ON CERTAIN INFINITESIMAL HOMOTHETIC CONDITIONS IN AFFINE GEOMETRY

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Abstract. We prove new results in which the homothety between a smooth convex body with positive Gauss curvature and certain small parameter deformation of it implies that the convex body is an ellipsoid. We also introduce a new convex body determined by an affine invariant construction similar to that of the convex floating body.

1. Introduction

Many developments across several areas of mathematics rely on characterizations of balls and ellipsoids among convex sets in Euclidean d-dimensional space. While many such characterizations are known, numerous others remain conjectural and open despite sustained efforts. In this paper, we establish several *infinitesimal versions* of some conjectural characterizations of ellipsoids in within the class of smooth convex bodies with positive Gauss curvature. The meaning of infinitesimal will be clarified shortly. For now, it suffices to say that our characterizations depend on the smallness of a parameter $\delta > 0$, which measures, in a certain sense, how close the characterization criteria are to the boundary of a strictly convex smooth body. Thus, our focus is on how the geometry of the boundary influences the validity of these open problems. The characterizations remain open in the general settings not because the characterization criteria fails away from the boundary, but because the boundary-based techniques do not extend farther in the interior of the body.

The techniques follow an earlier approach initiated by the second author in [17], [18], and reconsidered, more recently, by the second author with collaborators in [4]. The paper consists of two main parts. In the first part, and next section, we show that, ellipsoids are the only smooth, strictly convex bodies of very small uniform density for which the floating body and the body whose boundary is the surface of centers are homothetic with respect to a point included in the interior of both. The floating body and the surface of centers are classical notions, [6], [7], [16], that attracted renewed attention in [3], [9], [14], to cite just a few. In the second part of the paper, we will introduce a new affine invariant construction for smooth convex bodies with strictly positive Gauss curvature and prove a couple of characterizations of ellipsoids related to it.

2. Floating bodies and Surfaces of Centers in Small Regime

A convex body is a compact convex set in $\mathbb{R}^d, d \geq 2$, with non-empty interior. We denote by V_d the usual Lebesgue measure. In this paper, we consider $K \subset \mathbb{R}^d$, $d \geq 2$ to be a $\mathcal{C}^\infty$ smooth, strictly convex body with positive Gauss curvature, containing the origin in its interior, and we consider $\delta > 0$ to be some fixed positive real number assumed to be very small. The assumption on the smoothness of the boundary can be reduced to a lower order of differentiability, $\mathcal{C}^4$, which will be clear from the technique employed in proving the results.

Let $\xi \in \mathbb{S}^{d-1}$ be a unit vector, and let

$$H_t^-(\xi) = \{x \in \mathbb{R}^d \mid x \cdot \xi \leq t\}$$

be the half-space of outer normal ξ whose boundary lies at distance t from the origin.

2.1. **The Convex Floating Body.** Recall that the convex floating body K_δ of K (see [6], [16]), if it exists, is defined as the envelope of half-spaces cutting a cap of volume δ from K, that is

$$K_\delta = \bigcap_{\xi \in \mathbb{S}^{d-1}} H^-_{t(\xi)}(\xi),$$

where $t(\xi)$ is such that

$$V_d(K \cap H^+_{t(\xi)}(\xi)) = \delta.$$

The convex floating body is the object of a homothety conjecture asserting that K homothetic to K_δ, for some fixed $\delta \in (0, V_d(K)/2)$, implies that K is an ellipsoid. The conjecture was posed by Schütt and Werner in [16] and they showed that if there exists a sequence of positive δ_n's converging to zero such that K is homothetic to K_{δ_n} for each term of the sequence, then K must be an ellipsoid. Later, in [17], Stancu proved that if δ is a small enough density, and the body is of class $\mathcal{C}^2$ with strictly positive Gauss curvature, then K is an ellipsoid. The restriction on the regularity was later removed by Werner and Ye in [19].

Throughout the paper, the homotheties between convex bodies $K_1, K_2 \subset \mathbb{R}^d$ are taken with respect to a fixed point $O \in \mathrm{int}(K_1) \subseteq \mathrm{int}(K_2)$. In other words, if we place O at the origin, then K_1 is homothetic to K_2 if and only if there exists $\lambda > 0$ such that $K_1 = \lambda K_2$.

Finally, let us mention two more recent partial results on the homothety conjecture for the convex floating body. It was shown in [2] that, in the plane, the homothety conjecture holds if K is centrally-symmetric and close to a Euclidean ball in the Banach-Mazur distance, but that the homothety conjecture is not true for planar convex bodies that are not centrally-symmetric. In its larger generality, the conjecture remains open for centrally-symmetric convex bodies.

The relation between the support functions of K and K_δ for small δ was derived in [17]. We may assume similarly to the method employed in [4] that there exists a linear transformation that does not affect the directions parallel to ξ in $\mathbb{R}^d$, so that the boundary of K is locally approximated by

$$x_d = -\frac{1}{2}\mathcal{K}^{1/(d-1)}(\xi)\sum_{i=1}^{d-1} x_i^2 + o(|x|^2). \tag{1}$$

This representation assumes that the point on ∂K with support plane of normal ξ is, momentarily, the origin. Then, the difference between the two support functions amounts to estimating the height of the slab $K \cap H^+_{t(\xi)}$ of volume δ, and the result is as follows:

$$h_{K_\delta}(\xi) = h_K(\xi) - c_d\,\mathcal{K}^{\frac{1}{d+1}}(\xi)\,\delta^{\frac{2}{d+1}} + o(\delta^{\frac{2}{d+1}}), \tag{2}$$

where $c_d := \left(2^{\frac{d-1}{d+1}}\omega_{d-1}^{\frac{2}{d+1}}\right)^{-1}$ is a constant depending on the dimension.

2.2. **The Surface of Centers.** The surface of centers, [7], is the locus of centers of mass of the caps cut off from K in defining the floating body

$$x_{C_\delta}(\xi) = \frac{1}{\delta}\int_{K\cap H^+_{t(\xi)}(\xi)} x\,dx.$$

This surface, also called surface of buoyancy, is of uttermost significance in studying the flotation of 3-dimensional objects in a liquid of constant density. In particular, if C_δ is a sphere, then K floats in equilibrium in every direction. The tangent plane to C_δ at $x_{C_\delta}(\xi)$ has normal ξ and is, thus, parallel to the hyperplane bounding the half-space $H^-_{t(\xi)}(\xi)$. There are many other properties of this surface that connect both with the practical aspect of the problem and with many geometric questions, see, for example, [14] for a recent result related to Ulam's 19th problem in the Scottish book and [3] for a survey touching on many related questions.

Furthermore, as the surface of centers is known to be the locus of the centers of mass of the slices $K \cap H^+_{t(\xi)}(\xi)$ of constant volume, the smallness of δ enables us to derive the relation between the support functions of K and C_δ up to some small error term.

The fact that the center of mass of the cap is also the point of support of C_δ for the hyperplane of outer normal ξ, implies that solely the x_d component, in the sense of the previous approximation, of the center of mass is relevant. For simplicity, in what follows, we will be using C_δ to also denote the convex body whose boundary is the surface of centers. We thus have the following estimate on the support function of C_δ:

$$\begin{aligned}
h_{C_\delta}(\xi) &= \frac{1}{\delta}\int_{h_{K_\delta}(\xi)}^{h_K(\xi)} t\, V_{d-1}\left(K\cap H_t(\xi)\right)\, dt \\
&= \frac{\omega_{d-1}}{\delta}\int_{h_{K_\delta}(\xi)}^{h_K(\xi)} \left[t\left[\frac{2\left(h_K(\xi)-t\right)}{\sqrt[d-1]{\mathcal{K}(\xi)}}\right]^{\frac{d-1}{2}} + o\left(\left(h_K(\xi)-t\right)^{\frac{d+1}{2}}\right)\right] dt \\
&= h_K(\xi) - \frac{\omega_{d-1}}{\delta}\left[\frac{2^{\frac{d-1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\int_{h_{K_\delta}(\xi)}^{h_K(\xi)} \left(h_K(\xi)-t\right)^{\frac{d+1}{2}} dt + o\left(\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\right)\right] \\
&= h_K(\xi) - \frac{\omega_{d-1}}{\delta}\left[\frac{2^{\frac{d-1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\left[\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\cdot\frac{2}{d+3}\right] + o\left(\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\right)\right] \\
&= h_K(\xi) - \frac{1}{\delta}\frac{c_d^{-\frac{d+1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\left[\left(c_d\,\mathcal{K}^{\frac{1}{d+1}}(\xi)\,\delta^{\frac{2}{d+1}}\right)^{\frac{d+3}{2}}\cdot\frac{2}{d+3}\right] + o\left(\delta^{\frac{2}{d+1}}\right).
\end{aligned}$$

Rewriting the last expression, we have

$$h_{C_\delta}(\xi) = h_K(\xi) - \frac{2}{d+3}\,c_d\,\mathcal{K}^{\frac{1}{d+1}}(\xi)\,\delta^{\frac{2}{d+1}} + o\left(\delta^{\frac{2}{d+1}}\right), \tag{3}$$

where c_d is the same constant as in the description of the support function of K_δ relative to the support function of K, similarly deduced in [17].

Proposition 2.1. *Let $K\subset\mathbb{R}^d$, $d\geq 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature. If, for some small $\delta>0$, we have that the convex body of boundary C_δ is homothetic to the floating body K_δ with respect to a point in the interior of the domain bounded by C_δ, then K is an ellipsoid.*

This answers, in all dimensions, but in the small δ-regime, a question posed by Ryabogin. We note that Zawalski in [20] has solved a planar case of this question under an additional condition that any three consecutive chords tangent to the floating body close up to form a triangle.

Proof. In order to describe the homothety factor λ such that $K_\delta=\lambda C_\delta$, we will proceed as in [4] and use polar bodies, for which, $K_\delta^\circ=\frac{1}{\lambda}C_\delta^\circ$, and thus

$$\lambda^{-d} = \frac{V_d(K_\delta^\circ)}{V_d(C_\delta^\circ)}.$$

We start by estimating the volumes of the two polar bodies:

$$V_d(C_\delta^\circ) = \frac{1}{d}\int_{\mathbb{S}^{d-1}}\frac{1}{h_{C_\delta}^d(\xi)}\,d\mu(\xi) = V_d(K^\circ) + \frac{2c_d\delta^{\frac{2}{d+1}}}{d+3}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d+1]{\mathcal{K}(\xi)}}{h_K^{d+1}(\xi)}\,d\mu(\xi) + o(\delta^{\frac{2}{d+1}}), \tag{4}$$

and

$$(5)\quad V_d(K_\delta^\circ) = \frac{1}{d}\int_{\mathbb{S}^{d-1}} \frac{1}{h_{C_\delta}^d(\xi)}\,d\mu(\xi) = V_d(K^\circ) + c_d\delta^{\frac{2}{d+1}} \int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,d\mu(\xi) + o(\delta^{\frac{2}{d+1}}).$$

Consequently,

$$(6)\qquad \lambda^{-d} = 1 + c_d\delta^{\frac{2}{d+1}}\frac{d+1}{d+3}\,\frac{1}{V_d(K^\circ)}\int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,d\mu(\xi) + o(\delta^{\frac{2}{d+1}}),$$

and, thus,

$$(7)\qquad \lambda = 1 - c_d\delta^{\frac{2}{d+1}}\frac{d+1}{d(d+3)}\,\frac{1}{V_d(K^\circ)}\int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,d\mu(\xi) + o(\delta^{\frac{2}{d+1}}).$$

The homothety $h_{K_\delta}(\xi) = \lambda h_{C_\delta}(\xi)$, together with (3) and (7) imply that, for each $\xi \in \mathbb{S}^{d-1}$,

$$(8)\qquad \sqrt[d+1]{\mathcal{K}} = \frac{1}{d}\,\frac{h_K(\xi)}{V_d(K^\circ)}\int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,d\mu(\xi) + o(\delta^{\frac{2}{d+1}}).$$

Recall now a few elements of the dual mixed volume theory. For any two star bodies $L_{1,2}$ in $\mathbb{R}^d$ containing the origin, and any $i \in \mathbb{R}$, the i-th dual mixed volume of $L_{1,2}$ is defined (see [12]) by

$$\tilde{V}_i(L_1, L_2) = \frac{1}{d}\int_{\mathbb{S}^{d-1}} \rho_{L_1}^{d-i}\rho_{L_2}^{i}(\xi)\,d\mu(\xi),$$

where $\rho_{L_j} : \mathbb{S}^{d-1} \to (0,\infty)$ denotes the radial function of the star body L_j, $j = 1, 2$. Of particular interest for us is the case $i = -1$ for which it follows, via Hölder's inequality, the mixed volume inequality

$$\tilde{V}_{-1}^d(L_1, L_2) \geq V_d^{d+1}(L_1)V_d^{-1}(L_2).$$

Equality is reached above if and only if L_1 and L_2 are dilates of each other and, so, the radial functions ρ_{L_1} and ρ_{L_2} are multiples of each other. Therefore, given that the reciprocal of the support function of a convex body K containing the origin is the radial function of its polar body, ρ_{K°, equation (8) can be re-written in terms of a dual mixed volume and radial functions as follows

$$(9)\qquad \rho_{L^\circ}^{-1}(\xi) = \rho_{K^\circ}^{-1}(\xi)\cdot \tilde{V}_{-1}(K^\circ, L^\circ)V_d(K^\circ)^{-1} + o(\delta^{\frac{2}{d+1}}),$$

where L° is the star body whose radial function is, pointwise on $\mathbb{S}^{d-1}$, equal to $1/\sqrt[d+1]{\mathcal{K}}$.

Due to the dual mixed volume inequality above, if we assume that L° is not a dilation of K°, the inequality is strict and (9) implies, after re-arranging terms, that, for δ sufficiently small

$$(10)\qquad \frac{\rho_{L^\circ}(\xi)}{V_d(L^\circ)} < \frac{\rho_{K^\circ}(\xi)}{V_d(K^\circ)}.$$

We thus have two star bodies of unitary volume, namely $\frac{1}{V_d^{1/d}(K^\circ)}K^\circ$ and $\frac{1}{V_d^{1/d}(L^\circ)}L^\circ$, such that one is enclosed in the other. Unless the two bodies coincide, this is impossible.

Therefore, equality holds in (9) and L° is homothetic to K°. Thus, there exists $\lambda' > 0$ such that $\rho_{K^\circ}(\xi) = \lambda' \rho_{L^\circ}(\xi)$ for all $\xi \in \mathbb{S}^{d-1}$. This means that the convex body K is such that its support and, respectively, curvature functions satisfy pointwise on the unit sphere the equation

$$\lambda' h_K(\xi) = \sqrt[d+1]{\mathcal{K}}(\xi), \quad \forall \xi \in \mathbb{S}^{d-1}. \tag{11}$$

The fact that a convex body K satisfying (11) must be an ellipsoid is precisely the conclusion of the renowned Petty's Lemma in [13], concluding also our proposition.

□

3. Affine surface area bodies

In a similar manner with the definition of the floating body of a convex body K, we consider here the affine surface area body of a smooth convex body with strictly positive curvature as the intersection of half planes that cut a cap of given affine area from K. Note that, since along the portion of hyperplane included in K the affine surface area is zero, the affine surface area of the cap is precisely the affine surface area of the boundary of K that lies in $H^+_{t(\xi)}(\xi)$.

Definition 3.1. Let K be a smooth convex body in $\mathbb{R}^d$ with strictly positive Gauss curvature, with total affine surface area $\Omega(K)$, and let δ be a positive real number, $0 < \delta < \Omega(K)/2$. Then

$$A_\delta = \bigcap_{\xi \in \mathbb{S}^{d-1}} H^-_{t(\xi)}(\xi),$$

where $t(\xi)$ is such that

$$\Omega(K \cap H^+_t(\xi)) = \delta.$$

□

Interestingly, in the recent preprint [20]mentioned earlier, Zawalski studies the question of characterizing the homothety of a planar floating body of a sufficiently smooth convex body K of arbitrary density for which any three consecutive chords of K tangent to K_δ close up to form a triangle. Zawalski's result concludes that K is an ellipsoid and, in the process, it is shown that any three such chords divide the affine length of K in three equal parts.

In the context of our newly defined convex construction, the above condition implies that K_δ is also $A_{\Omega(K)/3}$ for that particular density δ of K. However, as we will show next, except for ellipsoids, even in the plane, the floating body is not, in general, an affine surface area body of K.

Lemma 3.2. *The floating body K_δ of a convex body K is not, in general, the affine surface area body $A_{\delta'}$ of K for some, possibly different, δ'.*

Proof. To see this, we consider the planar convex body K which is the convex hull of two arcs of circle of radius 1, and respectively 2, with centers one unit apart, whose flat sides are slightly smoothed to obtain a convex body with strictly positive curvature, as in Figure 1 below.

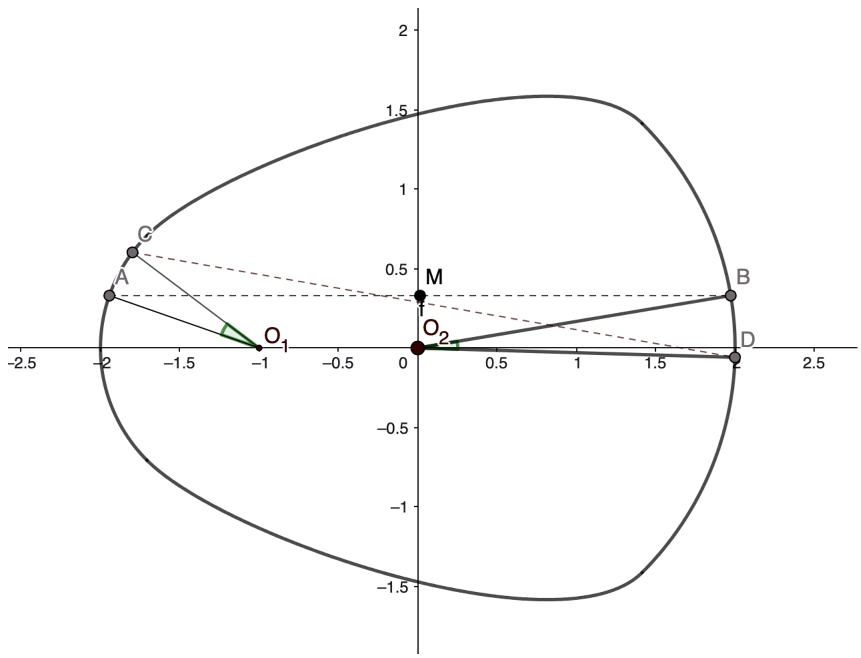

FIGURE 1. A planar convex body for which $A_{\delta'}$ is not a floating body

We know that, for any δ, the floating body K_δ of K is such that the chords cutting an area δ from K are tangent to K_δ at their midpoint. However, as we construct the affine length body $A_{\delta'}$ of K as the intersection of halfspaces cutting a fixed proportion of affine length from ∂K, the midpoint of some chords lies in the complement of the intersection.

Indeed, suppose that the chords AB and CD of the family that determine $A_{\delta'}$, for some $\delta' > 0$, have their end points on the respective arcs of the circle. As each chord cuts the same proportion of affine length from the boundary of K, the arcs AC and, respectively, BD have equal affine length. By direct calculation, the affine length of an arc of circle of radius r is $(r)^{\frac{2}{3}}\alpha$ where the angle α is the angle at the center of the circle subtending the arc.

This means that having chosen a fixed point C on ∂K, we can find D by comparing the affine lengths of the two arcs and evaluating the appropriate angle subtending BD. Since we choose the radius of the larger circle to be twice the radius of the smaller one, we get the following relation between the measures of the two angles at the centers of the circles subtending the arcs

$$m(\angle AO_1C) = \sqrt[3]{4}\, m(\angle AO_2C).$$

Using this, we can find the appropriate bounding halfspace determining A_δ. Using *GeoGebra*, [8], we find that the midpoint M of the horizontal chord AB is above the second chord, implying that $A_{\delta'}$ is not a floating body because it does not contain that point. The midpoint calculation can also be performed directly in this situation.

□

Finally, it is nice to note that for any fraction of affine length $\Omega(K)/m$, where $m \geq 3$ integer, we can consider the intersection of all m-polygons that subtend equal affine length from the boundary of a planar smooth convex body K with positive Gauss curvature to obtain $A_{\Omega(K)/m}$. In this sense, the smooth strictly convex body K resembles the caustic of an affine invariant billiard for which the polygons are periodic orbits, which is a potential topic of research that we would like to pursue.

Here, we address further the homothety question between K and one of its affine surface area body.

Theorem 3.3. *Let $K \subset \mathbb{R}^d, d \geq 2$, be a smooth convex body with strictly positive Gauss curvature. If for some sufficiently small $\delta > 0$, the body A_δ is homothetic to K, then K is an ellipsoid.*

Proof. We will be considering the earlier approximation of the boundary of the convex body K which, after a rigid motion, is described locally around a point p where the Gauss curvature of the boundary is $\mathcal{K}$, up to second order terms by a paraboloid P of position vector

$$X(x_1, \ldots, x_{d-1}) = \left(\underbrace{x_1, \ldots, x_{d-1}}_{x}, -\frac{1}{2}\mathcal{K}^{\frac{1}{d-1}} \left(x_1^2 + \cdots + x_{d-1}^2\right) + o(|x|^2) \right).$$

Up to little o, we have that, for any $i = 1, \ldots, d-1$,

$$X_i = \frac{\partial X}{\partial x_i} = e_i - \mathcal{K}^{\frac{1}{d-1}}\, x_i e_d,$$

where e_i denotes the standard i-th vector of the orthonormal basis in $\mathbb{R}^d$.

Therefore, the surface area element of the smooth convex body is, locally, up to a small error, that of the approximating paraboloid

$$dS = \sqrt{1 + \mathcal{K}^{\frac{2}{d-1}} |x|^2}\; dx_1 \cdots dx_{d-1}. \tag{12}$$

If f denotes the function $f(x) = -\frac{1}{2}\mathcal{K}^{\frac{1}{d-1}}(x_1^2 + \cdots + x_{d-1}^2)$, the Gauss curvature of the approximating paraboloid with the same (local) orientation as K is given by

$$\tilde{\mathcal{K}}(x) = \frac{\det(\text{Hess}(f))}{(1+|\nabla f|^2)^{(d+1)/2}} = \frac{\mathcal{K}}{\left(1+\mathcal{K}^{\frac{2}{d-1}}(x_1^2+\cdots+x_{d-1}^2)\right)^{(d+1)/2}}. \tag{13}$$

Recall that the affine surface area of convex bodies is not a continuous functional in general, but it is upper semi-continuous on the class of all d-dimensional convex bodies. Here, we are in an even more restrictive class of smooth convex bodies with strictly positive Gauss curvature. Let $\sigma : U \to \mathbb{R}^d$ be a smooth chart around the point p of the boundary of K and let

$$R_\delta = \{x = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} \mid (x_1 - x_1^0)^2 + \ldots + (x_{d-1} - x_{d-1}^0)^2 \le \delta^2\}$$

with centre $x^0 = (x_1^0, \dots, x_{d-1}^0)$ which we may take to be 0, $\sigma(x_1^0, \dots, x_{d-1}^0) = p$, and radius δ is contained in U. Then

$$\mathcal{K}(x^0) - \varepsilon \;<\; \mathcal{K}(x) \;<\; \mathcal{K}(x^0) + \varepsilon \qquad \text{whenever } x \in R_\delta,$$

so

$$\begin{aligned}\delta &= \int_{K\cap H^+_{t(\xi)}(\xi)} \mathcal{K}^{\frac{1}{d+1}}(u)\; dS_K(u) \approx \int_{P\cap H^+_{t(\xi)}(\xi)} \mathcal{K}_P(x)^{\frac{1}{d+1}}\, dS_P(x)\\ &= \int_{P\cap H_{t(\xi)}(\xi)} \left(\frac{\mathcal{K}}{(1+\mathcal{K}^{\frac{2}{d-1}}|x|^2)^{\frac{d+1}{2}}}\right)^{\frac{1}{d+1}} \sqrt{1+\mathcal{K}^{\frac{2}{d-1}}|x|^2}\, dx,\\ &= \omega_{d-1} R^{d-1} \mathcal{K}^{\frac{1}{d+1}},\end{aligned}$$

where R is the radius of $P \cap H_t(\xi)$. Solving for R, we get that

$$R \approx \frac{1}{\omega_{d-1}^{\frac{1}{d-1}}} \delta^{\frac{1}{d-1}} \mathcal{K}^{-\frac{1}{d^2-1}}, \tag{14}$$

and multiplying by $\frac{1}{2}\mathcal{K}^{1/(d-1)}$ to estimate the height of the cap, we obtain

$$h \approx \frac{1}{2}\,\omega_{d-1}^{-\frac{2}{d-1}}\, \delta^{\frac{2}{d-1}}\, \mathcal{K}^{\frac{1}{d-1}-\frac{2}{d^2-1}}.$$

Having estimated the distance from the hyperplane to the boundary point of outer normal u to be $c_d \delta^{\frac{2}{d-1}} \sqrt[d+1]{\mathcal{K}}$, where $c_d = \frac{1}{2}\left(\omega_{d-1}^{\frac{2}{d-1}}\right)^{-1}$, we conclude the following relation between the support functions of K and that of A_δ in the direction of the unit outer normal u at the point p:

$$h_{A_\delta}(u) = h_K(u) - c_d \delta^{\frac{2}{d-1}} \sqrt[d+1]{\mathcal{K}}(u) + o(\delta^{\frac{2}{d-1}}). \tag{15}$$

Therefore, by the same argument used in [4], the homothety hypothesis implies that there exists $\lambda > 0$ such that in all directions $u \in \mathbb{S}^{d-1}$ we have

$$\lambda h_K(u) = \sqrt[d+1]{\mathcal{K}}(u),$$

which by Petty's Lemma, [13], implies the desired result that K is an ellipsoid.

□

Lemma 3.4. *Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature and let A_δ denote the δ-affine surface area body of K for δ in some small interval $(0, \delta_0)$. Then, the affine surface area of K, $\Omega(K)$, satisfies*

$$\Omega(K) = \frac{1}{c_d} \lim_{\delta \searrow 0} \frac{V_d(K) - V_d(A_\delta)}{\delta^{\frac{2}{d-1}}}. \tag{16}$$

Proof. Let f be a continuous function on the unit sphere $\mathbb{S}^{d-1}$. Suppose that f defines a perturbation of a convex body L containing the origin in its interior to be the convex body, denoted L_t, of support function $h_t(u) = h_L(u) + tf(u), \forall u \in \mathbb{S}^{d-1}$, where $t \in (-\delta, \delta)$ is taken to be very small so that h_t remains positive. Aleksandrov, [1], showed that the following variational formula holds

$$\frac{dV_d(L_t)}{dt} = \lim_{t \to 0} \frac{V_d(L_t) - V_d(L)}{t} = \int_{\mathbb{S}^{d-1}} f(u)\, dS_L(u). \tag{17}$$

In our case, we have that $h_{A_\delta} = h_K + tf + o(t)$, where $t = \delta^{\frac{2}{d-1}} > 0$ and $f = \sqrt[d-1]{K}$. This approach was already considered, and proved to still hold even if h_t is not a support function, by Leichtweiss in [10], [11]. The reader may look also at the discussion of this point in [4]. So, the lemma follows as a direct consequence of Aleksandrov variational formula and equation (17).

□

Lastly, following an idea of [18], we now prove that the affine surface area body increases the volume product.

Theorem 3.5. *Let $K \subset \mathbb{R}^d$ be a smooth convex body with strictly positive Gauss curvature containing the origin in its interior. There exists a constant δ_K such that for any $\delta \in (0, \delta_K)$, the inequality*

$$V_d(A_\delta) \cdot V_d(A_\delta^\circ) \geq V_d(K) \cdot V_d(K^\circ)$$

holds, with equality if and only if K is an ellipsoid.

Proof. To simplify the notation, we will denote by $t := \delta^{\frac{2}{d+1}}$, $h(u) := h_K(u)$, and $h_t(u) := h_{A_\delta}$. This transforms our equation (17) into

$$h_t(u) = h(u) - c_d t \sqrt[d+1]{\mathcal{K}(u)} + o(t),$$

which implies that

$$V_d(A_\delta^\circ) = \frac{1}{d}\int_{S^{d-1}} h_t(u)^{-d}\, d\mu_{S^{d-1}} = V_d(K^\circ) + tc_d \int_{S^{d-1}} h^{-(d+1)}\ \sqrt[d+1]{\mathcal{K}(u)}\, d\mu_{S^{d-1}} + o(t),$$

where the last equality comes from a binomial expansion. On the other hand, the previous lemma that we proved also gives us that

$$V_d(A_\delta) = V_d(K) - c_d\, \Omega(K)\, t + o(t),$$

implying that

$$V_d(A_\delta)\cdot V_d(A_\delta^\circ) = V_d(K)\cdot V_d(K^\circ) + tc_d\left[V_d(K)\int_{S^{d-1}} h^{-(d+1)}\ \sqrt[d+1]{\mathcal{K}(u)}\, d\mu_{S^{d-1}} - V_d(K^\circ)\Omega(K)\right] + o(t).$$

We thus get that

$$\frac{1}{c_d}\lim_{t\to 0^+}\frac{V_d(A_\delta)\cdot V_d(A_\delta^\circ) - V_d(K)\cdot V_d(K^\circ)}{t} =$$

$$V_d(K)\int_{S^{d-1}} h^{-(d+1)}\ \sqrt[d+1]{\mathcal{K}(u)}\, d\mu_{S^{d-1}} - \Omega(K)V_d(K^\circ),$$

and we can rewrite it as follows

$$= V_d(K^\circ)\int_{S^{d-1}} h^{-(d+1)}\ \sqrt[d+1]{\mathcal{K}(u)}\, d\mu_{S^{d-1}} \left[\frac{\overbrace{\frac{1}{d}\int_{S^{d-1}} h\mathcal{K}(u)^{-1}\, d\mu_{S^{d-1}}}^{V_d(K)}}{\underbrace{\frac{1}{d}\int_{S^{d-1}} h^{-d}\, d\mu_{S^{d-1}}}_{V_d(K^\circ)}} - \frac{\overbrace{\int_{S^{d-1}} \mathcal{K}(u)^{\frac{-d}{d+1}} d\mu_{S^{d-1}}}^{\Omega(K)}}{\int_{S^{d-1}} h^{-(d+1)}\ \sqrt[d+1]{\mathcal{K}(u)}\, d\mu_{S^{d-1}}}\right].$$

Defining $g(u) = h^{d+1}(u)\mathcal{K}(u)^{-1}$, $F(x) = x^{\frac{-1}{d+1}}$ for $x > 0$ and $dw = h^{-d} d\mu_{S^{d-1}}$, we get that F is a positive and strictly decreasing function, and that dw is a volume form. With this, we can rewrite the expression in the brackets as

$$\frac{\int_{S^{d-1}} g dw}{\int_{S^{d-1}} dw} - \frac{\int_{S^{d-1}} F(g) g dw}{\int_{S^{d-1}} F(g) dw}.$$

For the resulting expression, we can use the generalized Hölder's inequality due to Andrews, [5] which gives us that its value is greater or equal to zero, and it is zero if only if g is constant. But, by Petty's Lemma again, [13] if g is constant, the body K is an ellipsoid. Therefore,

$$V_d(A_\delta)\cdot V_d(A_\delta^\circ) \geq V_d(K)\cdot V_d(K^\circ)$$

with equality if and only if K is an ellipsoid.

□

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