

Supporting Second Graders' Understanding of Base-Ten Structure: Challenges and Solutions

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Abstract

Supporting Second Graders' Understanding of Base-Ten Structure: Challenges and Solutions

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The present dissertation pursued two primary goals: first, to investigate second-grade students' understanding of the hierarchical structure of base ten in both symbolic and non-symbolic contexts and second, to enhance their knowledge of base-ten structure. Despite the importance of structural knowledge of base ten for later mathematics learning, children do not readily acquire such knowledge in school. The present research focused on the nature of children's understanding of the relations between base-ten units. Three studies examined second graders' understanding of base-ten structure through their unitizing performance. Unitizing is defined as viewing one quantity in more than one way, such as interpreting a numeral using two different units (e.g., 263 as 263 ones and as 26.3 tens).

In Study 1, children completed tasks assessing both unitizing performance and principle-of-position knowledge. Results showed that children displayed strong principle-of-position knowledge but limited unitizing knowledge when interpreting numerals. Study 2 investigated unitizing performance in a non-symbolic context, namely when solving measurement division problems using manipulatives. Children demonstrated promising unitizing performance in this context. Moreover, the type of manipulative influenced their performance. Finally, Study 3 examined the effectiveness of an instructional intervention for second graders designed to improve unitizing performance with numerals. Its design was informed by the findings in Study 2 and by research on analogies in mathematics instruction. Measurement division problems served as the source of the analogy and numerals as the target. Two types of analogies were compared: explicit analogies, where cognitive supports highlighted the structural similarities between the source and target, and implicit analogies, where no such links were provided. Results showed that analogy-based instruction improved children's unitizing performance when interpreting numerals. A visual analysis of learning trajectories suggests an advantage of

providing explicit analogies in instruction, a finding that was not captured by the statistical analyses.

Together, the overall findings offer several educational implications. They emphasize the need to teach the structure of base ten more explicitly and consistently in early mathematics. They also encourage the use of pedagogical tools, such as using measurement division and analogies, to support second graders' understanding of the hierarchical relation between base-ten units.

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Contribution of Authors

All the three studies conducted in this dissertation were designed in collaboration with my two co-supervisors, Dr. Helena P. Osana and Dr. Anne Lafay. More specifically, we collaborated on the creation of the materials and measures, as well as the design and delivery of the intervention for the third study. I was responsible for recruiting and testing participants, delivering the instructional interventions, and scoring, analyzing, and interpreting the data.

The data collected for Study 1 will be presented in April 2026 at the Cognitive Development Society in Montreal, QC, Canada. The reference for the presentation is the following: Osana, H. P., Barilaro, M., Rodrigues, M. L., & Lafay, A. (2026, April 9-11). *Unpacking second graders' knowledge of the multiplicative relations among base-ten units* [Poster presentation]. Cognitive Development Society, Montreal, QC, Canada.

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Chapter 1: General Introduction

Introduction

The three studies presented in this dissertation intend to deepen our current understanding of second-grade students' knowledge of base-ten structure and to investigate the effectiveness of interventions grounded in relational reasoning. Numerals are omnipresent in daily life. From identifying the correct bus number to being on time, measuring quantities in cooking recipes to understanding a restaurant bill, interpreting survey results, or taking the right dosage of medication, individuals constantly rely on their understanding of numerals to navigate everyday situations. More importantly, a flexible understanding of numerals is crucial. Consider, for example, a father who needs to give his daughter a medication prescribed at 50 grams. The scale available only displays weight in kilograms. With a flexible knowledge of numerals, the father can readily convert the quantity and understand that 50 grams is equivalent to 0.05 kilograms. Another illustration is in the realm of financial information. Individuals may require flexible knowledge to interpret non-canonical expressions of numerals, especially when manipulating prices, salaries, rents, or bank balances (e.g., a rent of 12 hundred dollars per month). As such, a strong understanding of numerals is essential for making sense of quantities in real-life contexts.

In mathematics, a flexible understanding of numerals is based on a deep and strong knowledge of the base-ten numeration. Base-ten numeration is essential for understanding quantities. This mathematical system involves a set of rules to represent quantities using numerals and to structure these quantities through a system of embedded units (e.g., groups of ones, tens, and hundreds). More specifically, in the primary grades, children are required to develop a strong knowledge of base-ten numeration (Programme de formation de l'école québécoise [PFEQ], Ministère de l'Éducation, du Loisir et du Sport, 2006). This knowledge encompasses first learning about the syntax of numerals, which are the rules that govern how any quantity is expressed symbolically in numerals (Cheung & Ansari, 2021; Mix et al. 2019; Ross, 1989). Second, children need to acquire an understanding of the hierarchical structure of base-ten numeration, characterized by a system of multiple levels of embedded units (Fuson, 1990). With a strong knowledge of both base-ten syntax and structure, children are more likely to develop future mathematical knowledge and skills (Carpenter et al., 1998; Mix et al., 2023; Moeller et al., 2011).

Despite the importance of developing both the syntax and structure of base-ten numeration, the literature is sparse on children's knowledge of the base-ten structure. Some studies suggested that children struggle to state the equivalence of different numeration units (e.g., knowing that one thousand is composed of 100 tens; Soury-Lavergne et al., 2020). Other studies evidenced children's difficulty to identify the total number of a given numeration unit in a numeral (e.g., the total number of tens in a three-digit numeral; Tempier, 2016). These studies revealed that children have a limited understanding of the base-ten structure. Notably, no study to date has directly compared children's knowledge of base-ten syntax to their understanding of base-ten structure, an important comparison that could shed light on the relation between both constructs and extend current understandings of how children think about base-ten numeration. Moreover, no study has investigated children's knowledge of the base-ten structure across both symbolic (i.e., numerals) and non-symbolic (i.e., word problems and manipulatives) contexts. Such exploration is needed to identify the factors that might influence children's relational understanding of base ten. Finally, only a few studies have proposed an instructional intervention targeting base-ten syntax and structure (Institute of Education Sciences, 2010) or structure only (Tempier, 2024). The limited validation of instructional approaches provides insufficient knowledge of how to support children's relational reasoning in base ten. The three studies presented in this dissertation aim to address these gaps in the research literature by (1) comparing second-grade students' knowledge of base-ten syntax and structure, (2) investigating the factors that influence children's structural knowledge of base ten in both symbolic and non-symbolic contexts, and (3) exploring the effectiveness of an instructional interventions designed to improve children's relational understanding of base ten.

Base-Ten Numeration

Theoretical Background

In most contemporary societies, quantities are represented symbolically using the Hindu-Arabic numeral system, which relies on a limited set of ten digits (i.e., 0 to 9). Referred to as numerals, those symbolic representations of quantities are governed by specific rules common to all positional systems, including base-ten. Base-ten numeration entails a syntax and a structure.

The syntax of base-ten refers to the rules that allow numerals to be decoded for meaning. Each digit in a numeral is associated with a specific denomination (e.g., ones, tens, hundreds), also called numeration units (Chambris, 2008; Houdement & Tempier, 2019). The value of each

digit depends on its position in the numeral (Ross, 1989), also referred to as the principle of position (Cheung & Ansari, 2021). In a three-digit numeral, the rightmost digit represents the ones, the next digit to the left represents the tens, and the next digit to its left the hundreds. Each numeration unit is associated with a specific power of ten: The ones are associated with 10^0 , the tens to 10^1 , and the hundreds to 10^2 . The interpretation of a numeral requires connecting the value of each digit and the value of the numeration unit associated. This ability is called the count+unit syntax (Mix et al., 2022). More specifically, the count+unit syntax allows for interpreting 347 as three groups of 10^2 , four groups of 10^1 , and seven groups of 10^0 . Although research has shown that principle-of-position knowledge takes time to develop across the primary grades (Cheung & Ansari, 2021), usually by the second grade, students demonstrate a relatively strong understanding of this principle (Mix et al., 2022; Osana & Lafay, 2019).

For children to develop a complete and flexible understanding of base ten, they must also acquire knowledge of the structure of base-ten. The base-ten system involves a hierarchical structure of multiple levels of numeration units (Fuson, 1990). A specific multiplicative relation links numeration units together: a power-of-10 relation (Fuson, 1990). For example, one ten is constituted of 10^1 ones, or 10 ones. In addition, one hundred is composed of 10^1 tens, also referred to as 10 tens, but also 10^2 ones, or 100 ones. Learning about the relations between numeration units requires knowledge of the ten-for-one conception (Fuson, 1990), defined as understanding that 10 of any numeration is equivalent to one of the next numeration unit. For example, 10 tens are equivalent to one hundred.

The ten-for-one conception is based on children's unitizing abilities. Unitizing has been defined as the cognitive process that allows simultaneous interpretation of a quantity as a collection of single lower-level units and any one higher-level unit (Fosnot & Dolk, 2001; Lamon, 1996; Simon, 2024). Unitizing a quantity in a base-ten context thus allows children to consider one collection of 10 things as the same as one thing of a higher-level numeration unit. More importantly, unitizing in a base-ten context is the process of interpreting any quantity according to the numeration unit used. For example, 245 can be seen simultaneously as 24.5 tens but also as 245 ones. As such, unitizing provides evidence of children's flexible understanding of the structure of base ten and can therefore serve as powerful indicator of children's understanding of this structure.

However, research has shown that children struggle when learning about base-ten structure (Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016). Soury-Lavergne et al. (2020) showed that children presented difficulty in finding equivalencies between numeration units (e.g., that 100 tens is equivalent to one thousand). Additionally, Tempier (2016) found that children struggled to interpret one numeral according to a given numeration unit, such as finding the total number of tens in 528. Finally, Barilaro et al. (2026) demonstrated that first- and second-grade students faced challenges when identifying the total number of a given numeration unit in visual representations of three-digit quantities. Notably, none of these studies investigated children's unitizing skills in base ten, which can be used as an indicator of base-ten structure understanding. Therefore, the first study of this dissertation aimed to extend current understanding of how children unitize quantities using numeration units.

Outline of Study 1

The current literature on children's understanding of the base-ten structure remains limited, and no study specifically investigated children's unitizing performance. Additionally, to my knowledge, no prior study has jointly examined children's understanding of both the syntax and the structure of base-ten numeration. Investigating these two constructs together could yield a more comprehensive conceptual model of children's knowledge of the base-ten system. Therefore, the first study in this dissertation aims to explore children's knowledge of both the syntax, operationalized as their principle-of-position knowledge, and the structure, assessed by their unitizing skills. The focus of my analysis in Study 1 is on students who present a strong principle-of-position knowledge but limited unitizing skills. These students are particularly interesting given that, although children generally show solid principle-of-position knowledge by the end of second grade, their ability to unitize often remains limited, even later in adulthood (Barilaro et al., 2024). Therefore, understanding how these students reason about structure could offer valuable pedagogical insights to better support students who struggle with unitizing in base-ten contexts.

Factors Influencing Children's Unitizing Abilities

Unitizing has been found to be challenging for young children in contexts other than base ten (Shipley & Shepperson, 1990; Sophian & Kailihiwa, 1998; Wege et al., 2023). In those studies, children were provided with non-symbolical representations of quantities (e.g., manipulatives) and were asked to identify the total number of groups. Additionally, given the

current literature on children's understanding of the base-ten structure (Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016), unitizing using numeration units in non-symbolical contexts might also be challenging. However, little is known about the conditions under which children can unitize quantities in non-symbolical contexts.

Measurement Division Problems

Carpenter et al. (1996) claimed that measurement division problems are a valuable pedagogical tool to support children's understanding of the base-ten structure. More specifically, in a measurement division problem, children are required to find the total number of groups that can be created from an initial quantity according to a specified number of items in each group (Carpenter et al., 2014). These types of problem are particularly relevant to an investigation on unitizing because children are required to see the initial quantity in two different ways: as a collection of single items and as a collection of groups composed of a specified number of single items. Additionally, when solving measurement division problems, children might not rely as much on their principle-of-position knowledge as they do when interpreting numerals. In other words, the context of a measurement division problem, if not in a base-ten context, might not activate their prior knowledge of base-ten numeration. Using measurement division problems could therefore be used as a tool to gain a deeper understanding of children's unitizing skills. However, in the early grades, children are not yet introduced to measurement division problems, which can impede their ability to solve such problems. Interestingly, Carpenter et al. (1993) found that children as young as kindergarten can solve measurement division problems when provided with manipulatives.

How Manipulatives can Influence Children's Reasoning

Manipulatives are external representations that are used to represent abstract mathematical concepts and with which children can physically interact when learning mathematics. Various authors have shown that the perceptual features of manipulatives can impact children's learning and performance in various mathematical domains, such as arithmetic, word-problem solving, or fractions (Carbonneau et al., 2013; Laski et al., 2015). For this reason, the features of manipulatives should be considered when they are used in instruction. While previous studies have investigated children's unitizing performance using external representations (Shipley & Shepperson, 1990; Sophian & Kailihiwa, 1998; Wege et al., 2023), they did not explicitly examine whether and how the features of manipulatives might impact

performance. Because prior studies on young children's unitizing demonstrated that children show a strong bias toward enumerating only single items when asked to unitize quantities, rather than groups of items, exploring the potential impact of the features of the external representations on performance is necessary.

First, the realism of the manipulatives might play a role in children's ability to identify a group. As shown in the literature (e.g., Carbonneau & Marley, 2015), the use of realistic manipulatives can shift children's attention away from the abstract concepts represented by the objects. For instance, Wege et al. (2023) asked children to unitize quantities presented with realistic manipulatives (e.g., animal counters). Their findings showed that children can sort the animal counters in groups and name the groups, but still focus on counting the discrete items when asked to count the groups. Wege et al. hypothesized that the realism of the manipulatives might have prompted children to focus on each individual item.

Second, some studies (Boyer et al., 2008; Jeong et al., 2007) found that the perceptual salience of discrete items in visual representations of continuous quantities directed children to count the items instead of attending to the continuous information that was required to solve proportional problems. Additionally, Barilaro et al. (2026) found that when children were presented with visual representations of three-digit quantities, they were able to identify the total number of tens contained in the quantity, but only when those tens were visible. The findings of Barilaro et al. highlight the importance of making the groups visible for children to see and count them.

Finally, Osana et al. (2026) found that physically constructing base-ten units supported children's representations of numerals using manipulatives. More specifically, the authors found that when using attachable beads, second graders created more accurate displays of two- and three-digit numerals than those who used string beads (i.e., beads already stacked together, where the individual ones were visible) or pipe cleaners (i.e., sticks of different lengths representing base-ten units, where the ones were not visible). The authors concluded that the physical actions of constructing the base-ten units with the attachable beads helped children develop a deeper understanding of place value than those who did not create the units themselves. Therefore, examining how the features of manipulatives influence children's ability to unitize quantities in base-ten contexts could yield valuable pedagogical insights for supporting the development of their unitizing skills.

Outline of Study 2

Measurement division problems embedded in base-ten contexts are expected to support children's unitizing because measurement division has the same hierarchical structure as the structure of base-ten numerals. Additionally, previous studies have demonstrated that the types of manipulatives that are used to solve mathematical problems can impact children's understanding of the concepts the objects are intended to represent (e.g., Carbonneau & Marley, 2015). In the context of unitizing, several perceptual features might play a role in children's ability to count composite units: the realism of the manipulatives, the perceptual salience of the units, and whether the manipulatives allow for the physical construction of groups. Therefore, Study 2 entails to investigate second-grade students' unitizing abilities when solving measurement division problems using various types of manipulatives. Tempier (2016) found that students tend to rely heavily on their principle-of-position knowledge when asked to interpret a numeral given a specific numeration unit. Interestingly, children appear to interpret numerals only through a syntax lens. Measurement division problems might limit the activation of prior knowledge of base-ten syntax, which in turn may offer a better picture of their unitizing skills. In the second grade, students have not yet been introduced to the concept of multiplication and division, making the solving of measurement division problems particularly difficult without the help of external representations. As shown in previous studies (Carpenter et al., 1993), children as young as those in kindergarten can solve such problems when given appropriate tools, such as manipulatives. Therefore, using measurement division problems along with different types of manipulatives might help identify the conditions under which children are most likely to successfully unitize in base ten.

Effective Mathematics Instruction

The Use of Analogies

The current literature on base-ten numeration has identified that students struggle to understand the base-ten structure (Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016). Given that knowledge of the base-ten structure is predictive of future mathematics achievement (Mix et al., 2023; Moeller et al., 2011), teachers should be encouraged to provide effective and targeted instructional interventions to support children's learning of the base-ten structure.

In the current literature on mathematical instructional intervention, analogies are considered an effective tool for helping children develop a deep and robust understanding of mathematical concepts. Analogical reasoning is a specific type of relational reasoning and have been revealed as a powerful cognitive tool for learning (Holyoak, 2013). This type of reasoning entails comparing a well-known situation (i.e., a source) to a to-be-learned situation (i.e., the target; Gentner, 1983). More specifically, when reasoning with analogies, individuals must identify the underlying conceptual similarities between the source and the target to abstract the structural commonalities between the two.

Analogies have been found to be widely used in mathematics instruction (Richland et al., 2004a). Two types of instructional uses of analogies have been tested in prior research: explicit and implicit analogies. Explicit analogies are characterized by the presence of explicit cognitive supports (Richland et al., 2007), while implicit analogies are characterized by the activation of the source prior to introducing the target without any explicit connections between the two (Sidney & Thompson, 2019). When presenting explicit analogies, teachers can use a variety of cognitive supports designed to help children direct their attention to the relevant structural similarities between the source and the target. Cognitive supports can take different forms, such as spatially aligning the source and the target (Matlen et al., 2020), and using gestures (Richland, 2015), relational language (Loewenstein & Gentner, 2005), and colors (Alibali et al., 2018). These cognitive supports are intended to help learners notice the structural similarities between the source and the target, resulting in learning of the target concepts. In contrast, implicit analogies are characterized by activating prior knowledge of the source before introducing the target (Sidney & Thompson, 2019). In these cases, the source and target are not simultaneously visible, and working on the source before introducing the target is intended to activate the relevant prior knowledge of the source, which then fosters mapping the conceptual structure of the prior knowledge to the target. Teachers using implicit analogies do not make explicit connections between the source and the target. Exploring the effectiveness of both explicit and implicit analogies in supporting children's learning about base-ten structure might provide valuable insight into educational practice, both with respect to unitizing, but also more generally across other mathematical domains.

Outline of Study 3

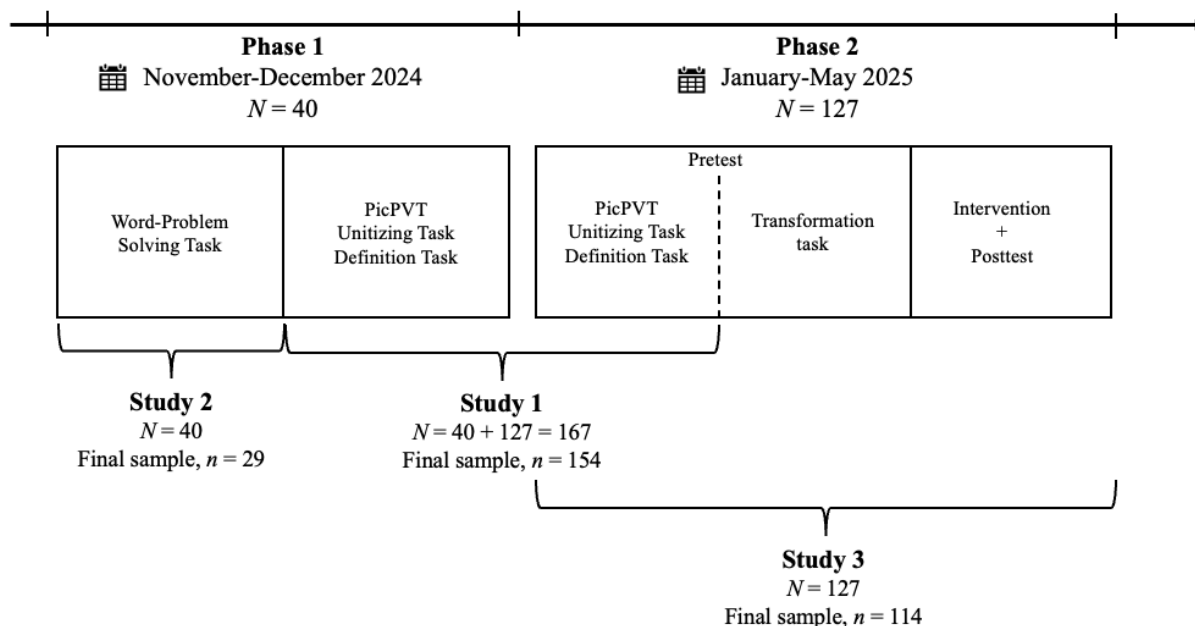
The third study of this dissertation investigates the effectiveness of an instructional intervention designed to improve children's unitizing abilities through the use of analogies. Solutions of measurement division word problems were used as the source analog, while numerals were used as the target. The purpose of the analogy was to support children's understanding of how to unitize numerals using different numeration units by recognizing the structural similarities to measurement division problems. Children participated in one pretest and one posttest session in which they completed tasks assessing their unitizing abilities and their principle-of-position knowledge. For the instructional intervention, children were randomly assigned to three conditions: an explicit condition, an implicit condition, and a control condition. In the explicit condition, the experimenter provided cognitive supports to map the solutions of the measurement division problems to the numerals. The explicit cognitive supports entailed gestures, relational language, colors, and spatial alignment. In the implicit condition, the experimenter presented measurement division problems and their solutions, before presenting numerals, without making any explicit connection between the solutions and the numerals. The presentation of the problems and their solutions prior to the numerals intend to activate children's prior knowledge about the structure of measurement division problems, similar to the structure of numerals. The results of the study promise important educational implications, such as informing teachers and educators about the effectiveness of using analogies in the context of learning the base-ten structure.

Distribution of the Sample of the Three Studies

The three studies presented in this dissertation were conducted with second graders in the province of Quebec (Canada) between November 2024 and May 2025. Figure 1 presents a visual summary of how the samples were constituted in each of the three studies. Recruitment took place in two phases. The first phase, conducted for Study 2, led to the recruitment of 40 participants who were tested in November and December 2024. The second phase occurred in 2025 and led to the recruitment of 127 students for Study 3, the intervention study. Participants for Study 1 ($N = 167$) were drawn from the combined samples of Studies 2 and 3.

Figure 1

Distribution of Participants Across the Three Studies



Note. Only 29 students from Study 2 were included in Study 1 based on the inclusion criteria described in Chapter 2. Additionally, two students recruited for Study 3 were excluded from both Studies 1 and 3: One did not consent to participate, and the other was absent on the day of the pretest. This resulted in a final sample of 154.

Overall Research Aim

I hope that the research conducted in the three studies presented in this dissertation can provide meaningful insights into children's understanding of base-ten structure and serves as a foundation for improving instructional approaches in teaching this foundational concept. Beyond supporting students through effective instruction, my goal was also to offer practical tools that teachers can readily incorporate in their mathematics classrooms. I strongly believe that teachers play a pivotal role in mathematics education, and I hope this dissertation contributes to supporting this important task.

Chapter 2: Study One

The Unseen Challenge in Mastering the Structure of Base Ten: Investigating the Nature of Second Graders' Unitizing

Introduction

Understanding quantities, whether represented symbolically (i.e., numerals) or non-symbolically, is essential in everyday life, from managing time, budgeting, and making travel plans, to making decisions about voting intentions (Ladel & Kortenkamp, 2016; Maass et al., 2022). In most modern societies, quantities are represented using the base-ten numeration system, a structured mathematical system encompassing specific rules and a hierarchical structure. Developing a strong knowledge of base ten is therefore fundamental to accurately interpreting and manipulating quantities in daily life. Learning base-ten numeration is also a key focus of the early mathematics curriculum (Programme de formation de l'école québécoise [PFEQ], Ministère de l'Éducation, du Loisir et du Sport, 2006), but it is complex and takes time to develop (Cheung & Ansari, 2021). In particular, developing an understanding of base ten relies on the acquisition of both its syntax and structure. The syntax refers to the rules for reading and writing numerals, whereas the structure of base ten is characterized by a hierarchical structure of embedded units (e.g., ones, tens, hundreds).

Several studies have shown that children tend to master the syntax of base-ten numeration by the second grade (Mix et al., 2022; Osana & Lafay, 2019). However, the few studies investigating individuals' knowledge of the structure demonstrated that this aspect of numeration is particularly challenging for children (Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier 2016) and adults (Barilaro et al., 2024; Thanheiser, 2009). This specific challenge could be detrimental for children given that structural knowledge of base ten provides the foundation for more advanced mathematical concepts, such as rational numbers (Steffe, 1991).

The present study aims to broaden current understandings of children's knowledge of base-ten numeration. In so doing, I will compare children's knowledge of base-ten syntax to their knowledge of the structure. Additionally, the present study will produce a fine-grained account of the challenges children experience when thinking and learning about the structure of base ten. Such an account will be useful for identifying potential pedagogical supports to improve children's understanding of the base-ten structure.

Base-Ten Numeration

Base-Ten Numeration Syntax: Principle of Position

Base-ten syntax refers to a set of rules that govern how quantities are represented as numerals, which are symbolic expressions using a finite set of digits from the Hindu-Arabic system (e.g., using the digits 3, 2, and 5 to represent the quantity three hundred and twenty-five as 325). Also referred to as principle of position (Cheung & Ansari, 2021), base-ten syntax is based on the idea that the specific value of a digit depends on its position in the numeral (Ross, 1989). The values (e.g., ones, tens, hundreds) are considered units, called “numeration units” by Chambris (2008) and Houdement and Tempier (2015). For example, in a three-digit numeral, the rightmost digit corresponds to the ones, the next digit to the left corresponds to the tens, and the next digit to the left, the hundreds. Each numeration unit is linked with a specific power of ten; the ones correspond to 10^0 , the tens to 10^1 , the hundreds to 10^2 , and so on. Interpreting a numeral, therefore, entails connecting the value of each digit to the value of its associated numeration unit, referred to as the count+unit syntax (Mix et al., 2022). The count+unit syntax specifies the precise value of any digit in a numeral: In 325, the digit 5 corresponds to five groups of 10^0 , the digit 2 to two groups of 10^1 , which is 20, and the digit 3 to three groups of 10^2 , which is 300. Principle-of-position knowledge permits an interpretation of the same digit having a different value depending on its position. The digit 2, for example, can have different values in 222; the rightmost 2 represents two groups of 10^0 , the 2 one place to the left represents two groups of 10^1 , and the leftmost digit represents two groups of 10^2 .

Individuals who have not yet mastered the syntax may produce errors when transcribing spoken number words into numerals. For instance, when hearing “three hundred and twenty-five,” a child might write something like “300205,” reflecting a tendency to transcribe each spoken word into a separate quantity, without applying syntactic, or principle-of-position, rules (Moeller et al., 2011; Moura et al., 2013). These so-called syntactic errors are commonly observed in the early grades, especially because of their developing understanding of base-ten syntax (Moura et al., 2013). Although the syntax is essential for writing and understanding numerals, children also need to acquire knowledge of the structure of the base-ten system, which supports the manipulation of quantities, whether symbolic or non-symbolic, and the development of future mathematical concepts (e.g., addition, subtraction, decimal numbers; Carpenter et al., 1998; Ladel & Kortenkamp, 2016; Tempier, 2024).

Base-Ten Numeration Structure: A System of Embedded Units

Base-ten numeration is characterized by a specific structure that relies on multiple levels of units (Fuson, 1990). Numeration units are linked by a specific multiplicative relation: a power-of-10 relation (Fuson, 1990). For example, one ten is composed of 10^1 ones (i.e., 10 ones), while a hundred consists of 10^1 tens (i.e., 10 tens) but also of 10^2 ones (i.e., 100 ones). When learning about numeration units, children usually acquire declarative knowledge of the relation between numeration units (e.g., one ten is equivalent to 10 ones; one hundred is equivalent to 100 ones). However, such declarative knowledge is not enough, and Fuson (1990) argued that a key concept in learning base-ten numeration is the ten-for-one conception—the understanding that grouping 10 numeration units at any given level creates one numeration unit of the next higher level (e.g., 10 ones together result in the creation of one ten).

The ten-for-one conception can be considered a particular case of unitizing. Unitizing is defined in the present research as simultaneously interpreting a group of discrete things as a collection of lower-level units (i.e., a collection of things) and one higher-level unit (i.e., one group; Fosnot & Dolk, 2001; Lamon, 1996; Simon, 2024). Fosnot and Dolk (2001) highlighted the importance of the developmental process through which children learn to simultaneously coordinate two levels of units (adjacent or not) to unitize quantities. In the context of base-ten numeration, the unitizing process depends on a selected numeration unit and involves interpreting a quantity based on that unit. For example, the quantity 365 can be seen as 3.65 units of a hundred and at the same time as 365 units of one. The ability to unitize numerals is therefore a meaningful indicator of children's flexible understanding of the base-ten structure. Understanding this hierarchical structure is fundamental for recognizing how numbers are constructed and for interpreting and manipulating quantities. For example, when asked to solve a subtraction problem involving regrouping (e.g., $47 - 29 =$), children can efficiently and meaningfully perform transformations based on the relations between numeration units to complete the computation (e.g., decomposing 47 into $30 + 17$ to subtract 9 from 17).

According to the Quebec school curriculum (Programme de formation de l'école québécoise [PFEQ], Ministère de l'Éducation, du Loisir et du Sport, 2006), second graders are expected to know the names of the numeration units (e.g., ones, tens, and hundreds) as well as the relations between them (e.g., a ten is composed of 10 ones). Accordingly, instruction should emphasize the principle of position, the relation between numeration units, and processes of

unitizing. However, researchers have found that teachers tend to place a strong emphasis on the principle of position while overlooking numeration units and their relations (Houdement & Tempier, 2019; Tempier, 2024). This teaching practice may lead children to interpret numerals solely through the lens of the principle of position and fail to recruit the relational structure between numeration units when needed. Given this pedagogical emphasis on syntax over structure, a critical question arises: What do second graders actually know and understand about base ten with respect to syntax and structure?

Children's Knowledge of Base-Ten Syntax and Structure

An important part of the literature on base-ten numeration has focused on studying children's understanding of syntax. Some studies have shown that principle-of-position knowledge takes time to develop, with estimates ranging from 5 to 7 years old (i.e., approximately kindergarten to second grade; Cheung & Ansari, 2021). Birgin et al. (2022) found that not all second graders reach an adequate level of principle-of-position knowledge, and struggle, for example, to transcribe number words (e.g., three tens and five ones) into a numeral, or the reverse. However, most studies revealed that children demonstrate a strong knowledge of the base-ten syntax by some point in the second grade (Mix et al., 2022; Osana & Lafay, 2019).

Nevertheless, as previously discussed, mastering base-ten numeration requires more than a strong knowledge of syntax. It also depends on structure, which appears to be particularly challenging for many children (Barilaro et al., 2026; Ladel & Kortenkamp, 2016; Soury-Lavergne et al., 2020; Tempier, 2016). First, researchers have documented specific difficulties experienced by children when learning the relation (i.e., the equivalence) between numeration units. Tempier (2016) found that third-grade students struggled to identify equivalencies between numeration units (e.g., only 49% of third-grade students identified that one hundred equals 10 tens, and only 37% converted three hundreds into 300 ones). Children's difficulties in relational reasoning between units were observed predominantly between hundreds and ones, and hundreds and tens. This finding was replicated by Soury-Lavergne et al. (2020), who showed that children from second to fifth grade had difficulties reporting the equivalencies between numeration units, such as decomposing one thousand into 100 tens. Therefore, these studies show that children appear to struggle with basic declarative knowledge of the relations between numeration units, which might be the first step to successfully unitizing quantities.

Second, research has shown that children from the first to fifth grade struggle to interpret a quantity using a given numeration unit (Barilaro et al., 2026; Chambris, 2008; Tempier, 2016). Barilaro et al. (2026) found that when first and second graders were asked to interpret visual representations of multi-digit quantities based on a given numeration unit (e.g., how many tens in a visual representation of 325), only 30% of students correctly accounted for embedded units. Similarly, Chambris (2008) reported that fewer than a quarter of fifth-grade students were able to interpret four-digit numerals in terms of hundreds (e.g., how many hundreds in 8734?), and Tempier (2016) found comparable results when third-grade students were asked to identify the number of tens in a three-digit numeral. Tempier (2016) observed children using their principle-of-position knowledge when asked to find the number of the tens in a numeral: The majority focused on the tens position rather than on the relations between the levels of numeration units. However, the author did not conduct a systematic analysis of children's response types, which makes it difficult to understand children's challenges when unitizing quantities.

In the current literature, no studies have directly compared children's knowledge of base-ten syntax and structure, which limits what is known about the development of their base-ten knowledge. Exploring the syntax and structure of base-ten numeration in tandem could produce relevant information about their development and could shed light on children's specific needs when learning about base-ten numeration in the classroom. Additionally, the few studies that have investigated children's knowledge of the base-ten structure only focused on their ability to decompose a numeral according to one specific given numeration unit (e.g., to identify the number of tens in 345). Little is known about children's unitizing in base ten, at least as assessed by their ability to interpret one numeral using two or more numeration units. Therefore, assessing children's knowledge of syntax through a principle-of-position task and structure through a unitizing task within the same study, and investigating factors that may explain persistent difficulties with structure understanding, may offer valuable insights to better support students' numeration knowledge.

Present Study

Understanding base-ten numeration relies on the development of both syntax and structure. The current literature indicates that while second graders generally demonstrate strong knowledge of base-ten syntax (Cheung & Ansari, 2021; Mix et al., 2022; Osana & Lafay, 2019), they often struggle with base-ten structure. More specifically, various studies have shown that

children experience challenges in their knowledge of relations between numeration units (Soury-Lavergne et al., 2020) as well as applying these relations to interpret quantities given a specific numeration unit in base-ten (Barilaro et al., 2026; Tempier, 2016). These difficulties can persist into adulthood (Barilaro et al., 2024; Thanheiser, 2009), resulting in difficulties understanding topics covered later in the elementary curriculum. To date, no studies has investigated children's profiles of base-ten understanding by jointly examining their knowledge of the syntax and structure. The following study aimed to examine children's knowledge of syntax using a task assessing their knowledge of the principle of position and their knowledge of structure using a task assessing their unitizing in base-ten contexts. More specifically, I focused on the nature of unitizing challenges for children with strong principle-of-position knowledge but poor unitizing performance. To do so, I compared children with strong principle-of-position knowledge and poor unitizing skills to children with strong knowledge of both and to children with weak knowledge of both. This comparison was conducted by an investigation of children's declarative knowledge of the relation between numeration units and their ability to apply this declarative knowledge to unitize base-ten quantities. The findings of the present study promise to inform instructional practices that intentionally and effectively support the development of base-ten structure knowledge.

Thus, the present study investigated second graders' understanding of base-ten numeration, with a specific focus on their principle-of-position and unitizing knowledge. More precisely, it addressed the following two research questions:

1. What is the relation between second graders' knowledge of base-ten syntax, assessed through a principle-of-position task, and their knowledge of base-ten structure, as assessed through a unitizing task? Is children's knowledge of base-ten syntax stronger than their knowledge of base-ten structure? Drawing on previous research (Barilaro et al., 2026; Houdement & Tempier, 2019; Osana & Lafay, 2019; Soury-Lavergne et al., 2020; Tempier, 2016), I hypothesized that the two constructs would be related, and that second-grade students would demonstrate higher principle-of-position than unitizing knowledge.
2. What distinguishes second graders who present strong principle-of-position knowledge but poor unitizing performance from (a) those with both strong principle-of-position and unitizing performance, and (b) those who show weak knowledge in both, on (1) their declarative knowledge of the relation between numeration units, and (2) the nature of

their responses when unitizing numerals? This research question will be addressed with an exploratory analysis that will focus on the unitizing difficulties experienced by children who demonstrate a strong principle-of-position knowledge. Several studies have focused on improving children's understanding of the principle of position (e.g., Mix et al., 2019), but relatively fewer have targeted children's knowledge of the base-ten structure (noteworthy exceptions are Institute of Education Sciences, 2010; Tempier, 2024). Gaining a better understanding of the challenges faced by second-grade students who possess strong principle-of-position knowledge but struggle with unitizing could inform the development of instructional approaches aiming to address their specific difficulties.

Method

Participants

Participants were second graders recruited in seven cities in the province of Quebec, Canada. Different recruitment procedures were used depending on the region in which the schools were located. In one city, the project was approved by a school board, and I contacted principals within the board by email and invited them to share information about the study with their second-grade teachers. In another city, I contacted second-grade teachers directly via email using the addresses available on the schools' website. Finally, in the five other cities, I asked schools' administrative assistants to distribute the study pamphlet to the second-grade teachers. Throughout the recruitment process, all teachers interested in participating were invited to contact me by email or phone to obtain more information about the study. The teachers who agreed to participate were asked to oversee the distribution and collection of consent forms and demographic questionnaires from parents. Once parental consent was obtained, I scheduled the meetings with the students in coordination with the teachers.

In total, 12 teachers across nine elementary schools participated in the study, and 165 second graders were enrolled. The socioeconomic status of each school is presented in Table 1. Schools can obtain a score from 1 (least disadvantaged) to 10 (most disadvantaged), and schools with scores of 8, 9, or 10 are considered disadvantaged by the Ministère de l'Éducation (2024). The score is based on the proportion of children from families where the mother has no high school diploma and the proportion of families with both unemployed parents.

Table 1*Socioeconomic Status of the Schools Participating in the Study*

City	School	Socioeconomic Status
City 1	School A	4 (advantaged)
	School B	2 (advantaged)
	School C	10 (disadvantaged)
City 2	School D	8 (disadvantaged)
City 3	School E	3 (advantaged)
City 4	School F	1 (advantaged)
City 5	School G	6 (advantaged)
City 6	School H	10 (disadvantaged)
City 7	School I	9 (disadvantaged)

Eleven children were excluded from the study as they did not complete the task assessing their declarative knowledge of relations between numeration units. This task was added to the protocol after these first 11 participants were met. Thus, the final sample comprised 154 children, with a mean age of 95.72 months (i.e., 7.9 years old; $SD = 5.17$; one missing datum). Among them, 79 were girls (51.3%), 73 were boys (47.4%), and two had missing data (1.3%). Most of the sample spoke only one language at home ($n = 117$, 76%), with a majority speaking French ($n = 113$, 73%). The remaining children spoke either two languages at home ($n = 32$, 21%) or three ($n = 5$, 3%). Details of the various spoken languages are presented in Table 2.

Table 2*Proportion of Children by Spoken Language*

Languages		<i>n</i> (%)
Monolingual		
	French	113 (73.4%)
	English	4 (2.6%)
	Nepali	1 (0.7%)
	Spanish	1 (0.7%)
Bilingual		

	French & English	9 (5.8%)
	French & Arab	8 (5.2%)
	French & Other	13 (8.4%)
Multilingual		
	French, English, & Other	5 (3.2%)

Regarding family income, most children ($n = 95$, 62%) came from a family with an income equal to or above \$80,000 a year. Twenty-six children (17%) were from families earning between \$20,000 and \$80,000 a year, and four children (3%) were from families with an income below \$20,000 a year. Parents for 29 participants (18%) did not report income information in the demographic questionnaire. In the sample, 25 students (16.2%) were identified as having mathematics difficulties based on their parents' responses to the demographic questionnaire, eight students (5.2%) were reported as having a Specific Learning Disorder with impairment in mathematics (Diagnostic and Statistical Manual of Mental Disorders, Fifth Edition, 2013), and there was one missing datum. Additionally, 21 students were identified as having an Attention-Deficit/Hyperactivity Disorder (13.6%).

Design and Procedure

A correlational research design was used to investigate whether children's principle-of-position knowledge differs from their unitizing skills. I met with each second grader individually. Before starting completing the tasks, I read aloud and explained an assent form to collect each second grader's assent to participate. The study obtained ethics approval from Concordia University.

Data collection took place from November 2024 to May 2025. Participants in this study were part of two separate studies that followed slightly different protocols (Study 2 and Study 3 of this dissertation). In total, 154 children were individually assessed by me. For 29 of them (Study 2), the session lasted approximately one hour, while for the remaining 125 children (Study 3) the session lasted approximately 30 minutes. All children completed the same three tasks, which are the focus of the present study: a task assessing their declarative knowledge of the relation between numeration units, a principle-of-position task, and a unitizing task. A fourth

task was also administered to each child, varying depending on the study protocol. As these additional tasks fall outside the scope of the present study, it is not reported here.

Children were randomly assigned to two isomorphic versions of the principle-of-position task (version 1 and version 2) and the unitizing task (version 1 and version 2). Of the 154 children, 92 were presented with version 1, and 62 with version 2.

Measures

Base-Ten Syntax: PicPVT

Children’s knowledge of the principle of position was assessed using the Picture Place Value Task (PicPVT; Osana & Lafay, 2019). Both versions of the PicPVT are presented in Appendix A (version 1) and Appendix B (version 2). Children were presented with a numeral accompanied by an image of dots. Children were asked to determine whether the image corresponded to the underlined digit in the displayed numeral. The images depicted either individual dots, groups of 10 dots, or groups of 100 dots. Half of the numerals shown were two-digit quantities (10 items) and the other half were three-digit quantities (10 items). Examples of both two- and three-digit quantities are presented in Figure 2.

Figure 2

Matching and Non-Matching Items on the PicPVT



There were 20 items on the PicPVT. On half of the items, the image matched the underlined digit in the presented numeral, as shown in Figure 2. For the remaining 10 items, the image did not correspond to the underlined digit in the numeral, as illustrated in Figure 2. Items were presented in the same pseudo-randomized order for all children. Participants received one

point for each correct answer and zero for each incorrect answer, for a minimum of zero points and a maximum of 20 points.

Base-Ten Structure: Definition Task

The Definition task was used to assess participants' declarative knowledge of the relation between numeration units. Specifically, the task evaluated whether the participants knew that there were 10 ones in a ten and that there were 100 ones and 10 tens in a hundred. The first question was a practice question to ensure they understood the task (e.g., "In a number, what is a one? Can you explain it in your own words?"). There were two trial questions. The first question was, "In a number, what is a ten? Can you explain it in your own words?" An optimal response included the relation between the ones and the tens (i.e., there are 10 ones in a ten). If the child did not mention the relation between those two numeration units, I gave the following prompt: "What is inside a ten? What do we need to make a ten?" Children received one point if they stated the correct relation between the ones and tens, and zero points for any other answer.

The second question was "In a number, what is a hundred? Can you explain it in your own words?" An optimal response included (a) the relation between the ones and the hundreds (i.e., there are 100 ones in a hundred), and (b) the relation between the tens and the hundreds (i.e., there are 10 tens in a hundred). If only one relation was mentioned, I gave the following prompt: "What else is inside a hundred? What do we also need to make a hundred?" If no relation was mentioned, I gave the following prompt: "What is inside a hundred? What do we need to make a hundred?" Children received one point for each correct relation mentioned, and thus could receive between zero, one, and two points. Children received two points if they correctly mentioned the relation between ones and hundreds and the relation between tens and hundreds. Children received one point if they only correctly mentioned one of the two relations. Finally, zero points were awarded if neither relation was mentioned. For the entire task, children could receive a minimum of zero points and a maximum of three points. I computed a definition mean score by taking the total number of points divided by three.

Base-Ten Structure: Unitizing Task

Description of the Task. The Unitizing task was used to assess children's ability to unitize a quantity presented in numerals. More specifically, this task assessed whether participants were able to interpret a quantity using two numeration units (for the two-digit items) or three numeration units (for the three-digit items). Both versions of the Unitizing task are

available on Appendix C (version 1), and Appendix D (version 2). At the beginning of the Unitizing task, all children were reminded of the relation between ones and tens (i.e., “There are 10 ones in a ten”) and between tens and hundreds (i.e., “There are 10 tens in a hundred”). After presenting the relations between numerations units, I explicitly encouraged children to remember these relations to complete the items on the task.

In each of 12 items, I showed a numeral and asked children to identify the total number of a specific numeration unit in the numeral (i.e., the number of ones, tens, or hundreds). Six of the items were two-digit numerals (e.g., 27) and six were three-digit numerals (e.g., 238). For each two-digit numeral, I asked participants about the number of ones (“In this number, how many ones are there in all?”) and the number of tens (“In this number, how many tens are there in all?”). For half of the two-digit numerals (i.e., three items), I asked the questions about the ones first, and for the other half, I asked the questions about tens first.

For each three-digit numeral, I asked participants to identify the total number of ones (“In this number, how many ones are there in all?”), tens (“In this number, how many tens are there in all?”), and hundreds (“In this number, how many hundreds are there in all?”). For two of the three-digit numerals, I asked the questions in the following order: ones, tens, and hundreds. For two of the other three-digit numerals, I asked the questions in the following order: tens, ones, and hundreds. Finally, for the final two three-digit numerals, I asked the questions in the following order: hundreds, tens, and ones.

Scoring. The task assessed unitizing, which refers to interpreting a quantity in two ways – that is, using two different numeration units. The following scoring procedure thus reflected this definition. For the two-digit items, participants received one point per item when they correctly answered both the question about the ones and the question about the tens. Participants obtained zero points if they answered either or both questions incorrectly. For example, for the numeral 24, participants obtained one point when they answered that there are 24 ones and two tens in 24. Zero points were awarded for any partial responses (i.e., only one of the two questions was correctly answered) or for incorrect responses on both questions. Participants could obtain between zero and six points for the two-digit items.

For the three-digit quantities, participants received one point if they correctly answered two or three of the questions asked on one item (i.e., about the number of ones, tens, and hundreds). Participants obtained zero points if they provided one or no correct response. For the

item 238, for example, participants could obtain one point if they accurately stated that there are 238 ones, 23 tens, and two hundreds. Participants could receive one point if they responded with two hundreds and 238 ones but made an error on the number of tens. For the three-digit items, participants could obtain between zero and six points. For the complete task, participants could obtain a total score ranging from zero to 12 points. A mean accuracy score, referred to as the unitizing score, was computed by dividing the total number of points obtained on all the items by 12.

Analytic Plan

First, I conducted a regression to examine the relation between children's performance at the PicPVT, at the Unitizing task, and at the Definition task. This analysis was followed by a *t*-test to investigate the difference between children's performance at the PicPVT and children's performance at the Unitizing task (Research Question 1). Second, I conducted an exploratory analysis to investigate children's unitizing difficulties (Research Question 2). The analysis entailed assigning children to different profiles of base-ten knowledge based on their performance on the PicPVT and the Unitizing task. Base-ten knowledge profiles were then compared on their knowledge of the relations between numeration units (assessed through the performance on the Definition task) and on their unitizing performance, assessed on the Unitizing task.

Results

Second Graders' Understanding of Base-Ten Numeration (Research Question 1)

Participants obtained a mean score of .28 ($SD = .41$) on the Unitizing task, a mean score of .89 ($SD = .16$) on the PicPVT, and a mean score of .61 ($SD = .35$) on the Definition task. A significant positive correlation was found between the PicPVT and unitizing, $r = .24, p = .003$; between the PicPVT and the definition score, $r = .30, p < .001$, and between unitizing and the definition score, $r = .32, p < .001$.

The scores on the Unitizing task were right-skewed, with a large proportion (62%) of children obtaining a score of zero. The unitizing score was therefore transformed into a binary variable with the absence of unitizing skills (i.e., score of zero) and the presence of unitizing skills (i.e., above zero score) as the two outcomes. A binary logistic regression was performed to test the effects of declarative knowledge of unit relations (the definition score) and principle-of-position knowledge (the PicPVT score) on the likelihood that children would engage in unitizing.

Overall, the model significantly predicted the odds of children demonstrating unitizing, $\chi^2(2) = 15.57, p < .001$. The model explained 13.1% of the variance in unitizing and correctly classified 65.6% of cases. Likelihood ratio tests indicated that both the PicPVT score ($p = .014$) and definition mean score ($p = .04$) significantly predicted unitizing. Specifically, for every one-unit increase in PicPVT score, the children were 29.5 times more likely to unitize (95% CI [1.96, 443.17]). In addition, for every one-unit increase in definition score, the children were 3.1 times more likely to unitize (95% CI [1.05, 8.86]). ROC analysis indicated that the model had modest discriminative ability, with an area under the curve (AUC) of .68, 95% CI [.59, .76].

A paired-samples t -test revealed that participants demonstrated significantly higher performance on the principle-of-position task compared to their performance on the unitizing task, $t(154) = 18.40, p < .001, d = 1.48$. This confirms what has been previously described in the literature: Children demonstrate a strong principle-of-position knowledge in second grade but demonstrate relatively weaker knowledge of base-ten structure.

Exploring Second Graders' Difficulty in Unitizing Quantities (Research Question 2)

In the following section, I explore the nature of children's challenges when asked to unitize a numeral. Children were assigned to profiles of base-ten understanding, and the profiles were compared on their declarative knowledge of the relations between numeration units (assessed through the Definition task) and the responses they provided on the Unitizing task. Therefore, the first part of this section details how the various profiles were created, followed by the comparison of children's performance on the Definition task by profile. Finally, in the final section, I provide a descriptive analysis of responses provided on the Unitizing task by profile.

Second Graders' Profiles of Base-Ten Understanding

Children were assigned to base-ten understanding profiles depending on their performance on the PicPVT and Unitizing tasks.

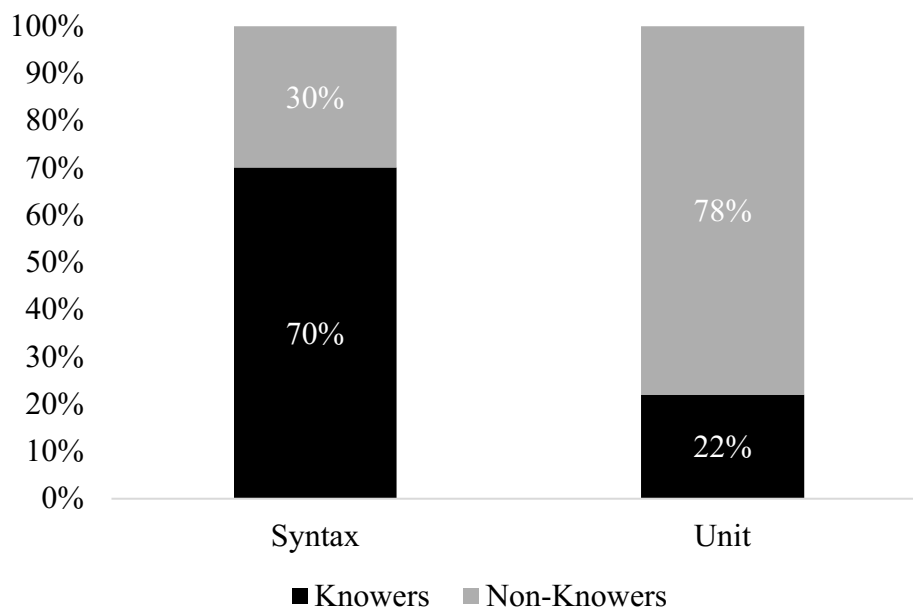
Performance Classification. On each task, children were assigned to Knower or Non-Knower categories depending on their performance on the PicPVT and Unitizing tasks. For the PicPVT, participants were categorized as Syntax Knowers if they achieved a score of 18/20 or higher. Exact binomial calculations indicated that the probability of attaining this score or higher by chance is approximately 0.02%, well below conventional significance thresholds, justifying its use as a conservative classification criterion. Those who scored below 18/20 were categorized as Syntax Non-Knowers. Because of the nature of the Unitizing task, it is not possible to

calculate probabilities. Therefore, I applied approximately the same cut-off as for the PicPVT, resulting in a cut-off of 92% (i.e., 11 correct answers out of 12). Children who obtained a score of 11/12 or more were categorized as Unit Knowers; the others were placed in the Unit Non-Knower category.

Figure 3 shows the proportion of participants placed in each category on each measure. The bar on the left in Figure 3 shows the proportion of participants placed into each category on the PicPVT. Among the 154 participants, 108 participants (70%) were categorized as Syntax Knowers, and the remaining 46 participants (30%) were labelled as Syntax Non-Knowers. The bar on the right in Figure 3 shows the proportion of participants placed into each category on the Unitizing task. An inverse pattern was found: Thirty-four participants (22%) were classified as Unit Knowers, with the remaining 120 participants (78%) categorized as Unit Non-Knowers.

Figure 3

Proportions of Participants in each Category for Principle of Position and Unitizing



Base-Ten Numeration Profiles. Children were assigned to one of four base-ten understanding profiles (see Table 3), depending on their classifications on each of the PicPVT and Unitizing tasks. Those who were categorized as Syntax Knowers and Unit Knowers were placed in the Complete Base-Ten profile. In contrast, Syntax Non-Knowers and Unit Non-Knowers were categorized in the No Base-Ten profile. Children labelled as Syntax Knowers but

as Unit Non-Knowers were placed in the Syntax Base-Ten profile, while those identified as Unit Knowers but Syntax Non-Knowers were assigned to the Structure Base-Ten profile.

Table 3

Profiles of Base-Ten Understanding

		Syntax	
		Syntax Knowers	Syntax Non-Knowers
Structure	Unit Knowers	Complete Base-Ten	Structure Base-Ten
	Unit Non-Knowers	Syntax Base-Ten	No Base-Ten

The proportion of participants in each profile is presented in Figure 4. As expected, most children were placed in the Syntax Base-Ten profile (51%, 79 out of 154 children). Thirty participants displayed a Complete Base-Ten profile (20%) and 42 children (27%) displayed a No Base-Ten profile. Three children were placed in the Structure Base-Ten profile (2%).

Figure 4

Proportions of Children's Base-Ten Understanding Profiles

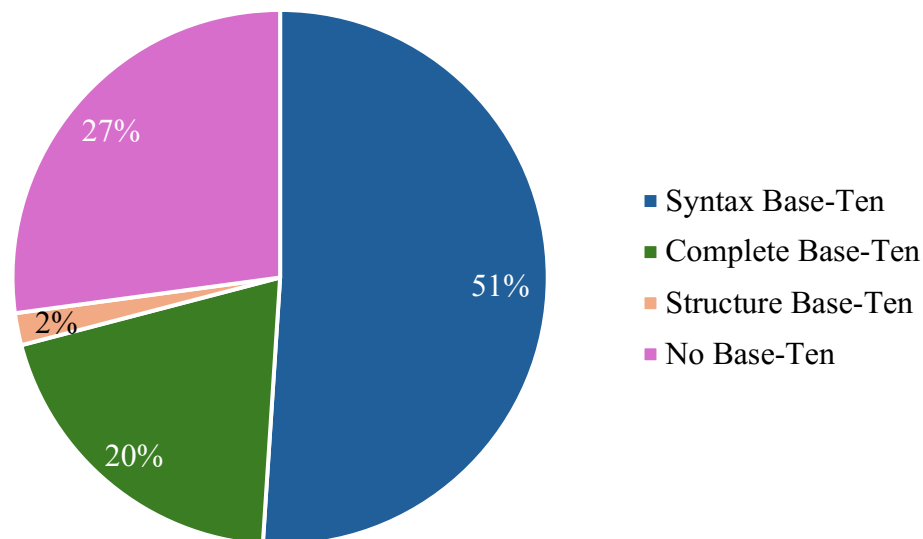


Table 4 presents the descriptive statistics for the PicPVT and the Unitizing task for each base-ten profile, providing a clearer picture of each group's performance.

Table 4

Means (and Standard Deviations) on the PicPVT and Unitizing Task for Each Base-Ten Profile

	Complete Base-Ten	Syntax Base-Ten	Structure Base-Ten	No Base-Ten
PicPVT	.98 (.03)	.97 (.04)	.78 (.08)	.66 (.14)
Unitizing	.98 (.04)	.11 (.24)	.94 (.05)	.06 (.21)

Children with a Structure Base-Ten profile were removed from the following analyses. First, they represented only a small proportion of the total sample (2%, $n = 3$). Second, these children did not fall within the scope of my second research question, which focused on understanding the specific challenges faced by children who demonstrate strong principle-of-position knowledge but poor unitizing performance. Children with a Structure Base-Ten profile who demonstrate strong unitizing performance despite having weak principle-of-position knowledge do not provide information that addresses the second research question.

Second Graders' Declarative Knowledge of the Relations Between Numeration Units

To investigate children's knowledge of the relation between numeration units, I performed an Analysis of Variance (ANOVA) with the base-ten understanding profiles as the between factor and the definition score as dependent variable. The three participants in the Structure Base-Ten profile were removed from the analysis, resulting in three between groups: Syntax Base-Ten ($n = 79$), Complete Base-Ten ($n = 30$), and No Base-Ten ($n = 42$). The results revealed a main effect of profile, $F(2, 148) = 9.61, p < .001$, partial eta-squared = .12. Follow-up pairwise comparisons using Fisher's Least Significant Difference (LSD) showed that children in the Complete Base-Ten profile showed better performance on the Definition task ($M = .81, SD = .31$) than children in the No Base-Ten profile ($M = .46, SD = .34, t(148) = 4.38, p < .001, d = 1.05$) and children in the Syntax Base-Ten profile ($M = .62, SD = .34, t(148) = 2.72, p = .007, d = .58$). Additionally, children with a Syntax Base-Ten profile demonstrated higher performance at the Definition task compared to children with a No Base-Ten profile, $t(148) = 2.43, p = .016, d = .47$.

Second Graders' Response Analysis

Children's responses on the Unitizing task were analyzed next to generate insights into the nature of their unitizing performance and shed light on any specific challenges.

Response Coding. The coding rubric used in the present study was informed by both prior literature and by the data. Specifically, Tempier (2016) reported that many students

interpreted numerals using their principle-of-position knowledge. I therefore initially examined the students' responses for such errors. Additional response types emerged from the data and were interpreted through the lens of the relational structure of base-ten numeration. The rubric presented in Table 5 consisted of four types of responses: Relational, Relational Attempt, No-Relation, and Other.

Table 5*Coding Rubric and Examples of Responses*

Description		Examples		
		Item 68	Item 238	
		Ones	Ones	Tens
Relational	Responses that entail accurately interpreting the numeral according to the numeration unit given.	68	238	23
No Relation	Responses that focus exclusively on the principle of position knowledge, by only considering the digit associated with the given numeration unit, without accounting for the units embedded within higher-level numeration units (i.e., tens in the hundreds).	8	8	3
Relational Attempt	Responses that demonstrate an attempt to relationally reason between level of units making a mistake doing so.	60 / 14	38 / 13	230 / 50 / 5
Other	Responses that show a misunderstanding of the question (e.g., answering about ones instead of tens), and responses that were uninterpretable.	6 / 20	2 / 40	8 / 100

First, responses that demonstrated an ability to consider the lower-level numeration units embedded in the higher-level numeration units were categorized as Relational. More specifically, when asked about the number of ones in a two-digit item, a Relational answer accounted for the ones embedded in the tens together with the ones in the ones place (e.g., 24 ones in 24). In a three-digit item, when asked about the tens, a Relational answer included the tens embedded in the hundreds (e.g., 23 tens in 238). When asked about the number of ones, a Relational answer entailed considering the ones embedded in both the tens and the hundreds (e.g., 238 ones in 238).

Second, responses that reflected an attempt to reason relationally across the levels of numeration units but contained an error were coded as Relational Attempt. In this category, children showed consideration of the lower-level units embedded in a higher-level one but made an error while doing so. For example, when asked to interpret 236 in terms of tens, some children answered 203 tens. This answer demonstrated an adequate interpretation of the 3 as three tens but a wrong interpretation of the 2 as 200 tens (instead of 20 tens). For the same item, other children answered 50 tens, an error suggesting that the child noticed the 20 tens embedded in the 2 hundreds but incorrectly interpreted the “3” as representing 30 tens, resulting in an answer of 50 (i.e., $20 + 30$).

Third, responses based solely on the principle of position, with no consideration of the relation between levels of numeration units, were labelled as No Relation responses. For example, when asked about the number of tens in 236, a No Relation response would be “3,” reflecting only the digit in the tens position without considering the tens embedded in the hundreds. Finally, responses that indicated a misunderstanding of the question (e.g., using the incorrect unit throughout the answer, reporting the number of ones instead of tens in a quantity), as well as those that were uninterpretable, were grouped into the Other category.

In the following analysis, I removed some items that presented ambiguities in assessing their knowledge. Specifically, for the two-digit items, questions about the number of tens were removed because the correct answer does not allow for differentiating children with and without a relational knowledge between levels of numeration units. For example, for the item 24, the precise number of tens is 2.4. However, children in second grade are expected to answer 2 because they are not yet introduced to decimal numbers. In this case, an answer of 2 does not effectively distinguish relational from non-relational understanding. Consequently, questions about the tens for two-digit items were removed. Similarly, questions about the hundreds were

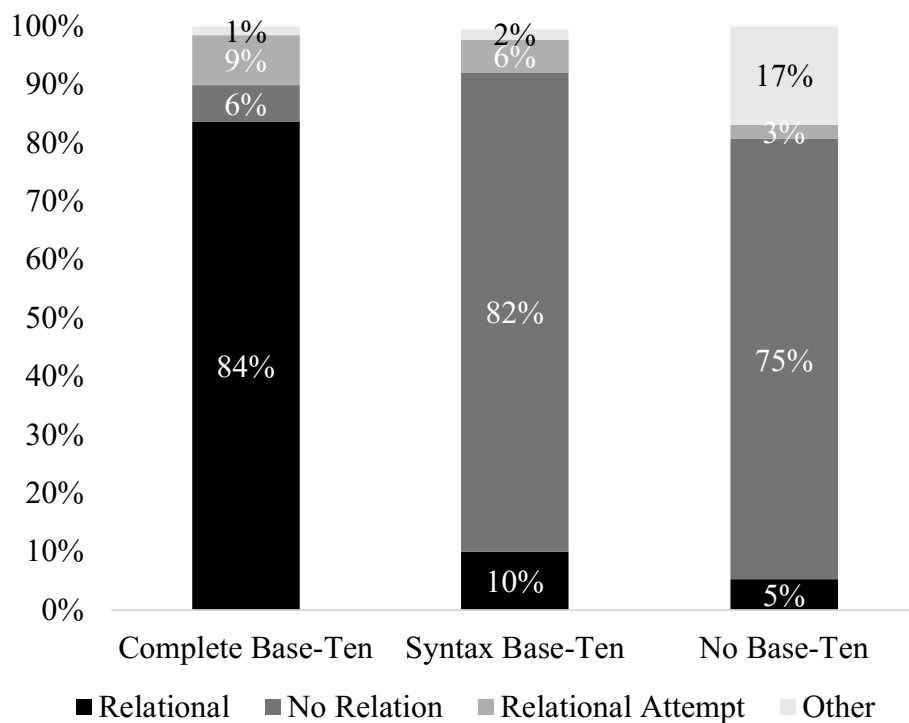
removed for the three-digit items because of the same interpretation issue. In total, six responses for the two-digit items and 12 responses for the three-digit items were considered in the following analysis (18 responses in total).

A trained research assistant and I coded all the data independently, resulting in 99% agreement. We resolved the discrepancies through discussion.

Response Analysis. Using items as the units of analysis, Figure 5 shows the proportion of response types provided on the Unitizing task by base-ten profile.

Figure 5

Proportions of Response Types by Base-Ten Understanding Profile



Note. The data were based on 2710 responses provided by 151 children (eight missing data).

The majority of responses provided by children in the Complete Base-Ten profile were Relational (84%), while most of responses provided by children in the Syntax and No Base-Ten profiles were No Relation responses (83% for children in the Syntax Base-Ten profile and 75% for those in the No Base-Ten profile). Additionally, children with a Complete Base-Ten profile provided about the same proportion of Relational Attempt responses (9%) as children in Syntax (6%) and No Base-Ten (3%) profiles. A noteworthy difference concerns the proportion of Other responses, constituting 1% of all responses from children with a Complete Base-Ten profile, 2%

with a Syntax Base-Ten profile, and 17% from children with a No Base-Ten profile. Children with a No Base-Ten profile produced a larger proportion of irrelevant answers than those with either a Complete or Syntax Base-Ten profile, confirming weaker syntax and structure knowledge. The responses analysis revealed that children with a Complete Base-Ten profile tended to produce more responses showing an understanding of the relation between numeration units than the other two groups. In contrast, children with a Syntax Base-Ten and a No-Base-Ten profiles were more likely to produce errors that focused solely on the digit in the position of the target numeration unit.

Discussion

The present study explored second graders' understanding of the syntax and structure of base-ten numeration. Participants were asked to complete a principle-of-position task, state the relation between numeration units, and interpret numerals using more than one unit (i.e., to unitize). This study had two objectives. First, unlike a large part of the literature on base-ten numeration, I was interested not only in children's principle-of-position knowledge, but also in their knowledge of the structure of the base-ten system, namely their unitizing performance. The first objective was therefore to compare children's principle-of-position to their unitizing skills, to confirm previous findings in one single study. Accuracy on both the principle-of-position and unitizing tasks was analyzed.

The second objective was to understand the nature of the challenges faced by children who demonstrate strong principle-of-position knowledge but poor unitizing abilities. To do so, I first compared the declarative knowledge of the relations between numeration units of children in three profiles: those with strong principle-of-position knowledge but poor unitizing skills, those with strong principle-of-position and unitizing skills, and those with relatively less knowledge of the principle of position and unitizing. Second, I conducted a response analysis to compare the three groups based on the types of responses children provided on the unitizing task. The present research promises to provide a deeper understanding of children's knowledge of base-ten numeration. Specifically, it aims to investigate the relation between principle-of-position knowledge and unitizing, the nature of children's challenges when asked to unitize quantities, and how children develop an understanding of the relations between numeration units. Additionally, this study promises several important educational implications.

Regarding the first research question, I first hypothesized that principle-of-position knowledge and unitizing skills would be related. This assumption was supported by the present data: Children's principle-of-position knowledge was predictive of their unitizing skills. Second, I hypothesized that second graders would display more advanced knowledge of the syntax of written numerals, as assessed through a principle-of-position task, than their knowledge of base-ten structure, as assessed with a unitizing task. The findings confirmed also this prediction: Second graders demonstrated higher levels of principle-of-position performance than unitizing performance. This result confirms what has already been shown about children's knowledge of syntax and structure. Several studies showed that in the second grade, children possess a strong knowledge of the principle of position (Mix et al., 2022; Osana & Lafay, 2019). For example, Osana and Lafay (2019) found similar results, with second graders obtaining more than 80% success on the same principle-of-position task as the one used in this study. Also, several studies showed that children in the first, second, and third grades struggle to interpret a quantity using a given numeration unit (e.g., finding the total number of tens in a three-digit quantity; Barilaro et al., 2026; Tempier, 2016). Tempier (2016) found that at the beginning of the third grade, children struggled to report the total number of tens in a three-digit numeral, which is similar to what was found in the current study. Barilaro et al. (2026) showed that only 30% of the first and second graders were able to interpret quantities represented visually given a specific numeration unit (e.g., reporting the total number of tens in three-digit quantities). Stronger performance on principle-of-position tasks than on unitizing tasks could be explained by the strong emphasis that is often placed on the principle of position to the detriment of base-ten structure in classroom instruction (Houdement & Tempier, 2019; Tempier, 2024).

Despite the confirmation of previous findings, my study is the first to directly compare children's knowledge of both base-ten syntax and structure. The results supported my predictions that indeed, students' principle-of-position knowledge is more advanced than their unitizing, assuming respectable construct validity of my measures. Directly comparing children's principle of position and their unitizing skills confirms what has been found in separate studies in the literature—that children need support when unitizing with numerals in base ten. My second research question aimed to extend current understandings of children's knowledge of base ten to identify specific gaps that can be addressed through instruction.

Regarding the second research question, I was interested in gaining a deeper understanding of the challenges experienced by children with who exhibit relatively weak unitizing skills. The results showed that half of second-grade students possessed strong principle-of-position knowledge but poor unitizing performance. As shown in previous research, the relatively strong emphasis on the principle of position in the early grades (Houdement & Tempier, 2019; Tempier, 2024) could have explained the observed pattern. The remaining students were about equally split into those who had strong knowledge of both and those who had weak knowledge of both.

To gain deeper insight into children's challenges when unitizing numerals, I conducted a comparison of children's declarative knowledge of the relations between numeration units as a function of base-ten profile. First, examining children's knowledge of unit relations was of interest because researchers have suggested that such knowledge (i.e., knowing that there are 10 ones in a ten, 10 tens in a hundred, and 100 ones in a hundred) is foundational for a relational understanding of the base-ten system (Chambris, 2008; Soury-Lavergne et al., 2020; Tempier, 2016). From a theoretical standpoint, without this declarative knowledge, children may be unable to determine the total number of tens in a three-digit numeral, as they would not know that a hundred contains 10 tens. The findings in the present study confirmed this hypothesis. The profile analysis showed that such declarative knowledge is strongest when children have a complete base-ten understanding, operationalized in this study as having strong principle-of-position and unitizing performance. These results suggest that declarative knowledge of the relations between numeration units might be important for the development of unitizing: The data showed that those with complete base-ten knowledge had stronger declarative knowledge than children who had low unitizing skills. Lacking declarative knowledge of the relations between numeration units could impede the development of unitizing because children might have difficulty seeing the embedded nature of numeration units. This hypothesis was confirmed by the regression analysis, showing that declarative knowledge of the relations between numeration units is predictive of the presence of unitizing skills.

Second, the data also showed that principle-of-position knowledge by itself is related to declarative knowledge of the relations between base-ten units. This relation was evidenced when I compared the declarative knowledge of the relations between base-ten units of children with strong principle-of-position knowledge and poor unitizing performance with that of children with

weak performance in both competencies. More precisely, children with strong principle-of-position knowledge showed higher knowledge of the relations between numeration units compared to those with weak principle-of-position knowledge. Nevertheless, exactly how principle-of-position knowledge is related to declarative knowledge cannot be answered by the data in the present study and should be considered in future research on children's base-ten numeration.

The second research question also addressed the nature of children's unitizing difficulties by examining the types of responses they provided on the unitizing task. This descriptive analysis specifically target the challenges that have been reported in the literature on children's difficulties when unitizing in base ten. First, the data revealed that most of the errors produced by second graders with low unitizing performance were responses that only considered the digit in the position of the target numeration unit (e.g., answering five when asked about the total number of tens in 253). This type of response demonstrates a failure to consider the embedded nature of base-ten, in which any numeration unit is composed of lower-level numeration units. Responses focusing only on the digit of the targeted numeration unit could be attributed to an overreliance on principle-of-position knowledge: Again, because of their extensive experience with the principle of position in school, children tend to interpret numerals in ways that are restricted to their knowledge of the value of each digit. Such an error could reflect a misconception about base-ten numeration and numbers in general by failing to consider that numbers are composed of nested numeration units. This finding aligns with two previous studies. Tempier (2016) noticed this type of response when the author asked children to find the total number of tens in a three-digit numeral, but this was observed without conducting a systematic analysis of children's response types. Barilaro et al. (2026) also reported that the majority of errors produced by first and second graders when asked to interpret visual representations of quantities were of this type.

Second, the response analysis revealed that among children with poor unitizing abilities, those with high levels of principle-of-position knowledge were more likely than those with lower principle-of-position knowledge to produce responses that reflected an understanding of the relations between numeration units. The regression analysis confirmed this observation, as higher principle-of-position knowledge was predictive of children's unitizing skills. Interestingly, what also emerged from the data was that among children with relatively low

performance on the unitizing task, those with weaker principle-of-position knowledge produced a higher proportion of irrelevant or uninterpretable responses than their peers with stronger principle-of-position knowledge. This pattern suggests that knowing the principle of position provides children with at least a partial understanding of numerals, allowing them to interpret each digit meaningfully, even if their understanding of the overall base-ten structure remains incomplete.

Finally, in the profile analysis, an unexpected finding emerged: a small number of children performed at a higher level on the unitizing task than on the principle-of-position task. This result was unexpected in the context of the Unitizing task used in the present study. More specifically, the Unitizing task required children to flexibly interpret numerals using more than one numeration unit. On the surface, one might assume that such a task would presuppose some understanding of the principle of position, since unitizing in this context presumably relies on knowing the positional value of digits in a numeral. For example, when asked to identify the total number of tens in a three-digit numeral, children are expected to know that the second digit from the left represents groups of ten and that the digit on its right (i.e., the hundreds) is also composed of tens. Unlike unitizing tasks involving concrete representations of quantities (e.g., groups of animal toys; Wege et al., 2023) that could provide visual cues to the units, interpreting numerals requires familiarity with symbolic representations and positional conventions. Accurately identifying the total number of ones, tens, or hundreds in a numeral relies on knowing the position that corresponds to each numeration unit (e.g., ones, tens, hundreds). Thus, it is possible that the few children who demonstrated higher unitizing knowledge than principle-of-position knowledge did not adequately understand what was required on the PicPVT task. The task required children to judge whether the underlined digit in a numeral matched the number of dots displayed below it, thereby engaging them in a direct comparison between two different types of representations. Such a task contrasts with how the principle of position is typically assessed in the literature, where tasks usually involve either symbolic notations or visual representations of quantities, but not both simultaneously (e.g., Mix et al., 2019). It is also important to note that only three children presented strong unitizing abilities despite poor knowledge of the principle of position. Therefore, this finding should be interpreted with caution, as it may reflect a spurious result or an artifact of the measure.

Contributions

The present study contributes in several ways to the literature on children's understanding of base-ten numeration. A first key contribution lies in the direct comparison of children's principle-of-position knowledge and their unitizing skills. Although previous studies have suggested that young children often lack knowledge of the structure of base ten (Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016), this claim had not been investigated through a direct comparison of performance on tasks assessing syntax and structure. The present findings provide such evidence by demonstrating that second graders performed significantly better on tasks assessing principle-of-position knowledge than on tasks requiring them to unitize numerals. Although prior research has documented that instruction tends to focus on teaching the principle of position (Houdement & Tempier, 2019; Tempier, 2024), this study suggests that such an instructional practice may hinder children's learning of the relational structure of base-ten numeration.

This study also contributes to the current literature by highlighting the importance of knowing the relation between numeration units. The results revealed that children with a complete understanding of base ten, as demonstrated by strong principle-of-position knowledge and strong unitizing skills, have greater declarative knowledge of the relations between numeration units than those with strong principle-of-position knowledge but poor unitizing abilities. Interestingly, among children with poor unitizing abilities, those with strong principle-of-position knowledge presented higher knowledge of the relations between numeration units compared to those with poor principle-of-position knowledge. Despite the correlational design of the current study, this result contributes significantly to the literature by showing that learning about relations between numeration units could be critical for developing an understanding of base-ten structure.

Finally, the response analysis of performance on the unitizing task uncovered the nature of children's relational thinking about numeration units. To date, although no previous study has examined children's base-ten unitizing in the same way, Tempier (2016) observed that many students demonstrated an overreliance on the principle of position when asked to unitize. The present study is an extension of Tempier's work by offering a more systematic investigation of his finding: The most common error I observed when children were asked to interpret numerals

given a specific unit entailed focusing only on the position that corresponded to the target unit while ignoring the relations between the levels of numeration units in the quantity.

In addition, another category of errors emerged from the data that had not yet been identified in prior work. The data revealed that some children engaged in relational reasoning but made mistakes while doing so. Such responses may represent a transitional phase in the development of unitizing abilities in which children begin to grasp the structure of base-ten numeration but have not yet fully mastered how to coordinate multiple levels of numeration units. This finding is in line with previous research showing that students can display various ways of thinking simultaneously before they shed less sophisticated conceptions for more advanced ones (Siegler, 2016).

Strengths and Limitations

This study presents three major strengths. First, to my knowledge, this is the first study to investigate children's understanding of base-ten numeration by comparing children's principle-of-position knowledge and unitizing skills. Since base-ten numeration encompasses both knowledge of its syntax and its structure, such a comparison offers valuable insights into the developmental trajectory of children's learning of both forms of knowledge. Additionally, given that base-ten numeration is predictive of future mathematical abilities (Carpenter et al., 1998; Mix et al., 2023; Moeller et al., 2011), gaining a deep understanding of what children know about the base-ten system is essential for informing pedagogical practices and supporting effective instruction. Second, this study focuses on unitizing as a reflection of children's understanding of the relational structure of base-ten numeration. To my knowledge, this is the first time such an operationalization has been used to assess children's structural understanding of base ten, particularly with the scoring approach applied in the study. More specifically, since unitizing is defined as the ability to simultaneously interpret one quantity using two different units (Lamon, 1996; Simon, 2024), the scoring method adopted in this study focused on whether children could correctly interpret the same numeral in two ways (e.g., ones, and tens, or tens and hundreds). The approach is innovative compared to previous studies (Barilaro et al., 2026; Tempier, 2016), which focused on children's ability to interpret a quantity using only one numeration unit. I argue that the task and scoring approach used in the present study offer a more accurate reflection of children's flexible and conceptual understanding of base-ten structure as it is more sensitive to their ability to switch between various units of reference.

Nevertheless, this study presents some limitations. First, only one aspect of children's unitizing was assessed: their ability to decompose numerals. Decomposing numerals entails identifying, within higher-level units, the number of embedded lower-level units. For example, in a base-ten context, decomposing a three-digit numeral refers to the ability to identify the total number of tens. In this case, children need to consider both the individual tens and the tens embedded in the hundreds. Administering a task in which students were required to compose higher-level units from lower-level ones could have provided more complete data on their understanding of the base-ten structure. For example, Ladel and Kortenkamp (2016) offered a task that required students to find the numeral that corresponded to a specific expression. For example, children needed to find the number associated with 3 hundreds + 24 tens + 12 ones. To solve this problem, children needed to compose the higher-level units from the given lower-level units (ones, and tens). Such a task might have prompted students to engage in more relational reasoning, thereby reducing the negative influence of their principle-of-position knowledge.

Educational Implications

The present findings have several important educational implications. First, the data show that second graders demonstrated a stronger principle-of-position knowledge than their unitizing performance. This underscores the importance of providing students with meaningful opportunities to explore the structure of base ten through tasks involving unitizing. In particular, the results support recommendations that teachers may wish to model how a single multi-digit numeral can be interpreted in different ways. Carpenter et al. (1996) suggested that measurement division word problems could serve as a valuable instructional tool to support children's understanding of the structure of base-ten. Such problems, especially when embedded in base-ten contexts, require children to form groups of 10 and count the number of groups formed. These problems provide opportunities for children to reflect on how the "measurement" of quantities depends on the unit used, a central idea behind base-ten structure. More broadly, learning about the structure of base-ten numeration, and especially how numeration units are interconnected, can offer children an excellent opportunity to develop a deep understanding of units. Such knowledge is foundational for the development of more advanced and often challenging mathematical skills, including multiplication, fractions, and rational numbers (Sophian, 2004; Steffe, 1991; Steffe & Olive, 2010).

Second, the results indicated that children with strong unitizing skills tend to demonstrate higher declarative knowledge of the relations between numeration units than those with weaker unitizing abilities. These findings suggest the importance of instructional practices that make these relations explicit. Previous studies have highlighted the role of manipulatives that make the targeted concept perceptually salient in supporting children's understanding of mathematical concepts (e.g., Laski et al., 2015). As shown in Barilaro et al. (2026), when tens were made visible in the visual representations of multi-digit quantities, students were more able to identify the number of tens embedded in the hundreds. Therefore, using manipulatives that make the relations between numeration units visually salient could support children's learning.

Finally, while the present study does not suggest that teachers should stop teaching the principle of position, the results highlight the critical role teachers must play in helping children transition from a superficial understanding of numerals, focused primarily on the principle of position, to relational reasoning with numeration units. Supporting children in this process is important given that they demonstrate an over-reliance on their principle-of-position knowledge, confirmed by the response analysis. The strong influence of the principle of position in children's thinking about numerals could impede them from engaging in relational thinking about numeration units and interfere with the development of a more advanced learning in mathematics.

Connecting Study 1 to Study 2

Study 1 investigated children's knowledge of base-ten syntax and structure by directly comparing their performance on a principle-of-position task and a unitizing task. Additionally, the study aimed to investigate the nature of the challenges presented by children with poor unitizing performance. More specifically, declarative knowledge of the relations between numeration units and types of unitizing responses were compared between three groups of children: those with strong principle-of-position knowledge but poor unitizing performance, those with strong knowledge in both competencies, and those with low competencies in both. The results revealed that second graders performed better on the principle-of-position task than on the unitizing task. Interestingly, children with strong unitizing performance demonstrated a higher understanding of the relations between numeration units than children with poor unitizing performance. The children in the latter group exhibited a substantial number of errors influenced by their prior knowledge of the principle of position. These findings highlight a specific challenge in acquiring base-ten structure: overcoming the obstacles presented by their prior knowledge of the principle of position.

However, the tasks included in Study 1 were entirely symbolic, which may have prompted the activation of their principle of position knowledge given that this is how they are typically taught to think about numerals in school (Houdement & Tempier, 2019). Carpenter et al. (1996) suggested that measurement division problems might serve as an effective pedagogical tool for supporting children's understanding of base-ten structure. Furthermore, prior research has shown that when children work in non-symbolic contexts, they can display unexpectedly strong knowledge and performance, otherwise hidden by traditional symbolic tasks (e.g., Sherman & Bisanz, 2009). Study 2 was therefore designed to investigate children's ability to unitize quantities, that is to identify and count composite units, when solving measurement division word problems. Since children in the second grade are not typically introduced to measurement division problems, manipulatives were provided to help solve the problems. Prior studies suggested that young children can solve measurement division word problems when given appropriate tools (e.g., Carpenter et al., 1993). However, no study has yet investigated how the features of manipulatives may influence children's ability to unitize quantities in a context of measurement division problem. Therefore, Study 2 aims to investigate not only children's unitizing when solving measurement division problems, but also to explore which manipulative

features may impact unitizing. The findings may offer promising insights into how such problems could serve as a mean to support children's acquisition of base-ten structure in instructional contexts.

Chapter 3: Study Two
Exploring the Factors Influencing Second Graders' Unitizing Performance in Base-Ten
Contexts: A Qualitative Study

Introduction

Acquiring a flexible understanding of numbers is fundamental for navigating daily life situations. For instance, individuals need to understand that when someone refers to the year “nineteen hundred and eighty-two,” they are referring to the year 1982. Since they are omnipresent, developing a flexible understanding of numbers is essential for interpreting financial information or making informed political decisions. More specifically, in many everyday activities, individuals must reason about the relations between base-ten units. Indeed, Yeong et al. (2015) claimed that “a unit cannot be separated from a number, and a number is dependent on its unit” (p. 173), highlighting the idea that understanding a number requires reasoning about the units that help make sense of the number’s meaning. Whether converting between currencies or adjusting measurements in recipes, individuals must fluently switch their unit of reference to interpret quantities appropriately.

Understanding numbers involves knowledge of base-ten numeration, which is composed of a syntax and a structure. Base-ten syntax refers to the rules that govern how quantities are represented using numerals (i.e., the symbolic, written form of a number; Ross, 1989). Also called the principle of position, base-ten syntax allows individuals to determine the value of each digit depending on its position in a numeral (Cheung & Ansari, 2021). Base-ten structure, on the other hand, corresponds to a system of embedded numeration units related by powers of ten (Fuson, 1990). Unitizing, defined in the present research as the interpretation of a quantity using at least two different numeration units simultaneously (Lamon, 1996; Simon, 2024), is one way to signal understanding of the base-ten structure. However, the literature tends to show that children struggle to acquire flexible knowledge of the base-ten structure (Barilaro et al., 2026; Tempier, 2016), and more specifically, to reason about the relations between base-ten units (Barilaro, 2026a). In Barilaro (2026a), I showed that second-grade students struggled to unitize numerals and tended to rely heavily on their principle-of-position knowledge when interpreting them, obscuring how multiple levels of units are related to each other. When children are presented with numerals and asked to identify the total number of a specific numeration unit

(e.g., tens), their prior knowledge of the principle of position is typically activated instead of any relational knowledge of numeration units.

Carpenter et al. (1996) suggested that measurement division problems are a valuable tool to support children's developing understanding of base-ten structure. Measurement division problems can be solved by children as early as kindergarten when they are provided with a variety of tools, such as concrete objects (or "manipulatives"). When embedded in base-ten contexts, the structure of measurement division mirrors the hierarchical structure of base ten, making measurement division problems particularly relevant for exploring unitizing in non-symbolic contexts. However, little is known about the factors that may either hinder or facilitate children's unitizing when solving such problems. The present study aims to explore the influence of two factors on unitizing performance: the number of composite unit levels in the problem and the type of manipulatives used to represent the quantities. Examining these two factors promises to shed light on the conditions that best support children's unitizing performance. Deepening current understandings of the conditions under which children's unitizing is supported promises to inform future instruction aimed at enhancing children's learning of base-ten structure.

Base-Ten Numeration: Children's Unitizing

Knowledge of base-ten numeration is characterized by syntax and structure. Base-ten syntax, also called the principle of position (Cheung & Ansari, 2021), refers to the idea that in a numeral, the position of each digit determines its value. Developing knowledge of the principle of position is crucial for interpreting and writing quantities using numerals. However, knowing the syntax is only one part of base-ten numeration, and children must also develop knowledge of base-ten structure. The structure relies on the idea that base ten is a system of intertwined numeration units (e.g., ones, tens, hundreds; Fuson, 1990; Houdement & Tempier, 2019). Each numeration unit is related to another by a power-of-ten relation. For example, one ten is composed of 10^1 ones (i.e., 10 ones), and one hundred is composed of 10^1 tens (i.e., 10 tens) or 10^2 ones (i.e., 100 ones). Fuson (1990) argued that children must acquire what she called the "ten-for-one conception," defined as the idea that 10 numeration units at any level (e.g., 10 hundreds) are equivalent to one numeration unit at the next higher level (e.g., one thousand). The ten-for-one conception allows individuals to interpret one group of 10 numeration units in two different ways: as 10 numeration units at one given level or as one numeration unit at the next higher level.

The ten-for-one conception can be seen as a particular case of unitizing. Unitizing is defined as the process of viewing one quantity simultaneously as a collection of basic units (i.e., single entities at any level in the quantity) and as one composite unit (i.e., one group) considered at the next higher level in the quantity (Lamon, 1996; Simon, 2024). For example, a ten can be considered as the targeted basic unit, while a hundred is the composite unit composed of 10 tens. Any quantity can be unitized in any number according to the reference unit used, and children must understand that the result of the unitizing process will be different depending on the unit that is used (Simon, 2024; Sophian, 2004). In the base-ten context, unitizing entails using numeration units (e.g., tens and hundreds) as reference units. For example, the quantity 345 can be seen as a collection of 345 ones but at the same time also as a collection of 3.45 hundreds. Unitizing is critical for later understanding in several mathematical domains, including multiplicative reasoning and rational numbers (Sophian, 2004; Steffe & Olive, 2010).

Base-ten structure appears to be challenging for children, as shown by a handful of previous studies (Barilaro, 2026a; Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016). More specifically, Tempier (2016) found that third graders struggled to interpret numerals in terms of one specific numeration unit, particularly when they needed to take the unit embedded in higher-level ones into account (e.g., finding the total number of tens in a three-digit numeral). In addition, Barilaro et al. (2026) found that first- and second-grade students struggled to identify the total number of a specific numeration unit (e.g., tens) in visual representations of three-digit quantities. Finally, to my knowledge, only one study directly investigated children's unitizing performance on a task that required using two different numeration units to interpret the same numeral (Barilaro, 2026a). For example, in Barilaro (2026a), I found that second-grade students struggled to interpret 327 as 32 tens and simultaneously as 327 ones. In the same study, an exploratory response analysis revealed that children tended to rely heavily on their principle-of-position knowledge when unitizing in base ten. For example, when I asked the second graders to find the total number of tens in 234, most of them answered three, focusing only on the value of the digit in the tens position. The overreliance on the principle of position could be a result of the emphasis on the principle of position in the elementary curriculum (Houdement & Tempier, 2019; Tempier, 2024).

Supporting Unitizing in Base-Ten Contexts

As shown in Barilaro (2026a), unitizing can be challenging for children when they are asked to simultaneously interpret numerals using two different numeration units (e.g., the total number of tens, and the total number of hundreds in 327). Unitizing has been found to be particularly difficult outside of base-ten contexts as well (Shipley & Shepperson, 1990; Sophian & Kailihiwa, 1998; Wege et al., 2023), but in the context of base ten, unitizing might be even more challenging because children's prior knowledge of the principle of position. Providing non-symbolic situations, defined in this study as contextual word problems and manipulatives, to represent quantities rather than only numerals, however, may help prevent the activation of the principle of position. As demonstrated in previous studies (e.g., Sherman & Bisanz, 2009), non-symbolic representations can sometimes reveal a deeper understanding of mathematical concepts than symbolic notation. To my knowledge, no research has been conducted on the factors that might support children's understanding of the base-ten structure in non-symbolic contexts, leaving open many questions on ways to support their developing understanding of numeration.

Measurement Division Problems

Carpenter et al. (1996) suggested that using measurement division problems can promote children's understanding of the relation between numeration units. A measurement division problem is one with a multiplicative structure where the total quantity and the size of the group is given. The solution entails finding the number of groups (i.e., composite units) that can be formed with the information given about the total quantity and the number of lower-level units in each group. When embedded in base-ten contexts, measurement division problems involve composite units that are grouped by powers of ten. Consider, for example, the following problem. "Hugo has 30 candies. He puts all the candies in bags. One bag contains 10 candies. How many full bags does Hugo have in all?" In this paper, I refer to "basic units" as those that are used to express the total quantity (Carpenter et al., 2014). Thus, in this problem, children are asked to identify the number of composite units (i.e., bags) that are formed from groups of 10 basic units (i.e., candies). When solving this problem, a child must understand that 30 represents the basic units, which are expressed in ones, and from that initial quantity, three composite units, expressed in tens, can each be formed from 10 basic units. The structure of a measurement division problem embedded in a base-ten context is therefore similar to the structure of base-ten numeration because the units are powers of ten and they are multiplicatively related by powers of

ten. Providing a problem of this type may engage children in relational reasoning to a greater degree than when interpreting symbolic representations of quantity.

Typically, however, children in the second grade are not yet introduced to concepts of multiplication and division, which may limit their performance on measurement division problems in symbolic contexts. Nevertheless, prior research has demonstrated that such problems are accessible to children as early as kindergarten when they are prompted to represent the quantities either concretely (Carpenter et al., 1993) or visually (Downton & Maffia, 2025). More specifically, Carpenter et al. (1993) found that 71% of their kindergarten sample correctly solved measurement division problems by directly modeling the situation using manipulatives (i.e., counters). However, although little is known about the influence of manipulative type when solving such problems embedded in base-ten contexts, some researchers have explored the use of manipulatives when unitizing quantities in other contexts (Wege et al., 2023). The authors suggested that the type of manipulative used could influence children's interpretation of quantities using more than one unit of reference, a claim supported by the literature described below.

Influence of Manipulative Type on Children's Unitizing

The perceptual features of manipulatives can influence children's learning and performance in mathematics (Carbonneau et al., 2013; Laski et al., 2015). When solving a measurement division problem, children can use manipulatives to represent both basic and composite units. Importantly, children must understand what each manipulative represents in the context of the problem because this understanding guides how they will use and count those manipulatives. Children's appropriate counting of basic and composite units might be influenced by the three following factors: the realism of the manipulatives, the salience of the units, and whether or not the composite units are physically constructed by the child.

Realism. Prior research has shown that the realism of manipulatives plays a role in children's learning and transfer in mathematics (e.g., Carbonneau & Marley, 2015; Cheung et al., 2023). Realism is defined as the extent to which a manipulative resembles its real-world referent; the more realistic a representation, the more closely it matches its real-world counterpart (Fyfe et al., 2014). For example, to represent a fish, a realistic manipulative might be a fish-shaped counter, whereas a cube would be considered generic because of the reduced likeness to its real-world referent. Carbonneau and Marley (2015) found that in domain of whole-number

magnitude, realistic manipulatives can support preschoolers' transfer, particularly when they received low-levels of instructional guidance. Cheung et al. (2023) extended this result by examining the relation between the type of manipulative (i.e., realistic or generic) and the labels used to refer to the quantities in the problems (i.e., specific referent or generic label) in equal sharing problems. The authors found that kindergartners improved their understanding of equal sharing to a greater extent when they used realistic manipulatives associated with generic labels (e.g., a strawberry presented as a "thing") compared to when generic manipulatives associated with generic labels (e.g., a cube presented as a "thing"). Interestingly, the authors also found the reverse pattern, with better performance on equal sharing when using generic manipulatives with real-world referents (e.g., a cube presented as a strawberry) than when they used realistic manipulatives associated with the same assigned referents (e.g., a strawberry counter presented as a strawberry). In the case of measurement division problems, specific referents are often used to label the basic quantity (e.g., candies) and the groups (e.g., bags). The finding of Cheung et al. (2023) might suggest that when solving measurement division problems, in which specific referents are used, children's performance might be higher when generic manipulatives are used than when realistic ones are used.

Nevertheless, in the context of unitizing, several authors (Brooks et al., 2011; Shipley & Shepperson, 1990) have argued that for children's unitizing, what matters is whether they know what needs to be counted. For example, Brooks et al. (2011) showed that young children aged 4 and 5 years are influenced by their knowledge of the object being counted. More specifically, they found that children are more likely to appropriately count the items when the label used explicitly identifies a countable unit. For instance, when children were asked to count the total number of bicycles, they excluded detached wheels from their counts, and when asked to count the wheels, they identified and counted all individual wheels presented. This finding supports the use of realistic manipulatives (e.g., candies and bags) over generic ones (e.g., cubes) because realistic manipulatives might be immediately associated with the objects described in the problems. Investigating whether the realism of manipulatives influences children's ability to unitize quantities is therefore essential, especially when base-ten blocks, which are generic manipulatives, are frequently used in classrooms for teaching base-ten numeration.

Salience of the Units. Prior research has found that manipulatives that make a target concept perceptually salient can support its understanding (Laski et al., 2015). A manipulative is

said to make a concept salient when it visually presents the underlying structure of the concept it is intended to convey. In the context of unitizing, Wege et al. (2023) found that when interpreting groups of discrete objects (e.g., animal counters), five-year-old children tended to systematically count the individual items (i.e., the basic units) within each group rather than the groups themselves (i.e., the composite units). This suggests that the “one-ness” of the groups (defined as the visual salience of the group as one unit) was not sufficient, and that the visual prominence of the individual items (i.e., counters) directed children’s attention toward the basic units.

Several studies explored the impact of the visibility of the basic units and the composite units, both outside and within base-ten contexts. Two studies (Boyer et al., 2008; Jeong et al., 2007) revealed that when quantities are presented using continuous visual representations, children’s attention was directed toward the overall group. Children were more likely to count the basic units when these were made visible in the representations through demarcations, compared to when only the group was visible and the basic units were not. Continuous representations appear to direct children’s attention to composite units as “ones.” Similar findings emerged in the context of base-ten numeration. Lafay et al. (2023) showed that children were more accurate in their construction and interpretations of quantities when composite units (i.e., numeration units) were visually salient because children identified those composite units as singular groups. Likewise, Barilaro et al. (2026) found that the salience of the composite units matter when children interpret visual representations of multidigit quantities. Specifically, the authors asked first- and second-grade students to identify the total number of tens in visual representations of three-digit quantities. The task required children to identify the total number of tens in visual representations, taking into account both the individual tens and those embedded in the hundreds. Results indicated that children’s performance was higher when the tens in the hundreds were perceptually salient than when they were not. Interestingly, the visibility of the ones (i.e., basic units) did not influence children’s ability to identify the tens. Thus, Barilaro et al.’s (2026) findings suggest that making the composite units visible helps children perceive them as single groups, regardless of whether the basic units are visible.

In sum, what appears most important for unitizing in base-ten context is the visibility of the composite units. The current research suggests that providing attachable manipulatives might offer similar advantages by allowing children to stack the manipulatives to form one new

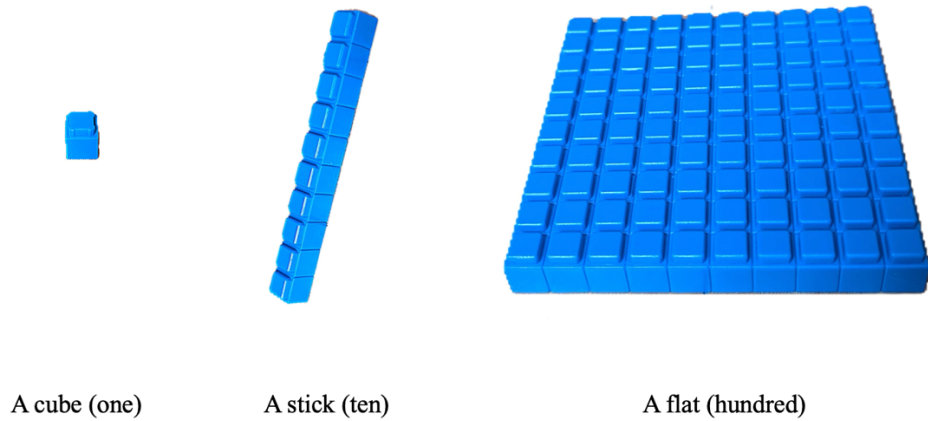
composite unit with a high degree of visibility, or “one-ness.” These findings highlight the importance of examining whether the salience of the one-ness of the groups in the manipulatives when solving measurement division problems can hinder or support children’s unitizing performance.

Pre-Composition of the Composite Units. Finally, children who have not yet been formally introduced to multiplication and division concepts are often prompted to create the groups themselves using manipulatives to solve measurement division problems (e.g., Carpenter et al., 1993). Interestingly, Manches and O’Malley (2016) found that 4- to 6-year-old children who were permitted to physically interact with manipulatives demonstrated more advanced conceptual strategies when solving an additive comparison task than those who were prevented from interacting with the manipulatives. Similarly, Osana et al. (2026) found that children who physically engaged in constructing numeration units demonstrated greater learning gains than those who used units that were pre-arranged in their respective base-ten groupings.

In many classroom settings, children are provided with base-ten blocks (see Figure 6), including cubes (i.e., to represent ones), sticks (i.e., tens), and flats (i.e., hundreds). A stick consists of ten cubes stacked together and is ten times longer than a single cube. A flat consists of ten sticks attached together and is ten times larger than a stick and 100 times larger than a cube. Because sticks and flats are usually pre-composed, in other words, already grouped in their respective base-ten units, children rarely experience constructing these composite units themselves. As a result, they may miss the opportunity to develop an understanding of how basic units combine to form composite units, which might limit their understanding of the hierarchical structure between numeration units. It is therefore important to explore whether using pre-composed manipulatives influences second graders’ unitizing performance relative to using composite units that were constructed by the students themselves. Findings could inform future instructional interventions by suggesting that children’s understanding of base-ten structure can be best supported when they create their own composite units.

Figure 6

Base-Ten Blocks



Present Study

The present study aimed to gain deeper insights into children's unitizing performance and, more specifically, the conditions under which children can successfully unitize quantities when solving measurement division problems. To do so, I adopted a qualitative approach to analyze children's reasoning because little is known about their unitizing performance in base-ten contexts, especially when solving measurement division problems. The paucity of research on children's unitizing in base ten thus precluded specific hypotheses. The analysis focused on an open-ended exploration of the nature of children's unitizing, the influence of their prior knowledge on base-ten numeration, and the influence of manipulative type on their performance.

As shown in previous studies, children often struggle to understand the structure of base-ten numeration, particularly when they are asked to unitize quantities represented with numerals (Barilaro, 2026a). One of the most common errors when interpreting a numeral using a given numeration unit is when learners focus only on the digit in the position of the targeted numeration unit (Barilaro, 2026a; Barilaro et al., 2026). This type of response reflects an overreliance on the principle of position, which tends to be the primary focus in elementary mathematics instruction (Houdement & Tempier, 2019; Tempier, 2024). As a result, children spontaneously activate their principle-of-position knowledge when presented with numerals, which negatively influences their ability to attend to the relations between numeration units (Barilaro, 2026a).

Carpenter et al. (1996) suggested that measurement division problems could be a valuable tool to support children's understanding of the base-ten structure. Measurement division problems require children to identify the number of composite units in a given quantity. Introducing measurement division problems before any formal instruction on multiplication and division may be beneficial, particularly if children are given manipulatives to solve them. Using concrete representations rather than symbolic ones might reduce the likelihood that the principle of position is activated and provide avenues for learning about the base-ten structure. However, the perceptual features of manipulatives might also influence unitizing performance. Therefore, their impact should be further investigated to better understand the conditions under which children can unitize using concrete objects.

The present study aimed to explore children's unitizing, specifically their ability to construct and count composite units and switch between various reference units, effectively reducing their reliance on the principle of position. In addition, the study sought to investigate whether realism, the perceptual salience of physical representations of units, and the pre-composition of the manipulatives used might influence children's unitizing performance. Second-grade students solved measurement division word problems grounded in familiar situations, such as fish grouped in nets. The problems were embedded in base-ten contexts, meaning that the units were related by powers of ten (e.g., one composite unit was composed of 10 basic units). To solve the problems, children were provided with different types of manipulatives: realistic objects, attachable cubes, or sets of base-ten blocks consisting of attachable cubes and pre-composed sticks and flats. Analyses focused on the verbal exchanges between the children and the experimenter and the children's physical actions with the manipulatives. The analysis was entirely qualitative, with no a priori hypotheses. The findings promise to inform the development of pedagogical tools and instructional interventions that can be tested in future research.

Method

Participants

Participants were 40 second graders from three classrooms in two urban elementary schools in the province of Quebec (Canada). To recruit participants, I contacted school secretaries by email and asked them to share the study pamphlet with their second-grade teachers and the school principals. Teachers interested in participating were invited to contact me by

email or phone to obtain more information about the study. Participating teachers oversaw the distribution and collection of information letters, consent forms, and demographic questionnaire to parents.

The data presented in this chapter are part of a larger study in which two types of measurement division problems were administered. The larger study involved 40 students who either solved problems in base ten ($n = 22$) or in other bases ($n = 18$). The present study was based on data from the 22 students who solved the problems embedded in base-ten contexts. From the sample of 22, two children were excluded because of technical issues with the video recording during data collection, resulting a lack of usable data for these participants. Therefore, the final sample in the present study consisted of 20 second-grade students.

The participants had a mean age of 91.70 months ($M = 7.69$ years, $SD = 3.94$). Among them were 12 girls (60%) and 8 boys (40%). The language spoken at home was French for 14 participants (70% of the sample), Spanish for one child (5%), both French and English for four children (20%), and both French and Lingala for one child (5%). As reported by the parents in the demographic questionnaire, six children had mathematical difficulties without a diagnosis of Specific Learning Disorder with impairment in mathematics (30%; Diagnostic and Statistical Manual of Mental Disorders, Fifth Edition, 2013), and three had Attention-Deficit/Hyperactivity Disorder (15%). For 11 children, the family income was equal or superior to \$80,000 (55%); for four children, the family income was between 20,000 and 80,000 (20%); and for two children, it was less than \$20,000 (10%). Information on family income was missing for three children (15%).

Design and Procedure

I met with each child individually in one session of approximately one hour during the months of November and December 2024. During the session, children completed four tasks: a task assessing principle-of-position knowledge, one assessing knowledge of the relations between numeration units, one assessing unitizing when presented with numerals, and the Word-Problem Solving task, which assessed the children's unitizing when presented with measurement division problems and manipulatives. Only the data from the administration of the Word-Problem Solving task were used for analysis in the present study. The task was always administered after the three other tasks. All sessions were video recorded.

Word-Problem Solving Task

Materials

Word Problems. The word problems were measurement division problems designed specifically for this study (see Table 6 for English translations). The problems in the original French are available in Appendix E. The word problems were designed to assess children's unitizing performance, operationalized by requiring them to create and count composite units as if they were ones.

Table 6

Word Problems Used in the Word-Problem Solving Task (Translated from French)

One-Level Problems	Two-Level Problems
<i>MD (30,10): candies, bags</i> Hugo has 30 candies. He puts all the candies in bags. Ten candies fit in a bag. How many full bags of candies does Hugo have? How many candies does Hugo have in all?	<i>MD (240,10,10): candies, bags, boxes</i> Hugo has 240 candies. He puts all the candies in bags and puts all the bags in boxes. One bag contains 10 candies, and one box contains 10 bags. How many full bags does Hugo have? How many full boxes does Hugo have? How many candies does Hugo have in all?
<i>MD (50,10): fish, nets</i> Alexane catches 50 fish. She puts all the fish in nets. One net contains 10 fish. How many full nets does Alexane have? How many fish does Alexane have in all?	<i>MD (462,10,10): fish, nets, crates</i> Manon catches 462 fish. She puts all the fish in nets and puts all the nets in crates. One net contains 10 fish, and one crate contains 10 nets. How many full nets does Manon have? How many full crates does Manon have? How many fish does Manon have in all?
<i>MD (47,10): seashells, boxes</i> Leo finds 47 seashells. He puts all the seashells in boxes. One box contains 10 seashells. How many full boxes does Leo	<i>MD (327,10,10): candies, bags, boxes</i> Leo has 327 candies. He puts all the candies in bags and puts all the bags in boxes. One bag contains 10 candies, and one box contains 10 bags. How many full bags does

have? How many seashells does Leo have in all?

Leo have? How many full boxes does Leo have? How many candies does Leo have in all?

MD (24,10): apples, baskets

Manon picks 24 apples. She puts all the apples into baskets. One basket contains 10 apples. How many full baskets does Manon have? How many apples does Manon have in all?

Note. Each problem is labeled with a specific code. “MD” stands for measurement division. The first number in parenthesis indicated the total number of basic units. This is followed by one or two numbers that represent the number of items within each composite unit involved in the problem. The code ends with the referents used in the problem (two or three referents).

The measurement division problems were embedded in base-ten contexts. In other words, all the units were base-ten units and were related by powers of ten. Ten units together were equivalent to one composite unit of the next-higher level. Participants solved seven word problems. Four were *one-level problems*, and three were *two-level problems*. *One-level problems* involved one level of composite unit constructed from the basic units presented in the problem (see Table 6, first column). For example, in the *MD (30, 10): candies, bags* problem, the composite unit (i.e., the bag) is composed of 10 basic units (i.e., 10 candies). *Two-level problems* involved two levels of composite units: (a) one composite unit containing 10 basic units presented in the problem, and (b) the other containing 10 of the first composite unit created (see *two-level problems* in Table 6). For example, in the *MD (240,10,10): candies, bags, boxes* problem, the first composite unit is the bag, composed of 10 basic units (i.e., 10 candies); the second composite unit is the box, containing 10 of the first composite units (i.e., 10 bags). All problems were embedded in real-life contexts, using real-world referents (e.g., candies and bags) that served as basic units (e.g., candies) and composite units (e.g., bags), respectively.

Manipulatives. For each problem, participants were provided with one of three sets of manipulatives presented in Figure 7. Participants were instructed to use the manipulatives to solve the problem. The manipulative sets varied along three dimensions: realism, attachability, and whether the composite units were pre-composed.

Figure 7

Manipulative Sets Used in the Word-Problem-Solving Task

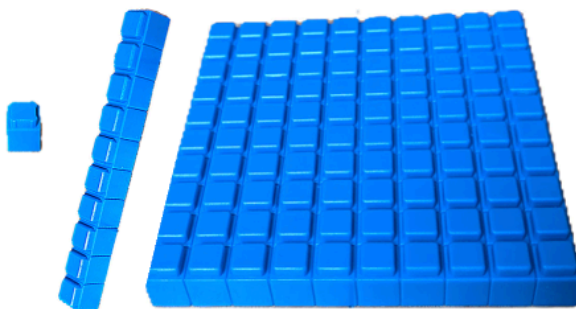
Set 1: Realistic / Non-Attachable / Not Pre-Composed



Set 2: Generic / Attachable / Not Pre-Composed



Set 3: Generic / Attachable / Pre-Composed



Children were randomly assigned to one of the sets of manipulatives in Figure 7. In Set 1, the objects included realistic objects that matched the context of the problem being solved (i.e., candies, apples, fish, and shell counters). Sets 2 and 3 were composed of generic objects consisting of cubes, sticks, and flats. In Set 2, the cubes were stackable (see second row of Figure 7). These cubes could be assembled to form sticks of any length. Finally, Set 3 consisted of base-ten blocks, which included stackable cubes, sticks, and flats (see third row of Figure 7). Sticks were pre-composed of 10 cubes stacked together, making them 10 times larger than a single cube. These sticks could be further assembled to create flats. Additionally, the set included pre-composed flats, each 10 times larger than a stick and 100 times larger than a cube. Like the pre-composed sticks, the flats were not detachable.

Procedure

Each problem was typed on a sheet of paper and placed in front of the participant. I presented each problem one at a time and read each one out loud. For each problem, I gave a set of manipulatives to participants and asked them to use the manipulatives to solve the problem and to answer all the questions asked. Requiring students to rely on manipulatives allowed me to observe their counting processes, which provided a way for me assess their ability to shift the unit of reference when counting. In other words, my goal was to observe whether children could count the same quantity represented with manipulatives using two different reference units (e.g.,

candies or bags). They were not permitted to write or draw anything on the sheet of paper on which the problem was written and were instructed to state their answers orally.

For the *one-level problems*, I gave children one of the three sets of manipulatives presented in Figure 7. Children who were assigned the third set of manipulatives were only provided with cubes and pre-composed sticks because the flats were not needed to solve *one-level problems*. I asked children to answer two questions: The first question was about the number of composite units (e.g., How many full bags does Hugo have?), and the second was about the number of basic units (e.g., How many candies does Hugo have in all?) The two questions were written on separate lines on the sheet of paper, and I revealed only one question at a time. I asked each subsequent question only if the child correctly answered the previous one and clearly demonstrated how they used the manipulatives to find their answer.

For the *two-level problems*, I provided only base-ten blocks (i.e., Set 3) to children because of the size of the quantities involved in the *two-level problems*. I asked children to answer three questions in the following order: a question about the first composite units (e.g., How many full bags does Hugo have in all?), a question about the second composite units (e.g., How many full boxes does Hugo have?), and a third question about the total number of basic units (e.g., How many candies does Hugo have in all?). The three questions were written on separate lines of text on the sheet of paper, which allowed me to reveal one question at a time. Similar to the administration of the *one-level problems*, I asked each question only if the child correctly answered the previous one, and children were prompted to use the manipulatives to find the answer and to justify any answer provided.

While some children spontaneously found the correct answers for both *one-level* and *two-level problems* using the manipulatives, others encountered difficulties. I provided different forms of support depending on the nature of the challenges. First, if a child did not use the manipulatives, I reminded them to do so, regardless of whether their answer was correct. Second, if a child provided an incorrect answer, I guided them to understand what each manipulative represented and how to count them depending on the unit of reference (e.g., one object represents a candy, and a group of 10 objects corresponds to a bag of 10 candies).

I presented the problems in random order to the children, and the number of problems completed by each child varied depending on individual pace. Therefore, none of the children solved all seven problems presented earlier. Five children solved only *one-level problems*: One

of them completed a single *one-level problem*, while the remaining four completed two. Among the remaining participants ($n = 15$), two completed one problem of each type, while two others completed two problems of each type. Four participants completed one *one-level problem* and two *two-level problems*, six completed two *one-level problems* and one *two-level problems*, and one participant completed three *one-level problems* and one *two-level problem*. In total, 34 *one-level problems* and 21 *two-level problems* were included in the present analysis.

Data Collection and Analysis

Data sources included the transcripts from each session and field notes. I analyzed the data using a thematic analysis (Terry et al., 2017). First, I familiarized myself with the data by reading each transcript several times and writing analytic memos to capture initial observations. Second, these analytic memos were used to label portions of the transcripts based on elements relevant to children's unitizing such as strategies, challenges, and conditions that supported successful unitizing. Codes were generated inductively from participants' utterances and were informed by the current literature on unitizing (e.g., Barilaro et al., 2026; Wege et al., 2023). Once codes were identified, I returned to the transcripts and reviewed them several times to identify segments of data corresponding to each code. This process allowed me to refine the definition of the codes and clarify the various utterances or actions associated with each code. Finally, I developed themes based on the identified codes. Themes were then reviewed and refined during the analysis.

My results focus on children's conceptual understanding and challenges in creating and counting composite units, including when switching reference units. Additionally, although measurement division problems were expected to reduce children's reliance on the principle of position, I examined the influence of their prior knowledge of base-ten numeration, particularly because base-ten blocks, a tool commonly used in teaching base ten, were provided during the task. Investigating whether the use of base-ten blocks might have shaped children's unitizing performance because of their prior knowledge of the numeration units represented by the manipulatives would provide insight into children's reasoning about quantities.

Results

The purpose of the present study is to investigate children's unitizing in base-ten contexts when solving measurement division problems. Because unitizing appears to be particularly challenging when working with numerals, investigating it in the context of solving measurement

division problems with manipulatives may provide valuable insights into children's unitizing performance. The problems involved one or two levels of numeration units to manipulate, and children were provided with various types of manipulatives.

My observations from the data and the current literature on unitizing in base ten allowed for several themes to emerge from the data. The first theme captured how children constructed composite units using manipulatives and how children enumerated those composite units. Across the data set, children constructed composite units in many ways, and their construction was shaped by the physical constraints of the manipulatives. The various ways of representing composite units influenced how they were subsequently counted. The second theme concerns children's ability to count basic units, and more specifically, whether they demonstrate a shift in their counting strategies, moving from counting composite units to focusing on basic units. Finally, the third theme reflects children's activation of their prior knowledge on base-ten numeration. More specifically, across the data, children's reliance on their prior knowledge of base ten appeared to depend on the type of manipulatives used.

Below are examples of children's reasoning, justifications, and actions demonstrating their understanding of unitizing and how they responded to prompts and guidance. Pseudonyms were used in all transcripts to preserve participants' anonymity. All the transcripts are presented in French to preserve their original meaning, particularly because of the nuances present in the children's justifications. Each transcript is numbered, and English translations are provided in Appendix F.

Various terms will be used in the following sections. Basic units refer to the initial quantity, expressed in ones. In *one-level problems*, the composite unit refers to the group formed by combining 10 basic units (i.e., tens). In *two-level problems*, two levels of composite units are involved: the first composite unit (i.e., a group of 10 basic units), and the second composite unit (i.e., a group of 10 first composite units). The term "referent" refers to the object used in the problem context, such as candies, bags, or boxes. Finally, numeration units (e.g., ones, tens, hundreds) are used when children refer to the structure of base-ten numeration rather than to the referents or the manipulatives.

Students' Construction and Counting of Composite Units

In the following section, the data for *one-level* and *two-level problems* were analyzed separately because the manipulatives differed, leading to distinct challenges in each case.

One-Level Problems

Constructing Composite Units. To construct composite units, children were provided either with stackable cubes, base-ten blocks (i.e., stackable cubes and sticks), or realistic objects (i.e., apple counters). Interestingly, depending on the material used, children created composite units in different ways, and the way they were created appeared to have influenced children's ability to count them. It is thus interesting to look at the various ways composite units were constructed and how the manipulatives appeared to have constrained their strategies.

First, when provided with stackable cubes only, children created composite units by grouping 10 cubes together. Two different grouping strategies were observed, with some children grouping the cubes on the table without stacking them and other children grouping the cubes by stacking them to form one stick of 10 cubes. Children who were assigned realistic manipulatives, in contrast, tended to simply group the items together on the table, presumably because the items could not be physically stacked. Finally, when working with base-ten blocks, children displayed different behaviors: Some ignored the pre-composed sticks and chose to stack 10 individual cubes together to form a stick (i.e., the composite unit), while most used the pre-composed sticks. These differences highlight how the physical affordances of the materials could influence children's strategies for building composite units. Depending on the manipulative type, children employed different ways of constructing composite units, which may have contributed to further difficulties when asked to count them as single entities (Boyer et al., 2008; Jeong et al., 2007; Tello et al., 2025).

Counting Composite Units. For the *one-level problems*, only one composite unit needed to be identified. Once children constructed the composite units, I asked them to identify the total number of composite units constructed. The ways the children counted the composite units provides valuable insights into the kinds of supports that might be effective in an instructional context.

Susie and Max both correctly identified the total number of composite units, but did not use the same justification for their answers. For example, Susie was asked to solve the problem *MD (47,10): seashells, bags* using realistic manipulatives (i.e., seashell counters). She first took 40 counters to represent the initial quantity and then created four groups of 10 counters on the table. I asked her to identify the total number of composite units (i.e., boxes).

Transcript 1

Researcher (R) : *Alors il y a combien de boîtes complètement remplies ?*

Susie : *Quatre.*

R : *Comment tu sais ? Tu m'expliques ?*

Susie : *Là il y en a quatre ici* (Susie gestured over the four groups of 10 counters created on the table).

Susie was able to quickly identify the four boxes by subitizing the four groups of 10 counters she had created on the table. While pointing to each of the four groups and stating the total, Susie demonstrated an understanding that each group could be considered as a single entity. Notably, she did not refer back to the basic unit (i.e., the individual seashells) when I asked her to justify her response. Instead, her reasoning showed a solid grasp of the idea that a group of items can be conceptualized and counted as a single thing.

Max, however, based her justification on the total number of basic units used to form those groups rather than working directly with composite units. She was asked to solve the problem *MD (50,10): fish, nets* and was provided with base-ten blocks. She represented the initial quantity of 50 fish by selecting five sticks.

Transcript 2

R : *Donc moi je veux savoir avec les 50 poissons combien de filets je peux faire ?*

Max : *filets je peux faire ... Je peux en faire 5.*

R : *Comment tu sais ?*

Max : *Parce qu'il y en a 50 et puis ici, là y'en a 4* (Max grabbed the five sticks in her hand and showed them to me), *puis là plus 5, ben plus 1 ça fait 5. Fait que y'en a 5.*

Here, Max justified her answer by referring directly to the initial quantity 50, which demonstrated a flexible understanding of unit levels: She was able to interpret each group both as 10 basic units and as one composite unit. This observation demonstrates a strong understanding of the ten-for-one conception (Fuson, 1990), also considered unitizing by Fosnot and Dolk (2001). From a unitizing perspective, this justification is particularly compelling, as it reveals that Max understood how the same quantity can be interpreted differently depending on the reference unit.

Although Susie and Max successfully counted the composite units, I observed several students struggle when I asked them to count the composite units. Consider Jordan, who was asked to solve the problem *MD (24,10): apples, baskets* using stackable cubes. He first took 24 stackable cubes to represent the initial quantity described in the problem. Then, to create the

composite units, he assembled 10 cubes together to form a stick. Jordan repeated this action one more time, resulting in two sticks and four cubes alone placed on the table in front of him.

Transcript 3

R : *Alors maintenant, est-ce que tu peux me dire combien de paniers on peut faire ?*

Jordan : 10.

R : *Alors comment tu sais ? Tu m'expliques ?*

Jordan : *Là* (Jordan showed the sentence in the problem text where it is indicated that one basket contains 10 apples).

In this interaction, when I asked Jordan to identify the total number of composite units (i.e., baskets), he referred to the total number of units within one composite unit. This shows that Jordan struggled to differentiate the number of composite units and the number of basic units embedded in each composite unit. Jordan understood that 10 basic units (i.e., apples) are in one composite unit (i.e., one basket), but when asked to count all the composite units, he failed to focus on the number of groups. This error could stem from a difficulty in identifying what needs to be counted, or a difficulty in knowing how to count the groups.

Lily also struggled to identify the number of composite units. She was asked to solve the problem *MD (50,10): fish, nets*. Lily used realistic manipulatives (i.e., fish counters) to represent the initial quantity and to create the composite units. In her case, the difficulty appeared to stem from a difficulty in differentiating the referents used in the problem, and especially the difference between the basic unit (i.e., fish) and the composite unit (i.e., nets).

Transcript 4

R : *Donc moi ce que je veux savoir c'est combien de filets elle va pouvoir faire ?*

Lily : 50. 50 poissons dans 50 filets.

R : *Il y a 50 filets ?*

Lily : *Bah on va pouvoir mettre 50 poissons dans 10 fil... euh...*

Lily answered “50 fish into 50 nets” when I asked her about the total number of nets. Her response revealed a confusion between the number of fish (i.e., basic units) and the number of nets (i.e., composite units), which cannot be mathematically equivalent in the context of the problem. Two types of support might assist Lily in this case. First, providing stackable cubes and encouraging her to physically construct a composite unit by assembling the cubes into groups of 10 might have helped her perceive the one-ness of the created composite unit. Second, providing manipulatives that explicitly represented the composite units as described in the problem, such as toy nets, might have supported her understanding of the difference between the two referents.

Using two distinct types of objects to represent basic and composite units might help children like Lily better grasp the distinction between each unit and the physical objects associated with them.

The hypothesis above was in fact supported by the work of another child, June, who also struggled to identify the number of composite units. June was asked to solve the problem *MD (47,10): seashells, boxes* and was given base-ten blocks. She began by representing the initial quantity with four sticks and seven cubes but then changed her representation to four groups of 10 non-stacked cubes placed next to a group of seven cubes. June struggled to find the number of composite units, so I helped her to associate the material with the appropriate referent used in the problem (i.e., one stick of 10 cubes is equivalent to one box of 10 seashells).

Transcript 5

R : *Une boîte contient 10 coquillages. Donc si je mets 10 coquillages ensemble, je pourrais remplir ...*

June : *Une boîte de coquillages.*

R : *Une boîte, oui, je pourrai remplir une boîte. Donc là, j'ai combien de boîtes ici* (I gestured over one group containing 10 cubes on the table)

June : *10*

R : *Alors, tu as 10 coquillages, oui, mais tu as combien de boîtes ? Tu peux remplir combien de boîtes avec 10 coquillages ?*

June : *10*

[Because June struggled to see the difference between counting the group and counting the individual cubes, I gave her little bags and asked her to create the groups by putting 10 cubes in each bag. She was then able to create the composite units using four bags].

R : *Alors, on a rempli combien de boîtes ?*

June : (June takes each bag in their hand and counts each bag) *Quatre*

R : *Oui, comment tu as fait pour compter ?*

June : *Parce que j'ai fait* (June points to the four bags, one by one), *il y en a 4.*

In this interaction, June initially struggled to identify the number of composite units when using base-ten blocks. However, when I provided her with a real bag—that is, an object that could more realistically represent the box mentioned in the problem—she was able to count the total number of bags without hesitation. This contrast suggests that representing a box using a neutral object such as a stick may pose difficulties for some children, particularly those who require more realistic manipulatives to make sense of the problem. This transcript clearly illustrates June's difficulty in connecting the stick to the concept of a box. By offering a more realistic

object, one that is closer in appearance and function to the composite unit in the problem, June was able to successfully find the total number. This example highlights how the type of manipulative can influence children's ability to mentally represent composite units and count them.

Two-Level Problems

For the *two-level problems*, children were required to construct and identify two levels of composite units. Constructing the first composite unit entailed grouping 10 basic units together (i.e., corresponding to the tens). The second composite unit can be created by grouping 10 of the first composite unit, resulting in the hundreds in a three-digit quantity. In the following sections, the findings on constructing both levels of composite units are reported first, followed by findings on how the children counted both levels of composite units. The findings on children's counting are reported separately for each level because they present distinct characteristics and challenges.

Constructing Composite Units. For the *two-level problems*, children were only provided the base-ten blocks, which consisted of a set of stackable cubes, stackable pre-composed sticks, and pre-composed flats. Children never represented the initial quantity using only the stackable cubes. Some of the children used the stackable cubes, sticks, and flats to represent the initial quantity, while other children used only stackable cubes and sticks in their representations.

The way the initial quantity was represented using the given material appeared to influence children's ability to identify the total number of the first composite unit (i.e., tens). Unlike *one-level problems*, *two-level problems* involved a hierarchical structure in which the first composite unit is embedded within the second composite unit (i.e., hundreds). More specifically, when using a flat to represent the total quantity, children must decompose the flat, either physically or mentally, into 10 sticks to determine the total number of sticks (i.e., first composite units).

Counting the First Composite Unit. When solving *two-level problems*, I first asked children to find the total number of tens (i.e., the first composite unit). Some students spontaneously identified the total number of the first composite units without any help or support. Sadie, for example, was asked to solve the problem *MD (462, 10, 10): fish, nets, crates*. She represented 462 by placing four flats, six sticks, and two cubes on the table. I then asked her to find the number of the first composite unit (i.e., the nets).

Transcript 6

Sadie : *J'ai 462.*

R : *Donc ok t'as fait la quantité, mais maintenant moi je veux vraiment savoir combien il y a de filets en tout ?*

Sadie : *Attends !*

R : *Je te laisse réfléchir, t'inquiète. Prends ton temps.*

Sadie : *Déjà y'a 40 dizaines (she counted all the sticks included in the four flats one by one). 46 filets remplis.*

Sadie first identified the 40 tens embedded in the four flats by counting each stick in the pre-composed flats one by one. She then added the six separate sticks, resulting in her answer of 46 nets. This transcript reflects an understanding of the equivalence between the referent used in the problem (i.e., net), the manipulatives (i.e., stick), and the corresponding numeration unit (i.e., ten) because she used the terms nets and tens interchangeably to refer to the sticks.

Sadie was one of only a few students who successfully identified the total number of the first composite units without any external support. The difficulties observed when other children were asked to identify the total number of tens appeared to stem from specific challenges. Consider Will, Susie, and Jasper, who were asked to solve the *MD (327, 10, 10): candies, bags, boxes* problem. They all represented the initial quantity using three flats, two sticks, and seven cubes. I identified three types of challenges, each illustrated below.

First, because of the way children constructed the initial quantity using cubes, sticks, and flats, some focused only on the sticks that were not embedded in the flats, overlooking the sticks that were embedded in the flats.

Transcript 7

R : *Donc est-ce que tu peux trouver le nombre de sachets qu'il a fait au total ?*

Will : 2 (Will grabbed the two sticks alone on the table and showed them to me). *À cause qu'ici y'a deux dizaines, après y'en a 10 dedans, dans un sac, fait que c'est sûr que y'en a deux remplis.*

In this transcript, Will focuses only on the two visible sticks on the table, ignoring those embedded within the flats. This type of error resembles a pattern found by Barilaro (2026a), namely the tendency to interpret a quantity by attending only to the sticks that represented the value of the tens digit in 327. Though there may be several explanations for Will's neglecting the sticks within the flats, one possibility is that the embedded sticks were not sufficiently visually salient (Barilaro et al., 2026).

Another difficulty seemed to emerge from a misunderstanding of the correspondence between the referent used in the problem and the manipulatives provided. More specifically, some children struggled to determine which manipulatives needed to be counted to correctly answer the question.

Transcript 8

R : *Okay, donc on cherche combien de sachets on peut faire ?*

Susie : 3

R : *Comment tu sais qu'il y en a 3 ?*

Susie : *Ça, c'est des des centaines et y'a 3* (Susie grabbed the three flats and showed them).

Susie identified the flat as representing the first composite unit, rather than the sticks. In this transcript, she demonstrated difficulty understanding what the manipulatives represented in the context of the problem, namely boxes represented by flats and bags represented by sticks. As a result, Susie counted the wrong manipulatives (i.e., the flats) as if they were ones instead of 10 sticks. The use of realistic manipulatives, unlike base-ten blocks, may have supported a more accurate and more direct mapping between the manipulatives and the referents in the problem, thereby helping the child better identify the correct representation that needed to be counted.

Finally, a last challenge appeared to be due to difficulties in switching the unit of reference when counting the manipulatives. In other words, some students counted different types of manipulatives (e.g., stick and flat) as if they all referred to the same composite unit. This suggests a struggle to modify their counting strategy when different manipulatives require different counting approaches. For example, when identifying the first composite unit, a stick should be counted as one composite unit, whereas a flat should be counted as 10 of those composite units. When children failed to differentiate between the stick and the flat, they tended to count both as one same composite unit each.

Transcript 9

R : *Donc on pourrait en faire combien des sachets ?*

Jasper : 5 (Jasper counted the two sticks and the three flats each time as one thing, one ten)

[I helped Jasper see that in a flat there are 10 sticks, 10 bags]

R : *Donc ça fait combien tout ça de sachets ?* (I pointed at the three flats)

Jasper : 30

R : *30 plus nos sachets qui sont sur le côté* (I pointed at the two sticks)
Jasper : *50*

In this exchange, Jasper initially counted the two sticks and the three flats as if they were all ones—that is, all at the same unit level. While he correctly treated one stick as a single unit when asked to find the total number of tens, he then counted each flat as a ten as well, rather than as 10 tens. Interestingly, Jasper repeated this misunderstanding in a different way later in the task. After I told him that each flat contained 10 sticks (i.e., 10 bags), he correctly identified 30 bags from the three flats. However, when I asked him to add the two sticks that were left alone on the table, Jasper again failed to switch his unit of reference and counted the two sticks as 20 bags. This clearly demonstrates confusion regarding what each block represented in the context of the problem. Similar errors have been documented in prior research (Barilaro, 2026a; Barilaro et al., 2026), both in the context of unitizing numerals and when finding the total number of tens in three-digit quantities represented visually.

Counting the Second Composite Unit. After identifying the total number of the first composite unit, children were then asked to identify the total number of the second composite units. In *two-level problems*, the second composite is formed by grouping 10 of the first composite units together. This task is particularly complex, as it requires children not only to have successfully identified the first composite units, but also to use them to construct the second composite unit. As a result, this step required children to engage more with deeply embedded relations, likely rendering the identification of the second composite unit a more demanding task than the first.

As for the first composite units, some children were able to spontaneously find the appropriate number of the second composite units without any support. Dustin was asked to solve *MD (327, 10, 10): candies, bags, boxes* problem. After Dustin correctly identified the total number of the first composite units (i.e., 32 bags), I asked him to find the total number of the second composite unit (i.e., boxes).

Transcript 10

R : *Combien de boîtes complètement remplies Léo a-t-il ?*

Dustin : *Trois*

R : *Comment tu sais ?*

Dustin : *A cause de trois boîtes* (Dustin pointed at the three flats placed on the table)

Interestingly, Dustin spontaneously found the number of the second composite units and justified his answer using the manipulatives. By pointing to the three flats on the table, he demonstrated an understanding of what the manipulatives represented and how they needed to be counted to find the appropriate solution.

Unlike Dustin, however, most children struggled when asked to identify the number of second composite units. The design of the base-ten blocks themselves seemed to contribute to some of the difficulties experienced by the children. As previously mentioned, children spontaneously and consistently represented the initial quantity by using cubes, sticks, and flats. Therefore, by virtue of the base-ten blocks' design, the second composite unit was already created on the table using the flats. Each flat should be counted as 10 units when I asked children about the first composite units, but the same flat had to be counted as one second composite unit when I asked about the second composite units. This dual representation of the flat added another challenge for children, who were required to shift what was being counted according to the composite unit that was required to solve the problem. Switching their referent when counting the exact same manipulative might be particularly challenging for students. Consider Taylor, who solved the *MD (240, 10, 10): candies, bags, boxes* problem. He began by representing the initial quantity using two flats and four sticks. After correctly finding the total number of the first composite units (i.e., 24), I asked Taylor to find the total number of the second composite units.

Transcript 11

R : *C'est combien de boîtes complètement remplis Hugo il a ? Et alors, on se souvient, une boîte contient 10 sachets, donc combien de boîtes il peut faire ?*

Taylor : *heu attends*

R : *Dans une boîte, il y a 10 sachets*

Taylor : *ça, c'est 100, 200* (Taylor grabbed two flats and showed them). *Attends, ça c'est 10* (Taylor grabbed one flat and showed it).

R : *ça c'est 10 quoi ?*

Taylor : *dizaines*

R : *10 sachets*

Taylor : *10 sachets*

R : *Okay*

Taylor : *ça c'est 10 sachets* (Taylor pointed at one flat)

R : *donc c'est combien de boîtes ?*

Taylor : *ça c'est 20 sachets* (Taylor pointed at a second flat)

R : *Donc ça c'est combien de boîtes*

Taylor : *heu 200 boîtes*

R : *ça c'est 10 sachets* (I pointed at one flat)

Taylor : *oui*

R : *10 sachets, 10 sachets ensemble ça fait combien de boîtes ?*

Taylor : *100*

Taylor struggled to interpret a flat as representing one box. Initially, he referred to a flat as “100” things, but later changed his answer to “10” things. When asked how many boxes one flat represented, Taylor answered 200 “boxes.” This response reflects a misunderstanding of what the manipulatives represent and how they should be counted when reasoning about different levels of units. Using pre-composed flats did not provide children the opportunity to physically construct the groups themselves, which may have partially accounted for the challenges I observed in correctly identifying the second composite units. In Taylor’s case, not having the opportunity to physically put 10 sticks together to create groups representing boxes may have prevented him from identifying the second composite unit. Realistic objects instead may also have supported the connection between the referents used in the problem and the manipulatives provided.

To test this hypothesis, I explored whether physically creating the second composite units would support identifying and counting them. Jasper, for example, was asked to solve the *MD* (327, 10, 10): *candies, bags, boxes* problem. He represented the initial quantity using three flats, two sticks, and seven cubes. After correctly identifying the total number of the first composite units (i.e., 32), I asked Jasper to find the total number of the second composite units.

Transcript 12

R : *Cette fois-ci, je te demande combien de boîtes complètement remplies Léo a-t-il ?*

Jasper : *Donc il faut que je compte mais mettons, ça c’est des cents* (Jasper pointed at the three flats)

R : *Alors, pour est-ce que tu sais qu’est-ce qu’il y a dans une boîte ?*

Jasper : *Dix*

R : *Dix quoi ?*

Jasper : *Bonbons*

R : *Alors c’est des sachets. Dans un sachet, il y a 10 bonbons. Mais dans une boîte, il y a 10 quoi ?*

Jasper : *Dix boîtes, heu 10 sachets.*

R : *Une boîte ça contient 10 sachets, super. Donc comment est-ce qu’on peut trouver combien on a de boîtes ?*

Jasper : *Heu*

R : *Rappelle-toi, t’as combien de sachets en tout déjà ?*

Jasper : *32*

R : *T’as 32 sachets. Donc si on réfléchit, on a 32 sachets et il faut qu’on fasse des groupes de combien ?*

Jasper : *De 10*

R : *Des groupes de 10 sachets. Alors je peux faire combien de groupes de 10 avec 32 ?*

Jasper : *Heu ...*

R : *Est-ce que tu vois ou pas ?*

Jasper : *Quoi ?*

R : *Est-ce que tu vois comment faire ?* (Jasper shook his head)

[I helped Jasper exchange the flats for sticks. The three flats were replaced by 30 sticks, resulting in a total of 32 sticks in front of him.]

R : *Okay, maintenant je veux que tu me dises avec ces 32 sachets, combien de boîtes tu peux faire ? Et tu peux faire les boîtes si tu as besoin.*

Jasper : *Donc faut qu'on prenne 10 trucs, 10 dizaines* (Jasper counted 10 sticks and placed 10 sticks together in one group on the table. Jasper continued to create two other groups of 10 sticks on the table).

R : *Alors t'as pu faire combien de boîtes ?*

Jasper : *Trois.*

Jasper initially struggled to remember that one box (i.e., the second composite unit) was composed of 10 bags (i.e., the first composite units). After identifying the relation between the first and the second composite units and determining the total number of first composite units, Jasper was not sure about what to do next with the manipulatives or what exactly needed to be counted. To support him, I encouraged Jasper to exchange the flats for sticks only. Once all the sticks were retrieved, he was able to construct the second composite units himself by grouping ten sticks together on the table. Interestingly, without any further guidance, Jasper spontaneously identified that he had formed three boxes. This observation is particularly interesting because it suggests that the act of physically constructing composite units may support children's understanding of the hierarchical relations between units (Osana et al., 2026).

Students' Counting of Basic Units

On each of the *one-level* and *two-level problems*, after the children identified the total number of composite units, I asked them to count the total number of basic units (i.e., the ones). This required children to change their way of counting the manipulatives, which allowed me to assess their interpretations of quantities using more than one unit. The ability to shift how quantities are interpreted depending on the unit of reference used reflects a flexible understanding of the hierarchical structure of numeration units.

On the *one-level problems*, most of the students were able to correctly switch their unit of reference and count all the manipulatives as if they were ones, regardless of manipulative type.

The same result was found for *two-level problems*, with most students able to identify the total number of the basic units. Consider Morgan, who solved the *MD (30, 10): candies, bags* problem. She used base-ten blocks to solve the problem and took three pre-composed sticks to represent the initial quantity. She counted the total number of first composite units (i.e., bags) by counting each stick one by one. Then, I asked Morgan to find the total number of basic units (i.e., candies).

Transcript 13

R : *Est-ce que tu peux me dire, combien de bonbons Hugo il a en tout ?*

Morgan : 30.

R : *Comment tu sais ?*

Morgan : 10 (Morgan is pointing at a first stick), 20 (Morgan is pointing at a second stick), 30 (Morgan is pointing at a third stick).

In this transcript, Morgan counted each stick by tens. She did not need to count each individual cube one by one, indicating her interpretation of a stick as both a single composite unit and as 10 basic units. The flexible switching between units demonstrates optimal unitizing performance and more specifically, an understanding of the ten-for-one conception (Fuson, 1990).

Not all children were able to correctly switch their unit of reference when counting the manipulatives, however. More specifically, I observed three types of errors when I asked children to count the number of basic units. The first error involved counting only the basic units used to create the composite units and ignoring the left-over manipulatives that were not used to form an additional full composite unit. Dustin was asked to solve the *MD (47, 10): seashells, boxes* problem. After having created the initial quantity using base-ten blocks, Dustin was able to identify the total number of composite units (i.e., bags). I then asked him to find the total number of basic units (i.e., seashells).

Transcript 14

R : *Et alors combien Léo il a de coquillages en tout ?*

Dustin : 40.

R : *Juste 40 ? Ah oui ? Et les petits coquillages qui étaient tout seuls, qui rentraient pas dans des boîtes, on les compte plus ?*

Dustin : Non.

When asked to identify the total number of seashells, that is, the number of basic units, Dustin only considered the 40 seashells used to create the four groups representing the four boxes (i.e., composite units). The seven cubes, left on the side of the table because they were not part of a complete composite unit, were ignored. Dustin appeared to focus only on the elements actively manipulated during the creation of the composite units, rather than on the total number of basic units represented by the manipulatives. This error might be explained by a difficulty managing the cognitive load inherent in the task. More specifically, Dustin initially focused on identifying the composite units and was then asked to shift his attention to consider the basic units. His tendency to focus solely on the basic units embedded within the previously identified composite unit may indicate difficulty in maintaining the total quantity in working memory. It is possible that the most recently processed information, the composite units, was the most readily accessible in his working memory. Another explanation could be that Dustin experienced difficulty switching between two counting strategies, which may stem from a deficit in cognitive flexibility (e.g., Bakker et al., 2022). These hypotheses should be examined in future research.

I observed the same type of error when children solved *two-level problems*, demonstrated by accounting only for the cubes embedded in the last composite units created (i.e., hundreds). Consider the following child, Riley, who was asked to solve *MD (462, 10, 10): fish, nets, crates* problem.

Transcript 15

R : *Et maintenant, est-ce que tu peux me dire combien de poissons Manon elle a en tout ?*

Riley : *400.*

R : *Juste 400 ?*

Riley : *Peut-être qu'il y en a plus ?*

In this transcript, Riley clearly focused only on the ones embedded in the four hundreds, the units that she had just identified when asked about the total number of crates.

The second type of error observed when counting basic units, in both *one-level* and *two-level problems*, consisted of considering only the manipulatives that were not part of any composite unit. Instead of counting all the basic units used to represent the total quantity, some children focused exclusively on the left-over ones that remained separate from the composite units (e.g., tens and hundreds). This error may reflect an overreliance on children's principle-of-position knowledge, where only the digit in the position of the target unit is considered. For

example, Jasper was asked to solve the *MD (24, 10): apples, baskets* problem. Using base-ten blocks, Jasper was able to create and count the composite units.

Transcript 16

R : *Elle a combien de pommes en tout Manon ?*

Jasper : 4.

R : *Elle en a 4 ?*

Jasper : *Dans le panier ?*

R : *En tout, elle a combien de pommes ?*

Jasper : 24.

When subsequently asked to identify the total number of apples, Jasper initially counted only the four individual cubes left aside that were not used to create the composite units. One explanation could be that Jasper presented difficulty in switching his view of the two sticks as 20 basic units, and therefore focused only on the cubes left aside. The use of sticks to represent composite units may have, in his case, obscured the basic units embedded within them. Perhaps the use of unattachable cubes or individual realistic objects might have supported Jasper to see the individual items also contained within the groups. After I emphasized “in all” in my question, Jasper spontaneously changed his response and included the 20 apples embedded in the composite units. This accurate switch in response after I emphasized the total could also be explained by a misunderstanding of the difference between a question that required to use principle of position (e.g., what is the digit in the tens position in 345, which is four) and a question that required children to reason relationally between levels of units (e.g., what is the total number of tens in 345, which is 34).

Finally, a last type of error involved referring back to the number of the last composite units created, which only occurred in the *two-level problems*. When solving the *MD (327, 10, 10): candies, bags, boxes* problem, Jasper created the initial quantity using three flats, two sticks, and seven cubes. After identifying the total number of the first composite units (i.e., 32 bags), and the total number of the second composite units (i.e., three boxes), I asked Jasper to identify the total number of basic units (i.e., candies).

Transcript 17

R : *Combien il a de bonbons en tout Léo ? Il a combien de bonbons en tout ?*

Jasper : *heu 32.*

R : *32 bonbons ?*

Jasper : 2 ?

Jasper remained focused on his previous answer regarding the total number of bags (i.e., 32), demonstrating a misunderstanding of what was being counted and what each manipulative was meant to represent. This error might also be explained by the increased cognitive load associated when solving *two-level problems*. More specifically, such problems required children to navigate between three different types of referents, which may have increased cognitive demands, leading to reduced performance.

Students' Prior Knowledge of Base-Ten Numeration

When solving measurement division problems, some children, especially those using base-ten blocks, relied their prior knowledge about base-ten numeration, and more specifically on their knowledge of the names of the numeration units, to label either the manipulatives or the referents in the problem. For instance, children sometimes referred to “bags” as “tens,” or described a stick as a “ten” rather than using the referent from the problem context. Previous studies have shown that when unitizing in base ten, children tend to activate their already-learned principle-of-position knowledge and interpret numerals using this prior knowledge (Barilaro, 2026a; Tempier, 2016). In this study, however, mentioning the numeration units appeared to be linked to the type of manipulatives provided. Specifically, children using base-ten blocks (i.e., stackable cubes, or stackable cubes and sticks) were more likely to use numeration unit terms to refer to the manipulatives, or to the referent in the problem, than those who used realistic manipulatives. For the *two-level problems*, children only received base-ten materials to solve the problem, which did not allow me to draw any conclusions on the influence of the types of manipulatives on their tendency to refer back to the numeration units.

For the *one-level problems*, Harper's case illustrates a specific challenge linked to the use of base-ten blocks and her knowledge of the manipulatives meaning (e.g., a stick refers to a ten). When solving the *MD (30, 10): candies, bags* problem, Harper used three sticks to represent the initial quantity (i.e., 30 candies). When asked to count the total number of composite units (i.e., bags), she correctly answered three. However, when asked to justify her response, Harper was unable to provide a clear explanation. I therefore attempted to ensure that she understood what each manipulative represented.

Transcript 18

R : *Tout ça c'est quoi ?* (I am pointing at one stick)

Harper : *une dizaine.*

R : *Okay et c'est quoi dans notre histoire ?*

Harper : *ça c'est une dizaine, puis on sait que c'est 10*
 R : *C'est 10 quoi ?*
 Harper : *10 unités collées ensemble.*
 R : *Oui c'est 10 unités, mais dans notre histoire c'est 10 quoi ?*

Harper identified the sticks as tens and the cubes as ones rather than as the referents mentioned in the problem (i.e., bags and candies, respectively). The tendency to interpret the manipulatives through the lens of numeration units likely reflects children's prior experiences with base-ten blocks, where sticks are typically labeled as tens and cubes as ones. In this particular case, Harper's prior knowledge of the numeration units represented by the manipulatives might have masked the connection between the manipulatives and the actual context of the word problem, thereby limiting her understanding of the relations between the physical objects and the problem context.

On the contrary, when provided with realistic objects to solve *one-level problems*, children often never mentioned the numeration units and based their justifications on the objects and the groups created. For example, Susie was provided with seashell counters to solve *MD (47, 10): seashells, boxes* problem. She first created four groups of 10 counters on the table. When asked about the number of the composite units (i.e., nets), Susie justified her answer using the counters.

Transcript 19

R : *Alors, il y a combien de boîtes complètement remplies ?*
 Susie : *Quatre.*
 R : *Comment tu sais ? Tu m'expliques ?*
 Susie : *Là y'en a quatre ici* (Susie gestured over the four groups created on the table) *Y'a 10, 10, 10, 10, égale 40* (Susie pointed at each group created on the table).

In this excerpt, Susie did not explicitly refer to any numeration unit. She justified her answer by pointing to at the groups of ten (i.e., the groups of seashell counters) on the table, without mentioning any numeration units.

Because children were only provided with base-ten blocks for the *two-level problems*, I was not able to determine whether the type of manipulative influenced the extent to which children relied on their prior knowledge of base-ten numeration. Nevertheless, as I observed when solving the *one-level problems*, children used terminology related to numeration units and justified their responses using the same terminology. Sadie's reasoning, for example, shows how her previous knowledge of the numeration units represented by each base-ten block might have

enabled her to make sense of the problem and accurately interpret the meaning of each manipulative. In the *MD (327, 10, 10): candies, bags, boxes* problem, Sadie constructed the initial quantity using three flats, two sticks, and seven cubes. I then asked her to identify the total number of the first composite units (i.e., the bags).

Transcript 20

Sadie : *ça fait 32*

R : *32 quoi ?*

Sadie : *32 sachets*

R : *Comment tu sais ?*

Sadie : *parce que ça c'est... vu que 10 dizaines c'est 100, puis 20 dizaines c'est 200, puis 30 dizaines c'est 300, puis ça c'est c'est les dizaines puis y'en a deux fait que 32.*

In this transcript, Sadie demonstrated an understanding of the relations between tens and hundreds. More specifically, she understood that when asked about the number of bags, she was actually being asked about the total number of tens. Her justification demonstrated that she could compose a hundred from 10 tens, which allows her to understand that a flat, identified by her as one hundred, is also composed of 10 tens. She also demonstrated an accurate switch between counting the tens in the hundreds by 10 (i.e., 10 tens, 20 tens, 30 tens makes 300) to counting the leftover tens by one (i.e., one ten, two tens).

Although prior knowledge of base-ten numeration can sometimes support children's reasoning when identifying composite units, it can also hinder their understanding of what is being counted. Max was asked to solve *MD (240, 10, 10): candies, bags, boxes* problem. In the following excerpt, Max tried to represent the initial quantity while making sense of the manipulatives used.

Transcript 21

R : *Pour l'instant on met de côté les boîtes et on va s'intéresser uniquement aux sachets. Et donc on se souvient, un sachet contient 10 bonbons. Et en tout, on a 240 bonbons.*

Max : *Fait que ça veut dire qu'il y aura deux centaines.*

R : *Peut-être.*

Max : *Y'en aura 2* (Max took two flats from a pile of flats)

R : *Est-ce que tu as besoin d'aide ?*

Max : *Ah voilà, y'en a deux centaines. Puis y'a quatre unités, fait que on enlève* (Max removed three cubes because she had seven cubes in front of her). *On en enlève seulement y'a quatre unités.*

R : *Y'a quatre unités ?*

Max : *Y'a quatre unités, puis y'a zéro... Ben y'a 24 dizaines puis y'a...*

R : *Y'a 24 dizaines ?*

Max : *Ben oui, puis y'a deux unités.*

R : *Alors ça c'est quoi ?* (I pointed at the initial quantity 240 written in the problem) 240 ?

Max : *oui mais y'a deux dizaines.*

R : *Deux centaines tu veux dire ?*

Max : *Oui et 24 dizaines, fait que ça veut dire qu'il y a pas de ça* (Max is removing the four cubes left in front of her), *parce qu'y en a zéro.*

Max appeared to struggle with the use of the numeration unit terms when working with the manipulatives. First, she confused ones and tens, initially attempting to represent the four tens using four cubes instead of sticks. Although she later correctly identified that there were 24 tens in the initial quantity, she seemed unable to extract meaningful information from this response. Additionally, she referred to the two hundreds inconsistently, calling them either ones or tens, revealing a limited understanding of the numeration units, the manipulatives, or the connection between the numeration units and the corresponding manipulatives. Her apparent confusion impeded her ability to make the appropriate connections between the units of reference in the problem and the manipulatives. In Max's case, using numeration unit terminology seemed to further confuse her, possibly explaining the errors she generated when using the manipulatives.

Discussion

The present study aimed to explore second-grade students' unitizing performance when solving measurement division problems. Additionally, the study examined how the type of manipulative provided to solve the problems influenced children's performance: I explored the role of the realism of the manipulatives, the perceptual salience of the units in the manipulatives, and whether children constructed the composite units themselves. Second-grade students solved measurement division problems embedded in base-ten contexts that required that either one (i.e., *one-level problems*) or two (i.e., *two-level problems*) composite units be identified. For the *one-level problems*, students were provided with three types of manipulatives: realistic objects, stackable cubes, and base-ten blocks. For the *two-level problems*, students were only provided with base-ten blocks. Children's strategies, verbal justifications, and answers were collected and analyzed qualitatively using a thematic analysis (Terry et al., 2017). Given that little is known about children's performance when unitizing quantities in base-ten contexts, the study was entirely exploratory and aimed to better understand the nature of children's thinking about units.

The observations in this study may help formulate specific hypotheses about children's thinking or about the impact of manipulatives in unitizing performance, hypotheses that can be tested in future experimental research. Ultimately, the primary objective was not to draw conclusions about the general population of second-grade students, but rather to inform the design of an instructional intervention designed to support children's unitizing in base-ten contexts.

Drawing from both the current literature on unitizing outside of base ten and patterns that emerged from the data, I identified three major themes. The first theme focused on patterns related to children's construction and counting of composite units. The second theme addressed their ability to shift the unit of reference when asked to enumerate basic units. The third theme explored the influence of children's prior knowledge about base-ten numeration and about base-ten blocks meaning on their unitizing performance. In the next sections, I discuss these themes in relation to children's unitizing performance, with particular attention to the role played by the manipulatives' features.

Children's Counting of Basic and Composite Units

When solving measurement division problems embedded in base-ten contexts, children were required to identify the total number of both composite units and basic units using various types of manipulatives. Finding both the number of composite units and basic units involved counting the manipulatives differently according to the unit of reference targeted, processes that in this study, correspond to unitizing. The findings revealed that, regardless of the manipulatives used, second-grade students could unitize quantities when these solving problems, either spontaneously or with support and guidance. This result contrasts with prior research indicating that younger children between the ages of 4 and 7 years tend to struggle to count composite units and often focus only on basic units (Shipley & Shepperson, 1990; Sophian & Kailihiwa, 1998; Wege et al., 2023). This discrepancy might be explained by population and task differences. First, the current study sampled older children, who may have had more prior exposure to the relations between units. Second, in the present study, unitizing was embedded within measurement division problems that provided a familiar and meaningful context, potentially supporting children's identification of both composite and basic units.

Previous research has shown that grounded and familiar non-symbolic contexts facilitate children's mathematical performance. For instance, Carpenter et al. (1981) and Sherman and Bisanz (2009) found that children displayed higher performance on non-symbolic problems (e.g.,

using grounded situations or manipulatives) compared to purely symbolic tasks (e.g., equations). Barilaro (2026a) found that the most frequent error produced by children when completing unitizing tasks that involved numerals was focusing on the digit in the targeted numeration unit position (e.g., four tens in 345), reflecting an overreliance on the principle of position, a concept heavily emphasized in classroom instruction (Houdement & Tempier, 2019). In contrast, the responses of the children in the present study accounted for the relations between different levels of units (e.g., finding the 34 tens in 345) to a greater extent than previous unitizing research might have predicted. This discrepancy between non-symbolic and symbolic tasks might reflect, as suggested by Carpenter et al. (1981), the influence of instructional practices rather than an inherent limitation in children's thinking. The results of the current study are promising because they suggest that using measurement division problems with manipulatives might better support children's understanding of base-ten structure than tasks where numerals are involved (Carpenter et al., 1996). Providing meaningful situations and manipulatives that resemble their real-world referents may help children engage in relational reasoning between numeration units in ways that are less constrained by their prior knowledge of symbolic tasks.

Impact of Manipulative Type

For *one-level problems*, children received realistic manipulatives, stackable cubes, and base-ten blocks (i.e., stackable cubes and pre-composed sticks). For the *two-level problems*, only base-ten blocks were provided. Children were also prompted to justify their answers using the manipulatives. Three main patterns emerged from the data regarding the possible impact of manipulative type. The patterns are discussed in turn below.

Perceptual Salience of the Composite Units

Children's ability to identify the total number of composite units using manipulatives appeared to be strongly influenced by the perceptual salience of the composite units. For a group of basic units to be perceived as a single composite unit, it must either carry inherent meaning (e.g., being identified as a countable singular object, such as a "net"; Brooks et al., 2011) or be perceived from the objects' visual characteristics as a continuous whole (e.g., Lafay et al., 2023). When using realistic manipulatives (e.g., fish counters), children created the composite units by grouping 10 items together on the table. As revealed by the data, the use of realistic manipulatives often impeded the identification of the total number of composite units (e.g., the number of nets) without explicit support. This finding aligns with Wege et al. (2023), who

showed that although children were able to group and name realistic manipulatives (e.g., animal counters) to form composite units, they struggled to count these groups as single entities. Previous research has suggested that realistic manipulatives carry strong semantic meaning, which may distract children from the mathematical goal of the task (McNeil et al., 2009). In the context of unitizing, when the basic units (but not the composite units) are represented realistically, attention might be drawn to the individual items and their everyday meaning (McNeil et al., 2009) rather than to the groups they form. An illustrative example from the current data highlights the importance of the visibility of the composite unit: One child, when provided with actual bags (closely matching the referents mentioned in the problem), was able to identify the total number of bags more easily than when only grouping the manipulatives on the table. This suggests that manipulatives that made the composite units perceptually salient may facilitate unitizing.

In contrast, the use of base-ten blocks, particularly pre-composed sticks and flats, appeared to more effectively support the perception of the “one-ness” of composite units. These manipulatives offer continuous representations of each unit (i.e., a stick represents better resembles a single higher-level unit than a loose group of ten cubes), which may help children conceptualize them as singular composite units rather than as a collection of individual basic units. This observation aligns with prior research (Barilaro et al., 2026; Lafay et al., 2023) that highlights the importance of the perceptual salience of composite units in fostering children’s ability to recognize and count composite units. These findings therefore suggest that using continuous representations of composite units may play a role in facilitating children’s unitizing performance by making the group perceptually salient as one single entity.

Prior Experience with Manipulatives

Using base-ten blocks to solve *one-level* and *two-level problems* appears to have prompted children to activate their prior knowledge of the manipulatives, and more particularly, the correspondences between the blocks and specific numeration units. Specifically, children called the cubes “ones,” the sticks “tens,” and the flats “hundreds.” The association between each block and one numeration unit could be due to an extensive use of base-ten blocks in schools to teach base-ten numeration (Moyer-Packenham et al., 2013) and the use of denomination-specific vocabulary. As shown in previous studies, prior knowledge of base-ten numeration—particularly the principle of position, which is often the focus of the classroom teaching—can negatively

influence children's interpretation of numerals (Barilaro, 2026a; Tempier, 2016). In the same way, using base-ten blocks prompted children to think about each block as representing one single numeration unit, resulting in difficulties for children to view the nested system of units. Several children focused solely on the denomination typically assigned to each manipulative (e.g., a flat is a hundred), without considering other possible interpretations of that same manipulative (e.g., a flat can also be seen as 10 tens or 100 ones). This rigid view of the quantities associated with specific manipulatives might prevent children from considering the tens embedded in the hundreds, for example, thereby masking the hierarchical structure of base-ten units. Nevertheless, because children already possess partial knowledge of how each block represents a numeration unit, explicit instruction on the various possible meanings conveyed by the manipulatives might help children recognize the structural similarities between measurement division problems solved using the blocks and the hierarchical structure of base ten.

Composition of Composite Units

The pre-composed sticks and pre-composed flats that were provided to children when solving *two-level problems* appeared to constrain the construction and identification of composite units. Children began their problem solving by representing the initial quantity using cubes, sticks, and flats (e.g., 327 candies were represented using three flats, two sticks, and seven cubes). To identify the total number of the first composite unit (e.g., tens), children had to consider both the visible sticks (e.g., two sticks) as well as those embedded within the flats (e.g., 30 sticks). However, when asked to identify the total number of the second composite units (e.g., hundreds), children had to shift their counting unit and interpret the flats not as groups of 10 sticks, but as a singular higher-level unit (e.g., three flats). In this context, a flat simultaneously represents 300 basic units (i.e., ones), 30 first composite units (i.e., tens), and three second composite units (i.e., hundreds). Children were therefore required to know the quantity represented by each object (e.g., cube, stick, and flat) depending on the referent unit used. Learning what mathematical concept is conveyed by a manipulative can be particularly challenging for children because of the objects' dual representation (Uttal et al., 1997). A manipulative can be seen as both an object on its own (e.g., a cube can be seen as just a cube) but also as a symbol of an abstract concept (e.g., a cube represents a one). In the current study, the dual representation might be even more challenging because a single object, here specifically the flat, conveys multiple mathematical meanings, depending on the reference unit used (e.g., 300

ones, 30 tens, and three hundreds, or even 15 tens and 150 ones). Understanding and manipulating those multiple meanings might increase cognitive demands on children, thereby hindering their ability to reason fluently about manipulatives.

The apparent negative impact of the pre-composed manipulatives on children's unitizing performance was evidenced by two types of errors. First, when identifying the total number of the first composite unit (i.e., tens), some children ignored the tens embedded within the flats and counted only the separate sticks. This mirrors the error previously reported in Barilaro (2026a), where children interpreted numerals by focusing exclusively on the digit in the tens place, a reflection of their principle-of-position knowledge. In this case, the pre-composed flats may have made the higher-level unit salient but may have obscured the lower-level units embedded within them. The second error observed entailed inaccurate counting when switching between sticks and flats. Consider the following example from the data, where a child was required to identify the total number of the first composite in the quantity 327 represented by three flats, two sticks, and seven cubes. When asked to identify the number of first composite units (i.e., the tens), one child correctly interpreted the three flats as 30 tens, but then erroneously counted the two sticks as 20 tens instead of two tens, resulting in an answer of 50 tens in total. This error again mirrors what was observed in Barilaro (2026a) when I asked children to unitize numerals. It suggests difficulty in using the correct unit when interpreting the flats and sticks. To successfully switch their unit when counting the manipulatives, children need to keep track of the embedded units, of the meanings of the manipulatives, and of the cumulative total. The number of processes required to complete the task might overload working memory, leading children to generate errors in switching their units when counting the manipulatives.

In the context of such errors, however, when children were given the opportunity to physically construct the second composite units themselves, either by using only sticks when creating the initial quantity or by exchanging each flat taken to create the initial quantity by 10 sticks, their counting of both the first and the second composite units was enhanced. First, to identify the first composite units, children now relied solely on sticks, which may have reduced the cognitive load associated with switching the unit of reference between flats and sticks. During the construction process and perhaps as a result of it, the perceptual salience of the sticks was increased, thereby reducing the dual representation of the manipulatives.

Second, to identify the second composite units, children were required to group 10 sticks together to construct each higher-level unit. As shown in previous studies (Castro-Alfonso et al., 2024; Manches & O'Malley, 2016), physically engaging with manipulatives supports children's learning of the concepts the manipulatives are intended to convey. In this case, constructing the second composite units may have strengthened children's understanding of the relation between tens (i.e., sticks) and hundreds (i.e., groups of 10 sticks). This finding also aligns with Osana et al. (2026) who showed that children who created the numeration units themselves learned more about place value than those who used pre-composed manipulatives. In the context of unitizing, physically building composite units appears to support children's understanding of the hierarchical structure of base-ten units and enhances their ability to identify both basic and composite units, together improving unitizing performance.

Contributions

This study makes several important contributions to the current literature. The first contribution is that second-grade students can unitize quantities in base-ten when solving measurement division problems. This finding deviates from prior research showing that younger children often struggle to identify composite units when working with manipulatives (Shipley & Shepperson, 1990; Sophian & Kailihiwa, 1998; Wege et al., 2023). In contrast, the present results showed that under certain conditions, second graders can recognize groups of single items as one composite unit while at the same time accounting for the individual elements within each group. Furthermore, by showing that children can successfully unitize in base-ten contexts when solving measurement division problems, the present study provides empirical support for Carpenter et al.'s (1996) claim that such problems can foster understanding of the relations between numeration units. These findings highlight the potential benefits of using measurement division word problems to support children's understanding of the relational structure of base ten.

Strengths and Limitations

One strength of the present study lies in the use of the thematic analysis, which enabled a detailed examination of the strategies, reasoning processes, and justifications produced by the children when unitizing. This analytic approach allowed for conclusions beyond whether children succeeded or failed; it revealed how children reasoned about units and their relations. Second, the use of different types of manipulatives provided an empirical basis upon which to

test specific hypotheses in future research on the role of manipulatives in children's unitizing performance. By offering a range of manipulatives, I was able to document possible affordances of specific features on the construction and identification of composite units. Finally, the guidance and support I provided throughout the session represented an additional strength of the study. This semi-structured, interactive approach made it possible for me to probe children's reasoning, clarify their interpretations of tasks and manipulatives, and identify which elements of the tasks supported their understanding and which were more challenging. Such insights would have been more difficult to obtain using a more structured protocol.

Several limitations of the present study must nevertheless be acknowledged. First, the qualitative nature of the present analysis and the relatively small sample limit any generalization drawn from this research. Second, an experimental design would likely have produced more robust findings, especially regarding the influence of different types of manipulatives on children's unitizing. Additionally, for the *two-level problems*, children used base-ten blocks exclusively, limiting the conclusions that can be drawn on the affordances of different manipulative types on children's unitizing performance. The *two-level problems* all involved three-digit quantities, meaning that in this study, the children were required to represent quantities above 200. Because representing quantities in this range using individual counters can be challenging for students, only base-ten blocks were provided to facilitate the construction of these quantities. A consideration for future research would be to explore alternative ways of representing large quantities to better understand how various types of manipulatives may support or hinder children's unitizing performance.

Educational Implications

The present findings uncover several important pedagogical implications. First, the study highlights the potential of using measurement division problems embedded in base-ten contexts to support instruction on base-ten structure. These problems appeared to provide meaningful opportunities for the participants to engage with the structure of base ten in grounded contexts, possibly reducing barriers when subsequently unitizing in symbolic contexts. Working with measurement division problems might be more effective in focusing students' attention to the relations between numeration units than symbolic representations. Second, the present results suggest that physically constructing composite units using manipulatives supports children's identification of these units more effectively than when working with pre-composed composite

units. The act of creating composite units may help children understand the relations between units by making the embedding of basic units within composite units explicit. It may also facilitate an awareness of the proportionality between the two levels of units (e.g., 10 basic units together from a new composite unit that is ten times larger as one basic unit). Finally, the type of manipulatives provided when solving measurement division problems appears to play a role in children's unitizing performance. In particular, realistic objects used to represent basic units may obscure the relations between the measurement division structure and the base-ten system. On the other hand, base-ten blocks may be especially supportive in other ways. First, the one-ness of the sticks and flats may facilitate the identification of composite units. Second, because base-ten blocks are often familiar to students, appropriate instruction may help them more readily connect the structure of measurement division problems to the structure of base-ten numeration.

Connecting Studies 1 and 2 to Study 3

Study 2 explored second-grade students' unitizing performance when solving measurement division problems using various types of manipulatives. Children solved measurement division problems embedded in base-ten contexts using either realistic objects, stackable cubes, or base-ten blocks (that contained stackable cubes, pre-composed sticks, and pre-composed flats). Students' strategies, justifications, and answers were analyzed qualitatively through a thematic analysis. Several findings from Study 2 study have the potential to inform both instructional practices and future research. First, measurement division problems with manipulatives made it possible to observe successful unitizing in second-grade students, confirming Carpenter's (1996) recommendations for using such tasks in instruction on base-ten numeration. The study also highlighted the role of manipulatives in shaping students' unitizing performance. Manipulatives that make the one-ness of the composite units salient seemed to support children's counting those units as if they were ones. For example, sticks, composed either from stackable cubes or already pre-composed, were easily identified as one thing by the students. Conversely, realistic manipulatives appeared to draw children's attention toward individual entities, rather than the composite units, presumably because they were perceptually salient. Furthermore, the results suggested that students improved their counting of the composite units when they physically constructed them using manipulatives. Finally, the use of base-ten blocks, a familiar classroom manipulative, prompted children to activate their prior knowledge of how these blocks represent different numeration units. Although the activation of their prior knowledge of the blocks sometimes interfered with their ability to link the manipulatives to the problem context (as the problems are not referring to the numeration units), in other cases, it helped them to connect the structure of the problem to that of base-ten numeration. In sum, the findings from Study 2 offer valuable insights for designing instructional interventions aimed at supporting unitizing involving symbolic representations of quantity.

Additionally, Study 1 revealed that second-grade students struggled to unitize numerals. They often relied on their principle-of-position knowledge, which impeded their relational reasoning about base ten units. Thus, in light of Study 1 and Study 2, and following Carpenter et al.'s (1996) suggestion on the use of measurement division word problems, Study 3 was designed to leverage children's reasoning about measurement division contexts to support their understanding of base-ten structure.

Recent research in mathematics education has emphasized the effectiveness of analogical reasoning in supporting children's mathematics learning (Gray & Holyoak, 2021; Richland et al., 2012). Using analogies in mathematics classrooms involves a source analog that is familiar to the students and has the same underlying structure as a target analog. In instruction based on analogical reasoning, children learn by abstracting the underlying conceptual structure between the two analogs (Gentner, 1983). Measurement division problems in base-ten contexts, and more precisely, the units involved such problems, share the same hierarchical structure as the system of numeration units in base ten. Therefore, in Study 3, measurement division problems were used as source analogs to support students' understanding of numerals, which were the target analogs.

Additionally, the findings of Study 2 directly informed the design of the instructional intervention in Study 3. First, students were provided with base-ten blocks to make the one-ness of composite units more salient. Study 2 showed that because children already knew the quantitative referents of base-ten blocks, using them might have facilitated the connection between the mathematical structure described in the problem and the structural relations between numeration units. In addition, stackable cubes were used for *one-level problems*, and stackable cubes and pre-composed sticks for *two-level problems*. The choice of these two specific sets of manipulatives for the *one-level* and *two-level problems* respectively was informed by Study 2, which showed that students more accurately identified and counted composite units when they physically constructed them than when they used pre-composed manipulatives. In *one-level problems*, using only stackable cubes enable the construction of the tens, while in *two-level problems*, providing cubes and pre-composed sticks allowed students to build hundreds. Study 3, therefore, tested the effectiveness of an analogy-based instructional intervention that used measurement division problems as the source analog to support children's performance when unitizing numerals.

Chapter 4: Study Three

Supporting Second Graders' Unitizing in Base-Ten Numeration: A Comparison of Explicit and Implicit Analogy-Based Instructions

Introduction

Mathematics is essential not only for everyday activities, such as cooking, telling time, or managing personal finances, but also for fostering informed and engaged citizenship (Maass et al., 2022). However, mathematics is often perceived as a difficult and intimidating subject in school, and many students also view it as irrelevant because they struggle to connect mathematical concepts to real-life situations (Boaler, 1993; Hiebert, 1984; Perlmutter et al., 1997). One possible explanation for this difficulty lies in the way mathematical concepts are often taught, as isolated procedures or facts, without explicitly supporting students to reason relationally across concepts. Relational reasoning, which involves identifying and applying relations between concepts and ideas, is essential for deep mathematical understanding (Miller Singley & Bunge, 2014; Richland et al., 2012), and the National Council of Teachers of Mathematics (2019) recognizes the importance of grounding mathematics learning in students' experience and prior knowledge.

In the context of base-ten numeration, teachers in the early years often tend to place extensive emphasis on the so-called principle of position (Houdement & Tempier, 2019; Tempier, 2024). The principle of position refers to the underlying syntax of a numeral, which enables the representation of any quantity using Hindu-Arabic symbols (Cheung & Ansari, 2021). According to the principle, the value of each digit is determined by its position in the numeral: The rightmost digit represents the numeration unit of ones (i.e., 10^0), the next digit to the left represents the numeration unit of tens (i.e., 10^1), the third digit represents the numeration unit of hundreds (i.e., 10^2), and so on (Cheung & Ansari, 2021; Mix et al., 2019; Ross, 1989). When teachers focus solely on the principle of position, however, the relational structure of base-ten numeration as a system of embedded units tends to get overlooked (Fuson, 1990; Houdement & Tempier, 2019; Tempier, 2024).

In line with recommendations of the National Council of Teachers of Mathematics (2019), instruction that emphasizes relations between mathematical concepts and builds on students' prior knowledge may offer a promising educational approach to foster relational reasoning. Analogies may serve as an effective instructional tool because analogical reasoning

constitutes a specific form of relational reasoning. Although researchers have shown that analogies are frequently used by teachers in their lessons (Richland et al., 2004a), the way teachers use analogies in their classrooms varies considerably. Specifically, teachers can provide various degrees of support when mapping the conceptual similarities between two analogs, degrees that can then impact the effectiveness of the analogies.

Accordingly, the present study aims to investigate the effectiveness of analogies that vary in the degree of instructional guidance for improving children's understanding of the relational structure of base-ten numeration. One way to assess structural knowledge of base ten is through unitizing, which involves decomposing or composing a numeral using more than one numeration unit (Lamon, 1996; Simon, 2024). As shown in previous studies, many students persistently struggle when it comes to unitizing quantities in base-ten contexts, which highlights the need for instructional interventions to support children's learning (Barilaro, 2026a; Barilaro et al., 2026; Soury-Lavergne et al., 2020; Tempier, 2016).

Theoretical Background: Analogical Reasoning in Instruction

Analogical reasoning is one of the most powerful cognitive processes that is activated not only during learning, but also in various aspects of daily life, including debate, politics, and creative thinking (Holyoak, 2013; Holyoak & Thagard, 1997; Smith & Ward, 2013). Using analogies, individuals can make sense of new situations (i.e., the target) by comparing them to a well-known situation (i.e., the source; Gentner, 1983). Analogical reasoning is only possible when the source and the target, though differing in their superficial elements, share a similar conceptual structure (Gentner & Toupin, 1986). When reasoning by analogy, individuals must engage in a comparison process between the source and the target. First, they must identify the conceptual similarities between the source and the target, while at the same time inhibit any superficial features that may differ (Gentner, 1983). Second, individuals are required to map the source structure to the target, allowing them to make sense of the target (Gentner, 1983). As demonstrated in previous studies, a strong knowledge of the source analog is essential for successful analogical reasoning to occur (Gentner, 1983).

Children tend to experience difficulties with relational reasoning, particularly when connecting mathematical concepts to each other (Hiebert, 1984; Osana & Pitsolantis, 2013) or linking mathematics to everyday life (Baki et al., 2009; Perlmutter et al., 1997). Additionally, various studies showed that children, and novices more generally, rarely notice the structural

similarities between two analogs spontaneously, or do so less frequently than adults or experts (Goldwater et al., 2021; Novick, 1988; Richland et al., 2004a). Such observed difficulties may be attributed to a substantial cognitive load imposed when reasoning by analogy, found to be heavily reliant on executive functions, such as working memory and inhibition (Morrison et al., 2001; Richland et al., 2004a; Thibaut et al., 2010). When identifying the conceptual similarities between the source and the target, children need to maintain all the attributes and functions of both the source and the target in working memory. At the same time, individuals must inhibit the irrelevant superficial features during comparison to focus only on the conceptual similarities. Thus, depending on the features of the source and target, the cognitive load imposed by analogical reasoning may increase and interfere with the reasoning process.

In mathematics, evidence demonstrates that teachers routinely use analogies as pedagogical tools to support children's relational reasoning in the classroom (Richland et al., 2004a). Using analogies in mathematics instruction has been shown to be effective because analogies help children connect prior mathematical knowledge to new concepts, especially when the source analog shares a similar structure with the target concept. For example, Sidney and Alibali (2015) compared two different types of source analogs to learn about fraction division—whole number division and fraction addition. Whole number division shares the same conceptual structure as does fraction division. Fraction addition, on the other hand, takes on an additive structure, and only shares superficial features with fraction division (i.e., the types of numbers in the problems). Sidney and Alibali found that fifth- and six-grade students gained more conceptual understanding of fraction division when they were provided with a whole number division analog than when they were provided with fraction addition, a superficial analog.

However, for the analogy to be effective, teachers must minimize the cognitive demands that are inherent in analogical reasoning. Research has explored the types of support that can reduce such cognitive demands, resulting in two ways that the load can be alleviated. Some authors argue that providing explicit cognitive supports helps direct children's attention to the relevant structural similarities between the source and the target of the analogy (Gray & Holyoak, 2021; Richland et al., 2007; Vendetti et al., 2015). By doing so, the cognitive load associated with identifying and process the shared structure between the two analogs is reduced. Yet other researchers suggest that activating relevant prior knowledge before introducing the target concept is sufficient to support children's reasoning because it serves to reduce the amount

of information they need to process (Day & Goldstone, 2012; Sidney & Thompson, 2019). In the literature, these two approaches are commonly referred to as explicit and implicit analogies, respectively.

Explicit Analogies: Directing Children's Attention

Explicit analogies are those used by teachers who guide students to compare analogs and to pay attention to the relevant structural relation between the source and the target. Providing explicit analogies may be effective because they make the relational structure between the source and the target salient, allowing learners to make sense of the target concepts. In mathematics instruction, several studies revealed that using explicit analogies increased children's identification of the similarities and differences between two analogs (Richland & Simms, 2015), reduced their misconceptions about the target mathematical concepts (Begolli & Richland, 2016), improved learning and transfer (Rittle-Johnson & Star, 2007), and supported relational reasoning (Gray & Holyoak, 2021).

Teachers can support children's analogical reasoning by explicitly directing their attention to the common structure between two analogs using various cognitive supports (Gray & Holyoak, 2021; Richland et al., 2007; Vendetti et al., 2015). When teachers are using explicit analogies, the cognitive supports provided are used to highlight explicitly the structural similarities between the two analogs. First, research has shown that presenting the source and the target simultaneously (Christie & Gentner, 2010; Rittle-Johnson & Star, 2007) and aligning them spatially (Matlen et al., 2020) facilitate the identification of their shared structure. Second, teachers can use gestures to guide children's focus toward relevant features and to scaffold the application of the source structure to the target (Richland, 2015). Third, the use of relational language (e.g., same as, like, similar to) during comparisons has been found to enhance children's perception of structural commonalities (Gentner et al., 2009; Gentner, 2016; Gick & Holyoak, 1983; Loewenstein & Gentner, 2005). Finally, color has been found to direct children's attention to the relevant information and, by extension, to promote learning (Alibali et al., 2018). In sum, spatial alignment, gesture, relational language, and color are four types of cognitive support that can direct children's attention to the shared conceptual structure across the two analogs.

Implicit Analogies: Activating Relevant Prior Knowledge

Other authors (Sidney, 2020; Sidney & Thompson, 2019) have argued that using explicit supports can be overwhelming for students and can derail them from using effective reasoning strategies. More specifically, the large amount of information provided to draw children's attention to conceptual similarities may increase the cognitive load associated with analogical reasoning. Indeed, children must simultaneously process the comparison of both the source and the target while at the same time integrating all the supports and inhibiting the perceptual differences between the two analogs (Richland et al., 2004b). These authors thus advocated for the use of implicit analogies through the activation of relevant prior knowledge (i.e., knowledge and familiarity with the source).

In analogical reasoning, the source is a key component (Gentner, 1988), considered as prior knowledge. Teachers can activate children's knowledge of the source through warm-up exercises focused the source before introducing the mathematical intended concept (i.e., target of the analogy). The activation of the relevant prior knowledge of the source would serve to reduce the cognitive demand associated with the direct comparison of both the source and the target. In this context, implicit analogies refer to those in which the relevant prior knowledge (i.e., the source) is activated to support learning about a new concept (i.e., the target) without any direct instructional support such as visuo-spatial cues, relational language, or gestures.

Several studies have demonstrated the effectiveness of using implicit analogies in the teaching of mathematical concepts (Sidney, 2020). In the study described earlier, for example, Sidney and Alibali (2015) found that activating relevant prior knowledge (i.e., whole number division) before instruction on a target concept (i.e., fraction division) resulted in greater learning outcomes when the children were not explicitly prompted to connect the two domains compared to when they were. Similarly, Sidney (2020) showed that children who received a brief reminder about whole number division before a lesson on fraction division demonstrated higher understanding and performance about fraction division when the connection between the two was left implicit compared to children who received explicit supports. Together, these findings suggest that activating relevant prior knowledge can support children in making relevant connections without the need for explicit prompts.

Target Domain: Unitizing in Base-Ten Numeration

In the present study, I investigated the effectiveness of using explicit and implicit analogies in instruction to support children's knowledge of the relational structure of base-ten numeration. Base-ten numeration is based on a system of numeration units (e.g., ones, tens, or hundreds) linked to each other by a power-of-ten relation (e.g., one hundred is 10^2 times as large as one, meaning there are 100 ones in a hundred; Fuson, 1990). Individuals with a strong knowledge of the power-of-ten relation know that a group of 10 ones is equivalent to one ten, and that one hundred is equivalent to 10 tens or to 100 ones.

Knowledge of the base-ten structure can be assessed through unitizing, the process of interpreting a quantity using at least two units of reference simultaneously. Unitizing in the context of base ten refers viewing a quantity as a collection of lower-level units (e.g., 10 ones) as such and also as a single higher-level unit (e.g., one ten; Lamon, 1996). This process enables individuals to either compose quantities, which involves creating a higher-level unit from lower-level units (e.g., creating one hundred from 10 tens), or decompose quantities, which is the reverse process of enumerating lower-level units embedded in higher-level units (e.g., identifying the 10 tens embedded within one hundred; Simon, 2024). For example, the quantity 325 can be interpreted as 32.5 tens or as 3.25 hundreds. When children possess strong unitizing abilities, they can interpret any quantity with respect to a predefined numeration unit.

Overall, there is evidence to suggest that unitizing in base ten is challenging for children. First, Barilaro (2026a) found that when children were asked to decompose a numeral into different numeration units (i.e., total number of tens and total number of ones), the majority of second graders failed to account for the tens embedded in the hundreds, showing weak knowledge of the base-ten structure. Interestingly, a response analysis revealed that the most common error entails to only considering the digit in the numeration unit target position (e.g., answering two tens in total in 427). Tempier (2016) reported similar findings when he asked third graders to identify the number of tens in a three-digit numeral. In another study, Barilaro et al. (2026) presented first and second graders with visual representations of quantities using rectangles and cubes that made embedded units perceptually salient. Despite the visibility of the embedded units, 70% of these students ignored them when asked to find the total number of units in the representation. Finally, Tempier (2016) and Ladel and Kortenkamp (2016) showed that children struggled when unitizing that required composing. They found that children

experienced challenges when identifying the numeral (e.g., 495) that resulted from composing numeration units (e.g., 3 hundreds + 17 tens + 25 ones). Together, these studies highlight the challenges children encounter with the structure of base-ten, and particularly when unitizing, regardless of whether the tasks involved symbolic or visual representations.

Despite observed difficulties when unitizing symbolic quantities in base-ten contexts, children can be successful when provided with concrete representations and contextual cues. Barilaro (2026b) found that children's unitizing performance in base-ten contexts was enhanced when they are presented with manipulatives and word problems grounded in familiar contexts. More specifically, I found that when given manipulatives to solve word problems grounded in familiar situations, second graders succeed in unitizing when decomposing quantities in base-ten contexts. Children solved measurement division problems (e.g., Hugo has 24 candies. He puts all his candies in bags that contain 10 candies. How many bags did Hugo fill?). Measurement division problems require creating composite units of 10 lower-level units and counting those composite units as if they were ones (e.g., creating groups of 10 candies in each and counting the number of groups). Children were required to use different types of manipulatives to solve the problems, such as realistic objects or base-ten blocks (i.e., one, ten and hundred blocks). The results showed that second graders were able to create the required number of groups and successfully count them as single units, especially when they had the opportunity to create the composite units themselves. These results highlight the potential of using measurement division word problems as a source analog when learning how to unitize numerals in base ten. Highlighting the common structure between measurement division and base-ten numeration as a pedagogical tool might support the development of children's unitizing in base-ten contexts.

Present Study

The current study aims to investigate the effectiveness of an instructional intervention involving either explicit or implicit analogies to support children's learning of the base-ten structure. Prior research has shown that the use of analogies can be effective as a pedagogical tool in mathematics instruction (Richland & Simms, 2015). More specifically, children's learning of mathematical concepts is facilitated by drawing an analogy from a familiar source concept that shares the same conceptual structure as the one being taught. Additionally, although children usually struggle to unitize quantities presented in numerals (Barilaro, 2026a), prior research has shown that second graders demonstrated promising performance in unitizing

quantities when presented with grounded measurement division problems (Barilaro, 2026b). Thus, using word problems as the source analogy when learning how to unitize in base ten could be an effective instructional approach.

The following study aimed to provide an instructional intervention based on analogies to support children's learning of how to unitize when decomposing a numeral in the context of base ten. The analogs used in the instruction were a grounded word problem and its solution (i.e., the source), and the numeral used to represent the total quantity (i.e., the target). The link between the source and the target in the instruction was either explicit or implicit, according to the presence (or the absence) of various cognitive supports identified in the literature. Through the use of analogies between grounded word problems and numerals, I hypothesized that children would transfer their knowledge from the measurement division problem structure to base ten.

I randomly assigned second graders to three conditions: explicit, implicit, and control. Participants completed measures at pretest and posttest: One task assessed their principle-of-position knowledge, one learning task assessing unitizing when decomposing a numeral, and one transfer task assessing unitizing when composing a numeral. The study was guided by the following research questions:

1. Does instruction based on analogical reasoning result in improved unitizing abilities as measured by gains in performance on unitizing when decomposing a numeral (learning measure) and by gains in performance on unitizing when composing a numeral (transfer measure)?
2. Does instruction involving explicit analogies lead to greater improvement in unitizing when decomposing a numeral (learning measure) and in unitizing when composing a numeral (transfer measure) than instruction involving implicit analogies?

Based on theoretical analyses of the benefits of using analogies in mathematics instruction (Gray & Holyoak, 2021; Rittle-Johnson & Star, 2007), I hypothesized for the first research question that participants in both analogy conditions would outperform participants in the control condition on both learning and transfer measures. Regarding the second research question, I hypothesized that participants in the explicit condition would demonstrate higher performance than participants in the implicit condition. More specifically, using explicit analogies might diminish the negative influence of the principle of position, thereby enhancing their performance. Some studies revealed that children show an overreliance on their principle-

of-position knowledge when interpreting a numeral according to a given numeration unit, which draws their attention away from the relational structure of numeration units (Barilaro, 2026a; Tempier, 2016). In the case of base-ten numeration structure, because children tend to incorrectly apply their principle-of-position knowledge when interpreting quantities, explicit analogies will help them overcome this tendency more so than implicit analogies. I thus predicted that for both learning and transfer, children who receive instruction that makes mappings between analogies explicit would outperform those who receive instruction that leaves the mappings implicit, especially in the context of base ten, for which children's principle of position knowledge can negatively influence their structural base-ten understanding.

Additionally, I conducted an exploratory descriptive analysis to examine how the type of instruction influenced participants' response types when unitizing quantities from before to after the instruction. Barilaro (2026a) showed that students produced various types of responses when asked to unitize numerals. The descriptive response analysis promises to provide valuable insights into the relation between instructional approach (i.e., the used of explicit or implicit analogies) and the development of children's knowledge of the base-ten structure.

Method

Participants

Second graders in five cities in the province of Quebec, Canada, participated in the study. Three different procedures were used to recruit students depending on the area. In city 1, second-grade teachers were contacted directly through email using the addresses available on their schools' website. In city 2, contacting school boards for approval was required before approaching schools for participation. Once the approval was obtained, I contacted the school principals within the boards. Finally, in the remaining three cities, school secretaries were contacted via email and asked to share the study information with their second-grade teachers. In all five cities, teachers who were interested in participating were invited to either reply to my email or contact me using the information provided in a pamphlet that was shared to the school secretaries or to the school board. In city 1, three schools accepted to take part in the project, while for the remaining four cities, only one school per area accepted to take part. A total of seven schools were involved in the study.

The socio-economic status of each school is presented in Table 7. This status is determined by a score based on the proportion of children from families where the mother has no

high school diploma and the proportion of families with unemployed parents (Ministère de l'Éducation, 2024). Schools can obtain a score from 1 (least disadvantaged) to 10 (most disadvantaged), and schools with scores of 8, 9, or 10 are considered disadvantaged by the Ministère de l'Éducation.

Table 7

Socioeconomic Status of the Participating Schools

		Socio-Economic Status
City 1	School A	4 (advantaged)
	School B	2 (advantaged)
	School C	10 (disadvantaged)
City 2	School D	8 (disadvantaged)
City 3	School E	3 (advantaged)
City 4	School F	1 (advantaged)
City 5	School G	6 (advantaged)

In total, nine teachers from seven schools participated. Teachers communicated with their students' parents about the study. They were asked to send and collect consent forms and demographic questionnaires. Parental consent was obtained for 127 second graders. After receiving consent from the students' parents, the meetings with the students were planned in agreement with the teacher. The participants in this study partially overlapped with those in Barilaro (2026a). Specifically, Barilaro (2026a) reported only the pretest data, whereas the present study reports the intervention data.

During data collection, 13 children were excluded: One child did not assent at the beginning of the first session, one child was absent on the day of the pretest, one child was absent on the day of the posttest, one child was administered the wrong tasks version at posttest, and six children missed at least one of the two intervention sessions. Additionally, any child showing difficulty solving the word problems used as the source analog during the first instructional session was excluded from the study. Successfully solving the word problems used in the first instructional session is crucial in the present study as those word problems were then used as source analog in the second instructional session. As demonstrated in previous studies, a strong knowledge of the source analog is essential for a successful analogy to occur (Gentner,

1983). After the first instructional session, three children were excluded from the final sample due to persistent difficulties in manipulating the manipulatives to represent the quantity and in counting the manipulatives, despite receiving feedback.

The final sample consisted of 114 children, with 55 girls (48.2 %), 57 boys (50.0 %), and two children who did not report their gender (1.8 %) on the demographic questionnaire. Participants had a mean age of 96.8 months (8.1 years old), $SD = 4.9$ (with one missing datum). The majority of the sample spoke one language at home ($n = 85$, 74.6%), while 25 children were bilingual (21.9%) and four were multilingual (3.5%). Table 8 presents details about the various languages spoken at home. Finally, parents reported that 21 children (18.4%) had difficulties in mathematics without a diagnosis of Specific Learning Disorder with impairment in mathematics (Diagnostic and Statistical Manual of Mental Disorders, Fifth Edition, 2013). In the sample, 17 children (14.9%) presented with Attention-Deficit/Hyperactivity Disorder according to the parents' responses on the questionnaire.

Table 8

Proportion of Children by Spoken Language

Languages	<i>n</i> (%)
Monolingual	
French	83 (72.8%)
English	3 (2.6%)
Nepali	1 (0.9%)
Bilingual	
French & English	5 (4.4%)
French & Arab	8 (7.0%)
French & Other	10 (8.8%)
Multilingual	
French & English & Other	4 (3.5%)

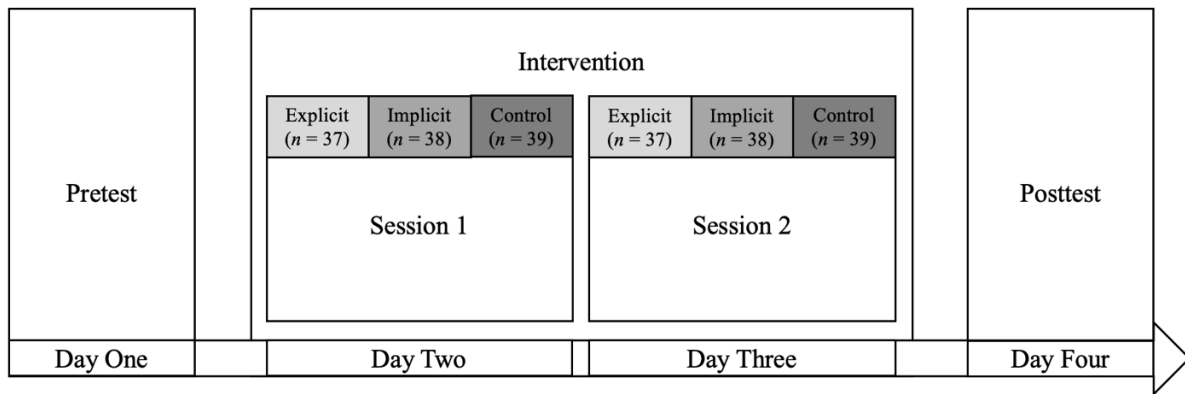
Design

The design of the study is presented in Figure 8. I used a pretest-instruction-posttest design. Participants were randomly assigned to one of three instructional conditions: (a) explicit condition ($n = 37$), (b) implicit condition ($n = 38$), and (c) control condition ($n = 39$). Each

participant took part in four meetings: the first during which the pretest was administered, two for the instructional sessions (i.e., Session 1 and Session 2), and the final meeting for the posttest.

Figure 8

Study Design



At both the pretest and posttest, participants completed four tasks: a principle-of-position task (PicPVT task), a task assessing participants' declarative knowledge of the relation between numeration units (Definition task), and two tasks assessing participants' unitizing abilities when decomposing numerals (Unitizing-Decomposing task) and when composing numerals (Unitizing-Composing task). The Definition task was not considered for the following study but was presented in Barilaro (2026a). To minimize test-retest effects, I created two isomorphic versions of each task (version 1 and version 2) for pretest and posttest administrations. In addition, to control for potential order effects, I counterbalanced the order in which children completed the tasks at pretest and posttest. As completing the PicPVT prior to the other tasks could have influenced participants' subsequent responses, I counterbalanced the task across pre- and posttest sessions: in the pretest, children either completed the PicPVT at the beginning or at the end of the session, and the opposite order was used in the posttest. More specifically, at pretest, half of the participants ($n = 56$) followed order A (i.e., PicPVT, Definition, Unitizing-Decomposing, and Unitizing-Composing) and the other half ($n = 58$) followed order B (i.e., Definition, Unitizing-Decomposing, Unitizing-Composing, and PicPVT). At posttest, participants who followed order A at pretest completed order B at posttest and vice versa. In the explicit condition, 21 participants completed order A and 16 completed order B at pretest. In the implicit condition, 15 participants completed order A and 23 completed order B at pretest.

Finally, in the control condition, 20 participants completed order A and 19 completed order B at pretest.

Procedure

I conducted the four meetings in person with all participants in a quiet room at their school. I individually administered the pretest and posttest to each child, with each session lasting approximately 20 minutes. I also delivered the instructional intervention for the three conditions in two sessions lasting approximately 30 minutes. The instructional sessions took place in groups of three to four children. Each meeting was conducted on different days. However, for three participants (two children were in the control condition and one was in the implicit condition), the pretest was scheduled on the same day as the first instructional sessions because of scheduling constraints. Given that the tasks on the pretest were unrelated to the content of the activities in the first instructional session, I considered this deviation negligible. Over the course of data collection, teachers were asked to maintain their regular mathematics curriculum. Data collection occurred between February 2025 and May 2025.

Because of scheduling constraints, I could not control for the time between each meeting. Table 9 presents the mean number of days both within and across conditions between the pretest and session 1, session 1 and session 2, and session 2 and the posttest. To rule out potential confounding effects related to time between meetings, I conducted a 3(condition: explicit, implicit, control) by 3(time: interval between pretest and session 1, interval between session 1 and session 2, interval between session 2 and posttest) mixed Analysis of Variance (ANOVA) with the number of days as the dependent variable. The analysis revealed a main effect of time, $F(2, 222) = 130.43, p < .001$, partial eta-squared = .10, with a larger time between the pretest and the first session, compared to both the interval between session 1 and session 2, $t(222) = 3.44, p < .001$, and the interval between session 2 and posttest, $t(222) = 3.56, p < .001$. No main effect of condition was found nor a condition by time interaction, both $F_s < 1$.

Table 9

Mean (and Standard Deviation) of Number of Days Between Meetings

Condition	Interval in Days		
	Pretest – Session 1	Session 1 – Session 2	Session 2 – Posttest
Explicit	4.08 (5.98)	1.97 (1.64)	1.57 (.96)

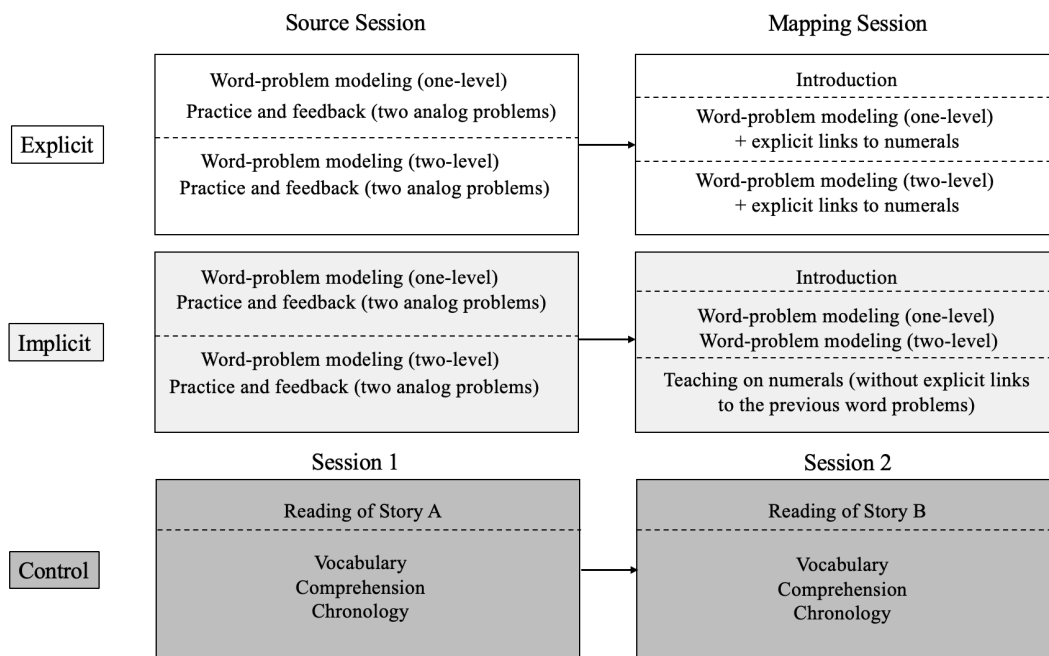
Implicit	3.95 (5.87)	1.63 (.94)	1.55 (1.03)
Control	2.62 (4.42)	1.67 (.96)	1.79 (1.03)
Across All	3.54 (5.45)	1.75 (1.22)	1.64 (1.01)
Conditions	Range: 1-20 days	Range: 1-6 days	Range: 1-4 days

Instructional Intervention

The instructional intervention was delivered in all three conditions in two sessions (referred to as Source session and Mapping session for the explicit and implicit conditions). Figure 9 presents the organization of the content in the intervention by condition and session.

Figure 9

Content and Structure of the Intervention by Instructional Session and Condition



For the intervention, the first session was dedicated to practice unitizing base-ten quantities in grounded word problems. As the grounded word problems were then used as the source of the analogy to understand how to unitize numerals in a base-ten context, the first session aimed to verify that each child understood those grounded word problems. The second session of the intervention was designed to map, explicitly or implicitly, the word problem and its solution to the associated numerals, to learn how to unitize numerals in base-ten contexts.

Source Session (Session 1)

The procedure and content of the first session in both the explicit and implicit conditions were identical. Two types of measurement division problems were presented: (a) *one-level problems* in which children were asked to construct and count one level of composite units (i.e., group objects to create a next-level composite unit); and (b) *two-level problems* in which they were asked to create and count two levels of composite units (i.e., regroup objects to first create a next-level composite unit, and then using that next-level composite unit, to create a higher-level composite unit). In Phase 1, three *one-level problems* were presented to children, while in Phase 2, three *two-level problems* were used. The problems presented in each phase were grounded in two contexts: an apple picking context and a fishing context.

In Phase 1, I began by modeling how to solve the following *one-level problem* using manipulatives: “Alexane picks 42 apples. She puts all her apples in baskets. A basket contains 10 apples. How many full baskets does Alexane have?” (translated from French). The problem was displayed on a PowerPoint slide to the group. I demonstrated how to use manipulatives (i.e., cubes only) to represent the total quantity (i.e., 42 apples) and how to create the composite units (i.e., the baskets). I showed children how to regroup 10 cubes together until four groups of 10 cubes were visible on the table, along with two remaining cubes. Finally, I demonstrated how to count the composite units as one thing (i.e., “this is one basket, two baskets, three baskets, and four baskets”). I then told the children that the answer to the question was four baskets.

Then, I gave children a booklet containing two *one-level practice problems*, analogous to the one modeled, and presented in the first row of Table 10. I also provided each child with a box containing small blue cubes and I introduced the cubes as tools to represent an object. I asked children to use the cubes to answer both *one-level practice problems* and write the solution to each problem in a booklet. Children were expected to use the cubes to represent the initial quantity in the problem (i.e., 53 for the first problem and 47 for the second problem). Then, from the quantity created using the cubes, children constructed and counted the number of composite units created (i.e., five baskets for the first problem and four nets for the second problem).

To check on children’s performance in solving each of the two *one-level practice problems*, I monitored children’s use of the cubes as well as their answers for each problem. I was looking for a correct answer as well as an appropriate use of the cubes. I considered an appropriate use of the cubes when children placed the cubes in groups of ten and counted the

groups as if they were ones. When a child provided a correct answer and demonstrated an appropriate use of cubes, I either allowed the child to move on to the next problem (if they solved the first practice item) or asked them to clear the cubes and wait for the others to finish solving their problems (if they solved the second practice item). If the answer or the use of the cubes was incorrect, I gave corrective feedback. For instance, when children struggled to create and identify the correct number of one-level composite units (i.e., baskets or nets), I explained how one cube represents one object in the problem (either an apple or a fish) and explained how when 10 cubes were grouped together, the entire group represents either a basket or a net. I then guided them through counting these units as if they were ones to find the correct solution.

Table 10

English Versions of the Source Session Practice Problems in the Explicit and Implicit Conditions

Word Problem Level	First Practice Item	Second Practice Item
One-Level	Kamala picks 53 apples. She puts all her apples in baskets. One basket contains 10 apples. How many full baskets does Kamala have?	Noah catches 47 fish. He puts all his fish in nets. One net contains 10 fish. How many full nets does Noah have?
Two-Level	Kamala picks 384 apples. She puts all her apples in baskets and puts all her baskets in boxes. One basket contains 10 apples, and one box contains 10 baskets. How many full baskets does Kamala have? How many full boxes does Kamala have?	Noah catches 439 fish. He puts all his fish in nets and then puts all his nets in crates. One net contains 10 fish, and one crate contains 10 nets. How many full nets does Noah have? How many full crates does Noah have?

When all the students in the group solved the first two *one-level problems*, I moved on to Phase 2, in which I presented a measurement division problem requiring constructing and counting two levels of composite units (i.e., *two-level problem*). As in Phase 1, the following *two-level problem* was displayed on a PowerPoint slide: “Alexane picks 325 apples. She puts all her apples into baskets and all her baskets into boxes. A basket contains 10 apples, and a box

contains 10 baskets. How many full baskets does Alexane have? How many full boxes does Alexane have?” I demonstrated how to use cubes and bars to represent the target quantity (i.e., 32 bars and five cubes to represent all the apples). When creating the quantity, I explicitly counted the total number of cubes—in other words, I counted 10 for each bar and one for each cube (e.g., 10, 20, 30 for three bars). I stated that a bar is considered a basket because it contains 10 cubes, also considered as 10 apples. Then, I asked the first question (i.e., How many full baskets does Alexane have?) and explained that I will count the number of bars to know the number of baskets (as one bar represents one basket). I then demonstrated how to count the bar (i.e., the basket) as one thing. Children were invited to count out loud with me when I counted the bars from 1 to 32. Then, I said that the correct answer was 32 baskets. Next, I asked, “How many full boxes does Alexane have?” I reminded the children that the 32 baskets are all put in boxes, and that one box contains 10 baskets. I demonstrated how to create a box by grouping 10 bars together on the table. I then showed how to count a group of 10 bars as one thing (i.e., one box). I then explicitly told the children that the answer to the question was three boxes.

Following the modeling of the *two-level word problem*, children received another booklet containing two *two-level practice problems* that were analogous to the one modeled (see second row of Table 10). I also provided each child with a box containing small blue cubes and bars. The bars were introduced as a group of 10 cubes, that can be used to represent a group. Children were asked to solve the problems using the manipulatives and to write the answers in their booklet. Similar to the *one-level practice problems*, children received corrective feedback and guidance, if needed.

Mapping Session (Session 2)

Explicit Condition. The mapping session in the explicit condition began with an introduction using a PowerPoint presentation in which I explained the relations between ones, tens, and hundreds units. For each numeration unit, I showed the students the manipulatives that can be used to represent it (i.e., one cube for a one, one bar for a ten, and one flat for a hundred). Additionally, when defining a ten, I explained that a ten is composed of 10 ones. The same type of information was given for a hundred: I stated that a hundred is composed of 100 ones, but also of 10 tens. To ensure children’s understanding of the relation between the various numeration units, I asked the participants three questions about the relation between ones, tens, and hundreds (e.g., “Can someone tell me what is in a ten?”). During this group activity, I acknowledged

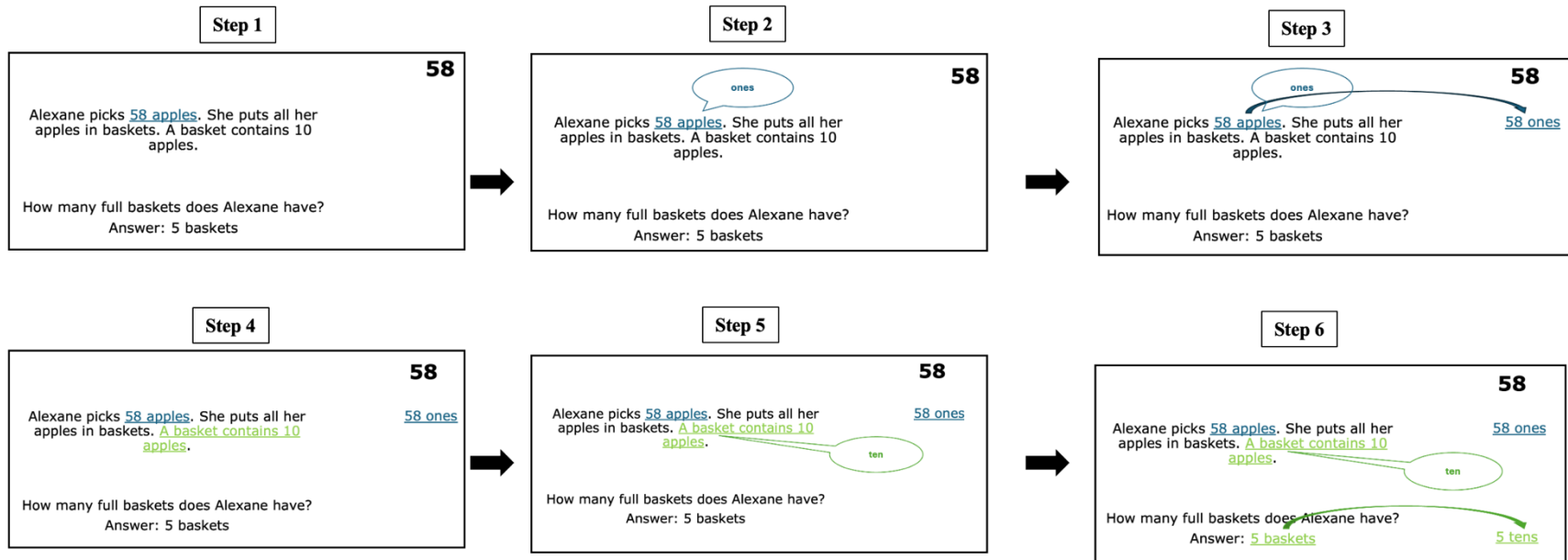
correct responses from any child and provided corrective feedback on incorrect answers so that the entire group received accurate information.

I then presented the following problem on a PowerPoint slide: “Alexane picks 58 apples. She puts all her apples in baskets. A basket contains 10 apples. How many full baskets does Alexane have?”. Similar to Session 1, I modeled how to solve this problem using cubes. After showing how to create and count the groups (i.e., baskets), I provided the answer to the problem (i.e., five baskets). Then, I explicitly mapped the word problem and its solution to the numeral associated (see Figure 9 for the various steps involved in this mapping). Colors and spatial alignments were used to facilitate the mapping between the word problem and its solution to the various numbers of numeration units (see Figure 9 for the colors and the spatial alignment). Additionally, I used relational language (e.g., “same as,” “such as,” “like”) and gestures (e.g., pointing to the five baskets in the problem and pointing to digit “5” in the numeral) to explicitly map the problem to the numeral.

First, I showed a slide containing the solved problem on the left and the numeral associated with this problem (i.e., 58) on the right (see Figure 10, Step 1). In this mapping process, I first showed how the apples are considered as ones (Figure 10, Step 2) and showed how 58 apples in the problem are the same as 58 ones in the numeral 58 (Figure 10, Step 3). Then, in Step 4, I reminded children about the relation between the basket and apples (i.e., one basket contains 10 apples), while in Step 5, I showed how one basket is equivalent to one ten, because it contains 10 apples, also equivalent to 10 ones. Finally, in the last step (Step 6 in Figure 10), I mapped the answer to the problem, five baskets, to the five tens total that are contained in 58.

Figure 10

PowerPoint Slides for the Connection Between the Problem and the Numeral 58 for the Explicit Condition



Finally, I demonstrated how to solve the following *two-level word problem*, using cubes and bars, following the same approach as for the *one-level word problem*: “Romain collects 429 sweets. He puts all his sweets into bags and then puts all his bags into cartons. A bag contains 10 sweets, and a box contains 10 bags. How many full bags does Romain have? How many full boxes does Romain have?” After the demonstration, I stated that there are 42 bags in total and four boxes in total. Then, I showed a PowerPoint slide in which the problem was presented on the left and the numeral associated with it (i.e., 429) was presented on the right. By linking the numeral to the *two-level problem* using relational language, colors, alignment, and gestures, I demonstrated how there are 429 ones, 42 tens, and four hundreds in total in 429.

Implicit Condition. The mapping session in the implicit condition started with the same introduction to numeration units as in the explicit condition. After the introduction, I modeled how to solve the same *one-level* and *two-level word problems* as in Session 2 in the explicit condition. However, in this condition, no explicit connection or link was made between each word problem and their associated numerals. I first demonstrated how to solve both the *one-level word problem* and the *two-level word problem*. After modeling how to solve the two problems, I showed the numerals associated, i.e., 58 and 429 without making any connection between the previous problems. When presenting 58, I only stated that there are 58 ones and five tens in total. For the quantity 429, I stated that there are 429 ones, 42 tens, and four hundreds in total.

Control Condition

The children in the control condition read two different stories, one in each session. The story was presented on a sheet of paper, and they completed activities related to vocabulary, comprehension, and event chronology. During the reading activity, children were invited to read the text out loud. Each child took turns reading the sentences. After reading the story, children completed a vocabulary activity in which they had to find the definition of six target words from the text (e.g., “quest”). In this activity, children had to choose one definition among three options for each word (e.g., “mission,” “flower,” or “house”), either by circling the correct answer or by drawing a line to match it with the corresponding word. This was followed by a comprehension activity, in which children answered six questions about the text (e.g., “What creatures does the young girl encounter on her quest?”). They gave their responses orally before writing them down on an answer sheet. Finally, in the chronology activity, the children received six summary sentences from the text and were asked to paste them in chronological order on their answer

sheet. The stories and the activities for each session are available in Appendix I for the first session, and Appendix J for the second session.

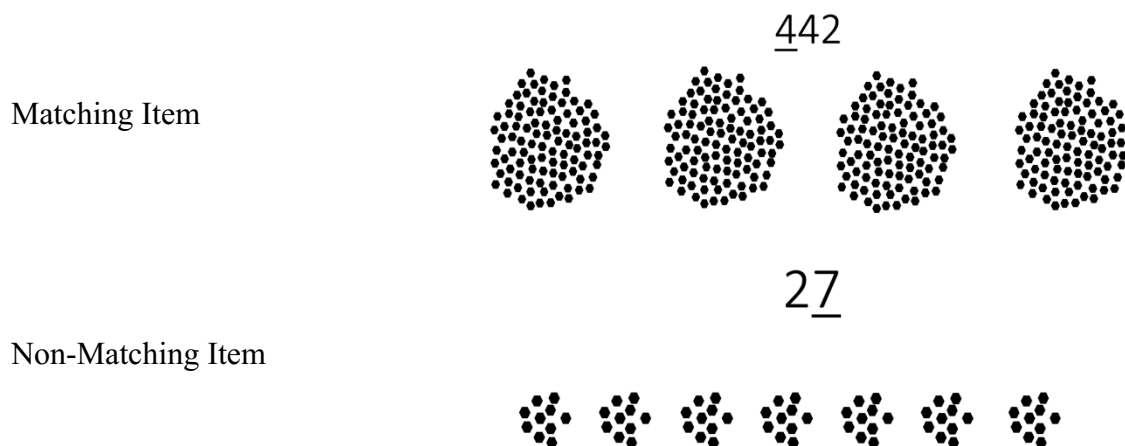
Measures

Picture Place Value Task

The Picture Place Value Task (PicPVT; Osana & Lafay, 2019) was used to assess participants' knowledge of the principle of position. Sample items are presented in Figure 11 and the complete task is available in the appendices (Appendix A for version 1, and Appendix B for version 2). On each item, participants were shown a multi-digit numeral. Each numeral had one digit (i.e., either the ones, tens, or hundreds) underlined. Participants were asked to judge whether the sets of dots were equivalent to the value of the underlined digit. Then, they were asked to provide an answer orally by either saying “yes” or “no.”

Figure 11

Example of Two Matching and Non-Matching Items of the PicPVT



The PicPVT consisted of 20 items: Ten were two-digit numerals, in which either the tens digit (five items) or the ones digit (five items) was underlined. The remaining 10 items were three-digit numerals, among which either the hundreds digit (five items), the tens digit (three items), or the ones digit (two items) was underlined. Half of the items (i.e., 10 items) presented sets that matched the underlined digit (see Figure 11), and the other half presented sets that did not match the underlined digit (see Figure 11). Participants received one point for each correct answer and zero for each incorrect answer. The PicPVT score was the mean number of points obtained across all items.

Unitizing-Decomposing Task

The Unitizing-Decomposing task assessed participants' ability to unitize when decomposing a multidigit numeral into different numeration units. The complete task is available in the appendices (Appendix C for version 1, and Appendix D for version 2). Participants were shown 12 multi-digit numerals and for each, they were asked to identify the total number of a specific target unit (either ones, tens, or hundreds). Half of the items were two-digit numerals (six items), and the other half were three-digit numerals (six items). For each of the two-digit numerals, children were always asked about the number of ones (i.e., "In this number, how many ones are there in all?") and the number of tens (i.e., "In this number, how many tens are there in all?"). For three of the two-digit quantities, questions about ones were asked first, and for the other three quantities, questions about tens were asked first. For each three-digit numeral, the same questions about the ones and tens were asked and an additional question about the hundreds was asked (i.e., "In this number, how many hundreds are there in all?"). For two of the three-digit numerals, questions were asked in the following order: ones, tens, and hundreds. For two other three-digit numerals, questions were asked in the following order: tens, ones, and hundreds. For the final two three-digit numerals, questions were asked in the following order: hundreds, tens, and ones.

As unitizing is defined as the ability to simultaneously view a quantity in terms of both lower-level and higher-level units, the scoring reflects this dual representation of a quantity. A two-digit numeral can be decomposed as a collection of ones and a collection of tens. Thus, if a child correctly interpreted the quantity in terms of ones and tens, they received one point. If a child only gave the correct interpretation for one numeration unit (and not both) or for none, they received zero points. For two-digit items, children could receive between zero and six points. A three-digit numeral can be decomposed as a collection of ones, a collection of tens, and a collection of hundreds. A child received one point if they correctly provided two or three correct answers. If a child only mentioned one or none of the correct interpretations (i.e., ones, tens or hundreds), they received zero points. For the three-digit items, children could receive between zero and six points. Decomposing mean score was the proportion of correct responses.

Unitizing-Composing Task

The Unitizing-Composing task, created specifically for this study and inspired by Tempier (2016) and Ladel and Kortenkamp (2016), assessed participants' ability to compose a

numeral (e.g., 447) from an equation with non-standard combinations of numeration units (e.g., 3 hundreds + 23 tens + 17 ones). Participants were given a sheet of paper containing 16 items that presented non-standard forms and were asked to write the correct numeral for each. Children were allowed to draw or write on their sheet to find the answers. The complete task is available in the appendices (Appendix G for version 1, and Appendix H for version 2).

Among the 16 items, five required a one-level transformation (i.e., one-level items)—that is, between tens and hundreds (e.g., 3 hundreds + 12 tens + 6 ones =) or between ones and tens (e.g., 2 tens + 19 ones =). Additionally, three items required the child to perform two transformations (i.e., two-level items), one between tens and hundreds and one between ones and tens (e.g., 1 hundred + 35 tens + 32 ones =). Finally, eight items required no transformations and relied only on applying the principle of position (i.e., no transformation items; e.g., 4 hundreds + 3 tens + 8 ones =).

Participants received one point for each correct answer. For the one-level items, participants could obtain between zero and five points; for the two-level items, they could obtain between zero and three points, while for the no-transformation items, they could receive between zero and eight points. The composing mean score was the proportion of correct responses.

Data Analysis Plan

All statistical analyses were conducted using IBM SPSS Statistics (version 30.0.0.0). Prior to conducting the main analyses, the data were screened for outliers, normality, and homogeneity of variance. For the Unitizing-Decomposing and Unitizing-Composing tasks, I computed gain scores by subtracting the pretest scores from the posttest scores (respectively the decomposing gain score, and the composing gain score). To examine the effect of the instruction on Unitizing-Decomposing and Unitizing-Composing tasks, I performed planned Helmert contrasts to address the two research questions. The instruction contrast compared the mean of the control condition against the mean of the explicit and implicit conditions combined (Research Question 1). The analogy type contrast compared the means of the explicit and implicit conditions (Research Question 2).

In addition, I conducted an exploratory analysis of the understanding profiles trajectories from before and after the instruction between conditions, to investigate the evolution of children's knowledge of base-ten relational structure. Responses were categorized following the

coding rubric developed in Barilaro (2026a), and profiles were elaborated based on proportion of response types produced.

Results

Effect of Instruction on Unitizing (Decomposing and Composing)

The data were at ceiling on the PicPVT and were thus excluded from subsequent analyses. The means and standard deviations of the decomposing mean score by item type (i.e., two-digit and three-digit numerals) and of the composing mean score by item type (i.e., no transformation, one-level transformation, and two-level transformation) at pretest and posttest are reported in Table 11.

Table 11

Means (and Standard Deviations) at Pretest and Posttest for the Unitizing-Decomposing and Unitizing-Composing Tasks

	Unitizing-Decomposing						Unitizing-Composing			
	Two-Digit Items		Three-Digit Items		No Items		One-Level Items		Two-Level Items	
	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
Condition	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>	<i>M (SD)</i>
Control	.39 (.46)	.41 (.46)	.40 (.48)	.43 (.49)	.94 (.15)	.96 (.13)	.52 (.35)	.66 (.37)	.35 (.46)	.42 (.44)
Explicit	.36 (.46)	.61 (.43)	.36 (.46)	.68 (.45)	.90 (.19)	.95 (.14)	.54 (.36)	.57 (.34)	.36 (.44)	.40 (.47)
Implicit	.26 (.40)	.45 (.44)	.26 (.42)	.48 (.47)	.98 (.06)	.96 (.06)	.53 (.33)	.57 (.34)	.24 (.37)	.28 (.39)

Firstly, I conducted a 3(condition: control, explicit, implicit) $\times$ 2(item type: two-digit, three-digit) mixed ANOVA, with condition as the between-groups factor and item type as the within-groups factor on the decomposing gain score. Only a main effect of condition was observed, $F(2, 111) = 6.17, p = .003$, partial eta-squared = .10. No main effect of item type was found, $F(1, 111) = 2.73, p = .10$, and no interaction between item type and condition was found, $F(2, 111) = .67, p = .52$.

The first contrast was the instruction contrast, which compared the decomposing gain score of children in both the explicit (+1) and implicit (+1) conditions combined to children in the control condition (-2). A significant contrast effect was found with mean decomposing gain score in the explicit and implicit conditions statistically higher than the mean gain score in the control condition, $t(111) = -3.35, p = .001, d = .99$. The analogy contrast, comparing the explicit condition (-1) against the implicit condition (+1), was not significant, meaning that there was no statistical difference in the decomposing gain score between the explicit and implicit conditions, $t(111) = 1.11, p = .27$.

Second, I run a 3(condition: control, explicit, implicit) $\times$ 3(item type: no transformation, one-level transformation, two-level transformation) mixed ANOVA, with condition as the between-groups factor and item type as the within-groups factor on the composing gain score. No main effect of condition was found, $F < 1$. Similarly, no main effect of item type, $F(1, 92) = 3.21, p = .08$, and no interaction between condition and item type, $F < 1$, were found.

Finally, no significant effects were observed for the composing gain score. Neither the instruction contrast, $t(92) = .74, p = .46$, nor the analogy contrast, $t(92) = .49, p = .63$, was significant.

Exploratory Data Analysis

In this section, I explored the responses produced by children on the Unitizing-Decomposing task at pretest and posttest to further examine the effect of instruction on their understanding of the relational structure between ones, tens, and hundreds.

Response Coding

The codes used for the exploratory analysis were identical as those used in Barilaro (2026a). A brief description of those codes is provided in Table 12, but a more detailed description is available on Barilaro (2026a). The coding rubric entailed the four following codes: Relational, No Relation, Relational Attempt, and Other.

Table 12

Coding Rubric and with Sample Responses for a Two-Digit Item (68) and a Three-Digit Item (238)

		Examples		
		Item 68	Item 238	
		Ones	Ones	Tens
Description				
Relational	Responses that entail accurately interpreting the numeral according to the numeration unit given.	68	238	23
No Relation	Responses that focus exclusively on the principle of position knowledge, by only considering the digit associated with the given numeration unit, without accounting for the units embedded within higher-level numeration units (i.e., tens in the hundreds).	8	8	3
Relational Attempt	Responses that demonstrate an attempt to relationally reason between level of units making a mistake doing so.	60 / 14	38 / 13	230 / 50 / 5
Other	Responses that show a misunderstanding of the question (e.g., answering about ones instead of tens), and responses that were uninterpretable.	6 / 20	2 / 40	8 / 100

As in Barilaro (2026a), I removed the items for which the interpretation presented some ambiguities. More specifically, for the two-digit items, I excluded all questions about the total number of tens. For the item 24, for example, an answer of two tens does not allow for a clear distinction between a child relying on the principle-of-position knowledge and a child reasoning relationally between levels of numeration units. Had the children been older and familiar with decimal numbers, this item could have been included, as they might have answered 2.4 tens in total, reflecting relational reasoning. Thus, for the two-digit items, only questions about ones were considered (six items). The same issue applied on items requiring the total number of hundreds in a three-digit numeral. So, for the three-digit numbers, only questions about ones and tens were considered (12 items).

A research assistant and I coded all the data independently, resulting in a 99% agreement. We resolved the discrepancies through discussion.

Responses Analysis

For each participant, a mean proportion of each response type was computed by item type and condition. The proportions of each response type are presented in Table 13.

Table 13

Mean Proportion of Response Type by Item Type and Condition

	Control (<i>n</i> = 39)		Explicit (<i>n</i> = 37)		Implicit (<i>n</i> = 38)	
	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
2-digit items						
Relational	.40 (.47)	.44 (.49)	.37 (.46)	.71 (.44)	.30 (.43)	.57 (.47)
No Relation	.57 (.47)	.54 (.50)	.60 (.46)	.29 (.44)	.68 (.44)	.43 (.47)
Relational Attempt	.00 (.00)	.00 (.03)	.02 (.14)	.00 (.00)	.00 (.00)	.00 (.00)
Other	.03 (.14)	.01 (.06)	.00 (.00)	.00 (.00)	.03 (.14)	.00 (.00)
3-digit items						
Relational	.32 (.41)	.41 (.47)	.28 (.37)	.64 (.41)	.22 (.36)	.45 (.44)
No Relation	.57 (.46)	.54 (.48)	.62 (.44)	.31 (.41)	.62 (.42)	.44 (.43)
Relational Attempt	.07 (.14)	.02 (.06)	.07 (.17)	.04 (.12)	.09 (.20)	.08 (.19)
Other	.04 (.10)	.03 (.10)	.03 (.06)	.02 (.08)	.07 (.18)	.04 (.10)

Note. Due to rounding, the proportion sums might not equal to one.

Profile Analysis

To investigate participants' learning trajectories, I categorized participants into profiles based on the types of responses produced on the Unitizing-Decomposing Task. A profile analysis can provide valuable insights into the individual paths taken by each child, capturing how their understanding evolved from pretest to posttest within each condition.

Participants who provided five (or more) responses of the same type out of six two-digit items (i.e., 83%) and ten (or more) responses of the same type out of 12 three-digit items (i.e., 83%) were placed in a profile that reflected the response type they used most consistently. Participants who did not meet the consistency threshold for any single response type were placed in the Mixed profile. Five profiles were thus identified: Relational, Relational Attempt, No Relation, Mixed, and Other. The proportions of participants in each profile are presented in Table 14 by time, condition, and item type.

Table 14

Proportions (and Frequencies) of Participants in Each Profile by Time, Condition, and Item Type

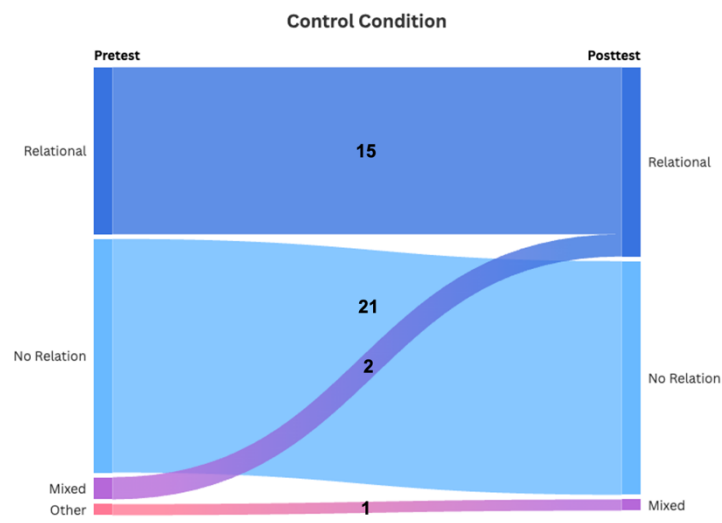
	Control		Explicit		Implicit	
	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
Two-Digit Items						
Relational	38.5 (15)	43.6 (17)	35.1 (13)	70.3 (26)	28.9 (11)	55.3 (21)
Relational Attempt	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)
No Relation	53.8 (21)	53.8 (21)	56.8 (21)	27.0 (10)	65.8 (25)	39.5 (15)
Mixed	5.1 (2)	2.6 (1)	8.1 (3)	2.7 (1)	2.6 (1)	5.3 (2)
Other	2.6 (1)	0.0 (0)	0.0 (0)	0.0 (0)	2.6 (1)	0.0 (0)
Three-Digit Items						
Relational	23.1 (9)	41.0 (16)	16.2 (6)	54.1 (20)	13.2 (5)	36.8 (14)
Relational Attempt	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	2.6 (1)	2.6 (1)
No Relation	48.7 (19)	51.3 (20)	54.1 (20)	21.6 (8)	52.6 (20)	29.0 (11)
Mixed	28.2 (11)	7.7 (3)	29.7 (11)	24.3 (9)	29.0 (11)	29.0 (11)
Other	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	2.6 (1)	2.6 (1)

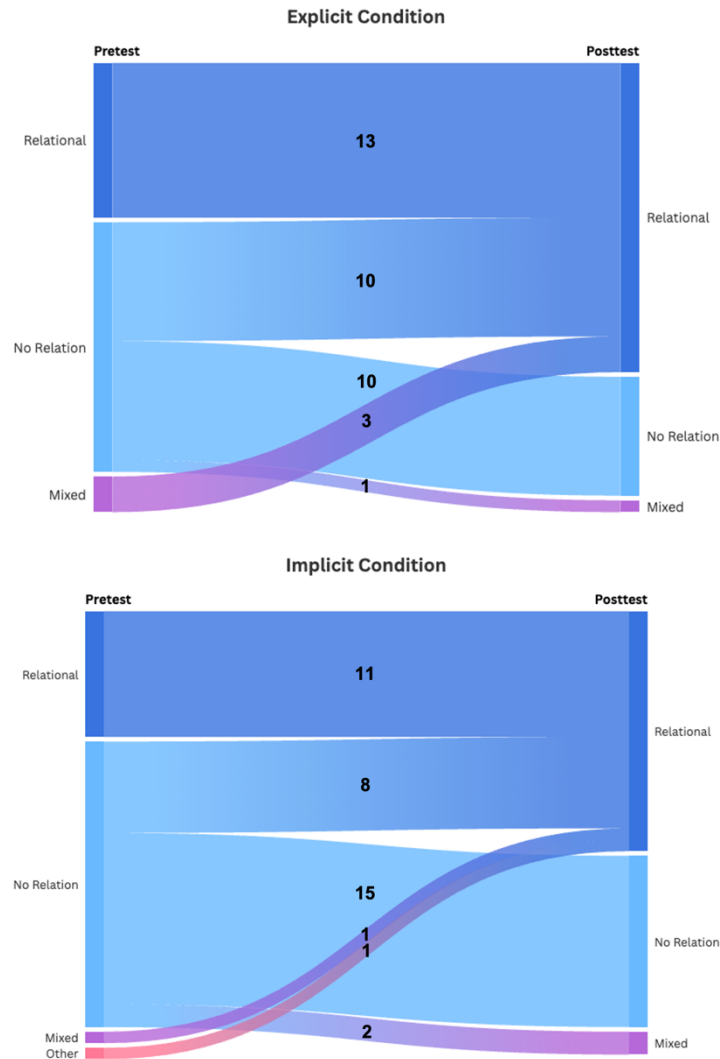
The frequencies of the participants in each profile at pretest and posttest were examined using Sankey diagrams, allowing for a visual analysis of the participants' profile trajectories by condition over time.

Two-Digit Items. The Sankey diagrams (see Figure 12) illustrate participants' trajectories across conditions, highlighting differences among the three conditions. In the explicit condition, almost half of the participants in the No Relation profile at pretest moved to the Relational profile after the intervention (48%, $n = 10$), one participant moved to the Mixed profile (5%), and the remaining stayed in the No Relation profile (47%, $n = 10$). In the implicit condition, however, of those who were in the No Relation profile at pretest, about a third moved to a Relational profile after the intervention (32%, $n = 8$), two moved to the Mixed profile (8%), and the majority stayed in the No Relation profile (60%, $n = 15$). In the control condition, all the participants in the No Relation profile at pretest stayed in the same profile a posttest. For both the explicit and implicit conditions, participants in the Mixed profile at pretest moved to the Relational profile at posttest (three and one participant, respectively). For the two-digit items, the exploratory analysis revealed that, more students in the explicit condition shifted from a No Relation profile to a Relational profile after the intervention compared to those in the implicit condition.

Figure 12

Trajectory of Participants' Profiles for Two-Digit Items by Condition (Frequencies per Trajectory Displayed)



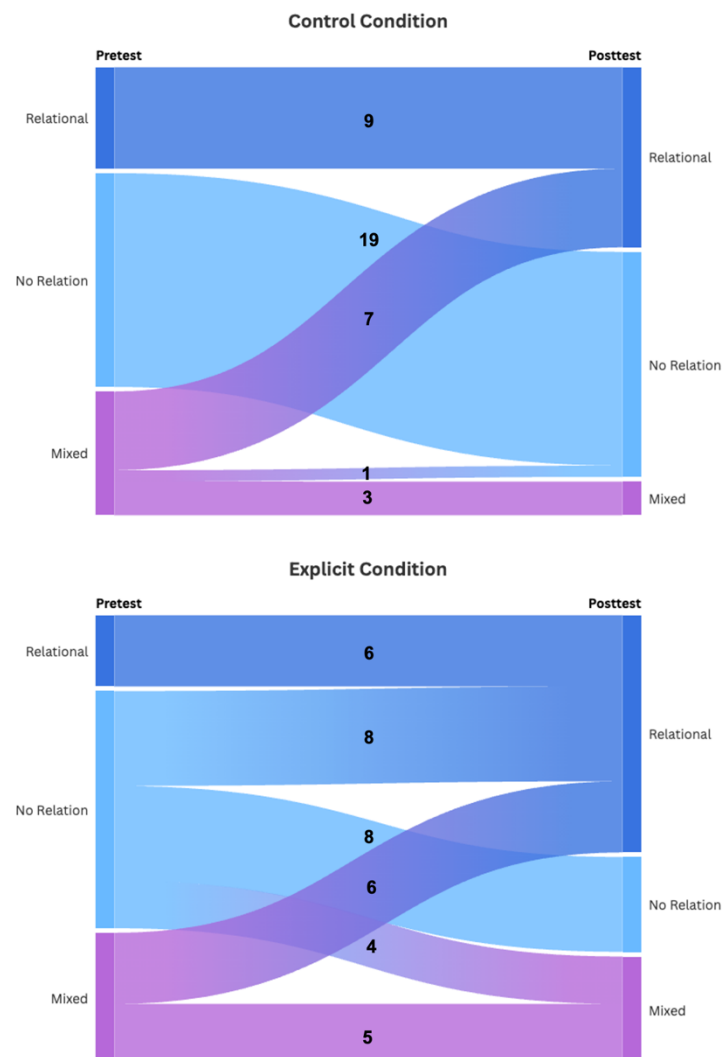


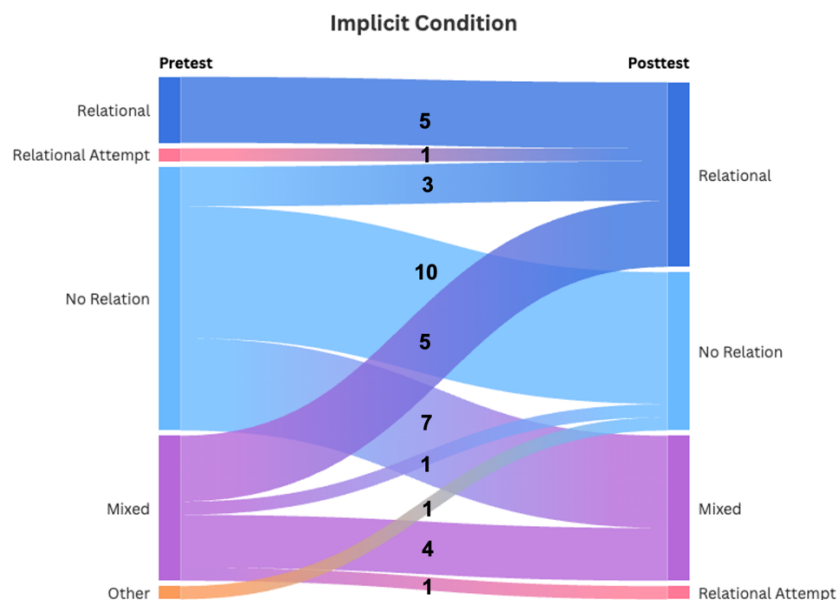
Three-Digit Items. Similar to the patterns in the data for the two-digit items, the Sankey diagrams provide a visual representation of participants' trajectories from pretest to posttest, depicting differences between explicit, implicit, and control conditions for the three-digit items (see Figure 13). Specifically, the data showed that among the participants in the No Relation profile at pretest, only 15% ($n = 3$) shifted to a Relational profile in the implicit condition compared to 40% ($n = 8$) in the explicit condition. Additionally, approximately the same proportion of participants moved from the Mixed profile at pretest to the Relational profile at posttest in both the implicit (46%, $n = 5$) and explicit (54%, $n = 6$) conditions. For the three-digit numerals, the trajectory analysis evidenced that, for the explicit condition, a several students from a No Relation and a Mixed profiles moved to a Relational profile after the intervention. By contrast, for the implicit condition, only a small number of students moved from a No Relation to

a Relational profile, and the trajectories were more dispersed compared to the two other conditions.

Figure 13

Trajectory of Participants' Profiles for Three-Digit Items by Condition (Frequencies per Trajectory Displayed)





Discussion

The objective of the present study was to investigate the effectiveness of an instructional intervention involving analogies to improve second graders' knowledge of the relational structure of base ten. The instruction aimed to improve children's unitizing performance when decomposing and composing symbolic representations of quantities. Second graders were randomly assigned to three instructional conditions: instruction involving explicit analogies, instruction involving implicit analogies, and control. They completed isomorphic measures at pretest and posttest: one assessing unitizing performance when decomposing a multidigit numeral (e.g., finding the total number of tens in 345) and the other assessing unitizing when composing a numeral from a non-canonical equation (e.g., finding the equivalent numeral corresponding to the non-canonical equation 2 hundreds + 45 tens + 13 ones). At pretest and posttest, I recorded children's accuracy for each task and documented their answers when asked to decompose a multidigit numeral.

In terms of accuracy, I predicted that second graders who received instruction based on analogical reasoning would improve their unitizing performance when decomposing and composing a numeral relative to children who did not receive instruction on unitizing (Research Question 1). The prediction was partially supported. First, when decomposing a numeral, the results revealed that children who received instruction based on either type of analogy—explicit or implicit—improved their unitizing abilities. As shown by previous research (e.g., Richland & Begolli, 2016), analogies in a mathematical context are an effective pedagogical tool. In the

present study, I designed an intervention based on the analogy between the multiplicative structure of the measurement division problems and the multiplicative relations between numeration units in base ten. The effectiveness of using analogy-based instruction to improve children's knowledge of the base-ten structure could be explained in two ways. First, the grounded measurement division problems used in this study shared a similar structure with the multiplicative structure of base ten, making them a relevant source for instruction on base-ten unitizing. Second, using the word problems as the source analogy was suitable because they represent a meaningful and familiar context for introducing the concept of unitizing. Barilaro (2026b) showed that second graders were able to successfully unitize quantities when they were grounded in measurement division word problems. Therefore, analogy-based instruction using measurement division word problems and numerals effectively supported children's understanding of the structure of base-ten numeration system.

In contrast, when composing a numeral, there were no condition effects on children's unitizing. The lack of transfer is likely due to the content and focus of the instruction. Specifically, the instructional sessions focused on highlighting—either explicitly or implicitly—the similarities between the structure and solution to measurement division problems embedded in base-ten and the various interpretations of a numeral using different numeration units. The instruction did not provide any opportunities for children to engage with symbolic representations—especially of the type used on the transfer task—and did not provide children with opportunities to perform addition involving quantities expressed in different units. Additionally, the transfer task required children to manipulate a representation of quantity that is particularly uncommon in school, namely using two-digit numerals to represent a single numeration unit (e.g., 23 tens). As shown by Houdement and Tempier (2022), children often learn to interpret numerals using a base-ten charts, in which no more than one digit must be placed in any numeration unit position (e.g., 23 is 2 tens and 3 ones). Therefore, in the absence of instruction on how to manipulate those non-canonical representations of quantities, children appeared unable to transfer their understanding of unitizing numerals to the transfer task.

Regarding the second research question, I predicted that instruction based on the use of explicit analogies would lead to greater gains in children's unitizing when decomposing and when composing a numeral compared to an instruction involving implicit analogies. The data did not show any condition effects on unitizing gains when decomposing a numeral nor when

composing a numeral. Children in both the explicit and implicit conditions improved their unitizing performance on two- and three-digit quantities from before to after the intervention. The design of the current study could explain why the explicit condition did not lead children to display higher decomposing performance than the implicit condition. First, previous studies have shown that individuals are more likely to transfer knowledge from a source to a target when the source analog has the same structure as the target concept (Sidney & Alibali, 2015; Yu et al., 2024). In one study, Sidney and Alibali (2015) showed that when a source analog has the same structure as the target concept, children can learn without any specific mapping supports. Importantly, the chosen source, here measurement division problems embedded in base-ten contexts, share the same structure of embedded units as the structure of the base-ten system. Second, a suitable source analog must be familiar to students, and they must have some prior knowledge about it (Gentner, 1983). In the present study, measurement division problems were grounded in familiar contexts enabling children to make sense of the situation described in the problem, and therefore understanding the structure of the units involved. Additionally, to enhance students' familiarity with the source analog, the first session of the instructional intervention focused on modeling and practicing solving measurement division problems. Therefore, the suitability of the source analog in the present study might have increased the impact of the implicit condition, leading to comparable unitizing performance in both conditions.

Despite the absence of statistically significant gains in performance between the explicit and implicit conditions, the analysis of students' learning trajectories provides valuable insights into possible developmental trajectories as a function of instruction. More specifically, the statistical analysis used in the present study masked the different learning trajectories of the students in the three conditions from before and after the instruction. The exploratory trajectory-based analysis might therefore suggest condition effects that may have been obscured by the statistical analysis. Notably, for the two-digit numerals, half of the students in the explicit condition and a quarter in the implicit condition shifted from a positional focus before instruction to one that reflected an understanding of the relations between numeration units after instruction. This observation may suggest that the instruction in the explicit condition is more effective than that in the implicit condition in facilitating a shift from surface-level strategies to relational thinking. This observation was even more pronounced for three-digit numerals: Almost half of students in the explicit condition whose responses were initially based on the principle of

position shifted to responses that demonstrated an understanding of the relations between numeration units. This finding is compared to only a sixth of students in the implicit condition who demonstrated the same shift. This difference between conditions may reflect the robustness of principle-of-position knowledge, especially when manipulating three-digit quantities. Without receiving instruction that incorporated explicit links between the solutions to word problems and corresponding numerals, children might have struggled to extract the underlying structure of the ones, tens, and hundreds. These findings suggest that explicit analogies might be more effective than implicit analogies in helping second graders navigate complex mathematical structures, particularly when prior knowledge—in this case, the principle of position—is deeply entrenched and possibly obstructs further learning. Therefore, using explicit links might be more beneficial than implicit links for fostering relational understanding in second-grade students, particularly when learning about three-digit quantities.

Contributions to the Literature

The present findings contribute to the current literature in several ways. To begin, to my knowledge, this is the first study to investigate the effectiveness of an empirically-validated instructional approach on second graders' unitizing in base-ten. Tempier (2024) used a comparison task to improve fourth graders' understanding of the relation between numeration units. In this task, students were asked to compare two quantities, represented either with base-ten blocks, with numerals (e.g., 250), or with numerals and associated numeration units (e.g., 25 hundreds). To compare the two quantities, the teacher involved in the study showed students how to convert one or both representations by using a common numeration unit, allowing to determine which quantity was the largest. students needed to convert one or both representations to determine which quantity was the largest. However, this task was conducted in a single classroom, and the evolution of students' strategies was reported descriptively without any statistical analyses. In contrast, the present study revealed, both statistically and with a descriptive learning trajectory analysis, that second graders improved their understanding of the relation between numeration units when provided with instruction based on analogies. This finding is consequential because understanding of the relations between numeration units in the early elementary years, such as in second grade, is foundational to future success in arithmetic (Carpenter et al., 1998; Ladel & Kortenkamp, 2016) and could support children's learning of

more advanced mathematical concepts involving units, such as multiplication and fractions (Sophian, 2004; Steffe & Olive, 2010; Yeong et al., 2015).

Second, the present study used grounded measurement division word problems and concrete representations (i.e., base-ten blocks) to promote understanding of the base-ten structure as assessed through unitizing in base ten. Measurement division problems have been used in previous studies to support students' understanding of division (e.g., Gaidoschik et al., 2025) but have not been commonly used to enhance children's knowledge of base-ten structure. The present study suggests that using measurement division problems in conjunction with base-ten blocks might be effective in supporting children's understanding of the base-ten structure, thus providing empirical evidence for Carpenter et al.'s (1996) recommendation to use such problems for learning about the structure of base-ten.

Additionally, the current study is the first to test the effect of explicit analogies over implicit analogies on the development of unitizing in base-ten numeration at the elementary level. Sidney (2020), for example, found that implicit analogies were more effective than explicit analogies for middle-school students receiving instruction on fraction division. In their study, students benefitted from activating the source (i.e., whole number division) without guidance or explicit mapping to the target (i.e., fraction division). In contrast, the current study did not reveal a clear advantage of implicit analogies over explicit analogies. In fact, the exploratory analysis of students' learning trajectories suggested that explicit analogies might be more beneficial than implicit analogies when supporting students' reasoning about base-ten structure. This work contributes to the literature by highlighting that the impact of both explicit and implicit analogies may be domain-specific and age-dependent. Elementary school students, in particular, may require more instructional support and guided mappings than older students.

Finally, the study brings an important contribution to the literature regarding children's developmental trajectories of relational thinking in base-ten. This is the first study to my knowledge to investigate children's profiles of base-ten structure understanding, and more importantly, to examine the evolution of those profiles after an intervention targeting the relational structure of base ten. The exploratory analysis showed that with a targeted instruction using suitable pedagogical tools (i.e., grounded word problem, concrete representations, and analogies), children can improve their relational thinking as evidenced by their trajectories

toward a more normative profile. This is particularly the case when I gave them explicit analogies, constituting an important contribution to the literature.

Strengths and Limitations

A strength of the study lies in the careful design of the instructional intervention, which was grounded in prior research on analogical reasoning. First, prior work emphasized that analogies are most effective when learners are familiar with and possess strong prior knowledge of the source analog (Gentner, 1983). Therefore, selecting an appropriate source analog is likely to increase its effectiveness in instruction. Barilaro (2026b) found that when second graders are presented with grounded word problems and manipulatives, they are more likely to successfully unitize quantities using numeration units. Building on these findings, the first instructional session in the present study was devoted to ensuring that children mastered the grounded problems, used as the source analogs, which likely increased the children's familiarity with them. This familiarity was intended to increase both the likelihood, and the quality, of the mappings between the analogs in the second instructional session. Second, in both the explicit and implicit conditions, the content of the second instructional session remained identical except for support provided for the mapping between the word problems, their solutions, and the associated numerals. This design ensured that only the manipulated factor (i.e., the nature of the link) was under investigation.

Additionally, because the instruction was delivered to children in small groups, the study has relatively strong ecological validity. This implementation mirrors typical classroom practices, where small group activities are commonly used. Finally, I was the sole administrator of the intervention and followed a standardized script for each condition. This allowed me to ensure high implementation fidelity and to minimize the risk of experimenter bias across sessions.

One limitation of the study concerns the duration of the instructional intervention. Only two instructional sessions were delivered, with the first one focusing primarily on building familiarity with the source analog. Although some children demonstrated improvement in their ability to unitize quantities, it is likely that a longer intervention would have led to greater gains in understanding unitizing in base ten. Another limitation relates to the sensitivity of the measures used to capture the effects of the intervention. Although the study aimed to improve children's unitizing when decomposing and composing numerals, the assessment tasks may not

have adequately reflected those skills. For instance, while the Interpretation task was intended to assess children's knowledge of unitizing when decomposing a numeral, an additional task requiring the identification of the number of numeration units within another unit (e.g., 30 tens = ____ ones) might have been more sensitive to children's knowledge of the relation between numeration units. Finally, the sample consisted predominantly of children from a relatively high socioeconomic status. As a result, the findings may not reflect those in the general population.

Educational Implications

The findings of this study carry important implications for classroom practice. First, the results encourage teachers to use analogies in instruction, including in lessons that target the structure of base-ten numeration. More specifically, teachers can enhance children's unitizing skill when decomposing numerals by leveraging their capacity to create and manipulate numeration units when solving grounded word problems with concrete objects. However, before using grounded word problems and concrete objects as source analogs, teachers may wish to ensure that students possess sufficient prior knowledge of the problems and the objects so that they may engage meaningfully with the source analog. In other words, children must be familiar and fluent with the source analogs to accurately recognize and apply them. Without this foundational knowledge, students may struggle to abstract the underlying conceptual structure between the source and the target and consequently fail to grasp the concept it is intended to convey. Therefore, the results suggest that teachers might wish to select source analogs carefully and verify their students' familiarity with and understanding of these sources before drawing analogical connections.

Finally, the exploratory findings highlight the influence of explicit versus implicit analogies in shaping children's learning trajectories in unitizing. More specifically, the results of the present study support the use of analogies, particularly when teaching unitizing with three-digit numerals. As revealed by the exploratory analysis—specifically students' learning trajectories—explicit analogies appear to help students move away from an over-reliance on the principle of position to relational thinking, which is ultimately what researchers recommend and what teachers should aim to achieve in their classrooms. Importantly, when implicit analogies are used, fewer students appear to move beyond their principle-of-position knowledge. Further, the trajectories observed in the implicit analogy condition showed great variability, with some children moving in directions that were not always aligned with the learning objective. This

result suggests that, especially when instruction involves three-digit numerals, implicit analogies may not be sufficient to guide students toward relational knowledge of base ten. Therefore, in contexts where existing knowledge may interfere with the development of new concepts, teachers are strongly encouraged to use explicit analogies to help steer students' learning trajectories in the desired direction.

Chapter 5: General Discussion

The three studies presented in this dissertation aimed to investigate second-grade students' understanding of base-ten structure assessed through their performance on unitizing tasks and to test the effectiveness of an instructional intervention designed to enhance their unitizing performance. This line of research is essential given that relational knowledge of base ten supports individuals in making sense of numerals, which are widely used in everyday life. Consider, for instance, how the same quantity can be presented differently depending on the objective: One media outlet might report that 12 hundred people attended a protest, while another might state that one thousand and two hundred people were present. Even if those two quantities are equivalent, the first reporter focused on hundreds, which might be used as a technique to make the crowd look smaller than it is. A relational understanding of base-ten numeration would allow individuals to be more confident in interpreting the quantities presented than without it. A flexible understanding of numerals implies recognizing that the same quantity can be expressed in different ways, depending on the unit of reference used.

Such relational understanding is not only relevant for daily life but also foundational for future mathematical competencies, including arithmetic (Carpenter et al., 1998; Ladel & Kortenkamp, 2016), multiplicative reasoning, and conceptual understanding of rational numbers (Sophian, 2004; Steffe & Olive, 2010; Yeong et al., 2015). In the early school years, children are introduced to base-ten numeration, which involves learning about both syntax and structure (Programme de formation de l'école québécoise, Ministère de l'Éducation, du Loisir et du Sport, 2006). Despite the goals of the current mathematics curriculum, emerging evidence suggests that many children and adults struggle to understand the hierarchical structure of the base-ten system (Barilaro et al., 2024; 2026; Tempier, 2016; Thanheiser, 2009). This dissertation offers promising contributions to the field by shedding light on how children reason about base-ten units and under what conditions their understanding can be supported. The present work paves the way for more effective curricula and targeted instruction that fosters flexible knowledge of base-ten numeration.

Synthesis of Key Findings

Three major findings emerged from the studies presented in this dissertation. First, the studies revealed that second-grade students experienced difficulties when asked to unitize quantities in base-ten contexts, especially when the quantity is presented symbolically (i.e.,

numerals). More specifically, children struggle to interpret a numeral using two different numeration units (e.g., finding the total number of tens and the total number of ones in a three-digit numeral; Barilaro, 2026a). However, when solving measurement division problems embedded in base-ten contexts with manipulatives, children appeared to demonstrate stronger unitizing performance, as demonstrated by their counting of both basic and composite units (Barilaro, 2026b). Manipulatives appeared to support children in attending to the relations between units, and more importantly, in supporting their ability to shift their interpretation of the quantity depending on the unit of reference (e.g., whether they were asked about the total number of the basic, i.e., the ones, or the total number of the composite units, i.e., the tens). These results are promising, as they suggest that second-grade students can engage in unitizing in non-symbolic contexts. Engaging with units in non-symbolic contexts might serve as a springboard for later unitizing in symbolic contexts (i.e., numerals).

Second, the analysis of children's responses when unitizing quantities, whether represented in numerals or with manipulatives, provided deeper insight into their understanding of base-ten structure and the difficulties they may encounter when learning it. Interestingly, two types of errors emerged across both symbolic and non-symbolic contexts. The first, and most frequent, error consisted in focusing solely on the digit in the place of the targeted unit (e.g., stating that there are six tens in total in 264; Barilaro, 2026a), revealing a rigid and superficial interpretation of multidigit quantities. The second error involved correctly interpreting part of the quantity but then applying the wrong unit to another part of the quantity, incorrectly switching the unit of reference when counting (Barilaro, 2026a, 2026b). For example, when asked to find the total number of tens in 345, a child might identify the three hundreds as 30 tens, but then mistakenly add the four tens as 40 tens, producing a total of 70 tens. These two errors reflect distinct challenges: The first points to a limited understanding of the base-ten structure, and the second suggests an emerging, but not yet stable, understanding of the relation between units.

Finally, the second and third studies yielded compelling results regarding second-grade students' learning of unitizing (Barilaro, 2026b, 2026c). First, the results of Study 2 revealed that children who spontaneously struggled to unitize benefited from individualized guidance, including prompts and explanations tailored to their specific difficulties. Notably, changing the manipulatives (such as providing actual bags to represent composite units, or exchanging a flat for 10 sticks) helped support children's reasoning about the hierarchical structure of the units

involved in the problem. Study 2 uncovered interesting findings about the types of support that may benefit children's unitizing and should be further examined in future research. Additionally, Study 3 showed that drawing analogies between measurement division problems embedded in base-ten contexts (i.e., a familiar and grounded situation) and unitizing numerals (i.e., the to-be-learned concept), supported gains in children's unitizing performance with numerals. Importantly, a descriptive analysis of children's learning trajectories revealed that cognitive supports (e.g., gestures, relational language, spatial alignment, and colors) were particularly helpful in drawing students' attention to the structural similarities between problems and numerals. Instruction designed to target unitizing numerals that built on familiar and grounded problems alongside concrete manipulatives led to significant improvement in unitizing performance.

The research in this dissertation confirmed that second-grade students struggle to unitize numerals, but also revealed that they showed promise when unitizing in the context of measurement division word problems with manipulatives. Instructional interventions based on analogies represent a promising pedagogical direction: When provided with appropriate instructional support, children demonstrated measurable improvement in their understanding of base-ten structure.

Theoretical Contributions

Together, the three studies contribute significantly to the current literature on base-ten numeration, and particularly to research on children's understanding of base-ten structure. First, across the three studies, base-ten structure was operationalized through a unitizing task that assessed children's ability to interpret one numeral using two different numeration units. This approach to assess base-ten structure is novel and extends the current literature on approaches to assessing base-ten numeration. Although previous studies have examined children's understanding of base-ten structure (e.g., Soury-Lavergne et al., 2020; Tempier, 2016), the authors' assessments were not based on unitizing processes. For example, Tempier (2016) asked students to identify the total number of one specific numeration unit in a numeral (e.g., total number of tens in 529) but did not require them to interpret the same numeral using another numeration unit, a key characteristic of unitizing. Overall, the studies by Soury-Lavergne et al. (2020) and Tempier (2016) demonstrated that children struggled to understand base-ten structure, a finding confirmed in the present research: Second-grade students exhibit challenges

in reasoning relationally about between numeration units. Moreover, the present findings suggest that unitizing may be underrepresented both in the current literature and in classroom instruction, as demonstrated by the difficulties evidenced by children in the present tasks.

The second major theoretical contribution of the present dissertation is that children's unitizing performance is likely context-dependent, similar to other areas of children's mathematical development as reported in previous studies (Carpenter et al., 1981; Lawler, 1981; Sherman & Bisanz, 2009). More specifically, the current studies revealed a difference in children's unitizing performance depending on whether quantities were presented symbolically (i.e., using numerals; Barilaro, 2026a) or non-symbolically (i.e., using manipulatives; Barilaro, 2026b). Study 1 showed that most second-grade students struggled to unitize numerals using different numeration units. In contrast, Study 2 provided promising evidence of children's unitizing competencies when solving measurement division problems using manipulatives and given targeted individual support. These findings are consistent with prior research showing that children tend to display higher performance when mathematical tasks are presented in non-symbolic contexts compared to symbolic ones (Carpenter et al., 1981; Sherman & Bisanz, 2009). The discrepancy observed when solving symbolic and non-symbolic problems might be explained from the way symbolic problems are taught in school (Carpenter et al., 1981). More specifically, instructional practices might encourage rigid reasoning strategies, which in turn may hinder children's solving of symbolic problems. This hypothesis is particularly relevant to the present research. As shown in Study 1, children often relied on their prior knowledge of the principle of position when interpreting numerals, which led to systematic errors. This overreliance on the principle of position might stem from instructional practices that emphasize the syntax of base ten over its structure (Houdement & Tempier, 2019; Tempier, 2024). In contrast, presenting quantities through non-symbolic contexts (e.g., measurement division problems and manipulatives), as in Study 2, may help children overcome their prior knowledge, thereby supporting stronger unitizing performance.

Additionally, across the three studies, capturing the nature of children's unitizing responses when interpreting numerals (Barilaro, 2026a, 2026c) or when using manipulatives (Barilaro, 2026b) provided a deeper understanding of how children reason about quantities in base-ten contexts. Tempier (2016) identified a common error when interpreting numerals: Children often focus on the digit in the position of the targeted numeration unit (e.g., there are

six tens in total in 264). This error demonstrated an overreliance on their principle-of-position knowledge, obstructing children's view of the base-ten system as one characterized by nested numeration units. This error was also consistently found in the present dissertation, particularly when children interpreted numerals. This pattern might stem from how base-ten numeration is typically taught in classrooms (Houdement & Tempier, 2019; Tempier, 2024). Such typical instructional practice may result in a rigid interpretation of numerals that hinders children's ability to attend to the relations between numeration units. This aligns with Carpenter et al. (1981), who argued that formal instruction can sometimes negatively impact children by preventing flexible understanding of mathematical concepts.

A second major error identified in the present studies entailed an incorrect shift in counting strategies between two levels of numeration units. For example, when asked to identify the total number of tens in 345, children might correctly identify the three hundreds as 30 tens, but when adding the four tens, they incorrectly interpret them as 40 tens, resulting in an incorrect answer of 70 tens in total. This type of error was found with both numerals and manipulatives and appears to reflect a transitional understanding: Children recognize that numeration units are related but fail to correctly and consistently interpret each level of units. Several cognitive factors could explain this factor. First, it may reflect limited cognitive flexibility, which is necessary to shift between different numeration units and has been shown to influence mathematical performance (e.g., Bakker et al., 2022). Second, working memory limitations could make it difficult for children to hold and coordinate multiple levels of numeration units at the same time. This is particularly true when the concept is not yet mastered, leading to an increased cognitive load (Sweller, 1994). Finally, the switching error is consistent with developmental perspectives that described learning trajectories as non-linear, often characterized by regressions and overlapping waves rather than development occurring in separate and distinct "stages" (e.g., Siegler, 2016). Together, the presence of these two distinct errors contributed to a more nuanced understanding of how unitizing develops in second-grade students. The findings also highlight the potential negative impact of overemphasizing the principle of position in early instruction and the need to support children in developing a more flexible and relational understanding of base-ten structure.

Finally, Study 3 highlighted that second-grade students can benefit from instruction on relational reasoning to enhance their unitizing performance. This finding is important, as it

suggests that children's difficulties are likely not because of a lack of developmental readiness, but rather from insufficient or untargeted instruction in this domain. These findings are inconsistent with those reported by Fletcher and Pine (2009), who found that even after instruction designed to overcome children's tendency to focus on counting discrete items, 4 and 5-year-old children persisted in their strategy of counting discrete items rather than groups of items. The discrepancy between their results and the current findings might stem from the fact that, in the present dissertation, unitizing was embedded within base-ten contexts. Because base-ten numeration is usually introduced through formal instruction in school, the implications of the present findings for mechanisms of unitizing development differ from those relevant to early numeracy in younger populations.

Educational Implications

Several important educational implications emerge from the three studies presented in this dissertation. First, the findings highlight the need for targeted instruction on base-ten structure starting in the early grades, particularly when the focus is on unitizing. Teaching only the principle of position instead of, or as a proxy for, relational knowledge of base ten may foster rigid reasoning about quantities and hinder the development of structural understanding. Instruction should not only emphasize the syntax of numerals but also promote an understanding of the hierarchical structure of base-ten units. Unitizing might be a productive basis on which to design and deliver such instruction.

The findings support the need to refine current mathematics curricula in the province of Quebec (Programme de formation de l'école québécoise, Ministère de l'Éducation, du Loisir et du Sport, 2006) to include specific and clear learning objectives related to base-ten structure. The results of Study 3 further demonstrate that when instruction focuses specifically on the relations between numeration units, students improve their understanding of base-ten structure, even within just two sessions. This suggests that an incomplete conceptual understanding of base-ten units could likely be a result of instruction that does not adequately expose children to unit concepts.

Second, both Study 2 and Study 3 offer promising directions for future pedagogical practices. Study 2 showed that measurement division problems, when paired with manipulatives, can support students' reasoning about base-ten structure. However, the types of manipulatives used appeared to have different affordances for learning. First, manipulatives that make the "one-

ness” of each composite unit perceptually salient, meaning that the group is easily seen as a single entity (e.g., a stick representing a group of 10 items, rather than 10 separate items grouped together), appear to be beneficial for students’ unitizing. Second, allowing children to physically construct the composite units with manipulatives themselves also supports their ability to count composite units. Teachers should therefore not only provide measurement division word problems and manipulatives, but also actively draw children’s attention to the structural similarities between non-symbolic and symbolic contexts involving numerals. Study 3 demonstrated that analogy-based instruction can enhance students’ unitizing performance when drawing attention to the underlying structural similarities between measurement division word problems and base-ten numerals. Teachers are therefore encouraged to use analogies to create meaningful bridges between non-symbolic and symbolic contexts, especially when unitizing is the instructional objective, and to offer repeated opportunities for children to reason flexibly about base-ten quantities.

Taken together, these findings emphasize that with targeted instruction and appropriate pedagogical tools, children can enhance their understanding of base-ten structure. Similar to the strong predictive power of early numeracy skills on children’s later mathematics achievement (e.g., Jordan et al., 2007; Jordan et al., 2009; Mazzocco & Thompson, 2005), supporting unitizing in early grades promises to lay a critical foundation for later mathematical concepts, including arithmetic, multiplicative reasoning, and rational numbers.

Limitations and Future Directions

The present studies present several limitations that should be considered in future research. First, this dissertation focused exclusively on unitizing as a key indicator of children’s understanding of base-ten structure. As conceptualized in the literature (Fosnot & Dolk, 2001; Lamon, 1996; Simon, 2024), unitizing is the cognitive process of interpreting the same quantity using two or more units (e.g., 345 can be seen as both 345 ones and 34.5 tens). Adhering to this characterization, unitizing was assessed using two different tasks in this dissertation. The first task, inspired by Tempier (2016), entailed decomposing a numeral in terms of a specific numeration unit (e.g., finding the total number of tens in 268). The second task, inspired by Ladel and Kortenkamp (2016), entailed composing a numeral from a decomposed non-canonical equation (e.g., determining the numeral corresponding to the expression 2 hundreds + 35 tens + 16 ones). These tasks were limited to assessing unitizing, which is only one component of base-

ten structure knowledge. To capture children's understanding of base-ten structure more broadly, other types of tasks should be considered. For example, Soury-Lavergne et al. (2020) assessed children's ability to place a numeral associated with a specific numeration unit into a base-ten chart and to find the equivalent quantity expressed in ones (e.g., placing 45 hundreds in a base-ten chart and determining that it corresponds to 4500 ones). Similarly, Tempier (2024) developed a comparison task in which two representations of quantities were presented. These quantities were expressed using base-ten blocks (e.g., two sticks and 14 cubes), numerals (e.g., 34), or numerals combined with a specific numeration unit (e.g., 13 tens). Students were asked to compare the two quantities, regardless of their format, and to identify which was larger. The task required children to flexibly interpret each quantity using a common unit. For instance, when comparing two hundreds to 14 tens, children might convert both into ones (i.e., 200 vs. 140), convert the hundreds into tens (i.e., 20 tens vs. 14 tens), or the tens into hundreds (i.e., 2 hundreds vs. 1 hundred). The strategies used in such tasks might provide valuable insights into how children reason about quantities. Extending the range of components that make up structural knowledge of base ten, as done by Mix et al. (2022) when investigating base-ten syntax, would allow for a more comprehensive understanding of children's knowledge and how the components are related to one another. Such examinations might result in a testable model of base-ten unitizing, potentially uncovering developmental pathways.

Second, the three studies in this dissertation focused exclusively on second-grade students. According to the mathematics curriculum in the province of Quebec (Programme de formation de l'école québécoise, Ministère de l'Éducation, du Loisir et du Sport, 2006), instruction on base-ten numeration begins in the first grade. However, some studies have shown that even kindergarten students display an emerging sense of base-ten numeration before receiving any formal instruction (Byrge et al., 2014; Mix et al., 2023). Outside the context of base ten, Sophian (2004) argued that young children should be given the opportunity to unitize quantities by prompting them to count not only individual entities, but also groups when presented with manipulatives. Such opportunities might foster an early understanding of units and support the development of later mathematical competencies, including a structural understanding of base-ten numeration. Future research should explore whether familiarizing students with units, and more specifically with base-ten units, at an earlier stage could enhance their later understanding of base ten.

Finally, the instructional intervention conducted in Study 3 represents only one of many possible approaches to supporting children's learning of base-ten structure. Various classroom tools could be explored to further extend and reinforce structure understanding. For instance, Houdement and Tempier (2022) found that base-ten charts are commonly used to teach base-ten numeration, but children often infer (or are taught) a rule from these charts: Only one digit can be placed in each column corresponding to a numeration unit. This conception might hinder their ability to recognize the relations between numeration units. Designing and testing instruction ~~protocol~~ in which the presentation and use of base-ten charts explicitly highlight these relations could offer simple, yet effective, shifts in instructional practice. Similarly, base-ten blocks are widely used in classrooms to teach base-ten numeration (Moyer-Packenham et al., 2013). Yet, their use often remains limited to illustrating the principle of position without explicitly drawing attention to the embedded system of base-ten units. Future studies could investigate how base-ten blocks, and other materials commonly used in elementary classrooms, might be used more effectively to support structural understanding of base ten. For example, such studies could allow children to actively construct higher-level numeration units and examine how these opportunities might foster flexible reasoning about numbers. This approach would closely align with current classroom practices, potentially making its implementation more feasible for teachers.

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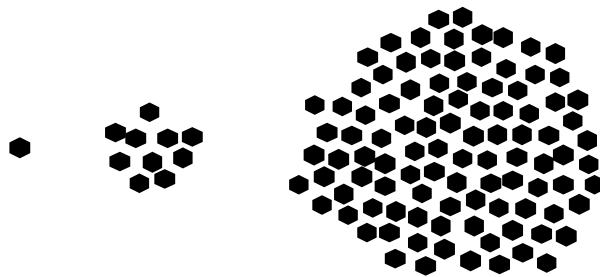
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Appendix A
PicPVT Version 1

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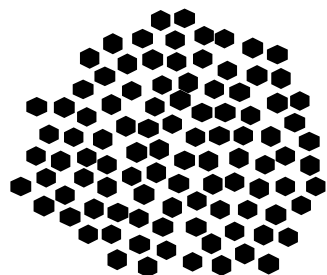
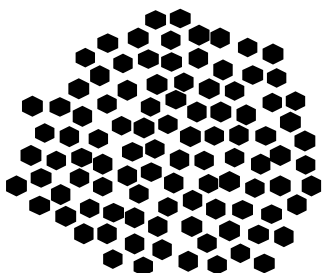
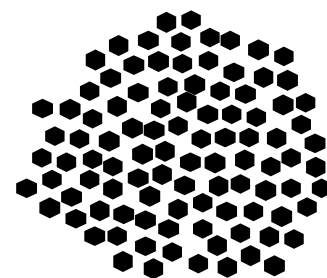
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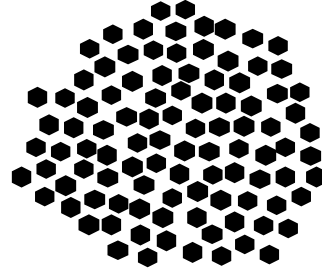
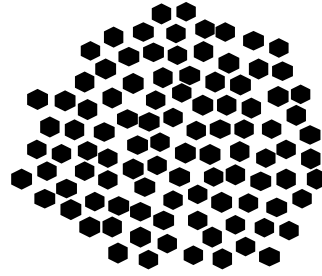
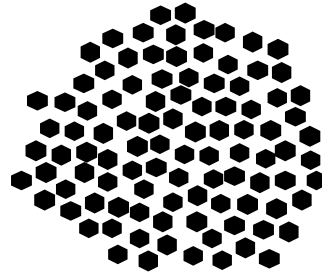
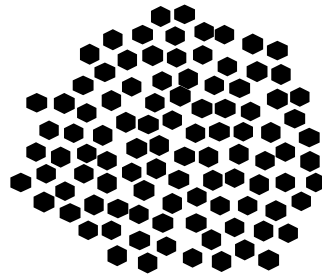
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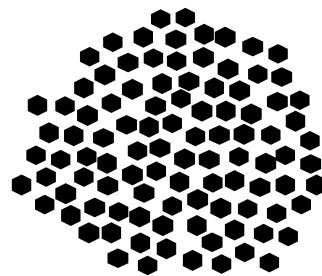
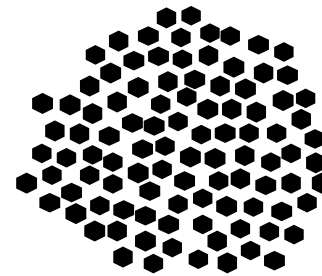
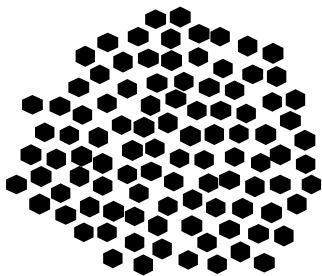
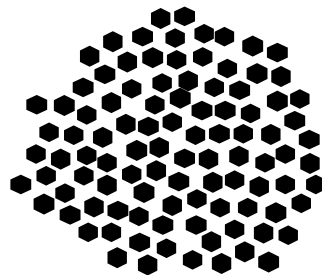
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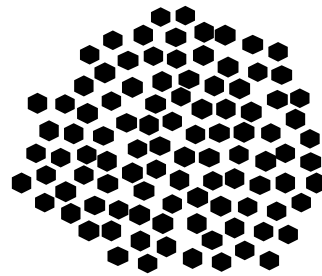
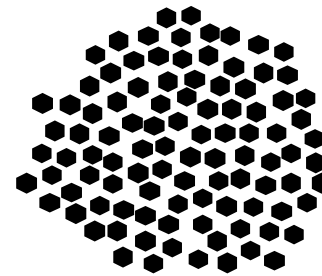
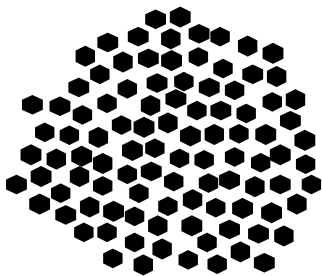
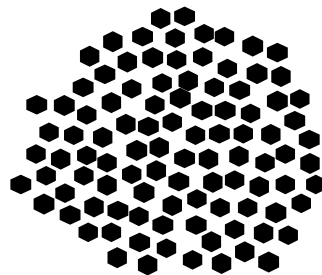
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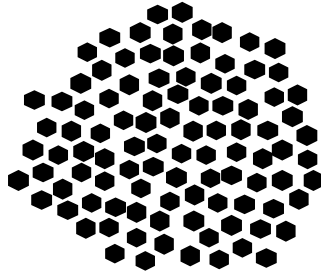
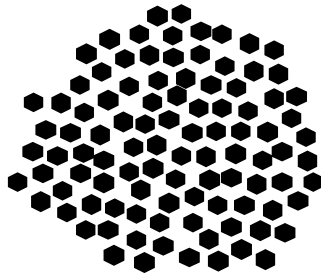
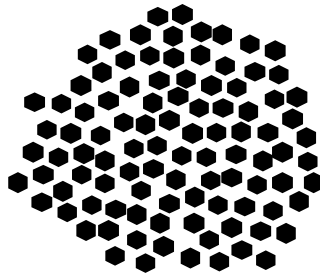
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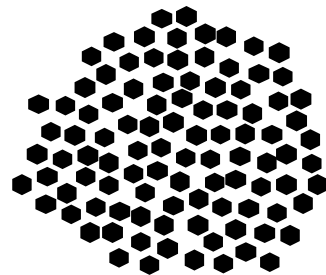
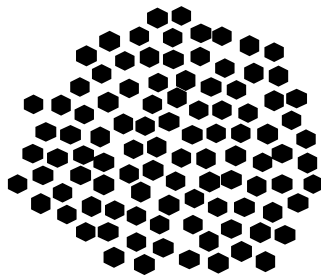
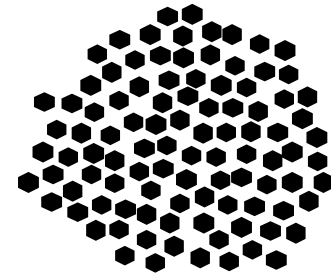
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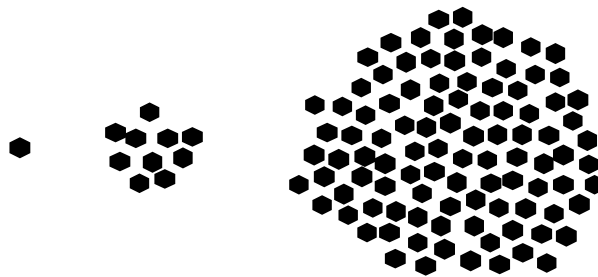


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Appendix B
PicPVT Version 2

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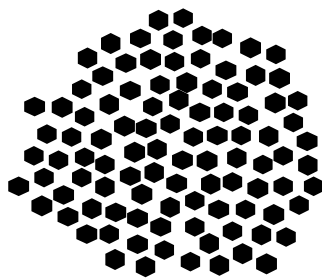
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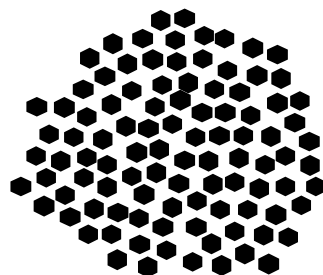
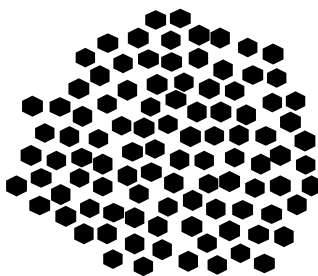


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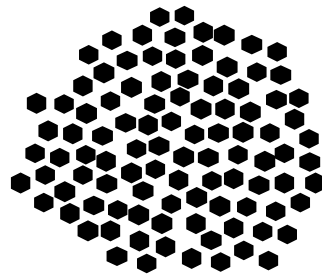
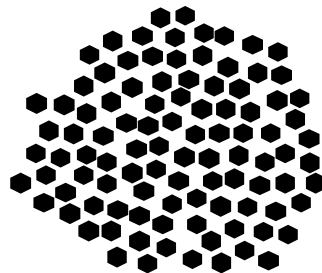
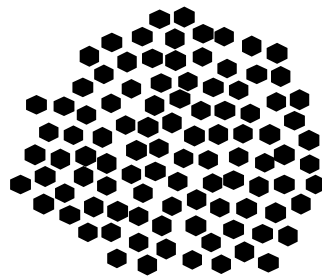
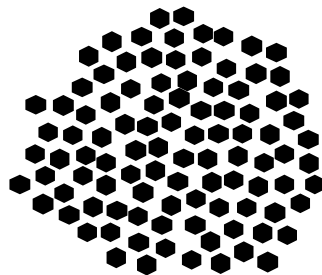


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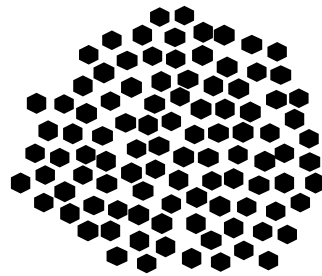
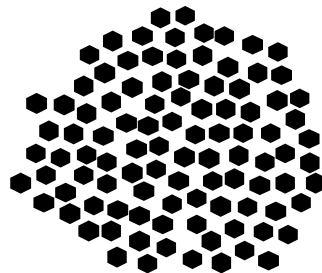
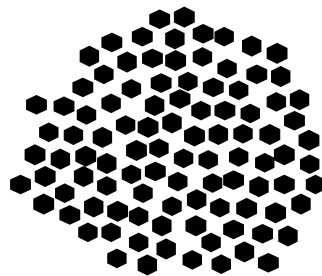
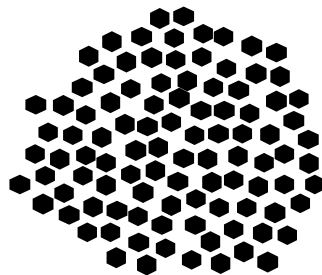
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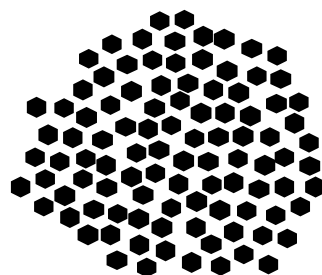
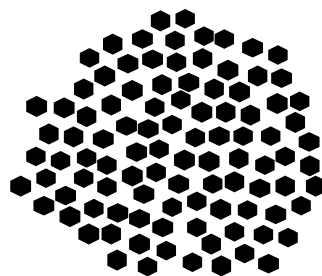
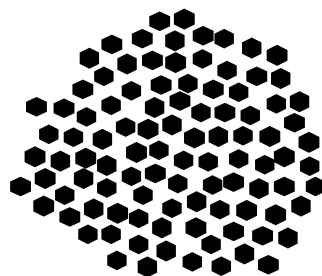
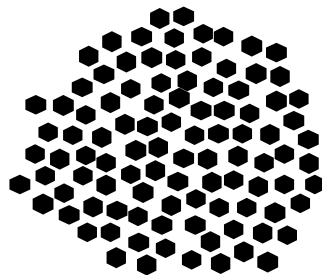
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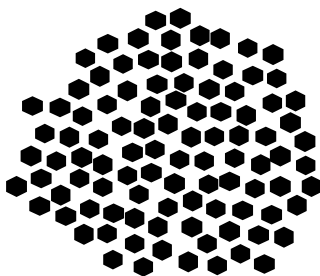
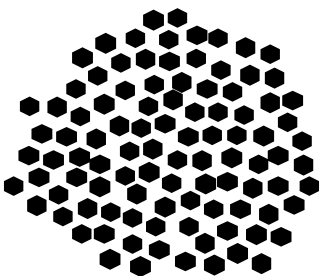
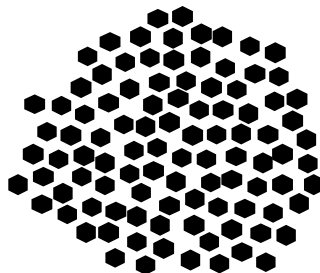
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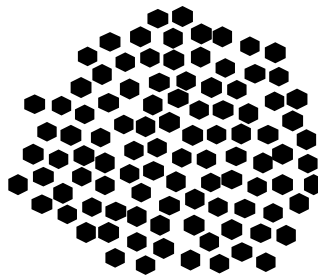


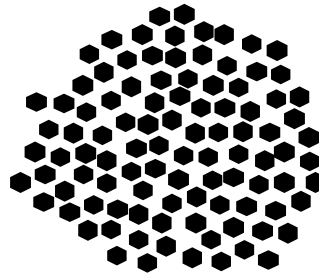
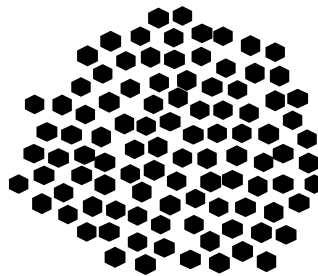
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Appendix C

Unitizing Task – Version 1

Items used in Version 1 of the Unitizing task (called Unitizing-Decomposing task in Study 3)

24	32
27	49

57

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253

238

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462

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645

Appendix D

Unitizing Task – Version 2

Items used in Version 2 of the Unitizing task (called Unitizing-Decomposing task in Study 3)

29	25
35	47

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Appendix E

French Version of the Word Problems Used in Study 2

One-Level Problems	Two-Level Problems
<i>MD (30,10) : bonbons, sachets</i> Hugo donne 30 bonbons. Il met tous ces bonbons dans des sachets. Un sachet contient 10 bonbons. Combien de sachets complètement remplis Hugo a-t-il ? Combien de bonbons Hugo a-t-il ?	<i>MD (240,10,10) : bonbons, sachets, boîtes</i> Hugo donne 240 bonbons. Il met tous ses bonbons dans des sachets et met ensuite les sachets dans des boîtes. Un sachet contient 10 bonbons et une boîte contient 10 sachets. Combien de sachets complètement remplis Hugo a-t-il ? Combien de boîtes complètement remplies Hugo a-t-il ? Combien de bonbons Hugo a-t-il ?
<i>MD (50,10) : poissons, filets</i> Alexane attrape 50 poissons. Elle met tous ses poissons dans des filets. Un filet contient 10 poissons. Combien de filets complètement remplis Alexane a-t-elle ? Combien de poissons Alexane a-t-elle ?	<i>MD (462,10,10) : poissons, filets, boîtes</i> Manon attrape 462 poissons. Elle met tous ses poissons dans des filets et met ensuite tous les filets dans des boîtes. Un filet contient 10 poissons et une boîte contient 10 filets. Combien de filets complètement remplis Manon a-t-elle ? Combien de boîtes complètement remplies Manon a-t-elle ? Combien de poissons Manon a-t-elle ?
<i>MD (47,10) : coquillages, boîtes</i> Léo trouve 47 coquillages. Il met tous ses coquillages dans des boîtes. Une boîte contient 10 coquillages. Combien de boîtes complètement remplies Léo a-t-il ? Combien de coquillages Léo a-t-il ?	<i>MD (327,10,10) : bonbons, sachets, boîtes</i> Léo donne 327 bonbons. Il met tous ses bonbons dans des sachets et met ensuite tous les sachets dans des boîtes. Un sachet contient 10 bonbons et une boîte contient 10 sachets. Combien de sachets complètement remplis Léo a-t-il ? Combien de boîtes complètement remplies Léo a-t-il ? Combien de bonbons Léo a-t-il ?

MD (24,10) : pommes, paniers

Manon cueille 24 pommes. Elle met toutes
ses pommes dans des paniers. Un panier
contient 10 pommes. Combien de paniers
complètement remplis Manon a-t-elle ?
Combien de pommes Manon a-t-elle ?

Appendix F

English Transcripts for Study 2

Transcript 1

Researcher (R) : *So how many boxes are completely full?*

Susie : *Four*

R : *How do you know? Can you explain?*

Susie : *There are four here* (Susie gestured over the four groups of 10 counters created on the table)

Transcript 2

R : *So, I want to know, with the 50 fish, how many nets can I make?*

Max : *Nets I can make... I can make 5.*

R : *How do you know?*

Max : *Because there are 50, and then here, there are four* (Max grabbed the five sticks in her hand and showed them to me), *and then plus 5, well plus 1 makes 5. So there are 5.*

Transcript 3

R : *So now, can you tell me how many baskets we can make?*

Jordan : *10*

R : *How do you know? Can you explain?*

Jordan : *Here* (Jordan showed the sentence in the problem text where it is indicated that one basket contains 10 apples)

Transcript 4

R : *So, what I want to know is how many nets she can make.*

Lily : *50. 50 fish in 50 nets*

R : *There are 50 nets?*

Lily : *Well, we can put 50 fish into 10 ne- uh...*

Transcript 5

R : *One box contains 10 seashells. So, if I put 10 seashells together, I could fill...*

June : *A box of seashells.*

R : *A box, yes, I could fill on box. So, here, how many boxes do I have?* (I gestured over one group containing 10 cubes on the table)

June : *10*

R : *So, you have 10 seashells, yes, but how many boxes do you have? How many boxes can you fill with 10 seashells?*

June : *10*

[Because June struggled to see the difference between counting the group and counting the individual cubes, I gave her little bags and asked her to create the groups by putting 10 cubes in each bag. She was then able to create the composite units using four bags].

R : *So, how many boxes did we fill?*
June : (June takes each bag in their hand and counts each bag) *Four*
R : *Yes, how did you count?*
June : *Because I did this* (June points to the four bags, one by one), *there are four.*

Transcript 6

Sadie : *I have 462.*
R : *Okay, so, you've found the quantity, but now what I really want to know is how many nets there are in total.*
Sadie : *Wait!*
R : *Take your time, don't worry. Think about it.*
Sadie : *First, there are 40 tens* (she counted all the sticks included in the four flats one by one). *46 full nets.*

Transcript 7

R : *So, can you find how many bags he made in total?*
Will : *2* (Will grabbed the two sticks alone on the table and showed them to me). *Because here there are two tens, and then there 10 inside, in one bag, so there definitely two full ones.*

Transcript 8

R : *Okay, so we're trying to find how many bags we can make?*
Susie : *3*
R : *How do you know there are 3?*
Susie : *Those are hundreds, and there are 3* (Susie grabbed the three flats and showed them).

Transcript 9

R : *So, how many bags could we make?*
Jasper : *5* (Jasper counted the two sticks and the three flats each time as one thing, one ten)

[I helped Jasper see that in a flat there are 10 sticks, 10 bags]

R : *So altogether, how many bags does that make?* (I pointed at the three flats)
Jasper : *30*
R : *30 plus the bags we have on the side* (I pointed at the two sticks)
Jasper : *50*

Transcript 10

R : *How many full does Leo have?*
Dustin : *Three*
R : *How do you know?*
Dustin : *Because there are three boxes* (Dustin pointed at the three flats placed on the table)

Transcript 11

R : *How many full boxes does Hugo have? And remember, one box contains ten bags, so how many boxes can he make?*

Taylor : *Uh...wait.*

R : *In one box, there are 10 bags*

Taylor : *this is 100, 200* (Taylor grabbed two flats and showed them). *Wait, this is 10* (Taylor grabbed one flat and showed it).

R : *This is 10 what?*

Taylor : *Tens*

R : *10 bags*

Taylor : *10 bags*

R : *Okay*

Taylor : *this is 10 bags* (Taylor pointed at one flat)

R : *so how many boxes is that?*

Taylor : *this is 20 bags* (Taylor pointed at a second flat)

R : *so how many boxes is that?*

Taylor : *uh... 200 boxes*

R : *this is 10 bags* (I pointed at one flat)

Taylor : *yeah*

R : *10 bags, 10 bags together, how many boxes does that make?*

Taylor : *100*

Transcript 12

R : *This time, I'm asking you how many full boxes Leo has?*

Jasper : *So I need to count, but like...those are hundreds* (Jasper pointed at the three flats)

R : *Do you remember what goes inside one box?*

Jasper : *Ten*

R : *Ten what?*

Jasper : *Candies*

R : *So those are bags. In one bag, there are 10 candies. But in one box, there are 10 what?*

Jasper : *Ten boxes, uh, 10 bags.*

R : *A box contains 10 bags, great. So how could we figure out how many boxes we have?*

Jasper : *Uh...*

R : *Remember, how many bags do you have in total?*

Jasper : *32*

R : *You have 32 bags. So, if we think about it, we have 32 bags, and we need to make groups of how many?*

Jasper : *Of 10*

R : *Groups of 10 bags. So how many groups of ten can we make with 32?*

Jasper : *Uh...*

R : *Do you see it, or not really?*

Jasper : *What?*

R : *Do you see how to do it?* (Jasper shook his head)

[I helped Jasper exchange the flats for sticks. The three flats were replaced by 30 sticks, resulting in a total of 32 sticks in front of him.]

R : *Okay, now I want you to tell me, with these 32 bags, how many boxes you can make? Et tu peux faire les boîtes si tu as besoin.*

Jasper : *So, we need to take 10 things, 10 tens* (Jasper counted 10 sticks and placed 10 sticks together in one group on the table. Jasper continued to create two other groups of 10 sticks on the table)

R : *So, how many boxes were you able to make?*

Jasper : *Three.*

Transcript 13

R : *Can you tell me how many candies Hugo has in total?*

Morgan : *30*

R : *How do you know?*

Morgan : *10* (Morgan is pointing at a first stick), *20* (Morgan is pointing at a second stick), *30* (Morgan is pointing at a third stick).

Transcript 14

R : *So how many seashells does Leo have in total?*

Dustin : *40*

R : *Only 40? Really? And the small seashells that were left on their own, that didn't fit into boxes, we don't count those anymore?*

Dustin : *No*

Transcript 15

R : *And now, can you tell me how many fish Manon has in total?*

Riley : *400*

R : *Only 400?*

Riley : *Maybe there are more?*

Transcript 16

R : *How many apples does Manon have in total?*

Jasper : *4*

R : *She has four?*

Jasper : *In the basket?*

R : *In total, how many apples does she have?*

Jasper : *24*

Transcript 17

R : *How many candies does Leo have in total? How many candies altogether?*

Jasper : *uh... 32*

R : *32 candies?*

Jasper : *2?*

Transcript 18

R : *What is this?* (I am pointing at one stick)

Harper : *A ten*

R : *Okay, and what is it in our story?*

Harper : *That's a ten, and we know it's 10*

R : *10 what?*

Harper : *10 units stacked together*

R : *Yes, it's 10 units, but in our story, it's 10 what?*

Transcript 19

R : *So, how many boxes are completely full?*

Susie : *Four.*

R : *How do you know? Can you explain?*

Susie : *There are four here* (Susie gestured over the four groups created on the table) *There is 10, 10, 10, 10, that makes 40* (Susie pointed at each group created on the table).

Transcript 20

Sadie : *That makes 32*

R : *32 what?*

Sadie : *32 bags*

R : *How do you know?*

Sadie : *because this is... Since 10 tens is 100, then 20 tens is 200, then 30 tens is 300, and then those are the tens, and there two, so 32.*

Transcript 21

R : *For now, let's put the boxes aside and focus only on the bags. And remember, one bag contains 10 candies. And altogether, we have 240 candies.*

Max : *So that means there will two hundreds*

R : *Maybe*

Max : *There will be 2* (Max took two flats from a pile of flats)

R : *Do you need some help?*

Max : *Oh, right, there are two hundreds. And then there are four units, so we take some away* (Max removed three cubes because she had seven cubes in front of her). *We only leave four units.*

R : *There are four units?*

Max : *There are four units, and then there are zero... Well, there are 24 tens, and there are...*

R : *There are 24 tens?*

Max : *Yeah, and then there are two units.*

R : *So, what is this?* (I pointed at the initial quantity 240 written in the problem) *240?*

Max : *Year, but there are two tens.*

R : *You mean two hundreds?*

Max : *Yes, and 24 tens, so that means there isn't any of that* (Max is removing the four cubes left in front of her), *because there are zero.*

Appendix G
Unitizing-Composing – Version 1

Écris un nombre à droite qui correspond à ce qui est écrit à gauche :

- 3 dizaines + 6 unités = ...
- 2 dizaines + 19 unités = ...
- 2 centaines + 27 dizaines + 3 unités = ...
- 2 centaines + 4 dizaines + 3 unités = ...
- 3 centaines + 12 dizaines + 6 unités = ...
- 4 dizaines + 5 unités = ...
- 3 centaines + 23 dizaines + 17 unités = ...
- 2 centaines + 14 dizaines + 29 unités = ...
- 1 centaine + 5 dizaines + 7 unités = ...
- 0 dizaine + 47 unités = ...
- 2 dizaines + 3 unités = ...
- 3 centaines + 9 dizaines + 2 unités = ...
- 0 dizaine + 53 unités = ...
- 1 centaine + 35 dizaines + 32 unités = ...
- 2 centaines + 36 dizaines + 4 unités = ...
- 3 dizaines + 31 unités = ...

Appendix H

Unitizing-Composing – Version 2

Écris un nombre à droite qui correspond à ce qui est écrit à gauche :

- 3 dizaines + 1 unité = ...
- 3 dizaines + 18 unités = ...
- 2 centaines + 19 dizaines + 5 unités = ...
- 2 centaines + 3 dizaines + 7 unités = ...
- 3 centaines + 24 dizaines + 2 unités = ...
- 4 dizaines + 7 unités = ...
- 3 centaines + 17 dizaines + 25 unités = ...
- 2 centaines + 36 dizaines + 18 unités = ...
- 1 centaine + 4 dizaines + 2 unités = ...
- 0 dizaine + 39 unités = ...
- 2 dizaines + 5 unités = ...
- 3 centaines + 5 dizaines + 8 unités = ...
- 0 dizaine + 62 unités = ...
- 1 centaine + 24 dizaines + 31 unités = ...
- 2 centaines + 34 dizaines + 8 unités = ...
- 2 dizaines + 34 unités = ...

Appendix I

Story and Activities for the First Session of the Control Condition

Élisabeth et le Monde Magique

Dans un petit village situé au cœur d'une forêt dense, vit une jeune fille nommée Élisabeth. À huit ans, elle adore écouter les récits des anciens du village : des histoires de créatures magiques et de mondes enchanteurs. Curieuse et à la recherche d'aventures, elle passe ses journées à explorer les bois. Un jour, lors d'une promenade, Élisabeth découvre une clairière secrète entourée d'arbres majestueux. Au centre, une vieille porte en bois se dresse, recouverte de mystérieuses écritures. Intriguée, elle pose la main sur la poignée et, avec précaution, ouvre la porte. À sa grande surprise, elle entre dans un monde féérique : des arbres immenses s'élèvent vers le ciel, leurs branches étincellent de fils d'or et d'argent, et des fleurs lumineuses illuminent le sol.

Émerveillée, Élisabeth se retourne en entendant une voix douce et mélodieuse : « Bienvenue, Élisabeth, dans le monde de la magie et des légendes. » Une fée élégante apparaît devant elle. « Je suis la fée Morgane, gardienne de ce royaume. Seuls les cœurs purs peuvent franchir cette porte. Mais le royaume est en danger. Une sorcière maléfique, appelée l'Ombre, plonge notre monde dans les ténèbres. Toi seule peux nous aider à rétablir la lumière. » La fée explique qu'Élisabeth doit retrouver la Pierre de Lumière, un objet magique capable de détruire les ténèbres. Pour cela, elle doit traverser des régions mystérieuses et relever des épreuves. Élisabeth accepte la quête.

Elle commence son voyage à travers des forêts sombres, des rivières étincelantes et des collines escarpées. Sur son chemin, elle rencontre des créatures fantastiques : des elfes, des nains et des licornes. Chacune lui donne une mission : résoudre une énigme, affronter un défi ou prouver son courage. Ces rencontres lui enseignent la patience, la ruse et la persévérance. Après un long voyage, Élisabeth atteint une montagne. Au sommet se trouve une grotte mystérieuse. C'est là qu'elle découvre la Pierre de Lumière, brillant d'une lumière éclatante. Remplie de détermination, elle redescend pour affronter l'Ombre.

Dans le sombre royaume de la sorcière, les arbres sont morts et les fleurs fanées. L'Ombre, une figure terrifiante, apparaît dans un nuage de fumée noire. La sorcière lance de puissants sorts, mais Élisabeth, armée de la Pierre de Lumière, repousse chaque attaque. Avec courage, elle libère une énergie intense qui dissipe les ténèbres et fait disparaître l'Ombre. Le monde magique retrouve sa splendeur. Les arbres reverdissent, les rivières scintillent à nouveau, et les créatures célèbrent la victoire d'Élisabeth. Morgane, la fée, la remercie et lui offre un pendentif en forme de fée.

Élisabeth traverse la porte et se retrouve dans la clairière, le cœur rempli de souvenirs magiques. De retour au village, elle raconte son aventure. Les habitants sont émerveillés, et les enfants commencent à croire que les contes de fées peuvent être réels. Élisabeth continue aujourd'hui de vivre des aventures extraordinaires, d'explorer d'autres mondes et de prouver que la magie existe pour ceux qui ont l'envie d'y croire.

Élisabeth et le Monde Magique

Activité n°1 :

Relie chacun des mots suivants à la bonne définition.

Une forêt dense

- Une forêt qui se situe dans la montagne
- Une forêt qui a beaucoup d'arbres
- Une forêt qui est magique

Une clairière

- Une créature magique
- Une grande étendue d'herbe
- Une maison

Avec précaution

- Sans réfléchir
- Brutalement
- En faisant attention

Une quête

- Une mission
- Une fleur
- Une maison

Un royaume

- Une école pour apprendre la magie
- Une pièce remplie de trésors
- Un endroit dirigé par un roi ou une reine

Un pendentif

- Une créature magique qui brille
- Un instrument pour chanter
- Un bijou que l'on met autour du cou

Activité n°2 :

Réponds à chacune des questions suivantes en te basant sur le texte :

1. Quel est le nom de l'enfant qui découvre le monde magique ?

2. Quelles sont les créatures que la jeune fille rencontre durant sa quête ?

3. Qui est l'Ombre ?

4. Comment s'appelle l'objet qu'Elisabeth doit trouver pour affronter l'Ombre ?

5. Quel objet, la fée Morgane, donne-t-elle à Elisabeth pour la remercier ?

Activité n°3 :

Colle les phrases suivantes dans l'ordre sur ta feuille

Elisabeth rencontre plein de créatures magiques et fantastiques, comme des elfes, des nains et des licornes.
Elisabeth découvre une clairière magique.
La fée Morgane annonce à Elisabeth qu'elle va devoir sauver le royaume d'une sorcière maléfique.
Elisabeth trouve la Pierre de Lumière dans une grotte cachée.
Elisabeth affronte l'Ombre et parvient à la faire disparaître grâce à la Pierre de Lumière.
Le monde de la magie et des légendes retrouve toutes ses couleurs et Morgane donne un pendentif en forme de fée à Elisabeth.

Appendix J

Story and Activities for the Second Session of the Control Condition

Léa et le Trappeur

Il était une fois, dans un petit village au pied d'une montagne enneigée, une jeune femme nommée Léa. Léa est une exploratrice, qui aime découvrir de nouvelles choses et relever des défis. Un jour, alors qu'elle contemple la beauté de la montagne depuis le village, un messenger vient à sa rencontre. « Chère Léa, j'ai besoin de ton aide. Mon ami le trappeur est coincé en haut de la montagne, blessé. Il a besoin d'aide pour redescendre en sécurité. Peux-tu lui porter secours ? » demande le messenger, visiblement inquiet. Sans hésiter, Léa accepte la mission. Elle sait que l'ascension de la montagne sera difficile, mais elle est prête à affronter tous les dangers pour sauver l'ami du messenger.

Accompagnée de son fidèle chien Lupo, Léa entreprend l'ascension de la montagne enneigée. Le vent souffle fort, tourbillonnant des flocons de neige autour d'eux. La jeune femme avance avec prudence, utilisant son bâton pour se frayer un chemin à travers la neige épaisse.

En chemin, elle rencontre un compagnon de route inattendu, un renard malicieux nommé Rigel. Malgré sa méfiance initiale, Léa comprend que Rigel peut lui être utile dans cette expédition. Le renard connaît bien la montagne et peut flairer les dangers avant qu'ils n'arrivent. Alors que le groupe progresse, des ennemis surgissent, comme une meute de loups affamés. Léa fait preuve de courage et d'intelligence pour les éloigner, utilisant des ruses apprises au fil de ses explorations passées.

Lorsqu'ils atteignent enfin le sommet, Léa et ses compagnons découvrent le trappeur blessé, gelé et affaibli. Sans perdre une seconde, Léa met en pratique ses connaissances en premiers secours pour le soigner. Grâce à sa volonté, le trappeur est sauvé. Cependant, le plus grand défi reste à venir : redescendre de la montagne en toute sécurité. Les conditions météorologiques se détériorent, rendant le chemin glissant et dangereux. Léa et ses compagnons unissent leurs forces et restent solidaires pour surmonter les obstacles qui se dressent sur leur route.

Après maintes péripéties, Léa et ses compagnons parviennent enfin à regagner le village, sains et saufs. Le trappeur retrouve ses forces grâce aux bons soins de Léa, et le messenger est soulagé de retrouver son ami en vie. En repensant à cette aventure éprouvante, Léa réalise combien l'entraide, le courage et la volonté sont essentiels pour surmonter les obstacles de la vie. Elle apprend une précieuse leçon : même dans les moments les plus sombres, il suffit de croire en soi et en ses compagnons pour trouver la force d'avancer.

Léa et le Trappeur

Activité n°1 :

Relie chacun des mots suivants à la bonne définition.

Un trappeur

- Personne qui vend des fromages
- Personne qui piège et attrape des animaux
- Une forêt qui est magique

Une ascension

- Une escalade
- Une forêt
- Une montagne

Méfiance

- Un arbre
- Un doute
- Un saut

Une ruse

- Une araignée
- Une astuce utilisée pour tromper quelqu'un
- Une bande-dessinée

Des péripéties

- Des aventures
- Des décisions
- Du repos

L'entraide

- Faire fuir les loups
- Se faire des amis
- S'aider mutuellement

Activité n°2 :

Réponds à chacune des questions suivantes en te basant sur le texte :

1. Quels animaux accompagnent Léa dans son aventure ?

2. Pourquoi le trappeur est-il en danger sur la montagne ?

3. Quel est le nom du chient de Léa ?

4. Pourquoi la montagne est-elle difficile à escalader ?

5. Qu'est-ce que Léa utilise pour marcher dans la neige ?

Activité n°3 :

Colle les phrases suivantes dans l'ordre sur ta feuille.

Léa reçoit un message lui disant qu'un trappeur est blessé et coincé dans la montagne.
Léa emmène son chien afin de gravir la montagne et affronter les dangers pour sauver le trappeur.
Léa et Lupo rencontrent Rigel, un renard qui connaît la montagne comme personne et qui pourra les aider à éviter les dangers.
Plusieurs loups menacent Léa et ses amis, et Léa réussit à les faire fuir grâce à plusieurs ruses.
Une fois le trappeur sauvé, Léa et ses amis ont peur de ne pas pouvoir redescendre à cause de la météo.
Avec cette aventure, Léa comprend l'importance de se faire confiance et de compter sur ses amis pour surmonter les obstacles.