3^{-1}\mathfrak{B}C_1^{-1} \\ &= M_3C_1\Lambda_1C_1^{-1} \\ &= M_3M_1 \\ &= M_2^{-1}. \end{aligned} \tag{3.57}$$

Hence, the matrices A, B satisfy the relation of the fundamental group of a one-holed torus

$$AB^{-1}A^{-1}B = M_2^{-1} \ ,$$

so they define a representation of $\pi^1(\mathcal{T}^1[\ell_2], p) \to SL(2, \mathbb{R})$. Again, since all the matrices A, B, M_2 are hyperbolic and their axes in $\mathbb{H}$ are disjoint, the subgroup of $SL(2, \mathbb{R})$ they generate is discrete and acts freely on $\mathbb{H}$. Hence the corresponding representation is Fuchsian. □

Let $\mathrm{Ax}(M_3)$ be the axis of M_3, and γ be the separating contour on $\mathcal{T}$ given by $\gamma = \mathrm{Ax}(M_3)/\langle M_3\rangle$. We have the following relation between the real Fenchel-Nielsen twist τ_γ and the log-toric variable β_γ when $\mathcal{T}$ is transformed into a one-holed torus $\mathcal{T}^1[\ell_2]$.

Proposition 3.11.8. *Let $\mathcal{T}$ be a trinion with monodromies (3.48). Transforming $\mathcal{T}$ into a one-holed torus $\mathcal{T}^1[\ell_2]$ by gluing along the axis of M_3, the real Fenchel-Nielsen twist τ_γ is related to the log-toric variable β_γ by*

$$\tau_\gamma = \beta_\gamma + \ln \Delta^2(\lambda, \lambda_2; \lambda) \ , \tag{3.58}$$

where

$$\Delta^2(\lambda, \lambda_2; \lambda) = \frac{(\lambda^2 + \lambda_2)(\lambda_2\lambda^2 + 1)}{(\lambda^2 - 1)^2\lambda_2^2} \ . \tag{3.59}$$

Proof. The real Fenchel-Nielsen twist τ_γ is defined as the hyperbolic distance between:

- the intersection point p_1 of the axis of M_3 with the unique geodesic mutually orthogonal to the axes of $M_2' = BM_2B^{-1}$ and M_3, and
- the intersection point p_2 of the axis of M_3 with the unique geodesic mutually orthogonal to the axes of M_2 and M_3.

The fixed points of the relevant hyperbolic generators are:

$$\begin{aligned}
\mathrm{Fix}(M_2) &= \left\{f_2^{(+)}, f_2^{(-)}\right\} = \{1/\lambda_2, 1\} \ , \\
\mathrm{Fix}(M_2') &= \left\{f_2^{(+)\prime}, f_2^{(-)\prime}\right\} \\
&= \left\{\frac{b_\gamma^2\lambda^2(\lambda^2\lambda_2+1)(\lambda_2+1)+\lambda_2(1-\lambda^2)^2}{b_\gamma^2\lambda_2(\lambda^2-1)(\lambda^2\lambda_2+1)}, \frac{b_\gamma^2\lambda^2(\lambda^2+\lambda_2)(\lambda_2+1)+\lambda_2(1-\lambda^2)^2}{b_\gamma^2(\lambda^2-1)(\lambda^2+\lambda_2)}\right\} \ , \\
\mathrm{Fix}(M_3) &= \left\{f_3^{(+)}, f_3^{(-)}\right\} = \left\{\frac{\lambda^2(\lambda_2+1)}{\lambda_2(\lambda^2-1)}, \infty\right\} \ .
\end{aligned}$$

Assuming $b_\gamma > 0$, one checks that

$$0 < f_2^{(+)} < 1 < f_3^{(+)} < f_2^{(+)\prime} < f_2^{(-)\prime} \ .$$

Hence, the coordinates of the intersection points are

$$p_1 = i\sqrt{\left(f_2^{(-)\prime} - f_3^{(+)}\right)\left(f_2^{(+)\prime} - f_3^{(+)}\right)} = i\sqrt{\frac{(\lambda^2+\lambda_2)(\lambda_2\lambda^2+1)}{\lambda_2^2(\lambda^2-1)^2}} \ ,$$

$$p_2 = i\sqrt{\left(f_2^{(-)} - f_3^{(+)}\right)\left(f_2^{(+)} - f_3^{(+)}\right)} = i\sqrt{\frac{(\lambda^2-1)^2}{b_\gamma^2(\lambda^2+\lambda_2)(\lambda_2\lambda^2+1)}} \ .$$

Therefore, the Fenchel-Nielsen twist is

$$\tau_\gamma = \ell(p_2, p_1) = \beta_\gamma + \ln\frac{(\lambda^2+\lambda_2)(\lambda_2\lambda^2+1)}{(\lambda^2-1)^2\lambda_2^2} = \beta_\gamma + \ln\Delta^2(\lambda, \lambda_2; \lambda) \ .$$

□

3.11.5 Generating function on $\mathcal{M}_{0,4}[\ell]$ and $\mathcal{M}_{1,1}[\ell_2]$

Consider first the moduli space of four-holed spheres with fixed boundary lengths $\boldsymbol{\ell} = (\widetilde{\ell}_1, \widetilde{\ell}_2, \widehat{\ell}_1, \widehat{\ell}_2)$. We denote this space by

$$\mathcal{M}_{0,4}[\boldsymbol{\ell}] \ .$$

The Weil–Petersson form ω_{WP} on $\mathcal{M}_{0,4}[\boldsymbol{\ell}]$ can be expressed in log-canonical coordinates $\{\ell_\gamma = \ln\lambda,\ \beta_\gamma = \ln b_\gamma\}$ as

$$\omega_{WP} = \mathrm{d}\beta_\gamma \wedge \mathrm{d}\ell_\gamma = \mathrm{d}(\beta_\gamma\, \mathrm{d}\ell_\gamma) \ . \tag{3.60}$$

In Fenchel–Nielsen coordinates $\{\ell_\gamma, \tau_\gamma\}$ its expression is [66]:

$$\omega_{WP} = \mathrm{d}\tau_\gamma \wedge \mathrm{d}\ell_\gamma = \mathrm{d}(\tau_\gamma \, \mathrm{d}\ell_\gamma) \ . \tag{3.61}$$

Subtracting (3.60) from (3.61), we obtain

$$\mathrm{d}((\tau_\gamma - \beta_\gamma)\, \mathrm{d}\ell_\gamma) = 0 \ ,$$

which implies the existence of a function $g_0(\lambda_0)$ such that

$$\frac{\tau_\gamma - \beta_\gamma}{\lambda} = \frac{\mathrm{d}g_0}{\mathrm{d}\lambda} \ .$$

Using Proposition 3.11.5, $g_0(\lambda)$ is given by the following antiderivative (integration constants are irrelevant)

$$\begin{aligned} g_0(\lambda) &= \int \frac{\ln \Delta(\widehat{\lambda}_1, \widehat{\lambda}_2; \lambda) + \ln \Delta(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \lambda)}{\lambda} \mathrm{d}\lambda \\ &= H(\widehat{\lambda}_1, \widehat{\lambda}_2; \lambda) + H(\widetilde{\lambda}_1, \widetilde{\lambda}_2; \lambda) - \mathrm{Li}_2(1 - \lambda^2) - \ln(\lambda^2) \ln(\lambda^2 - 1) \ , \end{aligned} \tag{3.62}$$

where we have introduced the auxiliary function

$$H(x, y; z) = -\frac{1}{2}\Big[\mathrm{Li}_2(-xyz) + \mathrm{Li}_2\Big(-\frac{xz}{y}\Big) + \mathrm{Li}_2\Big(-\frac{yz}{x}\Big) - \mathrm{Li}_2\Big(-\frac{z}{xy}\Big)\Big] - \ln(y)\ln(z) \ , \tag{3.63}$$

and $\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} \, \mathrm{d}t$ denotes Euler's dilogarithm. Note that all arguments of the dilogarithms in the expressions above are in $(0, 1)$ or negative since all the lambda's are greater than one, ensuring that the standard branch of Euler's dilogarithm $\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} \, \mathrm{d}t$ is well-defined and real-valued.

Now consider the moduli space $\mathcal{M}_{1,1}[\ell_2]$ of one-holed tori with fixed boundary length ℓ_2. The Weil–Petersson form ω_{WP} in log-canonical coordinates $\{\ell_\gamma, \beta_\gamma\}$ is

$$\omega_{WP} = \mathrm{d}\beta_\gamma \wedge \mathrm{d}\ell_\gamma \ ,$$

while in Fenchel–Nielsen coordinates $\{\ell, \tau\}$ it takes the form [65]:

$$\omega_{WP} = \mathrm{d}\tau_\gamma \wedge \mathrm{d}\ell_\gamma \ .$$

By the same reasoning as above, there exists a generating function $g_1(\lambda)$ satisfying

$$\frac{\tau_\gamma - \beta_\gamma}{\lambda} = \frac{\mathrm{d}g_1}{\mathrm{d}\lambda} \; .$$

According to Proposition 3.11.8, $g_1(\lambda)$ is given by

$$\begin{aligned} g_1(\lambda) &= \int \frac{\ln \Delta^2(\lambda, \lambda_2; \lambda)}{\lambda} \mathrm{d}\lambda \\ &= -\mathrm{Li}_2(-\lambda_2\lambda^2) - \mathrm{Li}_2(-\frac{\lambda^2}{\lambda_2}) - \mathrm{Li}_2(1-\lambda^2) - \ln\lambda^2 \ln(\lambda^2 - 1) - 2\ln\lambda_2 \ln\lambda \; . \end{aligned} \tag{3.64}$$

Remark 3.11.9. *The generating functions g_0 and g_1 are equal under the identification*

$$\lambda_0 = \widehat{\lambda}_1 = \widetilde{\lambda}_1 = \lambda \; , \qquad \widehat{\lambda}_2 = \widetilde{\lambda}_2 = \lambda_2 \; .$$

Therefore, the two expressions (3.62) and (3.64) are consistent.

3.11.6 Generating function on $\mathcal{M}_g$

Consider the moduli space $\mathcal{M}_g$ obtained by a complete trinion decomposition, together with its associated trinion graph Γ_{trin}. Recall that on an open dense subset of $\mathcal{M}_g$ the form Ω must coincide with the Weil-Petersson form ω_{WP}. One can compare the expression (3.51) for Ω in log-canonical coordinates with the one of ω_{WP} in Fenchel-Nielsen coordinates given by [66]:

$$\omega_{WP} = \sum_{j=1}^{3g-3} \mathrm{d}\tau_{\gamma_j} \wedge \mathrm{d}\ell_{\gamma_j} = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \left(\sum_{e \perp v} \mathrm{d}\tau_e \wedge \mathrm{d}\ell_e \right) \; . \tag{3.65}$$

In order to describe explicitly the symplectomorphism between the two sets of coordinates, one must distinguish three possible configurations of edges adjacent to given vertex $v^{(j)}$ of Γ_{trin}. Those three different configurations are shown in Figure 3.12. Let e_n, e_r, e_m be the three edges incident to $v^{(j)}$, with counter-clockwise ordering (e_n, e_r, e_m). When the vertex $v^{(j)}$ is in configuration (I) and (II), the edges e_n, e_r, e_m are called *standard edges*. When the vertex $v^{(j)}$ is in configuration (III), the edge e_m is a standard edge and e_n is called a *loop edge*.

Proposition 3.11.10. *Fixing the trinion graph Γ_{trin}, the Goldman form Ω (3.51) coincides with the Weil-Petersson form ω_{WP} (3.65) on an open dense subset of $\mathcal{M}_g$.*

Proof. The proof is by inspection of the contribution $\omega_{WP}^{v^{(j)}}$ of a vertex $v^{(j)}$ (for every possible

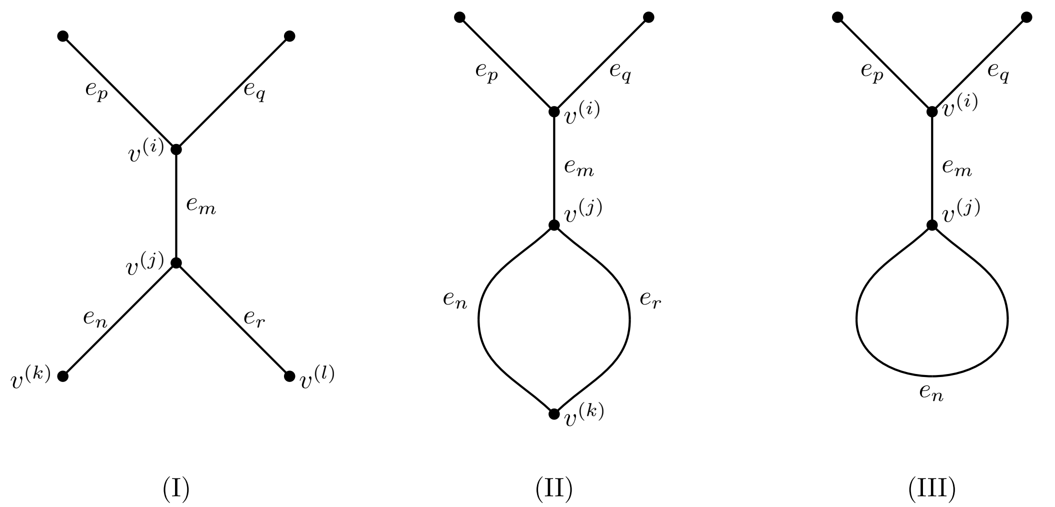

Figure 3.12: The three possible configurations of a vertex $v^{(j)}$ of the trinion graph Γ_{trin}.

configurations (I), (II), (III), shown in Figure 3.12) to the Weil-Petersson symplectic ω_{WP}. Consider first a vertex $v^{(j)}$ in configuration (I), linked to three other vertices $v^{(i)}, v^{(k)}, v^{(l)}$. Let $e_m = [v^{(j)}, v^{(i)}]$ be the standard edge corresponding to the gluing of the trinion $\mathcal{T}^{(j)}$ with $\mathcal{T}^{(i)}$.
According to Proposition 3.11.5, we have

$$\tau_{e_m} = \beta_{e_m} + \ln\Delta(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + \ln\Delta(\lambda_{e_q}, \lambda_{e_p}; \lambda_{e_m}) \ , \tag{3.66}$$

where the two edges $\{e_n, e_r\}$ with ordering (e_n, e_r, e_m) are incident to $v^{(j)}$, and the two edges $\{e_p, e_q\}$ with ordering (e_q, e_p, e_m) are incident to $v^{(i)}$ (see Figure 3.12, (I)).
Similarly, for the standard edges $e_n = [v^{(j)}, v^{(k)}]$ and $e_r = [v^{(j)}, v^{(l)}]$:

$$\tau_{e_n} = \beta_{e_n} + \ln\Delta(\lambda_{e_r}, \lambda_{e_m}; \lambda_{e_n}) + \ln\Delta(\lambda_{e_s}, \lambda_{e_t}; \lambda_{e_n}) \ , \tag{3.67}$$

$$\tau_{e_r} = \beta_{e_r} + \ln\Delta(\lambda_{e_m}, \lambda_{e_n}; \lambda_{e_r}) + \ln\Delta(\lambda_{e_v}, \lambda_{e_w}; \lambda_{e_r}) \ . \tag{3.68}$$

Only the first two terms in (3.66)–(3.68) correspond to the contribution of the vertex $v^{(j)}$. Inserting these expressions into (3.65) and keeping only the terms depending on $\lambda_{e_m}, \lambda_{e_n}, \lambda_{e_r}$, we get

$$
\begin{aligned}
\omega_{WP}^{v^{(j)}} &= \mathrm{d}\tau_{e_m} \wedge \mathrm{d}\ell_{e_m} + \mathrm{d}\tau_{e_n} \wedge \mathrm{d}\ell_{e_n} + \mathrm{d}\tau_{e_r} \wedge \mathrm{d}\ell_{e_r} \\
&= \mathrm{d}\beta_{e_m} \wedge \mathrm{d}\ell_{e_m} + \mathrm{d}\beta_{e_n} \wedge \mathrm{d}\ell_{e_n} + \mathrm{d}\beta_{e_r} \wedge \mathrm{d}\ell_{e_r} + \mathrm{d}\ell_{e_n} \wedge \mathrm{d}\ell_{e_r} + \\
&+ \mathrm{d}\ell_{e_r} \wedge \mathrm{d}\ell_{e_m} + \mathrm{d}\ell_{e_m} \wedge \mathrm{d}\ell_{e_n} \\
&= \sum_{\substack{e,\widetilde{e} \perp v^{(j)} \\ e<\widetilde{e}}} \left(\mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\widetilde{e}} + \mathrm{d}\beta_e \wedge \mathrm{d}\ell_e\right) \ .
\end{aligned}
$$

This calculation remains valid for a vertex $v^{(j)}$ in configuration (II), linked to only two other vertices $v^{(i)}$ and $v^{(k)}$ (Figure 3.12, (II)).
Let $v^{(j)}$ be linked to another vertex $v^{(i)}$ by a standard edge $e_m = [v^{(j)}, v^{(i)}]$ and to itself by a loop edge $e_n = [v^{(j)}, v^{(j)}]$ (Figure 3.12, (III)). Then using (3.56) and (3.58)

$$\tau_{e_m} = \beta_{e_m} + \ln \Delta(\lambda_{e_n}, \lambda_{e_n}; \lambda_{e_m}) + \ln \Delta(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) \ , \tag{3.69}$$

$$\tau_{e_n} = \beta_{e_n} + \ln \Delta^2(\lambda_{e_n}, \lambda_{e_m}; \lambda_{e_n}) \ . \tag{3.70}$$

Inserting (3.69)-(3.70) in (3.65) one can check the following

$$
\begin{aligned}
\omega_{WP}^{v^{(j)}} &= \mathrm{d}\tau_{e_m} \wedge \mathrm{d}\ell_{e_m} + \mathrm{d}\tau_{e_n} \wedge \mathrm{d}\ell_{e_n} \\
&= \mathrm{d}\beta_{e_m} \wedge \mathrm{d}\ell_{e_m} + \mathrm{d}\beta_{e_n} \wedge \mathrm{d}\ell_{e_n} \\
&= \sum_{\substack{e,\widetilde{e} \perp v^{(j)} \\ e<\widetilde{e}}} \left(\mathrm{d}\ell_e \wedge \mathrm{d}\ell_{\widetilde{e}} + \mathrm{d}\beta_e \wedge \mathrm{d}\ell_e\right) ,
\end{aligned}
\tag{3.71}
$$

since the first terms in the last sum cancel each other.
Therefore, summing over all the vertices of Γ_{trin}, taking all three configurations into account, forms (3.65) and (3.51) indeed coincide.

□

Consider a symplectic potential (or Liouville form) θ_{FN} for ω_{WP} in Fenchel-Nielsen coordinates such that $\mathrm{d}\theta_{FN} = \omega_{WP}$. A possible choice (see equation (3.65)) is:

$$\theta_{FN} = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{e \perp v} \tau_e \mathrm{d}\ell_e \ .$$

Similarly, for the same trinion graph, a symplectic potential in log-canonical coordinates (cf. equation (3.51)) can be written as

$$\theta_{LC} = \theta_{LC}^{(1)} + \theta_{LC}^{(2)} \ ,$$

$$\theta_{LC}^{(1)} = \frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{e \perp v} \beta_e \mathrm{d}\ell_e \,, \qquad \theta_{LC}^{(2)} = -\frac{1}{2} \sum_{v \in V(\Gamma_{trin})} \sum_{\substack{e, \tilde{e} \perp v \\ e < \tilde{e}}} \ell_{\tilde{e}} \mathrm{d}\ell_e \,.$$

Hence, there is a generating function $G(\Gamma_{trin})$ on $\mathcal{M}_g$ depending only on lengths $\ell(e)$ such that:

$$\theta_{FN} - \theta_{LC} = \sum_{v \in V(\Gamma_{trin})} \sum_{e \perp v} \frac{\partial G(\Gamma_{trin})}{\partial \ell_e} \mathrm{d}\ell_e \,. \tag{3.72}$$

The generating function $G(\Gamma_{trin})$ can be expressed as follows.

Theorem 3.11.11. *Given the edges configurations shown in Figure 3.12, the generating function $G(\Gamma_{trin})$ can be written as:*

$$G(\Gamma_{trin}) = \frac{1}{2} \sum_{e_m = [v^{(j)}, v^{(i)}]} \Big[g_0^{(j)}(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) \Big] + \frac{1}{2} \sum_{e_n = [v^{(j)}, v^{(j)}]} g_1^{(j)}(\lambda_{e_m}; \lambda_{e_n}) \,, \tag{3.73}$$

where standard edges contributions are given by

$$\begin{aligned} g_0^{(j)}(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) &= -\frac{1}{2} \Big[\Phi(\lambda_{e_n}, \lambda_{e_r}, \lambda_{e_m}) + \mathrm{Li}_2(1 - \lambda_{e_m}^2) + \ln(\lambda_{e_m}^2) \ln(\lambda_{e_m}^2 - 1) \Big] \,, \\ g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) &= -\frac{1}{2} \Big[\Phi(\lambda_{e_p}, \lambda_{e_q}, \lambda_{e_m}) + \mathrm{Li}_2(1 - \lambda_{e_m}^2) + \ln(\lambda_{e_m}^2) \ln(\lambda_{e_m}^2 - 1) \Big] \,, \end{aligned}$$

and loop edges contributions by

$$g_1^{(j)}(\lambda_{e_m}; \lambda_{e_n}) = -\Phi(\lambda_{e_n}, \lambda_{e_m}, \lambda_{e_n}) - \mathrm{Li}_2(1 - \lambda_{e_n}^2) - \ln(\lambda_{e_n}^2) \ln(\lambda_{e_n}^2 - 1) - \ln \lambda_{e_m} \ln \lambda_{e_n} + \frac{\ln^2 \lambda_{e_n}}{2} \,.$$

Here the auxiliary function Φ is defined as

$$\Phi(x, y, z) = \mathrm{Li}_2(-xyz) + \mathrm{Li}_2\Big(-\frac{xz}{y} \Big) + \mathrm{Li}_2\Big(-\frac{yz}{x} \Big) - \mathrm{Li}_2\Big(-\frac{z}{xy} \Big) \,.$$

Proof. We inspect the contribution of each edge to $G(\Gamma_{trin})$.
Let $v^{(j)}$ be in configuration (I) (see Figure 3.12). Consider the standard edge $e_m = [v^{(j)}, v^{(i)}]$. Recall the ordering of edges (e_n, e_r, e_m) at $v^{(j)}$ and (e_q, e_p, e_m) at $v^{(i)}$. We want to write the term in (3.72) corresponding to the edge e_m. This term can be splitted into two. Fixing $e = e_m$ in the difference $\theta_{FN} - \theta_{LC}^{(1)}$, and using Proposition 3.11.5, we obtain

$$(\tau_{e_m} - \beta_{e_m}) \mathrm{d} \ln \lambda_{e_m} = \frac{\ln \Delta(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + \ln \Delta(\lambda_{e_q}, \lambda_{e_p}; \lambda_{e_m})}{\lambda_{e_m}} \mathrm{d}\lambda_{e_m} \,. \tag{3.74}$$

Integrating expression (3.74) we get

$$\int \frac{\ln \Delta(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + \ln \Delta(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m})}{\lambda_{e_m}} \mathrm{d}\lambda_{e_m} = -\frac{1}{2} \left[\Phi(\lambda_{e_n}, \lambda_{e_r}, \lambda_{e_m}) + \Phi(\lambda_{e_p}, \lambda_{e_q}, \lambda_{e_m})\right] - \\ - \mathrm{Li}_2(1 - \lambda_{e_m}^2) - \ln(\lambda_{e_m}^2) \ln(\lambda_{e_m}^2 - 1) - \ln \lambda_{e_n} \ln \lambda_{e_m} - \ln \lambda_{e_q} \ln \lambda_{e_m} \ . \tag{3.75}$$

Now, fixing $e = e_m$ in $-\theta^{(2)}_{LC}$ we have

$$\sum_{\substack{\widetilde{e} \perp v^{(j)} \\ e_m < \widetilde{e}}} \ell_{\widetilde{e}} \, \mathrm{d}\ell_{e_m} + \sum_{\substack{\widetilde{e} \perp v^{(i)} \\ e_m < \widetilde{e}}} \ell_{\widetilde{e}} \, \mathrm{d}\ell_{e_m} = (\ln \lambda_{e_n} + \ln \lambda_{e_q}) \mathrm{d} \ln \lambda_{e_m} \ . \tag{3.76}$$

Integrating expression (3.76) and summing with (3.75) we recover

$$g_0^{(j)}(\lambda_{e_n}, \lambda_{e_r}; \lambda_{e_m}) + g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) \ .$$

This computation remains unmodified for $v^{(j)}$ in configuration (II).
Let $v^{(j)}$ be in configuration (III). Setting $\lambda_{e_r} = \lambda_{e_n}$ in the preceding computation, we obtain that the standard edge e_m contributes to the generating function by

$$g_0^{(j)}(\lambda_{e_n}, \lambda_{e_n}; \lambda_{e_m}) + g_0^{(i)}(\lambda_{e_p}, \lambda_{e_q}; \lambda_{e_m}) \ .$$

Let us finally consider the term of (3.72) corresponding to the loop edge e_n itself. Due to Proposition 3.11.8, we have:

$$(\tau_{e_n} - \beta_{e_n}) \frac{\mathrm{d}\lambda_{e_n}}{\lambda_{e_n}} = \frac{\ln \Delta^2(\lambda_{e_n}, \lambda_{e_m}; \lambda_{e_n})}{\lambda_{e_n}} \mathrm{d}\lambda_{e_n} \ ,$$

which, after integration with respect to λ_{e_n}, gives

$$\int \frac{\ln \Delta^2(\lambda_{e_n}, \lambda_{e_m}; \lambda_{e_n})}{\lambda_{e_n}} \mathrm{d}\lambda_{e_n} = -\Phi(\lambda_{e_n}, \lambda_{e_m}, \lambda_{e_n}) - \mathrm{Li}_2(1-\lambda_{e_n}^2) - \ln(\lambda_{e_n}^2) \ln(\lambda_{e_n}^2 - 1) - 2 \ln \lambda_{e_m} \ln \lambda_{e_n} \ . \tag{3.77}$$

Fixing $e = e_n$ in $-\theta^{(2)}_{FG}$ in this situation we obtain:

$$\sum_{\substack{\widetilde{e} \perp v^{(j)} \\ e_m < \widetilde{e}}} \ell_{\widetilde{e}} \, \mathrm{d}\ell_{e_m} = (\ln \lambda_{e_n} + \ln \lambda_{e_m}) \mathrm{d} \ln \lambda_{e_n} \ . \tag{3.78}$$

Integrating (3.78) and summing with (3.77) we get $g_1(\lambda_{e_m}; \lambda_{e_n})$.
Summing over all edges in Γ_{trin} we recover expression (3.73) for $G(\Gamma_{\mathrm{trin}})$. □

3.12 Higher $SL(N)$ groups

The generalization of our construction to an arbitrary $SL(N)$ character variety (in both complex and real cases) of $\pi_1(\mathcal{C})$ for compact genus g Riemann surfaces is pretty straightforward if one uses the system of higher rank Fock-Goncharov coordinates [21]. Technically, in the higher $SL(N)$ case the jumps matrices on edges S_e become anti-diagonal; each such matrix contains $N-1$ coordinates while matrices A also contain some of the coordinates (one coordinate for $SL(3)$ case, three for $SL(4)$ etc) [7]. The logarithms of eigenvalues of the monodromy matrices around the separating contours and the logarithms of the ratios of the corresponding toric variables form an additional set of $N-1$ conjugated pairs of coordinates for the Goldman symplectic form. The technical details for an arbitrary $SL(N)$ case are very similar to [7].

3.13 Conclusion and future research directions

In this chapter, we have constructed explicitly new systems of log-canonical coordinates on $SL(2,\mathbb{C})$ character varieties of compact Riemann surfaces as combinations of shear type coordinates and length-twist type coordinates. For complete trinion decomposition characterized by a chosen trinion graph Γ_{trin}, the relationship to Fenchel-Nielsen coordinates on $\mathcal{M}_g$ has been established through a generating function Γ_{trin}. These new systems of log-canonical coordinates as well as generating functions may find several potential applications.

First, consider a one-holed torus $\mathcal{T}^1[\ell_2]$ as in section 11.4, together with some choice of $SL(2,\mathbb{R})$ monodromies $\{A,B\}$ satisfying $AB^{-1}A^{-1}B = M_2$. Consider the Hamiltonian of the two-body Ruijsenaars system given by (cf. [58] [28], [51] equation (42)):

$$H(p,q) = 2\cosh q\sqrt{\frac{4\sinh^2 p - m - 2}{4\sinh^2 p - 4}}\ ,$$

according to the authors of [51], this Hamiltonian coincides with $\mathrm{tr}A$ on $\mathcal{M}_{1,1}[\ell_2]$ (the parameter m is chosen such that $\mathrm{tr}M_2 = m$). The action-angle coordinates in their work satisfy $\{p,q\}_G = 1$, so they must coincide with Fenchel-Nielsen coordinates (maybe with a different convention for the twist τ_γ). Thus, the generating function $g_1(\lambda)$ given by expression (3.64) should be related to the Hamilton-Jacobi action generating the canonical transformation $(\ell_\gamma = \ln\lambda, \beta_\gamma) \mapsto (q,p)$. This correspondence may be generalized to $\mathcal{M}_g$, for which the generating function $G(\Gamma_{trin})$ could probably be interpreted as a Hamilton-Jacobi action for some $SL(2,\mathbb{R})$ Hitchin-type integrable system.

Then we have also remarked that for $\mathcal{V}_2$, choosing the trinion graph to be the "theta graph" (see Figure 3.15, left, in Appendix), the generating function $G(\widetilde{\Gamma}_{trin})$ given by (3.82) closely resembles the 'Neumann-Zagier potential' (cf equation (19) in [50]). The possible connection between the generating function $G(\Gamma_{trin})$ and classical limits of quantum invariants on trivalent graphs is unexpected and remains at this stage conjectural. The question also remains open whether the combination of dilogarithms appearing in $G(\Gamma_{trin})$ has an interpretation in terms of hyperbolic volumes.

Finally, there exist some analogues of the Fenchel-Nielsen coordinates for $SL(3,\mathbb{C})$ character varieties that have been studied very recently [54]. It would be interesting to relate this construction to our framework applied to such a case.

3.14 Appendix

In this appendix we explicitly compute the forms of Theorem 3.7.1 and Theorem 3.8.1 on $\mathcal{M}_2$ (with one, two and three contours) and on $\mathcal{M}_3$ (with one, two, three and four contours). For complete trinion decompositions some explicit expressions of the generating function of Theorem 3.11.11 are also given.

3.14.1 $\mathcal{M}_2$

One contour

For $\mathcal{M}_2$ the idea is to select a separating contour γ which cuts a double-torus into two one-holed tori $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$, endowed with graphs $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ as shown in Figure 3.13 (the internal edges and the contour γ are not represented), whose associated shear type coordinates are $\{\widetilde{\zeta}_{e_1}, \widetilde{\zeta}_{e_2}, \widetilde{\zeta}_{e_3}\}$ and $\{\widehat{\zeta}_{e_1}, \widehat{\zeta}_{e_2}, \widehat{\zeta}_{e_3}\}$, respectively.

Proposition 3.14.1. *In log-canonical coordinates $\{\widetilde{\zeta}_{e_2}, \widetilde{\zeta}_{e_3}, \widehat{\zeta}_{e_2}, \widehat{\zeta}_{e_3}, \ell_\gamma, \beta_\gamma\}$ the Weil-Petersson form on $\mathcal{M}_2$ can be written :*

$$\omega_{WP} = 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_3}+2\mathrm{d}\widehat{\zeta}_{e_2}\wedge\mathrm{d}\widehat{\zeta}_{e_3}+\mathrm{d}\ell_\gamma\wedge\left(\mathrm{d}\widetilde{\zeta}_{e_2}+\mathrm{d}\widetilde{\zeta}_{e_3}\right)+\mathrm{d}\ell_\gamma\wedge\left(\mathrm{d}\widehat{\zeta}_{e_2}+\mathrm{d}\widehat{\zeta}_{e_3}\right)+\mathrm{d}\beta_\gamma\wedge\mathrm{d}\ell_\gamma\ . \quad (3.79)$$

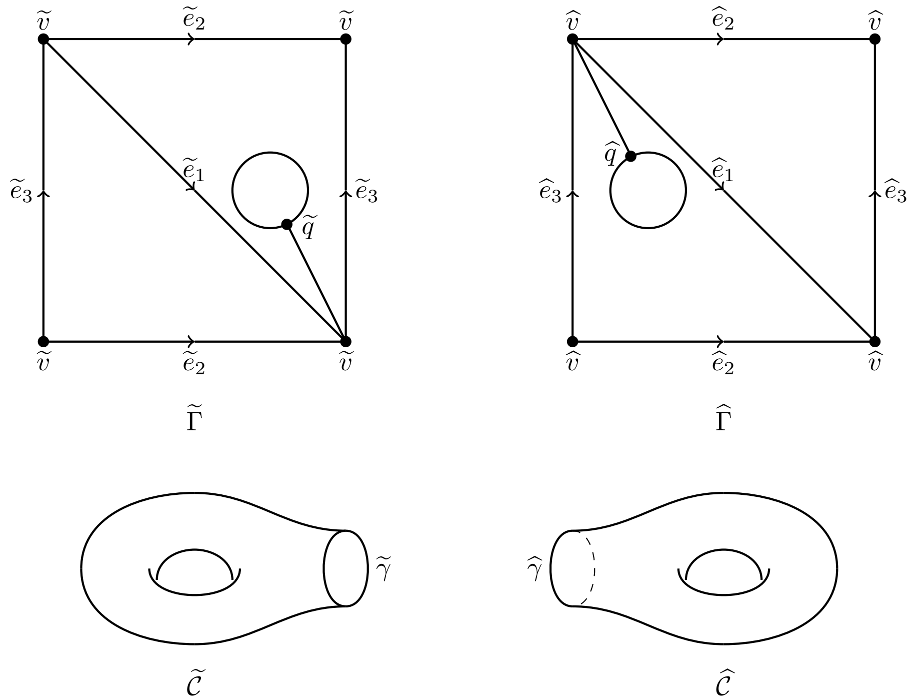

Figure 3.13: Single contour splitting for $\mathcal{M}_2$ and associated graphs.

Or, equivalently, the Poisson structure is given by the following Poisson tensor:

$$\mathbb{P}_{ij} = \begin{bmatrix} 0 & -2 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Proof. Using equation (3.40) for the vertex $\widetilde{v}$ of $\widetilde{\Gamma}$ in Figure 3.13, together with the twist-length type contribution $2\mathrm{d}\beta_{\widetilde{\gamma}} \wedge \mathrm{d}\ell_{\widetilde{\gamma}}$ coming from $\widetilde{q}$, without any gluing constraint, we get:

$$\Omega(\widetilde{\Gamma}) = 2\mathrm{d}\widetilde{\zeta}_{e_1} \wedge \mathrm{d}\widetilde{\zeta}_{e_2} + 2\mathrm{d}\widetilde{\zeta}_{e_1} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + 2\mathrm{d}\widetilde{\zeta}_{e_2} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + 2\mathrm{d}\beta_{\widetilde{\gamma}} \wedge \mathrm{d}\ell_{\widetilde{\gamma}}.$$

Similarly, using (3.41) for the vertex $\widehat{v}$ of $\widehat{\Gamma}$, together with the twist-length type contribution

$2\mathrm{d}\beta_{\widehat{\gamma}} \wedge \mathrm{d}\ell_{\widehat{\gamma}}$ coming from $\widehat{q}$, we obtain:

$$\Omega(\widehat{\Gamma}) = 2\mathrm{d}\widehat{\zeta}_{e_1} \wedge \mathrm{d}\widehat{\zeta}_{e_2} + 2\mathrm{d}\widehat{\zeta}_{e_1} \wedge \mathrm{d}\widehat{\zeta}_{e_3} + 2\mathrm{d}\widehat{\zeta}_{e_2} \wedge \mathrm{d}\widehat{\zeta}_{e_3} + 2\mathrm{d}\beta_{\widehat{\gamma}} \wedge \mathrm{d}\ell_{\widehat{\gamma}}.$$

Therefore we have:

$$\begin{cases} \widetilde{c}_{12} = \widetilde{c}_{13} = \widetilde{c}_{23} = 2 \\ \widehat{c}_{12} = \widehat{c}_{13} = \widehat{c}_{23} = 2 \end{cases}$$

Hence we get ; $\widetilde{c}_{23} - \widetilde{c}_{13} + \widetilde{c}_{12} = 2$, $\widehat{c}_{23} - \widehat{c}_{13} + \widehat{c}_{12} = 2$, so applying Theorem 3.7.1 equation (3.79) follows. □

Two contours

Here one of the contour, γ_2, is non-separating. The resulting pieces $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ by cutting along γ_1 and γ_2 are a one-holed torus and a three-holed sphere (i.e. a trinion) respectively, see Figure 3.14. We can associate to $\widetilde{\mathcal{C}}$, endowed with $\widetilde{\Gamma}$, shear-type coordinates $\{\widetilde{\zeta}_{e_1}, \widetilde{\zeta}_{e_2}, \widetilde{\zeta}_{e_3}\}$ whereas $\widehat{\mathcal{C}}$, with $\widehat{\Gamma}$, comes with shear-type coordinates $\{\widehat{\zeta}_{e_1}, \widehat{\zeta}_{e_2}, \widehat{\zeta}_{e_3}\}$, but all of them can be replaced by length-twist coordinates $\{\ell_{\gamma_1}, \ell_{\gamma_2}, \beta_{\gamma_1}, \beta_{\gamma_2}\}$ due to the constraints.

Proposition 3.14.2. *In log-canonical coordinates $\{\widetilde{\zeta}_{e_2}, \widetilde{\zeta}_{e_3}, \ell_{\gamma_1}, \ell_{\gamma_2}, \beta_{\gamma_1}, \beta_{\gamma_2}\}$ the Weil-Petersson form on $\mathcal{M}_2$ can be written :*

$$\omega_{WP} = 2\mathrm{d}\widetilde{\zeta}_{e_2} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + \mathrm{d}\ell_{\gamma_1} \wedge \left(\mathrm{d}\widetilde{\zeta}_{e_2} + \mathrm{d}\widetilde{\zeta}_{e_3}\right) + \mathrm{d}\beta_{\gamma_1} \wedge \mathrm{d}\ell_{\gamma_1} + \mathrm{d}\beta_{\gamma_2} \wedge \mathrm{d}\ell_{\gamma_2}\ .$$

Proof. As in the previous example we still have (the contribution from q_1^+ is now given by $2\mathrm{d}\beta_{\widetilde{\gamma}_1} \wedge \mathrm{d}\ell_{\widetilde{\gamma}_1}$):

$$\Omega(\widetilde{\Gamma}) = 2\mathrm{d}\widetilde{\zeta}_{e_1} \wedge \mathrm{d}\widetilde{\zeta}_{e_2} + 2\mathrm{d}\widetilde{\zeta}_{e_1} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + 2\mathrm{d}\widetilde{\zeta}_{e_2} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + 2\mathrm{d}\beta_{\widetilde{\gamma}_1} \wedge \mathrm{d}\ell_{\widetilde{\gamma}_1}\ , \tag{3.80}$$

whereas equation (3.41) applied to the vertices v_1^-, v_2^-, v_2^+ of $\widehat{\Gamma}$ (together with twist-length type contributions coming from q_1^-, q_2^-, q_2^+) gives:

$$\Omega(\widehat{\Gamma}) = \mathrm{d}\widehat{\zeta}_{e_3} \wedge \mathrm{d}\widehat{\zeta}_{e_1} + \mathrm{d}\widehat{\zeta}_{e_1} \wedge \mathrm{d}\widehat{\zeta}_{e_2} + \mathrm{d}\widehat{\zeta}_{e_2} \wedge \mathrm{d}\widehat{\zeta}_{e_3} + 2\mathrm{d}\beta_{\widehat{\gamma}_1} \wedge \mathrm{d}\ell_{\widehat{\gamma}_1} + 2\mathrm{d}\beta_{\widehat{\gamma}_2} \wedge \mathrm{d}\ell_{\widehat{\gamma}_2} + 2\mathrm{d}\beta_{\widehat{\gamma}_3} \wedge \mathrm{d}\ell_{\widehat{\gamma}_3}\ . \tag{3.81}$$

The constraints for γ_1 reads:

$$\begin{aligned} 2(\widetilde{\zeta}_{e_1} + \widetilde{\zeta}_{e_2} + \widetilde{\zeta}_{e_3}) &= \ell_{\widetilde{\gamma}_1}\ , \\ \widehat{\zeta}_{e_1} + \widehat{\zeta}_{e_3} &= \ell_{\widehat{\gamma}_1}\ , \\ \ell_{\widetilde{\gamma}_1} = \ell_{\widehat{\gamma}_1} &= \ell_{\gamma_1}\ , \end{aligned}$$

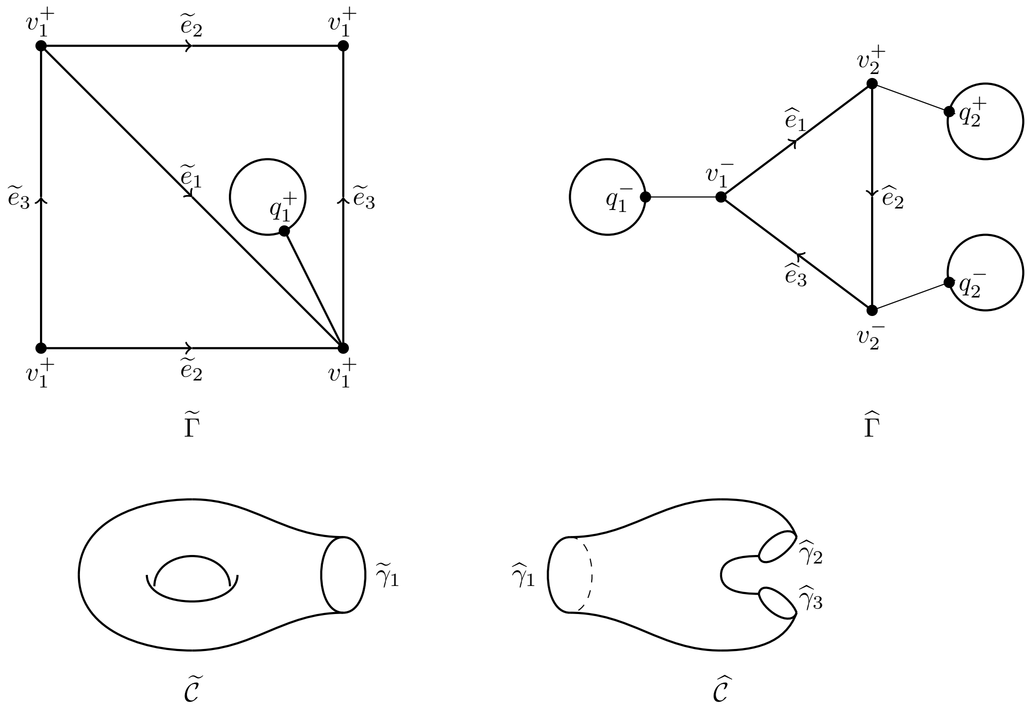

Figure 3.14: Two contour splitting for $\mathcal{M}_2$ and associated graphs.

and the constraints for the non-separating contour γ_2 are given by:

$$\begin{aligned}
\widehat{\zeta}_{e_1} + \widehat{\zeta}_{e_2} &= \ell_{\widehat{\gamma}_2} \ , \\
\widehat{\zeta}_{e_3} + \widehat{\zeta}_{e_2} &= \ell_{\widehat{\gamma}_3} \ , \\
\ell_{\widehat{\gamma}_2} = \ell_{\widehat{\gamma}_3} &= \ell_{\gamma_2} \ .
\end{aligned}$$

Hence:

$$\begin{aligned}
&\widetilde{\zeta}_{e_1} = \frac{1}{2}\ell_{\gamma_1} - \widetilde{\zeta}_{e_2} - \widetilde{\zeta}_{e_3} \ , \qquad \widehat{\zeta}_{e_2} = \ell_{\gamma_2} - \frac{1}{2}\ell_{\gamma_1} \ , \\
&\widehat{\zeta}_{e_1} = \frac{1}{2}\ell_{\gamma_1} \ , \qquad \widehat{\zeta}_{e_3} = \frac{1}{2}\ell_{\gamma_1} \ .
\end{aligned}$$

Inserting those constraints in (3.80) and (3.81) we obtain:

$$\begin{aligned}
\Omega(\widetilde{\Gamma}) &= 2\mathrm{d}\widetilde{\zeta}_{e_2} \wedge \mathrm{d}\widetilde{\zeta}_{e_3} + \mathrm{d}\ell_{\gamma_1} \wedge \left(\mathrm{d}\widetilde{\zeta}_{e_2} + \mathrm{d}\widetilde{\zeta}_{e_3}\right) + 2\mathrm{d}\beta_{\widetilde{\gamma}_1} \wedge \mathrm{d}\ell_{\gamma_1} \ , \\
\Omega(\widehat{\Gamma}) &= 2\mathrm{d}\beta_{\widehat{\gamma}_1} \wedge \mathrm{d}\ell_{\gamma_1} + 2\mathrm{d}\beta_{\widehat{\gamma}_2} \wedge \mathrm{d}\ell_{\gamma_2} + 2\mathrm{d}\beta_{\widehat{\gamma}_3} \wedge \mathrm{d}\ell_{\gamma_2} \ .
\end{aligned}$$

Summing up $\Omega(\widetilde{\Gamma})$ and $\Omega(\widehat{\Gamma})$ while setting $\beta_{\gamma_1} = 2(\beta_{\widetilde{\gamma}_1} + \beta_{\widehat{\gamma}_1})$ and $\beta_{\gamma_2} = 2(\beta_{\widehat{\gamma}_2} + \beta_{\widehat{\gamma}_3})$ we get the expression of Proposition 3.14.2. $\square$

Three contours

In this situation, we get complete decompositions into trinions. The two possible trinion graphs $\widetilde{\Gamma}_{trin}$ and $\widehat{\Gamma}_{trin}$ are shown in Figure 3.15. The log-canonical coordinates associated to $\widetilde{\Gamma}_{trin}$ (resp. $\widehat{\Gamma}_{trin}$) are denoted $\{\ell_{\widetilde{e}_i}, \beta_{\widetilde{e}_i}\}_{i=1}^3$ (resp. $\{\ell_{\widehat{e}_i}, \beta_{\widehat{e}_i}\}_{i=1}^3$).

Proposition 3.14.3. *The Weil-Petersson form ω_{WP} on $\mathcal{M}_2$ coincides with the following forms in coordinates $\{\ell_{\widetilde{e}_i}, \beta_{\widetilde{e}_i}\}_{i=1}^3$, $\{\ell_{\widehat{e}_i}, \beta_{\widehat{e}_i}\}_{i=1}^3$:*

- $\Omega(\widetilde{\Gamma}_{trin}) = \mathrm{d}\beta_{\widetilde{e}_1} \wedge \mathrm{d}\ell_{\widetilde{e}_1} + \mathrm{d}\beta_{\widetilde{e}_2} \wedge \mathrm{d}\ell_{\widetilde{e}_2} + \mathrm{d}\beta_{\widetilde{e}_3} \wedge \mathrm{d}\ell_{\widetilde{e}_3}$,

- $\Omega(\widehat{\Gamma}_{trin}) = \mathrm{d}\beta_{\widehat{e}_1} \wedge \mathrm{d}\ell_{\widehat{e}_1} + \mathrm{d}\beta_{\widehat{e}_2} \wedge \mathrm{d}\ell_{\widehat{e}_2} + \mathrm{d}\beta_{\widehat{e}_3} \wedge \mathrm{d}\ell_{\widehat{e}_3}$.

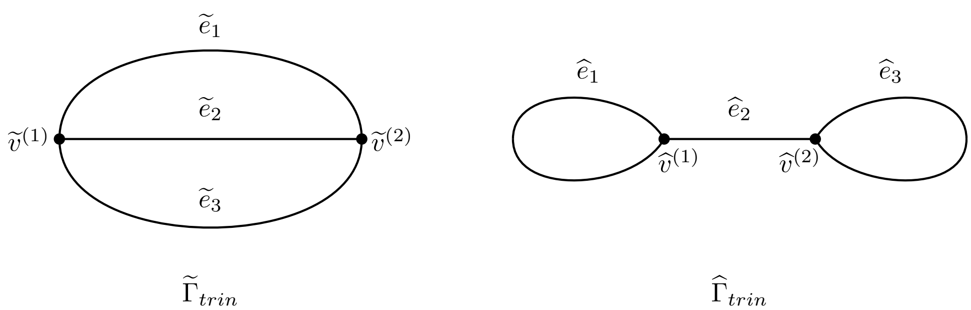

Figure 3.15: The two trinion graphs $\widetilde{\Gamma}_{trin}$ and $\widehat{\Gamma}_{trin}$.

Proof. The proof is by direct application of Theorem 3.9.2 to $\widetilde{\Gamma}_{trin}$ and $\widehat{\Gamma}_{trin}$.
Using formula (3.51) for $\widetilde{\Gamma}_{trin}$ one can get:

$$\Omega(\widetilde{\Gamma}_{trin}) = \frac{1}{2}\Big[\mathrm{d}\ell_{\widetilde{e}_3} \wedge \mathrm{d}\ell_{\widetilde{e}_2} + \mathrm{d}\ell_{\widetilde{e}_1} \wedge \mathrm{d}\ell_{\widetilde{e}_3} + \mathrm{d}\ell_{\widetilde{e}_2} \wedge \mathrm{d}\ell_{\widetilde{e}_1}\Big] + \frac{1}{2}\Big[\mathrm{d}\ell_{\widetilde{e}_1} \wedge \mathrm{d}\ell_{\widetilde{e}_2} + \\ + \mathrm{d}\ell_{\widetilde{e}_2} \wedge \mathrm{d}\ell_{\widetilde{e}_3} + \mathrm{d}\ell_{\widetilde{e}_3} \wedge \mathrm{d}\ell_{\widetilde{e}_1}\Big] + \mathrm{d}\beta_{\widetilde{e}_1} \wedge \mathrm{d}\ell_{\widetilde{e}_1} + \mathrm{d}\beta_{\widetilde{e}_2} \wedge \mathrm{d}\ell_{\widetilde{e}_2} + \mathrm{d}\beta_{\widetilde{e}_3} \wedge \mathrm{d}\ell_{\widetilde{e}_3} \\ = \mathrm{d}\beta_{\widetilde{e}_1} \wedge \mathrm{d}\ell_{\widetilde{e}_1} + \mathrm{d}\beta_{\widetilde{e}_2} \wedge \mathrm{d}\ell_{\widetilde{e}_2} + \mathrm{d}\beta_{\widetilde{e}_3} \wedge \mathrm{d}\ell_{\widetilde{e}_3}\ .$$

Similarly, for $\widehat{\Gamma}_{trin}$:

$$\Omega(\widehat{\Gamma}_{trin}) = \frac{1}{2}\Big[\mathrm{d}\ell_{\widehat{e}_2} \wedge \mathrm{d}\ell_{\widehat{e}_1} + \mathrm{d}\ell_{\widehat{e}_1} \wedge \mathrm{d}\ell_{\widehat{e}_1} + \mathrm{d}\ell_{\widehat{e}_1} \wedge \mathrm{d}\ell_{\widehat{e}_2}\Big] + \frac{1}{2}\Big[\mathrm{d}\ell_{\widehat{e}_2} \wedge \mathrm{d}\ell_{\widehat{e}_3} +$$
$$\mathrm{d}\ell_{\widehat{e}_3} \wedge \mathrm{d}\ell_{\widehat{e}_3} + \mathrm{d}\ell_{\widehat{e}_3} \wedge \mathrm{d}\ell_{\widehat{e}_2}\Big] + \mathrm{d}\beta_{\widehat{e}_1} \wedge \mathrm{d}\ell_{\widehat{e}_1} + \mathrm{d}\beta_{\widehat{e}_2} \wedge \mathrm{d}\ell_{\widehat{e}_2} + \mathrm{d}\beta_{\widehat{e}_3} \wedge \mathrm{d}\ell_{\widehat{e}_3}$$
$$= \mathrm{d}\beta_{\widehat{e}_1} \wedge \mathrm{d}\ell_{\widehat{e}_1} + \mathrm{d}\beta_{\widehat{e}_2} \wedge \mathrm{d}\ell_{\widehat{e}_2} + \mathrm{d}\beta_{\widehat{e}_3} \wedge \mathrm{d}\ell_{\widehat{e}_3} \ .$$

□

Proposition 3.14.4. *For the graph $\widetilde{\Gamma}_{trin}$, the generating function $G(\widetilde{\Gamma}_{trin})$ is given by*

$$G(\widetilde{\Gamma}_{trin}) = -3\left[\mathrm{Li}_2\left(\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}}{\widetilde{\lambda}_{e_3}}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_3}\widetilde{\lambda}_{e_1}}{\widetilde{\lambda}_{e_2}}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}}{\widetilde{\lambda}_{e_1}}\right)\right] -$$
$$- \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_1}\right) - \ln\left(\widetilde{\lambda}^2_{e_1}\right)\ln\left(\widetilde{\lambda}^2_{e_1}-1\right) - \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_2}\right) - \ln\left(\widetilde{\lambda}^2_{e_2}\right)\ln\left(\widetilde{\lambda}^2_{e_2}-1\right) -$$
$$- \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_3}\right) - \ln\left(\widetilde{\lambda}^2_{e_3}\right)\ln\left(\widetilde{\lambda}^2_{e_3}-1\right) + \frac{3}{2}\ln^2\left(\widetilde{\lambda}_{e_1}\right) + \frac{3}{2}\ln^2\left(\widetilde{\lambda}_{e_2}\right) + \frac{3}{2}\ln^2\left(\widetilde{\lambda}_{e_3}\right) -$$
$$- 2\ln\left(\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}\right) , \tag{3.82}$$

and for the graph $\widehat{\Gamma}_{trin}$ by

$$G(\widehat{\Gamma}_{trin}) = \frac{3}{2}\left[\mathrm{Li}_2(-\widehat{\lambda}^2_{e_1}\widehat{\lambda}_{e_2}) + \mathrm{Li}_2\left(-\frac{\widehat{\lambda}^2_{e_1}}{\widehat{\lambda}_{e_2}}\right) + \mathrm{Li}_2(-\widehat{\lambda}^2_{e_3}\widehat{\lambda}_{e_2}) + \mathrm{Li}_2\left(-\frac{\widehat{\lambda}^2_{e_3}}{\widehat{\lambda}_{e_2}}\right)\right] + \frac{5}{2}\mathrm{Li}_2\left(-\widehat{\lambda}_{e_2}\right) -$$
$$- \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_1}\right) - \ln\left(\widetilde{\lambda}^2_{e_1}\right)\ln\left(\widetilde{\lambda}^2_{e_1}-1\right) - \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_2}\right) - \ln\left(\widetilde{\lambda}^2_{e_2}\right)\ln\left(\widetilde{\lambda}^2_{e_2}-1\right) -$$
$$- \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_3}\right) - \ln\left(\widetilde{\lambda}^2_{e_3}\right)\ln\left(\widetilde{\lambda}^2_{e_3}-1\right) + 2\ln^2\widehat{\lambda}_{e_2} - 4\ln\widehat{\lambda}_{e_2} + \frac{3}{2}\ln^2\left(\widetilde{\lambda}_{e_1}\right) + \frac{3}{2}\ln^2\left(\widetilde{\lambda}_{e_3}\right) . \tag{3.83}$$

Proof. According to Theorem 3.11.11, for the graph $\widetilde{G}_{trin}$ which contains only standard edges, we have:

$$G(\widetilde{\Gamma}_{trin}) = g_0^{(1)}(\widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_2}; \widetilde{\lambda}_{e_1}) + g_0^{(2)}(\widetilde{\lambda}_{e_2}, \widetilde{\lambda}_{e_3}; \widetilde{\lambda}_{e_1}) + g_0^{(1)}(\widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_3}; \widetilde{\lambda}_{e_2}) + g_0^{(2)}(\widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_1}; \widetilde{\lambda}_{e_2}) +$$
$$+ g_0^{(1)}(\widetilde{\lambda}_{e_2}, \widetilde{\lambda}_{e_1}; \widetilde{\lambda}_{e_3}) + g_0^{(2)}(\widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_2}; \widetilde{\lambda}_{e_3})$$
$$= -\frac{1}{2}\Big[\Phi(\widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_2}, \widetilde{\lambda}_{e_1}) + \Phi(\widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_2}) + \Phi(\widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_2}) + \Phi(\widetilde{\lambda}_{e_3}, \widetilde{\lambda}_{e_1}; \widetilde{\lambda}_{e_2}) +$$
$$+ \Phi(\widetilde{\lambda}_{e_2}, \widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_3}) + \Phi(\widetilde{\lambda}_{e_1}, \widetilde{\lambda}_{e_2}, \widetilde{\lambda}_{e_3})\Big] - \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_1}\right) - \ln\left(\widetilde{\lambda}^2_{e_1}\right)\ln\left(\widetilde{\lambda}^2_{e_1}-1\right) -$$
$$- \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_2}\right) - \ln\left(\widetilde{\lambda}^2_{e_2}\right)\ln\left(\widetilde{\lambda}^2_{e_2}-1\right) - \mathrm{Li}_2\left(1-\widetilde{\lambda}^2_{e_3}\right) - \ln\left(\widetilde{\lambda}^2_{e_3}\right)\ln\left(\widetilde{\lambda}^2_{e_3}-1\right) . \tag{3.84}$$

We recall

$$\Phi(x,y,z) = \mathrm{Li}_2(-xyz) + \mathrm{Li}_2\Big(-\frac{xz}{y}\Big) + \mathrm{Li}_2\Big(-\frac{yz}{x}\Big) - \mathrm{Li}_2\Big(-\frac{z}{xy}\Big)\ .$$

Using the identity

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{\ln^2(-z)}{2}\ ,$$

one can rewrite the auxilliary function $\Phi(x,y,z)$ as (the constant $-\frac{\pi^2}{6}$ is irrelevant):

$$\Phi(x,y,z) = \mathrm{Li}_2(-xyz) + \mathrm{Li}_2\Big(-\frac{xz}{y}\Big) + \mathrm{Li}_2\Big(-\frac{yz}{x}\Big) + \mathrm{Li}_2\Big(-\frac{xy}{z}\Big) + \frac{1}{2}\ln^2\left(\frac{xy}{z}\right)\ .$$

Therefore, the first term in (3.84)

$$-\frac{1}{2}\Big[\Phi(\widetilde{\lambda}_{e_3},\widetilde{\lambda}_{e_2},\widetilde{\lambda}_{e_1}) + \Phi(\widetilde{\lambda}_{e_1},\widetilde{\lambda}_{e_3},\widetilde{\lambda}_{e_2}) + \Phi(\widetilde{\lambda}_{e_1},\widetilde{\lambda}_{e_3},\widetilde{\lambda}_{e_2}) + \Phi(\widetilde{\lambda}_{e_3},\widetilde{\lambda}_{e_1};\widetilde{\lambda}_{e_2}) + \\ + \Phi(\widetilde{\lambda}_{e_2},\widetilde{\lambda}_{e_1},\widetilde{\lambda}_{e_3}) + \Phi(\widetilde{\lambda}_{e_1},\widetilde{\lambda}_{e_2},\widetilde{\lambda}_{e_3})\Big]$$

can be rewritten

$$-3\left[\mathrm{Li}_2\left(\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}}{\widetilde{\lambda}_{e_3}}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_3}\widetilde{\lambda}_{e_1}}{\widetilde{\lambda}_{e_2}}\right) + \mathrm{Li}_2\left(-\frac{\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}}{\widetilde{\lambda}_{e_1}}\right)\right] + \\ + \frac{1}{2}\left[\ln^2\left(\frac{\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}}{\widetilde{\lambda}_{e_3}}\right) + \ln^2\left(\frac{\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_2}}{\widetilde{\lambda}_{e_2}}\right) + \ln^2\left(\frac{\widetilde{\lambda}_{e_2}\widetilde{\lambda}_{e_3}}{\widetilde{\lambda}_{e_1}}\right) + \ln^2\left(\frac{\widetilde{\lambda}_{e_1}\widetilde{\lambda}_{e_3}}{\widetilde{\lambda}_{e_2}}\right)\right]\ .$$

Simplifying the second sum of four logarithmic terms and summing with the remaining terms in (3.84) we obtain equation (3.82).

Now for the graph $\widehat{\Gamma}_{trin}$ that contains two loop edges and one standard edge, we have, according to Theorem 3.11.11

$$
\begin{aligned}
G(\widehat{\Gamma}_{trin}) &= g_1^{(1)}(\widehat{\lambda}_{e_2};\widehat{\lambda}_{e_1}) + g_0^{(1)}(\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_1};\widehat{\lambda}_{e_2}) + g_0^{(2)}(\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_3};\widehat{\lambda}_{e_2}) + g_1^{(2)}(\widehat{\lambda}_{e_2};\widehat{\lambda}_{e_3}) \\
&= -\left[\Phi(\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_2},\widehat{\lambda}_{e_1}) + \frac{1}{2}\Phi(\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_2}) + \Phi(\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_2}) + \Phi(\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_2},\widehat{\lambda}_{e_3})\right] - \\
&- \mathrm{Li}_2\left(1-\widehat{\lambda}_{e_1}^2\right) - \ln\left(\widehat{\lambda}_{e_1}^2\right)\ln\left(\widehat{\lambda}_{e_1}^2-1\right) - \mathrm{Li}_2\left(1-\widehat{\lambda}_{e_2}^2\right) - \ln\left(\widehat{\lambda}_{e_2}^2\right)\ln\left(\widehat{\lambda}_{e_2}^2-1\right) - \\
&- \mathrm{Li}_2\left(1-\widehat{\lambda}_{e_3}^2\right) - \ln\left(\widehat{\lambda}_{e_3}^2\right)\ln\left(\widehat{\lambda}_{e_3}^2-1\right) - \ln\widehat{\lambda}_{e_1}\ln\widehat{\lambda}_{e_2} - \ln\widehat{\lambda}_{e_3}\ln\widehat{\lambda}_{e_2} + \frac{1}{2}\ln^2\widehat{\lambda}_{e_1} + \\
&+ \frac{1}{2}\ln^2\widehat{\lambda}_{e_3}\ .
\end{aligned} \tag{3.85}
$$

Using again (3.14.1), one can rewrite

$$
-\left[\Phi(\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_2},\widehat{\lambda}_{e_1}) + \frac{1}{2}\Phi(\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_1},\widehat{\lambda}_{e_2}) + \Phi(\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_2}) + \Phi(\widehat{\lambda}_{e_3},\widehat{\lambda}_{e_2},\widehat{\lambda}_{e_3})\right]
$$

as

$$
\begin{aligned}
&\frac{3}{2}\left[\mathrm{Li}_2(-\widehat{\lambda}_{e_1}^2\widehat{\lambda}_{e_2}) + \mathrm{Li}_2\left(-\frac{\widehat{\lambda}_{e_1}^2}{\widehat{\lambda}_{e_2}}\right) + \mathrm{Li}_2(-\widehat{\lambda}_{e_3}^2\widehat{\lambda}_{e_2}) + \mathrm{Li}_2\left(-\frac{\widehat{\lambda}_{e_3}^2}{\widehat{\lambda}_{e_2}}\right)\right] + \frac{5}{2}\mathrm{Li}_2\left(-\widehat{\lambda}_{e_2}\right) + 2\ln^2\widehat{\lambda}_{e_2} + \\
&+ \ln^2\widehat{\lambda}_{e_1} + \ln^2\widehat{\lambda}_{e_3} - 4\ln\widehat{\lambda}_{e_2}\ .
\end{aligned} \tag{3.86}
$$

Summing this last equation with the other terms of (3.85) we recover (3.83). □

3.14.2 $\mathcal{M}_3$

One contour

In this case $\widetilde{\mathcal{C}}$ is a one-holed double torus and $\widehat{\mathcal{C}}$ is a one-holed torus, the associated graphs $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ are shown in Figure 3.16. We associate to $\widetilde{\mathcal{C}}$, endowed with $\widetilde{\Gamma}$, shear-type coordinates $\{\widetilde{\zeta}_{e_1},\widetilde{\zeta}_{e_2},\widetilde{\zeta}_{e_3},\widetilde{\zeta}_{e_4},\widetilde{\zeta}_{e_5},\widetilde{\zeta}_{e_6},\widetilde{\zeta}_{e_7},\widetilde{\zeta}_{e_8},\widetilde{\zeta}_{e_9}\}$ and to $\widehat{\mathcal{C}}$, with $\widehat{\Gamma}$, shear-type coordinates $\{\widehat{\zeta}_{e_1},\widehat{\zeta}_{e_2},\widehat{\zeta}_{e_3}\}$.

Proposition 3.14.5. *In log-canonical coordinates*

$$
\{\widetilde{\zeta}_{e_2},\widetilde{\zeta}_{e_3},\widetilde{\zeta}_{e_4},\widetilde{\zeta}_{e_5},\widetilde{\zeta}_{e_6},\widetilde{\zeta}_{e_7},\widetilde{\zeta}_{e_8},\widetilde{\zeta}_{e_9},\widehat{\zeta}_{e_2},\widehat{\zeta}_{e_3},\ell_\gamma,\beta_\gamma\}
$$

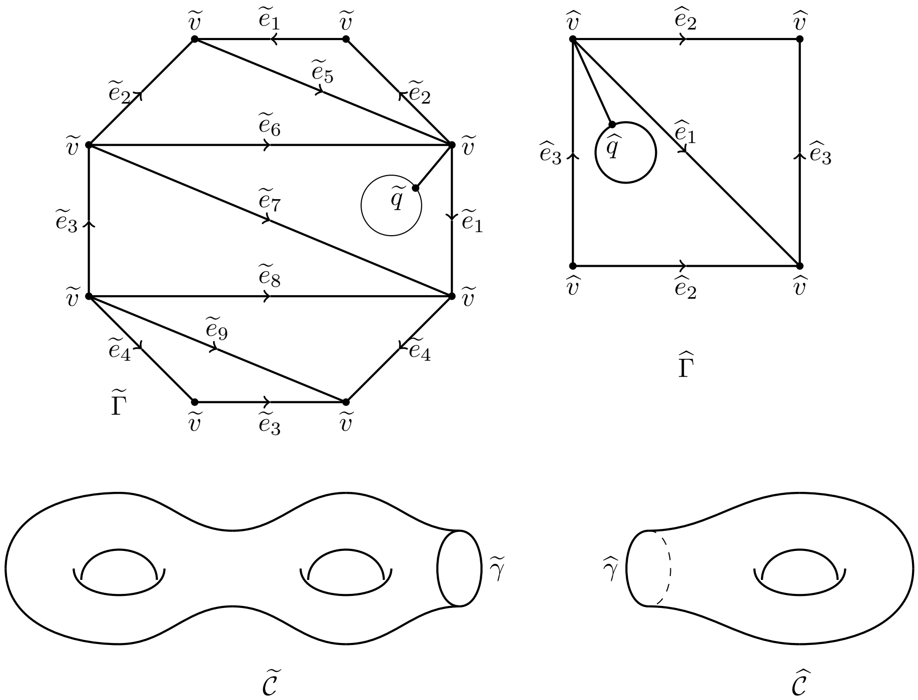

Figure 3.16: Single contour splitting for $\mathcal{M}_3$ and associated graphs.

the Weil-Petersson form on $\mathcal{M}_2$ can be written :

$$\begin{aligned}\omega_{WP} = {}& 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_3} + \mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_4} + 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_5} + 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_6} + 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_8} + \\ & + 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} + 2\mathrm{d}\widetilde{\zeta}_{e_2}\wedge\mathrm{d}\widetilde{\zeta}_{e_4} - 3\mathrm{d}\widetilde{\zeta}_{e_3}\wedge\mathrm{d}\widetilde{\zeta}_{e_5} + 2\mathrm{d}\widetilde{\zeta}_{e_3}\wedge\mathrm{d}\widetilde{\zeta}_{e_6} - 2\mathrm{d}\widetilde{\zeta}_{e_3}\wedge\mathrm{d}\widetilde{\zeta}_{e_8} + 2\mathrm{d}\widetilde{\zeta}_{e_3}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} - \\ & - 2\mathrm{d}\widetilde{\zeta}_{e_4}\wedge\mathrm{d}\widetilde{\zeta}_{e_5} - 2\mathrm{d}\widetilde{\zeta}_{e_4}\wedge\mathrm{d}\widetilde{\zeta}_{e_7} - 2\mathrm{d}\widetilde{\zeta}_{e_4}\wedge\mathrm{d}\widetilde{\zeta}_{e_8} + 2\mathrm{d}\widetilde{\zeta}_{e_4}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} + 2\mathrm{d}\widetilde{\zeta}_{e_5}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} - 2\mathrm{d}\widetilde{\zeta}_{e_6}\wedge\mathrm{d}\widetilde{\zeta}_{e_8} - \\ & - 2\mathrm{d}\widetilde{\zeta}_{e_7}\wedge\mathrm{d}\widetilde{\zeta}_{e_8} + 2\mathrm{d}\widetilde{\zeta}_{e_7}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} + 4\mathrm{d}\widetilde{\zeta}_{e_8}\wedge\mathrm{d}\widetilde{\zeta}_{e_9} + 2\mathrm{d}\widehat{\zeta}_{e_2}\wedge\mathrm{d}\widehat{\zeta}_{e_3} + \mathrm{d}\ell_\gamma\wedge\left(\mathrm{d}\widehat{\zeta}_{e_2} + \mathrm{d}\widehat{\zeta}_{e_3}\right) + \\ & + \mathrm{d}\ell_\gamma\wedge\left(\mathrm{d}\widetilde{\zeta}_{e_2} + 2\mathrm{d}\widetilde{\zeta}_{e_3} + 2\mathrm{d}\widetilde{\zeta}_{e_4} + \mathrm{d}\widetilde{\zeta}_{e_5} + \mathrm{d}\widetilde{\zeta}_{e_6} + 2\mathrm{d}\widetilde{\zeta}_{e_7} + 2\mathrm{d}\widetilde{\zeta}_{e_8} + 2\mathrm{d}\widetilde{\zeta}_{e_9}\right) + \mathrm{d}\beta_\gamma\wedge\mathrm{d}\ell_\gamma\end{aligned} \tag{3.87}$$

Proof. The proof is by direct application of expression (3.40) and Theorem 3.7.1. For sake of brevity we only write the non-zero coefficients $\widetilde{c}_{ij}$ for $\Omega(\Gamma_1)$:

$$\begin{cases} \widetilde{c}_{12} = \widetilde{c}_{15} = \widetilde{c}_{16} = \widetilde{c}_{25} = \widetilde{c}_{26} = \widetilde{c}_{28} = \widetilde{c}_{34} = \widetilde{c}_{39} = \widetilde{c}_{49} = \widetilde{c}_{57} = \widetilde{c}_{58} = \widetilde{c}_{67} = \widetilde{c}_{69} = \widetilde{c}_{79} = 2 \\ \widetilde{c}_{38} = \widetilde{c}_{46} = \widetilde{c}_{48} = \widetilde{c}_{78} = -2 \\ \widetilde{c}_{24} = 3 \\ \widetilde{c}_{13} = \widetilde{c}_{14} = \widetilde{c}_{17} = \widetilde{c}_{18} = \widetilde{c}_{19} = \widetilde{c}_{23} = \widetilde{c}_{29} = \widetilde{c}_{59} = \widetilde{c}_{89} = 4 \\ \widetilde{c}_{35} = \widetilde{c}_{45} = -4 \end{cases}$$

Whereas for $\Omega(\Gamma_2)$ the same non-zero coefficients are given by $\widehat{c}_{12} = \widehat{c}_{13} = \widehat{c}_{23} = 2$. We let to the reader the straightforward but long computation of each $\widetilde{c}_{ij} - \widetilde{c}_{1j} + \widetilde{c}_{1i}$ in order to check (3.87). □

Two contours

Choose two separating contours γ_1 and γ_2. As depicted in Figure 3.17, bottom, we obtain by cutting along these contours a one-holed torus $\mathcal{C}^{(1)}$, a two-holed torus $\mathcal{C}^{(2)}$ and another one-holed torus $\mathcal{C}^{(3)}$, with respective graphs and shear-type variables: $\Gamma^{(1)}$ and $\{\zeta^{(1)}_{e_1}, \zeta^{(1)}_{e_2}, \zeta^{(1)}_{e_3}\}$, $\Gamma^{(2)}$ and $\{\zeta^{(2)}_{e_1}, \zeta^{(2)}_{e_2}, \zeta^{(2)}_{e_3}, \zeta^{(2)}_{e_4}, \zeta^{(2)}_{e_5}, \zeta^{(2)}_{e_6}\}$, $\Gamma^{(3)}$ and $\{\zeta^{(3)}_{e_1}, \zeta^{(3)}_{e_2}, \zeta^{(3)}_{e_3}\}$.

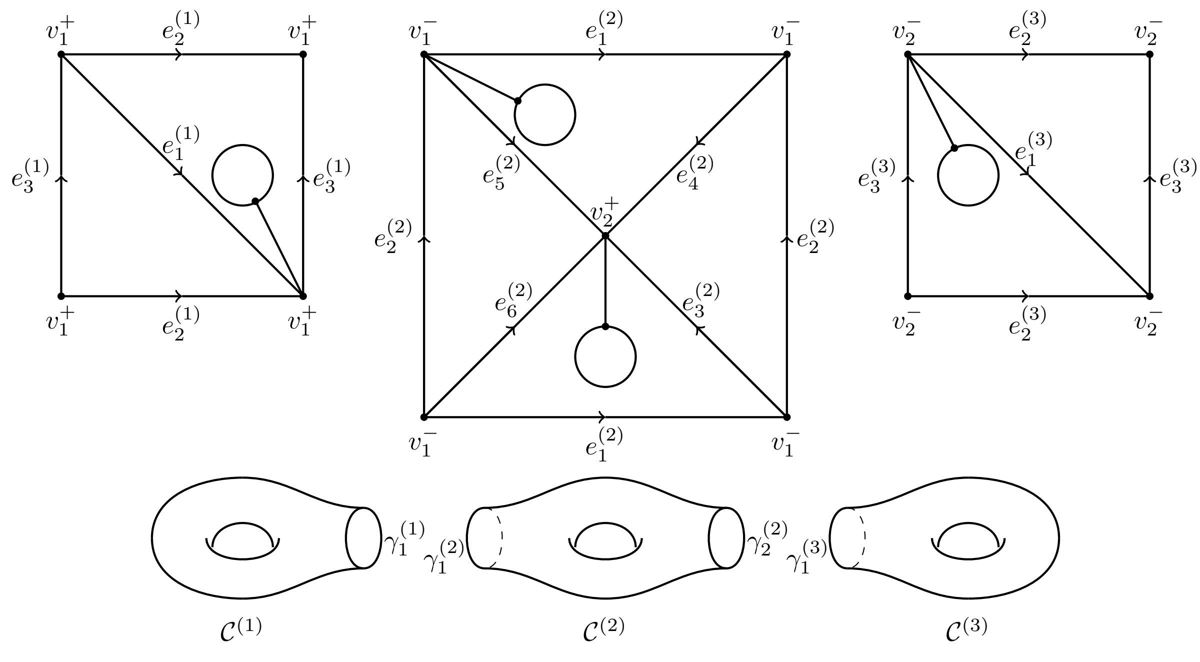

Figure 3.17: Two contour splitting for $\mathcal{M}_{3,0}$ and associated graphs.

Proposition 3.14.6. *The Weil-Petersson symplectic form on $\mathcal{M}_3$ in log-canonical coordinates* $\{\zeta^{(1)}_{e_2}, \zeta^{(1)}_{e_3}, \zeta^{(2)}_{e_2}, \zeta^{(2)}_{e_3}, \zeta^{(2)}_{e_4}, \zeta^{(2)}_{e_6}, \zeta^{(3)}_{e_2}, \zeta^{(3)}_{e_3}, \ell_{\gamma_1}, \beta_{\gamma_1}, \ell_{\gamma_2}, \beta_{\gamma_2}\}$ *is given by:*

$$\begin{aligned}\omega_{WP} = & 2\mathrm{d}\zeta^{(1)}_{e_2}\wedge\mathrm{d}\zeta^{(1)}_{e_3} - 2\mathrm{d}\zeta^{(2)}_{e_2}\wedge\mathrm{d}\zeta^{(2)}_{e_4} + 2\mathrm{d}\zeta^{(2)}_{e_2}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + 2\mathrm{d}\zeta^{(2)}_{e_3}\wedge\mathrm{d}\zeta^{(2)}_{e_4} + 2\mathrm{d}\zeta^{(2)}_{e_3}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + \\ & + 2\mathrm{d}\zeta^{(2)}_{e_4}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + 2\mathrm{d}\zeta^{(3)}_{e_2}\wedge\mathrm{d}\zeta^{(3)}_{e_3} + \mathrm{d}\ell_{\gamma_1}\wedge\left(\mathrm{d}\zeta^{(1)}_{e_2}+\mathrm{d}\zeta^{(1)}_{e_3}\right) + \mathrm{d}\ell_{\gamma_1}\wedge\left(\mathrm{d}\zeta^{(2)}_{e_2}+\mathrm{d}\zeta^{(2)}_{e_3}-\mathrm{d}\zeta^{(2)}_{e_6}\right) + \\ & + \mathrm{d}\ell_{\gamma_2}\wedge\left(\mathrm{d}\zeta^{(2)}_{e_3}-\mathrm{d}\zeta^{(2)}_{e_2}-\mathrm{d}\zeta^{(2)}_{e_4}+\mathrm{d}\zeta^{(2)}_{e_6}\right) + \mathrm{d}\ell_{\gamma_2}\wedge\left(\mathrm{d}\zeta^{(3)}_{e_2}+\mathrm{d}\zeta^{(3)}_{e_3}\right) + \mathrm{d}\beta_{\gamma_1}\wedge\mathrm{d}\ell_{\gamma_1} + \mathrm{d}\beta_{\gamma_2}\wedge\mathrm{d}\ell_{\gamma_2}\end{aligned} \tag{3.88}$$

Proof. In this situation, by equation (3.40) (plus the twist-length contributions) we have:

$$\begin{aligned}\Omega(\Gamma^{(1)}) &= 2\,\mathrm{d}\zeta^{(1)}_{e_1}\wedge\mathrm{d}\zeta^{(1)}_{e_2} + 2\,\mathrm{d}\zeta^{(1)}_{e_1}\wedge\mathrm{d}\zeta^{(1)}_{e_3} + 2\,\mathrm{d}\zeta^{(1)}_{e_2}\wedge\mathrm{d}\zeta^{(1)}_{e_3} + 2\mathrm{d}\beta_{\gamma_1^{(1)}}\wedge\mathrm{d}\ell_{\gamma_1^{(1)}}\ , \\ \Omega(\Gamma^{(2)}) &= \mathrm{d}\zeta^{(2)}_{e_1}\wedge\mathrm{d}\zeta^{(2)}_{e_2} - \mathrm{d}\zeta^{(2)}_{e_1}\wedge\mathrm{d}\zeta^{(2)}_{e_4} + \mathrm{d}\zeta^{(2)}_{e_1}\wedge\mathrm{d}\zeta^{(2)}_{e_5} - \mathrm{d}\zeta^{(2)}_{e_1}\wedge\mathrm{d}\zeta^{(2)}_{e_6} \\ &\quad + \mathrm{d}\zeta^{(2)}_{e_2}\wedge\mathrm{d}\zeta^{(2)}_{e_3} - \mathrm{d}\zeta^{(2)}_{e_2}\wedge\mathrm{d}\zeta^{(2)}_{e_4} - \mathrm{d}\zeta^{(2)}_{e_2}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + \mathrm{d}\zeta^{(2)}_{e_3}\wedge\mathrm{d}\zeta^{(2)}_{e_4} \\ &\quad - \mathrm{d}\zeta^{(2)}_{e_3}\wedge\mathrm{d}\zeta^{(2)}_{e_5} + \mathrm{d}\zeta^{(2)}_{e_3}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + \mathrm{d}\zeta^{(2)}_{e_4}\wedge\mathrm{d}\zeta^{(2)}_{e_5} + \mathrm{d}\zeta^{(2)}_{e_4}\wedge\mathrm{d}\zeta^{(2)}_{e_6} \\ &\quad + \mathrm{d}\zeta^{(2)}_{e_5}\wedge\mathrm{d}\zeta^{(2)}_{e_6} + 2\mathrm{d}\beta_{\gamma_1^{(2)}}\wedge\mathrm{d}\ell_{\gamma_1^{(2)}} + 2\mathrm{d}\beta_{\gamma_2^{(2)}}\wedge\mathrm{d}\ell_{\gamma_2^{(2)}}\ , \\ \Omega(\Gamma^{(3)}) &= 2\,\mathrm{d}\zeta^{(3)}_{e_1}\wedge\mathrm{d}\zeta^{(3)}_{e_2} + 2\,\mathrm{d}\zeta^{(3)}_{e_1}\wedge\mathrm{d}\zeta^{(3)}_{e_3} + 2\,\mathrm{d}\zeta^{(3)}_{e_2}\wedge\mathrm{d}\zeta^{(3)}_{e_3} + 2\mathrm{d}\beta_{\gamma_1^{(3)}}\wedge\mathrm{d}\ell_{\gamma_1^{(3)}}\ .\end{aligned} \tag{3.89}$$

The constraints for the first contour γ_1 read:

$$\begin{aligned}2(\zeta^{(1)}_{e_1}+\zeta^{(1)}_{e_2}+\zeta^{(1)}_{e_3}) &= \ell_{\gamma_1^{(1)}} \\ 2\zeta^{(2)}_{e_1}+2\zeta^{(2)}_{e_2}+\zeta^{(2)}_{e_3}+\zeta^{(2)}_{e_4}+\zeta^{(2)}_{e_5}+\zeta^{(2)}_{e_6} &= \ell_{\gamma_1^{(2)}} \\ \ell_{\gamma_1^{(1)}} = \ell_{\gamma_1^{(2)}} &= \ell_{\gamma_1}\end{aligned}$$

Whereas for the second contour γ_2 we have:

$$\begin{aligned}\zeta^{(2)}_{e_3}+\zeta^{(2)}_{e_4}+\zeta^{(2)}_{e_5}+\zeta^{(2)}_{e_6} &= \ell_{\gamma_2^{(2)}} \\ 2(\zeta^{(3)}_{e_1}+\zeta^{(3)}_{e_2}+\zeta^{(3)}_{e_3}) &= \ell_{\gamma_1^{(3)}} \\ \ell_{\gamma_2^{(2)}} = \ell_{\gamma_1^{(3)}} &= \ell_{\gamma_2}\end{aligned}$$

Therefore one can eliminate the variables $\zeta^{(1)}_{e_1}$, $\zeta^{(2)}_{e_1}$, $\zeta^{(2)}_{e_5}$ and $\zeta^{(3)}_{e_1}$ according to:

$$\begin{aligned}\zeta^{(1)}_{e_1} &= \frac{1}{2}\ell_{\gamma_1}-\zeta^{(1)}_{e_2}-\zeta^{(1)}_{e_3}\ , \qquad & \zeta^{(3)}_{e_1} &= \frac{1}{2}\ell_{\gamma_2}-\zeta^{(3)}_{e_2}-\zeta^{(3)}_{e_3}\ , \\ \zeta^{(2)}_{e_1} &= \frac{1}{2}(\ell_{\gamma_1}-\ell_{\gamma_2})-\zeta^{(2)}_{e_2}\ , \qquad & \zeta^{(2)}_{e_5} &= \ell_{\gamma_2}-\zeta^{(2)}_{e_3}-\zeta^{(2)}_{e_4}-\zeta^{(2)}_{e_6}\ .\end{aligned}$$

Inserting these expressions into (3.89), summing $\Omega(\Gamma^{(1)})$ with $\Omega(\Gamma^{(2)})$ and $\Omega(\Gamma^{(3)})$ while defining $\beta_{\gamma_1} = 2(\beta_{\gamma_1^{(1)}} + \beta_{\gamma_1^{(2)}})$ and $\beta_{\gamma_2} = 2(\beta_{\gamma_2^{(2)}} + \beta_{\gamma_1^{(3)}})$ one can obtain (3.88). □

Three contours

Let us think about the 3-torus as a sphere with three handles as shown on Figure 3.18. Cutting these handles along three separating contours $\gamma_1, \gamma_2, \gamma_3$ we get four pieces: $\mathcal{C}^{(3)}$ a trinion and three one-holed tori $\mathcal{C}^{(1)}$, $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$. Denote the associated graphs by $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}$.

Proposition 3.14.7. *The Weil-Petersson form on $\mathcal{M}_3$ in log-canonical coordinates $\{\zeta_{e_2}^{(1)}, \zeta_{e_3}^{(1)}, \zeta_{e_2}^{(2)}, \zeta_{e_3}^{(2)}, \zeta_{e_2}^{(4)}, \zeta_{e_3}^{(4)}, \ell_{\gamma_1}, \ell_{\gamma_2}, \ell_{\gamma_3}, \beta_{\gamma_1}, \beta_{\gamma_2}, \beta_{\gamma_3}\}$ is given by:*

$$\begin{aligned}\omega_{WP} = {} & 2\mathrm{d}\zeta_{e_2}^{(1)} \wedge \mathrm{d}\zeta_{e_3}^{(1)} + 2\mathrm{d}\zeta_{e_2}^{(2)} \wedge \mathrm{d}\zeta_{e_3}^{(2)} + 2\mathrm{d}\zeta_{e_2}^{(4)} \wedge \mathrm{d}\zeta_{e_3}^{(4)} + \mathrm{d}\ell_{\gamma_1} \wedge \left(\mathrm{d}\zeta_{e_2}^{(1)} + \mathrm{d}\zeta_{e_3}^{(1)}\right) + \\ & + \mathrm{d}\ell_{\gamma_2} \wedge \left(\mathrm{d}\zeta_{e_2}^{(2)} + \mathrm{d}\zeta_{e_3}^{(2)}\right) + \mathrm{d}\ell_{\gamma_3} \wedge \left(\mathrm{d}\zeta_{e_2}^{(4)} + \mathrm{d}\zeta_{e_3}^{(4)}\right) + \mathrm{d}\ell_{\gamma_2} \wedge \mathrm{d}\ell_{\gamma_1} + \mathrm{d}\ell_{\gamma_3} \wedge \mathrm{d}\ell_{\gamma_2} + \mathrm{d}\ell_{\gamma_1} \wedge \mathrm{d}\ell_{\gamma_3} + \\ & + \mathrm{d}\beta_{\gamma_1} \wedge \mathrm{d}\ell_{\gamma_1} + \mathrm{d}\beta_{\gamma_2} \wedge \mathrm{d}\ell_{\gamma_2} + \mathrm{d}\beta_{\gamma_3} \wedge \mathrm{d}\ell_{\gamma_3} \ .\end{aligned} \tag{3.90}$$

Proof. Applying equation (3.40), and taking into account the twist-length contributions, let us write:

$$\begin{aligned}\Omega(\Gamma^{(1)}) &= 2\mathrm{d}\zeta_{e_1}^{(1)} \wedge \mathrm{d}\zeta_{e_2}^{(1)} + 2\mathrm{d}\zeta_{e_1}^{(1)} \wedge \mathrm{d}\zeta_{e_3}^{(1)} + 2\mathrm{d}\zeta_{e_2}^{(1)} \wedge \mathrm{d}\zeta_{e_3}^{(1)} + 2\mathrm{d}\beta_{\gamma_1^{(1)}} \wedge \mathrm{d}\ell_{\gamma_1^{(1)}} \ , \\ \Omega(\Gamma^{(2)}) &= 2\mathrm{d}\zeta_{e_1}^{(2)} \wedge \mathrm{d}\zeta_{e_2}^{(2)} + 2\mathrm{d}\zeta_{e_1}^{(2)} \wedge \mathrm{d}\zeta_{e_3}^{(2)} + 2\mathrm{d}\zeta_{e_2}^{(2)} \wedge \mathrm{d}\zeta_{e_3}^{(2)} + 2\mathrm{d}\beta_{\gamma_1^{(2)}} \wedge \mathrm{d}\ell_{\gamma_1^{(2)}} \ , \\ \Omega(\Gamma^{(3)}) &= \mathrm{d}\zeta_{e_2}^{(3)} \wedge \mathrm{d}\zeta_{e_1}^{(3)} + \mathrm{d}\zeta_{e_3}^{(3)} \wedge \mathrm{d}\zeta_{e_2}^{(3)} + \mathrm{d}\zeta_{e_1}^{(3)} \wedge \mathrm{d}\zeta_{e_3}^{(3)} + 2\mathrm{d}\beta_{\gamma_1^{(3)}} \wedge \mathrm{d}\ell_{\gamma_1^{(3)}} + 2\mathrm{d}\beta_{\gamma_2^{(3)}} \wedge \mathrm{d}\ell_{\gamma_2^{(3)}} + \\ & \qquad + 2\mathrm{d}\beta_{\gamma_3^{(3)}} \wedge \mathrm{d}\ell_{\gamma_3^{(3)}} \ , \\ \Omega(\Gamma^{(4)}) &= 2\mathrm{d}\zeta_{e_1}^{(4)} \wedge \mathrm{d}\zeta_{e_2}^{(4)} + 2\mathrm{d}\zeta_{e_1}^{(4)} \wedge \mathrm{d}\zeta_{e_3}^{(4)} + 2\mathrm{d}\zeta_{e_2}^{(4)} \wedge \mathrm{d}\zeta_{e_3}^{(4)} + 2\mathrm{d}\beta_{\gamma_1^{(4)}} \wedge \mathrm{d}\ell_{\gamma_1^{(4)}} \ .\end{aligned} \tag{3.91}$$

The constraints for γ_1 reads:

$$\begin{aligned}2(\zeta_{e_1}^{(1)} + \zeta_{e_2}^{(1)} + \zeta_{e_3}^{(1)}) &= \ell_{\gamma_1^{(1)}} \ , \\ \zeta_{e_1}^{(3)} + \zeta_{e_3}^{(3)} &= \ell_{\gamma_1^{(3)}} \ , \\ \ell_{\gamma_1^{(1)}} = \ell_{\gamma_1^{(3)}} &= \ell_{\gamma_1} \ .\end{aligned}$$

Those for γ_2 are given by:

$$2(\zeta_{e_1}^{(2)} + \zeta_{e_2}^{(2)} + \zeta_{e_3}^{(2)}) = \ell_{\gamma_1^{(2)}} \ ,$$
$$\zeta_{e_1}^{(3)} + \zeta_{e_2}^{(3)} = \ell_{\gamma_2^{(3)}} \ ,$$
$$\ell_{\gamma_1^{(2)}} = \ell_{\gamma_2^{(3)}} = \ell_{\gamma_2} \ .$$

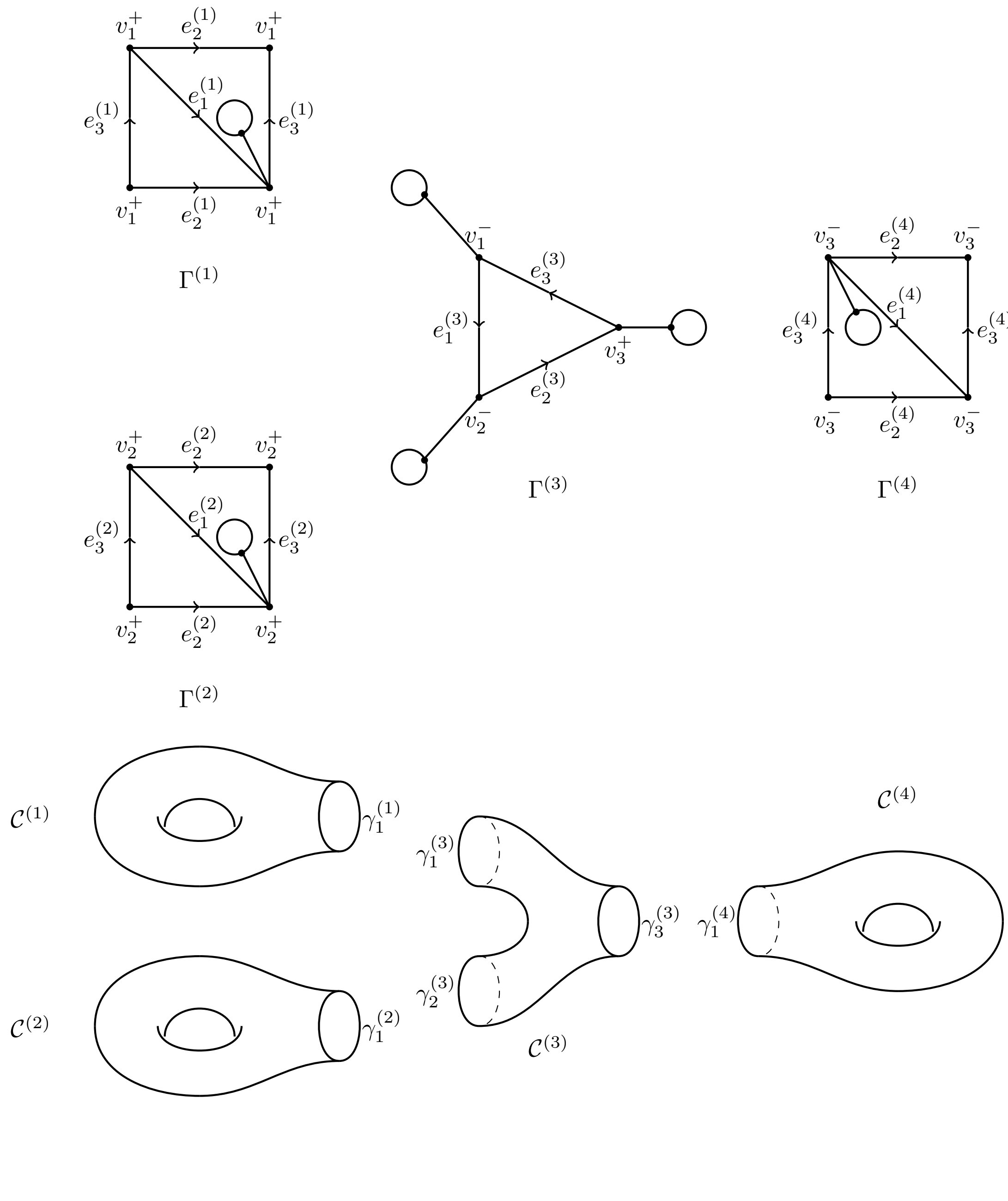

Figure 3.18: Three contour splitting for $\mathcal{M}_{3,0}$ and associated graphs.

And for γ_3 we have:

$$2(\zeta_{e_1}^{(4)} + \zeta_{e_2}^{(4)} + \zeta_{e_3}^{(4)}) = \ell_{\gamma_1^{(4)}} \ ,$$
$$\zeta_{e_2}^{(3)} + \zeta_{e_3}^{(3)} = \ell_{\gamma_3^{(3)}} \ ,$$
$$\ell_{\gamma_1^{(4)}} = \ell_{\gamma_3^{(3)}} = \ell_{\gamma_3} \ .$$

From these constraints we get:

$$\zeta_{e_1}^{(1)} = \frac{1}{2}\ell_{\gamma_1} - \zeta_{e_2}^{(1)} - \zeta_{e_3}^{(1)} \ , \qquad \zeta_{e_1}^{(2)} = \frac{1}{2}\ell_{\gamma_2} - \zeta_{e_2}^{(2)} - \zeta_{e_3}^{(2)} \ , \qquad \zeta_{e_1}^{(4)} = \frac{1}{2}\ell_{\gamma_3} - \zeta_{e_2}^{(4)} - \zeta_{e_3}^{(4)} \ ,$$
$$\zeta_{e_1}^{(3)} = \frac{1}{2}\left(\ell_{\gamma_1} + \ell_{\gamma_2} - \ell_{\gamma_3}\right) \ , \qquad \zeta_{e_2}^{(3)} = \frac{1}{2}\left(\ell_{\gamma_2} - \ell_{\gamma_1} + \ell_{\gamma_3}\right) \ , \qquad \zeta_{e_3}^{(3)} = \frac{1}{2}\left(\ell_{\gamma_1} - \ell_{\gamma_2} + \ell_{\gamma_3}\right) \ .$$

Inserting these expressions into (3.91) while setting $\beta_{\gamma_1} = 2(\beta_{\gamma_1^{(1)}} + \beta_{\gamma_1^{(3)}})$, $\beta_{\gamma_2} = 2(\beta_{\gamma_1^{(2)}} + \beta_{\gamma_2^{(3)}})$ and $\beta_{\gamma_3} = 2(\beta_{\gamma_1^{(4)}} + \beta_{\gamma_3^{(3)}})$ we get (3.90). □

Four contours

With four contours we recover complete trinion decompositions. As an example, take the trinion graph to be the "tetrahedron graph" Γ_{tetra} shown in Figure 3.19.

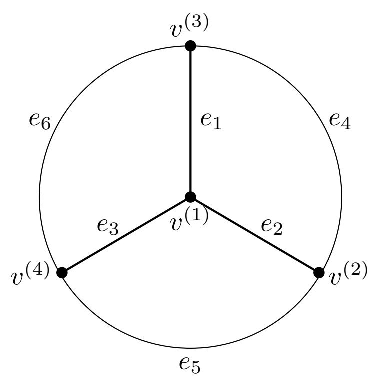

Figure 3.19: Tetrahedron graph Γ_{tetra}.

Proposition 3.14.8. *The Weil-Petersson form ω_{WP} on $\mathcal{M}_3$ coincides with the following* *form in coordinates* $\{\ell_{e_j}, \beta_{e_j}\}_{j=1}^{6}$

$$\begin{aligned}\omega_{WP} = {} & \mathrm{d}\ell_{e_2} \wedge \mathrm{d}\ell_{e_1} + \mathrm{d}\ell_{e_1} \wedge \mathrm{d}\ell_{e_3} + \mathrm{d}\ell_{e_3} \wedge \mathrm{d}\ell_{e_2} + \mathrm{d}\ell_{e_4} \wedge \mathrm{d}\ell_{e_2} + \mathrm{d}\ell_{e_2} \wedge \mathrm{d}\ell_{e_5} + \\ & + \mathrm{d}\ell_{e_5} \wedge \mathrm{d}\ell_{e_4} + \mathrm{d}\ell_{e_1} \wedge \mathrm{d}\ell_{e_4} + \mathrm{d}\ell_{e_4} \wedge \mathrm{d}\ell_{e_6} + \mathrm{d}\ell_{e_6} \wedge \mathrm{d}\ell_{e_1} + \mathrm{d}\ell_{e_3} \wedge \mathrm{d}\ell_{e_6} + \\ & + \mathrm{d}\ell_{e_6} \wedge \mathrm{d}\ell_{e_5} + \mathrm{d}\ell_{e_5} \wedge \mathrm{d}\ell_{e_3} + \mathrm{d}\beta_{e_1} \wedge \mathrm{d}\ell_{e_1} + \mathrm{d}\beta_{e_2} \wedge \mathrm{d}\ell_{e_2} + \mathrm{d}\beta_{e_3} \wedge \mathrm{d}\ell_{e_3} + \\ & + \mathrm{d}\beta_{e_4} \wedge \mathrm{d}\ell_{e_4} + \mathrm{d}\beta_{e_5} \wedge \mathrm{d}\ell_{e_5} + \mathrm{d}\beta_{e_6} \wedge \mathrm{d}\ell_{e_6}\ .\end{aligned} \tag{3.92}$$

Proof. The proof is by direct computation using Theorem 3.9.2, equation (3.51), applied to Γ_{tetra}. The form $\Omega(\Gamma_{tetra})$ coincides with ω_{WP} on $\mathcal{M}_3$ by Proposition 3.11.10. □

Proposition 3.14.9. *For the tetrahedron graph* Γ_{tetra}*, the generating function* $G(\Gamma_{tetra})$ *takes the following expression*

$$\begin{aligned} G(\Gamma_{tetra}) = & -\frac{3}{2}\left[\mathrm{Li}_2\left(\lambda_{e_1}\lambda_{e_2}\lambda_{e_3}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_1}\lambda_{e_2}}{\lambda_{e_3}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_3}\lambda_{e_1}}{\lambda_{e_2}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_2}\lambda_{e_3}}{\lambda_{e_1}}\right)\right] - \\ & - \mathrm{Li}_2\left(1-\lambda_{e_1}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_2}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_3}^2\right) + \frac{\ln^2\left(\lambda_{e_1}\right)}{4} + \frac{\ln^2\left(\lambda_{e_2}\right)}{4} + \frac{\ln^2\left(\lambda_{e_3}\right)}{4} - \\ & - \frac{\ln\left(\lambda_{e_1}\lambda_{e_2}\lambda_{e_3}\right)}{2} - \frac{3}{2}\left[\mathrm{Li}_2\left(\lambda_{e_2}\lambda_{e_4}\lambda_{e_5}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_2}\lambda_{e_4}}{\lambda_{e_5}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_2}\lambda_{e_5}}{\lambda_{e_4}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_4}\lambda_{e_5}}{\lambda_{e_2}}\right)\right] - \\ & - \mathrm{Li}_2\left(1-\lambda_{e_2}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_4}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_5}^2\right) + \frac{\ln^2\left(\lambda_{e_2}\right)}{4} + \frac{\ln^2\left(\lambda_{e_4}\right)}{4} + \frac{\ln^2\left(\lambda_{e_5}\right)}{4} - \\ & - \frac{\ln\left(\lambda_{e_2}\lambda_{e_4}\lambda_{e_5}\right)}{2} - \frac{3}{2}\left[\mathrm{Li}_2\left(\lambda_{e_3}\lambda_{e_5}\lambda_{e_6}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_3}\lambda_{e_5}}{\lambda_{e_6}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_3}\lambda_{e_6}}{\lambda_{e_5}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_5}\lambda_{e_6}}{\lambda_{e_3}}\right)\right] - \\ & - \mathrm{Li}_2\left(1-\lambda_{e_3}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_5}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_6}^2\right) + \frac{\ln^2\left(\lambda_{e_3}\right)}{4} + \frac{\ln^2\left(\lambda_{e_5}\right)}{4} + \frac{\ln^2\left(\lambda_{e_6}\right)}{4} - \\ & - \frac{\ln\left(\lambda_{e_3}\lambda_{e_5}\lambda_{e_6}\right)}{2} - \frac{3}{2}\left[\mathrm{Li}_2\left(\lambda_{e_1}\lambda_{e_4}\lambda_{e_6}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_1}\lambda_{e_4}}{\lambda_{e_6}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_4}\lambda_{e_6}}{\lambda_{e_1}}\right) + \mathrm{Li}_2\left(-\frac{\lambda_{e_1}\lambda_{e_6}}{\lambda_{e_4}}\right)\right] - \\ & - \mathrm{Li}_2\left(1-\lambda_{e_1}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_4}^2\right) - \mathrm{Li}_2\left(1-\lambda_{e_6}^2\right) + \frac{\ln^2\left(\lambda_{e_1}\right)}{4} + \frac{\ln^2\left(\lambda_{e_4}\right)}{4} + \frac{\ln^2\left(\lambda_{e_6}\right)}{4} - \\ & - \frac{\ln\left(\lambda_{e_1}\lambda_{e_4}\lambda_{e_6}\right)}{2}\ . \end{aligned} \tag{3.93}$$

Proof. Applying Theorem 3.11.11 to the graph G_{tetra} which has only standard edges we get

$$\begin{aligned} G(\Gamma_{tetra}) = {} & g_0^{(1)}(\lambda_{e_3}, \lambda_{e_2}; \lambda_{e_1}) + g_0^{(3)}(\lambda_{e_4}, \lambda_{e_6}; \lambda_{e_1}) + g_0^{(1)}(\lambda_{e_1}, \lambda_{e_3}; \lambda_{e_2}) + g_0^{(2)}(\lambda_{e_5}, \lambda_{e_4}; \lambda_{e_2}) + \\ & + g_0^{(1)}(\lambda_{e_2}, \lambda_{e_1}; \lambda_{e_3}) + g_0^{(4)}(\lambda_{e_6}, \lambda_{e_5}; \lambda_{e_3}) + g_0^{(2)}(\lambda_{e_2}, \lambda_{e_5}; \lambda_{e_4}) + g_0^{(3)}(\lambda_{e_6}, \lambda_{e_1}; \lambda_{e_4}) + \\ & + g_0^{(2)}(\lambda_{e_4}, \lambda_{e_2}; \lambda_{e_5}) + g_0^{(4)}(\lambda_{e_3}, \lambda_{e_6}; \lambda_{e_5}) + g_0^{(3)}(\lambda_{e_1}, \lambda_{e_4}; \lambda_{e_6}) + g_0^{(4)}(\lambda_{e_5}, \lambda_{e_3}; \lambda_{e_6})\ . \end{aligned} \tag{3.94}$$

By a direct computation very similar to the proof for the expression of $G(\widetilde{\Gamma}_{trin})$ in Proposition 3.14.4 we get equation (3.93). □

Chapter 4

Future research directions

In Chapter 2, we analyzed degenerations of Fay's identities to higher orders in the local parameter τ_a near the point a, deriving the new identity (2.2). As an application, we employed this identity to construct solutions of the form (2.4.1) to the Schwarzian KP equation (2.3). These results naturally suggest the following future research directions:

- The first direction is to start with the four-point Fay trisecant identity and study terms of order greater than three in τ_a in the Taylor expansion of the Abel map. This leads to n-th order Taylor expansions of $V(c) = \Theta_{ac}\Theta^*_{cb}$, as discussed in equations (2.45) and (2.46). However, the number of terms grows very rapidly, and simplifying them using known degenerations requires heavy algebraic manipulations. A more systematic approach, based on conceptually stronger techniques, would therefore be desirable. Even the case of order τ_a^4 would already be very interesting, as it could potentially produce integrable PDEs in four dimensions, of which very few examples are known.

- Another possible generalization is to start from the more general n-point version of Fay's identity. Dubrovin ([16]) has shown that the n-point generalization of the identity (2.14) encodes the full KP and KdV hierarchies. A natural question is then: what is the n-point generalization of (2.35)? Does it encode the full SKP and SKdV hierarchies in a similar way?

Chapter 3 focused on constructing a new symplectic parameterization of $SL(2,\mathbb{C})$ character varieties of compact Riemann surfaces. This illustrates the effectiveness of the framework of [7] for such a problem. Although the existing literature already contains many results for $SL(2,\mathbb{C})$, much less is known for $SL(N,\mathbb{C})$. Extending this construction to arbitrary $SL(N,\mathbb{C})$ character varieties of compact Riemann surfaces would therefore be a natural next step. Some technical issues must be addressed, in particular the correct ordering of the N

eigenvalues of monodromy matrices along separating contours. This may require the use of flag varieties [21]. Finally, as noted in Chapter 3, the generating function relating the new coordinates to Fenchel–Nielsen coordinates may possess some applications for integrable systems. These applications would certainly deserve further exploration. There exist also other generating functions related to “flip of the separating contour γ” (see [51]) and also to “flip of a triangulation edge” (see [7]), and the question of knowing how these different functions are related would also be very interesting.

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